Approximation of the functions (Part One)

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Introduction

- Assume that we have a set of values $(x_i,\ y_i), \quad i = 1, 2, 3, \ldots, n$ of functions $y = f(x)$, the value of x being equally spaced i.e

$$x_i = x_0 + ih, \quad i = 1, 2, 3, \ldots, n.$$

$$x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \quad x_3 = x_0 + 3h, \ldots, x_n = x_0 + nh.$$

- Suppose that we are required to recover the values of $f(x)$ for some intermediate values of x, or to obtain the derivative of $f(x)$ for some x in the range $x_0 \leq x \leq x_n$.
- The methods for the solution of these problems are based on the concept of the differences of a function which we now proceed to define.

Interpolation formula

When the values of x are equally spaced as above, interpolation formula are based on three types of differences:

1. Forward differences;
2. Backward differences;
3. Central differences.

Forward differences

- If we subtract from each value of y (except y_0) the proceeding value of y, we get $y_1 - y_0, \quad y_2 - y_1, \quad y_3 - y_2, \ldots y_n - y_{n-1}$.
- If $y_0, y_1, y_2, \ldots, y_n$ denotes a set of values of y, then $y_1 - y_0, \quad y_2 - y_1, \quad y_3 - y_2, \ldots y_n - y_{n-1}$ are called the first differences of y.

Forward Differences

- If we subtract from each value of y (except y_0) the proceeding value of y, we get $y_1 - y_0, \quad y_2 - y_1, \quad y_3 - y_2, \ldots y_n - y_{n-1}$ respectively known as the first differences of y.
- Denoting these differences by $\triangle y_0, \quad \triangle y_1, \quad \triangle y_2, \quad \triangle y_3, \ldots \quad \triangle y_n$.
- We have: $\triangle y_0 = y_1 - y_0, \quad \triangle y_1 = y_2 - y_1, \quad \triangle y_2 = y_3 - y_2, \quad \triangle y_3 = y_4 - y_3, \ldots \quad \triangle y_n = y_n - y_{n-1}$.
- Where $\triangle$ is called the **forward difference operator,** and $\triangle y_0, \quad \triangle y_1, \quad \triangle y_2, \quad \triangle y_3, \ldots \quad \triangle y_n$ are called the first forward differences.
- In general, the first forward differences are given by

$$\triangle y_i = y_{i+1} - y_i, \quad i = 0, 1, 2, 3, \ldots, n.$$

Forward Differences

- The differences of the first forward differences are called the second forward differences and are denotes by

$$\triangle^2 y_0, \quad \triangle^2 y_1, \quad \triangle^2 y_2, \quad \triangle^2 y_3, \ldots \quad \triangle^2 y_n.$$

- Now, the second forward differences are defines as the differences of the first differences, that is:

$$\triangle^2 y_0 = \triangle(y_1 - y_0) = \triangle y_1 - \triangle y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0.$$

$$\triangle^2 y_1 = \triangle(y_2 - y_1) = \triangle y_2 - \triangle y_1 = (y_3 - y_2) - (y_2 - y_2) = y_3 - 2y_2 + y_1.$$

- In general

$$\triangle^2 y_n = \triangle y_{n+1} - \triangle y_n = y_{n+2} - 2y_{n+1} + y_n.$$

- Here, $\triangle^2$ is called the second forward difference operator.

Forward Differences

- Similarly, one can define **third forward differences are:**

$$\triangle^3 y_0 = \triangle^2 (\triangle y_0) = \triangle^2 y_1 - \triangle^2 y_0 = y_3 - 3y_2 + 3y_1 - y_0.$$

$$\triangle^3 y_1 = \triangle^2 (\triangle y_1) = \triangle^2 y_2 - \triangle^2 y_1 = y_4 - 3y_3 + 3y_2 - y_1.$$

- Generally, we have

$$\triangle^3 y_n = \triangle^2 (\triangle y_n) = \triangle^2 y_{n+1} - \triangle^2 y_n = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n.$$

- The **fourth forward differences** can be defined as

$$\begin{aligned} \triangle^4 y_0 &= \triangle^3 y_1 - \triangle^3 y_0 = (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0) \\ &= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 \end{aligned}$$

- Hence,

$$\triangle^4 y_n = \triangle^3 y_{n+1} - \triangle^3 y_n = y_{n+4} - 4y_{n+3} + 6y_{n+2} - 4y_{n+1} + y_n.$$

Forward Differences

- It is therefore, clear that any higher order forward differences can easily be expressed in terms of the ordinates, since the coefficients occurring on the right side are the binomial coefficients.
- Generally, we have

$$\triangle^3 y_n = \triangle^2 \left(\triangle y_n\right) = \triangle^2 y_{n+1} - \triangle^2 y_n = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n.$$

- In general, the n^{th} differences are defined as:

$$\triangle^n y_k = \triangle^{n-1} y_{k+1} - \triangle^{n-1} y_k.$$

- In function notation, the forward differences are as written below:

$$\begin{aligned} \triangle f(x) &= f(x+h) - f(x), \\ \triangle^2 f(x) &= f(x+2h) - 2f(x+h) + f(x), \\ \triangle^3 f(x) &= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x), \end{aligned}$$

and so on, where h is the interval of differencing.

Table of Forward Differences

The following table shows how the forward differences of all orders can be formed:

x	$y = f(x)$	$\triangle$	$\triangle^2$	$\triangle^3$	$\triangle^4$	$\triangle^5$
x_0	y_0					
x_1	y_1	$\triangle y_0$				
x_2	y_2	$\triangle y_1$	$\triangle^2 y_0$			
x_3	y_3	$\triangle y_2$	$\triangle^2 y_1$	$\triangle^3 y_0$		
x_4	y_4	$\triangle y_3$	$\triangle^2 y_2$	$\triangle^3 y_1$	$\triangle^4 y_0$	
x_5	y_5	$\triangle y_4$	$\triangle^2 y_3$	$\triangle^3 y_2$	$\triangle^4 y_1$	$\triangle^5 y_0$

The first term in the table y_0 is called the leading term and the differences $\triangle y_0, \triangle^2 y_0, \triangle^3 y_0, \ldots$ are called the leading differences. The above differences table is known as forward difference table or Diagonal difference table.

Properties of Forward Differences

- $\triangle(f(x) \pm g(x)) = \triangle f(x) \pm \triangle g(x)$ i.e $\triangle$ is linear.
- $\triangle(\alpha f(x)) = \alpha \triangle f(x)$, α being a constant.
- $\triangle^m \triangle^n f(x) = \triangle^{m+n} f(x)$, where m and n are positive integers.
- $\triangle [f(x).g(x)] \neq f(x).\triangle g(x)$.

<u>**Observation 1:**</u> We can express any higher order forward difference of y_0 in terms of the entire $y_0, y_1, y_2, \ldots, y_n$. From

$$\begin{aligned}
\triangle y_0 &= y_1 - y_o, \\
\triangle^2 y_0 &= y_2 - 2y_1 + y_0, \\
\triangle^3 y_0 &= y_3 - 3y_2 + 3y_1 - y_0, \\
\triangle^4 y_0 &= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0, \quad \textbf{and so on.}
\end{aligned}$$

Properties of Forward Differences

- We can see that the coefficients of the entries on the RHS are binomial coefficients. Therefore, in general

$$\triangle^n y_n = y_n - \binom{n}{1} y_{n-1} + \binom{n}{2} y_{n-2} - \binom{n}{3} y_{n-3} + \ldots + (-1)^n y_0. \quad (1)$$

- Observation 2: We can express any value of y in terms of leading entry y_0. We know that

$$y_1 - y_0 = \triangle y_0 \Rightarrow y_1 = y_0 + \triangle y_0; \quad \Rightarrow y_1 = (1+\triangle) y_0.$$

- Now,

$$y_2 - y_1 = \triangle y_1 \Rightarrow y_2 = y_1 + \triangle y_1; \Rightarrow y_2 = (1+\triangle) y_1; \quad y_2 = (1+\triangle)^2 y_0.$$

- Similarly, $y_3 = (1+\triangle)^3 y_0$ and so on. In general,

$$y_n = (1+\triangle)^n y_0 = y_0 + \binom{n}{1} \triangle y_0 + \binom{n}{2} \triangle^2 y_0 + \binom{n}{3} \triangle^3 y_0 + \ldots + \triangle^n y_0. \quad (2)$$

Backward Differences

- The differences $y_1 - y_0,\ y_2 - y_1,\ y_3 - y_2, \ldots, y_n - y_{n-1}$ are called first backward differences if they are denoted by $\nabla y_1,\ \nabla y_2,\ \nabla Y_3, \ldots, \nabla y_n$, respectively.
- So that

$$\begin{aligned} \nabla y_1 &= y_1 - y_0, \\ \nabla y_2 &= y_2 - y_1, \\ \nabla y_3 &= y_3 - y_2, \ldots \\ \nabla y_n &= y_n - y_{n-1}, \end{aligned}$$

where ∇ is the backward difference operator.

Backward Differences

- Now, the second backward differences are defined as the differences of the first differences, i.e

$$\begin{aligned}\nabla^2 y_2 &= \nabla(\nabla y_2) = \nabla(y_2 - y_1) \\ &= \nabla y_2 - \nabla y_1 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0. \\ \nabla^2 y_3 &= \nabla(\nabla y_3) = \nabla(y_3 - y_2) \\ &= \nabla y_3 - \nabla y_2 = (y_3 - y_2) - (y_2 - y_1) = y_3 - 2y_2 + y_1.\end{aligned}$$

- In general,

$$\nabla^n y_k = \nabla^{n-1} y_k - \nabla^{n-1} y_{k-1}. \tag{3}$$

Table of Backward Differences

The backward difference table is given as

x	$y=f(x)$	∇	∇^2	∇^3	∇^4	∇^5
x_0	y_0					
x_1	y_1	∇y_1				
x_2	y_2	∇y_2	$\nabla^2 y_2$			
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$		
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$	
x_5	y_5	∇y_5	$\nabla^2 y_5$	$\nabla^3 y_5$	$\nabla^4 y_5$	$\nabla^5 y_5$

In function notation, these are written as

$$\begin{aligned}
\nabla f(x) &= f(x) - f(x-h), \\
\nabla f(x+h) &= f(x+h) - f(x), \\
\nabla^2 f(x+2h) &= f(x+2h) - f(x+h) + f(x), \\
\nabla^3 f(x+3h) &= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x).
\end{aligned}$$

And so on, where h is the interval of differencing.

Example for Forward difference

Example

Construct a forward difference table from the following data

x	0	1	2	3	4
y	1	1.5	2.2	3.1	4.6

Evaluate $\triangle^3 y_1$, y_x, and y_5.

Solution

The forward difference table is given below.

x	y	$\triangle$	$\triangle^2$	$\triangle^3$	$\triangle^4$
$x_0 = 0$	$y_0 = 1$				
$x_1 = 1$	$y_1 = 1.5$	$\triangle y_0 = 0.5$			
$x_2 = 2$	$y_2 = 2.2$	$\triangle y_1 = 0.7$	$\triangle^2 y_0 = 0.2$		
$x_3 = 3$	$y_3 = 3.1$	$\triangle y_2 = 0.9$	$\triangle^2 y_1 = 0.2$	$\triangle^3 y_0 = 0$	
$x_4 = 4$	$y_4 = 4.6$	$\triangle y_3 = 1.5$	$\triangle^2 y_2 = 0.6$	$\triangle^3 y_1 = 0.4$	$\triangle^4 y_0 = 0.4$

Now,

$\triangle^3 y_1 = y_4 - 3y_3 + 3y_2 - y_1 = \triangle^2 y_2 - \triangle^2 y_1 = 4.6 - 3(3.2) + 3(2.2) - 1.5 = 0.4$

Solution

We known that

$y_x = y_0 + \binom{x}{1}\triangle y_0 + \binom{x}{2}\triangle^2 y_0 + \binom{x}{3}\triangle^3 y_0 + \binom{x}{4}\triangle^4 y_0,$

$y_x = y_0 + \frac{x!}{(x-1)!1!}\triangle y_0 + \frac{x!}{2!(x-2)!}\triangle^2 y_0 + \frac{x!}{3!(x-3)!}\triangle^3 y_0 + \frac{x!}{4!(x-4)!}\triangle^4 y_0,$

$y_x = 1+x(0.5)+\frac{1}{2}x(x-1)(0.2)+\frac{1}{3!}x(x-1)(x-2)(0)+\frac{1}{4!}x(x-1)(x-2)(x-3)(0.4)$

$$\begin{aligned} y_x &= 1 + \frac{x}{2} + \frac{1}{10}(x^2 - x) + \frac{1}{60}(x^4 - 6x^3 + 11x^2 - 6x) \\ \therefore y_x &= \frac{1}{60}\left[x^4 - 6x^3 + 17x^2 + 18x + 60\right] \end{aligned}$$

Therefore,

$$y_5 = \frac{1}{60}\left[(5)^4 - 6(5)^3 + 17(5)^2 + 18(5) + 60\right] = 7.5.$$

Example 2

Example

Find the polynomial of degree three which has the values equal to $1, 15, 85$ and 259 corresponding to the values $0, 2, 4$ and 6 of the argument.

Central Differences

- The central difference operator δ is defined by the relations

$$\begin{aligned} y_1 - y_0 &= \delta y_{\frac{1}{2}}, \\ y_2 - y_1 &= \delta y_{\frac{3}{2}}, \\ y_3 - y_2 &= \delta y_{\frac{5}{2}}, \ldots \\ y_n - y_{n-1} &= \delta y_{n-\frac{1}{2}}. \end{aligned}$$

- For the higher order central differences, we have

$$\begin{aligned} \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}} &= \delta^2 y_1, \\ \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}} &= \delta^2 y_2, \ldots \\ \delta^2 y_2 - \delta^2 y_1 &= \delta^3 y_{\frac{3}{2}}, \quad \textbf{and so on.} \end{aligned}$$

Table for Central Differences

The central differences are tabulated below.

x	$y=f(x)$	δ	δ^2	δ^3	δ^4	δ^5
x_0	y_0					
x_1	y_1	$\delta y_{\frac{1}{2}}$				
x_2	y_2	$\delta y_{\frac{3}{2}}$	$\delta^2 y_1$			
x_3	y_3	$\delta y_{\frac{5}{2}}$	$\delta^2 y_2$	$\delta^3 y_{\frac{3}{2}}$		
x_4	y_4	$\delta y_{\frac{7}{2}}$	$\delta^2 y_3$	$\delta^3 y_{\frac{5}{2}}$	$\delta^4 y_2$	
x_5	y_5	$\delta y_{\frac{9}{2}}$	$\delta^2 y_4$	$\delta^3 y_{\frac{7}{2}}$	$\delta^4 y_3$	$\delta^5 y_{\frac{5}{2}}$

We can see from the table that central differences on the same horizontal line have the same suffix. Also, all odd differences have a fractional suffix, and the even differences have an integer suffix.

Comparison of three types of differences

- Note 1: From the three tables, we can see that only the notation changes, not the differences. For example,

$$y_1 - y_0 = \triangle y_0 = \nabla y_1 = \delta y_{\frac{1}{2}}.$$

- Note 2: If we write $y = f(x)$ as $y = f_x$ or $y = y_x$, then the entries corresponding to x, $x+h$, $x+2h, \ldots$, are y_x, y_{x+h}, $y_{x+2h}, \ldots$, respectively, and

$$\triangle y_x = y_{x+h} - y_x, \quad \triangle^2 y_x = \triangle y_{x+h} - \triangle y_x, \quad \text{and so on.}$$

- Similarly,

$$\begin{aligned} \nabla y_x &= y_x - y_{x-h}, \\ \delta y_x &= y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}, \quad \text{and so on.} \end{aligned}$$

Exercise II.1

1. Prove the following results:
   - $\triangle\nabla = \nabla\triangle = \triangle - \nabla\delta^2$
   - $\triangle + \nabla = \frac{\triangle}{\nabla} - \frac{\nabla}{\triangle}$
2. Tabulate the forward differences for the given data

x	1	2	3	4	5	6	7	8	9
y	1	8	27	64	125	216	343	512	729

3. Form a backward differences table of the function $f(x) = x^3 - 3x^2 - 5x - 7$ for $x = -1, 0, 1, 2, 3, 4, 5$.
4. Show that
   - $y_3 = y_2 + \triangle y_1 + \triangle^2 y_0 + \triangle^3 y_0$.
   - $\triangle^2 y_8 = y_8 - 2y_7 + y_6$.
5. If $y_0 = 3, y_1 = 12, y_2 = 81, y_3 = 2000, y_4 = 100$ show that $\triangle^4 y_0 = -7459$.

Interpolation with equal intervals

Definition

- Interpolation is a technique of obtaining the value of a function for any intermediate values of the independent variable i.e argument within an interval, when the values of the arguments are given.
- Here, x(argument): $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots, x_n$ and y_x(entry): $y_0, y_1, y_2, \ldots, y_n$.
- The process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called interpolation.
- The process of finding the value of a function outside the given range of arguments is called extrapolation. However, the term **interpolation** is applied to both process.

Newton's Forward Interpolation Formula

- Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots, y_n$ for $(n+1)$ values of $x_0, x_1, x_2, \ldots, x_n$, of the independent variable x (argument).
- Let these values of x be equidistant, i.e $x_i = x_0 + ih$, $i = 0, 1, 2, 3, \ldots, n$ and let $y(x)$ be the polynomial in x of n^{th} degree, such that $y_i = f(x_i)$, $i = 0, 1, 2, 3, \ldots, n$. **Suppose we need to evaluate $y(x)$ near the beginning of table of values.**
- Therefore,
  $y(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)(x - x_1) + A_3(x - x_0)(x - x_1)(x - x_2)$
  $+ \ldots + A_n(x - x_0)(x - x_1)\ldots(x - x_n)$.
- Putting $x = x_0, x_1, x_2, \ldots, x_n$ successively in the above equation , we get
  $y_0 = A_0, \quad y_1 = A_0 + A_1(x_1 - x_0)$,
  $y_2 = A_0 + A_1(x_2 - x_0) + A_2(x_2 - x_1)(x_2 - x_1)$ and so on.

Cont...

- From these

$$
\begin{aligned}
A_0 &= y_0 \\
A_1 &= \frac{y_1 - A_0}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\triangle y_0}{h} \\
A_2 &= \frac{y_2 - A_0 - A_1(x_2 - x_0}{(x_2 - x_0)(x_2 - x_1} = \frac{y_2 - y_0 - A_1(2h)}{2h^2} \\
&= \frac{y_2 - y_0 - \frac{\triangle y_0}{h}(2h)}{2h^2} = \frac{y_2 - y_0 - 2\triangle y_0}{2h^2} \\
&= \frac{y_2 - y_0 - 2(y_1 - y_0)}{2h^2} = \frac{y_2 - 2y_1 + y_0}{2h^2} \\
\therefore A_2 &= \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{1}{2!h^2}\triangle^2 y_0
\end{aligned}
$$

Cont...

- Similarly, $A_3 = \frac{1}{3!h^3}\triangle^3 y_0,$ and so on.
- Putting these values in the equation of $y(x)$, we get $y(x) = y_0 + \frac{\triangle y_0}{h}(x-x_0) + \frac{\triangle^2 y_0}{2!h^2}(x-x_0)(x-x_1) + \frac{\triangle^3 y_0}{3!h^3}(x-x_0)(x-x_1)(x-x_2)\ldots$
- Putting $p = \frac{x-x_0}{h}$, i.e $x = x_0 + ph.$ where p is a real number, above equation takes the form $y_p = y_0 + p\triangle y_0 + \frac{p(p-1)}{2!}\triangle^2 y_0 + \frac{p(p-1)(p-2)}{3!}\triangle^3 y_0 + \ldots + \frac{p(p-1)(p-2)\ldots(P-(n-1))}{n!}\triangle^n y_0.$
- Where $y_p = y(x_0 + ph).$ This equation is known as **Newton's Forward Interpolation formula.**

Newton's Backward Interpolation Formula

- Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \dots, y_n$ for $(n+1)$ values of $x_0, x_1, x_2, \dots, x_n$, of the independent variable x.
- Let these values of x be equidistant, i.e $x_i = x_0 + ih$, $i = 0, 1, 2, 3, \dots, n$ and let $y(x)$ be the polynomial in x of n^{th} degree, such that $y_i = f(x_i)$, $i = 0, 1, 2, 3, \dots, n$.
- Suppose that it is required to evaluate $y(x)$ near the end of the table of values, then we can assume that

$$y(x) = A_0 + A_1(x - x_n) + A_2(x - x_n)(x - x_{n-1}) + \dots + A_n(x - x_n)(x - x_{n-1}) \dots (x - x_1 \quad (4)$$

- Putting $x = x_n, x_{n-1}, x_{n-2}, \dots, x_1$ successively in (4), we get

$$\begin{aligned} y_n &= y(x_n) = A_0, \\ y_{n-1} &= y(x_{n-1}) = A_0 + A_1(x_{n-1} - x_n) \\ y_{n-2} &= A_0 + A_1(x_{n-2} - x_n) + A_2(x_{n-2} - x_n)(c_{n-2} - x_{n-1}), \end{aligned}$$

and so on. These equations give

Newton's Backward Interpolation Formula

- These equations give

$$
\begin{aligned}
A_0 &= y_n, \\
A_1 &= \frac{y_{n-1} - A_0}{x_{n-1} - x_n} = \frac{y_{n-1} - y_n}{x_{n-1} - x_n} = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} = \frac{\nabla y_n}{h}, \\
A_2 &= \frac{y_{n-2} - y_n - A_1(x_{n-2} - x_n)}{(x_{n-2} - x_n)(x_{n-2} - x_{n-1})} = \frac{y_{n-2} - y_n - A_1(-2h)}{2h^2} \\
&= \frac{y_{n-2} - y_n - \frac{\nabla y_n}{h}(-2h)}{2h^2} = \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2} \\
\therefore A_2 &= \frac{1}{2!h^2}\nabla^2 y_n.
\end{aligned}
$$

- Similarly,

$$A_3 = \frac{1}{3!h^3}\nabla^3 y_n, \quad \text{and so on.}$$

Newton's Backward Interpolation Formula

- Putting these values in Equation (4), we have
$y(x) = y_n + \frac{1}{h}(x - x_n)\nabla y_n + \frac{1}{2!h^2}\nabla^2 y_n(x - x_n)(x - x_{n-1}) + \frac{1}{3!h^3}\nabla^3 y_n(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \dots$
- Let $p = \frac{x - x_n}{h}$, i.e $x = x_n + ph$, where p is a real number. Then the above equation takes the form
$y_p =$
$y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots + \frac{p(p+1)(p+2)\dots(p+n-1)}{n!}\nabla^n y_n,$
- where $y_p = y(x_n + ph)$, This equation is known as **Newton's Backward Interpolation Formula.**
- **Note:** Since the formula involves the backward differences, it is called backward interpolation formula since is used to interpolate the values of y near the end of a set of tabular values.

Example 3

Example

The following data give I, the indicated HP and V, the speed in knots developed by ship.

V	8	10	12	14	16
I	1000	1900	3250	5400	8950

Find I when $V = 9$, using Newton's forward interpolation formula.

Solution

- We note that $v = 9$ is near to the beginning of the table. Hence, to get the corresponding I, we use Newton's forward interpolation formula. The forward differences are calculated and tabulated as follows:

V	I	$\triangle$	$\triangle^2$	$\triangle^3$	$\triangle^4$
8	1000				
10	1900	900			
12	3250	1350	450		
14	5400	2150	800	350	
16	8950	3550	1400	600	250

Solution

- Here, $V_0 = 8$, $I_0 = 1000$, $\triangle I_0 = 900$, $\triangle^2 I_0 = 450$, $\triangle^3 I_0 = 350$, $\triangle^4 I_0 = 250$. Hence, the interpolation polynomial will be of degree 4. That is

$$I = I_0 + p\triangle I_0 + \frac{p(p-1)}{2!}\triangle^2 I_0 + \frac{p(p-1)(p-2)}{3!}\triangle^3 I_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\triangle^4 I_0$$

- Let I_p be the value of I when $V = 9$. Then $p = \frac{V-V_0}{h} = \frac{9-8}{2} = \frac{1}{2} = 0.5$. Therefore,

$$I_p = 1000+(0.5)(900)+\frac{(0.5)(0.5-1)}{2!}(450)+\frac{(0.5)(0.5-1)(0.5-2)}{3!}(350) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}(250) = 1405.8594$$

Example 4

Example

The amount A of a substance remaining in a reaction system after an interval of time t in a certain chemical experiment is tabulated below:

$t(min)$	2	5	8	11
$A(gm)$	94.8	87.9	81.3	75.1

Obtain the value of A where $t = 9$ using Newton's backward interpolation formula.

Solution

- Since the value of $t = 9$ is near the end of the table, to get the corresponding value of A we use Newton's backward interpolation formula.
- The backward differences are calculated and tabulated below.

t	A	∇	∇^2	∇^3
2	94.8			
5	87.9	-6.9		
8	81.3	-6.6	0.3	
11	75.1	-6.2	0.4	0.1

- Here,

$$t_n = 11, A_n = 75.1, \nabla A_n = -6.2, \nabla^2 A_n = 0.4, \nabla^3 A_n = 0.1$$

Solution

- Hence, the interpolation polynomial will be of degree 3. That is

$$A = A_n + p\nabla A_n + \frac{p(p+1)}{2!}\nabla^2 A_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 A_n,$$

- let A_p be the value of A when $t = 9$. Then

$$p = \frac{t - t_n}{h} = \frac{9 - 11}{3} = -\frac{2}{3}.$$

- Therefore,

$$\begin{aligned} A_p = 75.1 &+ \left(\frac{-2}{3}\right)(-6.2) + \frac{1}{2!}\left(\frac{-2}{3}\right)\left(\frac{-2}{3}+1\right)(0.4) \\ &+ \frac{1}{3!}\left(\frac{-2}{3}\right)\left(\frac{-2}{3}+1\right)\left(\frac{-2}{3}+2\right)(0.1) \end{aligned}$$

$$A_p = 79.183951.$$

Example 5

Example

Find a polynomial which takes the following data

x	1	3	5	7	9	11
y	3	14	19	21	23	28

and hence compute y_x at $x = 2, 12$.

Solution

- The forward differences table is given by

x	y	$\triangle$	$\triangle^2$	$\triangle^3$	$\triangle^4$
1	3				
3	14	11			
5	19	5	-6		
7	21	2	-3	3	
9	23	2	0	3	0
11	28	5	3	3	0

- Taking

$$x_0 = 1, y_0 = 3, p = \frac{x - x_0}{h} = \frac{x - 1}{2}.$$

Cont....

- Using Newton's forward interpolation formula, we get

$$y_p = y_0 + p\triangle y_0 + \frac{1}{2!}p(p-1)\triangle^2 y_0 + \frac{1}{3!}p(p-1)(p-2)\triangle^3 y_0 + \frac{1}{4!}p(p-1)(p-2)(p-3)\triangle^4 y_0$$

$$= 3 + \frac{x-1}{2}(11) + \frac{1}{2!}\frac{x-1}{2}\left(\frac{1}{2}(x-1)-1\right)(-6) + \frac{1}{3!}\frac{1}{2}(x-1)\left[\frac{1}{2}(x-1)-1\right]\left[\frac{1}{2}(x-1)-2\right](3)$$

$$\begin{aligned} y_p &= 3 + \frac{11}{2}(x-1) - \frac{3}{4}(x^2-4x+3) + \frac{1}{16}(x^3-9x^2+23x-15) \\ \therefore y_p &= \frac{1}{16}\left[x^3-21x^2+159x-91\right]. \end{aligned}$$

Cont....

- Again take $x_n = 11, y_n = 28, p = \frac{x-11}{2}$. Using Newton's backward interpolation formula,

$$y_p = y_n + p\nabla y_n + \frac{1}{2!}p(p+1)\nabla^2 y_n + \frac{1}{3!}p(p+1)(p+2)\nabla^3 y_n$$

$$\begin{aligned} &= 28 + \frac{5}{2}(x-11) + \frac{1}{2!}\frac{1}{2^2}(x-11)(x-9)(3) \\ &\quad + \frac{1}{3!}\frac{1}{2^3}(x-11)(x-9)(x-7)(3) \end{aligned}$$

$$\begin{aligned} &= 28 + \frac{5}{2}(x-11) + \frac{1}{16}(x-11)(x-9)(x-1) \\ \therefore y_p &= \frac{1}{16}\left(x^3 - 21x^2 + 159x - 91\right). \end{aligned}$$

Cont....

- So we can use any one of the formula to find the polynomial. Therefore,

$$y_x = \frac{1}{16}\left(x^3 - 21x^2 + 159x - 91\right).$$

- Now,

$$y_2 = \frac{1}{16}\left((2)^3 - 21(2)^2 + 159(2) - 91\right) = 9.4375.$$
$$y_{12} = \frac{1}{16}\left((12)^3 - 21(12)^2 + 159(12) - 91\right) = 32.5625.$$

Equidistant Terms with one or more Missing Values

Definition

When one or more of the values of the function $y = f(x)$ corresponding to the equidistant values of x are missing. We can find these missing values using finite difference operator E and $\triangle$. The method is best illustrated by the following example.

Example

Find the missing value in the following table.

x	16	18	20	22	24	26
y	43	89	-	155	268	388

Solution

Since five values are given it is possible to express y as a polynomial of fourth degree.

- Hence, the fifth differences of y are zeros. Taking the origin for x at 16, from the given table we have
  $y_0 = 43,\ y_1 = 89,\ y_3 = 155,\ y_4 = 268,\ y_5 = 388$ and we have to find y_2.
- We know that $\triangle^5 y_0 = 0$ for all values of x,

$$\triangle^5 y_0 = 0 \quad \text{i.e} \quad (E-1)^5 y_0 = 0.$$

$$\text{i.e} \quad \left(E^5 - \binom{5}{1}E^4 + \binom{5}{2}E^3 - \binom{5}{3}E^2 + \binom{5}{4}E - 1\right) y_o = 0$$

$$\text{or} \quad \left(E^5 - 5E^4 + 10E^3 - 10E^3 + 5E - 1\right) y_0 = 0$$

$$\text{i.e} \quad E^5 y_0 - 5E^4 y_0 + 10E^3 y_0 - 10E^2 y_0 + 5E y_0 - y_0 = 0.$$

Solution Cont...

- Hence, $y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$
- Substituting the given values,

$$388 - 5(268) + 10(155) - 10y_2 + 5(89) - 43 = 0 \quad \Rightarrow y_2 = 100.$$

- Therefore,

x	16	18	20	22	24	26
y	43	89	100	155	268	388

Example

Example

Find the missing values in the following table of values of x and y.

x	0	1	2	3	4	5	6
y	-4	-2	-	-	220	546	1148

Hint: There being given five values and two missing values, we may have $\triangle^5 y_0 = 0$ and $\triangle^6 y_0 = 0$.

Exercise II.2

1. From the following data find y at $x = 43$ using Newton's forward interpolation formula.

x	40	50	60	70	80	90
y	184	204	226	250	176	304

2. The population of a certain town in decennial census was as given below. Estimate the population for the year 1895.

Year (x)	1891	1901	1911	1921
Population in thousands (y)	46	66	81	101

3. The area A of a circle of diameter d is given for the following values

d	80	85	90	95	100
A	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter $d = 105$.

Exercise II.2

4. From the following table, estimate the values of $f(22)$ and $f(42)$.

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

5. Find the polynomial which takes the following data

x	4	6	8	10
y	1	3	8	16

Hence, calculate y at $x = 5$.

6. Obtain the estimate of the missing value in the following table

x	1	2	3	4	5
y	2	5	7	-	32

7. Given $y_0 = 3, \quad y_1 = 12, \quad y_2 = 81, \quad y_3 = 200, \quad y_4 = 100$. Find $\triangle^4 y_0$ without forming the difference table.

Central Difference Interpolation Formula

Definition

The central difference formula are most suited for **interpolation near the middle for a tabulated set.** The most important central difference formula are those due to Stirling, Bassel and Everett.

For convenience, we state the central difference formula by taking the central ordinate as y_0 corresponding to $x = x_0$: