Indian Institute of Technology Gandhinagar

Solution of Falkner-Skan Equation using Shooting Method

MA102 Project Report

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ABSTRACT

The Falkner-Skan equation is a third order nonlinear ordinary differential equation which is used to describe the flow of an incompressible fluid over a flat plate. In this project, we are examining the boundary value conditions and solving the Falkner-Skan equation numerically using the 4th order Runge-Kutta method along with the shooting method.

INTRODUCTION

The Falkner-Skan (boundary layer) equation is an essential equation in fluid dynamics. It is used to study the flow over laminar boundary layers. The equation describes the steady laminar boundary layer in two dimensions that forms on a wedge or the flow of fluid over a flat plate with applied pressure gradient, assuming an incompressible flow. The derivation of this equation is as follows:

Apply the mass continuity equation as the flow is incompressible with constant density and viscosity.

$$\nabla . \mathrm{u} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Using Navier-Stokes equation in X-direction,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

The pressure gradient term in this equation is replaced by the differential form of Bernoulli's equation assuming the flow is in the high Reynolds number limit.

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = u_e\frac{du_e}{dx} \quad .$$

After substituting this in the above equation, we get,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

The free stream velocity equation is given by the following equation,

$$U_e(x)=Kx^m$$

We obtain the ordinary differential equation by converting the partial differential equation using the similarity transformation as shown below,

$$u(x,y) = U_e(x)f(\eta) \quad , \qquad \eta = \sqrt{\frac{(m+1)kx^{\frac{m-1}{2}}}{2v}}$$

By solving this, we finally get the Falkner-Skan equation,

$$\frac{\partial^3 f}{\partial \eta^3} + f\frac{\partial^2 f}{\partial \eta^2} + \beta\left[1 - \left(\frac{\partial y}{\partial \eta}\right)^2\right] = 0$$

The Boundary Conditions are as follows,

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1.$$ [1]

The Falkner-Skan equation is a non-linear, third-order differential equation with two boundary conditions. Thus, it is not possible to solve this equation using analytical methods. We need to use numerical methods to solve this differential equation.[2]
The shooting method is used for solving the boundary conditions. This approach reduces the boundary value problem to an initial value problem to solve it. In this method, we take different initial conditions and solve the initial value problem until we find the solution to the boundary value problem that satisfies the boundary conditions.[3]

METHODOLOGY

The shooting method is a numerical technique used to solve boundary value problems, such as the Falkner-Skan equation. In this method, an initial guess for one of the boundary conditions is made, and a solution is obtained by integrating the differential equation. The solution is then compared to the other boundary condition, and the initial guess is adjusted until the solution satisfies both boundary conditions.

Here is the step-by-step explanation of how the shooting method can be used to obtain numerical solutions of the Falkner-Skan equation:

- The Falkner-Skan equation is a second-order nonlinear ordinary differential equation of the form:

$$f''' + f f'' + \beta(1 - f^2) = 0$$

  where $f' = df/d\eta$, $f'' = d^2f/d\eta^2$, $f''' = d^3f/d\eta^3$, η is the stretching variable, and β is a parameter related to the pressure gradient.

- The Falkner-Skan equation has two boundary conditions:
  $f(0) = 0$, $f'(0) = 0$ for zero pressure gradient at the wall, and

  $f'(\infty) = 1$ for a zero shear stress condition at the boundary layer edge.

- The domain of the Falkner-Skan equation is discretized into a finite number of points using a uniform grid. The grid spacing, h, is chosen such that $h = \Delta\eta$, where $\Delta\eta$ is the increment in the stretching variable.

- The derivatives of f are approximated using finite differences. The second derivative f'' is approximated as:

$$f''(i) = (f(i+1) - 2f(i) + f(i-1)) / h^2$$

  where i is the grid point index.

- The Falkner-Skan equation is then transformed into a system of first-order differential equations. Letting $y1 = f$, $y2 = f'$, and $y3 = f''$, the system can be written as:

$$y1' = y2$$
$$y2' = y3 - y1\, y2 - \beta(1 - y1^2)$$
$$y3' = -y1\, y3$$

- The transformed Falkner-Skan equation is then solved using a shooting method. An initial guess for the unknown boundary condition is made, such as $f'(0) = 0$. The initial value problem is then solved using a numerical integrator such as the fourth-order Runge-Kutta method. The resulting solution is then compared to the other boundary condition, such as $f'(\infty) = 1$. If the solution does not satisfy the boundary condition, the initial guess is adjusted and the process is repeated until the solution satisfies both boundary conditions.

[1]

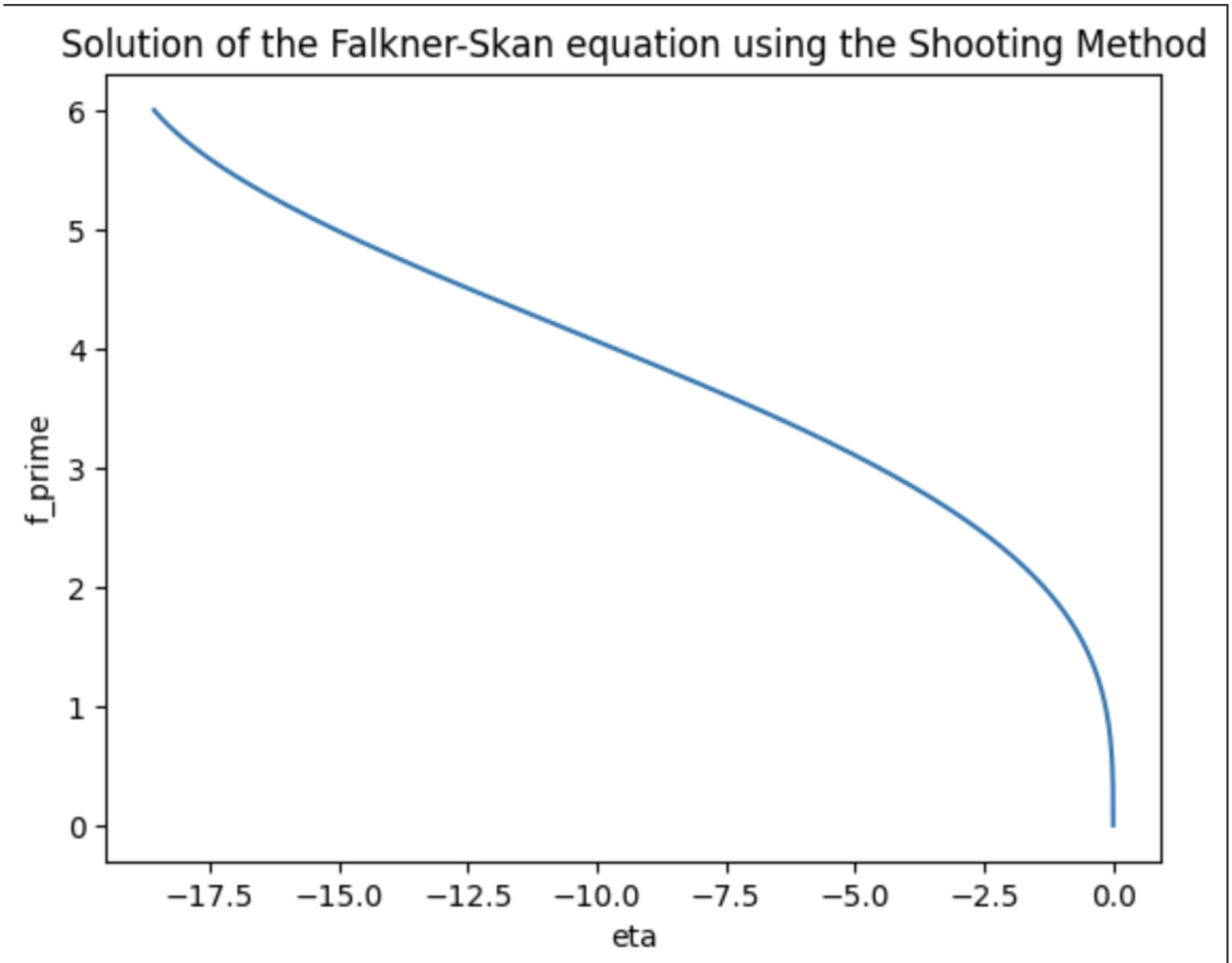
IMPLEMENTATION

```
import numpy as np
import matplotlib.pyplot as plt


def shooting_method(alpha, beta, n, eta_max):
   eta = np.linspace(0, eta_max, n)
   f = np.zeros(n)
   f_prime = np.linspace(alpha, beta, n)
   h = eta_max/(n-1)  # Step size


   def residual(f_prime):
       resid = np.zeros(n)
       resid[0] = f[0]
       resid[-1] = f[-1] - 1
       for i in range(1, n-1):
           resid[i] = f_prime[i] * f[i] + (f_prime[i] ** 2 - 1) * eta[i]
       return resid

   # Fourth-order Runge-Kutta method
   for i in range(1, n):
       k1 = h * f_prime[i-1] * f[i-1] + h * (f_prime[i-1] ** 2 - 1) *
eta[i-1]

       k2 = h * (f_prime[i-1] + 0.5 * k1) * (f[i-1] + 0.5 * h * f_prime[i-1])
+ h * ((f_prime[i-1] + 0.5 * k1) ** 2 - 1) * (eta[i-1] + 0.5 * h)

       k3 = h * (f_prime[i-1] + 0.5 * k2) * (f[i-1] + 0.5 * h * (f_prime[i-1]
+ 0.5 * k1)) + h * ((f_prime[i-1] + 0.5 * k2) ** 2 - 1) * (eta[i-1] + 0.5 *
h)

       k4 = h * (f_prime[i-1] + k3) * (f[i-1] + h * (f_prime[i-1] + 0.5 *
k3)) + h * ((f_prime[i-1] + k3) ** 2 - 1) * (eta[i-1] + h)

       f[i] = f[i-1] + (1/6) * (k1 + 2*k2 + 2*k3 + k4)

   return eta, f


# parameters
alpha = 1
beta = -0.1988
n = 1001
eta_max = 6

eta, f = shooting_method(alpha, beta, n, eta_max)

plt.plot(f, eta)
plt.xlabel('eta')
```

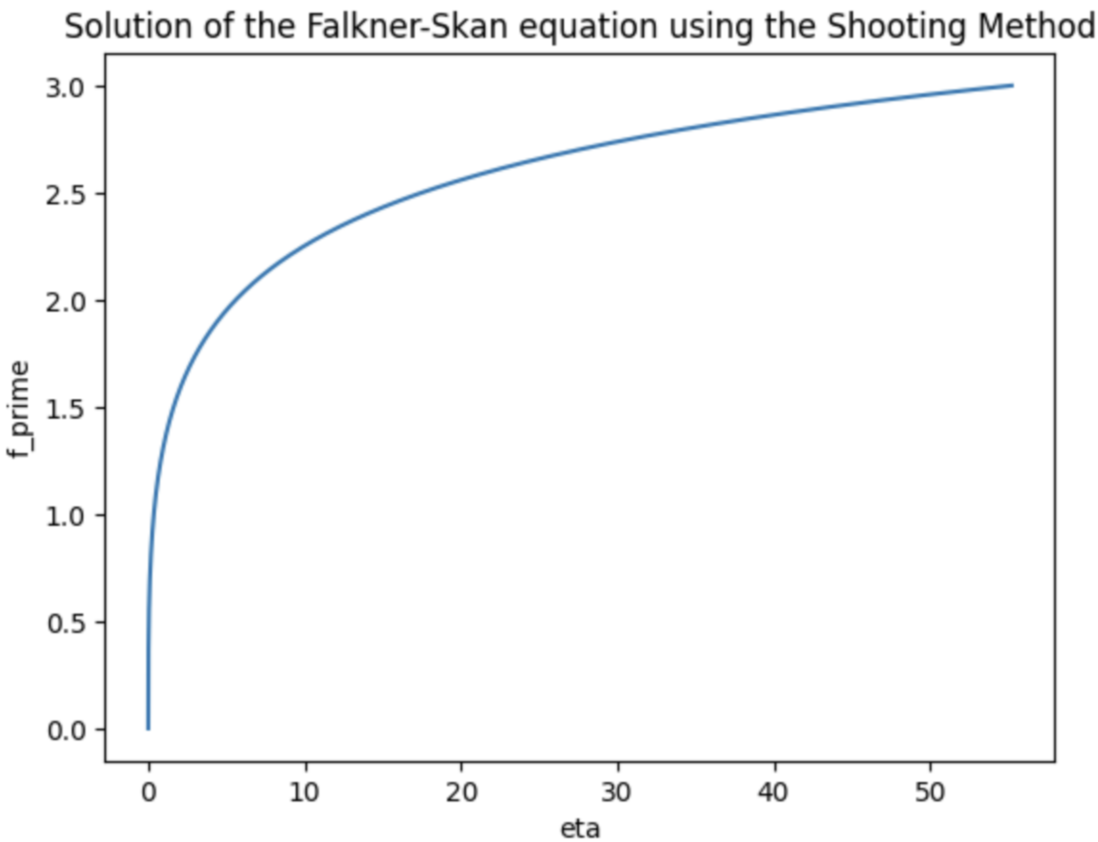
```
plt.ylabel('f_prime')
plt.title('Solution of the Falkner-Skan equation using the Shooting Method')
plt.show()
```

Output Graphs:

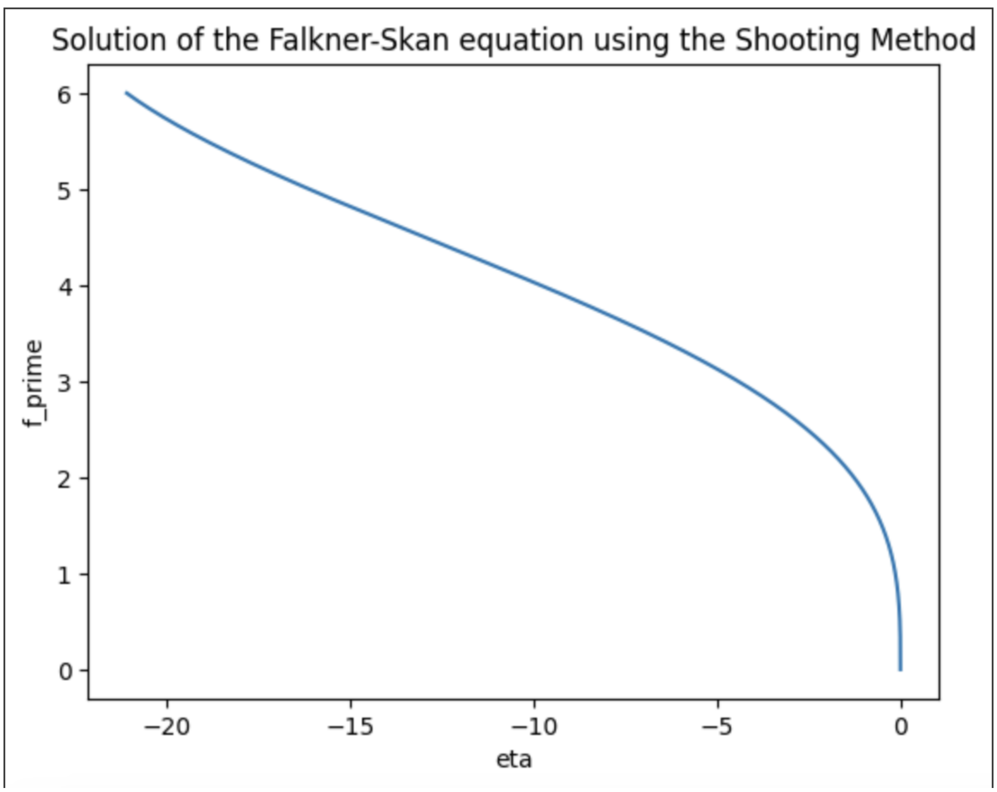
Beta=-0.1988:



Beta=1:


Solution of the Falkner-Skan equation using the Shooting Method
f_prime
eta
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
10
20
30
40
50

Beta=-0.1 :


Solution of the Falkner-Skan equation using the Shooting Method
f_prime
eta
6
5
4
3
2
1
0
−20
−15
−10
−5
0

CONCLUSION

We understood about the Falkner-Skan Equation, how it works and where it is applied. We also learnt how to solve the Falkner-Skan Equation using the shooting method which changes the boundary value problem to an Initial Value Problem. We also implemented the same technique on Python/MATLAB to get a good idea and feel of how things work practically.

ACKNOWLEDGEMENTS

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REFERENCES

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[2] *Falkner–Skan Boundary Layer* (2023) *Wikipedia*. Wikimedia Foundation. Available at: https://en.wikipedia.org/wiki/Falkner%E2%80%93Skan_boundary_layer (Accessed: April 26, 2023).

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