# ME303M: Heat and Mass Transfer: Fall 2022

## Analysis of Two-Dimensional Heat Conduction: Report

Group number: 13

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### Problem Statement:

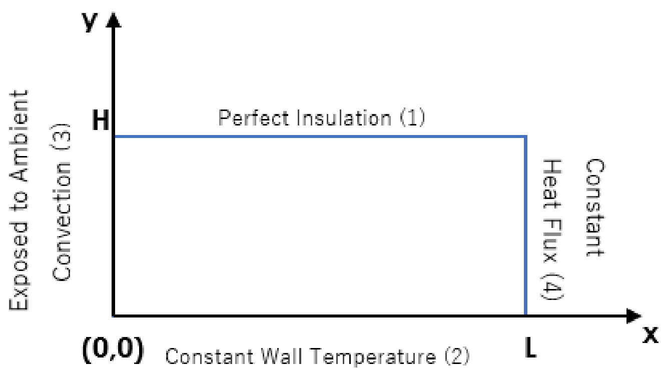
A thin 2D slab is cut from a metal block at z = z0. There is an internal heat source/sink given by Q'''gen. The slab is initially maintained at 200˚F. The walls of the slab are subjected to the following conditions:

| Sl no | Parameter | Value/Condition |
|---|---|---|
| 1 | Metal | Steel |
| 2 | Slab Dimensions (L x H) | 0.08 m x 0.005 m |
| 3 | Heat Generation (Q'''gen) | $-550xy + \ln(32y^{0.38} + 50x)$ |
| 4 | Bottom Wall | Constant Wall Temperature |
| 5 | Top Wall | Perfectly insulated |
| 6 | Right Wall | Exposed to Ambient Convection |
| 7 | Left Wall | Constant Heat Flux |

Given: $T_{amb}$=-10˚C, h=10 W/m2K, $q_0''=3.26y^2$ kW/m2, CWT=32˚C

The origin is placed at the bottom left corner.

### Boundary Conditions:



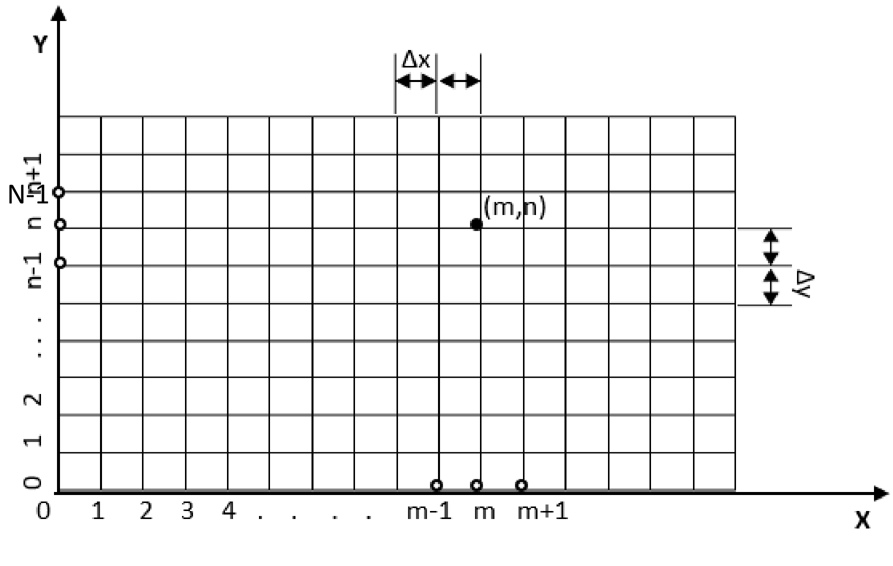

1) Top Wall: $\dot{q}=0\,;\ \frac{\partial T(x,H,t)}{\partial y} = 0$

2) Bottom wall: $T(x,0,t) = 32^0C$

3) Left Wall: $-k\frac{\partial T(0,y,t)}{\partial x} = h\big(T_\infty - T(0,y,t)\big)$

4) Right Wall: $q_{(L,y,t)} = -k\frac{\partial T(L,y,t)}{\partial x} = -3.26y^2$

### Formulation: Finite Difference Method



Nodal network for finite difference formulation

The rectangular slab is divided into a rectangular mesh with nodal points spaced Δx and Δy apart in x- and y- directions, respectively. Nodes are indicated using double subscript notation (m,n), where m=0,1,2....(M-1) and n=0,1,2,....(N-1) are the node counts in x- and y- directions.

For node (m,n)

- x-coordinate: mΔx
- Y-coordinate: nΔy

Since the body is planar, we assume Δz=1 (unit thickness)

Applying the Law of Conservation on Energy to a volume element of size (Δx.Δy.1) which is a general interior node centered about (m,n), we get;

$$\sum\dot{Q}_{all} + \dot{Q}gen = \frac{\Delta E_{ele}}{\Delta t}$$

Assuming the direction of heat conduction to be towards the node,

$$\dot{Q}_{\mathrm{right}} + \dot{Q}_{\mathrm{left}} + \dot{Q}_{\mathrm{up}} + \dot{Q}_{\mathrm{down}} + \mathrm{Q}\dot{g}en = \frac{\Delta E_{ele}}{\Delta t}$$

Heat transfer Area $A_x$= Δy.1 and $A_y$= Δx.1; Substituting from Fourier Law of Heat Conduction,

$$k.\Delta y.\frac{\left(T_{(m-1,n)} - T_{(m,n)}\right)}{\Delta \mathrm{x}} + k.\Delta y.\frac{\left(T_{(m+1,n)} - T_{(m,n)}\right)}{\Delta \mathrm{x}} + k.\Delta \mathrm{x}.\frac{\left(T_{(m,n\text{-}1)} - T_{(m,n)}\right)}{\Delta \mathrm{y}} +$$

$$k.\Delta \mathrm{x}.\frac{\left(T_{(m,n+1)} - T_{(m,n)}\right)}{\Delta \mathrm{y}} + \dot{\mathrm{q}}_{gen}.\Delta x.\Delta y = \rho C_p \Delta x.\Delta y.1\frac{\left(T^{i+1}_{(m,n)} - T^{i}_{(m,n)}\right)}{\Delta \mathrm{t}}$$

Taking square mesh Δx = Δy $=l$

$$\left(T_{(m-1,n)}-T_{(m-1,n)}\right)+\left(T_{(m+1,n)}-T_{(m-1,n)}\right)+\left(T_{(m,n\text{-}1)}-T_{(m,n)}\right)+\left(T_{(m,n+1)}-T_{(m,n)}\right)$$
$$+\frac{\dot{\mathrm{q}}_{gen}}{\mathrm{k}}l^2=\frac{l^2}{\alpha\Delta\mathrm{t}}\left(T^{i+1}_{(m,n)}-T^{i}_{(m,n)}\right)$$

where,

k – Thermal Conductivity of slab

α – Thermal Diffusivity

ρ – Density

$C_p$ – Specific Heat Capacity

Using the explicit method, we obtain

$$\boldsymbol{T^{i+1}_{(m,n)}=\ \tau(T^{i}_{(m-1,n)}+T^{i}_{(m+1,n)}+T^{i}_{(m,n-1)}+T^{i}_{(m,n+1)})+(1-4\tau)\,T^{i}_{(m,n)}+\ \frac{\dot{q}_{gen}}{k}l^2\tau}$$

where $\tau\ =\frac{\alpha\Delta\mathrm{t}}{l^2}$

For steady-state analysis, $T^{i+1}_{(m,n)}-T^{i}_{(m,n)}\ =0$; Therefore

$$\boldsymbol{T^{\mathrm{s}}_{(m,n)}=\frac{1}{4}[T_{(m-1,n)}+T_{(m+1,n)}+T_{(m,n-1)}+T_{(m,n+1)}+\ \frac{\dot{q}_{gen}}{k}l^2]}$$

Similarly, the Law of conservation of Energy is used to obtain the finite difference equations at every node.

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Nodes at the bottom surface are at **constant temperature T**$=\boldsymbol{32^0C}$ per the given boundary conditions. This is true for all time t.

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For nodes at the left wall, i.e. ∀(m,n) s.t. m=0, the equations are:

$$h\Delta y\left(T_\infty-T_{(m,n)}\right)+k\,\Delta y\frac{\left(T_{(m+1,n)}-T_{(m,n)}\right)}{\Delta\mathrm{x}}+\ k.\frac{\Delta\mathrm{x}}{2}.\frac{\left(T_{(m,n\text{-}1)}-T_{(m,n)}\right)}{\Delta\mathrm{y}}+k.\frac{\Delta\mathrm{x}}{2}.\frac{\left(T_{(m,n+1)}-T_{(m,n)}\right)}{\Delta\mathrm{y}}$$
$$+\dot{\mathrm{q}}_{gen}.\frac{\Delta x}{2}.\Delta y\ =\ \rho C_p\frac{\Delta x}{2}.\Delta y.1\frac{\left(T^{i+1}_{(m,n)}-T^{i}_{(m,n)}\right)}{\Delta\mathrm{t}}$$

Taking square mesh Δx = Δy $=l$

$$\boldsymbol{T^{i+1}_{(m,n)}=\ \tau(2T^{i}_{(m+1,n)}+T^{i}_{(m,n-1)}+T^{i}_{(m,n+1)})+(1-(4+\frac{2hl}{k})\tau)\,T^{i}_{(m,n)}+\ \frac{\dot{q}_{gen}}{k}l^2\tau+\ \frac{2hT_\infty l}{k}\tau}$$

For steady-state analysis, $T^{i+1}_{(m,n)}-T^{i}_{(m,n)}\ =0$; Therefore

$$\boldsymbol{T^{\mathrm{s}}_{(m,n)}=\frac{1}{(4+\frac{2hl}{k})}[2T_{(m+1,n)}+T_{(m,n-1)}+T_{(m,n+1)}+\ \frac{\dot{q}_{gen}}{k}l^2+\frac{2hT_\infty l}{k}]}$$

For nodes at the right wall, i.e. ∀(m,n) s.t. m=M-1, the equations are:

$$q_0\Delta y + k\Delta y\frac{(T_{(m\text{-}1,n)}-T_{(m,n)})}{\Delta \mathrm{x}} + k.\frac{\Delta \mathrm{x}}{2}.\frac{(T_{(m,n\text{-}1)}-T_{(m,n)})}{\Delta \mathrm{y}} + k.\frac{\Delta \mathrm{x}}{2}.\frac{(T_{(m,n+1)}-T_{(m,n)})}{\Delta \mathrm{y}} + \dot{q}_{gen}.\frac{\Delta \mathrm{y}}{2}.\Delta \mathrm{x} = \rho C_p\frac{\Delta x}{2}.\Delta y.1\frac{\left(T^{i+1}_{(m,n)}-T^{i}_{(m,n)}\right)}{\Delta \mathrm{t}}$$

Taking square mesh Δx = Δy $=l$

$$\boldsymbol{T^{i+1}_{(m,n)} = \tau(2T^{i}_{(m\text{-}1,n)} + T^{i}_{(m,n-1)} + T^{i}_{(m,n+1)}) + (1-4\tau)\,T^{i}_{(m,n)} + \frac{\dot{q}_{gen}}{k}l^2\tau + \frac{2q_0 l}{k}\tau}$$

For steady-state analysis, $T^{i+1}_{(m,n)} - T^{i}_{(m,n)} = 0$; Therefore

$$\boldsymbol{T^{\mathrm{s}}_{(m,n)} = \frac{1}{4}[2T_{(m-1,n)} + T_{(m,n-1)} + T_{(m,n+1)} + \frac{\dot{q}_{gen}}{k}l^2 + \frac{2q_0 l}{k}]}$$

For nodes at the top wall, i.e. ∀(m,n) s.t. n=N-1, the equations are:

$$k.\frac{\Delta y}{2}.\frac{(T_{(m-1,n)}-T_{(m,n)})}{\Delta x} + k.\frac{\Delta y}{2}.\frac{(T_{(m+1,n)}-T_{(m,n)})}{\Delta x} + k.\Delta x.\frac{(T_{(m,n-1)}-T_{(m,n)})}{\Delta y}$$

$$+\dot{q}_{gen}.\Delta x.\frac{\Delta y}{2} = \rho C_p\Delta x.\frac{\Delta y}{2}.1\frac{\left(T^{i+1}_{(m,n)} - T^{i}_{(m,n)}\right)}{\Delta \mathrm{t}}$$

Taking square mesh Δx = Δy $=l$

$$\boldsymbol{T^{i+1}_{(m,n)} = \tau(2T^{i}_{(m,n-1)} + T^{i}_{(m-1,n)} + T^{i}_{(m+1,n)} + T^{i}_{(m,n+1)}) + (1-4\tau)\,T^{i}_{(m,n)} + \frac{\dot{q}_{gen}}{k}l^2\tau}$$

For steady-state analysis, $T^{i+1}_{(m,n)} - T^{i}_{(m,n)} = 0$; Therefore

$$\boldsymbol{T^{\mathrm{s}}_{(m,n)} = \frac{1}{4}[T_{(m-1,n)} + T_{(m+1,n)} + 2T_{(m,n-1)} + \frac{\dot{q}_{gen}}{k}l^2]}$$

For node at corner, m=M-1, n=N-1, the equations are:

$$q_0\frac{\Delta y}{2} + k\frac{\Delta y}{2}\frac{(T_{(m\text{-}1,n)}-T_{(m,n)})}{\Delta \mathrm{x}} + k.\frac{\Delta \mathrm{x}}{2}.\frac{(T_{(m,n\text{-}1)}-T_{(m,n)})}{\Delta \mathrm{y}} + \dot{q}_{gen}.\frac{\Delta \mathrm{y}}{2}.\frac{\Delta \mathrm{x}}{2} = \rho C_p\frac{\Delta x}{2}.\frac{\Delta y}{2}.1\frac{\left(T^{i+1}_{(m,n)}-T^{i}_{(m,n)}\right)}{\Delta \mathrm{t}}$$

Taking square mesh Δx = Δy $=l$

$$\boldsymbol{T^{i+1}_{(m,n)} = 2\tau(T^{i}_{(m,n-1)} + T^{i}_{(m-1,n)}) + (1-4\tau)\,T^{i}_{(m,n)} + \frac{\dot{q}_{gen}}{k}l^2\tau + \frac{2q_0 l}{k}\tau}$$

For steady-state analysis, $T^{i+1}_{(m,n)} - T^{i}_{(m,n)} = 0$; Therefore

$$\boldsymbol{T^{s}_{(m,n)}} = \frac{1}{4}[2\boldsymbol{T}_{(m-1,n)}+2\boldsymbol{T}_{(m,n-1)} + \frac{\dot{q}_{gen}}{k}\boldsymbol{l}^2 + \frac{2q_0 l}{k}]$$

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For node at corner, m=0, n=N-1, the equations are:

$$h\frac{\Delta y}{2}\left(T_\infty - T_{(m,n)}\right) + k\,\frac{\Delta y}{2}\frac{\left(T_{(m+1,n)} - T_{(m,n)}\right)}{\Delta x} + k.\frac{\Delta x}{2}.\frac{\left(T_{(m,n\text{-}1)} - T_{(m,n)}\right)}{\Delta y} + \dot{q}_{gen}.\frac{\Delta x}{2}.\frac{\Delta y}{2}$$
$$= \rho C_p\frac{\Delta x}{2}.\frac{\Delta y}{2}.1\frac{\left(T^{i+1}_{(m,n)} - T^{i}_{(m,n)}\right)}{\Delta t}$$

$$\boldsymbol{T^{i+1}_{(m,n)}} = 2\boldsymbol{\tau}(\boldsymbol{T^{i}_{(m+1,n)}} + \boldsymbol{T^{i}_{(m,n-1)}}) + (\mathbf{1} - (\mathbf{4} + \frac{\mathbf{2}\boldsymbol{hl}}{\boldsymbol{k}})\boldsymbol{\tau})\,\boldsymbol{T^{i}_{(m,n)}} + \frac{\dot{\boldsymbol{q}}_{\boldsymbol{gen}}}{\boldsymbol{k}}\boldsymbol{l}^2\boldsymbol{\tau} + \frac{\mathbf{2}\boldsymbol{hT_\infty l}}{\boldsymbol{k}}\boldsymbol{\tau}$$

For steady-state analysis, $T^{i+1}_{(m,n)} - T^{i}_{(m,n)} = 0$; Therefore

$$\boldsymbol{T^{s}_{(m,n)}} = \frac{1}{(4+\frac{2hl}{k})}[2\boldsymbol{T}_{(m+1,n)}+2\boldsymbol{T}_{(m,n-1)} + \frac{\dot{q}_{gen}}{k}\boldsymbol{l}^2 + \frac{2hT_\infty l}{k}]$$

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**Methodology**

For all the nodes, time-dependent equations are obtained using the finite difference method. Furthermore, the initial temperature of all nodes is known (200°F and 32°C for those in contact with the isothermal wall. In case of Steady state analysis, average of 200°F and 32°C was taken as the first guess value of Temperature and the iterations ran there onwards.

Assumption: Alloy of Steel used – Stainless Steel (Type 304L)

Iteratively solving the equations for a small time step Δt will yield the required results.

- Exit Condition (Convergence Criteria) Used for Analysis:
  **If(error<tolerance)**
  **break;**

  where, error = $T^{i+1}_{(m,n)} - T^{i}_{(m,n)}$

  and the system was anlyzed for various tolerance values.

- Both Transient and Steady state analysis was done for 2 cases:
  - **Case 1:** Assumption: Properties are constant

- **Case 2:** The variation of properties with temperature has been taken into account while computing. The thermodynamic property values have been lineraly interpolated from a technical report on Thermophysical properties of stainless steels by Choong S Kim, Argonne National Laboratory, Illinois.

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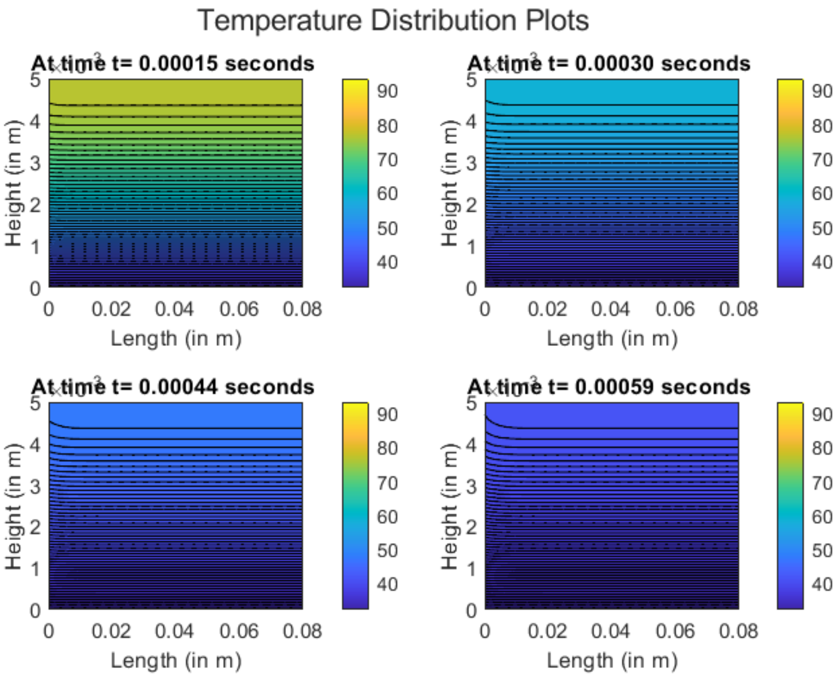
## Results

### Case 1:

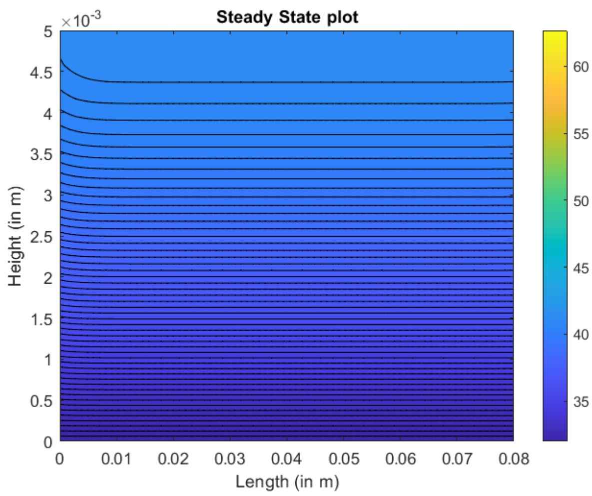
a) Transient Heat Conduction:

- Time taken to reach Steady state**: 0.0006 s** at Tol = 0.01˚C
- Temperature distribution plots obtained at various time instants:

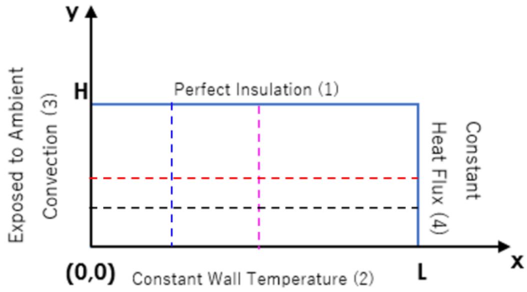


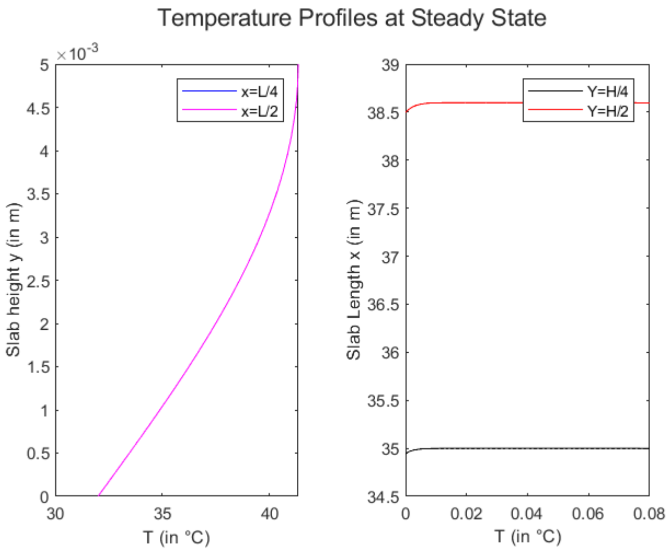

b) Steady Heat Conduction

- No of iterations to reach Steady state: 1337
- Temperature Distribution Contour at Steady State




- Temperature Profiles at steady state at various sections:

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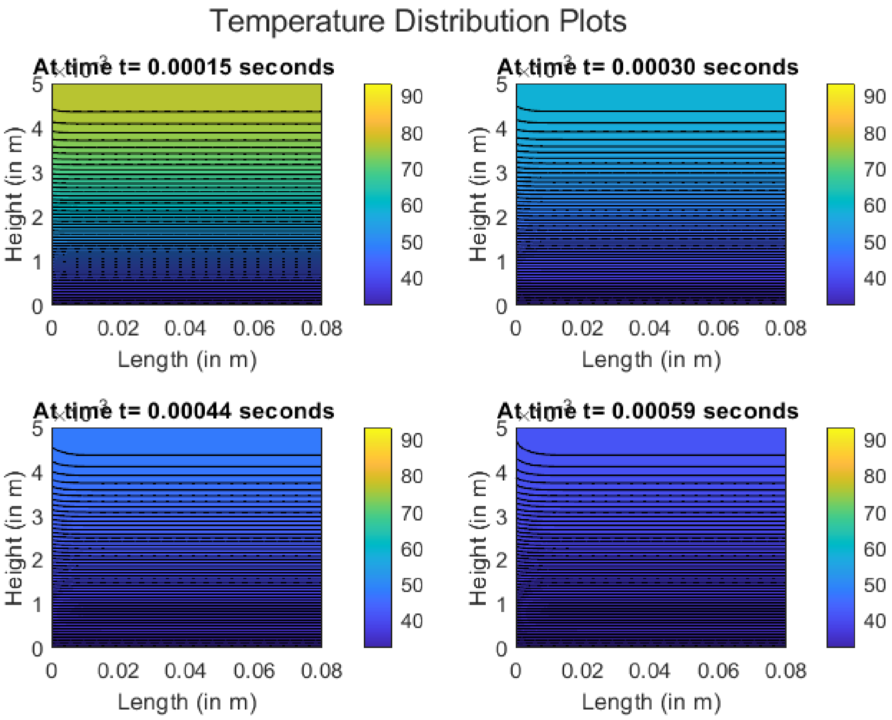
**Case 2**:

a) Transient Heat Conduction:

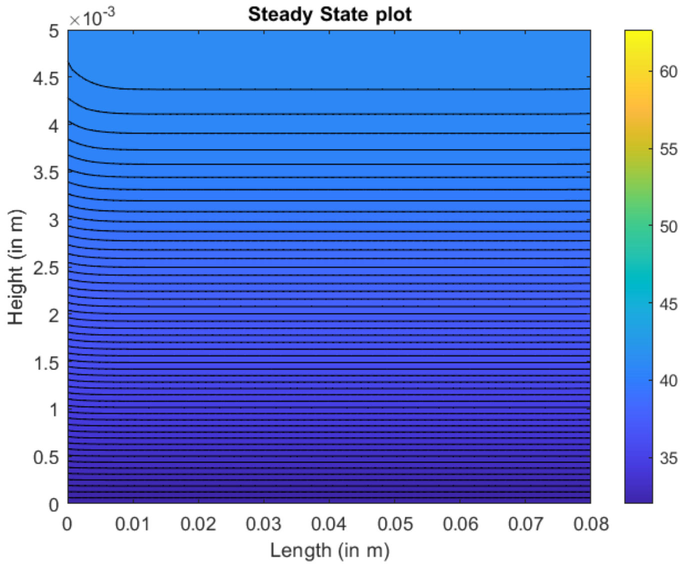
Time taken to reach Steady state: **0.00059 s** at Tol=0.01˚C

Temperature distribution plots obtained at various time instants:

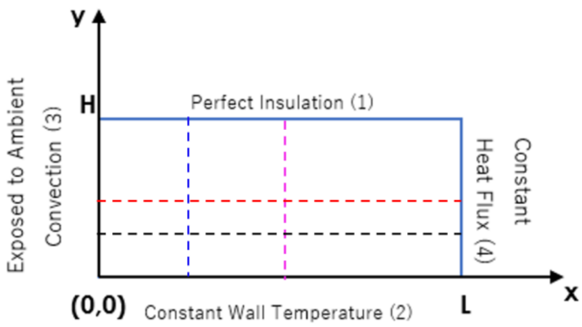


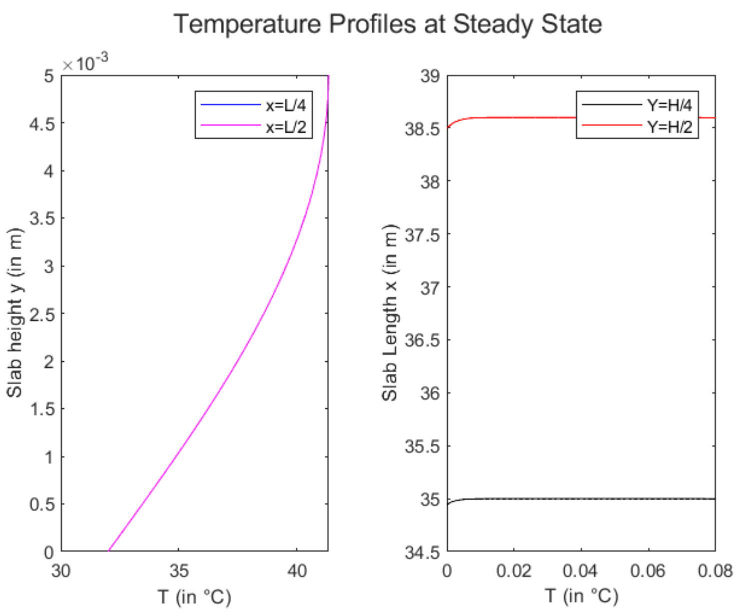
b) Steady Heat Conduction

- No of iterations to reach Steady state: 1337
- Temperature Distribution Contour at Steady State




- Temperature Profiles at steady state at various sections:

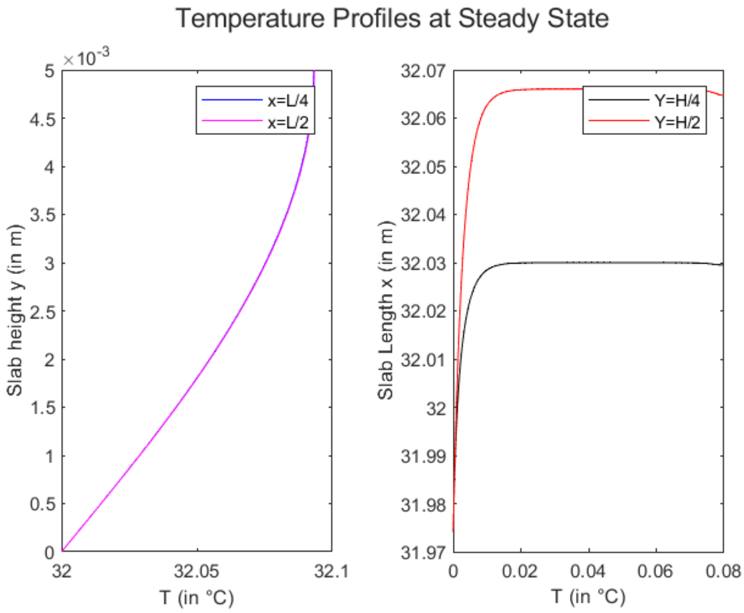



It can be observed that the temperature profile at section x=L/2 and x=L/4 overlap. The slb has cooled down from 200˚F, which is around 93.3˚C to temperatures between 30 and 40˚C

## Tolerance vs No of Iterations

| Tolerance (in ˚C) | No of Iterations | |
|---|---|---|
| | Transient | Steady |
| 0.1 | 149 | 75 |
| 0.01 | 1984 | 1337 |
| 0.001 | 4134 | 3486 |
| 0.0001 | 6283 | 5636 |

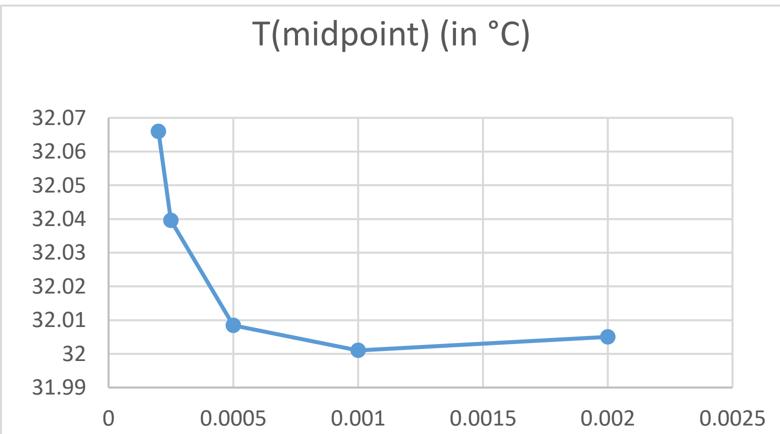



This is the temperature profile curve at steady state when the tolerance value was set to be 0.0001˚C. Comparing with the plots obtained at tol=0.01˚C, we can clearly make out that making the tolerance smaller and smaller gives more accurate results.

## Mesh size vs Output

The temperature of the midpoint of slab is being tracked.

| Mesh size l (in m) | T(midpoint) (in ˚C) |
|---|---|
| 0.002 | 32.005 |
| 0.001 | 32.001 |
| 0.0005 | 32.0084 |
| 0.00025 | 32.0396 |
| 0.0002 | 32.066 |




Could not perform analysis of smaller meshes due to limited computational ability of PC.

## Inferences

- The steady state temperatures computed for case 1 and case 2 are almost identical. The time taken to reach steady state in both the cases differ by just 0.00001s. We can infer that the values of thermodynamic properties of Stainless Steel (Type 304L) are almost constant
- As seen from the temperature distribution contours at various time instants, the temperature change rate decreases with time. This is in accordance with Newton's Law of cooling, which states the heat transfer rate is proptional to the temperature difference between the two bodies between which heat is transferred.
- The computation time increases as the tolerance value is decreased. Smaller tolerance gives better results.
- As the mesh size decreases, the output become better. But after reaching a specific mesh size, the output values remain constant even on further refining. This is visible in the plot obtaied. This is called grid independence.

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