THREE-DIMENSIONAL COUPLED DYNAMICS OF A BUOY AND MULTIPLE MOORING LINES: FORMULATION AND ALGORITHM

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Summary

The equations of the coupled three-dimensional motion of a submerged buoy and multiple mooring lines are formulated using Kane's formalism. The lines are modelled using lumped masses and tension-only springs including structural damping. Surface waves are described by Stokes's second-order wave theory. The hydrodynamic loads due to viscous drag are applied via a Morison's equation approach using the instantaneous relative velocities between the fluid field and the bodies (buoy and lines). The detailed algorithm is presented and the equations are solved using a robust implementation of the Runge–Kutta method provided in MATLAB. The mathematical model and associated algorithm are validated by comparison with special cases of an elastic catenary mooring line and other published data.

1. Introduction

The current trend in the oil industry towards the development of deep-water fields has created the need for reliable methods of analysis to address the problems associated with such developments. It is likely that oil production systems in the near future will consist of the basic floater–mooring–riser configuration and the coupling of the dynamics of the floating structure and the mooring/riser system is particularly significant for deep-water installations. An uncoupled analysis of buoy-line dynamics was given by Leonard *et al.* (**1**). Coupled analyses were presented by Mavrakos *et al.* (**2**) for two dimensions, and for three dimensions by Sun (**3**) and Tjavaras *et al.* (**4**), assuming only translational buoy motions. In these works the differential equations of the line are formulated and solved numerically. As reported in Tjavaras *et al.* (**4**) it is necessary in these models to include the bending stiffness of the line, albeit small, in order to avoid an ill-posed problem when the tension becomes small. An alternative approach is to model the mooring line by lumped masses connected by springs, as in the works of Huang (**5**) and Buckham *et al.* (**6**). The lumped mass approach is attractive because of its intuitive simplicity and ability to tackle problems with complex geometry and varying material properties and constitutive behaviour. Problems such as line touchdown can be modelled in a straightforward manner and the large motion dynamics of deep-water systems is captured.

It is the purpose of this paper to present a method of analysis, based on Kane's formalism (**7**, **8**) for the three-dimensional coupled dynamics of a subsurface buoy and multiple mooring lines using a lumped mass–spring model for the lines, and to present the algorithm for writing the equations of motion in a form ready for efficient numerical solution. No difficulties are encountered when

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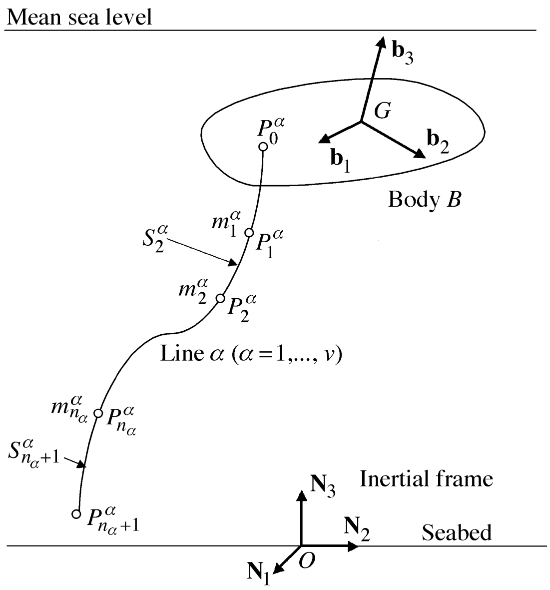

Fig. 1 Buoy and typical line

the mooring line becomes slack, in which case the tension is set at zero. Bending and torsion are not modelled in the present work but it is possible to include these effects by using appropriate rotational springs at the lumped masses. We assume that the hydrodynamic loads are due primarily to added-mass effects and viscous drag. In this regard, we allow for loading due to an arbitrary fluid velocity and acceleration field which is assumed to be undisturbed by the system. This allows for the inclusion of wave and current effects via the use of the Morison *et al.* approach (**9**). The possible load due to vortex shedding is not considered in the analysis but can be included by the use of appropriate lift coefficients.

A brief outline of the paper is as follows. Section 2 defines the geometry of the problem and associated generalized cooordinates and generalized speeds. The system kinematics is derived in section 3. The kinetics is addressed in sections 4 to 8. This involves derivations of inertia forces as well as internal and external loads due to both hydrodynamic and non-hydrodynamic effects. In section 9, the equations of motion of the system are assembled in a form amenable for numerical solution. Section 10 presents three test problems for validation purposes. Discussion and detailed results of further simulations are outside the scope of this paper and will be presented elsewhere.

2. System configuration

A diagram of the system to be analysed is given in Fig. 1. The origin of inertial coordinates is an arbitrary point O on the seabed and the inertial frame is denoted by N with unit vectors $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3$. Buoy B has a body-fixed frame at its centre of mass G with unit vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ parallel to its central principal axes. Line α ($\alpha = 1, \dots, \nu$) is attached to B at point P_0^α. The line is modelled by n_α lumped masses m_j^α ($j = 1, \dots, n_\alpha$) at points $P_1^\alpha, P_2^\alpha, \dots, P_{n_\alpha}^\alpha$. Without loss of generality, the system is kinematically constrained by specifying the motion of end-points $P^{\alpha}_{n_\alpha+1}$ as

$$\mathbf{OP}^{\alpha}_{n_\alpha+1} = \sum_{i=1}^{3} {}^{N}c_i^{\alpha}(t)\mathbf{N}_i \qquad (\alpha = 1, \dots, \nu), \tag{2.1}$$

where ${}^{N}c_i^{\alpha}(t)$ ($\alpha = 1, \dots, \nu$; $i = 1, 2, 3$) are prescribed functions of time t. Fixing these end-points would represent the multi-point mooring system. The system to be analysed consists of the following subsystems:

- rigid body B with six degrees of freedom;
- mooring lines L^{α} with $3n_\alpha$ degrees of freedom ($\alpha = 1, \dots, \nu$).

The total number of degrees of freedom is

$$m = 6 + 3\sum_{\alpha=1}^{\nu} n_\alpha. \tag{2.2}$$

2.1 *Orientation of body B*

An arbitrary orientation of body B can be specified by employing space-three 1-2-3 orientation angles $\theta_i (i = 1, 2, 3)$ defined as follows (Kane *et al.* (**8**)). Beginning with $\mathbf{b}_i$ aligned with $\mathbf{N}_i$ ($i = 1, 2, 3$) we rotate B successively about $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3$ by angles $\theta_1, \theta_2, \theta_3$ respectively. The body-fixed unit vectors $\mathbf{b}_i$ are then related to the inertial unit vectors $\mathbf{N}_i$ by

$$\begin{pmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \\ \mathbf{N}_3 \end{pmatrix} = \left[{}^{N}C^{B}\right] \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix}, \tag{2.3}$$

where the orthogonal transformation matrix $[{}^{N}C^{B}]$ is called the space-three 1-2-3 rotation matrix and is given by Kane *et al.* (**8**) as

$$\left[{}^{N}C^{B}\right] = \begin{pmatrix} c_2c_3 & s_1s_2c_3 - s_3c_1 & c_1s_2c_3 + s_3s_1 \\ c_2s_3 & s_1s_2s_3 + c_3c_1 & c_1s_2s_3 - c_3s_1 \\ -s_2 & s_1c_2 & c_1c_2 \end{pmatrix} \tag{2.4}$$

with $s_i = \sin\theta_i$, $c_i = \cos\theta_i$ ($i = 1, 2, 3$).

2.2 *Generalized coordinates*

2.2.1 *Orientation and position of body B*. Define

$$q_i^B = \theta_i \qquad (i = 1, 2, 3), \tag{2.5}$$
$$q_{3+i}^B = \mathbf{OG}\cdot\mathbf{b}_i \qquad (i = 1, 2, 3).$$

2.2.2 *Position of lumped masses on line α.* Let

$$\mathbf{GP}_0^\alpha = \sum_{i=1}^{3} p_i^\alpha \mathbf{b}_i \qquad (\alpha = 1, \ldots, \nu), \tag{2.6}$$

where p_i^α $(i = 1, 2, 3)$ are constants that specify the location of the attachment point P_0^α relative to the centre of mass G of the body. The position of the attachment point relative to the inertial origin O is given by

$$\mathbf{OP}_0^\alpha = \mathbf{OG} + \mathbf{GP}_0^\alpha = \sum_{i=1}^{3} (q_{3+i}^B + p_i^\alpha)\mathbf{b}_i. \tag{2.7}$$

We specify the positions of the lumped masses relative to the point P_0^α. Define

$$q_{3(j-1)+i}^\alpha = \mathbf{P}_0^\alpha \mathbf{P}_j^\alpha \boldsymbol{\cdot} \mathbf{b}_i \qquad (i = 1, 2, 3;\ j = 1, \ldots, n_\alpha). \tag{2.8}$$

The m generalized coordinates are q_i^B $(i = 1, \ldots, 6)$; q_r^α $(r = 1, \ldots, 3n_\alpha;\ \alpha = 1, \ldots, \nu)$. The position of the end-point $P_{n_\alpha+1}^\alpha$ is specified relative to O:

$$\mathbf{OP}_{n_\alpha+1}^\alpha = \sum_{i=1}^{3} {}^B c_i^\alpha(t)\mathbf{b}_i, \tag{2.9}$$

where ${}^B c_i^\alpha(t)$ are functions of time t determined from the prescribed position of $P_{n_\alpha+1}^\alpha$ in inertial coordinates by the relation

$$\begin{pmatrix} {}^N c_1^\alpha \\ {}^N c_2^\alpha \\ {}^N c_3^\alpha \end{pmatrix} = \left[{}^N C^B\right] \begin{pmatrix} {}^B c_1^\alpha \\ {}^B c_2^\alpha \\ {}^B c_3^\alpha \end{pmatrix}. \tag{2.10}$$

2.3 *Generalized speeds*

Define the generalized speeds (Kane and Levinson (**7**)) as

$$\begin{aligned} u_i^B &= \boldsymbol{\omega}^B \boldsymbol{\cdot} \mathbf{b}_i \qquad && (i = 1, 2, 3), \\ u_{3+i}^B &= \mathbf{v}^G \boldsymbol{\cdot} \mathbf{b}_i \qquad && (i = 1, 2, 3), \\ u_{3(j-1)+i}^\alpha &= \mathbf{v}^{P_j^\alpha} \boldsymbol{\cdot} \mathbf{b}_i \qquad && (\alpha = 1, \ldots, \nu;\ j = 1, \ldots, n_\alpha;\ i = 1, 2, 3), \end{aligned} \tag{2.11}$$

where $\boldsymbol{\omega}^B$ is the angular velocity of B, $\mathbf{v}^G$ is the velocity of the centre of mass G of body B and $\mathbf{v}^{P_j^\alpha}$ is the velocity of point P_j^α.

2.4 *Orthogonal triad of unit vectors associated with each line segment*

The kth segment on line α is defined by points P_{k-1}^α and P_k^α and is denoted by S_k^α $(\alpha = 1, \ldots, \nu;$ $k = 1, \ldots, n_\alpha + 1)$. An orthogonal triad of unit vectors, denoted by $\mathbf{t}_k^\alpha, \mathbf{s}_k^\alpha, \mathbf{h}_k^\alpha$, is illustrated

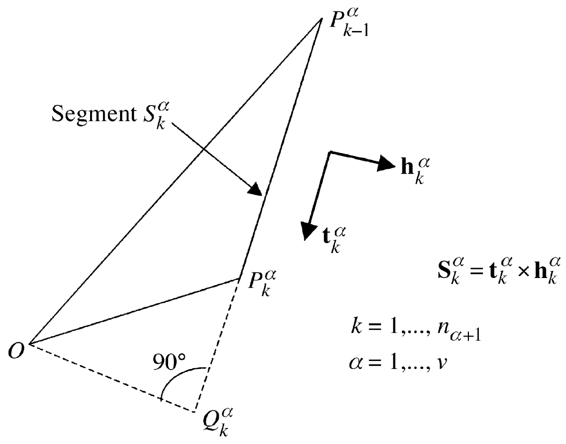

Fig. 2 Unit vectors for line segments

in Fig. 2. The position vectors of the lumped masses and the line end-points relative to the inertial origin O may be written as

$$\mathbf{OP}^{\alpha}_k = \mathbf{OP}^{\alpha}_0 + \mathbf{P}^{\alpha}_0\mathbf{P}^{\alpha}_k = \sum_{i=1}^{3} Y^{\alpha}_{i,k}\mathbf{b}_i \quad (\alpha = 1,\ldots,\nu;\, k = 0,\ldots,n_\alpha + 1), \tag{2.12}$$

where we define for $\alpha = 1,\ldots,\nu$ and for $i = 1, 2, 3$

$$Y^{\alpha}_{i,k} = \begin{cases} q^B_{3+i} + p^{\alpha}_i & \text{for } k = 0, \\ q^B_{3+i} + p^{\alpha}_i + q^{\alpha}_{3k-3+i} & \text{for } k = 1,\ldots,n_\alpha, \\ {}^B c^{\alpha}_i & \text{for } k = n_\alpha + 1. \end{cases} \tag{2.13}$$

For segment S^{α}_k, let $Z^{\alpha}_{i,k} = \mathbf{P}^{\alpha}_{k-1}\mathbf{P}^{\alpha}_k \cdot \mathbf{b}_i$ $(k = 1,\ldots,n_\alpha + 1;\ i = 1, 2, 3)$. Then we have for $\alpha = 1,\ldots,\nu$ and for $i = 1, 2, 3$

$$Z^{\alpha}_{i,k} = \begin{cases} q^{\alpha}_i & \text{for } k = 1, \\ q^{\alpha}_{3(k-1)+i} - q^{\alpha}_{3(k-2)+i} & \text{for } k = 2,\ldots,n_\alpha, \\ Y^{\alpha}_{i,n_\alpha+1} - Y^{\alpha}_{i,n_\alpha} & \text{for } k = n_\alpha + 1. \end{cases} \tag{2.14}$$

The length of segment S^{α}_k is denoted by $Z^{\alpha}_{4,k}$ and is found as

$$Z^{\alpha}_{4,k} = \left[\sum_{i=1}^{3} (Z^{\alpha}_{i,k})^2\right]^{1/2}. \tag{2.15}$$

The unit tangent vector directed from P^{α}_{k-1} to P^{α}_k is

$$\mathbf{t}^{\alpha}_k = \frac{\mathbf{P}^{\alpha}_{k-1}\mathbf{P}^{\alpha}_k}{|\mathbf{P}^{\alpha}_{k-1}\mathbf{P}^{\alpha}_k|} = \frac{\sum_{i=1}^{3} Z^{\alpha}_{i,k}\mathbf{b}_i}{Z^{\alpha}_{4,k}}. \tag{2.16}$$

From the diagram in Fig. 2, the unit normal in the same plane as $OP^{\alpha}_{k-1}\ P^{\alpha}_{k}$ is computed as

$$\mathbf{h}^{\alpha}_{k} = \frac{\mathbf{OQ}^{\alpha}_{k}}{|\mathbf{OQ}^{\alpha}_{k}|} = \frac{\sum_{i=1}^{3} Z^{\alpha}_{4+i,k}\mathbf{b}_i}{Z^{\alpha}_{8,k}}, \tag{2.17}$$

where for $k = 1, \ldots, n_{\alpha} + 1;\ i = 1, 2, 3,$

$$Z^{\alpha}_{4+i,k} = \mathbf{OQ}^{\alpha}_{k}\cdot\mathbf{b}_i = \mathbf{OP}^{\alpha}_{k} - (\mathbf{OP}^{\alpha}_{k}\cdot\mathbf{t}^{\alpha}_{k})\mathbf{t}^{\alpha}_{k} = Y^{\alpha}_{i,k} - \left(\sum_{s=1}^{3} Y^{\alpha}_{s,k} t^{\alpha}_{s,k}\right) t^{\alpha}_{i,k}, \tag{2.18}$$

$$t^{\alpha}_{i,k} = \mathbf{t}^{\alpha}_{k}\cdot\mathbf{b}_i = \frac{Z^{\alpha}_{i,k}}{Z^{\alpha}_{4,k}}, \tag{2.19}$$

$$Z^{\alpha}_{8,k} = |\mathbf{OQ}^{\alpha}_{k}| = \left[\sum_{i=1}^{3}(Z^{\alpha}_{4+i,k})^2\right]^{1/2}. \tag{2.20}$$

To complete the triad, the unit vector $\mathbf{s}^{\alpha}_{k}$ is given by $\mathbf{s}^{\alpha}_{k} = \mathbf{t}^{\alpha}_{k} \times \mathbf{h}^{\alpha}_{k}$.

3. Kinematics

3.1 *Velocity*

We need to express the quantities $\dot{q}^{B}_{r}, \dot{q}^{\alpha}_{r}$ in terms of the generalized speeds, where the dots indicate differentiation with respect to time t. First we have the standard relations for space-three 1-2-3 rotation angles given by Kane *et al.* (**8**) as

$$\begin{aligned} \dot{q}^{B}_{1} &= u^{B}_{1} + \frac{s_2}{c_2}(u^{B}_{2}s_1 + u^{B}_{3}c_1), \\ \dot{q}^{B}_{2} &= u^{B}_{2}c_1 - u^{B}_{3}s_1, \\ \dot{q}^{B}_{3} &= \frac{1}{c_2}(u^{B}_{2}s_1 + u^{B}_{3}c_1). \end{aligned} \tag{3.1}$$

Next, using the expression for $\mathbf{v}^{G}$ in terms of generalized speeds (see equation (2.11)) and the relation

$$\mathbf{v}^{G} = \frac{{}^{B}d}{dt}(\mathbf{OG}) + \boldsymbol{\omega}^{B} \times \mathbf{OG}$$

we deduce that

$$\begin{aligned} \dot{q}^{B}_{4} &= u^{B}_{4} - u^{B}_{2}q_6 + u^{B}_{3}q_5, \\ \dot{q}^{B}_{5} &= u^{B}_{5} + u^{B}_{1}q_6 - u^{B}_{3}q_4, \\ \dot{q}^{B}_{6} &= u^{B}_{6} - u^{B}_{1}q_5 + u^{B}_{2}q_4. \end{aligned} \tag{3.2}$$

The notation ${}^{B}d/dt$ indicates time differentiation with respect to the B reference frame and a similar notation will be used for differentiation with respect to the inertial frame N.

Using the relations

$$\mathbf{v}^{P_0^\alpha} = \mathbf{v}^G + \boldsymbol{\omega}^B \times \mathbf{GP}_0^\alpha$$

and

$$\mathbf{v}^{P_j^\alpha} = \mathbf{v}^{P_0^\alpha} + \frac{{}^B d}{dt}(\mathbf{P}_0^\alpha \mathbf{P}_j^\alpha) + \boldsymbol{\omega}^B \times \mathbf{P}_0^\alpha \mathbf{P}_j^\alpha$$

we can similarly obtain the following expressions for the $\dot{q}_r^\alpha$:

$$\begin{aligned}
\dot{q}_{3j-2}^\alpha &= u_{3j-2}^\alpha - u_4^B - u_2^B(p_3^\alpha + q_{3j}^\alpha) + u_3^B(p_2^\alpha + q_{3j-1}^\alpha),\\
\dot{q}_{3j-1}^\alpha &= u_{3j-1}^\alpha - u_5^B + u_1^B(p_3^\alpha + q_{3j}^\alpha) - u_3^B(p_1^\alpha + q_{3j-2}^\alpha),\\
\dot{q}_{3j}^\alpha &= u_{3j}^\alpha - u_6^B - u_1^B(p_2^\alpha + q_{3j-1}^\alpha) + u_2^B(p_1^\alpha + q_{3j-2}^\alpha),\\
&\qquad \alpha = 1, \ldots, \nu;\ j = 1, \ldots, n_\alpha.
\end{aligned} \tag{3.3}$$

3.2 *Acceleration*

The angular acceleration of B is

$$\boldsymbol{\Delta}^B = \frac{{}^N d}{dt}(\boldsymbol{\omega}^B) = \frac{{}^B d}{dt}(\boldsymbol{\omega}^B) + \boldsymbol{\omega}^B \times \boldsymbol{\omega}^B = \sum_{i=1}^{3} \dot{u}_i^B \mathbf{b}_i. \tag{3.4}$$

The acceleration of G is

$$\begin{aligned}
\mathbf{a}^G &= \frac{{}^N d}{dt}(\mathbf{v}^G) = \frac{{}^B d}{dt}(\mathbf{v}^G) + \boldsymbol{\omega}^B \times \mathbf{v}^G\\
&= \mathbf{b}_1\big(\dot{u}_4^B + u_2^B u_6^B - u_3^B u_5^B\big) + \mathbf{b}_2\big(\dot{u}_5^B - u_1^B u_6^B + u_3^B u_4^B\big)\\
&\quad + \mathbf{b}_3\big(\dot{u}_6^B + u_1^B u_5^B - u_2^B u_4^B\big)
\end{aligned} \tag{3.5}$$

and the accelerations of the lumped masses are found in a similar fashion as

$$\begin{aligned}
\mathbf{a}^{P_j^\alpha} &= \mathbf{b}_1\big(\dot{u}_{3j-2}^\alpha + u_2^B u_{3j}^\alpha - u_3^B u_{3j-1}^\alpha\big) + \mathbf{b}_2\big(\dot{u}_{3j-1}^\alpha - u_1^B u_{3j}^\alpha + u_3^B u_{3j-2}^\alpha\big)\\
&\quad + \mathbf{b}_3\big(\dot{u}_{3j}^\alpha + u_1^B u_{3j-1}^\alpha - u_2^B u_{3j-2}^\alpha\big), \quad \alpha = 1, \ldots, \nu;\ j = 1, \ldots, n_\alpha.
\end{aligned} \tag{3.6}$$

3.3 *Partial velocities*

Following Kane and Levinson (**7**), the partial velocities are the coefficients of the generalized speeds in the expressions for the velocities of the system components and are written by inspection.

3.3.1 *Rigid body B.* Partial velocities $\boldsymbol{\omega}_r^B$ and $\mathbf{v}_r^G$:

$$\boldsymbol{\omega}_r^B = \begin{cases} \mathbf{b}_r & (r = 1, 2, 3),\\ \mathbf{0} & (r = 4, 5, 6), \end{cases} \tag{3.7}$$

$$\mathbf{v}_r^G = \begin{cases} \mathbf{0} & (r = 1, 2, 3),\\ \mathbf{b}_{r-3} & (r = 4, 5, 6). \end{cases} \tag{3.8}$$

3.3.2 *Connection points* P_0^α ($\alpha = 1, \dots, \nu$). Partial velocities $\mathbf{v}_r^{P_0^\alpha}$:

$$\mathbf{v}_r^{P_0^\alpha} = \begin{cases} p_2^\alpha \mathbf{b}_3 - p_3^\alpha \mathbf{b}_2 & (r = 1), \\ p_3^\alpha \mathbf{b}_1 - p_1^\alpha \mathbf{b}_3 & (r = 2), \\ p_1^\alpha \mathbf{b}_2 - p_2^\alpha \mathbf{b}_1 & (r = 3), \\ \mathbf{b}_1 & (r = 4), \\ \mathbf{b}_2 & (r = 5), \\ \mathbf{b}_3 & (r = 6). \end{cases} \tag{3.9}$$

3.3.3 *Lumped masses at* $P_j^\alpha (\alpha = 1, \dots, \nu;\ j = 1, \dots, n_\alpha)$. Partial velocities $\mathbf{v}_r^{P_j^\alpha}$:

$$\mathbf{v}_r^{P_j^\alpha} = \begin{cases} \mathbf{b}_1 & (r = 3j - 2), \\ \mathbf{b}_2 & (r = 3j - 1), \\ \mathbf{b}_3 & (r = 3j), \\ \mathbf{0}, & \text{otherwise}. \end{cases} \qquad r \in \{1, \dots, 3n_\alpha\}. \tag{3.10}$$

4. Generalized inertia forces

The non-hydrodynamic generalized inertia force F_r^{*B} on body B is

$$F_r^{*B} = \boldsymbol{\omega}_r^B \cdot \mathbf{T}^* + \mathbf{v}_r^G \cdot (-M_0 \mathbf{a}^G) \qquad (r = 1, \dots, 6), \tag{4.1}$$

where M_0 is the mass of the body and $\mathbf{T}^*$ is the inertia torque which is defined as (**7**)

$$\begin{aligned} \mathbf{T}^* = &-[\dot{u}_1{}^B I_1 - u_2^B u_3^B (I_2 - I_3)]\mathbf{b}_1 \\ &- [\dot{u}_2{}^B I_2 - u_3^B u_1^B (I_3 - I_1)]\mathbf{b}_2 \\ &- [\dot{u}_3{}^B I_3 - u_1^B u_2^B (I_1 - I_2)]\mathbf{b}_3. \end{aligned} \tag{4.2}$$

Here, the unit vectors $\mathbf{b}_i$ are chosen parallel to the central principal axes of B and I_1, I_2, I_3 are the moments of inertia of B about $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ respectively. The hydrodynamic inertia forces contribute to what is known as the added-mass effects of the buoy motion in water and will be discussed later. Equation (4.1) may be written in the form

$$\{F^{*B}\} = -[V^B]\{\dot{u}^B\} - [W^B]\{\phi^B\}, \tag{4.3}$$

where $[V^B]$ and $[W^B]$ are 6×6 diagonal matrices with diagonal entries defined as

$$V_{11}^B = I_1, \quad V_{22}^B = I_2, \quad V_{33}^B = I_3, \quad V_{44}^B = V_{55}^B = V_{66}^B = M_0,$$
$$W_{11}^B = I_3 - I_2, \quad W_{22}^B = I_1 - I_3, \quad W_{33}^B = I_2 - I_1, \quad W_{44}^B = W_{55}^B = W_{66}^B = M_0.$$

For a spherical buoy, the off-diagonal entries are zero. The vector $\{\dot{u}^B\}$ is a 6×1 column vector with entries $\dot{u}_r^B (r = 1, \ldots, 6)$ and $\{\phi^B\}$ is a 6×1 column vector with entries

$$\begin{aligned}
&\phi_1^B = u_2^B u_3^B, \quad \phi_2^B = u_3^B u_1^B, \quad \phi_3^B = u_1^B u_2^B, \\
&\phi_4^B = u_2^B u_6^B - u_3^B u_5^B, \\
&\phi_5^B = u_3^B u_4^B - u_1^B u_6^B, \\
&\phi_6^B = u_1^B u_5^B - u_2^B u_4^B.
\end{aligned} \tag{4.4}$$

For line L^α with lumped masses m_j^α the non-hydrodynamic generalized inertia force is

$$F_r^{*\alpha} = \sum_{j=1}^{n_\alpha} \mathbf{v}_r^{P_j^\alpha} \cdot (-m_j^\alpha \mathbf{a}^{P_j^\alpha}) \quad (r = 1, \ldots, 3n_\alpha; \alpha = 1, \ldots, \nu). \tag{4.5}$$

This may be written in matrix form as

$$\{F^{*\alpha}\} = -[V^\alpha]\{\dot{u}^\alpha\} - [V^\alpha]\{\phi^\alpha\}, \tag{4.6}$$

where $\{\dot{u}^\alpha\}$ is the $3n_\alpha \times 1$ column vector of $\dot{u}_r^\alpha$ values, $[V^\alpha]$ is a $3n_\alpha \times 3n_\alpha$ diagonal matrix and $\{\phi^\alpha\}$ is a $3n_\alpha \times 1$ column vector defined as

$$\begin{aligned}
V_{3j-2,\,3j-2}^\alpha = V_{3j-1,\,3j-1}^\alpha = V_{3j,\,3j}^\alpha &= m_j^\alpha, \\
\phi_{3j-2}^\alpha &= u_2^B u_{3j}^\alpha - u_3^B u_{3j-1}^\alpha, \\
\phi_{3j-1}^\alpha &= -u_1^B u_{3j}^\alpha + u_3^B u_{3j-2}^\alpha, \\
\phi_{3j}^\alpha &= u_1^B u_{3j-1}^\alpha - u_2^B u_{3j-2}^\alpha,
\end{aligned}$$

$$j = 1, \ldots, n_\alpha; \alpha = 1, \ldots, \nu.$$

5. Gravity, buoyancy and touchdown

The generalized active forces due to gravity, buoyancy and touchdown act in the same direction and are treated together. We will model the seabed normal reaction forces on the lines at touchdown but we assume that the buoy B does not experience touchdown. Seabed friction is not modelled. The generalized active force due to gravity and buoyancy on buoy B (mass M_0 and volume V_0) is

$$F_r^{GB/B} = -M_0^b g \mathbf{N}_3 \cdot \mathbf{v}_r^G = -M_0^b g \begin{cases} 0 & (r = 1, 2, 3), \\ C_{31} & (r = 4), \\ C_{32} & (r = 5), \\ C_{33} & (r = 6), \end{cases} \tag{5.1}$$

where $M_0^b = M_0 - \rho_f V_0$ and C_{ij} refers to the elements of matrix $[{}^N C^B]$, equation (2.4). If we denote the volume of the portion of line L^α associated with lumped mass m_j^α by V_j^α, the net force on lumped mass m_j^α due to gravity and buoyancy is $-m_j^{\alpha b} g \mathbf{N}_3$, where $m_j^{\alpha b} = m_j^\alpha - \rho_f V_j^\alpha$. To allow for the possibility of contact between any portion of the mooring lines and the seabed (‘touchdown’) we assume that the seabed normal reaction force is directly proportional to the depth of lumped-mass penetration into the bed surface. Hence the vertical touchdown reaction force on m_j^α is given by

$$\tfrac{1}{2}k_E(|h_j^\alpha| - h_j^\alpha)\mathbf{N}_3 = \begin{cases} \mathbf{0} & \text{if } h_j^\alpha \geqslant 0, \\ k_E|h_j^\alpha|\mathbf{N}_3 & \text{if } h_j^\alpha < 0, \end{cases}$$

where $h_j^\alpha = \mathbf{OP}_j^\alpha \cdot \mathbf{N}_3$ is the height of point P_j^α above the seabed and k_E is a seabed stiffness coefficient. Using equations (2.3) and (2.12) we can write h_j^α as

$$h_j^\alpha = \sum_{i=1}^{3} C_{3i} Y_{i,j}^\alpha \qquad (j = 1, \ldots, n_\alpha). \tag{5.2}$$

The net force on m_j^α due to gravity, buoyancy and seabed normal reaction at touchdown is written as $\boldsymbol{\chi}_j^\alpha = \chi_j^\alpha \mathbf{N}_3$, where

$$\chi_j^\alpha = -m_j^{\alpha b} g + \tfrac{1}{2}k_E(|h_j^\alpha| - h_j^\alpha). \tag{5.3}$$

The generalized active force due to $\boldsymbol{\chi}_j^\alpha$ is

$$F_r^{GBT/L^\alpha} = \sum_{j=1}^{n_\alpha} \chi_j^\alpha \mathbf{N}_3 \cdot \mathbf{v}_r^{P_j^\alpha}. \tag{5.4}$$

Using (2.3) and (3.10) we write this as

$$\begin{aligned} F_{3j-2}^{GBT/L^\alpha} &= \chi_j^\alpha C_{31}, \\ F_{3j-1}^{GBT/L^\alpha} &= \chi_j^\alpha C_{32}, \\ F_{3j}^{GBT/L^\alpha} &= \chi_j^\alpha C_{33} \end{aligned} \tag{5.5}$$

$(j = 1, \ldots, n_\alpha)$.

6. Line tension

The stiffness $k^{S_j^\alpha}$ of segment S_j^α of line L^α is defined in the usual way as

$$k^{S_j^\alpha} = \frac{A_{0,j}^\alpha E_j^\alpha}{l_j^\alpha},$$

where $A_{0,j}^\alpha$, E_j^α, l_j^α are, respectively, the area of cross-section, modulus of elasticity and unstretched length of the segment. The instantaneous length of the segment is denoted by $Z_{4,j}^\alpha$ (equation (2.15)). We allow for line tension but not for compression. To this end, we define the elongation of segment S_j^α as

$$Z_{9,j}^\alpha = \tfrac{1}{2}[(Z_{4,j}^\alpha - l_j^\alpha) + |Z_{4,j}^\alpha - l_j^\alpha|] \quad (j = 1, \ldots, n_\alpha + 1), \tag{6.1}$$

which is identically zero if the instantaneous segment length becomes less than the unstretched length. The magnitude of the tension in segment S_j^α is thus

$$B_j^\alpha = k^{S_j^\alpha} Z_{9,j}^\alpha.$$

The line tensions act on B at points P_0^α ($\alpha = 1, \ldots, \nu$) in the directions of the unit vectors $\mathbf{t}_1^\alpha$. The generalized active force due to line tension on body B is therefore

$$F_r^{T/B} = \sum_{\alpha=1}^{\nu} a_{r\alpha} B_1^\alpha \quad (r = 1, \ldots, 6), \tag{6.2}$$

where $a_{r\alpha} = \mathbf{t}_1^\alpha \cdot \mathbf{v}_r^{P_0^\alpha}$ and is evaluated as

$$a_{r\alpha} = \begin{cases} p_2^\alpha t_{3,1}^\alpha - p_3^\alpha t_{2,1}^\alpha & (r = 1), \\ p_3^\alpha t_{1,1}^\alpha - p_1^\alpha t_{3,1}^\alpha & (r = 2), \\ p_1^\alpha t_{2,1}^\alpha - p_2^\alpha t_{1,1}^\alpha & (r = 3), \\ t_{1,1}^\alpha & (r = 4), \\ t_{2,1}^\alpha & (r = 5), \\ t_{3,1}^\alpha & (r = 6), \end{cases} \tag{6.3}$$

and $t_{i,k}^\alpha = \mathbf{b}_i \cdot \mathbf{t}_\alpha^k$ ($i = 1, 2, 3; k = 1, \ldots, n_{\alpha+1}; \alpha = 1, \ldots, \nu$), given by equation (2.19).

Using similar arguments we write the generalized active force due to tension on the lumped masses in line L^α as

$$F_r^{T/L^\alpha} = -\sum_{j=1}^{n_\alpha+1} A_{jr}^\alpha B_j^\alpha, \tag{6.4}$$

where

$$A_{jr}^\alpha = \begin{cases} \mathbf{t}_1^\alpha \cdot \mathbf{v}_r^{P_1^\alpha} & (j = 1), \\ \mathbf{t}_j^\alpha \cdot (\mathbf{v}_r^{P_j^\alpha} - \mathbf{v}_r^{P_{j-1}^\alpha}) & (j = 2, \ldots, n_\alpha), \\ -\mathbf{t}_{n_\alpha+1}^\alpha \cdot \mathbf{v}_r^{P_{n_\alpha}^\alpha} & (j = n_\alpha + 1) \end{cases} \tag{6.5}$$

($r = 1, \ldots, 3n_\alpha$). Using the definitions of the partial velocities, equation (3.10), this is evaluated for $r = 1, \ldots, 3n_\alpha$ as

$$A^\alpha_{1r} = \begin{cases} t^\alpha_{1,1} & (r = 1), \\ t^\alpha_{2,1} & (r = 2), \\ t^\alpha_{3,1} & (r = 3), \\ 0, & \text{otherwise}, \end{cases}$$

$$A^\alpha_{jr} = \begin{cases} -t^\alpha_{1,j} & (r = 3j-5), \\ -t^\alpha_{2,j} & (r = 3j-4), \\ -t^\alpha_{3,j} & (r = 3j-3), \\ t^\alpha_{1,j} & (r = 3j-2), \\ t^\alpha_{2,j} & (r = 3j-1), \\ t^\alpha_{3,j} & (r = 3j), \\ 0, & \text{otherwise}, \end{cases} \qquad (j = 2, \ldots, n_\alpha), \tag{6.6}$$

$$A^\alpha_{n_\alpha+1,r} = \begin{cases} -t^\alpha_{1,n_\alpha+1} & (r = 3n_\alpha - 2), \\ -t^\alpha_{2,n_\alpha+1} & (r = 3n_\alpha - 1), \\ -t^\alpha_{3,n_\alpha+1} & (r = 3n_\alpha), \\ 0, & \text{otherwise}. \end{cases}$$

7. Mooring line structural damping

In any line segment, the damping force on the end masses is of the form $\pm C_s(\mathbf{v}_R\cdot\mathbf{t})\mathbf{t}$, where $\mathbf{v}_R$ is the velocity of one mass relative to the other, $\mathbf{t}$ is the unit tangent vector along the segment and C_s is a structural damping coefficient. We write the velocity of P^α_k in the form

$$\mathbf{v}^{P^\alpha_k} = \sum_{i=1}^{3} \xi^\alpha_{i,k}\mathbf{b}_i \qquad (k = 0, \ldots, n_{\alpha+1};\ \alpha = 1, \ldots, \nu). \tag{7.1}$$

Here

$$\xi^\alpha_{i,0} = \begin{cases} u^B_4 + u^B_2 p^\alpha_3 - u^B_3 p^\alpha_2 & (i = 1), \\ u^B_5 - u^B_1 p^\alpha_3 + u^B_3 p^\alpha_1 & (i = 2), \\ u^B_6 + u^B_1 p^\alpha_2 - u^B_2 p^\alpha_1 & (i = 3), \end{cases}$$

$$\xi^\alpha_{i,k} = u^\alpha_{3(k-1)+i} \quad (i = 1, 2, 3;\ k = 1, \ldots, n_\alpha),$$

$$\xi^\alpha_{i,n_\alpha+1} = {}^B v^\alpha_i \quad (i = 1, 2, 3), \tag{7.2}$$

where ${}^B v^\alpha_i$ $(i = 1, 2, 3)$ are the components of the specified velocity of point $P^\alpha_{n_\alpha+1}$ in the B frame and are computed from the known inertial velocity components ${}^N\dot{c}^\alpha_i$ using the transformation matrix $[{}^N C^B]$, that is,

$$\begin{pmatrix} {}^N\dot{c}^\alpha_1 \\ {}^N\dot{c}^\alpha_2 \\ {}^N\dot{c}^\alpha_3 \end{pmatrix} = \left[{}^N C^B\right] \begin{pmatrix} {}^B v^\alpha_1 \\ {}^B v^\alpha_2 \\ {}^B v^\alpha_3 \end{pmatrix}. \tag{7.3}$$

It is then possible to write the force on points P_k^α and P_{k-1}^α due to structural damping in segment S_k^α in the form

$$\mathbf{D}^{P_k^\alpha/S_k^\alpha} = g_k^\alpha \mathbf{t}_k^\alpha = -\mathbf{D}^{P_{k-1}^\alpha/S_k^\alpha} \quad (k = 1, \ldots, n_\alpha), \tag{7.4}$$

where

$$\begin{aligned} g_k^\alpha &= -C^{S_k^\alpha}(\mathbf{v}^{P_k^\alpha} - \mathbf{v}^{P_{k-1}^\alpha})\cdot\mathbf{t}_k^\alpha \ \operatorname{sign}(Z_{9,k}^\alpha) \\ &\quad - C^{S_k^\alpha} \ \operatorname{sign}(Z_{9,k}^\alpha)\sum_{i=1}^{3}(\xi_{i,k}^\alpha - \xi_{i,k-1}^\alpha)t_{i,k}^\alpha \quad (k = 1, \ldots, n_{\alpha+1}) \end{aligned} \tag{7.5}$$

and $C^{S_k^\alpha}$ is the damping coefficient for segment S_k^α. The generalized active force due to line structural damping on body B is

$$F_r^{SD/B} = -\sum_{\alpha=1}^{\nu} a_{r\alpha} g_1^a \qquad (r = 1, \ldots, 6;\ \alpha = 1, \ldots, \nu) \tag{7.6}$$

and on line L^α

$$F_r^{SD/L^\alpha} = \sum_{k=1}^{n_\alpha+1} A_{kr}^\alpha g_k^\alpha. \tag{7.7}$$

The quantities $a_{r\alpha}$ and A_{kr}^α are given by (6.3) and (6.6) respectively.

7.1 *Viscous drag*

If the dimensions of body B are small compared to the length of the surface waves we can assume that the fluid velocity field is not affected by the presence of the body. We assume that the rotational damping torque $\mathbf{T}_D$ due to fluid drag can be written in the form

$$\mathbf{T}_D = -\tfrac{1}{2}\rho_f A_B C_\omega R_B^3 \mid \boldsymbol{\omega}^B \mid \boldsymbol{\omega}^B, \tag{7.8}$$

where A_B is the surface area of the body, R_B is the typical radial body dimension and C_ω is a rotational damping coefficient.

For the drag resisting translational motion we need the velocity of the body relative to the fluid:

$$\mathbf{V}_R^G = \mathbf{v}^G - \mathbf{U}_F^G, \tag{7.9}$$

where $\mathbf{U}_F^G$ is the fluid velocity at the location of the body's centre of mass G. The drag on the body is

$$\mathbf{F}_D^B = \sum_{i=1}^{3} \mathbf{F}_D^{(i)}, \tag{7.10}$$

where the drag in direction $\mathbf{b}_i$ is given by Morison's formula (Chakrabarti (**9**)):

$$\mathbf{F}_D^{(i)} = -\tfrac{1}{2}\rho_f A_B^{(i)} C_D^{(i)} |\mathbf{V}_R^G\cdot\mathbf{b}_i|(\mathbf{V}_R^G\cdot\mathbf{b}_i)\mathbf{b}_i. \tag{7.11}$$

Here $A_B^{(i)}$ is the projected surface area of the body normal to $\mathbf{b}_i$ and $C_D^{(i)}$ is the associated drag coefficient. The generalized active force due to viscous drag on body B is

$$\begin{aligned}F_r^{D/B} &= -\tfrac{1}{2}\rho_f A_B C_\omega R_B^3[(u_1^B)^2 + (u_2^B)^2 + (u_3^B)^2]^{\frac{1}{2}} u_r^B,\\ F_{3+r}^{D/B} &= -\tfrac{1}{2}\rho_f A_B^{(r)} C_D^{(r)} |u_{3+r}^B - {}^B U_r^G|(u_{3+r}^B - {}^B U_r^G) \quad (r = 1, 2, 3).\end{aligned} \tag{7.12}$$

The quantities ${}^B U_r^G$ are the components of $\mathbf{U}_F^G$ in the body-fixed frame.

Consider segment S_k^α, diameter d_k^α, unstretched length l_k^α $(k = 1, \ldots, n_\alpha + 1)$. Assume that the segment S_k^α has a velocity equal to the velocity of its mid-point and is given by

$$\mathbf{V}^{S_k^\alpha} = \tfrac{1}{2}\left(\mathbf{v}^{P_k^\alpha} + \mathbf{v}^{P_{k-1}^\alpha}\right) \quad (k = 1, \ldots, n_\alpha + 1). \tag{7.13}$$

Let the fluid velocity at the segment mid-point be $\mathbf{U}_F^{S_k^\alpha}$. Let $C_{DT}^{S_k^\alpha}$, $C_{DN}^{S_k^\alpha}$ be the tangential and normal drag coefficients for segment S_k^α. The associated areas are

$$A_T^{S_k^\alpha} = \pi d_k^\alpha\, l_k^\alpha \qquad \text{and} \qquad A_N^{S_k^\alpha} = l_k^\alpha\, d_k^\alpha. \tag{7.14}$$

The velocity of S_k^α relative to the fluid is

$$\mathbf{V}_R^{S_k^\alpha} = \mathbf{V}^{S_k^\alpha} - \mathbf{U}_F^{S_k^\alpha} \tag{7.15}$$

and its evaluation will be discussed below. The viscous drag on segment S_k^α is, by Morison's equation (**9**),

$$\begin{aligned}\mathbf{F}_D^{S_k^\alpha} &= -\tfrac{1}{2}\rho_f A_T^{S_k^\alpha}\, C_{DT}^{S_k^\alpha} |\mathbf{V}_R^{S_k^\alpha}\cdot\mathbf{t}_k^\alpha|(\mathbf{V}_R^{S_k^\alpha}\cdot\mathbf{t}_k^\alpha)\mathbf{t}_k^\alpha\\ &\quad - \tfrac{1}{2}\rho_f A_N^{S_k^\alpha}\, C_{DN}^{S_k^\alpha} |\mathbf{V}_R^{S_k^\alpha}\cdot\mathbf{h}_k^\alpha|(\mathbf{V}_R^{S_k^\alpha}\cdot\mathbf{h}_k^\alpha)\mathbf{h}_k^\alpha\\ &\quad - \tfrac{1}{2}\rho_f A_N^{S_k^\alpha}\, C_{DN}^{S_k^\alpha} |\mathbf{V}_R^{S_k^\alpha}\cdot\mathbf{s}_k^\alpha|(\mathbf{V}_R^{S_k^\alpha}\cdot\mathbf{s}_k^\alpha)\mathbf{s}_k^\alpha.\end{aligned} \tag{7.16}$$

For segments S_1^α, $S_{n_\alpha+1}^\alpha$, we apply drag forces $\mathbf{F}_D^{S_1^\alpha}$, $\mathbf{F}_D^{S_{n_\alpha+1}^\alpha}$ to masses m_1^α, $m_{n_\alpha}^\alpha$ at P_1^α, $P_{n_\alpha}^\alpha$ respectively. For segments S_k^α $(k = 2, \ldots, n_\alpha)$ we apply $\frac{1}{2}\mathbf{F}_D^{S_k^\alpha}$ to masses m_k^α, m_{k-1}^α at points P_k^α, P_{k-1}^α. Let F_r^{D/S_k^α} $(r = 1, \ldots, 3n_\alpha)$ be the generalized active force due to viscous drag on segment S_k^α, defined by

$$\begin{aligned}F_r^{D/S_1^\alpha} &= \mathbf{F}_D^{S_1^\alpha}\cdot\mathbf{v}_r^{P_1^\alpha},\\ F_r^{D/S_k^\alpha} &= \tfrac{1}{2}\mathbf{F}_D^{S_k^\alpha}\cdot\left(\mathbf{v}_r^{P_k^\alpha} + \mathbf{v}_r^{P_{k-1}^\alpha}\right) \qquad (k = 2, \ldots, n_\alpha),\\ F_r^{D/S_{n_\alpha+1}^\alpha} &= \mathbf{F}_D^{S_{n_\alpha+1}^\alpha}\cdot\mathbf{v}_r^{P_{n_\alpha}^\alpha}.\end{aligned} \tag{7.17}$$

To facilitate the evaluation of F_r^{D/S_k^α} we note that $\mathbf{F}_D^{S_k^\alpha}\cdot\mathbf{b}_i$ may be written in the form

$$\mathbf{F}_D^{S_k^\alpha}\cdot\mathbf{b}_i = \eta_{i,k}^\alpha + \beta_{i,k}^\alpha + \gamma_{i,k}^\alpha \qquad (i = 1, 2, 3;\, k = 1, \ldots, n_\alpha + 1), \tag{7.18}$$

where

$$\eta_{i,k}^{\alpha} = -\tfrac{1}{2}\rho_f A_T^{S_k^\alpha}\ C_{DT}^{S_k^\alpha}|\mathbf{V}_R^{S_k^\alpha}\cdot\mathbf{t}_k^\alpha|(\mathbf{V}_R^{S_k^\alpha}\cdot\mathbf{t}_k^\alpha)\, t_{i,k}^\alpha, \tag{7.19}$$

$$\beta_{i,k}^{\alpha} = -\tfrac{1}{2}\rho_f A_N^{S_k^\alpha}\ C_{DN}^{S_k^\alpha}|\mathbf{V}_R^{S_k^\alpha}\cdot\mathbf{h}_k^\alpha|(\mathbf{V}_R^{S_k^\alpha}\cdot\mathbf{h}_k^\alpha)\, h_{i,k}^\alpha, \tag{7.20}$$

$$\gamma_{i,k}^{\alpha} = -\tfrac{1}{2}\rho_f A_N^{S_k^\alpha}\ C_{DN}^{S_k^\alpha}|\mathbf{V}_R^{S_k^\alpha}\cdot\mathbf{s}_k^\alpha|(\mathbf{V}_R^{S_k^\alpha}\cdot\mathbf{s}_k^\alpha)\, s_{i,k}^\alpha \tag{7.21}$$

for $i = 1, 2, 3$; $k = 1, \ldots, n_\alpha + 1$ and $t_{i,k}^\alpha = \mathbf{b}_i\cdot\mathbf{t}_k^\alpha$, $h_{i,k}^\alpha = \mathbf{b}_i\cdot\mathbf{h}_k^\alpha$, $s_{i,k}^\alpha = \mathbf{b}_i\cdot\mathbf{s}_k^\alpha$.

The generalized active force due to viscous drag on line L^α is

$$F_r^{D/L^\alpha} = \sum_{k=1}^{n_\alpha+1} F_r^{D/S_k^\alpha} \qquad (r = 1, \ldots, 3n_\alpha). \tag{7.22}$$

We now determine the fluid velocity and acceleration fields at the segment mid-points. Since $\mathbf{OP}_k^\alpha = \sum_{i=1}^3 Y_{i,k}^\alpha \mathbf{b}_i$ $(k = 0, \ldots, n_\alpha + 1)$, the position vector of the mid-point of segment S_k^α is

$$\begin{aligned}\mathbf{OS}_k^\alpha &= \tfrac{1}{2}\big(\mathbf{OP}_{k-1}^\alpha + \mathbf{OP}_k^\alpha\big) \qquad (k = 0, \ldots, n_\alpha + 1),\\ &= \tfrac{1}{2}\sum_{i=1}^{3}(Y_{i,k-1}^\alpha + Y_{i,k}^\alpha)\mathbf{b}_i. \end{aligned} \tag{7.23}$$

Define

$$[^N OS]^\alpha = [^N C^B]\ [^B OS]^\alpha, \tag{7.24}$$

where the kth column of matrix $[^B OS]^\alpha$ is the vector $\mathbf{OS}_k^\alpha$ given in equation (7.23). Then the columns of matrix $[^N OS]^\alpha$ are the position vectors in inertial coordinates of the mid-points of segments S_k^α. The fluid velocity and acceleration fields are calculated by function subroutines based on Stokes's second-order wave theory (**9**). These subroutines calculate the fluid velocity and acceleration vectors $\mathbf{U}_F$, $\mathbf{a}_F$ in inertial coordinates at an arbitrary position $\mathbf{x}$ and time t. To evaluate the segment–fluid relative velocity $\mathbf{V}_R^{S_k^\alpha}$ we use equation (7.1) to write, for $k = 1, \ldots, n_\alpha + 1$,

$$\mathbf{V}^{S_k^\alpha} = \big(\tfrac{1}{2}\mathbf{v}^{P_k^\alpha} + \mathbf{v}^{P_{k-1}^\alpha}\big) = \tfrac{1}{2}\sum_{i=1}^{3}\big(\xi_{i,k}^\alpha + \xi_{i,k-1}^\alpha\big)\mathbf{b}_i, \tag{7.25}$$

$$\mathbf{V}_R^{S_k^\alpha} = \mathbf{V}^{S_k^\alpha} - \mathbf{U}_F^{S_k^\alpha}, \tag{7.26}$$

where $\mathbf{U}_F^{S_k^\alpha}$ must be expressed in the B frame, that is,

$$\{^B V_R\}^{S_k^\alpha} = \{^B V\}^{S_k^\alpha} - \{^B U_F\}^{S_k^\alpha} \tag{7.27}$$

$$= \{^B V\}^{S_k^\alpha} - [^N C^B]^T\{^N U_F\}^{S_k^\alpha}. \tag{7.28}$$

A similar procedure is used to calculate the relative velocity $\mathbf{V}_R^G$ between the centre of mass G of body B and the fluid in frame B.

8. Hydrodynamic pressure forces

As before, we assume that the fluid velocity field is unaffected by the presence of body B. The fluid acceleration at G is $\mathbf{a}_F^G = D\mathbf{U}_F/Dt$ evaluated at G where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}. \tag{8.1}$$

Let $[{}^N A]$ be the added-mass matrix of body B in the inertial frame. Define the inertia matrix $[{}^N E]$ in the inertial frame by

$$[{}^N E] = \rho_f V_0[I] + [{}^N A], \tag{8.2}$$

where V_0 is the volume of the body B and $[I]$ is the 3×3 identity matrix. The hydrodynamic pressure force $\mathbf{H}^B$on body B may be written (Landau and Lifshitz (**10**)) as

$$\mathbf{H}^B = \mathbf{H}^{I/B} + \mathbf{H}^{A/B}, \tag{8.3}$$

where $\mathbf{H}^{I/B}$is due to fluid inertia and $\mathbf{H}^{A/B}$ is due to added mass. Let the vectors $\mathbf{H}^{I/B}$ and $\mathbf{H}^{A/B}$ have inertial components $\{{}^N H^{I/B}\}$ and $\{{}^N H^{A/B}\}$. Then

$$\{{}^N H^{I/B}\} = [{}^N E]\{{}^N a_F^G\}, \tag{8.4}$$
$$\{{}^N H^{A/B}\} = -[{}^N A]\{{}^N a^G\}, \tag{8.5}$$

where $\{{}^N a_F^G\}$ is the fluid acceleration at G in the absence of the body and $\{{}^N a^G\}$ is the acceleration of G, both in the inertial frame. In frame B, we write equations (8.4) and (8.5) as

$$\{{}^B H^{I/B}\} = [{}^B E]\ \{{}^B a_F^G\}, \tag{8.6}$$
$$\{{}^B H^{A/B}\} = -[{}^B A]\ \{{}^B a^G\}. \tag{8.7}$$

The vector $\{{}^B a^G\}$ is given by (3.5). Matrix $[{}^B A]$ is the added-mass matrix of B in frame B and is known from tables (regular shapes). Matrix $[{}^B E]$ is computed as

$$[{}^B E] = [{}^N C^B]^T[{}^N E][{}^N C^B] = \rho_f V_0[I] + [{}^B A] \tag{8.8}$$

and $\{{}^B a_F^G\}$ is found from

$$\left\{{}^B a_F^G\right\} = \left[{}^N C^B\right]^T\left\{{}^N a_F^G\right\}. \tag{8.9}$$

The generalized active force on B due to fluid inertia is $F_r^{I/B} = \mathbf{H}^{I/B}\cdot\mathbf{v}_r^G$ and is given by

$$F_r^{I/B} = 0, \quad F_{3+r}^{I/B} = {}^B H_r^{I/B} \quad (r = 1, 2, 3). \tag{8.10}$$

The generalized inertia force on B due to added-mass effects is $F_r^{*A/B} = \mathbf{H}^{A/B}\cdot\mathbf{v}_r^G$. In this case we need to isolate the $\dot{u}_r^B$ and we obtain

$$\{F^{*A/B}\} = -[M^{A/B}]\{\dot{u}^B\} - [M^{A/B}]\{\phi^B\}, \tag{8.11}$$

where $[M^{A/B}]$ is a 6×6 diagonal matrix defined by

$$[M^{A/B}] = \mathrm{diag}(0, 0, 0, {}^{B}A_{11}, {}^{B}A_{22}, {}^{B}A_{33}) \tag{8.12}$$

and $\{\phi^B\}$ is the 6×1 vector defined in (4.4). In (8.12) we have neglected the added inertia terms due to body rotation in the fluid. Subscripts 11, 22 and 33 refer to the principal body axes. The quantities ${}^{B}A_{ii}$ are the components of the added-mass matrix of body B in the B frame. We remark that for a spherical body ${}^{B}A_{ii}$ is half of the displaced mass of water ($i = 1, 2, 3$).

For line L^α, the transformation matrix between the local S_k^α frame and the body-fixed B frame is

$$\left[{}^{S_k^\alpha}C^B\right] = \begin{pmatrix} t_{1,k}^\alpha & t_{2,k}^\alpha & t_{3,k}^\alpha \\ h_{1,k}^\alpha & h_{2,k}^\alpha & h_{3,k}^\alpha \\ s_{1,k}^\alpha & s_{2,k}^\alpha & s_{3,k}^\alpha \end{pmatrix}.$$

Then we write

$$\begin{pmatrix} \mathbf{t}_k^\alpha \\ \mathbf{h}_k^\alpha \\ \mathbf{s}_k^\alpha \end{pmatrix} = \left[{}^{S_k^\alpha}C^B\right] \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{pmatrix}.$$

The added-mass matrix $[{}^{B_k^\alpha}A^{S_k^\alpha}]$ of segment S_k^α in the B frame is related to its local S_k^α frame representation $[{}^{S_k^\alpha}A^{S_k^\alpha}]$ by

$$\left[{}^{B}A^{S_k^\alpha}\right] = \left[{}^{S_k^\alpha}C^B\right]^T \left[{}^{S_k^\alpha}A^{S_k^\alpha}\right] \left[{}^{S_k^\alpha}C^B\right]. \tag{8.13}$$

The matrix $[{}^{S_k^\alpha}A^{S_k^\alpha}]$ is given by

$$A_{11}^{S_k^\alpha} = 0, \quad A_{22}^{S_k^\alpha} = A_{33}^{S_k^\alpha} = \rho_f V^{S_k^\alpha} = \rho_f \frac{\pi}{4}(d^2\ell)^{S_k^\alpha} \tag{8.14}$$

for a cylindrical body, where subscripts 11, 22 and 33 refer respectively to the tangential and normal directions as described in section 2.4. The hydrodynamic pressure force on segment S_k^α is

$$\mathbf{H}^{S_k^\alpha} = \mathbf{H}^{I/S_k^\alpha} + \mathbf{H}^{A/S_k^\alpha}, \tag{8.15}$$

where the terms correspond to fluid acceleration and added-mass effects, respectively. Considering the first term in (8.15), the components of $\mathbf{H}^{I/S_k^\alpha}$ in the B frame are

$$\left\{{}^{B}H^{I/S_k^\alpha}\right\} = \left[{}^{B}E^{S_k^\alpha}\right]\left\{{}^{B}a_F^{S_k^\alpha}\right\}, \tag{8.16}$$

where

$$\left[{}^{B}E^{S_k^\alpha}\right] = \rho_f \frac{\pi}{4}(d^2\ell)^{S_k^\alpha}[I] + \left[{}^{B}A^{S_k^\alpha}\right] \tag{8.17}$$

and

$$\left\{{}^{B}a_F^{S_k^\alpha}\right\} = \left[{}^{N}C^B\right]^T \left\{{}^{N}a_F^{S_k^\alpha}\right\}. \tag{8.18}$$

Here, the vectors $\{{}^N a_F^{S_k^\alpha}\}$ and $\{{}^B a_F^{S_k^\alpha}\}$ are the fluid accelerations at the mid-point of segment S_k^α in the inertial and B frames respectively, the former being computed by an independent routine as mentioned previously. Let F_r^{I/S_k^α} be the generalized active force due to $\mathbf{H}^{I/S_k^\alpha}$ on segment S_k^α. Then

$$F_r^{I/S_1^\alpha} = \mathbf{H}^{I/S_1^\alpha} \cdot \mathbf{v}_r^{P_1^\alpha}. \tag{8.19}$$

For $k = 2, \dots, n_\alpha$,

$$F_r^{I/S_k^\alpha} = \tfrac{1}{2}\mathbf{H}^{I/S_k^\alpha} \cdot (\mathbf{v}_r^{P_k^\alpha} + \mathbf{v}_r^{P_{k-1}^\alpha}) \quad (r = 1, \dots, 3n_\alpha) \tag{8.20}$$

and for $k = n_{\alpha+1}$,

$$F_r^{I/S_{n_\alpha+1}^\alpha} = \mathbf{H}^{I/S_{n_\alpha+1}^\alpha} \cdot \mathbf{v}_r^{P_{n_\alpha}^\alpha}. \tag{8.21}$$

The generalized active force on L^α due to fluid inertia is then

$$F_r^{I/L^\alpha} = \sum_{k=1}^{n_\alpha+1} F_r^{I/S_k^\alpha} \quad (r = 1, \dots, 3n_\alpha). \tag{8.22}$$

We now consider the second term in (8.15). In keeping with the lumped-mass approximation, we apply the forces $\mathbf{H}^{A/S_k^\alpha}$ to the points P_k^α. Specifically, the added-mass forces of segment 1 and half of segment 2 are lumped at point P_1^α. Similarly, the added-mass forces of segment $n_{\alpha+1}$ and half of segment n_α are lumped at point $P_{n_\alpha}^\alpha$. The added-mass effects of the other segments are lumped in halves at the ends. Denoting by $\mathbf{H}^{A/P_k^\alpha}$ the hydrodynamic pressure forces corresponding to the added mass of the segments lumped at points P_k^α, we have

$$\begin{aligned}
\mathbf{H}^{A/P_1^\alpha} &= \mathbf{H}^{A/S_1^\alpha} + \tfrac{1}{2}\mathbf{H}^{A/S_2^\alpha}, \\
\mathbf{H}^{A/P_k^\alpha} &= \tfrac{1}{2}(\mathbf{H}^{A/S_k^\alpha} + \mathbf{H}^{A/S_{k+1}^\alpha}) \quad (k = 2, \dots, n_\alpha - 1), \\
\mathbf{H}^{A/P_{n_\alpha}^\alpha} &= \tfrac{1}{2}\mathbf{H}^{A/S_{n_\alpha}^\alpha} + \mathbf{H}^{A/S_{n_\alpha+1}^\alpha}.
\end{aligned} \tag{8.23}$$

In general, $\mathbf{H}^{A/P_k^\alpha}$ has components in frame B

$$\{{}^B H^{A/P_k^\alpha}\} = -\left[Q^{P_k^\alpha}\right]\{{}^B a^{P_k^\alpha}\}, \tag{8.24}$$

where

$$\begin{aligned}
\left[Q^{P_1^\alpha}\right] &= \left[{}^B A^{S_1^\alpha}\right] + \tfrac{1}{2}\left[{}^B A^{S_2^\alpha}\right], \\
\left[Q^{P_k^\alpha}\right] &= \tfrac{1}{2}\left[{}^B A^{S_k^\alpha}\right] + \tfrac{1}{2}\left[{}^B A^{S_{k+1}^\alpha}\right] \quad (k = 2, \dots, n_\alpha - 1), \\
\left[Q^{P_{n_\alpha}^\alpha}\right] &= \tfrac{1}{2}\left[{}^B A^{S_{n_\alpha}^\alpha}\right] + \left[{}^B A^{S_{n_\alpha+1}^\alpha}\right],
\end{aligned} \tag{8.25}$$

and $\{{}^B a^{P_k^\alpha}\}$ is given by (3.6). We can write

$$\mathbf{a}^{P_k^\alpha} = \mathbf{\Omega}^{P_k^\alpha} + \mathbf{\Psi}^{P_k^\alpha}, \tag{8.26}$$

where

$$\boldsymbol{\Omega}^{P_k^\alpha} = \dot{u}_{3k-2}^\alpha \mathbf{b}_1 \dot{u}_{3k-1}^\alpha \mathbf{b}_2 + \dot{u}_{3k}^\alpha \mathbf{b}_3, \tag{8.27}$$

$$\begin{aligned}\boldsymbol{\Psi}^{P_k^\alpha} &= (u_2^B u_{3k}^\alpha - u_3^B u_{3k-1}^\alpha)\mathbf{b}_1 \\ &\quad + (-u_1^B u_{3k}^\alpha + u_3^B u_{3k-2}^\alpha)\mathbf{b}_2 \\ &\quad + (u_1^B u_{3k-1}^\alpha - u_2^B u_{3k-2}^\alpha)\mathbf{b}_3.\end{aligned} \tag{8.28}$$

We can thus write

$$\mathbf{H}^{A/P_k^\alpha} = \mathbf{S}^{P_k^\alpha} + \mathbf{R}^{P_k^\alpha}, \tag{8.29}$$

where

$$\{S^{P_k^\alpha}\} = -\left[Q^{P_k^\alpha}\right]\{\Omega^{P_k^\alpha}\} \quad (k = 1, \dots, n_\alpha), \tag{8.30}$$

$$\{R^{P_k^\alpha}\} = -\left[Q^{P_k^\alpha}\right]\{\Psi^{P_k^\alpha}\} \quad (k = 1, \dots, n_\alpha). \tag{8.31}$$

Let the generalized forces due to $\mathbf{S}^{P_k^\alpha}$ and $\mathbf{R}^{P_k^\alpha}$ be X_r^{A/P_k^α}, Y_r^{A/P_k^α} respectively, defined as

$$X_r^{A/P_k^\alpha} = \mathbf{S}^{P_k^\alpha} \cdot \mathbf{v}_r^{P_k^\alpha}, \tag{8.32}$$

$$Y_r^{A/P_k^\alpha} = \mathbf{R}^{P_k^\alpha} \cdot \mathbf{v}_r^{P_k^\alpha}. \tag{8.33}$$

For line L^α, define

$$X_r^{A/L^\alpha} = \sum_{k=1}^{n_\alpha} X_r^{A/P_k^\alpha} \quad (r = 1, \dots, 3n_\alpha). \tag{8.34}$$

This may be written as

$$\{X\}^{A/L^\alpha} = -\left[M^{A/L^\alpha}\right]\{\dot{u}^\alpha\}, \tag{8.35}$$

where $[M^{A/L^\alpha}]$ is a block-diagonal matrix defined by

$$\left[M^{A/L^\alpha}\right] = \operatorname{diag}\left\{\left[Q^{P_1^\alpha}\right], \left[Q^{P_2^\alpha}\right], \dots, \left[Q^{P_{n_\alpha}^\alpha}\right]\right\}. \tag{8.36}$$

For line L^α define

$$Y_r^{A/L^\alpha} = \sum_{k=1}^{n_\alpha} Y_r^{A/P_k^\alpha} \quad (r = 1, \dots, 3n_\alpha). \tag{8.37}$$

This may be written as

$$\{Y\}^{A/L^\alpha} = -\left[M^{A/L^\alpha}\right]\{\phi^\alpha\}, \tag{8.38}$$

where $\{\phi^\alpha\}$ is defined by

$$\begin{pmatrix}\phi_{3k-2}^\alpha \\ \phi_{3k-1}^\alpha \\ \phi_{3k}^\alpha\end{pmatrix} = \{\Psi^{P_k^\alpha}\} \quad (k = 1, \dots, n_\alpha), \tag{8.39}$$

with $\{\Psi^{P_k^\alpha}\}$ given by equation (8.28). The generalized inertia force on subsystem L^α due to added mass is

$$F_r^{*A/L^\alpha} = X_r^{A/L^\alpha} + Y_r^{A/L^\alpha} \quad (r = 1, \ldots, 3n_\alpha). \tag{8.40}$$

From equations (8.35) and (8.38)

$$\{F_r^{*A/L^\alpha}\} = -\left[M^{A/L^\alpha}\right]\{\dot{u}^\alpha\} - \left[M^{A/L^\alpha}\right]\{\phi^\alpha\}. \tag{8.41}$$

8.1 *Externally applied forces and moments on body B*

Any system of applied forces and moments may be replaced by an equivalent force–couple system $\mathbf{F}^0$, $\mathbf{T}^0$, where force $\mathbf{F}^0$ passes through the centre of mass G of B. Let

$$\mathbf{F}^0 = \sum_{i=1}^{3} {}^N F_i^0 \mathbf{N}_i = \sum_{i=1}^{3} {}^B F_i^0 \mathbf{b}_i, \tag{8.42}$$

$$\mathbf{T}^0 = \sum_{i=1}^{3} {}^N T_i^0 \mathbf{N}_i = \sum_{i=1}^{3} {}^B T_i^0 \mathbf{b}_i. \tag{8.43}$$

The generalized active force due to $\mathbf{F}^0$ and $\mathbf{T}^0$ is

$$F_r^{E/B} = \mathbf{F}^0 \cdot \mathbf{v}_r^G + \mathbf{T}^0 \cdot \boldsymbol{\omega}_r^B \quad (r = 1, \ldots, 6), \tag{8.44}$$

that is,

$$F_r^{E/B} = {}^B T_r^0, \quad F_{3+r}^{E/B} = {}^B F_r^0 \quad (r = 1, 2, 3), \tag{8.45}$$

where the force and torque components in the B frame are calculated from

$$\{{}^B F^0\} = \left[{}^N C^B\right]^T \{{}^N F^0\}, \tag{8.46}$$

$$\{{}^B T^0\} = \left[{}^N C^B\right]^T \{{}^N T^0\}. \tag{8.47}$$

9. Equations of motion

Now, in order to formulate the equations of motion for the entire system we define the system generalized coordinates $\bar{q}_i$ $(i = 1, \ldots, m)$, where m is the number of degrees of freedom (equation (2.2)) as follows:

$$\bar{q}_i = q_i^B \quad (i = 1, \ldots, 6), \tag{9.1}$$

$$\bar{q}_{m_\alpha + k} = q_k^\alpha \quad (\alpha = 1, \ldots, \nu; \quad k = 1, \ldots, 3n_\alpha), \tag{9.2}$$

$$m_\alpha = \begin{cases} 6 & (\alpha = 1), \\ 6 + 3\sum_{r=1}^{\alpha-1} n_r & (\alpha = 2, \ldots, \nu). \end{cases} \tag{9.3}$$

Similarly, let the system generalized speeds be $\bar{u}_i$ $(i = 1, \ldots, m)$, where

$$\bar{u}_i = u_i^B \quad (i = 1, \ldots, 6), \tag{9.4}$$

$$\bar{u}_{m_\alpha + k} = u_k^\alpha \quad (\alpha = 1, \ldots, \nu;\ k = 1, \ldots, 3n_\alpha). \tag{9.5}$$

To write the equations of motion we assemble the components of the generalized inertia and active forces for the system. Matrices will be denoted by square brackets and column vectors by curly brackets.

The non-hydrodynamic generalized inertia forces for the system are denoted by F_r^{*NH} $(r = 1, \ldots, m)$, where

$$F_i^{*NH} = F_i^{*B} \quad (i = 1, \ldots, 6) \tag{9.6}$$

$$F_{m_\alpha + k}^{*NH} = F_k^{*\alpha} \quad (\alpha = 1, \ldots, \nu;\ k = 1, \ldots, 3n_\alpha). \tag{9.7}$$

We write (4.3) and (4.6) as

$$\{F^{*NH}\} = -\left[V^S\right]\{\dot{\bar{u}}\} - \left[W^S\right]\{\phi^S\}, \tag{9.8}$$

where $\bar{u}_r$ $(r = 1, \ldots, m)$ are the system generalized speeds,

$$\{\phi^S\} = (\{\phi^B\}, \{\phi^1\}, \ldots, \{\phi^\nu\})^T, \tag{9.9}$$

and $[V^S]$ and $[W^S]$ are $(m \times m)$ block diagonal matrices defined as

$$[V^S] = \mathrm{diag}([V^B], [V^1], \ldots, [V^\nu]), \tag{9.10}$$

$$[W^S] = \mathrm{diag}([W^B], [V^1], \ldots, [V^\nu]). \tag{9.11}$$

The generalized active forces for the system due to gravity, buoyancy and touchdown are denoted by F_r^{GBT} $(r = 1, \ldots, m)$, where

$$\begin{aligned} F_i^{GBT} &= F_i^{GB/B} \qquad (i = 1, \ldots, 6), \\ F_{m_\alpha + k}^{GBT} &= F_k^{GBT/L^\alpha} \quad (\alpha = 1, \ldots, \nu;\ k = 1, \ldots, 3n_\alpha). \end{aligned} \tag{9.12}$$

The generalized active forces for the system due to mooring line tension are denoted by F_r^T $(r = 1, \ldots, m)$, where

$$\begin{aligned} F_i^T &= F_i^{T/B} \qquad (i = 1, \ldots, 6), \\ F_{m_\alpha + k}^T &= F_k^{T/L^\alpha} \quad (\alpha = 1, \ldots, \nu;\ k = 1, \ldots, 3n_\alpha). \end{aligned} \tag{9.13}$$

The generalized active forces for the system due to structural damping in the mooring lines are denoted by F_r^{SD} $(r = 1, \ldots, m)$, where

$$\begin{aligned} F_i^{SD} &= F_i^{SD/B} \qquad (i = 1, \ldots, 6), \\ F_{m_\alpha + k}^{SD} &= F_k^{SD/L^\alpha} \quad (\alpha = 1, \ldots, \nu;\ k = 1, \ldots, 3n_\alpha). \end{aligned} \tag{9.14}$$

The generalized active forces due to viscous drag on the system are denoted by F_r^D $(r = 1, \dots, m)$, where

$$\begin{aligned} F_i^D &= F_i^{D/B} \qquad (i = 1, \dots, 6), \\ F_{m_\alpha+k}^D &= F_k^{D/L^\alpha} \quad (\alpha = 1, \dots, \nu;\ k = 1, \dots, 3n_\alpha). \end{aligned} \tag{9.15}$$

The generalized active forces for the system due to fluid inertia and added mass are denoted by F_r^I and F_r^{*A} respectively $(r = 1, \dots, m)$. These are defined by

$$\begin{aligned} F_i^I &= F_i^{I/B} \qquad (i = 1, \dots, 6), \\ F_{m_\alpha+k}^I &= F_k^{I/L^\alpha} \quad (\alpha = 1, \dots, \nu;\ k = 1, \dots, 3n_\alpha). \end{aligned} \tag{9.16}$$

From (8.11) and (8.41)

$$\{F^{*A}\} = -[M^A]\{\dot{\bar{u}}\} - [M^A]\{\phi^S\}, \tag{9.17}$$

where $\bar{u}_r (r = 1, \dots, m)$ are the system generalized speeds and $[M^A]$ is an $m \times m$ block diagonal matrix defined by

$$[M^A] = \mathrm{diag}([M^{A/B}], [M^{A/L^1}], \dots, [M^{A/L^\nu}]). \tag{9.18}$$

The generalized active force for the system due to externally applied forces and moments is

$$F_r^E = \begin{cases} F_r^{E/B} & (r = 1, \dots, 6), \\ 0 & (r = 7, \dots, m). \end{cases} \tag{9.19}$$

The total generalized inertia force for the system is

$$\begin{aligned} \{F^{*S}\} &= \{F^{*NH}\} + \{F^{*A}\} \\ &= -[V^S]\{\dot{\bar{u}}\} - [W^S]\{\phi^S\} - [M^A]\{\dot{\bar{u}}\} - [M^A]\{\phi^S\} \\ &= -[A^S]\{\dot{\bar{u}}\} - [B^S]\{\phi^S\}, \end{aligned} \tag{9.20}$$

where

$$[A^S] = [V^S] + [M^A], \tag{9.21}$$

$$[B^S] = [W^S] + [M^A]. \tag{9.22}$$

The total generalized active force for the system is

$$[F^S] = [F^{GBT}] + \{F^T\} + \{F^{SD}\} + \{F^D\} + \{F^I\} + \{F^E\}. \tag{9.23}$$

We are now able to write the system of $2m$ coupled nonlinear equations of motion of buoy B and its ν mooring lines (Kane and Levinson (**7**)) as

$$\{F^{*S}\} + \{F^S\} = \{0\}. \tag{9.24}$$

From (9.20) we have

$$\{\dot{\bar{u}}\} = [A^S]^{-1}(-[B^S]\{\phi^S\} + \{F^S\}). \tag{9.25}$$

Define the $2m \times 1$ column vector $\{x\}$ by

$$\{x\} = \left\{\frac{\{\bar{q}\}}{\{\bar{u}\}}\right\}. \tag{9.26}$$

The system of equations to be solved is then

$$\{\dot{x}\} = \left\{\frac{\{\dot{\bar{q}}\}}{\{\dot{\bar{u}}\}}\right\}. \tag{9.27}$$

The functions $\dot{\bar{q}}_r$ $(r = 1, \ldots, m)$ are supplied by the kinematic analysis presented in section 3 and the functions $\dot{\bar{u}}_r (r = 1, \ldots, m)$ are given by (9.25). When appropriate initial conditions are specified, equation (9.27) is solved by the implementation of the Runge–Kutta algorithm 'ode45' provided in MATLAB (**11**). No numerical difficulties are encountered for the test problems considered in the sequel. Discussion of the results of further simulations will be presented in subsequent work.

10. Test problems

10.1 *Problem* 1*: the elastic catenary*

To test the algorithm we consider the case of a single-point mooring system which is a special case of the present formulation with $\nu = 1$. For comparison purposes, we make use of the closed form solution for an elastic catenary presented by Irvine (**12**). This solution was rewritten for the geometry illustrated in Fig. 3. For an anchored line of unstretched length L_0, area of cross-section A_0, modulus of elasticity E, submerged weight per unit length $\rho_b g$, the stretched line profile as a function of unstretched arclength s is given by

$$x(s) = \frac{Hs}{EA_0} + \frac{H}{\rho_b g}\left[\sinh^{-1}\left(\frac{V - \rho_b g L_0 + \rho_b g s}{H}\right) - \sinh^{-1}\left(\frac{V - \rho_b g L_0}{H}\right)\right], \tag{10.1}$$

$$\begin{aligned} z(s) = {} & \frac{s}{EA_0}\left(V - \rho_b g L_0 + \frac{1}{2}\rho_b g s\right) \\ & + \frac{H}{\rho_b g}\left\{\left[1 + \left(\frac{V - \rho_b g L_0 + \rho_b g s}{H}\right)^2\right]^{\frac{1}{2}} - \left[1 + \left(\frac{V - \rho_b g L_0}{H}\right)^2\right]^{\frac{1}{2}}\right\}, \end{aligned} \tag{10.2}$$

where the top end is supported by a force whose horizontal and vertical components are H and V as shown in Fig. 3. The distances a and b (Fig.3) are obtained from equations (10.1)and (10.2) by putting $s = L_0$, that is,

$$a = \frac{HL_0}{EA_0} + \frac{H}{\rho_b g}\left[\sinh^{-1}\left(\frac{V}{H}\right) - \sinh^{-1}\left(\frac{V - \rho_b g L_0}{H}\right)\right], \tag{10.3}$$

$$b = \frac{L_0}{EA_0}\left(V - \frac{1}{2}\rho_b g L_0\right) + \frac{H}{\rho_b g}\left\{\left[1 + \frac{V^2}{H^2}\right]^{\frac{1}{2}} - \left[1 + \left(\frac{V - \rho_b g L_0}{H}\right)^2\right]^{\frac{1}{2}}\right\}. \tag{10.4}$$

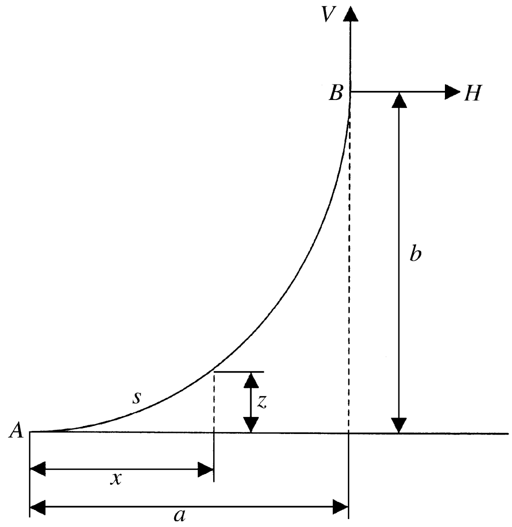

Fig. 3 The elastic catenary

The tension at any point is

$$T(s) = [H^2 + (V - \rho_b g L_0 + \rho_b g s)^2]^{\frac{1}{2}}. \tag{10.5}$$

We now consider the case of a solid spherical buoy, radius a_0, with one line attached to its centre starting from an initial condition in which the line is unstretched. A constant horizontal force $H = 1000$ N is applied to the sphere's centre. When the steady-state rest position is attained, the line is being held by the following force components H, V at its top end:

$$H = 1000\ (\mathrm{N}), \tag{10.6}$$

$$V = |(M_0 - \rho_f V_0)g|, \tag{10.7}$$

where M_0, V_0 are respectively the mass and volume of the sphere. Knowing these values of H and V we can determine the expected steady-state line profile and tension from (10.1), (10.2) and (10.5). A simulation was conducted with the following parameters: sphere radius $a_0 = 1$ m, sphere density $\rho_0 = 800$ kg m^{-3}, volume $V_0 = \frac{4}{3}\pi a_0^3$, mass $M_0 = \rho_f V_0$; line diameter $d_0 = 35$ mm, line length $L_0 = 13$ m, mass per unit length $\rho = 50$ kg m^{-1}, modulus of elasticity $E = 10^7$ N m^{-2}.

The line was modelled using 10 lumped masses. The steady-state line profile after a simulation of $t = 400$ s, using the presented algorithm, and the profile determined from the static solution equations (10.1), (10.2) are shown in Fig. 4 with no noticeable difference. A comparison between the steady-state tensions in the segments and the tensions computed from (10.5) at the segment mid-points is shown in the following table, where the segments are numbered consecutively from 1 to 11 starting at the buoy. The difference in segment 1 is about 3 per cent, in segments 2 to 10 at most 0·2 per cent while in segment 11, at the anchor, it is about 8 per cent. Alternative distribution of the line mass may give different accuracy at the line ends but the exploration of this point is outside the scope of the present paper.

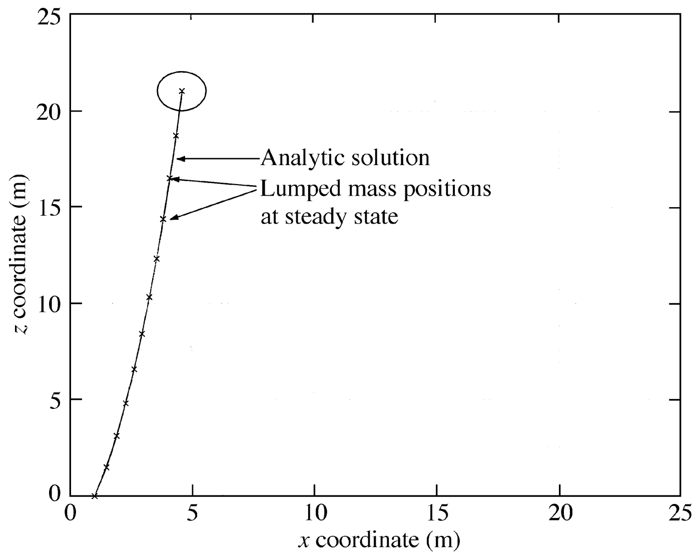

Fig. 4 Test problem 1: steady-state cable profile and analytic solution

Segment no.	Tension (N) in segment lumped-mass model	Tension (N) at segment centre analytic solution
11	3152·5	3428·2
10	3971·3	3975·2
9	4523·3	4527·4
8	5078·3	5083·2
7	5634·0	5641·4
6	6190·1	6201·5
5	6748·1	6762·9
4	7308·8	7325·4
3	7873·1	7888·7
2	8437·3	8452·7
1	9285·0	9017·2

10.2 *Problem* 2*: sphere and three mooring lines*

The initial configuration is shown in Fig. 5. Three mooring lines are symmetrically connected to a sphere at its centre and anchored at points A_1, A_2, A_3. The characteristics of the sphere and each line are as follows: sphere radius $a_0 = 1$m, sphere density $\rho_0 = 500$ kg m^{-3}, volume $V_0 = \frac{4}{3}\pi a_0^3$,

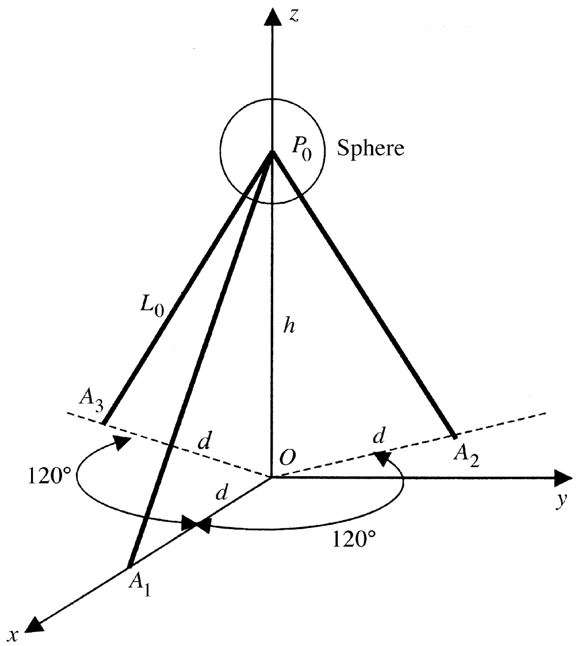

Fig. 5 Sphere and symmetrically arranged mooring lines

mass $M_0 = \rho_f V_0$; line diameter $d_0 = 35$ mm, line length $L_0 = 13$ m, mass per unit length $\rho = 50$ kg m^{-1}, modulus of elasticity $E = 10^6$ N m^{-2}.

The sphere is held with the lines initially unstretched and then released from rest at time $t = 0$, that is, referring to Fig. 5, the sphere is initially at height $h = 5$ m and the anchor points are at distances $d = 12$ m from the origin O. From symmetry, we expect the sphere to rise vertically and attain a steady-state equilibrium position in which each line assumes the elastic catenary shape given by (10.1) and (10.2). By solving (10.3) with $a = 12$ m using Newton's method, the value of H is found to be 685·15 N. Then, knowing that for each line in steady state $V = \frac{1}{3}|M_0 - \rho_f V_0 g|$, we determine the profile of each line using (10.1) and (10.2). For the simulation, each line is modelled by 10 lumped masses. A plot of the steady-state line profiles and the analytically determined profiles is shown in Fig. 6.

10.3 *Problem 3: single-point mooring of sphere in waves*

We consider a single-point mooring system consisting of a submerged sphere and a line attached to its centre subjected to wave loading and compare with the results published by Tjavaras *et al.* (**4**). The sphere has a diameter of 1·5 m, mass 1611 kg, added mass 907 kg and drag coefficient 0·2. The line has an unstretched length of 20 m, cross-sectional area $7{\cdot}85 \times 10^{-5}$ m^2, modulus of elasticity $6{\cdot}369 \times 10^8$ N m^{-2} and density 1140 kg m^{-3}. The system is moored in 25 m of water and is subject to an incident wave of period 5 s. The interaction between the buoy and the free surface is neglected. For a wave amplitude of 0·1 m the response is regular (non-chaotic) and a plot of the normalized

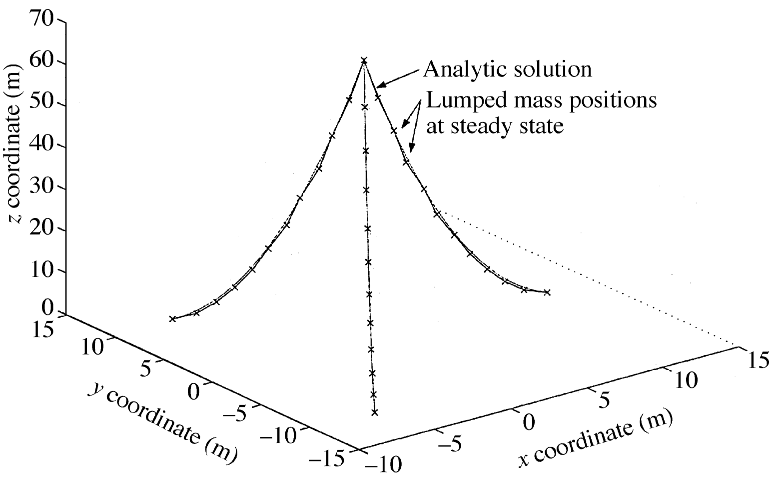

Fig. 6 Test problem 2: steady-state cable profiles and analytic solution

line tension at the buoy is shown in Fig. 7. Based on a visual inspection, this is the same result presented by Tjavaras *et al.* (**4**, Fig. 9). For example, the maximum and minimum tension values as well as the period of vibration and its beating pattern are the same.

11. Multi-line simulation results

Using the same parameters for the sphere and lines as in Problem 3 above, we conduct a simulation of a mooring system in waves, consisting of three mooring lines symmetrically arranged and initially unstretched as in Fig. 5 with $d = 12$ m, $h = 16$ m. The water depth is 25 m and the wave period is 5 s. The wave propagates in the positive x direction (OX) and has an amplitude of 0·25 m. The line tensions at the sphere are shown in Fig. 8. From symmetry, the tensions in lines P_0A_2 and P_0A_3 are the same. The tension in line P_0A_1 which lies in the plane normal to the wave front is generally higher. Discussion and results of further simulations are outside the scope of this paper and will be presented elsewhere.

12. Conclusions

A systematic analysis procedure (formulation and algorithm) for the three-dimensional dynamics of a submerged buoy and multiple mooring lines has been presented and validated by comparison with known results for special cases as well as with available published data. We remark that in the present formulation it is possible to simulate the motion of a towed body by specifying the motion of the line end-points. The method is based on Kane's formalism which is well known to provide an efficient way of formulating the equations of motion of multibody systems. Further development is needed to model the effects of bending and torsion for the purpose of studying, for example, the dynamics of marine risers.

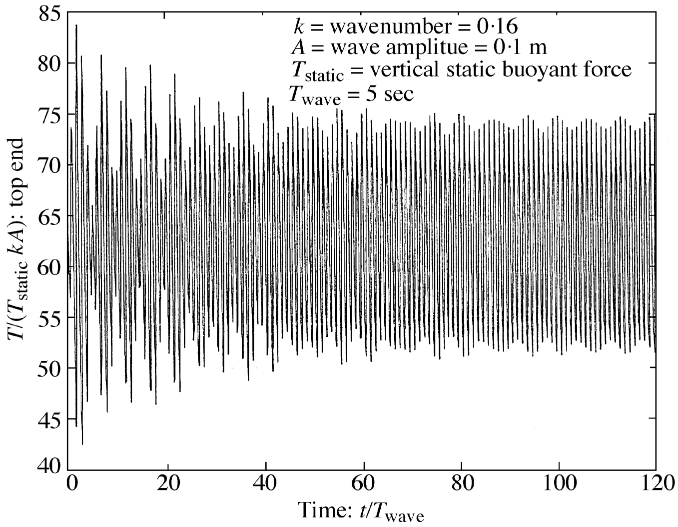

Fig. 7 Test problem 3: tension of cable at buoy

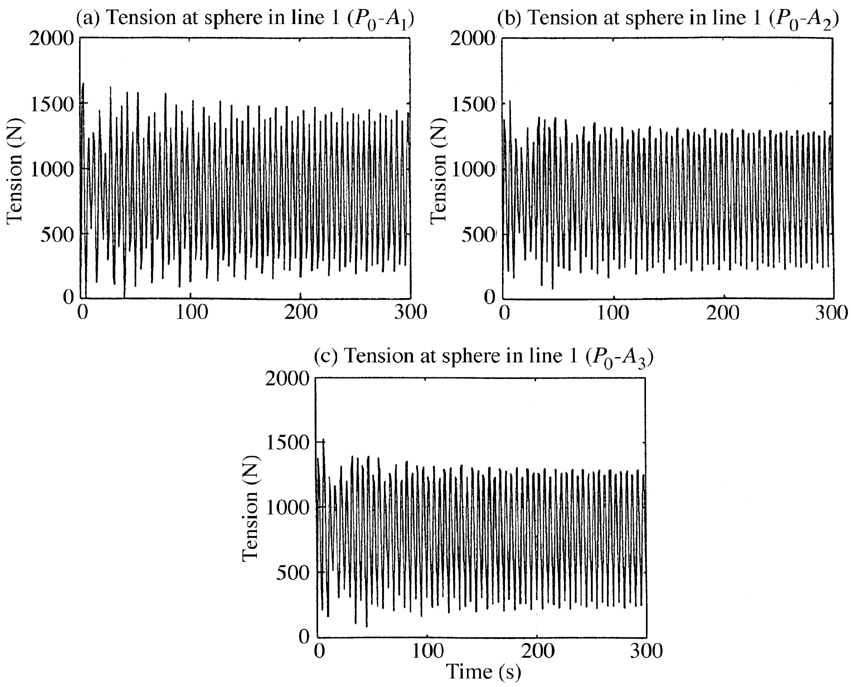

Fig. 8 Tensions in three-line mooring system under wave excitation

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