Ordinary Differential Equation © BinaryPhi

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Sec. 1: Separable Differential Equations

Form

$$\frac{dy}{dx} = f(x)g(y)\,. \tag{1}$$

Solution

1) When $g(y) \neq 0$,

$$\frac{dy}{g(y)} = f(x)dx\,,$$
$$\int \frac{dy}{g(y)} = \int f(x)dx + C\,.$$

This is the General Solution to this type of differential equations.

2) When $g(y) = 0$, of all functions of y, if exists y_0, as a constant, such that $g(y) = g(y_0) = 0$, where y_0 has the property that

$$\frac{dy}{dx} = 0.$$

We can say that $y = y_0$ is also one of the solutions to the differential equation.

Example

$$\tan(x)\,\frac{dy}{dx} = 1 + y\,.$$

[Solution] First, make this equation neat:

$$\frac{dy}{dx} = \frac{1}{\tan(x)}\,(1+y)\,,$$

which is the form of the separable differential equation.
Then assume the function of y to be not equal to 0:

$$\begin{aligned}
\frac{1}{1+y}\,dy &= \frac{1}{\tan(x)}\,dx\,,\\
\frac{1}{1+y}\,dy &= \frac{\cos(x)}{\sin(x)}\,dx\,,\\
\frac{1}{1+y}\,dy &= \frac{1}{\sin(x)}\,d\left(\sin(x)\right),\\
\int \frac{1}{1+y}\,dy &= \int \frac{1}{\sin(x)}\,d\left(\sin(x)\right) + c\,,\\
\ln|1+y| &= \ln|\sin(x)| + c\\
&= \ln\left|e^{c}\cdot\sin(x)\right|,\\
|1+y| &= \left|e^{c}\cdot\sin(x)\right|,\\
1+y &= \pm\, e^{c}\cdot\sin(x), C := \pm\, e^{c}\\
y &= C\cdot\sin(x) - 1\,,
\end{aligned}$$

where C is not equal to 0. Now, consider function $1+y$ to be equal to 0:

$$1 + y = 0 \Longrightarrow y = -1\,.$$

$y = -1$ is, in fact, a solution to this equation that $\tan(x)\,0 = 0$. In addition, the assumption of $y = -1$ matches the circumstance when $C = 0$.
Therefore, the general solution of this differential equation is

$$\boxed{y = C\cdot\sin(x) - 1, \forall C \in \mathbb{R}\,.}$$

Example

$$dx + xydy = y^2dx + ydy\,.$$