MANAGEMENT SCIENCE
Vol. 29, No. 11, November 1983
Printed in U.S.A.

SALES TERRITORY ALIGNMENT: A REVIEW AND MODEL*

ANDRIS A. ZOLTNERS† AND PRABHAKANT SINHA‡

The sales territory alignment problem may be viewed as the problem of grouping small geographic sales coverage units into larger geographic clusters called sales territories in a way that the sales territories are acceptable according to managerially relevant alignment criteria. This paper first reviews sales territory alignment models which have appeared in the marketing literature. A framework for sales territory alignment and several properties of a good sales territory alignment are developed in the course of the review. A general sales territory alignment model which accommodates these properties is developed. A solution procedure for the general model is presented. Finally, an actual implementation of the general model is described. The implementation provides a comparison of the general model with a similar model which has been frequently cited in the marketing literature.
(MARKETING MODELS; SALES FORCE PLANNING; MATHEMATICAL PROGRAMMING APPLICATION)

1. Introduction

Over the years, marketing scientists have developed numerous models for various sales management decisions. These models have addressed such decisions as sales force sizing, sales force organization, sales effort allocation to products and markets, salesperson time management, the establishment of commission rates and quotas, sales territory alignment, and salesperson selection.

Working with the sales managements of a number of corporations, we have found that more firms are willing to accept and implement a management science approach to the sales territory alignment decision than any of the other sales force decisions mentioned above. Three reasons for this observation are proposed. First, companies are faced regularly with the sales territory alignment decision. Sales territories need to be adjusted as new products are introduced and as markets shift. Changes in sales force size and sales force organization, as well as the desire to improve performance through better coverage, equitable workloads and reduced travel time bring about a need to realign sales territories. Second, the realignment of sales territories is a lengthy process for most firms, requiring many man-months of effort. Consequently, an organized approach for expediting the process is often appreciated. Third, models for the other sales force decisions are frequently perceived as too esoteric and require data which are currently collected by only a small percentage of companies.

It is interesting to note that in spite of the fact that sales territory alignment is an area where management science techniques are likely to be accepted and implemented, it is also an area in which the marketing science literature does not propose a generalized optimization approach to the problem. For example, Hess and Samuels [7, p. P-53] admit that their procedure, GEOLINE, "does not provide optimal sales territories" and Segal and Weinberger [13, p. 374] indicate that the geographic properties for alignments developed using their algorithm "are not guaranteed to be satisfactory" and that their algorithm has a potential of creating "disconnected" territories.

*Accepted by Leonard Jon Parsons; received February 4, 1982. This paper has been with the authors 3 months for 1 revision.
†Northwestern University.
‡Rutgers University.

0025-1909/83/2911/1237$01.25

An approach to sales territory alignment which improves upon the existing methodology is presented in this paper. A review of sales territory alignment models which have appeared in the marketing literature will be presented first. A framework for sales territory alignment and several properties of a good sales territory alignment are developed in the course of the review. A general sales territory alignment model which accommodates these properties is developed in §3. A solution procedure for this model is presented in §4. An actual implementation of the general model is described in §5. The implementation provides a comparison of the model developed in §3 with the Hess and Samuels model. Finally, several important model implementation issues are discussed in §6.

2. A Framework for Sales Territory Alignment and a Review of Sales Territory Alignment Approaches

The sales territory alignment problem may be viewed as the problem of grouping small geographic sales coverage units (SCUs) into larger geographic clusters called sales territories in a way that the sales territories are acceptable (or optimal) according to managerially relevant alignment criteria. The choice of SCUs depends upon the alignment application. They are usually defined in terms of a meaningful sales force planning unit for which the requisite data can be obtained. Counties, zip code areas, census tracts, SMSA's, DMA's, company trading areas and other predetermined account/prospect clusters are some examples of SCUs.

Several sales territory alignment approaches have appeared in the marketing literature. These approaches can be divided between those which depend entirely upon managerial heuristics and those which utilize a mathematical programming model. Totally heuristic approaches have been proposed by Easingwood [3], Lodish [9], [10] and Heschel [6]. Both Easingwood and Lodish develop and support a single alignment criterion for evaluating sales territory configurations. Easingwood advocates balancing workload while Lodish supports a sales territory alignment whose territories have equal marginal profitabilities. Heschel suggests that both market potential and sales territory size be used as design criteria.

The heuristic approaches have two major shortcomings. First, they are trial and error procedures which rely on an adjust-and-evaluate mechanism to arrive at reasonable territory configurations. For example, Easingwood [3, p. 527 and p. 530] recommends that sales management "successively adjust the boundaries until workload is uniformly allocated," Step 3 of Lodish's alignment procedure [9, p. 34] is "Management decides which men should have areas added and subtracted from their territories to lower the deficiencies or surpluses in Step 2" and Heschel [6, p. 41] states that "various territory alternatives were considered." As a result, these approaches provide sales management with data and an objective for sales territory alignment, but do not provide a methodology for actually designing sales territories. They are unsuitable for major territory restructuring and can overlook good territory alignments even when they are used for minor adjustments.

The second major shortcoming is that these approaches do not allow the model user to employ multiple alignment criteria. Even when an attractive criterion such as equalizing marginal profitability is employed, it is almost always desirable to maintain a minimum level of exposure in areas having low marginal profitability so that the firm's market share and consumer franchise are protected.

These trial-and-error approaches have one significant advantage. Since territory adjustments are made manually or through man-machine interactions, it is impossible to obtain disconnected territories or territories which disregard geographic obstacles such as mountains and waterways. Access routes to customers and prospects can be easily considered.

Two types of mathematical programming models have been employed for sales territory design. Shanker, Turner and Zoltners [14] have formulated a set-partitioning model for sales territory design. Alternatively, the models developed by Hess and Samuels [7], Segal and Weinberger [13], Zoltners [16], Richardson [12], and the model presented in this paper can be classified as SCU-assignment models. Of the two types of models, the SCU-assignment models are more effective. In sales territory alignment experiments conducted by the authors, the set-partitioning formulation was found to be cumbersome and computationally unattractive.

The following framework is useful for describing SCU-assignment models. Assume that m sales territories are to be designed from a total of n SCUs. Usually n is considerably larger than m. Let the territories be indexed by i ($i = 1, 2, \ldots, m$) and the SCUs by j ($j = 1, 2, \ldots, n$). The basic design decision is whether or not territory i includes SCU j, or equivalently, whether or not SCU j is assigned to sales territory i. This decision can be represented by a zero-one decision variable,

$$x_{ij} = \begin{cases} 1 & \text{if sales territory } i \text{ includes SCU } j, \\ 0 & \text{if sales territory } i \text{ does not include SCU } j. \end{cases}$$

A sales territory alignment is established whenever x_{ij} values are specified in a way that $\sum_{i=1}^{m} x_{ij} = 1$ for each SCU j. Since there are an extremely large number of feasible solutions to these constraints in a typical sales territory alignment implementation, there are also an extremely large number of potential sales territory configurations. Some alignments will be better than others. Evaluative measures or attributes are therefore required to discern good sales territory configurations from lesser ones. Examples of common attributes include workload, sales potential, sales volume, and distance measures. The criterion or criteria for evaluating various alignments will differ among companies depending upon their environment, available data, and orientation.

Generally, attribute values for each sales territory are determined by aggregation. Each SCU is evaluated in terms of the selected attributes. Since any potential sales territory is a collection of SCUs, a territory can be evaluated relative to the territory attribute value arrived at by aggregating the corresponding SCU attribute values. For example, if a_j is an attribute value for SCU j and the territory attribute aggregation function is linear then $\sum_{j=1}^{n} a_j x_{ij}$ is the attribute value for territory i.

The SCU-assignment models, developed to date, use linear aggregation functions. One aggregation function is used as an objective function and the others form the constraint set. The constraints are used to eliminate weak or undesirable territory configurations. The objective function serves to select an optimal solution from the set of territory alignments which are feasible relative to the constraints.

Hess and Samuels [7] were the first to apply a SCU-assignment model for sales territory alignment. Their model can be stated as follows:

$$(M_{HS}) \quad \text{minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} \left(a_j d_{ij}^2\right) x_{ij} \tag{1}$$

$$\text{subject to:} \quad \sum_{j=1}^{n} a_j x_{ij} = \left(\sum_{j=1}^{n} a_j \right) \Big/ m \quad \text{for } i = 1, 2, \ldots, m, \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} = 1 \quad \text{for } j = 1, 2, \ldots, n, \tag{3}$$

$$x_{ij} = 0 \text{ or } 1 \quad \text{for } i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n, \tag{4}$$

where a_j is the attribute for SCU j, d_{ij} is the Euclidean distance between the center of SCU j and the center of territory i.

Model (M_{HS}) is designed to build compact sales territories while equalizing a single attribute or "sales activity" among the territories. Constraints (2) insure that the resulting sales territories maintain the desired balance. Requiring that sales territories have almost equal workload or almost equal sales potential are two examples of frequently utilized alignment objectives. Constraints (3) and (4) insure that each SCU is assigned to exactly one sales territory. The objective function (1) was selected by Hess and Samuels because it has the tendency to assign SCUs having large values for a_j (e.g., sales potential, workload) to the closest territory, hence producing compact territories.

Model (M_{HS}) is over-constrained. It is unlikely that a reasonable feasible solution can be found for constraints (2)–(4). However, the model becomes a transportation model when the integrality assumption in constraints (4) is relaxed; that is, when constraints (4) are replaced by $0 \leq x_{ij} \leq 1$. Since efficient algorithms exist for the transportation model, Hess and Samuels observe that the linear programming relaxation (LP relaxation) of their model can be solved quickly. Further, there are many feasible solutions to the LP relaxation. However, they may be fractional. Fractional solutions imply that some SCUs are split among two or more sales territories. Fortunately, at most $m - 1$ SCUs will be split. Hess and Samuels suggest that sales territories be designed by solving the LP relaxation of their model and rounding the fractional solutions to integral values. The rounding rule employed in their procedure corresponds to assigning a split SCU to that territory having the largest fractional value for the decision variable. Ties are broken arbitrarily.

The Hess and Samuels LP relaxation plus rounding approach has several shortcomings. First, the solution may not be optimal. It may be embarrassing for the model user to develop a sales territory alignment which is not as good as an alignment developed by a sales manager or a sales analyst looking at a map. Second, the rounded solution may significantly disrupt the balance specified by constraints (2). This is bound to happen whenever the fractional variables correspond to SCUs which have large attribute values. For their seven applications, however, Hess and Samuels [7, p. P-51] report that most of the rounded solutions have created territories within $\pm 10\%$ of balance. Third, the Hess and Samuels approach does not guarantee that sales territories are contiguous or connected. A salesperson may be required to enter a different sales territory in an effort to reach another part of his/her territory. Fourth, because it is a single attribute approach, the Hess and Samuels approach does not accommodate multiple territory alignment criteria. Finally, Hess originally developed this approach for political redistricting [8]. Whereas Euclidean compactness is an important criterion for political redistricting, the same is not true for sales territory alignment. The accessibility of SCUs from a territory center via travel lanes (e.g., highways, airline routes) is considerably more important. The inability of the Hess and Samuels approach to accommodate travel lanes and nontraversable obstacles such as mountains and waterways represents a serious deficiency in their approach.

Two other SCU-assignment approaches have been developed which sought to address some of the shortcomings of the Hess and Samuels model. Segal and Weinberger [13] attempt to incorporate accessibility into the Hess and Samuels model (M_{HS}) by replacing $a_j d_{ij}^2$ by the distance of the shortest path between SCU j and the center of territory i via the network connecting adjacent traversable SCUs. Sales territories designed with this objective function tend to accommodate travel lanes and geographic obstacles. However, there is no guarantee that accessible sales territory alignments will be developed.

Instead of using the transportation model as the basic LP relaxation, Segal and Weinberger employ a minimum cost feasible flow relaxation. Unfortunately, this relaxation suffers from the same shortcomings as does the Hess and Samuels relaxation. SCUs may be split. In fact, more than $m-1$ fractional solutions are possible. To compensate, Segal and Weinberger offer an elaborate rounding procedure to deal with split SCUs. They do not, however, offer empirical results of the ensuing attribute imbalance. Finally, contiguity and multi-attribute balancing are not addressed.

Zoltners [16] recognized a need for multi-attribute sales territory alignment and has extended the Hess and Samuels model to incorporate multiple attribute balancing. Zoltners' model can be stated as follows:

$$(M_Z) \qquad \text{minimize} \quad \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_{ij}$$

$$\text{subject to:} \quad l_{ik} < \sum_{j=1}^{n} a_{ijk}x_{ij} < u_{ik} \qquad \text{for} \quad i=1,2,\ldots,m, \quad k=1,2,\ldots,h,$$

$$\sum_{i=1}^{m} x_{ij} = 1 \qquad \text{for} \quad j=1,2,\ldots,n,$$

$$x_{ij} = 0 \text{ or } 1 \qquad \text{for} \quad i=1,2,\ldots,m, \quad j=1,2,\ldots,n,$$

where

c_{ij} is a distance measure associated with the assignment of SCU j to center i,

k is an index for the h relevant territory design attributes,

a_{ijk} is the kth attribute value for SCU j conditioned on the assignment of SCU j to territory center i,

l_{ik} is a lower limit for the kth attribute for the sales territory centered at i,

u_{ik} is an upper limit for the kth attribute for the sales territory centered at i.

Using a model similar to (M_Z), Richardson [12] has designed sales territories for Pfizer, Inc. in terms of five different measures of sales potential. Richardson does not provide a formulation for his model. If he did, it would be similar to the following:

$$(M_R) \qquad \text{minimize} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} a_j^* d_{ij}^2 x_{ij}$$

$$\text{subject to:} \quad \sum_{j=1}^{n} a_{jk}x_{ij} = \left(\sum_{j=1}^{n} a_{jk}\right)\Big/ m$$

$$\text{for} \quad i=1,2,\ldots,m, \quad k=1,2,\ldots,h,$$

$$\sum_{i=1}^{m} x_{ij} = 1 \qquad \text{for} \quad j=1,2,\ldots,n,$$

$$x_{ij} = 0 \text{ or } 1 \qquad \text{for} \quad i=1,2,\ldots,m, \quad j=1,2,\ldots,n,$$

where

k is an index for the h $(=5)$ relevant territory design attributes,

a_{jk} is the kth attribute value for SCU j,

a_j^* is a weighted combination of the design attributes, e.g., $a_j^* = \sum_{k=1}^{h} w_k a_{jk}$,

d_{ij} is the Euclidean distance between the center of SCU j and the center of territory i.

Like the model builders who preceded him, Richardson does not solve (M_R) directly. He solves the LP relaxation. Unfortunately, there may be as many as hm split SCUs. Richardson does not provide details of how he deals with the massive SCU splitting which can occur nor does he speak of the degree of noncontiguity in the final territory alignment.

Richardson recognizes the importance of travel lanes and geography. He incorporates accessibility into his territory alignment by designing the shape of troublesome territories a priori. This is accomplished within the model by establishing multiple territory centers which reflect the desired shape of the troublesome sales territory. One disadvantage of prespecifying territory configurations is that better territory alignments may be overlooked.

To summarize, each of the SCU-assignment models mentioned above struggles with several sales territory design issues. These issues may be stated as properties of a good sales territory alignment. Specifically, a good sales territory alignment has been obtained when:

(P1) Each SCU is assigned to exactly one sales territory,

(P2) Sales territories are "almost" balanced relative to one or more territory attributes,

(P3) Sales territories are contiguous,

(P4) Sales territories are compatible with geographic considerations such as highways and nontraversable objects including mountains and waterways.

We have developed sales territories using the SCU-assignment models mentioned above and experienced difficulty developing sales territory alignments which satisfy properties (P1)–(P4). As a result, we developed a new model for sales territory alignment. The model, which is a generalization of the four SCU-assignment models presented above, was designed to satisfy these properties. The next section contains a description of our model.

3. A Model for Sales Territory Alignment

The sales territory alignment model, which is presented in this section, utilizes a hierarchical SCU-adjacency tree structure. The steps required to develop this tree structure are described before presenting the model.

A hierarchical SCU-adjacency tree is created for each sales territory center. The adjacency tree is a graph whose vertex is the SCU containing the territory center. Nodes of the graph represent other SCUs which are candidates for inclusion in the sales territory. Edges of the graph connect SCUs which are adjacent via a feasible road network. Figures 1 and 6 illustrate a geographic region comprised of 17 SCUs and the resulting hierarchical SCU-adjacency tree.

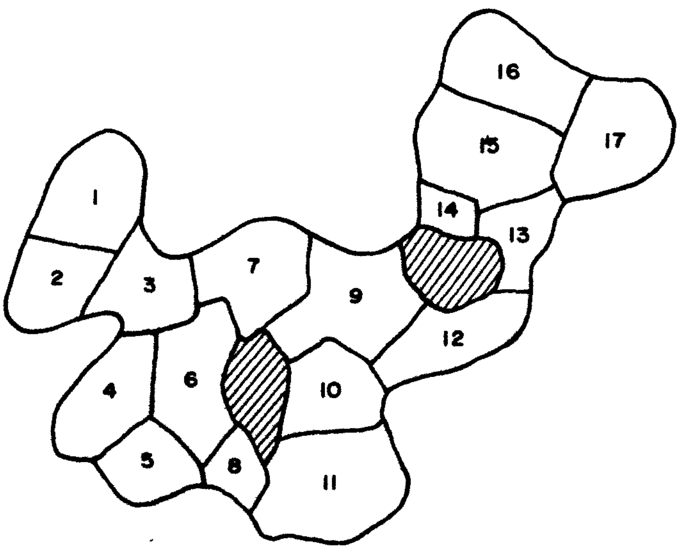

FIGURE 1. A Geographic Region Comprised of 17 SCUs.

(1) The sales territory center is in SCU 6

(2) The shaded areas represent nontraversable objects

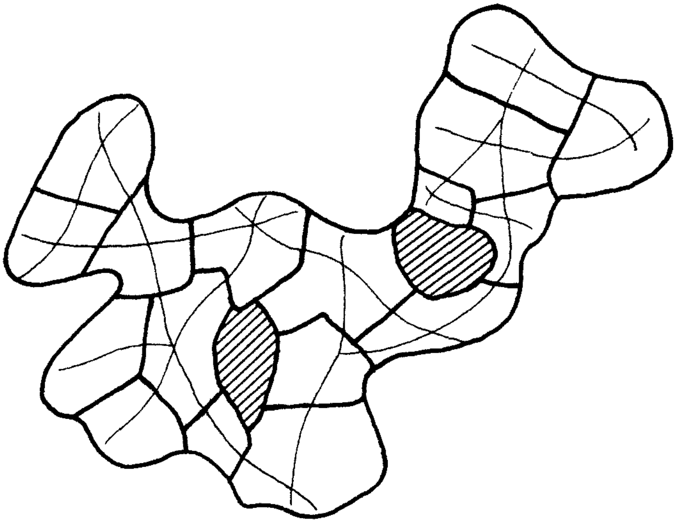

FIGURE 2. The Geographic Region with the Superimposed Road Network.

A hierarchical SCU-adjacency tree can be created for each sales territory center in the following way. First, a road and highway network is superimposed upon the geographic region. Figure 2 illustrates a road network for the geographic region presented in Figure 1. Notice that the incorporation of the road network allows for the implicit representation of nontraversable geographic obstacles such as mountains, lakes and rivers.

The authors have developed a computer readable data base of all U.S. interstate highways, all U.S. highways and all important state highways. Each road in the data base is represented by a long string of horizontal and vertical coordinates. This road data base was developed explicitly to test and implement the models presented in this paper.

A road graph can be constructed once the road network has been superimposed upon the geography. The nodes of the graph represent the SCUs and the edges represent roads which directly connect SCUs. Figure 3 illustrates a road graph integrating the SCUs in Figure 1 and the road network in Figure 2.

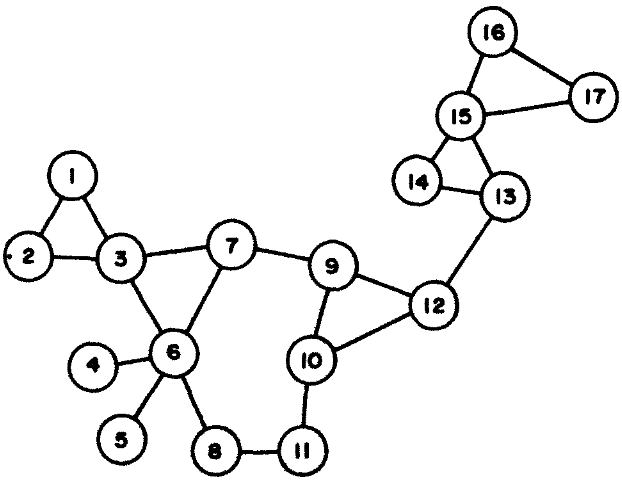

FIGURE 3. The Road Graph.

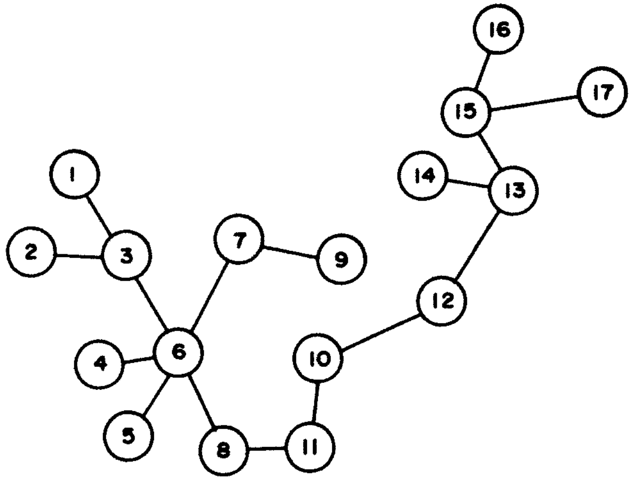

FIGURE 4. A Shortest Path SCU-Adjacency Tree.

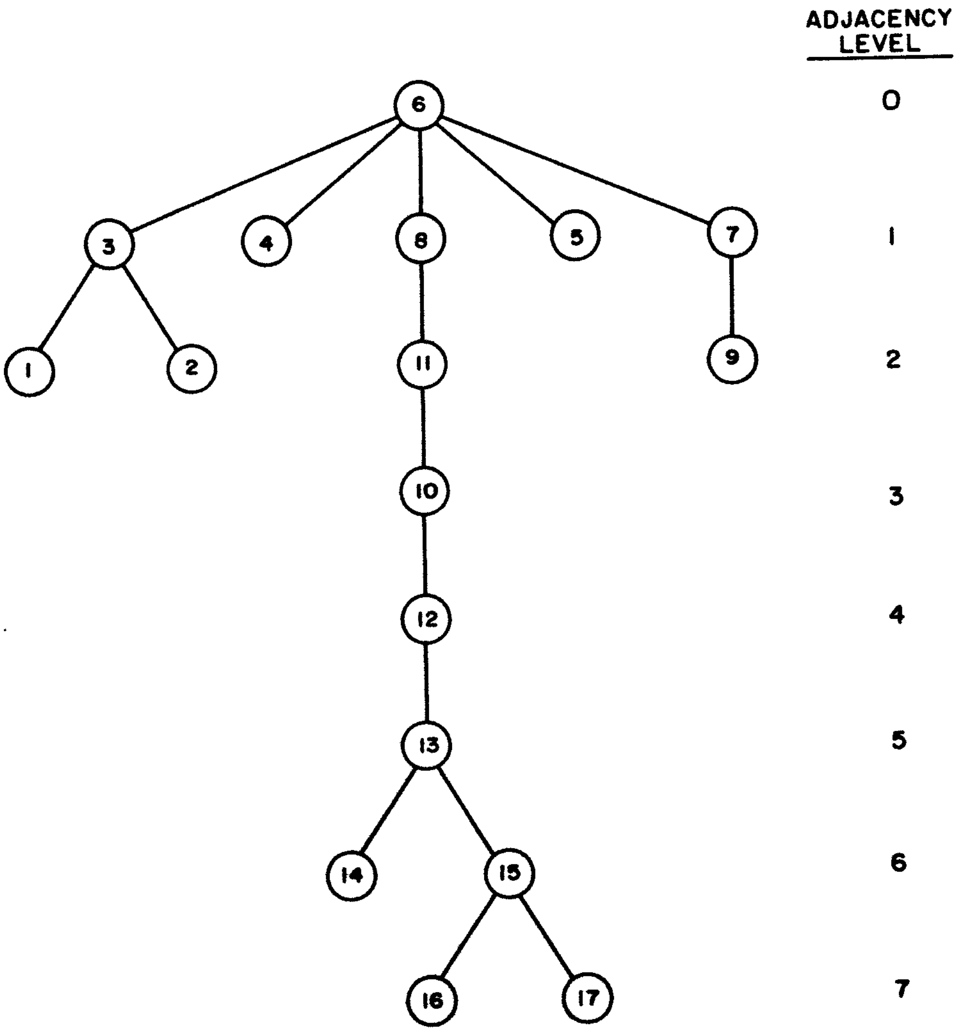

FIGURE 5. A Shortest Path SCU-Adjacency Tree and Adjacency Levels.

Using the road graph, the shortest paths (measured in travel time) between each territory center and the surrounding SCUs are calculated. A shortest path algorithm is used for this calculation. The number of passes through this algorithm is reduced significantly by the property that the shortest path between any two nodes, say ⓐ and ⓑ, contains the shortest path between any other two nodes lying on the shortest path between ⓐ and ⓑ. Computational effort is reduced further by eliminating the calculation of shortest paths from a territory center to SCUs which are so far removed from the center that it would be infeasible to include them in the sales territory originating at the center. As was mentioned earlier, Segal and Weinberger have also employed the shortest path concept. However, they create their graph using SCU adjacencies as opposed to superimposing a road network.

A shortest path SCU-adjacency tree can be established utilizing the set of shortest paths connecting the territory centers and SCUs. This tree, illustrated in Figure 4, characterizes the quickest routes from the territory center to each SCU. In Figure 5, the shortest path SCU-adjacency tree is redrawn with the territory center as the top node. This representation allows for a quick assessment of an SCU-adjacency level for each SCU. SCUs which are immediately adjacent to the territory center are at the first adjacency level. SCUs which are one removed are at the second level, and so forth.

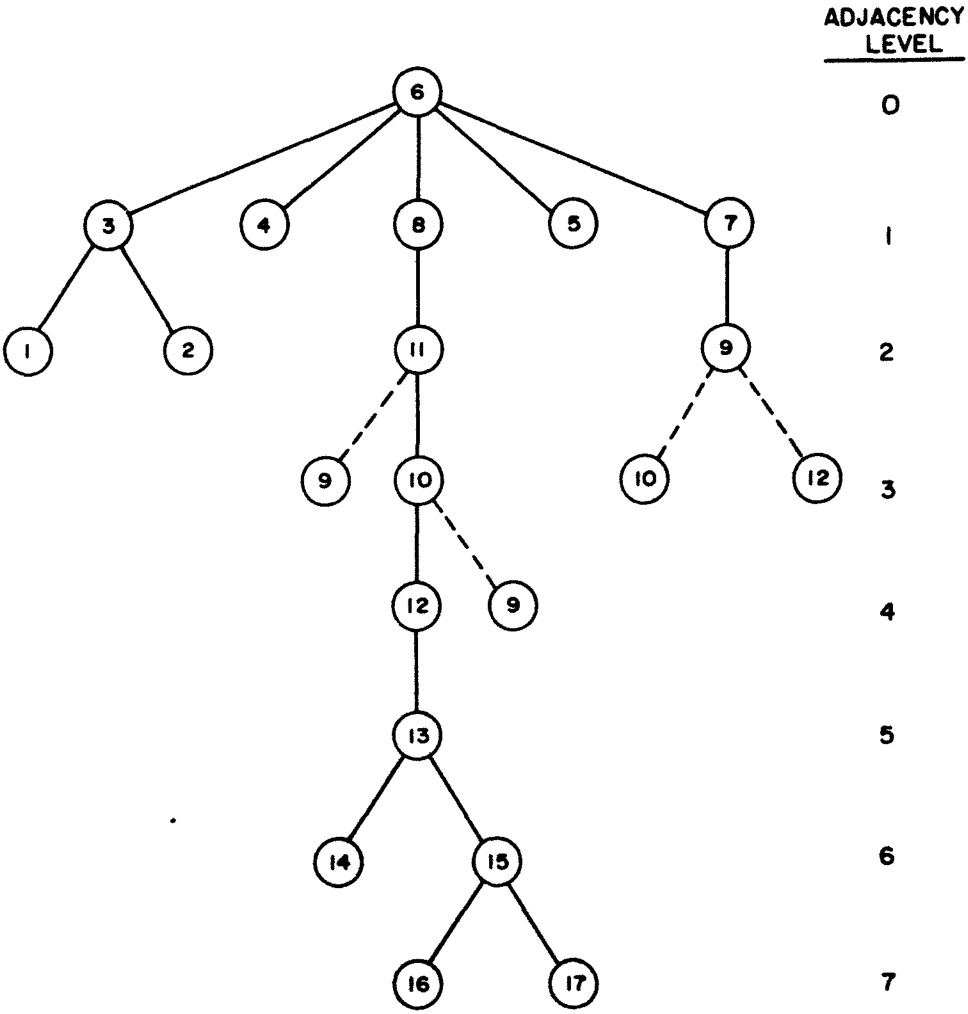

FIGURE 6. A Hierarchical SCU-Adjacency Tree.

Assuming that shortest paths will be used to represent distances between territory centers and SCUs, the following sales territory contiguity property can be stated in terms of the shortest path SCU-adjacency tree. Given that an SCU at the kth level is assigned to a territory center, sales territory contiguity is insured when its SCU predecesor at the $(k-1)$st level is also assigned to the territory center. This property is used in the model.

From a sales territory design perspective, the shortest path SCU-adjacency tree has one shortcoming. It establishes rigid access and contiguity constraints due to the fact that only one shortest path to a territory center is selected for each SCU. Consequently, sales territories which include a particular SCU must evolve along the shortest path connecting that SCU with the territory center. Notice that SCU 10 in Figure 4 can only be accessed by going below (south of) the geographic obstacle. On the other hand, situations could arise where it would be desirable to include SCU 10 in the territory originating at SCU 6 and include SCU 11 in a different territory. The shortest path SCU-adjacency tree does not allow this alternative. Additional edges must be added to the shortest path SCU-adjacency tree to accommodate such cases. To illustrate this point, several edges have been added in Figure 6. Additional edges can be derived by obtaining near optimal shortest paths (e.g., the second shortest, third shortest, etc.). However, in most cases additional edges would be supplied by experienced sales managers. The resulting graph, called the hierarchical SCU-adjacency tree, contains duplicate nodes (SCUs) on different branches of the tree. Fortunately, the hierarchical SCU-adjacency tree maintains the sales territory contiguity property mentioned above. Specifically, sales territory contiguity is insured whenever an SCU at the kth adjacency level is assigned to a territory center, then at least one of its SCU predecessors at the $(k-1)$st level is also assigned to the territory center.

A general sales territory alignment model is presented below. The sets A_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$), used in constraints (7), represented the set of SCUs which immediately precede SCU j on any branch of the hierarchical SCU-adjacency tree for center i.

$$(M_{SZ}) \quad \text{minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ijk*} x_{ij}, \tag{5}$$

$$\text{subject to:} \quad l_{ik} \leq \sum_{j=1}^{n} a_{ijk} x_{ij} \leq u_{ik}$$

$$\text{for } i = 1,2, \ldots, m,\ k = 1,2, \ldots, h,\ k \neq k^*, \tag{6}$$

$$x_{ij} \leq \sum_{p \in A_{ij}} x_{ip} \quad \text{for } i = 1,2, \ldots, m,\ j = 1,2, \ldots, n, \tag{7}$$

$$\sum_{i=1}^{m} x_{ij} = 1 \quad \text{for } j = 1,2, \ldots, n, \tag{8}$$

$$x_{ij} = 0 \text{ or } 1 \quad \text{for } i = 1,2, \ldots, m,\ j = 1,2, \ldots, n, \tag{9}$$

where

k is an index for the h relevant territory design attributes,

a_{ijk} is the kth attribute value for SCU j conditioned on the assignment of SCU j to territory center i,

l_{ik} is a lower limit for the kth attribute for the sales territory centered at i,

u_{ik} is an upper limit for the kth attribute for the sales territory centered at i.

The objective function (5) and constraints (6) employ linear territory attribute aggregation functions. The functions used in constraints (6) are more general than those stated in earlier SCU-assignment models. The additional i subscript enables the SCU attributes, represented by coefficient a_{ijk}, to be center dependent. Center dependent attributes are required to incorporate factors such as differences in travel time and salesperson effectiveness. Examples of relevant territory design attributes, along with attribute aggregation functions and their customary utilization within model (M_{SZ}), appear in Table I.

The solution to model (M_{SZ}) satisfies properties (P1)–(P4). Constraints (8) and (9) insure that property (P1) is satisfied. Constraints (6) insure that the sales territories are almost balanced relative to h territory attributes. The choices of l_{ik} and u_{ik} reflect the degree of the desired balance. In most cases, sales territory alignments for which $l_{ik} = u_{ik}$, for all i and k, are impossible to create. Consequently l_{ik} and u_{ik} are usually set a little below and above the desired attribute level. Constraints (7) insure that

TABLE I

Relevant Territory Design Attributes and their Customary Utilization within Model (M_{SZ})

Common Attributes	Coefficient Definition (a_{ijk} or a_{jk})	Normal Usage
Workload ·Number of accounts and prospects ·Number of sales calls ·Call effort	Workload can be either center dependent or center independent. For example, when travel time from the territory center is relevant, then the attribute coefficient should be defined in terms of a_{ijk}. If workload is measured as the number of sales calls or the number of accounts and prospects then a_{jk} is appropriate.	Workload aggregation functions usually appear as constraints (6). They can be incorporated into (M_{SZ}) to insure that the resulting sales territories have equitable workloads.
Sales Potential ·Industry sales ·Company sales	Sales potential measures are usually center independent; a_{jk} may be defined as the sales potential in SCU j.	Sales potential aggregation functions usually appear as constraints (6). They can be incorporated into (M_{SZ}) to insure that the resulting sales territories have balanced sales potentials.
Travel Time	This measure is center dependent. Let a_{ijk} be the time required by a sales person residing at center i to travel to SCU j. Travel time is calculated in terms of the road network described in §3. Travel speeds for roads comprising the shortest path from center i to SCU j are multiplied by the appropriate distance to arrive at a travel time. Most authors have employed Euclidean distances in territory alignment studies. Euclidean distances are not recommended because they do not adequately reflect geographical considerations.	Travel time may appear in the objective function (5) or in constraints (6). When appearing in the objective function, the model (M_{SZ}) aligns sales territories which minimize total travel time. As a constraint, travel time may appear as an upper bound on territory construction. Specifically, a model user can incorporate a maximum travel time for any sales territory.

TABLE I (*continued*)

Common Attributes	Coefficient Definition (a_{ijk} or a_{jk})	Normal Usage
Disruption	Disruption may be defined as the number of SCUs which are reassigned as a result of a new sales territory alignment. Define: $a_{ijk} = \delta_{ij}$ or $a_{ijk} = p_j \delta_{ij}$ where $\delta_{ij} = \begin{cases} 0 & \text{if SCU } j \text{ is currently a part of the sales territory centered at } i \\ 1 & \text{if SCU } j \text{ is currently a part of a sales territory centered somewhere other than } i \end{cases}$ p_j is the workload or sales potential associated with SCU j	Disruption usually appears in the objective function. Model (M_{SZ}) may be used to create a sales territory alignment which minimizes disruption. When disruption is incorporated into constraints (6), sales territories can be developed that restrict the number of SCUs which are switched out of a sales territory to a prespecified maximum.
Weighted Combinations of SCU Attributes	The attribute coefficient can be defined as a weighted sum of other attributes. Specifically, $\tilde{a}_{ijk} = \sum_{r=1}^{s} w_r a_{ijr}$	A weighted combination of SCU attributes may appear in the objective function. There is little reason to place a weighted attribute aggregation function into the constraint set.
Distance Weighted Attribute Measure	The attribute coefficient can be defined as a single attribute, such as sales potential or workload, weighted by the distance from SCU j to the center of territory i. Specifically, $\tilde{a}_{ijk} = d_{ij} a_{ijr}$ where d_{ij} is the distance from SCU j to the center of territory i, r is the index of a prespecified SCU attribute	Some authors have incorporated a distance weighted attribute measure into the objective function. Such a measure has the advantage of assigning SCUs that have large sales potentials or workloads to the closest territory center. There is little reason to place a distance weighted-attribute aggregation function into the constraint set.

property (P3) holds. Finally, (P4) is satisfied since the road network, that was developed to construct the hierarchical SCU-adjacency tree, insures that sales territory configurations are compatible with geographic considerations.

Model (M_{SZ}) is a complex integer programming model. The computational effort required to solve it using general purpose integer programming algorithms is prohibitive. Consequently, a specialized solution procedure was developed. The next section describes an efficient solution approach that produces sales territory alignments satisfying properties (P1)–(P4).

4. A Solution Approach for (M_{SZ})

As stated earlier, model (M_{SZ}) is a complex integer programming model. The model can be quite large. For instance, a sales territory alignment decision involving 400 SCUs, 15 territory centers and 3 attributes has 6000 binary decision variables, 45 balancing constraints—(6), approximately 1200 contiguity constraints—(7), and 400 multiple-choice constraints—(8). Fortunately, the multiple-choice structure can be exploited. Good algorithms exist (e.g., [1], [15]) for the relaxation of (M_{SZ}) which is formed by dropping constraints (7). The algorithm we have developed to solve (M_{SZ}) utilizes this relaxation through the implicit treatment of constraints (7). Employing this technique, we have consistently obtained solutions within 2% of optimality with modest computational effort, e.g., two minutes on a Prime 550 minicomputer for the problem described above.

Our solution approach is demonstrated for the single attribute (i.e., $h = 1$), territory-center independent (i.e., $a_{ij} = a_j$ for all i), weighted-distance model (e.g., $a_{ijk*} = a_j d_{ij}$). Recall that the Hess and Samuels and Segal and Weinberger models fall into this category.

As documented by Hess and Samuels, this special case of (M_{SZ}) occurs frequently in practice. It has been incorporated into several sales force planning systems [2], [4], [11]. Further, this model serves as a good basis for comparing sales territory alignments developed by a linear programming based-approach (e.g., Hess and Samuels) with the one proposed next. §5 contains a comparison for a small number of sales districts.

Consider the following single-attribute, weighted distance model having territory-center independent attributes:

$$(M_1) \qquad \text{minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} (a_j d_{ij}) x_{ij} \tag{10}$$

$$\text{subject to:} \quad \sum_{j=1}^{n} a_j x_{ij} = b_i \qquad \text{for} \quad i = 1, 2, \ldots, m, \tag{11}$$

$$x_{ij} \leqslant \sum_{p \in A_{ij}} x_{ip} \qquad \text{for} \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \tag{12}$$

$$\sum_{i=1}^{m} x_{ij} = 1 \qquad \text{for} \quad j = 1, 2, \ldots, n, \tag{13}$$

$$x_{ij} = 0 \text{ or } 1 \qquad \text{for} \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \tag{14}$$

where b_i is the balancing objective; e.g., $b_i = (\sum_{j=1}^{n} a_j)/m$.

The following multi-step procedure is used to solve (M_1). First, constraints (12) are dropped. Notice that the resulting model is equivalent to (M_{HS}). Second, a Lagrange relaxation of (M_1) is formed by multiplying constraints (11) by Lagrange multipliers, λ_i

$(i = 1, 2, \ldots, m)$, and bringing these constraints into the objective function:

$$(M_2) \qquad \text{minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} a_j(d_{ij} - \lambda_i)x_{ij} + \sum_{i=1}^{m} \lambda_i b_i$$

$$\text{subject to:} \quad \sum_{i=1}^{m} x_{ij} = 1 \qquad \text{for} \quad j = 1, 2, \ldots, n,$$

$$x_{ij} = 0 \text{ or } 1 \qquad \text{for} \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n.$$

Model (M_2) can be rewritten as the following separable integer program:

$$(M_3) \qquad \sum_{j=1}^{n} a_j \quad \text{minimize} \quad \sum_{i=1}^{m} (d_{ij} - \lambda_i)x_{ij}$$

$$\sum_{i=1}^{m} x_{ij} = 1 \qquad \text{for} \quad j = 1, 2, \ldots, n,$$

$$x_{ij} = 0 \text{ or } 1 \qquad \text{for} \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n.$$

Third, model (M_3) is solved. This model has the following trivial solution:

$$x_{ij} = \begin{cases} 1 & \text{whenever} \quad d_{ij} - \lambda_i = \min_p \{d_{pj} - \lambda_p\}, \\ 0 & \text{otherwise} \qquad \text{for} \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n. \end{cases} \tag{15}$$

Fortunately, the solution specified by (15) satisfies the contiguity conditions (12). The following proposition establishes this result.

PROPOSITION. *The solution obtained from* (15) *satisfies constraints* (12). *That is, if SCU j is assigned to center i, then at least one $r \in A_{ij}$ is also assigned to center i.*

PROOF. Suppose not; i.e., assume that each $r \in A_{ij}$ is assigned to center $q(r)$ and that $q(r) \neq i$ for all r. Then

$$d_{q(r)r} - \lambda_{q(r)} < d_{ir} - \lambda_i \qquad \text{for all} \quad r \in A_{ij}. \tag{16}$$

Since SCU j is assigned to center i, it also follows that:

$$d_{ij} - \lambda_i < d_{q(r)j} - \lambda_{q(r)} \qquad \text{for each } q(r) \quad \text{where} \quad r \in A_{ij}. \tag{17}$$

Let $r^* \in A_{ij}$ be that SCU which is on the shortest path between SCU j and center i and let D_{jr^*} be the distance between SCU j and SCU r^*. Then,

$$\begin{aligned} d_{ij} - \lambda_i &= D_{jr^*} + d_{ir^*} - \lambda_i \\ &> D_{jr^*} + d_{q(r^*)r^*} - \lambda_{q(r^*)} \qquad \text{from (16)} \\ &= d_{q(r^*)j} - \lambda_{q(r^*)}. \end{aligned}$$

This contradicts (17).

Fourth, a subgradient search is used to search the set of λ's and develop a sales territory alignment. A subgradient search similar to the one proposed by Held, Wolfe and Crowder [5] has been developed. At each iteration the λ_i's are updated based on the extent of the violation in constraints (11). The search procedure is efficient and yields sales territories which are almost balanced. As Table II in the next section illustrates, territories are usually within 5% of complete balance.

Sales territory contiguity in the above model is a direct consequence of two conditions: (a) a single distance-weighted attribute measure was used in the objective function, and (b) the attribute measure was SCU specific and center independent. In its general form, model (M_{SZ}) may have neither of these properties. A similar alignment procedure can still be used for the general case. This algorithm enforces contiguity in the search process. Noncontiguous SCUs in solutions to the Lagrange relaxation are reassigned to their closest contiguous center. The hierarchical SCU-adjacency tree is used to establish the appropriate reassignments. Violations of constraints (6) are smoothed through successive choices of λ_i.

5. A Comparison of the Linear Programming-Based and Subgradient Optimization Approaches

Model (M_{SZ}) has been used to develop sales territory alignments for several companies. One implementation called for a single attribute, territory center-independent model. This implementation provided an opportunity to compare sales territories created by the linear programming-based procedures developed by Hess and Samuels and Segal and Weinberger with the subgradient optimization approach described in the last section.

Three sales districts were randomly selected for the comparison. Each sales district had a district manager and thirteen salespeople. Zip codes were used as SCUs. There were between 204 and 280 zip codes in these districts.

Model (M_{HS}) was used for the comparison. Constraints (2) were formulated to insure that the sales district sales potential would be balanced across sales territories and a weighted-distance objective function (1) was utilized. The alignment approaches were compared in terms of the four properties listed in §2:

(P1) *Single Assignment of SCUs*

The linear programming solution to (M_{HS}) can split SCUs among multiple territory centers. Between 10 and 12 SCUs $(m - 1 = 12)$ were split in these sales districts. A heuristic was employed to reassign split SCUs to a single center. On the other hand, the subgradient optimization procedure guarantees that SCUs will not be split.

(P2) *Sales Potential Balance*

Table II provides a comparison of the sales potential balance achieved by the two approaches. Statistics for the current sales territory alignments are also included as a frame of reference.

(P3) *Contiguity*

Both procedures created contiguous sales territory alignments for these three sales districts.

TABLE II

A Comparison of Sales Potential Balance for the Current Sales Territory Alignment and the Alignments Derived Using the Linear Programming-Based and Subgradient Optimization Approaches

Sales District	Current Alignment		L.P.-Based Alignment		Subgradient Alignment	
	Range*	Standard Deviation*	Range	Standard Deviation	Range	Standard Deviation
A	0.86–1.18	0.11	0.81–1.22	0.10	0.95–1.05	0.03
B	0.61–1.37	0.22	0.90–1.16	0.06	0.93–1.05	0.04
C	0.69–1.31	0.20	0.92–1.04	0.04	0.96–1.04	0.03

*The sales potential range and standard deviation are expressed as a percentage of the average sales potential. That is, a sales territory having a sales potential percentage of 1.00 has a sales potential of $(\sum_j a_j)/m$.

(P4) *Geographic Considerations*

Figures 7 and 8 reveal that sales territory alignments become more practical when the relevant road network is taken into account. Four troublesome sales territories in one of the sales districts were isolated. Figure 7 shows the territory alignment using the linear programming-based approach. Figure 8 demonstrates the sales territory configuration using the approach developed in this paper. Clearly, it will take less travel time to cover the accounts in these sales territories if the alignment in Figure 8 is adopted.

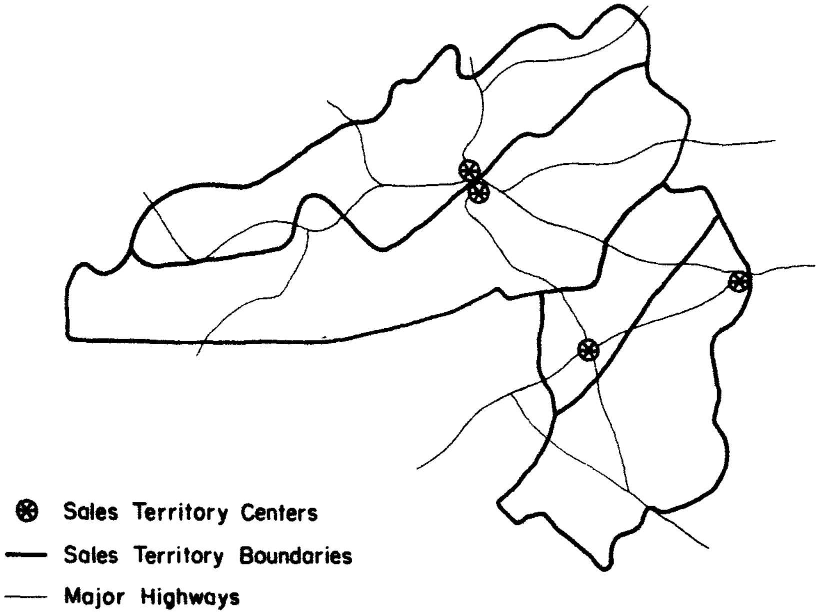

FIGURE 7. An Alignment for Four Sales Territories Using (M_{HS}).

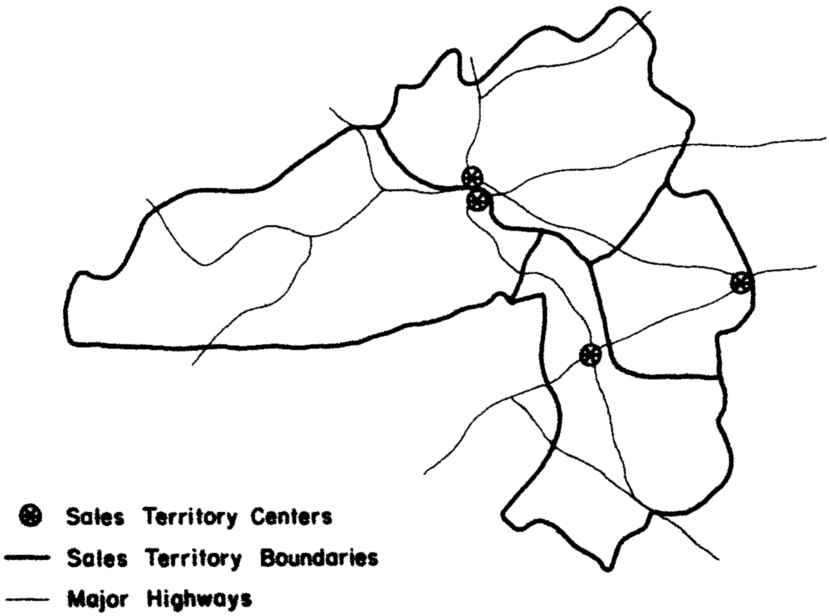

FIGURE 8. An Alignment for Four Sales Territories Using (M_1).

6. Implementation Issues

Two important implementation issues arise with regard to model (M_{SZ}). These issues are discussed in this section.

Model (M_{SZ}) is designed to be used to develop sales territories which satisfy properties (P1)–(P4). At the same time, the objective function provides a capability to optimize an attractive attribute such as: minimize disruption or minimize travel time. In this form, the model is consistent with the way most sales and marketing managers currently align sales territories. Most managers attempt to balance several key territory attributes such as sales potential and anticipated workload. They try to minimize travel time and keep relocation to a minimum. Unfortunately, most sales and marketing managers do not think of profit maximization when they align sales territories. However, several authors have recently described procedures for the development of profit maximizing sales territory alignments. These procedures require, as a territory alignment corequisite, an optimal deployment of sales force effort to the SCUs that comprise the geographic area which needs to be aligned.

The idea of integrating sales resource allocation and sales territory alignment was first proposed by Lodish [9]. Lodish suggested, essentially, a two-step procedure. First, a sales resource allocation model is used to derive profit maximizing workloads for the SCUs. Sales territories are then aligned to balance the profit maximizing workloads. Lodish argued that sales territory alignments derived using this procedure are likely to produce more profit than can be obtained from alignments derived from sales potential equalizing approaches.

Several other authors have also integrated sales resource allocation models and territory alignments models. Beswick and Cravens [2] and Glaze and Weinberg [4] have developed profit-maximizing sales force planning systems which couple a custom-designed resource allocation model with model (M_{HS}).

Based on the work of these authors, a generalized approach for developing profit-maximizing sales territories can be formulated. This approach is outlined in Figure 9. First, the total time available to the sales force is calculated. Total time relates directly to the size of the sales force. It can be stated in terms of the number of sales people, number of days or number of hours. Next, the total sales time is disaggregated into account and prospect call time, travel time and administrative time. This initial breakout of sales time can be based upon historical averages, adjusted for the anticipated travel time improvement resulting from an optimal territory alignment. Using the total account and prospect call time as an allocatible resource, a sales resource allocation model may be used to derive a profit-maximizing allocation of call time to the accounts and prospects comprising each SCU. (The interested reader can see Zoltners and Sinha [17] for a review of the sales resource allocation models that have appeared in the marketing literature.) A workload measure can be developed for each SCU by adding the SCU call frequency, produced by the sales resource allocation model, to an estimate of the travel and administrative effort required to adequately service the SCU. Model (M_{SZ}) is then used to develop territories which have a balanced workload. The model user is free to choose an appropriate objective function. Minimize disruption and minimize travel time are two good choices.

Finally, the sales territory alignment model solution must be checked for feasibility. This is accomplished by determining the total travel time, TRAVEL(OPT), associated with the (M_{SZ}) solution and comparing it with the travel time component, TRAVEL**, assumed in the initial disaggregation of the total sales force time. If TRAVEL** is significantly less than TRAVEL(OPT), then the current sales force size cannot accommodate the sales effort targets suggested by the sales resource allocation model. If TRAVEL** is significantly greater than TRAVEL(OPT), then more time can be allocated to calling and a more profitable territory alignment can be achieved.

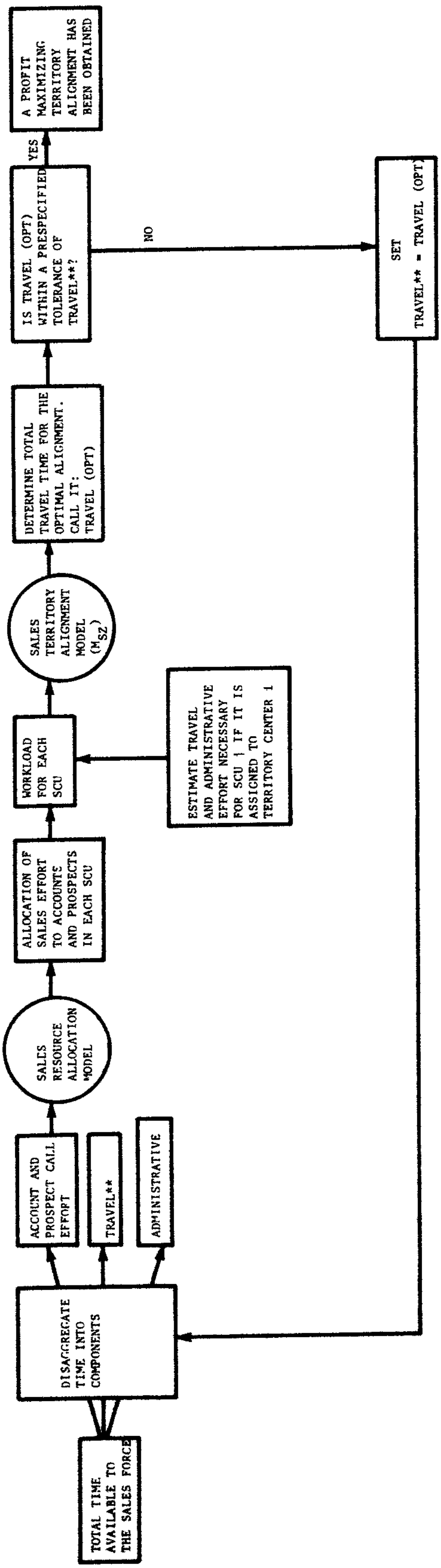

FIGURE 9. An Approach For Developing Profit Maximizing Sales Territory Alignments.

In either case, the models would be re-solved with a different breakout for call and travel time. This iterative procedure converges to an optimum in those cases where "minimize travel time" serves as the (M_{SZ}) objective function. Otherwise, a good solution is usually found after a few iterations.

The second implementation issue deals with sales territory centers. As stated, model (M_{SZ}) aligns sales territories relative to m fixed territory centers or home bases. Hess and Samuels [7] describe a center-seeking alignment approach which can be used to derive both territory centers and territory alignments. Their approach utilizes an iterative procedure. Model (M_{HS}) is solved at each iteration to align territories relative to a specified set of territory centers. Territory centers are updated at each iteration. At each step in the procedure, a new set of territory centers is defined as the set of centroids of the sales territories generated by (M_{HS}) at the prior iteration. The procedure terminates when the set of territory centroids does not change appreciably in two successive iterations. The Hess and Samuels center-seeking approach can employ (M_{SZ}) as well as (M_{HS}) at each iteration.

The center-seeking territory alignment approach has several practical shortcomings. Sales territories can be designed which do not contain cities in which sales people are willing to live or cities in which sales management would like to locate the field force. Most sales managers have strong preferences for home-base cities. Territory alignments that fail to recognize these preferences are usually not implemented. In those cases where a list of candidate sales territory centers is available, model (M_{SZ}) may be used to align territories relative to alternative territory center combinations. Management can then choose the most suitable sales territory alignment and home base configuration from among the combinations that were tested.

Recall that model (M_{SZ}) requires the creation of the hierarchical SCU-adjacency tree for each territory center. The computational effort needed to develop these trees increases linearly with the number of potential centers and could become prohibitive. However, notice that the hierarchical SCU-adjacency tree need only be created once for each potential territory center. The tree can be indexed by city name and stored in a hierarchical SCU-adjacency tree (HSAT) data base. The HSAT data base can be used across applications and over time within a single application. In fact, an up-to-date HSAT data base eliminates the need to maintain an up-to-date road and highway data base.

7. Conclusion

An optimization model for sales territory alignment has been presented in this paper. The model is designed to develop sales territory alignments which are contiguous, balanced relative to one or more alignment criteria and compatible with geographic considerations. The objective can be stated in terms of minimizing travel time, minimizing disruption or maximizing profitability.

In practice, an optimal sales territory alignment is rarely implemented without alteration. Sales and marketing managers typically change the optimal territory alignment in the face of nonquantifiable considerations which cannot be readily incorporated into (M_{SZ}). As an example, sales people may threaten to resign because a proposed territory alignment reduces their sales potential, eliminates favorite customers or requires more overnight trips. The development of decision support systems which are capable of incorporating both the quantitative and qualitative aspects of sales territory alignment is an important topic for further research in the area of sales territory alignment.[1]

[1]The authors thank Leonard M. Lodish and the referees for their insightful observations and suggestions.

References

1. Beale, E. M. L. and Tomlin, J. A., "Special Facilities in a General Mathematical Programming System for Non-Convex Problems Using Ordered Sets of Variables," in J. Lawrence, (Ed.), *Proceedings of the Fifth International Conference on Operations Research*, Venice, Tavistock Publications, London, 1968.
2. Beswick, C. A. and Cravens, D. W., "A Multistage Decision Model for Salesforce Management," *J. Marketing Res.*, Vol. 14 (May 1977), pp. 135–144.
3. Easingwood, C., "Heuristic Approach to Selecting Sales Regions and Territories," *Oper. Res. Quart.*, Vol. 24, No. 4 (December 1973), pp. 527–534.
4. Glaze, R. and Weinberg, C. B., "A Sales Territory Alignment Program and Account Planning System," in R. Bagozzi (Ed.), *Sales Management: New Developments from Behavioral and Decision Model Research*, Marketing Science Institute, Cambridge, Mass., 1979, pp. 325–343.
5. Held, M., Wolfe, P. and Crowder, H. P., "Validation of Subgradient Optimization," *Math. Programming*, Vol. 6 (1974), pp. 62–88.
6. Heschel, M. S., "Effective Sales Territory Development," *J. Marketing*, Vol. 41, No. 2 (April 1977), pp. 39–43.
7. Hess, S. W. and Samuels, S. A., "Experiences with a Sales Districting Model: Criteria and Implementation," *Management Sci.*, Vol. 18, No. 4, Part II (December 1971), pp. 41–54.
8. ———, Weaver, J. B., Siegfeldt, H. J., Whelan, J. N. and Zitlau, P. A., "Nonpartisan Political Redistricting by Computer," *Oper. Res.*, Vol. 13 (1965), pp. 998–1008.
9. Lodish, L. M., "Sales Territory Alignment to Maximize Profit," *J. Marketing Res.*, Vol. 12 (February 1975), pp. 30–36.
10. ———, "'Vaguely Right' Approach to Sales Force Allocations," *Harvard Bus. Rev.*, (January–February 1974), pp. 119–124.
11. Parasuraman, A. and Day, R. L., "A Management-Oriented Model for Allocating Sales Effort," *J. Marketing Res.*, Vol. 14 (February 1977), pp. 22–33.
12. Richardson, R. J., "A Territory Realignment Model—MAPS," Presented at the New Orleans ORSA/TIMS Meeting, May 1, 1979.
13. Segal, M. and Weinberger, D. B., "Turfing," *Oper. Res.*, Vol. 25, No. 3 (1977), pp. 367–386.
14. Shanker, R. J., Turner, R. E. and Zoltners, A. A., "Sales Territory Design: An Integrated Approach," *Management Sci.*, Vol. 22, No. 3 (November 1975), pp. 309–320.
15. Sinha, P. and Zoltners, A. A., "A Multiple Choice Integer Programming Algorithm," Working Paper No. 78-46, Northwestern University, Graduate School of Management, Evanston, Illinois, February 1978.
16. Zoltners, A. A., "A Unified Approach to Sales Territory Alignment," Working Paper #76-24, Northwestern University, August 1976, in R. Bagozzi, (Ed.), *Sales Management: New Developments from Behavioral and Decision Model Research*, Marketing Science Institute, Cambridge, Mass., 1979, pp. 360–376.
17. ——— and Sinha, P., "Integer Programming Models for Sales Resource Allocation," *Management Sci.*, Vol. 26, No. 3 (March 1980), pp. 242–260.