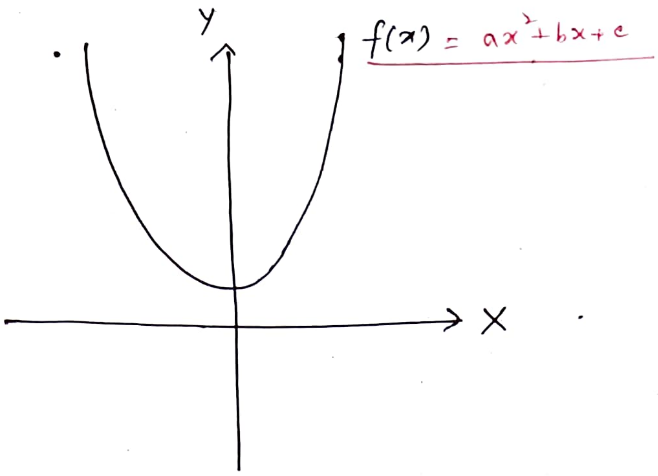
Quadratic Equation



$$\boxed{f(x) = ax^2 + bx + c}$$ Eqⁿ of a Quadratic Function

$$= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2\right]$$

$$= a\left[\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\right]$$

important

$$= a\left[\left\{x - \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\right\}\left\{x - \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\right\}\right]$$

Now,

since $f(x) = a\left[\left(x+\frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2\right]$

$$f\left(-\frac{b}{2a}\right) = -a \cdot \frac{b^2-4ac}{4a^2}$$

$$= -\left(\frac{b^2-4ac}{4a}\right)$$

$$f\left(-\frac{b}{2a}+\epsilon\right) = a\left[\epsilon^2 - \frac{b^2-4ac}{4a^2}\right]$$

$$f\left(-\frac{b}{2a}-\epsilon\right) = a\left[\epsilon^2 - \frac{b^2-4ac}{4a^2}\right]$$

So, clearly,

$$f\left(-\frac{b}{2a}+\epsilon\right) = f\left(-\frac{b}{2a}-\epsilon\right)$$

hence, $\boxed{x = -\frac{b}{2a}}$ is the Axis of Symmetry — ①.

$f(0) = c$ which is y-intercept. —— ②

x-cordinate of vertex —— ③

$= -b/2a$

Why?

$$f(x) = a\left[\left(x+\frac{b}{2a}\right)^2 + \left(\frac{4ac-b^2}{4a^2}\right)\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 + \left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2\right]$$

Graph of $f(x) = x^2 + 8x + 9$

$f(0) = 9$

Axis of Symmetry : $x = -\frac{b}{2a} = -\frac{8}{2} = -4$

$a = 1\ (>0)$

We have seen

$$f\left(-\frac{b}{2a} \pm \epsilon\right) = a\epsilon^2 - a \cdot \frac{b^2 - 4ac}{4a^2}$$

$$= a\epsilon^2 + f\left(-\frac{b}{2a}\right)$$

$$f\left(-\frac{b}{2a} \pm \epsilon\right) - f\left(-\frac{b}{2a}\right) = a\epsilon^2$$

So, $f\left(-\frac{b}{2a} \pm \epsilon\right) - f\left(-\frac{b}{2a}\right) > 0$ if $\boxed{a > 0}$

ie if $a>0$: $f\left(-\frac{b}{2a} \pm \epsilon\right) > f\left(-\frac{b}{2a}\right)$

so, $\boxed{f\left(-\frac{b}{2a}\right) \text{ represents min value}}$

& $f\left(-\frac{b}{2a} \pm \epsilon\right) < f\left(-\frac{b}{2a}\right)$ when $\boxed{a < 0}$

so. $\boxed{f\left(-\frac{b}{2a}\right) \text{ is the max. value}}$

Say roots

Consider Fundamental Theorem of Algebra :

Every algebraic equation with real or imaginary coeffecients has at least one real or imaginary root.

Theorem: A quadratic equation cannot have more than two distinct roots

Consider 2 roots to be α, β.

$$(x-\alpha)(x-\beta)$$

$$= x^2 - (\alpha+\beta)x + \alpha\beta \quad \text{———} \quad (1)$$

Consider general form: $ax^2 + bx + c$ ——— (2)

Equating,

$$x^2 - (\alpha+\beta)x + \alpha\beta = ax^2 + bx + c$$

$$\alpha+\beta = -\frac{b}{a}$$

$$\alpha\beta = \frac{c}{a}$$

Here,

$$\alpha+\beta = -8 \quad \text{———} \quad (3)$$

$$\alpha\beta = 9 \quad \text{———} \quad (4)$$

Taking (3) let $\alpha = -4+u$; $\beta = -4-u$

$$\alpha\beta = (-4+u)(-4-u)$$

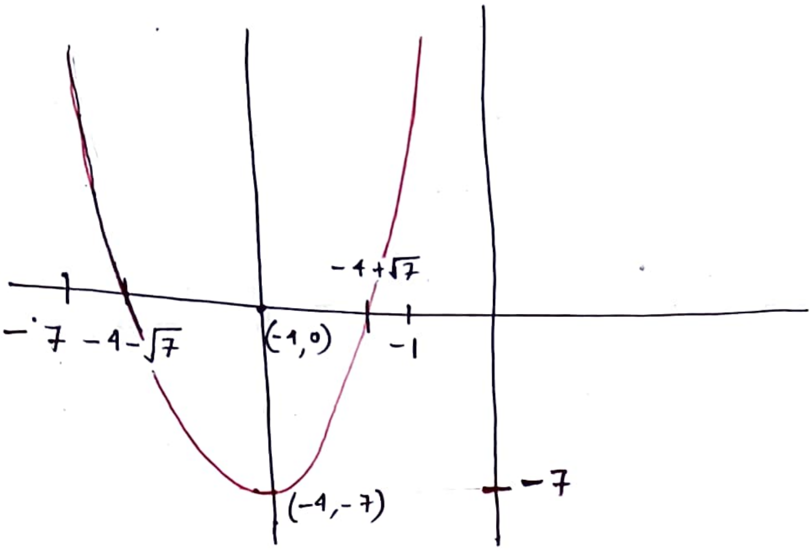
$$\Rightarrow 16 - u^2 = 9$$

$$\Rightarrow u^2 = 16 - 9 = 7$$

$$\Rightarrow u = \pm\sqrt{7}$$

So, $\alpha = -4+\sqrt{7}$

$\beta = -4-\sqrt{7}$



$$f(-4) = 16 + 8(-4) + 9$$
$$= 16 - 32 + 9$$
$$= 25 - 32 = -7$$

Result :

A quadratic equation cannot have more than 2 distinct roots

Proof :

$$ax^2 + bx + c = (x - \alpha)(px + q) \quad \text{[by fundamental theorem]}$$

where $p (\neq 0)$ and q are constants

By the fundamental theorem, $px + q = 0$ has atleast one root

Consider that root to be β.

$$px + q = (x - \beta) \cdot r$$

So, $ax^2 + bx + c = (x - \alpha)(x - \beta) \cdot r$

$$= rx^2 - (\alpha + \beta) r x + \alpha\beta r$$

Equating coeff of x^2 we get $a = r$

$$ax^2+bx+c = a(x-\alpha)(x-\beta)$$

Now let us take a quantity γ s.t $\gamma \neq \alpha$, $\gamma \neq \beta$.

Putting $x = \gamma$,

$$a\gamma^2+b\gamma+c = a(\gamma-\alpha)(\gamma-\beta) \neq 0 \left[\because a \neq 0\right]$$

So, $x = \gamma$ can't be a root.

So a quadratic equation cannot have more than 2 roots.

Note :

$$f(x) = \left[x-\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)\right]\left[x-\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\right]$$

$$\alpha = -\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$$

$$\beta = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

So, clearly if $b^2-4ac > 0$, α, β are real roots and different

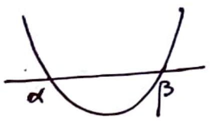
$b^2-4ac = 0$, α, β are equal and real roots

$b^2-4ac < 0$, α, β are imaginary roots.

Chart

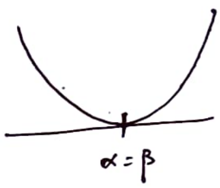
$$\boxed{f(x) = ax^2 + bx + c}$$

a	D	roots	sign of f(x)
$a>0$	$D>0$	$\alpha<\beta$ real roots.	see graph 1 below
$a>0$	$D=0$	real roots $\alpha=\beta$	see graph 2 below
$a>0$	$D<0$	Complex roots $\alpha, \bar{\alpha}$	$f(x)>0\ \forall x$
$a<0$	$D>0$	real roots $\alpha<\beta$	see graph 3 below
$a<0$	$D=0$	real roots $\alpha=\beta$	see graph 4 below
$a<0$	$D<0$	Complex roots $\alpha, \bar{\alpha}$	$f(x)<0\ \forall x$



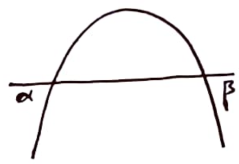
$f(x)>0 \begin{cases} x<\alpha \\ x>\beta \end{cases}$

$f(x)<0 : \alpha<x<\beta$



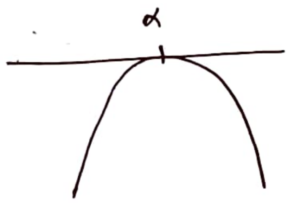
$f(x)>0 \begin{cases} x>\alpha \\ x<\alpha \end{cases}$

$f(x)=0 \quad x=\alpha$



$f(x)<0 \begin{cases} x<\alpha \\ x>\beta \end{cases}$

$f(x)>0 : \alpha<x<\beta$



$f(x)<0 \begin{cases} x<\alpha \\ x>\alpha \end{cases}$

$f(x)=0 \quad x=\alpha$

Roots

if $b^2-4ac>0$ and $b^2-4ac =$ a perfect Square : Real, Rational & Unequal

if $b^2-4ac>0$ and $b^2-4ac =$ not a perfect Square : Real, Irrational & Unequal.

Some Results about Roots

$$f(x) = ax^2 + bx + c$$

i. $c = 0$ $\qquad ax^2 + bx = 0$

$$\Rightarrow x(ax+b) = 0 \quad : \quad \boxed{x = 0 \text{ or } x = -\frac{b}{a}}$$

ii. Coefficient of $x = 0$ ie $b = 0$

$$ax^2 + c = 0$$

$$\Rightarrow x^2 = -\frac{c}{a}$$

$$\Rightarrow \boxed{x = \pm\sqrt{\frac{-c}{a}}}$$

So, when coefficient of $x = 0$, the roots are equal in Magnitude but opposite in sign.

iii. Coefficient of $x^2 = 0$

$$bx + c = 0$$

$$x = -c/b$$

So one root is $-c/b$. What about the other root?

Let $x = \frac{1}{y}$

$$a \cdot \frac{1}{y^2} + b \cdot \frac{1}{y} + c = 0$$

$$cy^2 + by + a = 0$$

$a = 0 \Rightarrow cy^2 + by = 0$

$$y = 0 \text{ or } y = -b/c$$

When $y = -b/c \Rightarrow x = -c/b$: 1 root

$y = 0 \Rightarrow x = \frac{1}{0}$ = Indeterminate

iv. $\underline{a = b = 0}$

$$ax^2 + bx + c = 0$$

Let $x = \frac{1}{y}$

$$cy^2 + by + a = 0$$

$$\Rightarrow cy^2 = 0 \quad (\text{when } a = b = 0)$$

So 2 roots for $cy^2 = 0$ are $0, 0$

$\frac{1}{0}$ = Indeterminate

So Both roots are Indeterminate when $a = b = 0$

v. $\underline{a = c = 0}$

Consider $cy^2 + by + a = 0$ where $x = \frac{1}{y}$

$$by = 0$$

One root is 0 and other root is indeterminate

vi. $\underline{b = c = 0}$

Both roots are Zero

vii. $\underline{a = b = c = 0}$

Identity

Summary

Condition	Root 1	Root 2
$c = 0$	0	$-\frac{b}{a}$
$b = 0$	$\sqrt{\frac{-c}{a}}$	$-\sqrt{\frac{-c}{a}}$
$a = 0$	$-c/b$	Indeterminate (I)
$a = b = 0$	I	I
$b = c = 0$	0	0
$a = c = 0$	0	I

$a = b = c = 0 \rightarrow$ Identity $\rightarrow$ more than 2 roots

Theorem :

In a Quadratic equation with real coeffecients $a, b, c \in R$ imaginary roots occur in Conjugate Pairs

Theorem

In a Quadratic equation with rational coeffecients $a, b, c \in Q$ irrational roots occur in Conjugate Pairs

$$Eq^n: \boxed{ax^2 + bx + c = 0}$$

Proof :

$$ax^2 + bx + c = 0 \qquad a, b, c \in Q$$

Now, Roots : $p + \sqrt{q}$ be a root

$$a(p + \sqrt{q})^2 + b(p + \sqrt{q}) + c = 0$$

$$a(p^2 + q^2 + 2p\sqrt{q}) + bp + b\sqrt{q} + c = 0$$

$$(ap^2 + aq^2 + bp + c) + (2ap + b)\sqrt{q} = 0$$

ie,

$$ap^2 + aq^2 + bp + c = 0 \quad \text{——— } (1)$$

$$2ap + b = 0 \quad \text{——— } (2)$$

Now. $\boxed{p = -\frac{b}{2a}}$

$$ap^2 + aq + bp + c = 0$$

$$a\left(\frac{b^2}{4a^2}\right) + aq + b\left(-\frac{b}{2a}\right) + c = 0$$

$$\frac{b^2}{4a} - \frac{b^2}{2a} + c + aq = 0$$

$$aq = \frac{b^2}{2a} - \frac{b^2}{4a} - c$$

$$q = \frac{b^2}{2a^2} - \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$q = \frac{b^2 - 4ac}{4a^2}$$

So, $\sqrt{q} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$

hence if $p + \sqrt{q}$ is a root $p - \sqrt{q}$ is also a root.

(OR)

replace $(p - \sqrt{q})$ in place of x

We get, $ap^2 + aq + bp + c - (2ap + b)\sqrt{q} = 0$

We know: $ap^2 + aq + bp + c = 0$

$$2ap + b = 0$$

Hence $p - \sqrt{q}$ is also a root.

Q. ~~find~~ Find Condition st. General Equation of second degree in x and y may be resolved into 2 linear factors.

Ans:

General Expression of 2nd degree in X and Y

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$\Rightarrow ax^2 + (2hy + 2g)x + (by^2 + 2fy + c) = 0$$

$$\Rightarrow ax^2 + 2(hy+g)x + (by^2 + 2fy + c) = 0$$

is the Quadratic form.

$$D = 4(hy+g)^2 - 4a(by^2 + 2fy + c)$$

$$= 4h^2y^2 + 4g^2 + 8hyg - 4aby^2 - 8afy - 4ac$$

$$= 4\left[(h^2 - ab)y^2 + 2(hg - af)y + g^2 - ac\right]$$

if $D = 0$ then roots are equal,

ie, $(h^2 - ab)y^2 + 2(hg - af)y + g^2 - ac = 0$

for D = +ve perfect square,

Roots of $(h^2 - ab)y^2 + 2(hg - af)y + g^2 - ac = 0$

must be equal.

$$4(hg - af)^2 - 4(g^2 - ac)(h^2 - ab) = 0$$

$$\Rightarrow af^2 + bg^2 + ch^2 - 2fgh - abc = 0$$

So,

Gen eqⁿ : $$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

$$ax^2 + 2gx + 2hxy + by^2 + 2fy + c = 0$$

To resolve it into 2 linear factors (ie, 2 real roots and rational)

Condition: $$af^2 + bg^2 + ch^2 - 2fgh - abc = 0 \quad (a \neq 0)$$

Some important results :

1. Ratio of roots of equation $ax^2+bx+c=0$ is equal to ratio of roots of $px^2+qx+r=0$. Prove $rpb^2 = caq^2$.

Ans :

$$ax^2+bx+c=0 \qquad\qquad px^2+qx+r=0$$

$$\alpha+\beta = -b/a \qquad\qquad \gamma+\delta = -\frac{q}{p}$$

$$\alpha\beta = c/a \qquad\qquad \gamma\delta = \frac{r}{p}$$

We also know, $\dfrac{\alpha}{\beta} = \dfrac{\gamma}{\delta}$

$$\frac{(\alpha+\beta)^2}{\alpha\beta} = \frac{\alpha^2+\beta^2+2\alpha\beta}{\alpha\beta} = \frac{\alpha}{\beta}+\frac{\beta}{\alpha}+2$$

$$= \frac{\gamma}{\delta}+\frac{\delta}{\gamma}+2$$

$$= \frac{(\gamma+\delta)^2}{\gamma\delta}$$

Thus,

$$\frac{b^2/a^2}{c/a} = \frac{q^2/p^2}{r/p}$$

$$\Rightarrow \frac{b^2 a}{c a^2} = \frac{q^2 p}{r p^2}$$

$$\Rightarrow \frac{b^2}{ca} = \frac{q^2}{rp}$$

$$\Rightarrow b^2 rp = q^2 ca$$

$$\Rightarrow \boxed{rp.\, b^2 = caq^2}$$

Remember

$$ax^2 + bx + c = 0$$

$$px^2 + qx + r = 0$$

if $r \cdot p \cdot b^2 = c \cdot a \cdot q^2$ then $\frac{\alpha}{\beta} = \frac{\gamma}{\delta}$.

2. $ax^2 + bx + c = 0$

$px^2 + qx + r = 0$

Condⁿ for Common Root:

$$ax^2 + bx + c = 0$$

$$px^2 + qx + r = 0$$

$$\frac{\alpha^2}{br - cq} = \frac{\alpha}{pc - ar} = \frac{1}{aq - bp}$$ where α = Common Root

$$\alpha = \frac{br - cq}{pc - ar} \quad \text{——— (1)}$$

$$\alpha = \frac{pc - ar}{aq - bp} \quad \text{——— (2)}$$

So, $(pc - ar)^2 = (br - cq)(aq - bp)$ is the Condition for Common Root.

ii. Condition for 2 common roots

$ax^2 + bx + c = 0$ $\qquad$ $px^2 + qx + r = 0$

$\alpha + \beta = -b/a$ $\qquad$ $\alpha + \beta = -q/p$

$\alpha\beta = c/a$ $\qquad$ $\alpha\beta = r/p$

So, $\frac{b}{a} = \frac{q}{p}$ —— (1)

$\frac{c}{a} = \frac{r}{p}$ —— (2)

from (1) $\frac{a}{p} = \frac{b}{q}$

from (2) $\frac{a}{p} = \frac{c}{r}$

Combining we get : $\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$ as the condition for 2 common roots.