2022 BPMT References

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1. (Olympiad; 3) How many three digit numbers have the property that the average of its three digits is exactly the value of the middle digit.

(A) 41 **(B)** 42 **(C)** 43 **(D)** 44 **(E)** 45

Reference: *2005 AMC 12A Problem 11*

Assume the three-digit number is $\overline{ABC}$, which means we have

$$\frac{A+B+C}{3} = B\,.$$

With some transformations, we have

$$A + C = 2B\,.$$

Since $\overline{ABC}$ is a three-digit numbers, $A \neq 0$. Additionally, A and C must be of the same parity. A has 9 choices. Correspondingly, C has 5 options to each value of A (If A is odd, B has to be odd: $1, 3, 5, 7, 9$; If A is even, B has to be even: $0, 2, 4, 6, 8$). Thus, there are $9 \times 5 = \boxed{45}$ options.

2. (Olympiad; 6) In a cycling tournament, there are over thousands of athletes needs a number for the race. Assume there are 100000 athletes and the number on their back range from 00000 to 99999, which is always kept in five digits. Now, the officials want that any two numbers from this range differ from one another by at least two digits. For instance, 00000 and 00001 are not allowed because the difference is only in one digit, while 00000 and 00011 are allowed. Find the maximum amount of such numbers they can use.

(A) 1000 **(B)** 9999 **(C)** 10000 **(D)** 10999 **(E)** 90000

Reference: *1990 USAMO Problem 1*

Considering the case of 00000 and 00001, we have the first four digits being the same though the last digit is different. In order to have at least one more different digit, one of the first four digits of 00001 has to be different from that of 00000. Without loss of generality, we assume one of the different digit is the last digit. Then, as long as we choose two different numbers from 0000 to 9999, we are guaranteed there are at least two digits of difference. This works for other cases, such as there are 10^{n-1} numbers when there are 10^n athletes. In conclusion, we have the maximum amount being $10^{5-1} = \boxed{10000}$.

3. (Olympiad; 6) Find the range (ranges) of x such that $0 \le x \le 2\pi$ and

$$2\cos(nx) \le \left|\sqrt{1-\sin(2nx)} - \sqrt{1+\sin(2nx)}\right| \le \sqrt{2}\,,$$

where n is a positive integer: $n \in \mathbb{Z}^+$.

(A) $\frac{3n\pi}{2}$ **(B)** $\frac{3\pi}{2n}$ **(C)** $\frac{3\pi}{4n}$ **(D)** $\frac{3\pi}{4}$ **(E)** $\frac{3\pi}{2}$

Reference: *1965 IMO Problem 1*

We can start by dealing with the left inequality, from which we see that $2\cos(nx)$ is smaller than a non-negative number. This means if $2\cos(nx)$ is negative, the inequality is always satisfied. When $x \in [0, 2\pi/n]$, which is the period of $2\cos(nx)$, $2\cos(nx) \le 0$ when $x \in [\pi/2n, 3\pi/2n]$.

$$\begin{aligned}
4\cos^2(nx) &\le \left(\sqrt{1-\sin(2nx)} - \sqrt{1+\sin(2nx)}\right)^2\\
4\cos^2(nx) &\le (1-\sin(2nx)) + (1+\sin(2nx)) - 2\sqrt{1-\sin^2(2nx)}\\
2\cos^2(nx) &\le 1 - \sqrt{1-\sin^2(2nx)}\\
2\cos^2(nx) &\le 1 - \left|\cos(2nx)\right|\\
2\cos^2(nx) &\le 1 - \left|\cos^2(nx) - \sin^2(nx)\right|
\end{aligned}$$

If $\cos(2nx) = \cos^2(nx) - \sin^2(nx) > 0$, $x \in (0, \pi/4n) \cup (3\pi/4n, 5\pi/4n) \cup (7\pi/4n, 2\pi/n)$. Notice that $(3\pi/4n, 5\pi/4n) \subset [\pi/2n, 3\pi/2n]$. Thus, it'll be much more convenient to carry on with only $x \in (0, \pi/4n) \cup (7\pi/4n, 2\pi/n)$. Thus, we have

$$\cos^2(nx) \le 1 - \cos^2(nx) + \sin^2(nx) = \sin^2(nx).$$

Since we assumed $\cos^2(nx) - \sin^2(nx) > 0$, there is no solution when $\cos(2nx) > 0$.
If $\cos(2nx) = \cos^2(nx) - \sin^2(nx) \le 0$, $x \in [\pi/4n, 3\pi/4n] \cup [5\pi/4n, 7\pi/4n]$. We have

$$\cos^2(nx) \le 1 - \sin^2(nx) + \cos^2(nx) = \cos^2(nx).$$

This inequality (equality) is always satisfied when $x \in [\pi/4n, 3\pi/4n] \cup [5\pi/4n, 7\pi/4n]$.
Combined, we get

$$x \in ([\pi/2n, 3\pi/2n]) \cup ([\pi/4n, 3\pi/4n] \cup [5\pi/4n, 7\pi/4n]) = [\pi/4n, 7\pi/4n]$$

(P.S. Please still remember we are only dealing with the period $x \in [0, 2\pi/n]$)
Then, we can focus on the right inequality:

$$\left(\sqrt{1-\sin(2nx)} - \sqrt{1+\sin(2nx)}\right)^2 \le 2 \Longrightarrow 2 - 2\left|\cos(2nx)\right| \le 2\,,$$

which is always true.
Therefore, when in $[0, 2\pi/n]$, the range is formed from $x \in [\pi/4n, 7\pi/4n]$: $3\pi/2n$.
In $[0, 2\pi]$, the ranges are $3\pi/2n \times n = \boxed{3\pi/2}$.

$$\sqrt{(x+3)^2+(y-2)^2}=\sqrt{(x-3)^2+y^2}$$

4. (Fundamental; 2) In the xy-plane, points that have coordinates satisfying the above equation are:

(A) a line **(B)** a circle **(C)** an ellipse **(D)** a parabola **(E)** one branch of a hyperbola

Reference: *GRE MATH GR1268 Problem 10*

We have

$$\begin{aligned}(x+3)^2+(y-2)^2 &= (x-3)^2+y^2\\ x^2+6x+9+y^2-4y+4 &= x^2-6x+9+y^2\\ 12x-4y+4 &= 0,\end{aligned}$$

which is the equation of $\boxed{\text{a line}}$.

5. (Calculus Fundamental; 3) How many real roots does the polynomial $8x^7+5x^3-2$ have?

(A) None **(B)** One **(C)** Two **(D)** Three **(E)** Seven

Reference: *GRE MATH GR0568 Problem 18*

Assume

$$f(x)=8x^7+5x^3-2.$$

What we have is

$$f'(x)=56x^6+15x^2,$$

which is always larger or equal to 0. For a function's derivative being larger or equal to 0, the function must be (weakly) increasing, thus crossing the x-axis only one time. Thus, there is $\boxed{\text{One}}$ root for the polynomial.

6. (Linear Algebra Fundamental; 4) Consider the two planes

$$x+3y-2z=7 \text{ and } 2x+y-3z=0$$

in $\mathbb{R}^3$. What is the set of the intersection of these planes?
Clue for those who didn't study Linear Algebra: Since two orthogonal vectors have dot product of 0, $2x+y-3z=0$ is the plane orthogonal to the vector $(2,1,-3)$, while $x+3y-2z=7$ can be seen as the plane $x+3y-2z=0$ shifting in the x-axis.

(A) No intersection
(B) $\{(0,3,1)\}$
(C) $\{(x,y,z) \mid x=t, y=3t, z=7-2t, t\in\mathbb{R}\}$
(D) $\{(x,y,z) \mid x=7t, y=3+t, z=1+5t, t\in\mathbb{R}\}$
(E) $\{(x,y,z) \mid x-2y-z=-7\}$

Reference: *GRE MATH GR0568 Problem 27*

The best solution of this problem is to substitute the equations of the two planes to each of the options. Thus, you will find $\boxed{\text{D}}$ gives you the satisfied condition.

7. (Olympiad; 6-7) Find the minimum perimeter among the eight sided polygons in the complex plane whose vertices are the zeros of $P(z)$, which

$$P(z) = z^8 + \left(8\sqrt{7} + 22\right) z^4 - \left(8\sqrt{7} + 23\right) .$$

(A) $8\sqrt{2}$ **(B)** 16 **(C)** $16\sqrt{2}$ **(D)** 9 **(E)** 1

Reference: *2011 AMC 12B Problem 24*

From

$$\begin{aligned} P(z) &= z^8 + \left(8\sqrt{7} + 22\right) z^4 - \left(8\sqrt{7} + 23\right) \\ &= (z^4 - 1)\left(z^4 + \left(8\sqrt{7} + 23\right)\right), \end{aligned}$$

we have

1. $z^4 = 1 \Longrightarrow z = e^{in\pi/2}$
2. $z^4 = -\left(8\sqrt{7} + 23\right)$

$$z^2 = i\left(\sqrt{7} + 4\right) \text{ or } -i\left(\sqrt{7} + 4\right)$$

$$z = e^{i(2n+1)\pi/4}\sqrt{\sqrt{7} + 4} = e^{i(2n+1)\pi/4}\left(\frac{\sqrt{7}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

Therefore,

$$\begin{aligned} \text{Perimeter} = 8l &= 8\sqrt{a^2 + b^2 - 2ab\cos\theta} \\ &= 8\sqrt{\left(\frac{\sqrt{7}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 + 1^2 - 2\cdot\left(\frac{\sqrt{7}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\cdot 1 \cdot \cos\left(\frac{\pi}{4}\right)} \\ &= 8\sqrt{4 + \sqrt{7} + 1 - 2\left(\frac{\sqrt{7}+1}{\sqrt{2}}\frac{\sqrt{2}}{2}\right)} \\ &= 16 . \end{aligned}$$

8. (Olympiad; 7) See triangle in Figure 1. $AB = 20, AC = 22, BC = 25, CD = 10$, E is on BC such that $\angle BAE = \angle CAD$. The value of the length of BE is in the form of p/q, what is p.

(A) 5000 **(B)** 4000 **(C)** 3000 **(D)** 2000 **(E)** 1000

Reference: *2005 AIME II Problem 14*

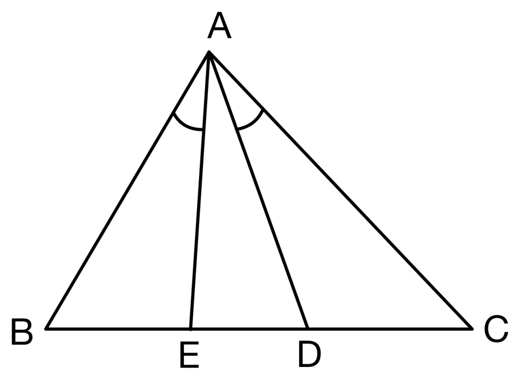

Figure 1: Problem 8

9. (High School Math; 3) A rectangular box $\mathcal{P}$, with

$$\text{surface area } \mathcal{S} = 112, \text{ sum of the lengths of edges } \mathcal{L} = 64,$$

is inscribed in a sphere of radius r. Find ANS $= r$.

ANS $= \boxed{006}$.

Reference: *2005 AMC 12A Problem 22*

10. (Olympiad; 6) Calculate

$$\sum_{k=0}^{\infty} \frac{\sin(k\pi/4)}{3^k} = \frac{a}{b}\left(c\sqrt{d}+e\right),$$

where a/b and $\sqrt{d}$ are of the simplest form. Find ANS $= a \times c \times d \times e\,.$

ANS $= \boxed{90}$.

Reference: *2021 MIT Math Prize For Girls Problem 7*

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\sin(k\pi/4)}{3^k} &= \sum_{k=0}^{7}\left[\frac{\sin(k\pi/4)}{3^k}\sum_{n=0}^{\infty}\frac{1}{3^{8n}}\right] \\
&= \left[\sum_{k=0}^{7}\frac{\sin(k\pi/4)}{3^k}\right]\sum_{n=0}^{\infty}\frac{1}{3^{8n}} \\
&= \left[\frac{\sin(0\pi/4)}{3^0}+\frac{\sin(1\pi/4)}{3^1}+\cdots+\frac{\sin(7\pi/4)}{3^7}\right]\frac{1}{1-3^{-8}} \\
&= \left[0+\frac{\sqrt{2}/2}{3}+\frac{1}{3^2}+\frac{\sqrt{2}/2}{3^3}+0+\frac{-\sqrt{2}/2}{3^5}+\frac{-1}{3^6}+\frac{-\sqrt{2}/2}{3^7}\right]\frac{3^8}{3^8-1} \\
&= \left[\frac{\sqrt{2}}{2}\left(\frac{1}{3}+\frac{1}{3^3}-\frac{1}{3^5}-\frac{1}{3^7}\right)+\left(\frac{1}{3^2}-\frac{1}{3^6}\right)\right]\frac{3^8}{3^8-1} \\
&= \frac{400\sqrt{2}+240}{3^7}\cdot\frac{3^8}{3^8-1} \\
&= \frac{1200\sqrt{2}+720}{3^8-1} = \frac{15\sqrt{2}+9}{82} = \frac{3}{82}\left(5\sqrt{2}+3\right).
\end{aligned}$$

$\therefore$ ANS $= \boxed{90}$.

11. (Olympiad; 6-8 Depends) Prove that for any positive integer k,

$$(k^2)!\cdot\prod_{j=0}^{k-1}\frac{j!}{(j+k)!}$$

is an integer. Clue: use Induction.

Reference: *2016 USAMO Problem 2*

12. (Olympiad; 8) Given a triangle ABC with a point P inside it, D, E, F are the feet of the perpendiculars from P to BC, CA, AB, respectively. For what point P does

$$\frac{BC}{PD}+\frac{CA}{PE}+\frac{AB}{PF}$$

has its least value. Clue: use Cauchy-Schwarz inequality.

Reference: *1981 IMO Problem 1*