2022 BINARYPhi Mathematics Test

by BinaryPhi Club

November 2022

Name: ____________ **Date:** ____________ **Score:** ____________

Instruction

Questions: The test is composed of 12 questions.

The first 8 questions are Multiple Choice Questions.

The last 4 questions are questions that you need to give a number $x \in [000, 999]$ or a proof.

Calculator: No calculators.

Time limit: 120 minutes.

Score: Q1-Q8 is 8 points for each correct answer.

Q9-Q12 is 14 points for each correct answer and reasonable steps or comprehensive proof.

Clues: You will see a number ranging from 0 to 10 in front of every problem.

They are indicators of the difficulty of the problem IN MY OPINION.

Difficulty: Not too hard, not too easy. (5 ± 0.5 Overall)

References: IMO and USAMO and AMC Problems.

But you can hardly find the answer to my problem directly from those problems.

So don't bother searching online, the reference will be included in the Answer Version.

Good Luck!

1. (Olympiad; 3) How many three digit numbers have the property that the average of its three digits is exactly the value of the middle digit.

(A) 41 **(B)** 42 **(C)** 43 **(D)** 44 **(E)** 45

2. (Olympiad; 6) In a cycling tournament, there are over thousands of athletes needs a number for the race. Assume there are 100000 athletes and the number on their back range from 00000 to 99999, which is always kept in five digits. Now, the officials want that any two numbers from this range differ from one another by at least two digits. For instance, 00000 and 00001 are not allowed because the difference is only in one digit, while 00000 and 00011 are allowed. Find the maximum amount of such numbers they can use.

(A) 1000 **(B)** 9999 **(C)** 10000 **(D)** 10999 **(E)** 90000

3. (Olympiad; 6) Find the range (ranges) of x such that $0 \le x \le 2\pi$ and

$$2\cos(nx) \le \left|\sqrt{1-\sin(2nx)} - \sqrt{1+\sin(2nx)}\right| \le \sqrt{2}\,,$$

where n is a positive integer: $n \in \mathbb{Z}^+$.

(A) $\frac{3n\pi}{2}$ **(B)** $\frac{3\pi}{2n}$ **(C)** $\frac{3\pi}{4n}$ **(D)** $\frac{3\pi}{4}$ **(E)** $\frac{3\pi}{2}$

$$\sqrt{(x+3)^2+(y-2)^2} = \sqrt{(x-3)^2+y^2}$$

4. (Fundamental; 2) In the xy-plane, points that have coordinates satisfying the above equation are:

(A) a line **(B)** a circle **(C)** an ellipse **(D)** a parabola **(E)** one branch of a hyperbola

5. (Calculus Fundamental; 3) How many real roots does the polynomial $8x^7 + 5x^3 - 2$ have?

(A) None **(B)** One **(C)** Two **(D)** Three **(E)** Seven

6. (Linear Algebra Fundamental; 4) Consider the two planes

$$x + 3y - 2z = 7 \text{ and } 2x + y - 3z = 0$$

in $\mathbb{R}^3$. What is the set of the intersection of these planes?
Clue for those who didn't study Linear Algebra: Since two orthogonal vectors have dot product of 0, $2x + y - 3z = 0$ is the plane orthogonal to the vector $(2, 1, -3)$, while $x + 3y - 2z = 7$ can be seen as the plane $x + 3y - 2z = 0$ shifting in the x-axis.

(A) No intersection
(B) $\{(0, 3, 1)\}$
(C) $\{(x, y, z) \mid x = t, y = 3t, z = 7 - 2t, t \in \mathbb{R}\}$
(D) $\{(x, y, z) \mid x = 7t, y = 3 + t, z = 1 + 5t, t \in \mathbb{R}\}$
(E) $\{(x, y, z) \mid x - 2y - z = -7\}$

7. (Olympiad; 6-7) Find the minimum perimeter among the eight sided polygons in the complex plane whose vertices are the zeros of $P(z)$, which

$$P(z) = z^8 + \left(8\sqrt{7} + 22\right) z^4 - \left(8\sqrt{7} + 23\right) .$$

(A) $8\sqrt{2}$ **(B)** 16 **(C)** $16\sqrt{2}$ **(D)** 9 **(E)** 1

8. (Olympiad; 7) See triangle in Figure 1. $AB = 20, AC = 22, BC = 25, CD = 10$, E is on BC such that $\angle BAE = \angle CAD$. The value of the length of BE is in the form of p/q, what is p.

(A) 5000 **(B)** 4000 **(C)** 3000 **(D)** 2000 **(E)** 1000

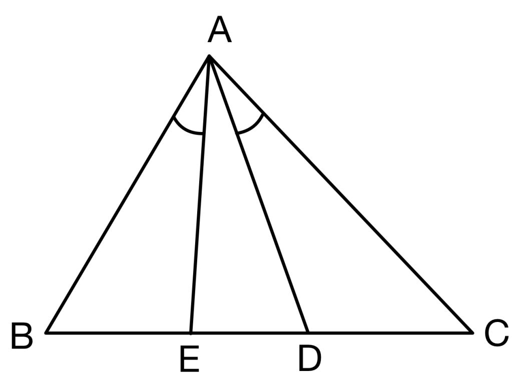

Figure 1: Problem 8

9. (High School Math; 3) A rectangular box $\mathcal{P}$, with

$$\text{surface area } \mathcal{S} = 112, \text{ sum of the lengths of edges } \mathcal{L} = 64,$$

is inscribed in a sphere of radius r. Find ANS $= r$.

10. (Olympiad; 6) Calculate

$$\sum_{k=0}^{\infty} \frac{\sin(k\pi/4)}{3^k} = \frac{a}{b}\left(c\sqrt{d} + e\right),$$

where a/b and $\sqrt{d}$ are of the simplest form. Find ANS $= a \times c \times d \times e$.

11. (Olympiad; 6-8 Depends) Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer. Clue: use Induction.

12. (Olympiad; 8) Given a triangle ABC with a point P inside it, D, E, F are the feet of the perpendiculars from P to BC, CA, AB, respectively. For what point P does

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$$

has its least value. Clue: use Cauchy-Schwarz inequality.