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Fractals

In the three-dimensional world that we live in, there is an abundance of patterns all around us. These patterns on earth can at times be explained using mathematical theory. Surprisingly in nature, there is a well-known pattern called fractals. The concept of fractals was popularized by Benoit Mandelbrot in his book, *The Fractal Geometry of Nature* which he published in 1982 [1]. In the book, Mandelbrot specifies that fractals tend to have an infinite pattern and happen to be very abundant in the earth's environment. By the characteristics of geometry, scaling and self-similarity fractals have the tendency of appearing in natural phenomena containing a certain mystery.

The work of fractal geometry was best known from Benoit Mandelbrot. Mandelbrot's passion for fractals started when he questioned himself about measuring the length of Britain's coastline [7]. Since coastlines are irregular and jagged, the act of acquiring an exact measurement is difficult using, "a catalog of lengths of simple geometric curves" (Mandelbrot 26) [2]. Equipped with this idea, Mandelbrot realized that the coastline could be measured by first taking a collection of yardsticks with length ε, denoted as $L(\varepsilon)$, and connecting them around a coastline (Mandelbrot 25) [2]. Depending on the case, this measurement would be off from the actual measurement since there are jagged edges on the border. To gain a more accurate value, Mandelbrot attained that by using the Richardson Effect, where we constantly divide $L(\epsilon)$ into a bigger collection of yardsticks with shorter lengths ϵ, we can accommodate to the jagged edges of the coastline. See figure below [10]:

With the broken segments of the yardsticks, one can concur that the approximation of the length of mentioned segments can be represented by the equation (Mandelbrot 29) [2]:

$$L(\epsilon) \approx F\epsilon^{1-D} \ , \ F \ constant$$

Upon further research with different coastlines, Mandelbrot found accurate lengths for the coastlines using different values of D. From that fact, Mandelbrot concluded that D is dependent on the shape of the coastline (Mandelbrot 29) [2] and denoted the value D to what is known as the fractal dimension.

Before Mandelbrot was able to connect fractal dimensions to other parts of nature, Mandelbrot had to construct a foundation of which fractals dimensions are defined in. When speaking about the dimension of fractals, one can say that fractals are dimensionally discordant (Mandelbrot 15) [2]. In other words, a fractal's topological dimension represented as D_t is always an integer, but the fractal dimension D doesn't have to be an integer such that:

$$D > D_t$$

(Mandelbrot 15) [2]. Mandelbrot was able to work with fractals by defining both D_t and D within the Euclidean space (Mandelbrot 15) [2], where Euclidean space can be described as a coordinate system which consists of a set of numbers, a set of two dimensional planes or three-dimensional planes [3]. Specifically, when working in the Euclidean span $\mathbb{R}^E$, Mandelbrot restricts D_t and D, such that (Mandelbrot 15) [2]:

$$0 < D_t < E \text{ and } 0 < D < E.$$

With the property that D and D_t exist in the Euclidean space, Mandelbrot proposed that if one was to exceed the topological dimension 1, the value D would be known as the fractal dimension (Mandelbrot 31) [2].

Following fractal dimension, the idea of symmetry comes into play. For fractals, symmetry lies within the context of self-scaling which in turn is described as self-similarity. Self-scaling in fractals exhibits a phenomena known as scale invariance; meaning that no matter how much one can quantify the length or size of the fractal, the fractal image will always remain identical to its previous image [4]. In addition, the concept of divergence falls under the umbrella of self-similarity. In the book, *The Fractal Geometry of Nature*, Mandelbrot defines divergence as, “some quantity that is commonly expected to be positive and finite turns out either to be infinite or … vanish”(Mandelbrot 19) [2]. With respect to nature, one possible example of divergence can occur if there was some outside force/energy which obstructs the fractal pattern causing a break in self-similarity. After looking at symmetry, fractal dimension and divergence one can observe a strong correlation between these ideas and fractals.

When observing nature, one can use geological studies to explain why certain events occur. Specifically, snowfall occurs when the temperature in the air is cold enough to form ice crystals in the clouds. These ice crystals then fall down and accumulate on the surface of the earth. If one was to closely zoom in on a snowflake, they will notice a fractal pattern. This pattern can be described by the famous Koch's Curve [6]. Helge von Koch, known mostly for his work in infinite linear equations and matrices, was able to reproduce this phenomenal pattern of a snowflake. The construct of a Koch's Curve can be summarized as, "starting with an equilateral triangle, removing the inner third of each side, building another equilateral triangle at the location where the side was removed, and then repeating the process indefinitely" [7]. In a technical sense, Koch's curve could be manufactured by the Lindenmayer system which will produce a fractal-like geometry with self-similarity. Aristid Lindenmayer, known for developing the Lindenmayer system in his work, *Mathematical Models for Cellular Interactions in Development*, allowed Aristid to model branching in plants (Prusinkiewicz 289-290) [9]. Specifically, a Lindenmayer system is a concept where one uses symbols to construct an order of instructions that builds objects [6]. For the Koch Curve, one uses what is known as an *initiator* and *generator* in the Lindenmayer system (Lindenmayer 1) [8]. The initiators role is to be the foundation of the shape, while the generator appends itself to the initiator when needed. Observe the diagram below (Lindenmayer 2) [8]:

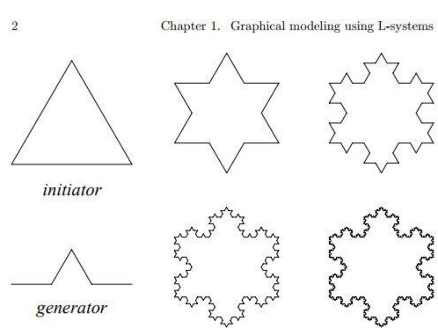


Figure 1.1: Construction of the snowflake curve

Specifically, for the Koch's snowflake, the initiator is in the shape of a triangle. Step by step, the generator will remove the bottom line of the initiator and append two lines of equal length to the opposite ends of the bottom part of the initiator as shown in figure 1.1 [6](Lindenmayer 1) [8]. This process can continue indefinitely where two initiators are then attached to those opposite ends of the generator and the removal of the bottom line from the initiator happens once again. By the construction of a Koch's snowflake, since the scaling of the initiator and generator is preserved, the overall shape is defined to be self-similar and fractal-like.

Mandelbrot had added to Koch's work by finding the similarity dimension of the fractal pattern. The similarity dimension is a measurement of the quantity of parts that lie in the set of the Euclidean space (Mandelbrot 37-38) [2]. To obtain the similarity dimension, Mandelbrot used the previous idea of the Lindenmayer system where *N* is defined as the quantity of the sides that the initiator holds, and *r* which defines the lengths of sides *N*. After defining the variables, he proved that the similarity dimension is defined as $D = \frac{\log(N)}{\log\left(\frac{1}{r}\right)}$ for the Koch's curve (Mandelbrot 39) [2]. By using different values of N and r, one can plug in those numbers to the expression $D = \frac{\log(N)}{\log\left(\frac{1}{r}\right)}$ and create a Koch's Curve by using the Lindenmayer system. For the resemblance of the snowflake, Mandelbrot concluded that a triadic Koch curve with the properties of $(N = 4, r = 1/3)$ were the best numbers to exhibit the fractal pattern of a snowflake (Mandelbrot 39) [2]. As previously stated from the foundation of fractal dimension, D will remain within the bounds $0 < D < E$ because the value $\frac{\log(4)}{\log(\frac{1}{\frac{1}{3}})} \approx 1.26185$ is in between the inequality. In summary, the Koch's curve can be represented by a fractal dimension D in the Euclidean space, but that dimension does not fully explain why a snowflake exhibits such a beautiful pattern.

Another natural phenomenon, which can be modeled with the Lindenmayer system, is the fern plant. By examining a fern leaf, one can see a strong resemblance of self-similarity and the pattern of fractal dimension in the overall structure. In fact, almost all plant models contain a high characteristic of self-similarity (Lindenmayer 176) [8]. Using an iterated function system, in accordance with the Lindenmayer system, one can properly model the fractal pattern of a fern leaf (Lindenmayer 178) [8]. An iterated function system is a set of mappings denoted T, such that the set is equal to:

$$\{T_1, T_2, \dots, T_n\}$$

where the transformations $T_1, T_2, \dots, T_n$ preserve lines, points and planes (Lindenmayer 178) [8]. With the IFS in hand, we can create a Lindenmayer system that will use these transformations to build a fractal, self-similar like pattern of the leaves. Observe the following Lindenmayer system (Lindenmayer 182) [8]:

$$\omega: \; A(1)$$

$$p_1: A(s): * \rightarrow F(s)\left[+A\left(\frac{S}{r}\right)\right]\left[-A\left(\frac{S}{r}\right)\right]F(s)A\left(\frac{S}{r}\right)$$

In this constructed Lindenmayer system, we observe two parts. The first part, ω, acts as an initializer when s = 1. This ω allows us to have a starting point in the construction of the overall pattern. Once we have an initializer, we use the second part of the Lindenmayer system p_1. This p_1 acts as the producer, meaning that p_1 builds the appending pattern onto the starting point ω. Using s as an iterator and r as a quantifier for the length of a line, $A\left(\frac{s}{r}\right)$ and $F(s)$ recursively build the pattern needed by creating different segments correlated by the transformation points

$T_1, \ldots, T_n$ (Lindenmayer 180) [8]. Specifically, for the fern leaf, there are four linear matrices that are defined as the transformation points (Lindenmayer 180) [8]:

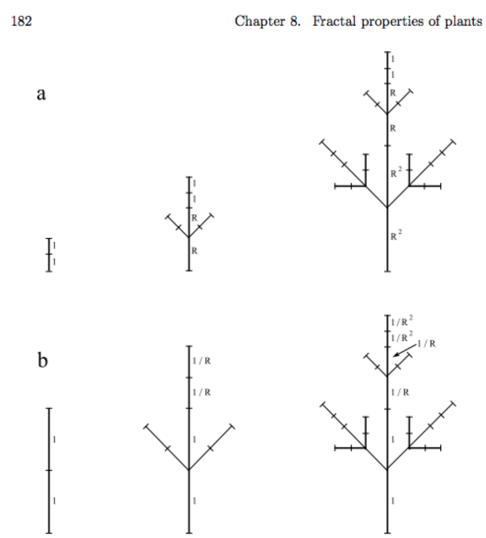
$$T_1 = \begin{bmatrix} 0.00 & 0.00 & 0.00 \\ 0.00 & 0.16 & 0.00 \\ 0.00 & 0.00 & 1.00 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 0.20 & 0.23 & 0.00 \\ -0.26 & 0.22 & 0.00 \\ 0.00 & 1.60 & 1.00 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} -0.15 & 0.26 & 0.00 \\ 0.28 & 0.24 & 0.00 \\ 0.00 & 0.44 & 1.00 \end{bmatrix}$$

$$T_4 = \begin{bmatrix} 0.85 & -0.04 & 0.00 \\ 0.04 & 0.85 & 0.00 \\ 0.00 & 1.60 & 1.00 \end{bmatrix}$$

This process of the Lindenmayer system is shown by the following diagram (Lindenmayer 182) [8]:



In the top left corner of the diagram, we have the starting base initialized ω. As the Lindenmayer system continues to build the structure with the producer p_1 , we can see that our quantifiable variable r is strategically used to build the branching segments of the fern leaf. In particular, the branching segments are denoted by the left $-A\left(\frac{s}{r}\right)$ and right $+A\left(\frac{s}{r}\right)$ parts of the stem (Lindenmayer 182) [8]. In addition, we can also observe the self-similarity and scaling that is occurring. As one continues to go up the main stem of the leaf, we can see that even though the image is small, the images going down are identical to the images going up the stem. By using different transformation values $T_1, \dots, T_n$, one could alter the pattern and manufacture other leaves that exist in the fern family. Through the iterated function system and the Lindenmayer system, one can illustrate most plant-based models using this technique because both systems go hand in hand for self-similarity and scaling. With this, we gain versatility in being able to show that fractal like patterns exist in almost all plants.

The biggest search that lies in the field of fractals is the unknown drive which causes plants and geology to have reoccurring patterns of fractals. Mathematicians have properly and successfully found models like the iterated function system and Lindenmayer system which are used hand in hand. By defining functions that are related to certain transformations of points in an IFS, we can reproduce fractals by putting those functions and transformations into a Lindenmayer system, which in turn can build the overall fractal pattern. However, the ability of producing fractals does not necessarily explain the mystery as to what causes such a phenomena. Some can say that the act of growth in the universe happens to naturally be of fractal form. On the contrary, there are plenty of objects in nature that don't exhibit fractals, in particular rocks. When studying fractals, one must understand the concepts of fractal dimension, geometry, self-similarity and scaling to see why nature has the tendency to hold fractal patterns within itself. Perhaps the reason why nature holds such a phenomena is through the characteristics that fractals hold, that nature has evolved to operate with self-similarity and recursion to have the ability to thrive on planet earth.

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