State-of-the-art of

Vehicular Traffic Flow Modelling

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Abstract

Nowadays traffic flow and congestion is one of the main societal and economical problems related to transportation in industrialised countries. In this respect, managing traffic in congested networks requires a clear understanding of traffic flow operations. That is, insights into what causes congestion, what determines the time and location of traffic breakdown, how does the congestion propagate through the network, etc., are essential.

For this purpose, during the past fifty years, a wide range of traffic flow theories and models have been developed to answer these research questions. This paper presents a overview of some fifty years of modelling vehicular traffic flow. A rich variety of modelling approaches developed so far and in use today will be discussed and compared. The considered models are classified based on the

level-of-detail with which the vehicular flow is described. For each of the categories, issues like modelling accuracy, applicability, generalisability, and model calibration and validation, are discussed.

Keywords: traffic flow theory, traffic flow modelling, microsimulation.

1   Introduction

Research on the subject of traffic flow modelling started some forty years ago, when Lighthill and Whitham (1955) presented a model based on the analogy of vehicles in traffic flow and particles in a fluid. Since then, mathematical description of traffic flow has been a lively subject of research and debate for traffic engineers. This has resulted in a broad scope of models describing different aspects of traffic flow operations, either by considering the time-space behaviour of individual drivers under the influence of vehicles in their proximity (*microscopic* models), the behaviour of drivers without explicitly distinguishing their time-space behaviour (*mesoscopic* models), or from the viewpoint of the collective vehicular flow (*macroscopic* models). In addition to the controversy between these microscopic, mesoscopic, and macroscopic modelling streams, several researchers have joined the debate on the macroscopic modelling approach most suitable for a correct description of traffic flow. Moreover, recent theoretical and empirical findings of Boris Kerner and co-workers resulted in increased public attention for the subject of macroscopic flow modelling (see Kerner *et al.* (1996), Kerner (1999)). Also, due to improved techniques and increased computational capacity to solve large-scale control problems, applications of realistic flow models in model-based control approaches have become feasible.

This contribution will present a concise summary of nearly fifty years of modelling vehicular traffic flow. A rich variety of modelling approaches developed so far and in use today will be discussed and compared. This comparison will mainly consider theoretical issues of model derivation and characteristics. However, some practical issues, such as model calibration, are discussed as well.

2   Categorisation of traffic flow models

Traffic flow models may be categorised according to various criteria (level of detail, operationalisation, representation of the processes). We discuss several classes of traffic flow models, the order of which is based on the *level-of-detail* classification (i.e. (sub-) microscopic, mesoscopic, and macroscopic modelling approaches).

2.1 Traffic flow modelling and simulation

Traffic operations on roadways can be improved by field research and field experiments of real-life traffic flow. However, apart from the scientific problem of reproducing such experiments, costs and safety play a role of dominant importance as well. Due to the complexity of the traffic flow system, analytical approaches may not provide the desired results. Therefore, traffic flow (simulation-) models designed to characterise the behaviour of the complex traffic flow system have become an essential tool in traffic flow analysis and experimentation.

Depending on the type of model, the application area of these traffic flow models is very broad, e.g.:

- Evaluation of alternative treatments in (dynamic) traffic management.

- Design and testing of new transportation facilities (e.g. geometric designs).

- Operational flow models serving as a sub-module in other tools (e.g. model-based traffic control and optimisation, and dynamic traffic assignment).

- Training of traffic managers.

The description of observed phenomena in traffic flow is however not self-evident. General mathematical models aimed at describing this behaviour using mathematical equations include the following approaches (cf. Papageorgiou (1998)):

1. Purely *deductive* approaches whereby known accurate physical laws are applied.

2. Purely *inductive* approaches where available input/output data from real systems are used to fit generic mathematical structures (ARIMA models, polynomial approximations, neural networks).

3. *Intermediate* approaches, whereby first basic mathematical model-structures are developed first, after which a specific structure is fitted using real data.

Table 1: Overview of traffic flow models.

detail level	MODEL NAME / REF.	DI				SC	RE	OP	AR
		v	$v0$	y	o				
(sub-) microscopic models	MIXIC (Van Arem and Hogema (1995))	+		+	+	d	s	s	ml
	SIMONE (Minderhoud (1999))	+	+	+	+	d	s	s	ml, d
	PELOPS (Ludmann (1998))	+		+	+	d	s	s	ml
	safe-distance models (May (1990))	+				c	d	a	sl
	stimulus-response models (Leutzbach (1988), May (1990))	+				c	d	a	sl
	psycho-spacing models (Wiedemann (1974))	+	+	+		c	s	s	ml
	FOSIM (Vermijs *et al.* (1995))	+	+	+	+	d	s	s	ml, d
	CA-models (Nagel (1996,1998), Wu and Brilon (1999), Esser *et al.* (1999))	+	+			d	s	s	n, u
	Particle pedestrian model (Hoogendoorn and Bovy (2000a)	+	+	+	+	d	s	s	o
	INTEGRATION (Van Aerde (1994)).	+				d	d	s	n
mesoscopic models	headway distr. models (Hoogendoorn and Bovy (1998a))			+	+	c	s	a	c
	reduced gas-kinetic model (Prigogine and Herman (1971))	+				c	d	a	al
	improved gas-kinetic model (Paveri-Fontana (1975))	+	+			c	d	a	al
	multilane gas-kinetic model (Helbing (1997b))	+	+		+	c	d	a	ml, d
	multiclass gas-kinetic model (Hoogendoorn and Bovy (2000b))	+	+	+		c	d	a	al
	multiclass multilane model (Hoogendoorn (1999))	+	+	+		c	d	a	al, ml, d
	cluster models (Botma (1978))	+	+			c	d	a	al
macroscopic models	LWR model (Lighthill and Whitham (1955))	+				c	d	a	al
	Payne-type models ((Payne (1971,1979))	+				c	d	a	al
	Helbing-type models (Helbing (1996,1997))	+			+	c	d	a	al
	Cell-Transmission Model (Daganzo (1994a,b,1999))	+				d	d	s	n
	METANET (Kotsialos *et al.* (1998,1999))	+				d	d	s	n
	semi-discrete model (Smulders (1990))					sd	s	a	al, d
	FREFLO (Payne (1979))	+				d	d	s	n
	MASTER (Treiber *et al.*(1999))	+				d	d	a	ml

DI: dimension (other than time / space): velocity v, desired velocity v^0, lateral position y (lanes), and <u>o</u>ther
SC: scale (<u>c</u>ontinuous, <u>d</u>iscrete, and <u>s</u>emi-<u>d</u>iscrete);
RE: process representation (<u>d</u>eterministic, <u>s</u>tochastic);
OP: operationalisation (<u>a</u>nalytical, <u>s</u>imulation);

AR: area of application (<u>c</u>ross-section, <u>s</u>ingle <u>l</u>ane stretches, <u>m</u>ulti<u>l</u>ane stretches, <u>ag</u>gregate <u>l</u>ane stretches, <u>dis</u>continuities, motorway <u>n</u>etwork, <u>ur</u>ban network, and <u>o</u>ther).

Papageorgiou (1998) convincingly argues that it is unlikely that traffic flow theory will reach the descriptive accuracy attained in other domains of science (e.g. Newtonian physics or thermodynamics). The only accurate physical law in traffic flow theory is the *conservation of vehicles equation*; all other model structures reflect either counter-intuitive idealisations or coarse approximations of empirical observations. Consequently, the challenge of traffic flow researchers is to look for useful theories of traffic flow that have sufficient descriptive power, where *sufficiency* depends on the application purpose of their theories.

In this article, we review some of the modelling achievements of five decades of traffic flow modelling research. To this end, the discussed traffic models are classified according to the following:

- Scale of the independent variables (continuous, discrete, semi-discrete);

- Level of detail (submicroscopic, microscopic, mesoscopic, macroscopic);

- Representation of the processes (deterministic, stochastic);

- Operationalisation (analytical, simulation);

- Scale of application (networks, stretches, links, and intersections).

Table 1 presents an overview of some well-known models, based on the proposed criteria. While not being exhaustive, the table provides insights into traffic modelling efforts during the last five decades of traffic flow operations related research. Let us discuss the criteria in some detail.

Level of detail. Traffic models may be classified according to the *level of detail* with which they represent the traffic systems. This categorisation can be operationalised by considering the *distinguished traffic entities* and the *description level* of these entities in the respective flow models. We propose the following classification:

1. *Submicroscopic simulation models* (high-detail description of the functioning of vehicles' subunits and the interaction with their surroundings).

2. *Microscopic simulation models* (high-detail description where individual entities are distinguished and traced).

3. *Mesoscopic models* (medium detail).

4. *Macroscopic models* (low detail).

A *microscopic simulation* model describes both the space-time behaviour of the systems' entities (i.e. vehicles and drivers) as well as their interactions at a high level of detail (individually). For instance, for each vehicle in the stream a lane-change is described as a detailed chain of drivers' decisions.

Similar to microscopic simulation models, the *submicroscopic* simulation models describe the characteristics of individual vehicles in the traffic stream. However, apart from a detailed description of driving behaviour, also *vehicle control behaviour* (e.g. changing gears, AICC operation, etc.) in correspondence to prevailing surrounding conditions is modelled in detail. Moreover, the functioning of specific parts (sub-units) of the vehicle is described.

A *mesoscopic model* does not distinguish nor trace individual vehicles, but specifies the *behaviour* of individuals, for instance in probabilistic terms. To this end, traffic is represented by (small) groups of traffic entities, the activities and interactions of which are described at a low detail level. For instance, a lane-change manoeuvre might be represented for an individual vehicle as an instantaneous event, where the decision to perform a lane-change is based on e.g. relative lane densities, and speed differentials. Some mesoscopic models are derived in analogy to gas-kinetic theory. These so-called gas-kinetic models describe the dynamics of velocity distributions.

Macroscopic flow models describe traffic at a high level of aggregation as a flow without distinguishing its constituent parts. For instance, the traffic stream is represented in an aggregate manner using characteristics as flow-rate, density, and velocity. Individual vehicle manoeuvres, such as a lane-change, are usually not explicitly represented. A macroscopic model may assume that the traffic stream is properly allocated to the roadway lanes, and employ an approximation to this end. Macroscopic flow models can be classified according the *number* of partial differential equations that frequently underlie the model on the one hand, and their *order* on the other hand.

Scale of the independent variables. Since almost all traffic models describe dynamical systems, a natural classification is the time-scale. We will distinguish two time scales, namely continuous and discrete. A *continuous* model describes how the traffic system's state changes continuously over time in response to continuous stimuli. *Discrete models* assume that state changes occur discontinuously over time at discrete time instants. Besides time, also other independent variables can be described by either continuous or discrete variables (e.g. position, velocity, desired velocity). Mixed models have also been proposed (e.g. Smulders (1990)).

Representation of the processes. In this respect, we will distinguish *deterministic* and *stochastic* models. The former models have no random variables implying that all actors in the model are defined by exact relationships. Stochastic models incorporate processes that include random variates.

For instance, a car-following model can be formulated as either a deterministic or a stochastic relationship by defining the driver's reaction time as a constant or as a random variable respectively.

Operationalisation. With respect to the operationalisation criterion, models can be operationalised either as analytical solutions of sets of equations, or as a simulation model. For a thorough contemplation on the uses and misuses of simulation in traffic, we refer to chapter 13 of May (1990).

Scale of application. The application scale indicates the area of application of the model. For instance, the model may describe the dynamics of its entities for a single roadway stretch, an entire traffic network, a corridor, a city, etc.

In the remainder of this article, we discuss the modelling approaches in more detail, in the order of the _level-of-detail_ classification ((sub-) microscopic, mesoscopic and macroscopic). We will also provide some examples of models for each of these modelling types. Aim of the discussion is to provide some insight into mechanisms of different modelling approaches.

3 Submicroscopic and microscopic traffic flow modelling

In this section, we will discuss submicroscopic and microscopic traffic flow models, the development of which started during the sixties with the so-called car-following models. We will briefly review different model refinements and alternative microscopic modelling approaches.

3.1 Car-following models

During the 1960's, research efforts focussed on the so-called _follow the leader models_. These models are based on supposed mechanisms describing the process of one vehicle following another. We will discuss three types of car-following models, namely _safe-distance models_, _stimulus-response models_, and _psycho-spacing models_.

3.1.1 Safe-distance models

Safe-distance car-following models describe the dynamics of a single vehicle in relation to its predecessor. In this respect, a very simple model is Pipes' rule (cf. Pipes (1953)): "_A good rule for following another vehicle at a safe distance is to allow yourself at least the length of a car between you and the vehicle ahead for every ten miles an hour (16.1km/hr) of speed at which you are travelling_". Using this driving-rule, we can determine the required _gross distance headway_ D_n of vehicle n driving with velocity v with respect to vehicle $n - 1$:

$$D_n(v) = L_n(1 + v/16.1) \qquad (1)$$

where L_n denotes the length of vehicle n. In Pipes' model, the minimal safe distance increases linearly with the velocity v of the vehicle. A similar approach was proposed by Forbes *et al.* (1958). Both Pipes' theory and Forbes' theory were compared to field measurements. It was concluded that according to Pipes' theory, the minimum headways are slightly less at low and high velocities than observed in empirical data. However, considering the models' simplicity, agreement with real-life observations is astonishing (cf. Pignataro (1973)).

Leutzbach (1988) discusses a more refined model describing the spacing of constrained vehicles in the traffic flow. He states that the overall reaction time T consists of:

- *perception time* (time needed by the driver to recognise that there is an obstacle);

- *decision time* (time needed to make decision to decelerate), and;

- *braking time* (needed to apply the brakes).

The braking distance is defined by the distance needed by a vehicle to come to a full stop, incorporating the reaction time of the driver, and the maximal deceleration. The latter is among other things a function of the weight of the friction with the road surface μ, and the acceleration due to gravity g. The *total safety distance model* assumes that drivers consider braking distances large enough to permit them to brake to a stop without causing a rear-end collision with the preceding vehicles if the latter vehicles come to a stop *instantaneously*. The corresponding safe distance headway equals:

$$D_n(v) = L_n + Tv + v^2/(2\mu g) \qquad (2)$$

Let us consider two successive vehicles with approximately equal braking distances. We assume that the spacing between the vehicles must suffice to avoid a collision when the first vehicle comes to a full stop (the so-called *reaction time distance model*). That is, if the first vehicle stops, the second vehicle only needs the distance it covers during the overall reaction time T with *unreduced speed*, yielding Forbes' model. Jepsen (1998) proposes that the gross-distance headway $D_n(v)$ effectively occupied by vehicle n driving with velocity v is a function of the vehicle's length L_n, a constant minimal distance between the vehicles d_{min}, the reaction time T and a speed risk factor F:

$$D_n(v) = (L_n + d_{min}) + v(T + vF) \qquad (3)$$

Experienced drivers have a fairly precise knowledge of their reaction time T. For novice drivers, rules of thumb apply ("stay two seconds behind the vehicle ahead", "keep a distance of half your velocity to the vehicle ahead"). From field studies, it is found that the delay of an unexpected event to a reme-

dial action is in the order of 0.6 to 1.5 seconds. The *speed-risk factor F* stems from the observation that experienced drivers do not only aim to prevent rear-end collisions. Rather, they also aim to minimise the potential damage or injuries of a collision, and are aware that in this respect their velocity is an important factor. This is modelled by assuming that drivers increase their *time* headway by some factor – the *speed-risk factor* – linear to *v*. Finally, the *minimal distance headway d_{min}* describes the minimal amount of spacing between motionless vehicles, observed at jam density.

Note that this occupied space equals the gross distance headway only if the following vehicle is constrained. In the remainder of this paper, this property is used. Otherwise, the car-following distance is larger than the safe distance needed. Dijker *et al*. (1998) discuss some empirical findings on user-class specific car-following behaviour in congested traffic flow conditions.

3.1.2   Stimulus-response car-following models

Drivers who follow try to conform to the behaviour of the preceding vehicle. This car-following process is based on the following principle:

$$response = sensitivity \times stimulus \tag{4}$$

In general, the *response* is the braking or the acceleration of the following vehicle, delayed by an overall reaction time *T*. Consider vehicle *n* following vehicle *n*−1. Let $x_n(t)$ denote the position of vehicle *n* at instant *t*. A well known model specification is (Chandler *et al*. (1958)):

$$a_n(t+T) = \gamma(v_{n-1}(t) - v_n(t)) \tag{5}$$

where $v_n(t)$ and $a_n(t)$ respectively denote velocity and acceleration of vehicle *n* at *t*, and γ denotes the driver's sensitivity. Thus, the *stimulus* is defined by the velocity difference between leader and follower. The following expression has been proposed for driver's *sensitivity* γ (Gazis *et al*. (1961)):

$$\gamma = c \cdot (v_n(t+T))^m / (x_{n-1}(t) - x_n(t))^l \tag{6}$$

Thus, the following vehicle adjusts its velocity $v_n(t)$ proportionally to both distances and speed differences with delay *T*. The extent to which this occurs depends on the values of *c*, *l* and *m*. In combining eqns. (5) and (6), and integrating the result, relations between the velocity $v_n(t+T)$ and the distance headway $x_{n-1}(t) - x_n(t)$ can be determined. Assuming stationary traffic conditions, the following relation between the *equilibrium velocity V^e* and the *density r* results:

$$V^e(r) = V^0(1 - (r/r_{jam})^{l-1})^{1/(1-m)} \tag{7}$$

for $m \neq 1$ and $l \neq 1$. We refer to Leutzbach (1988) for a more general expression.

In the model of Gazis *et al.* (1961), behaviour of free-flowing drivers is modelled very unrealistically: when the distance headway is very large, drivers still react to velocity differences. Also, slow drivers are *dragged along* when following faster vehicles. When vehicle-types are distinguished, this modelling assumption implies that a slow truck following a fast person-car increases its velocity until it drives at the same velocity as the leading person-car. In addition, the traffic is assumed homogeneous. That is, all model parameters are equal for all user-classes and all lanes of the roadway.

Since lane-changing processes cannot be easily described, car following models have been mainly applied to single lane traffic (e.g. tunnels, cf. Newell (1961)) and traffic stability analysis (Herman *et al.* (1959), May (1990; chapter 6)). That is, using car-following models the limits of local and asymptotic stability of the stream can be analysed. Montroll (1961) proposed applying a stochastic term to eq. (5) representing the *acceleration noise*, describing the difference between actual acceleration and the 'ideal' acceleration. A interesting application is the use of a simple follow-the-leader rule to derive a second-order macroscopic flow model (Payne (1971); see section 5.3):

$$V(x(t+T),t+T) = V^e(r(x+D,t)) \qquad (8)$$

where $x(t)$ denotes the location of the driver at instant t, $V(x,t)$ its velocity at x and t, V^e the equilibrium velocity as a function of the density r at $(x+D,t)$; T equals the reaction time and D equals the gross-distance headway with respect to the preceding vehicle. Eq. (8) shows that drivers adapt their velocity to the equilibrium velocity, which is a function of the traffic density at $x+D$. The equilibrium velocity represents a trade-off between the desired velocity of a driver (i.e. the velocity the driver aims to attain when conditions are free-flow), and a reduction in the velocity due to worsening traffic conditions.

3.1.3 Psycho-spacing models

For $l \neq 0$, the car-following eqn. (5) and (6) presume that the following driver reacts, on the one hand, to very small changes in the relative velocity $v_n(t) - v_{n+1}(t)$, even when the distance headway is large. On the other hand, the model assumes that the response equals zero as the velocity differences disappear, even if the distance between the consecutive vehicles is very small or large. To remedy this problem insights from perceptual psychology have been used to show that drivers are subject to certain limits in their perception of the stimuli to which they respond (cf. Todosiev and Barbosa (1964)).

The basic behavioural rules of such so-called *psycho-spacing* models are:

- At large spacings, the following driver is not influenced by velocity differences.

- At small spacings, some combinations of relative velocities and distance headways do not yield a response of the following driver, because the relative motion is too small.

Wiedemann (1974) developed the first psycho-spacing model. He distinguished *constrained* and *unconstrained driving* by considering *perception thresholds*. Moreover, *lane-changing* and *overtaking* are incorporated in his modelling approach. Psycho-spacing models are the foundation of a number of contemporary microscopic simulation models.

Recently, the ability of microscopic models to describe *transient traffic flow behaviour* like the *capacity drop*, and stability of so-called wide jams (jams in which the velocity of traffic is near zero, and which propagate upstream with a near constant velocity) has been addressed by Krauss *et al*. (1999). They show that in order for microscopic simulation models to exhibit the correct description of the capacity drop (and thus of wide jams), these models need to inhibit mechanisms that yield higher queue inflows than queue discharge rates (the so-called *slow-to-start* driving rule). This holds for any microscopic model for which the reaction time (or mean headways during free-flow conditions) is smaller than the 'jam escape time'. Simple car following models do not have this feature, and are therefore *not suited for correct description of congested traffic flow*.

3.2 Microscopic simulation models

The availability of fast computers has resulted in an increasing interest in complex micro-simulation models. These models distinguish and trace single cars and their drivers. Driver's behaviour is generally described by a large set of if-then rules (production-rule systems). From driver behaviour and vehicle characteristics, position, speed and acceleration of each car are calculated for each time step.

A large number of microscopic simulation models have been developed. In illustration, the SMARTEST project (see Algers *et al.* (1997)) identified 58 microscopic simulation models of which 32 were analysed. Some of these are *true* microscopic simulations, in the sense that they model the car-following behaviour and the lane-changing behaviour of each individual vehicle in the traffic flow. Usually, the car-following behaviour is based on the *psycho-spacing* modelling paradigm. To describe the lane-changing behaviour, microscopic models generally distinguish the decision to perform a lane-change, the choice of a target lane, and the acceptance of the available gap on this target lane.

Examples of microsimulation models are AIMSUN2 (Barceló *et al.* (1999)), FOSIM (Vermijs *et al.* (1995)), and the Wiedemann (1974) model. For an overview, we refer to Algers *et al*. (1997).

3.3 Submicroscopic simulation models

In addition to describing the time-space behaviour of the individual entities in the traffic system, submicroscopic simulation models describe the functioning of specific parts and processes of vehicles and driving tasks. For instance, a submicroscopic simulation model describes the way in which a driver applies the brakes, considering among other things the driver's reaction time, the time needed to apply the brake, etc. These submicroscopic simulation models are very suited to model the impacts of driver support system on the vehicle dynamics and driving behaviour. For example, the submicroscopic model SIMONE (Minderhoud (1999)) describes the functioning and the driver's operation of an Intelligent Cruise Control (ICC) system, influenced by the direct surroundings of the vehicle.

Other examples of submicroscopic models are MIXIC (Van Arem (1995)) and PELOPS (Ludmann (1998)). For a review on microscopic and submicroscopic simulation models, we refer to Ludmann (1998) and Minderhoud (1999).

3.4 Cellular automaton models

A more recent addition to the development of microscopic traffic flow theory are the so-called *Cellular Automaton* (CA) or *particle hopping* models. CA-models describe the traffic system as a lattice of cells of equal size (typically 7.5*m*). A CA-model describes in a discrete way the movements of vehicles from cell to cell (cf. Nagel (1996,1998)). The size of the cells are chosen such that a vehicle driving with a velocity equal to one moves to the next downstream cell during one time step. The vehicle's velocity can only assume a limited number of discrete values ranging from zero to v_{max}.

The process can be split-up into three steps:

- *Acceleration*. Each vehicle with velocity smaller than its maximum velocity v_{max}, accelerates to a higher velocity, i.e. $v \leftarrow \min(v_{max}, v+1)$.

- *Deceleration*. If the velocity is smaller than the distance gap d to the preceding vehicle, then the vehicle will decelerate: $v \leftarrow \min(v, d)$.

- *Dawdling* ("*Trödeln*"). With given probability p_{max}, the velocity of a vehicle decreases spontaneously: $v \leftarrow \max(v-1, 0)$.

Using this *minimal set* of driving rules, and the ability to apply parallel computing[*], the CA-model is very fast, and can consequently be used both to simulate traffic operations on large-scale motorway networks, as well as for traffic assignment and traffic forecasting purposes. The initial single-lane model of Nagel (1996) has been generalised to multilane multiclass traffic flow (cf. Nagel *et al.* (1999)). Due to the simplicity of computation, the CA-models are applicable to large traffic networks.

The classical CA car-following model is *space-oriented* and of a heuristic nature. Wu and Brilon (1999) proposed an alternative multilane CA-model, using *time-oriented* car-following rules. This model describes drivers' behaviour more realistically, while retaining all positive features of the original CA-model.

CA-models aim to combine advantages of complex micro-simulation models, while remaining computationally efficient. However, the car-following rules of both the space-oriented and time-oriented CA-models lack intuitive appeal and their exact mechanisms are not easily interpretable from the driving-task perspective. Moreover, they are too crude to describe and study microscopic details of traffic flow (e.g. overtaking and merging) sufficiently accurate from a single driver's perspective.

Verification of CA-models on German and American motorways and urban traffic networks (cf. Wu and Brilon (1999), Nagel *et al.* (1999), Esser *et al.* (1999)) shows fairly realistic results on a macroscopic scale, especially in the case of urban networks in terms of reproduction of empirical speed-density curves.

3.5   Particle models

In closure of this section on microscopic and submicroscopic simulation models, we briefly consider the so-called *particle models*. Although these models distinguish and trace the individual vehicles, their behaviour is described by aggregate equations of motion, for instance a macroscopic traffic flow model. These particle models can be considered as a specific type of numerical solution approach (so-called *particle discretisation methods*; cf. Hockney and Eastwood (1988)), applied to mesoscopic or macroscopic continuum traffic flow models. An example of a particle model is INTEGRATION (Van Aerde (1994)). Recently, Hoogendoorn and Bovy (2000a) derived a stochastic microscopic pedestrian simulation model based on gas-kinetic pedestrian flow equations.

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[*] When one relaxes the *parallel update* requirement, we generally do not speak of Cellular Automata models. However, the term *particle hopping model* still applies (cf. Nagel (1998)).

4 Mesoscopic traffic flow models

Mesoscopic flow models describe traffic flow at a medium detail level. Vehicles and driver behaviour are not distinguished nor described individually, but rather in more aggregate-terms (e.g. using probability distribution functions). However, the behaviour rules are described at an individual level. For instance, a gas-kinetic model describes velocity distributions at specific locations and time instants. The dynamics of these distributions are generally governed by various processes (e.g. acceleration, interaction between vehicles, lane-changing), describing the individual driver's behaviour. Three well known examples of mesoscopic flow models are the so-called *headway distribution* models, *cluster* models, and the *gas-kinetic continuum* models.

4.1 Headway distribution models

A *time-headway* is defined by the difference in passage times of two successive vehicles. In general, it is assumed that these time-headways are identically distributed independent random variates. Headway distribution models are *mesoscopic* in the sense that they describe the *distribution* of the headways of the individual vehicles, while neither explicitly considering nor tracing each vehicle separately.

Examples of headway distribution models are Buckley's semi-Poisson model (Buckley (1968)), and the Generalised Queuing Model (Branston (1976)). *Mixed headway distribution models* distinguish between *leading* and *following* vehicles: time headways of leading drivers and following drivers are taken from different probability distributions.

Headway distribution models have been criticised for neglecting the role of traffic dynamics. Moreover, these models assume that all vehicles are essentially the same. That is, the probability distribution functions are independent of traveller type, vehicle type, travel purpose, level of drivers' guidance, etc. To remedy this, Hoogendoorn and Bovy (1998a) developed a headway distribution model for respectively multiclass traffic flow and multiclass multilane traffic flow. Using a new estimation technique, they analyse multiclass traffic on both two-lane rural roads as well as two-lane motorways in the Netherlands.

4.2 Cluster models

Cluster models are characterised by the central role of *clusters* of vehicles. A cluster is a group of vehicles that share a specific property. Different aspects of clusters can be considered. Usually, the *size* of a cluster (the number of vehicles in a cluster) and the *velocity* of a cluster are of dominant im-

portance. Generally, the size of a cluster is dynamic: clusters can grow and decay. The within cluster traffic conditions, e.g. the headways, velocity differences, etc., are usually *not* considered explicitly: clusters are *homogeneous* in this sense. Usually, clusters emerge because of restricted overtaking possibilities due to e.g. overtaking prohibitions, or interactions with other vehicles making overtaking impossible, or due to prevailing weather or ambient conditions (see Botma (1978)).

4.3   Gas-kinetic continuum models of Prigogine and Herman

Instead of describing the traffic dynamics of individual vehicles, gas-kinetic traffic flow models describe the dynamics of the velocity distribution functions of vehicles in the traffic flow. In this section, we first recall the seminal models of Prigogine and Herman (cf. Prigogine (1961), Prigogine and Herman (1971)), after which a few extensions to this model type are dealt with.

The gas-kinetic models describe the dynamics of the reduced *Phase-Space Density* (PSD) $\tilde{\rho}(x,v,t)$ [*]. The reduced PSD can be interpreted as follows: at instant t the expected number of vehicles present at an infinitesimal region $[x,x+dx)$ driving with a velocity $[v,v+dv)$ equals $\tilde{\rho}(x,v,t)dxdv$. The concept of the reduced PSD is borrowed from statistical physics, and can be considered as a *mesoscopic generalisation* of the macroscopic traffic density $r(x,t)$. The reduced PSD reflects the *velocity distribution function* of a single-vehicle. Prigogine and Herman (1971) also introduce the *two-vehicle distribution function* $\tilde{\varphi}(x,v,x',v',t)$, which can be interpreted as follows: $\tilde{\varphi}(x,v,x',v',t)dxdvdx'dv'$ describes the expected number of *vehicle pairs*, where the vehicles are respectively located at $[x,x+dx)$ and $[x',x'+dx')$ while driving at velocity $[v,v+dv)$ and $[v',v'+dv')$.

Prigogine and Herman assumed that dynamic changes of the reduced PSD are caused by the following processes:

- *Convection*. Vehicles with a velocity v flowing into or out of the roadway segment $[x,x+dx)$ cause changes in the reduced PSD $\tilde{\rho}(x,v,t)$.

- *Acceleration towards the desired velocity*. Vehicles not driving at their desired velocity will accelerate if possible.

- *Deceleration due to interaction between drivers*. A vehicle that interacts with a slower vehicle will need to reduce its velocity when it cannot immediately overtake.

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[*] In the sequel the 'reducedness' will be explained.

Their deliberations yielded the following partial differential equation:

$$\partial_t \tilde{\rho} + v \partial_x \tilde{\rho} = (\partial_t \tilde{\rho})_{acc} + (\partial_t \tilde{\rho})_{int} \tag{9}$$

where we have used the shorthand notation $\partial_t \tilde{\rho} = \partial \tilde{\rho} / \partial t$, and where $(\partial_t \tilde{\rho})_{acc}$ reflects changes caused by acceleration towards the desired velocity and $(\partial_t \tilde{\rho})_{int}$ denote changes caused by interactions between vehicles.

Interaction process

Prigogine (1961) assumed that when an incoming fast moving car (velocity v) reaches a slow moving vehicle (velocity $w < v$), it either passes directly, or it slows down to the velocity w of the slow vehicle. The following assumptions are made:

- The *slow-down event* has a probability of $1 - \pi$, while the *immediate overtaking event* has probability π. Overtaking does *not* affect the velocity v of the fast vehicle.

- The velocity w of the slow vehicle is unaffected by interaction events.

- The lengths of vehicles can be neglected.

- The fast vehicle slows-down instantaneously.

- Interactions affecting more than two vehicles are neglected.

Based on these assumptions, the interaction term equals the so-called *collision equation*:

$$(\partial_t \tilde{\rho})_{int} = (1 - \pi) \int (w - v) \tilde{\varphi}(x, v, x, w, t) \mathrm{d}w \tag{10}$$

The assumption of *vehicular chaos* – vehicles are *uncorrelated* – implies:

$$\tilde{\varphi}(x, v, x', v', t) = \tilde{\rho}(x, v, t) \cdot \tilde{\rho}(x', v', t) \tag{11}$$

Thus, the collision equation (10) becomes:

$$(\partial_t \tilde{\rho})_{int} = (1 - \pi) \tilde{\rho}(x, v, t) \int (w - v) \tilde{\rho}(x, w, t) \mathrm{d}w \tag{12}$$

Acceleration process

The acceleration process describes relaxation of drivers' speed towards a traffic-condition-dependent equilibrium velocity. Prigogine and Herman (1971) proposed the following expression:

$$(\partial_t \tilde{\rho})_{acc} = -\partial_v (\tilde{\rho} \cdot (\tilde{V}^0(v \mid x, t) - v) / \tau) \tag{13}$$

where τ denotes the acceleration time and $\tilde{V}^0(v\,|\,x,t)$ is the *desired velocity* distribution.

Nelson (1995) and Nelson *et al.* (1997) improved the gas-kinetic models, by relaxing the assumption of Prigogine and Herman (1971) that speeds of slowing-down vehicles are *uncorrelated* with speeds of impeding vehicles. This is achieved by describing both slowing-down and speeding-up interactions by a quadratic Boltzmann term, based on suitable *mechanical* (driver-reaction) and vehicular *correlation* models, while neglecting passing interactions.

4.4   Improved gas-kinetic model of Paveri-Fontana

The interaction term has been criticised, mainly concerning the validity of the *vehicular chaos assumption*. It has been argued that the collision term "corresponds to an approximation in which correlation between nearby drivers is neglected", being only valid in situations where no vehicles are platooning (cf. Munjal and Pahl (1969)). Paveri-Fontana (1975) considers a hypothetical scenario where a free-flowing vehicle catches up with a slow moving queue. He considers two extreme cases:

1. The incoming vehicle passes the *whole* queue as if it were one vehicle.

2. It consecutively passes each single car in the queue *independently*.

Paveri-Fontana (1975) shows that the Prigogine and Herman formalism reflects the second case, while the real-life situation falls between these two extremes. He also shows that the term reflecting the acceleration process yields a desired velocity distribution that is dependent on the local number of vehicles. This is in contradiction to the well-accepted hypothesis that driver's personality is indifferent with respect to changing traffic conditions (the so-called *personality condition*; cf. Daganzo (1995)). To remedy this deficiency, Paveri-Fontana considers the *Phase-Space Density* (PSD) $\rho(x,v,v^0,t)$. The latter can be considered as a generalisation of the *reduced* Phase-Space-Density with an independent variable describing the desired velocity v^0. He proposed using equation (9) in which[*]:

$$(\partial_t\rho)_{\mathrm{acc}} = -\partial_v(\rho \cdot (v^0 - v)/\tau) \tag{14}$$

and:

$$(\partial_t\rho)_{\mathrm{int}} = -\rho(v,v^0)\int_{w<v}(1-\pi)\,|\,w-v\,|\,\tilde{\rho}(w)\mathrm{d}w + \tilde{\rho}(v)\int_{w>v}(1-\pi)\,|\,w-v\,|\,\rho(w,v^0)\mathrm{d}w \tag{15}$$

where the *reduced PSD* equals:

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[*] For convenience, we have dropped the dependence on (*x,t*) from notation.

$$\tilde{\rho}(v) = \tilde{\rho}(x,v,t) \stackrel{def}{=} \int \rho(x,v,v^0,t) \mathrm{d}v^0 \qquad (16)$$

Another approach to correct the interaction term is reported in Beylich (1978), where the number of vehicles in a platoon is considered as an attribute. This approach is similar to the approach of Hoogendoorn (1999), who describes traffic as a collection of platoons (see section 4.6).

Another issue, raised by among others Nelson *et al.* (1998) is that plausible speed-density relations can only be determined from the Prigogine-Herman model, based on the nontrivial assumption that the underlying distribution of desired speeds is *non-zero for vanishingly small speeds*. Nelson and Sopasakis (1998) investigate the situation when this assumption does not hold. It is found that at concentrations above some critical value, there is a two-parameter family of solutions, and hence a continuum of mean velocities for each concentration. This result holds for both constant values of the passing probability and the relaxation time, and for values that depend on concentration in the manner assumed by Prigogine and Herman. It is hypothesised that this result reflects the well-known tendency toward substantial scatter in observational data of traffic flow at high concentrations.

4.4.1 *Gas-kinetic multiclass traffic flow modelling*

Hoogendoorn and Bovy (2000b) developed gas-kinetic multiclass traffic flow models, describing the dynamics of the so-called Multiclass Phase-Space-Density (MUC-PSD) $\rho_u(x,v,v^0,t)$, where u indicates the user-class u from the set **U**, yielding:

$$\partial_t \rho_u + v \partial_x \rho_u = (\partial_t \rho_u)_{\mathrm{acc}} + (\partial_t \rho_u)_{\mathrm{int}} \qquad (17)$$

where:

$$(\partial_t \rho_u)_{\mathrm{acc}} = -\partial_v (\rho_u \cdot (v^0 - v)/\tau_u) \qquad (18)$$

describes the acceleration process of vehicles of class u towards their desired velocity v^0, given the acceleration time τ_u, and where:

$$(\partial_t \rho_u)_{\mathrm{int}} = -(1-\pi_u)\sum_s (I_{u,s}(x,t) - R_{u,s}(x,t)) \qquad (19)$$

where $I_{u,s}(x,t)$ and $R_{u,s}(x,t)$ are respectively defined by:

$$I_{u,s}(x,t) = \int_{w<v} |w-v| \rho_u(x,v,v^0,t)\rho_s(x,w,w^0,t)\,\mathrm{d}w\,\mathrm{d}w^0 \qquad (20)$$

and:

$$R_{u,s}(x,t) = \int\limits_{w>v} |w-v| \rho_u(x,w,v^0,t) \rho_s(x,v,w^0,t)\, dw\, dw^0 \tag{21}$$

Note that $I_{u,s}(x,t)$ and $R_{u,s}(x,t)$ describe the dynamic influences of vehicular interactions of class u with vehicles from the same ($s = u$) or other classes ($s \neq u$). The assumptions underlying the model of Hoogendoorn and Bovy (2000b) are equivalent to those of Prigogine (1961) (see section 4.3). Compared to the model of Paveri-Fontana (1975), distinction of user-classes results in an *asymmetric* slow-down process of fast vehicles, i.e. vehicles of relatively fast user-classes catch up with vehicles of slow user-classes more frequently than vice-versa.

4.5   Gas-kinetic multilane equations of Helbing

Helbing (1997a) presents a gas-kinetic model for multilane traffic flow operations. The approach is similar to the approach of Paveri-Fontana (1975), although lane-changing is explicitly considered. Let j denote the lane index. Then, let $\rho_j(x,v,v^0,t)$ denote the so-called multilane Phase-Space Density (ML-PSD) on lane j. Helbing proposes the following relation:

$$\partial_t \rho_j + v\partial_x \rho_j = (\partial_t \rho_j)_{ACC} + (\partial_t \rho_j)_{INT} + (\partial_t \rho_j)_{VC} + (\partial_t \rho_j)_{LC} + v_j^+ - v_j^- \tag{22}$$

In contrast to the model of Paveri-Fontana (1975) and Hoogendoorn and Bovy (2000b), the multilane model of Helbing considers three additional terms:

- The *velocity diffusion term* $(\partial_t \rho_j)_{VC}$, taking into account the individual fluctuations of the velocity due to imperfect driving.

- The *lane-changing term* $(\partial_t \rho_j)_{LC}$ modelling dynamic changes in the ML-PSD due to vehicles changing from and to lane j.

- The rate of vehicles entering and leaving the roadway $v_j^{\pm}$.

The interaction process is similar to the original model of Paveri-Fontana. Another multilane gas-kinetic model was proposed by Klar and Wegener (1998).

4.6   Generic gas-kinetic models and platoon-based multilane multiclass modelling

Recently, Hoogendoorn (1999) developed a traffic flow model mesoscopically describing the dynamics of generic traffic flow systems. The model describes the dynamics of the generic phase-space density (g-PSD) $\rho_a(\mathbf{x},\mathbf{v},\mathbf{w},t)$ where $\mathbf{x}$ and $\mathbf{v}$ are vectors in the n-dimensional space respectively denoting the location and velocity of the traffic entities. The vectors $\mathbf{w}$ and $\mathbf{a}$ respectively denote *continuous*

and *discrete* attributes of the traffic entities, for instance desired velocity, destination, user-class, or lane. The dynamic equations presented by Hoogendoorn (1999) describe how the g-PSD changes due to convection, acceleration, smooth adaptation of continuous attributes, event-based non-continuum processes (e.g. interaction of traffic entities), and condition-based non-continuum processes (e.g. postponed lane changing). Let us emphasise that the generic model of Hoogendoorn (1999) unifies the different theories of Prigogine and Herman (1971), Paveri-Fontana (1975), Helbing (1997a), and Hoogendoorn and Bovy (2000b) in that all these models are special cases of the generic model of Hoogendoorn (1999).

Hoogendoorn (1999) specifies the generic model equations to derive a gas-kinetic platoon-based model for *multilane multiclass traffic flow operations*, describing dynamics of vehicles of class u on roadway lane j, with are either freely flowing ($c = 1$) or platooning ($c = 2$) via the multilane multi-class (MLMC) PSD $\rho_{(u,j,c)}(x,v,v_0,t)$:

$$\partial_t \rho_{(u,j,c)} + v \partial_x \rho_{(u,j,c)} = (\partial_t \rho_{(u,j,c)})_{ACC} + (\partial_t \rho_{(u,j,c)})_{INT} + (\partial_t \rho_{(u,j,c)})_{ILC} + (\partial_t \rho_{(u,j,c)})_{CLC} \qquad (23)$$

By representing traffic as a *collection of platoons*, rather than largely independently moving entities, several drawbacks of earlier gas-kinetic models are remedied. For one, Hoogendoorn (1999) argues that gas-kinetic flow models *overestimate* the number of vehicle interactions due to the vehicular chaos assumption. Secondly, in real-life traffic flow, the acceleration behaviour of *platooning* vehicles is completely determined by the platoon-leader. Both modelling problems can be solved by separately describing traffic flow operations for free-flowing and platooning vehicles. Finally, finite-space requirements of vehicles are included (see also Helbing (1997a)).

5 Continuum macroscopic traffic flow models

Macroscopic traffic flow models assume that the *aggregate* behaviour of drivers depends on the traffic conditions in the drivers' direct environments. That is, they deal with traffic flow in terms of aggregate variables. Usually, the models are derived from the analogy between vehicular flow and flow of continuous media (e.g. fluids or gasses), yielding flow models with a limited number of equations that are relatively easy to handle. In this section, we discuss *continuum* macroscopic flow models. That is, we will consider models describing the dynamics of macroscopic variables (e.g. density, velocity, and flow) using partial differential equations. Since the model to be established in this paper is a macroscopic model, the discussion will be elaborate.

The independent variables of a continuous macroscopic flow model are location x and time instant t. To introduce the dependent traffic flow variables, consider a small segment $[x, x+dx)$ of a roadway

referred to as 'cell x'. Most macroscopic traffic flow models describe the dynamics of the density $r = r(x,t)$, the velocity $V = V(x,t)$, and the flow $m = m(x,t)$. The density $r(x,t)$ describes the *expected number of vehicles* on the roadway segment $[x,x+\mathrm{d}x)$ *per unit length* at instant t. The flow $m(x,t)$ equals the *expected number of vehicles* flowing past x during $[t,t+\mathrm{d}t)$ per time unit. The velocity $V(x,t)$ equals the *expected velocity of vehicle* defined by $m(x,t)/r(x,t)$. Some macroscopic traffic flow models also contain partial differential equations of the *velocity variance* $\Theta = \Theta(x,t)$, or the *traffic pressure* $P = P(x,t) = r\Theta$.

We will discuss three types of continuum macroscopic models, namely:

1. Lighthill-Whitham-Richards models (dynamic equations of r).

2. Payne-type models (dynamic equations of r and V).

3. Helbing-type models (dynamic equations of r, V, and Θ).

5.1 Conservation of vehicles

Let us assume that the dependent traffic flow variables are differentiable functions of time and space. Then, the following relations hold exactly:

$$m = rV \tag{24}$$

and the *conservation of vehicles equation*:

$$\partial_t r + \partial_x m = 0 \tag{25}$$

describing that the number of vehicles in cell x increases according to the balance of inflow at the boundaries x and $x+\mathrm{d}x$ of the cell.

Eqns. (24) and (25) constitute a system of two independent equations and three unknown variables. Consequently, to get a complete description of traffic dynamics, a third independent model equation is needed. In the ensuing of this section, several model specifications are considered.

5.2 Lighthill-Whitham-Richards (LWR) type models

The most straightforward approach is to assume that the expected velocity V can be described as a function of the density r:

$$V(x,t) = V^e(r(x,t)) \tag{26}$$

The resulting non-linear first-order partial differential equation was introduced by Lighthill and Whitham (1955):

$$\partial_t r + \partial_x (rV^e(r)) = 0 \tag{27}$$

A drawback of the LWR-model is that it does not yield a unique continuous solution. In addition, the admissible *generalised solutions* are not unique (cf. Lebacque (1996)). Generalised solutions of the non-linear equation (25) can be determined by studying the so-called *characteristic curves* along which information from the initial traffic conditions are transported (see Leutzbach (1988)). It can be shown that from each point on the line $t = 0$, a single characteristic curve originates. These curves $C = \{x(t), t\}$ are *straight lines*, defined by:

$$C = \{t \cdot a(r(x,0)), t\} \tag{28}$$

where $a(r) = \mathrm{d}(rV^e(r))/\mathrm{d}r$. When $\partial_x a(r(x,0)) < 0$, the characteristic curves intersect and a *shock wave* is formed is the speed of the kinematic wave. The shock wave velocity can then be determined using the so-called *Rankine-Hugoniot conditions* (see Hirsch (1990a)), stating that the velocity V_c of the shock, multiplied by the jump in the density shock $[r] = r_2 - r_1$ of the two regions 1 and 2 separated by the shock, equals the jump in the local flow-rate $m_2 - m_1$:

$$V_c = [m]/[r] = (m_2 - m_1)/(r_2 - r_1) \tag{29}$$

Clearly the direction of the shock wave depends on the sign of $m_2 - m_1$ (outflow from the shock minus inflow of the shock)[*]. The equations *do not yield continuous solutions*. Therefore, in practical applications, the *entropy* or (*vanishing*) *viscosity solutions* are determined (see Lebacque (1996)). These vanishing viscosity solutions are solutions of the partial differential eq. (25) to which a second order viscosity term $\delta \cdot (\partial_x^2 r_\delta)$ is added. The resulting solutions r_δ are approximate solutions of (25). The *vanishing* viscosity solution is defined by $r = \lim_{\delta \to 0} r_\delta$. It is observed that the viscosity solution is a unique generalised and physically feasible *solution* of eq. (25).

5.3   Payne-type models

Payne (1971) proposed the first continuum traffic flow model by a coupled system of two partial differential equations, that is, the system of eqns. (24) and (25) are extended by a partial differential

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[*] Note that since for a shock to occur we have $[r] > 0$, an *upstream moving shock* ($V_c < 0$) implies $m_2 < m_1$. Since this can only occur when upstream traffic conditions are *congested*, in the one-dimensional case an *upstream directed* characteristic implies *congested* traffic conditions.

equation describing the dynamics of the velocity *V*. Payne's model is derived from a simple car-following rule (8). Application of Taylor's expansion rule to the left-hand-side and the right-hand-side of (8) yields:

$$V(x(t+T), t+T) \approx V(x,t) + T \cdot V(x,t)\partial_x V(x,t) + T\partial_t V(x,t) \tag{30}$$

and:

$$V^e(r(x+D,t)) \approx V^e(r(x,t)) + D\partial_x r(x,t)\frac{\mathrm{d}}{\mathrm{d}r}V^e(r(x,t)) \tag{31}$$

Since, the density equals $r = 1/D$, by substituting (30) and (31) into (8), we find:

$$\partial_t V + \overbrace{V\partial_x V}^{C} = \overbrace{(V^e(r) - V)/T}^{R} - \overbrace{(c_0^2/r)\partial_x r}^{A} \tag{32}$$

where $c_0^2 = \xi/T > 0$ is the anticipation constant, with $\xi = -\mathrm{d}V^e/\mathrm{d}r$ describing the decrease-rate in the equilibrium velocity with increasing density. Assuming a constant decrease-rate implies a linear relation between the density r and the equilibrium velocity $V^e(r)$.

Payne identified *convection*, *relaxation*, and *anticipation* terms:

- *Convection* (term *C* of eq. (32)) describes changes in the mean velocity due to inflowing and out-flowing vehicles.

- *Relaxation* (term *R* of eq. (32)) describes the tendency of traffic flow to relax to an equilibrium velocity.

- *Anticipation* (term *A* of eq. (32)) describes the drivers' anticipation on spatially changing traffic conditions downstream.

In the following section, these processes are discussed in more detail. For an overview of other recent developments of Payne-type models, we refer to Liu *et al.* (1998).

Similar to the LWR-model, the state of the system at a point (x,t) can be determined by considering the characteristics emanating from the line $t = 0$. In contrast to the LWR-model, *two* characteristics (the so-called *Mach-lines*) emanate from each point $(x,0)$ which are not straight lines. Rather, these curves $C^\pm = \{x^\pm(s), s\}$ are defined by the following differential equations:

$$\mathrm{d}x^\pm = (V(x(t),t) \pm c_0)\mathrm{d}t \quad \text{with} \quad x(0) = x \quad \text{and} \quad V(x,0) = V_0(x) \tag{33}$$

From this result we can conclude that c_0 is comparable to the *local sonic velocity* for the propagation of disturbances in gasses. Moreover, since small perturbations are transported along these characteris-

tics, the absence of a path-line (defined by $dx = V(x(t),t)dt$) yields that disturbances in the flow are not transported together with the vehicles. Clearly, this also holds for the LWR-model, which is a serious drawback of both modelling approaches. We refer to Hoogendoorn (1999) for a detailed description of characteristic curves in vehicular traffic flow.

General form of Payne-type model

A more general model form of equation (32) is given by:

$$\dot{V} = \partial_t V + \overbrace{V \partial_x V}^{C} = \overbrace{(V^e(r) - V)/T}^{R} - \overbrace{(1/r)\partial_x P + (\eta/r)\partial_x^2 V}^{A} \tag{34}$$

where P is the *traffic pressure* and η is the *kinematic traffic viscosity*. The total time derivative $\dot{V}$ describes the rate of velocity changes experienced by a moving observer who observes the traffic flow while moving along with the stream at the same velocity V. Note that for the Payne model (32), we have $P = r \cdot c_0^2$ and $\eta = 0$. By introducing traffic viscosity, *approximate smooth* solutions of Payne's model result. In addition, the numerical treatment of these higher order models is simplified.

From equation (34) we observe that the total time derivative is composed of a true time derivative $\partial_t V$ and a *convection term C*. The latter term describes changes in the velocity V due to inflowing vehicles with a different velocity. The *relaxation term R* describes the tendency of the traffic stream to adjusts its velocity to an equilibrium value $V^e(r)$.

Several authors (e.g. Payne (1971), Kerner *et al.* (1996), Lyrintzis *et al.* (1994), and Liu *et al.* (1998)) argue that the traffic flow interpretation of the term A of equation (34) differs from the classical meaning of this term in kinetic theory of compressible media. These authors argue that term A describes drivers *anticipatory behaviour* on changing traffic conditions downstream, reflected by regions of spatially changing traffic pressure P and changing spatial acceleration $\partial_x V$.

When an observer moving along with the traffic observes a region of spatially increasing traffic pressure ($\partial_x P > 0$), the total time derivative $\dot{V}$ decreases, implying that the moving observer decelerates. The second order term $\eta(\partial_x^2 V)/r$ shows that in regions of spatial accelerations, i.e. $\partial_x^2 V > 0$, this *diffusion* term yields an increase in the velocity of the moving observer. That is, when the moving observer drives in a region of spatial acceleration, the driving will *go along with the other drivers*. Summarising, the pressure term $\partial_x P$ describes the local anticipation behaviour of drivers, whereas the diffusion term $\partial_x^2 V$ describes the higher-order tendencies of drivers. In opposition, Helbing (1996) argues that the second order term should be deprived of its classical meaning of viscosity; rather the diffusion term reflects changing drivers' states (brisk to careful driving).

Hoogendoorn (1999) argues that the latter interpretation of the anticipation term recalled in the above paragraph is flawed. He shows that the spatial derivative of the traffic pressure P must be interpreted in a similar way as the pressure term in equations describing the dynamics of a compressible continuous medium like a fluid or a gas. That is, it reflects the changes in the mean velocity of the cell caused by groups of vehicles having different velocities flowing into the cell.

In the kinetic theory of fluids and gasses, the so-called *viscosity term* reflects changes due to the *friction* between particles in the medium as well as between particles and boundaries. When particles move away from other particles (*accelerate*), friction causes the following particle to be 'dragged along' with the accelerating particle. Thus, *viscosity* in vehicular flow is not essentially different from viscosity in viscous continuous media.

Let us finally remark that two *competing* processes are conveyed by the relaxation term R in eq. (34). On the one hand an *active* process, conveying that drivers aim to traverse the roadway at their desired velocities. On the other hand, a *damping process*, conveying that drivers are slowed down due to interaction with other vehicles. Kerner *et al*. (1996) show that under specific circumstances, these competing processes result in the spontaneous occurrence of seemingly random traffic jams, so-called *phantom jams* that are also observed in real-life traffic flow (e.g. Kerner (1999)). Payne (1971) has shown that given certain circumstances, his model yields unstable behaviour. That is, in a certain density area the traffic model is *metastable*. In this region, small variations in the traffic density will yield regions of increasing traffic densities, leading to the occurrence of *start-stop* waves or localised traffic jams. This is a very important property of the Payne-type models.

Models of Philips, Kühne, and Kerner

Philips (1979) proposed to use a *density dependent relaxation time $T = T(r)$*. Moreover, he approximates the traffic pressure using $P(r) = r\Theta^e(r)$, with $\Theta^e(r) = \Theta^0(1 - r/r_{jam})$. Note that the velocity variance equals zero when the density equals the *jam-density r_{jam}*.

Kühne (1991) and Kerner *et al*. (1996) choose $P = r \cdot c_0^2$ and $\eta = \eta_0$. That is, both the velocity variance and the traffic viscosity are constants. The assumption of a constant velocity variance is not realistic. Rather, in equilibrium, the velocity variance *decreases* with *increasing* traffic density, i.e. $d\Theta^e(r)/dr < 0$ (see Helbing (1997a)). Finally, improvements to both Payne's original modelling efforts as well as several numerical schemes to solve the resulting modelling equations are due to Lyrintzis *et al*. (1994) and Liu *et al*. (1998).

Non-local traffic model of Helbing

A recent contribution to the field of macroscopic flow models is given by Treiber *et al.* (1999). The authors propose a flow model based on gas-kinetic principles. The relationship for the equilibrium velocity follows from the acceleration of vehicles on the one hand, and the interaction of vehicles on the other hand. The latter process incorporates a non-local interaction term that reflects the anticipatory behaviour of drivers:

$$V^e = V^0 - T(1 - p(r'))r'\widetilde{\Theta} \tag{35}$$

where V^0 is the expected desired velocity, $p(r')$ is the *immediate lane-changing probability*, and $\widetilde{\Theta}$ is an *interaction term* reflecting the influence on the velocity of vehicular interactions. The prime indicates that the corresponding variable should be considered at the *interaction point* $x' = x + \gamma \cdot (1/r_{jam} + TV)$, with $\gamma \approx 1$. The authors claim that the non-local interaction term is very favourable for robust numerical approximation.

Helbing *et al.* (1998) show how this non-locality yields the occurrence of *different traffic states* (types of congestion), depending on the combined values of the flow-rate on the main road and the on-ramp respectively. The paper shows that the model is generally in good agreement with Dutch motorway data. The paper also reveals clearly how both the self-formation of spatial clusters and the congestion due to active bottlenecks are correlated, that is, caused by similar mechanisms. However, due to the bottleneck, traffic breakdown under metastable or linearly unstable conditions will occur frequently (that is, with a very high probability).

5.4   Helbing-type models

Helbing (1996) has extended the Payne-type models by introducing an additional partial differential equation for the velocity variance Θ. His macroscopic model is derived from gas-kinetic equations and consists of the conservation of vehicles equation (25), the velocity dynamics (34), and the following equation describing the dynamics of the variance Θ:

$$\partial_t \Theta + V \partial_x \Theta = -2(P/r)\partial_x V + 2(\Theta^e - \Theta)/T - (1/r)\partial_x J \tag{36}$$

where the *flux of velocity variance* $J = J(x,t) = r(x,t)\Gamma(x,t)$ is defined by the product of the density and the skewness of the velocity distribution. Rather than being experimentally determined, the equilibrium velocity V^e and variance Θ^e are determined by considering the interaction process between

vehicles in the stream. The resulting expressions are functions of the density r, the velocity V and the velocity variance Θ, namely:

$$V^e(r,V,\Theta) = V^0 - T(1 - p(r))P \quad \text{and} \quad \Theta^e(r,V,\Theta) = C - T(1 - p(r))J \tag{37}$$

where $p(r)$ denotes the immediate overtaking probability while C is the *covariance* between the velocity and the desired velocity. The model equations are 'closed' by specifying expressions for p, C and J. Helbing (1996) also proposes techniques to incorporate the fact that vehicles occupy a non-vanishing amount of roadway space.

The way in which disturbances in the flow are transported can again be analysed by considering the characteristic curves (see Hoogendoorn (1999)). Helbing-type models have *three characteristic curves* (one path-line and two Mach-lines), along which small perturbations propagate. This implies that small disturbances are transported both along with the (mean) traffic flow as well as in the upstream and downstream directions with respect to this mean flow.

5.5   Semi-discrete and discrete macroscopic flow models

In this section, we will discuss some discrete macroscopic traffic flow models. Frequently, these models are established by application of a finite difference scheme to the continuous model equations described in the previous section. If so, it is of dominant importance that the numerical solution approach preserves the essential characteristics of the underlying continuum model. Additionally, when solutions of the continuum model are *not unique*, which is the case for all first-order models, care should be taken that the physically feasible solution is approximated by the numerical solution technique.

Most of these solution approaches involve numerical approximation in the spatial direction (dividing the roadway into small segments $i = [i\Delta x,(i+1)\Delta x)$), the temporal direction (dividing the time axis into small periods $k = [k\Delta t,(k+1)\Delta t)$), or both. Applicability of solution approaches depends on the mathematical characteristics of the underlying system of equations (e.g. order of the equations, number of equations).

5.5.1   Discrete Lighthill-Whitham-Richards model

Determination of analytical solutions for the LWR-model is possible using the *method of characteristics* (Hoogendoorn,1999)). Since application of this method can be cumbersome, several approaches have been used to determine numerical solutions to the LWR-model.

The *Cell-Transmission model* (Daganzo (1994a,b)) is a discrete flow model that uses carefully selected segment sizes, and a *piecewise linear* relation for $M^e(r) = rV^e(r)$. That is, given a time-step Δt, the lengths Δx of cells are chosen such that under free-flow conditions, all vehicles in cell i flow into the downstream cell $i+1$, i.e. $\Delta x = V^0 \cdot \Delta t$. The number of vehicles flowing out of segment i is bounded by the space left on segment $i+1$. Daganzo (1999) further improves the Cell-Transmission model.

Lebacque (1996) applied the *Godunov-scheme* to the LWR model. The author shows that the Cell-Transmission model is a special case of the general Godunov solution approach. The scheme has the nice interpretation that the *flow* out of the segment i is locally defined by the smallest of two quantities, namely the *local traffic demand* and *supply*.

Multiclass multilane generalisations

Daganzo (1997) presents a generalised theory to model motorways in the presence of two vehicle types and a subset of lanes reserved for one of the vehicle classes. It describes the case of a long homogeneous motorway, based on the Cell-Transmission model.

Hoogendoorn (1999) has developed a multilane multiclass traffic flow model based on mesoscopic principles. The model inherits a number of the properties of the underlying gas-kinetic equations (e.g. description using platoons, finite-space requirements).

5.5.2   Semi-discrete and discrete Payne-type models

By spatial discretisation, Smulders (1990) determines a macroscopic flow model based on the model of Payne (1971) that is continuous in time and discrete in the spatial direction. Furthermore, using the theory of *martingales* he introduces a stochastic component based on a counting process. The author approximates the flow m by a convex sum approximation of the average density r and velocity V of consecutive discrete segments i and $i+1$:

$$\mathrm{d}r_i = (m_{i-1} - m_i)\mathrm{d}t / \Delta x + (\mathrm{d}\xi_{i-1} - \mathrm{d}\xi_i) / \Delta x \tag{38}$$

where the approximation of the outflow of segment i equals:

$$m_i = (\alpha r_i + (1-\alpha)r_{i+1})(\alpha V_i + (1-\alpha)V_{i+1}) \tag{39}$$

and where ξ_i reflects the stochastic departure process of vehicles from segment i. The velocity dynamics are given by:

$$\mathrm{d}\, r_i = \overbrace{\frac{1}{\Delta x} V_{i-1}(V_{i-1} - V_i)\,\mathrm{d}\, t}^{C} + \overbrace{\frac{V^e(r_i) - V_i}{T}\,\mathrm{d}\, t}^{R} - \overbrace{\frac{c_0^2}{\Delta x}\frac{r_{i+1} - r_i}{r_i + c}\,\mathrm{d}\, t}^{A} + \overbrace{\mathrm{d}\, w_i}^{N} \tag{40}$$

where $c > 0$ is a constant.

With respect to the conservation equation (38), this approximation yields the conservation of vehicles. However, unless $\alpha = 1$ vehicles may flow out of an empty downstream segment i, possibly yielding negative density values. Moreover, if the downstream segment is congested (i.e. $r_{i+1} = r_{jam}$, and $V_{i+1} = 0$), vehicles will still flow out of the upstream cell into the saturated downstream section.

The approximation of the velocity dynamics is heuristically established by considering the traffic flow interpretation of the respective processes. In illustration, since the term $-c_0^2(\partial r/\partial x)/r$ is assumed to reflect drivers' anticipatory behaviour, it is approximated using a *forward* discretisation scheme, yielding term A of eq. (40). We note that Smulders (1990) has modified the latter term. The modified term has been validated successfully using empirical data.

Examples of *discrete Payne-type models* are the models of Payne (1979), Van Maarseveen (1982), Papageorgiou *et al.* (1989), Kotsialos *et al.* (1999), Lyrintzis *et al.* (1994), and Liu *et al.* (1998). In the latter two contributions, several numerical solution techniques applied in physics are applied to the different traffic flow models. Another method is applied by Kerner *et al.* (1996). Here, the equations (25) and (34) are transformed using $W = \partial_x V$ into a *quasi-linear* system of partial differential equations and successively solved using a central difference approach (cf. Hirsch (1990a,b)). Let us remark that most of these schemes need the second-order viscosity term for numerical stability.

A different approach is used by Van Aerde (1994), who numerically approximates solutions of the continuum model by a *particle discretisation method* thereby constructing the 'microscopic' simulation model INTEGRATION (see section 3.5). This particle discretisation approach has been extensively applied in other fields of research, for instance in hydrodynamic flow modelling and filtering. Recently, Hoogendoorn and Bovy (2000) applied a particle discretisation method to numerically solve a gas-kinetic pedestrian flow model, thereby establishing a two-dimensional stochastic microscopic pedestrian flow model.

A numerical solution procedure for Helbing's model is proposed in Helbing (1997a). The author suggests using an explicit two-step (*predictor-corrector*) *MacCormack* scheme. The stability of this scheme depends on the relative magnitude of the viscosity terms. Hoogendoorn (1999) proposes an *upwind scheme* based on *splitting of the flux-vector*, based on the *conservative model formulation*.

Hoogendoorn (1999) also proposes a scheme based on the Godunov-approach. Both numerical solution approaches to a large extent use physical properties of underlying partial differential equations.

6   Links between modelling approaches

Although being fundamentally different, relations between (sub-) microscopic, mesoscopic and macroscopic flow models are reported in the literature. Figure 1 provides an overview of these relations.

Klar and Wegener (1998) describe a hierarchy of models: the authors present a simple microscopic flow model that is used to determine gas-kinetic flow equations. These are subsequently transformed into a macroscopic traffic flow model.

The derivation of the original Payne model can be considered as an example of 'degeneration' of microscopic flow model to a macroscopic flow model. Application of the method of moments (e.g. Leutzbach (1988)) yields macroscopic equations from mesoscopic traffic flow models. Ysertant (1997) presents a particle method to derive macroscopic model equations for compressible fluids. Nagel (1998) shows the relation between CA-models and the simple wave model.

We have discussed derivation of microscopic models from macroscopic equations by Van Aerde (1984), using a particle discretisation method. Such methods have been applied to numerical approximation of dynamic models of flows in continuous media (Hockney and Eastwood (1988)). Application of particle discretisation methods to derive microscopic models from gas-kinetic equations has recently been reported by Hoogendoorn and Bovy (2000a), who applied the method to gas-kinetic equations describing pedestrian flows.
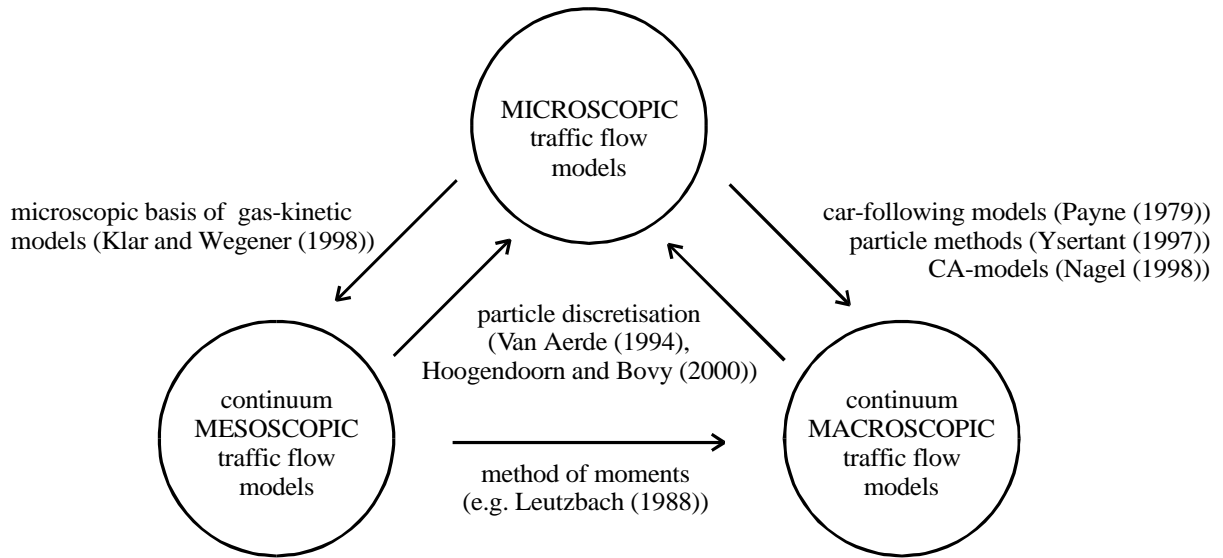
Figure 1: Relations between microscopic, mesoscopic, and macroscopic flow models.

Regarding the relation between microscopic and macroscopic traffic flow models, the work of Franklin (1961) and Del Castillo (1996) must also be mentioned. Franklin (1961) has developed a microscopic model that captures macroscopic features of traffic flow, such as shockwaves, using a *stimulus-response* car-following model. In the same vein, Del Castillo (1996) proposes a car-following model, the three parameters of which can be determined directly from speed-density data. This car-following model exhibits similar shockwave propagation behaviour as the model of Franklin (1961).

7   Issues on applicability of modelling approaches

When developing and using traffic flow models, the appropriate 'level-of-detail' of the modelling approach (i.e. submicroscopic, microscopic, mesoscopic, macroscopic) must be considered. This section provides some viewpoints from the literature on this issue. Based on these viewpoints, we conclude that the correct level-of-detail that should be considered is largely dependent on the envisaged model application. It has been suggested that macroscopic models should be used when the available model development time and resources are too limited for development of a microscopic model. Nevertheless, we argue that in some occasions, macroscopic modelling approaches *provide better results* than modelling approaches with a higher level-of-detail. In our opinion, development and maintenance costs are of less importance.

7.1 Non-linear and chaotic-like behaviour of traffic flow

Several authors have observed the non-linear of even chaotic-like behaviour of the traffic system (cf. Bovy and Hoogendoorn (1998), Pozybill (1998)). Among these behaviours is the metastable or unstable behaviour of traffic flow, implying that in heavy traffic a critical disturbance can be amplified and develop into a traffic jam. Empirical experiments performed by Forbes (1958), and Edie and Foote (1958,1960) have shown that a disturbance at the foot of an upgrade propagates from one vehicle to the next, while being amplified until at some point a vehicle came to a complete stop. This *instability effect* describes that once the density crosses some critical density, traffic flow becomes rapidly more congested *without any obvious reasons*. More empirical evidence of this instability and start-stop wave formation can be found in among others Verweij (1985), Ferrari (1989), and Leutzbach (1991). Kerner and Rebhorn (1997) and Kerner (1999) show empirically that local jams can persist for several hours, while maintaining their form and characteristic properties. In other words, the stable complex structure of a traffic jam can and does exist on motorways[*]. These findings show that traffic flow has some *chaotic-like properties*, implying that *microscopic disturbances* in the flow can result in the on-set of local traffic jams persisting for several hours.

Microscopic simulation models are founded on the assumption that the behaviour of each individual vehicle is a function of the traffic conditions in its direct environment. However, the (microscopic) behaviour of humans in real-life traffic – not in contrived "car-following experiments" – is hard to observe, measure and validate (cf. Daganzo (1994a)). This is unfortunate, given the observed chaotic-like behaviour of the collective traffic flow discussed in the previous paragraph. That is, *microscopic details* of the simulation models have to be just right for the microscopic simulation to realistically describe and predict the stop-start waves in traffic flow. Thus, from a model calibration perspective, the large number of sometimes-unobservable parameters play a compromising role.

Conversely, in macroscopic models, the number of parameters is relatively small and, more importantly, comparably easy to observe and measure. Calibration and validation of macroscopic models therefore requires less effort than calibration of microscopic or mesoscopic models. Moreover, *macroscopic* models are deemed to describe *macroscopic* characteristics of traffic flow more accurately. In addition, microscopic simulation tools do not provide insight into the *macroscopic* mechanisms of traffic flow (e.g. shock wave behaviour). On the contrary, macroscopic models provide such insights by means of mathematical analysis and manipulation.

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[*] Apart from the formation of stop-and-go waves and localised structures, Kerner and Rehborn (1997) describe a hysteric phase-transition from free-traffic to *synchronised flow* that mostly appears near on-ramps.

7.2 On-line model applicability

Since in a microscopic simulation model each vehicle is described by its own equations of motion, computer-time and memory requirements grow proportionally to the number of simulated vehicles. Consequently, this type of modelling is most suitable for *off-line* traffic simulations, although with the emergence of fast and cheap microcomputers, this argument will gradually become of less importance. Nevertheless, macroscopic models are computationally less demanding, thereby allowing simulations of very large traffic networks.

More important is the fact that due to the very nature of microsimulation models, the absence of an analytical relation between model input and output, renders these models unsuitable for direct application in model based control approaches. In other words, since optimal model-based control requires the availability of explicit input-output relations for fast (on-line) computation, microscopic simulation models are not easily applicable. In opposition, the solutions of macroscopic models are available in closed analytical form, and are excellently applicable for model-based traffic flow estimation, prediction, and control approaches.

7.3 Model generalisability: user-class and lane distinction

Let us finally discuss the ability to generalise a model with respect to different user-classes and roadway lanes. In this respect, most microscopic simulation models are able to implement different user-classes such as person-cars, trucks, and vans. Moreover, most microscopic models describe the behaviour of these different vehicles on multilane facilities. Therefore, generalisation with respect to user-classes and lanes is not an issue.

Several researchers have tried to generalise both gas-kinetic models (cf. Helbing (1997b), Hoogendoorn and Bovy (2000)) as well as macroscopic models (e.g. Michalopoulos *et al.* (1984), Daganzo (1997a)) towards a multiclass and/or multilane traffic description. Let us emphasise that the generalisation is *not self-evident*, and requires detailed analysis before the asymmetric interaction processes between classes and lanes can be described correctly. Recenlty, Hoogendoorn (1999) presented operational multilane multiclass gas-kinetic and macroscopic traffic flow models.

7.4 Model calibration and validation

An important issue in application of microscopic simulation models is the model calibration. In general, the lack of 'microscopic' data result necessitates to macroscopic or mesoscopic calibration that cannot produce the optimal parameters since the number of degrees of freedom is too large. Brack-

stone and MacDonald (1998) recommend using suitable data sources (e.g. instrumented vehicles), and dis-assembling the models and testing them in a step-by-step fashion. Whenever new behavioural rules are added to the model, they should be tested extensively, and preferably in *isolation*.

A powerful instrument to study the impacts of changes in the model parameters is sensitivity analysis. This is especially useful for complex microscopic models, in which effects of the parameters on the flow behaviour are hard to analyse mathematically.

In general, macroscopic models are relatively easy to calibrate using loop detector data (see Cremer and Papageorgiou (1981), and Helbing (1997a)). Mostly, speed-density relations derived from observations are required. In a recent paper, Kerner *et al.* (2000) show that traffic jam dynamics can be described and predicted using macroscopic models that feature only some characteristic variables, which are to a large extent independent on roadway geometry, weather, etc. This implies that macroscopic models can describe jam propagation reliably, without the need for in-depth model calibration.

7.5   Conclusions on applicability of modelling approaches

Microscopic simulation models are most suitable for *off-line* traffic simulations, for instance to perform detailed studies of geometric design and vehicle equipment (e.g. on-ramps, lane-merges, driver-support systems), or to gain insight into flow quantities that are difficult to determine empirically. However, their *application in on-line traffic control* is limited due to the large computation times and the absence of an explicit model input-output relation. Moreover, only few microscopic simulation models have been extensively calibrated and validated (see Algers *et al.* (1997)). Moreover, calibration generally pertains to aiming at reproducing macroscopic quantities such as speed-density curves by changing parameters describing driving behaviour.

Gas-kinetic models (mesoscopic) have been criticised for having too many variables to be solved in real-time, hampering their application to among other things on-line traffic control. However, several researchers (e.g. Prigogine and Herman (1971), Leutzbach (1990), Helbing (1996,1997a,b), Klar and Wegener (1998), and Hoogendoorn (1999) have used these gas-kinetic-type mesoscopic models for the derivation of their macroscopic models, for which gas-kinetic models are highly suitable. Finally, Hoogendoorn and Bovy (2000) have applied a particle discretisation approach to establish solutions to complex gas-kinetic pedestrian flow equations.

Macroscopic traffic flow models are suitable for large-scale simulations of traffic flow in networks. An example is the METANET model, which has for instance been applied to the peri-urban network of the Dutch city of Amsterdam, and the Paris network (see Kotsialos *et al.* (1999)).

Moreover, they are very suited for analysing and reproducing macroscopic characteristics in traffic flow, such a shock waves, and queue-lengths. This is justified by the observed chaotic-like nature of traffic flow. Moreover, recent developments in the direction of multiclass multilane traffic flow have improved model applicabity and accuracy. Nevertheless, macroscopic models are not very suited for analysing inherently microscopic characteristics of the flow. Most importantly, solutions to macroscopic models are usually formulated in closed analytical form, making them very suitable for application in model-based estimation, prediction, and control approaches.

8 Macroscopic modelling stream controversy

Concerning *macroscopic* modelling approach, the question which model type should be considered is not self-evident, given the ongoing debate between the LWR-model proponents and the Payne-type model (and Helbing-model) advocates. In this section we will discuss the main arguments presented by both streams to show the relative superiority of their respective modelling approaches. Let us remark that Lebacque and Lesort (1999) propose a methodology for theoretically comparing LWR-type models and Payne-type models. They propose a set of problems and situations that can be used as a test-case for model comparison. Although their approach has obvious limits, their work contributes to constitute an exhaustive model comparison framework.

8.1 Critique on LWR-type models

The simple continuum model has some shortcomings, given in the following list (see Liu *et al*. (1998) and Papageorgiou (1998)):

- *Steady-state speed-density relationship*. The LWR-models contain *stationary speed-density relations*, implying that the mean velocity adapts *instantaneously* to the traffic density rather than considering some delay. That is, the kinematic theory does not allow fluctuations around the equilibrium speed-density relationship.

- *Discontinuities in the density*. The kinematic wave-theory of Lighthill and Whitham shows shock wave formation by steeping velocity jumps to infinite sharp discontinuities in the density. In other words, it produces *discontinuous solutions* irrespective of the smoothness of initial conditions, due to the dominating convective term in the non-linear partial differential equation (25). These are in contradiction with smooth shocks observed in real-life traffic that can be described by Payne-type models.

- *Regular start-stop waves.* The LWR-theory is not able to describe regular start-stop waves with amplitude-dependent oscillation times that are observed in real-life traffic (e.g. Verweij (1985).

- *Traffic hysteresis.* In real-life traffic flow, hysteresis phenomena have been observed (cf. Treiterer and Myers (1974)), showing that the average headways of vehicles approaching a jam are smaller than vehicles flowing out of a jam. These hysteresis phenomena are not described by the LWR models. Conversely, the Payne-type models are able to describe traffic hysteresis (see Zhang (1999)).

- *Localised structures and phantom-jams.* Similarly, the LWR-models are not able to predict the occurrence of localised structures and phantom-jams, i.e. the LWR-theory does not describe the amplification of small disturbances in heavy traffic.

To include traffic hysteresis, different speed-density relations, and mechanisms to describe transients from one speed-density curve to another are to be included in the first-order model. Examples of these approaches are described in Newell (1965). To this end, Newell exploits the fact that acceleration flows and deceleration flows follow distinctively different paths in the speed-density plane. Another approach is described in Daganzo *et al*. (1999).

8.2   Critique on Payne-type models

The most fundamental criticisms regarding the Payne-type models have been formulated in Daganzo (1995). The author's criticisms stem from the dissimilarity between vehicular flow and the flow of molecules in a fluid or a gas:

- *Anisotropy*. In opposition to fluid particles responding to both stimuli from upstream and downstream, a driver-vehicle combination is an anisotropic particle. In other words, a driver will primarily react to traffic conditions downstream. In opposition to vehicular flow, particles in a fluid or a gas 'react' to stimuli from all directions.

- *Unaffected slow-vehicles*. The speed of slow vehicles should be virtually unaffected by faster vehicles. Conversely, the slow particles in a fluidic flow or gas flow are affected by faster particles.

- *Personality*. Unlike particles in a fluid or gas, driver-vehicle combinations have their own personalities that remain largely unaffected by traffic conditions. For instance, drivers are aggressive or timid, or drivers aim to drive at their desired velocities.

Daganzo (1995) shows that existing Payne-type models can result in negative speeds at the tails of congested regions. This is caused by the second order dissipation term. However, Liu *et al.* (1998) argue that the violation of the anisotropy condition is a result of the pressure term. They state that by observing the *inviscid flow equations* under congested conditions one characteristic curve moves *upstream*, while the other characteristic moves downstream (with a velocity which is larger than the average velocity of the flow). However, these characteristics describe the upstream moving influence of congestion downstream, and the fact that some vehicles drive faster than others do. They *do not reflect physical movements of vehicles in traffic flow*. In other words, the fact that a characteristic curve is directed *upstream* does not imply that vehicles move in that direction. Neither is it necessary that all vehicles have the same velocity. In fact, as is shown in Hoogendoorn (1999), c_0^2 can be interpreted as the *variance* in the velocities of the different vehicles.

Zhang (1998) shows that higher-order models do not correctly describe congestion spillback and the dynamics of shocks. To improve macroscopic modelling for higher-order models, the constant anticipation coefficient must either be replaced by a dynamically varying coefficient, thereby assuring that no characteristics precede the traffic flow when overtaking possibilities are small. Moreover, the ensure correct description of congestion spillback, a *non-local interaction* term must be present in the relaxation (interaction) term.

8.3 Finite space requirements, velocity variance, and finite reaction and braking times

In addition to Daganzo's criticism, Helbing (1996) argues that three other conditions need to be fulfilled for a valid macroscopic traffic flow model:

- *Finite space requirements:* vehicles are modelled by infinitely small particles, i.e. the *finite space requirements of vehicles* are seldom incorporated. As a consequence, on some occasions, the traffic density temporarily becomes larger than the bumper-to-bumper jam density.

- *Consideration of the velocity variance.* Most macroscopic traffic flow models neglect the essential role of the velocity variance. Kühne (1984a,b) observed that the velocity variance is an indicator for the occurrence of traffic breakdown.

- *Finite braking times and reaction times*: most macroscopic flow models neglect the finite reaction and braking times of driver-vehicle units.

The models of Helbing (1996,1997a), Hoogendoorn and Bovy (2000a), and Hoogendoorn (1999) consider these modelling issues.

9 Summary

This article provides an overview of the current state-of-the-art of vehicular traffic flow modelling. To provide a structured overview of these modelling achievements, the models have been classified according to level-of-detail (submicroscopic, microscopic, mesoscopic, macroscopic). Other criteria have been considered as well, namely scale of the independent variables (continuous, semi-discrete, discrete), representation of processes (deterministic, stochastic), operationalisation (analytical, simulation), and application area (e.g. links, stretches, networks).

With respect to model applicability, microscopic simulation models are ideally suited for off-line simulations, for instance to test roadway geometry. From the viewpoint of applicability to model-based estimation, prediction, and control, the absence of a closed analytical solution presents a problem that is not easily solved. Moreover, several authors argue that microscopic models are unsuitable to represent true macroscopic characteristics of traffic flow (queue lengths, capacities) due to unobservability of several parameters in microscopic flow models, and the nonlinear dependence of model outcomes on these parameters.

Gas-kinetic models have the advantage that they enable description behaviours of individual vehicles, without the need to describe their individual time-space behaviour. Nevertheless, the resulting equations have been criticised for having too many parameters and high dimensionality, hampering calibration and their real-time applicability. Nevertheless, they are ideally suited as a foundation to derive macroscopic flow models. Moreover, using particle dicretisation approaches, the gas-kinetic models can be microscopically discretised.

Macroscopic models are suited for large scale, network-wide applications, where macroscopic characteristics of the flow are of prime interest. Generally, calibration of macroscopic models is relatively simple (compared to microscopic and mesoscopic models). However, macroscopic models are generally too coarse to correctly describe microscopic details and impacts, for instance caused by changes in roadway geometry. Due to the availability of closed analytical solutions, there are however very suitable for application in model-based estimation, prediction, and control of traffic flow.

List of Frequently Used Symbols

$\tilde{\rho}(x, v, t)$:   reduced phase-space density at (x,t) (veh/m)

$\rho(x, v, v^0, t)$:   phase-space density at (x,t) (veh/m)

π (or p)   :   immediate overtaking probability

τ :   acceleration time (s)

$\Theta(x,t)$:   velocity variance at (x,t) (m^2/s^2)

c_0 :   local sound speed / wave velocity (m/s)

D :   gross distance headway with respect to its predecessor (m)

L :   length of vehicle (m)

$m(x,t)$:   traffic momentum / traffic flow at (x,t) (veh/s)

$P(x,t)$:   traffic pressure at (x,t) $(veh\text{-}m/s^2)$

$r(x,t)$:   traffic density at (x,t) (veh/m)

u :   user-class (e.g. person-car, trucks)

v :   velocity (m/s)

v^0 :   free speed (m/s)

$V(x,t)$:   mean velocity at (x,t) (m/s)

V^e :   equilibrium velocity (m/s)

V^0 :   mean free speed (m/s)

V_c :   shockwave speed (m/s)

t :   time instant (s)

T :   reaction time (s)

x :   location (m)

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