

ON MULTI-MARGINAL PARTIAL OPTIMAL TRANSPORT: RECTIFYING INFEASIBLE EXTENSION STRATEGIES AND EFFICIENT PRIMAL-DUAL METHODS

Quang Nguyen Dinh Thien

Dung Hoang Duc

University of Information Technology, Vietnam National University, Ho Chi Minh City, Vietnam

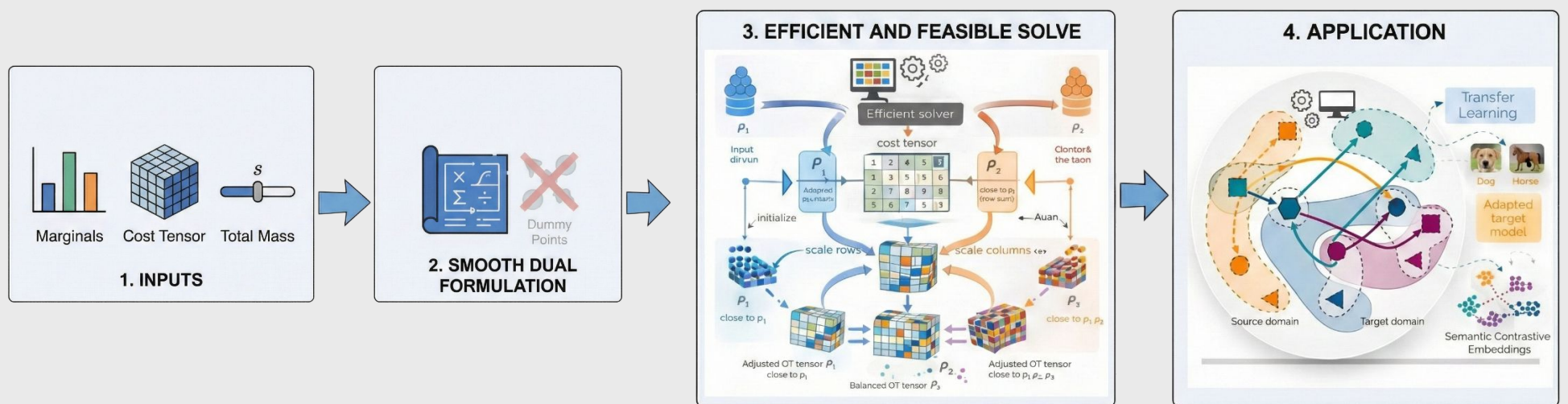
Motivation

- **Robustness:** MMPOT handles **noisy and unbalanced data** significantly better than standard Optimal Transport.
- **Research Gap:** Current "dummy point" extension strategies are mathematically flawed, leading to **infeasible solutions** that violate total mass constraints.
- **Computational Cost:** Previous methods suffer from high complexity $\mathcal{O}(1/\epsilon^4)$, making them impractical for large-scale problems.

Contribution

- **Novel Framework:** A new **Primal-Dual** formulation for MMPOT that eliminates the need for dummy point extensions.
- **Feasibility:** Theoretically rectifies the infeasibility of previous strategies, ensuring strict adherence to mass constraints.
- **Fast Algorithms:** Introduces **GreenhornMMPOT**, **PDAAM**, and **APDAGD** solvers achieving an optimal convergence rate of $\mathcal{O}(1/\epsilon)$.

Overview



Description

1. The Problem with Previous Methods

- **Infeasibility of Extension Strategies:** Current methods rely on adding "dummy points" to balance mass. However, the non-linearity of **Entropic Regularization** mathematically prevents accurate mass adjustment at the marginals.
- **Consequence:** This results in **infeasible solutions** that strictly violate the total mass transportation constraint (s).
- **Complexity Explosion:** To minimize approximation error, the extended cost entries must be extremely large ($A \approx \mathcal{O}(1/\epsilon)$). This causes the Sinkhorn complexity to surge to $\mathcal{O}(1/\epsilon^4)$ (compared to the standard $\mathcal{O}(1/\epsilon^2)$), or even grow exponentially when marginals $m \geq 4$.

2. Proposed Algorithms

Novel Primal-Dual Framework: Instead of tensor extension, we establish a **smooth dual formulation** directly for MMPOT. This eliminates approximation errors caused by dummy points.

New Solvers: We introduce three algorithms:

- **GreenhornMMPOT:** Greedy coordinate descent, highly efficient for early iterations.
- **PDAAM & APDAGD:** Accelerated algorithms that achieve the optimal theoretical convergence rate of $\mathcal{O}(1/\epsilon)$.
- **Rounding Algorithm:** A novel rounding method utilizing an "Enforcing Procedure" is proposed to project approximate solutions onto the feasible set, ensuring strict adherence to mass constraints.

3. Applications & Results

- **Application:** Partial Barycenter computation on noisy datasets (e.g., MNIST). MMPOT is shown to effectively filter outliers by transporting only a specific fraction of the total mass, unlike standard MMOT.
- **Performance:** The proposed algorithms (PDAAM, APDAGD) demonstrate significantly **faster convergence** in terms of both runtime and iteration count compared to the Extension Sinkhorn baselines.
- **Accuracy:** The solutions achieve lower objective value gaps while strictly satisfying the total mass constraints, verifying the theoretical feasibility analysis.

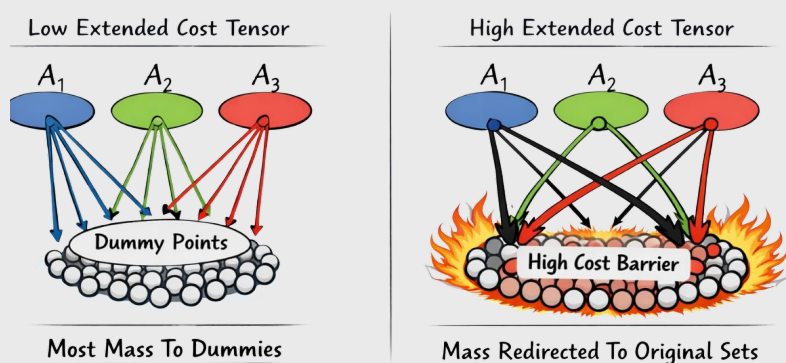


Figure 1. Illustration of mismatch mass behaviour in MMPOT

Algorithm	Complexity	Extended Cost	Optimal n -dep.	Optimal ϵ -dep.	All Mass Transport
First Ext. Sinkhorn	$\mathcal{O}\left(\frac{n^m m^5 \ C\ _\infty^2 \log n}{\epsilon^4}\right)$	✓	✓	×	×
Second Ext. Sinkhorn	$\mathcal{O}\left(\frac{n^m m^5 \ C\ _\infty^2 \log n}{\epsilon^4}\right)$	✓	✓	×	✓
GreenhornMMPOT	$\mathcal{O}\left(\frac{n^m m^4 \ C\ _\infty \log n}{\epsilon^2}\right)$	×	✓	×	✓
PDAAM	$\mathcal{O}\left(\frac{n^{m+0.5} m^3 \ C\ _\infty \log n}{\epsilon}\right)$	×	×	✓	✓
APDAGD	$\mathcal{O}\left(\frac{n^{m+0.5} m^3 \ C\ _\infty \log n}{\epsilon}\right)$	×	×	✓	✓

Figure 2. Comparison of entropic algorithm solvers for MMPOT