Propositions and Logical Connectives

Proposition : It is a declarative senstence that is either True or False but not both.

Symbols : P, Q, R symbolizes Propositional statements and are called Propositional Components

$\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ are Logical Connectives

eg :

$P \wedge Q \Rightarrow$ P and Q : Conjunction

$P \vee Q \Rightarrow$ P or Q : Disjunction.

$\neg P \Rightarrow$ not P : negation.

$P \rightarrow Q \Rightarrow$ if P then Q : Conditional

$P \leftrightarrow Q \Rightarrow$ P if and only if Q : Biconditional.

ex. of a Compound sentence which is more complex can be :

$$(P \wedge \neg Q) \vee (\neg S \rightarrow R)$$

Truth Tables :

We can evaluate truth tables from the truth values of its components.

Let us see the basic Truth Tables,

P	Q	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

and Negation:

P	$\neg P$
T	F
F	T

Let us construct a truth table for a more complex sentence say $\neg P \longrightarrow (Q \wedge P)$

P	Q	$\neg P$	$Q \wedge P$	$\neg P \longrightarrow Q \wedge P$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

<u>Tautology</u>: A Compound sentence is a <u>tautology</u> when its truth value is <u>true</u> regardless of the truth value of its components.

eg: $(P \vee \neg P)$ is a <u>tautology</u>

Contradiction: A compound sentence is a **Contradiction** when its truth value is **false** regardless of the truth value of its components.

eg: $(P \wedge \neg P)$ is a Contradiction.

Logical Equivalence:

Let ψ and φ be proportional sentences.

$\psi \Longleftrightarrow \varphi$, whenever the following holds:

For every truth assignment applied to the components of ψ and φ, the resulting truth values of ψ and φ are identical.

ex $\quad \neg(P \vee Q) \Longleftrightarrow P \wedge \neg Q$

Propositional Logic Laws:

1. **De Morgan's Law:**

$$\neg(\psi \vee \varphi) \Longleftrightarrow \neg\psi \wedge \neg\varphi$$

$$\neg(\psi \wedge \varphi) \Longleftrightarrow \neg\psi \vee \neg\varphi$$

2. **Commutative Laws**

$$\psi \wedge \varphi \Longleftrightarrow \varphi \wedge \psi$$

$$\psi \vee \varphi \Longleftrightarrow \varphi \vee \psi$$

3 <u>Associative Laws</u>

$$\psi \vee (\varphi \vee \chi) \iff (\psi \vee \varphi) \vee \chi$$

$$\psi \wedge (\varphi \wedge \chi) \iff (\psi \wedge \varphi) \wedge \chi$$

4. <u>Idempotent Laws</u>

$$\psi \wedge \psi \iff \psi$$

$$\psi \vee \psi \iff \psi$$

5 <u>Distributive Laws</u>

$$\psi \wedge (\varphi \vee \chi) \iff (\psi \wedge \varphi) \vee (\psi \wedge \chi)$$

$$\psi \vee (\varphi \wedge \chi) \iff (\psi \vee \varphi) \wedge (\psi \vee \chi)$$

$$(\varphi \vee \chi) \wedge \psi \iff (\varphi \wedge \psi) \vee (\chi \wedge \psi)$$

$$(\varphi \wedge \chi) \vee \psi \iff (\varphi \vee \psi) \wedge (\chi \vee \psi)$$

6. <u>Double Negation Law</u> : $\neg\neg\psi \iff \psi$

7. <u>Tautology Law</u> : $\psi \wedge (\text{a tautology}) \iff \psi$

8. <u>Contradiction Law</u> : $\psi \vee (\text{a contradiction}) \iff \psi$

9. <u>Conditional Laws</u> :

$$\psi \rightarrow \varphi \iff \neg\psi \vee \varphi \iff \neg(\psi \wedge \neg\varphi)$$

10. <u>Contrapositive Law</u> : $\psi \rightarrow \varphi \iff \neg\varphi \rightarrow \neg\psi$

11. <u>Bi Conditional Law</u> : $\psi \leftrightarrow \varphi \iff (\psi \rightarrow \varphi) \wedge (\varphi \rightarrow \psi)$

Now, we will no longer use truth tables to prove logical equivalence but use laws instead.

<u>exercise</u>: Prove: $(P \rightarrow R) \wedge (Q \rightarrow R) \Leftrightarrow (P \vee Q) \rightarrow R$.

$$(P \rightarrow R) \wedge (Q \rightarrow R) \Leftrightarrow (\neg P \vee R) \wedge (\neg Q \vee R)$$

(Conditional Law)

$$\Leftrightarrow (\neg P \wedge \neg Q) \vee R$$

(distributive Law)

$$\Leftrightarrow \neg (P \vee Q) \vee R \quad \text{(De Morgan)}$$

$$\Leftrightarrow (P \vee Q) \rightarrow R \quad \text{(Conditional Law)}$$

hence proved!

Let us now prove: $x \notin A \setminus B \Leftrightarrow x \notin A$ or $x \in B$

$$x \notin A \setminus B \Leftrightarrow \neg x \in A \setminus B \Leftrightarrow \neg (x \in A \wedge x \notin B)$$

(by def^n of $\notin$) (by def^n of $\setminus$)

$$\Leftrightarrow x \notin A \vee x \in B$$

(De Morgan)

ie, $\boxed{x \notin A \text{ or } x \in B}$

Let us prove another result,

$$P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R.$$

$$P \rightarrow (Q \rightarrow R) \Leftrightarrow \neg P \vee (Q \rightarrow R) \Leftrightarrow \neg P \vee (\neg Q \vee R)$$

(Conditional Laws)

$$\Leftrightarrow (\neg P \vee \neg Q) \vee R \quad \text{(Associative)}$$

$$\Leftrightarrow \neg (P \wedge Q) \vee R \quad \text{(De Morgan)}$$

$$\Leftrightarrow (P \wedge Q) \rightarrow R \quad \text{(Conditional law)}$$