I will use stan to model here, so that when people are beginning to use stan, they will have examples to start with.

Easy

E1

$$Pr(rain \mid Monday)$$

But also:

E2

Pr(Monday|rain) = probability that it is Monday, given that it is raining.

E3

1:

$$Pr(Monday \mid rain)$$

4:

$$Pr(M \mid r) = \frac{Pr(r \mid M)Pr(M)}{Pr(r)}$$

E4

I think that this means that even though there is a true proportion of water, so the probability of water is .7 doesn't have any real meaning. But we have imperfect ability to observe the true proportion of water, so this represents our belief about the proportion of water given what we have observed.

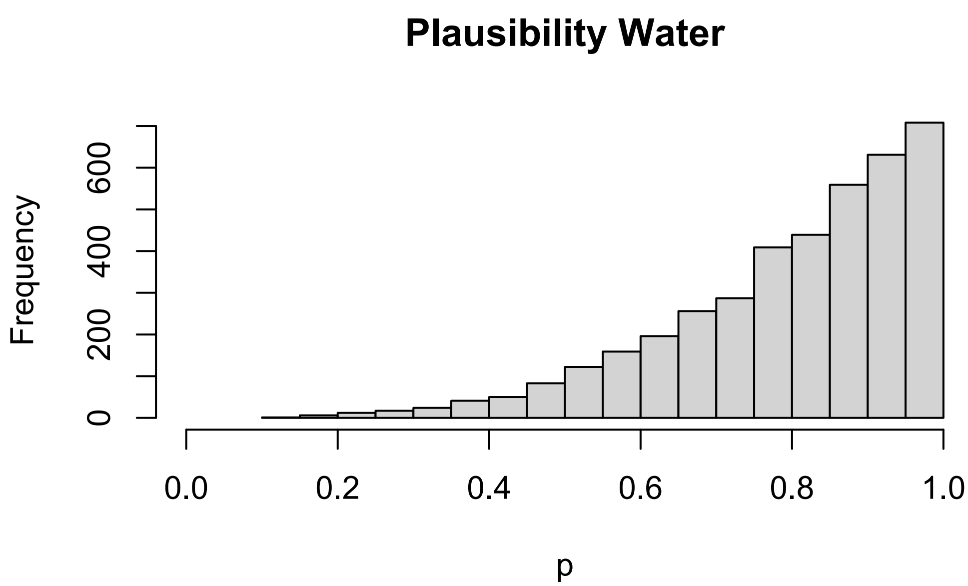
Medium

M1

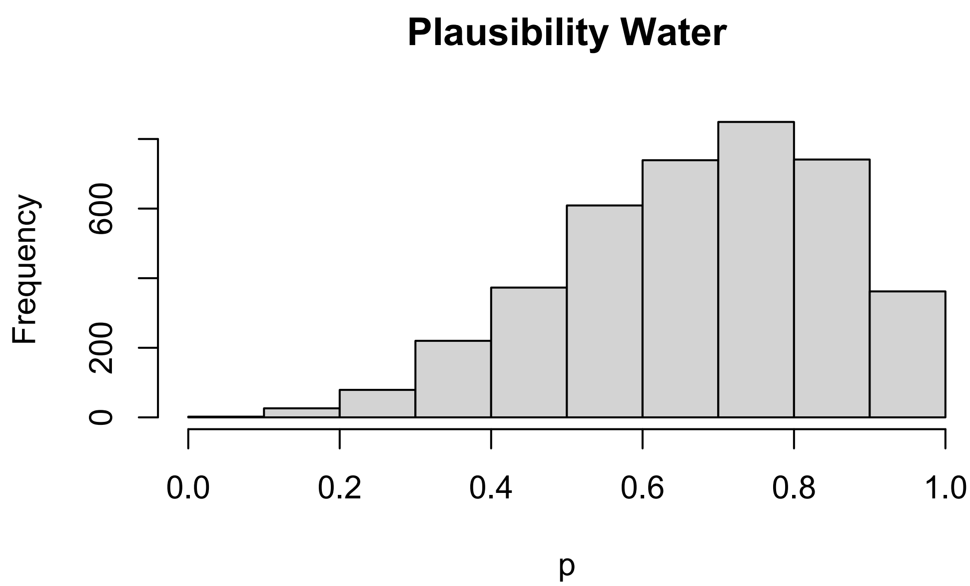
See stan_models/m1.stan for the stan file

```
# data
data_1 <- c(1,1,1)
data_2 <- c(1,1,1,0)
data_3 <- c(0,1,1,0,1,1,1)

# get the data into a list for stan
dat1 <- list(N = length(data_1), water = data_1)
dat2 <- list(N = length(data_2), water = data_2)
dat3 <- list(N = length(data_3), water = data_3)

# need to compile the stan model to C++
library(cmdstanr)

# check for syntax errors
```

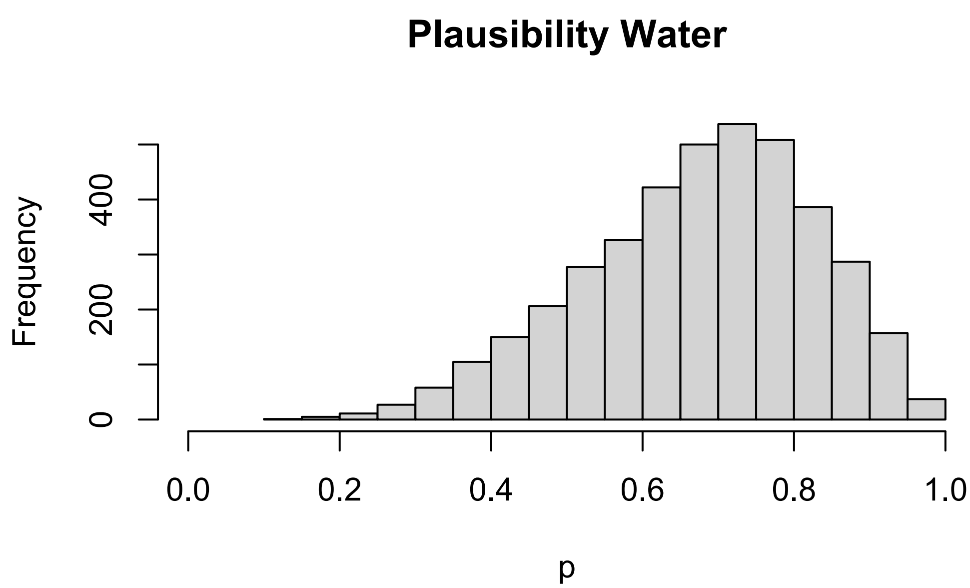
```
mod_m1 <- cmdstan_model("stan_models/m1.stan", compile = F)
mod_m1$check_syntax()
# if syntax is good, compile the model
mod_m1 <- cmdstan_model("stan_models/m1.stan")


# fit the model to the data, n.b. I only have show_messages=F because I don't
want them to be shown in the
# rendered document, I would normally want to see these.
fit1 <- mod_m1$sample(data = dat1, chains = 4, parallel_chains = 4, show_messages
= F)
fit2 <- mod_m1$sample(data = dat2, chains = 4, parallel_chains = 4, show_messages
= F)
fit3 <- mod_m1$sample(data = dat3, chains = 4, parallel_chains = 4, show_messages
= F)

# plot histograms of the posterior samples of p
hist(fit1$draws("p", format = "df")$p, xlab = "p", main = "Plausibility Water",
xlim = c(0,1))
```




```
hist(fit2$draws("p", format = "df")$p,  xlab = "p", main = "Plausibility Water",
xlim = c(0,1))
```

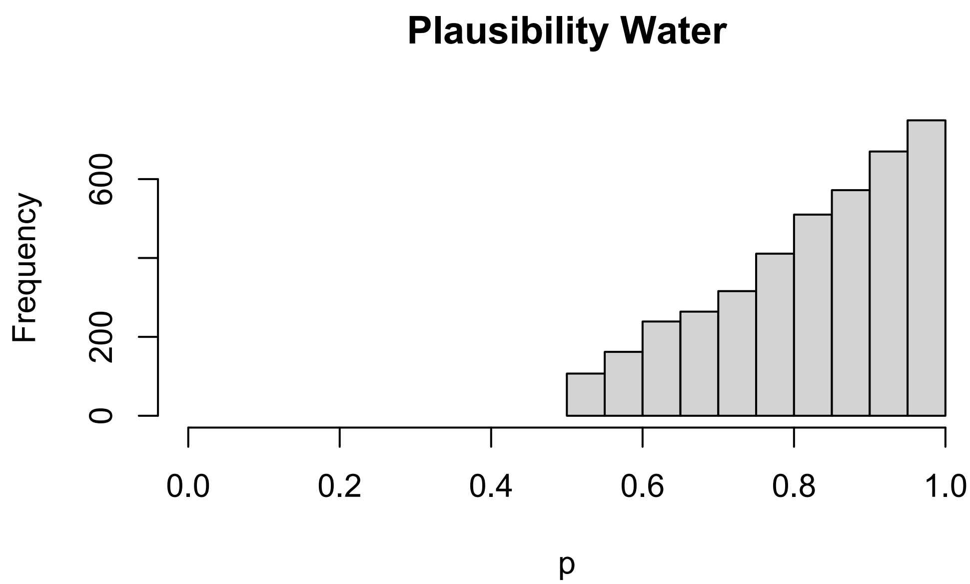



```r
hist(fit3$draws("p", format = "df")$p,  xlab = "p", main = "Plausibility Water",
xlim = c(0,1))
```

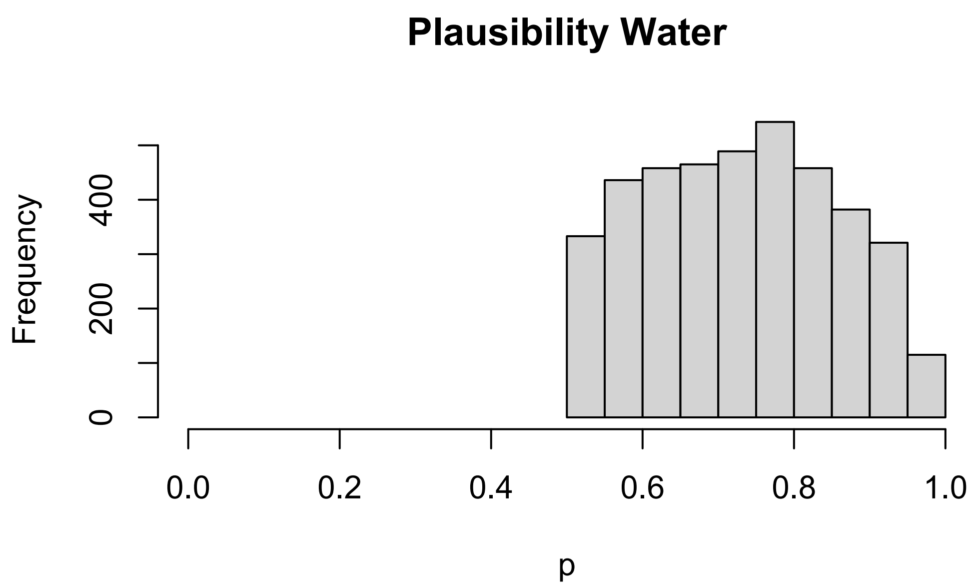


M2

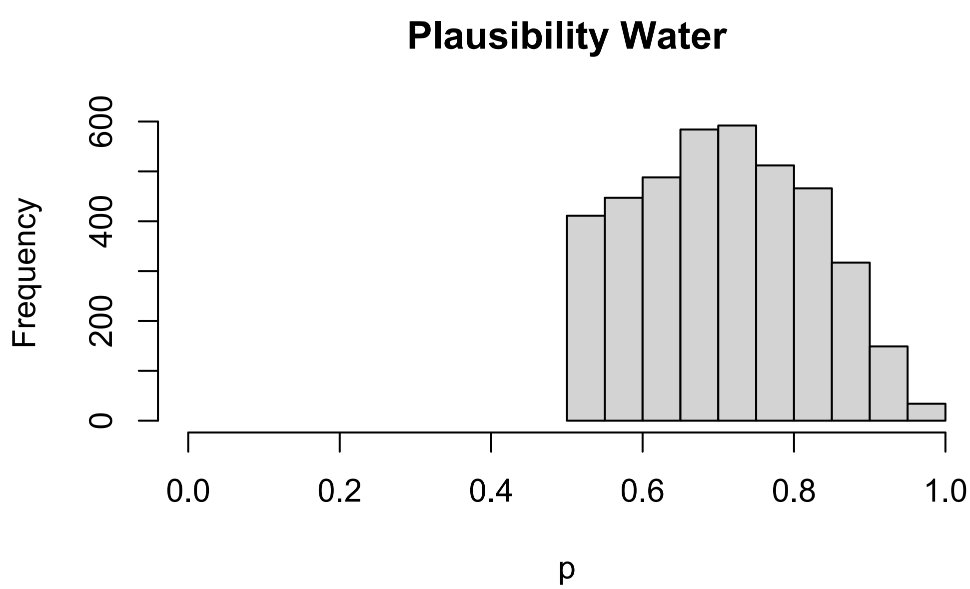
```r
mod_m2 <- mod_m1 <- cmdstan_model("stan_models/m2.stan", compile = F)
mod_m2$check_syntax()
# if syntax is good, compile the model
mod_m2 <- cmdstan_model("stan_models/m2.stan")

fit1 <- mod_m2$sample(data = dat1, chains = 4, parallel_chains = 4, show_messages
= F)
fit2 <- mod_m2$sample(data = dat2, chains = 4, parallel_chains = 4, show_messages
= F)
fit3 <- mod_m2$sample(data = dat3, chains = 4, parallel_chains = 4, show_messages
= F)

hist(fit1$draws("p", format = "df")$p, xlab = "p", main = "Plausibility Water",
xlim = c(0,1))
```




```r
hist(fit2$draws("p", format = "df")$p,  xlab = "p", main = "Plausibility Water",
xlim = c(0,1))
```




```r
hist(fit3$draws("p", format = "df")$p,  xlab = "p", main = "Plausibility Water",
xlim = c(0,1))
```



M3

$$\Pr(earth \mid land) = \frac{Pr(land \mid earth)Pr(earth)}{Pr(land \mid earth)Pr(earth) + Pr(land \mid mars)Pr(mars)}$$

$$= \frac{.3 \times .5}{.3 \times .5 + 1 \times .5}$$

$$= 0.23$$

M4

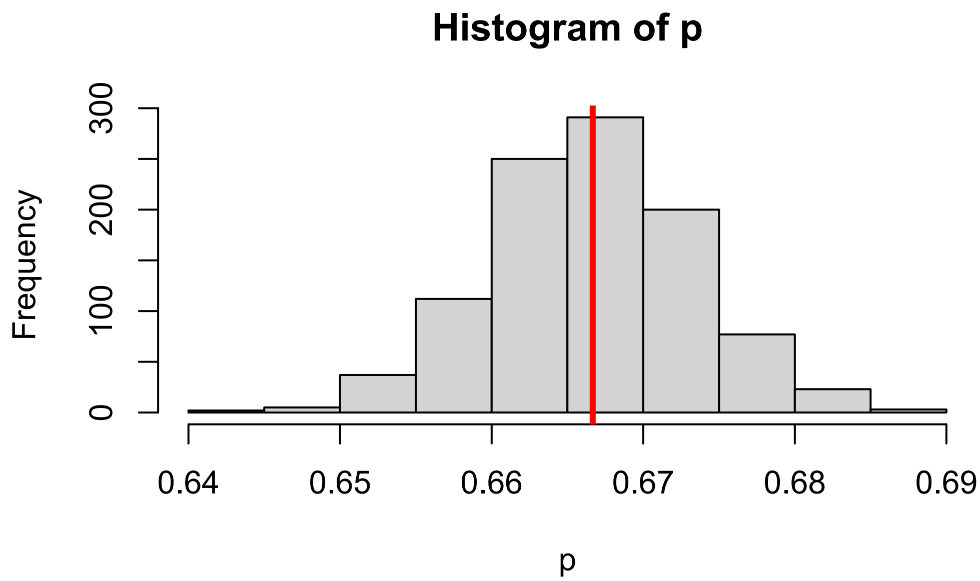
$$Pr\big(bb \mid b_{up}\big) = \frac{Pr\big(b_{up} \mid BB\big)Pr(BB)}{Pr\big(b_{up} \mid Pr(BB)\big)Pr(BB) + Pr(b_up \mid BW)Pr(BW)}$$

$$= \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3}}$$

$$= \frac{2}{3}$$

Simulation to double check.

```
p <- c()
for(i in 1:1000){
  # simulate drawing of ww/0, bw/1, bb/2
  b_sides <- c(0,1, 2)
  # draw a first card
  card <- sample(b_sides,1e4, replace = T)
  # simulate if black is up
  b_up <- rbinom(1e4,1,card/2)
  # simulate if black is down
  b_down <- rbinom(1e4, 1, ifelse(card==0,0,ifelse(card==1 & b_up==1,0,1)))
  # calculate proporsion of b_up AND b_down given black up
  prob <- sum(b_up==1 & b_down==1)/sum(b_up == 1)
  p[i] <- prob
}

hist(p)
abline(v = 2/3, col = "red", lwd = 3)
```



M5

Same as above but $Pr(BB) = \frac{1}{2}$ so $Pr\left(bb \mid b_{up}\right) = \frac{4}{5}$

M6

$$= \frac{1 \times \frac{1}{6}}{1 \times \frac{1}{6} + \frac{1}{2} \times \frac{1}{3}}$$

$$= \frac{1}{2}$$

M7

$$Pr(bb \mid b - w) = \frac{Pr(b - w \mid bb)Pr(bb)}{Pr(b - w \mid bb)Pr(bb) + Pr(b - w \mid bw)Pr(bw)}$$

$$= \frac{\left(1 \times \frac{3}{4}\right)\frac{1}{3}}{\left(1 \times \frac{3}{4}\right)\frac{1}{3} + \left(\frac{1}{2} \times \frac{1}{2}\right)\frac{1}{3}}$$

$$= \frac{3}{4}$$

Hard

H1

$$Pr(A \mid twin_1) = \frac{Pr(twin_1 \mid A)Pr(A)}{Pr(twin_1 \mid A)Pr(A) + Pr(twin_1 \mid B)Pr(B)}$$

$$= \frac{.1 \times .5}{.1 \times .5 + .2 \times .5} = \frac{1}{3}$$

$$Pr(B \mid twin_1) = \frac{Pr(twin_1 \mid A)Pr(A)}{Pr(twin_1 \mid A)Pr(A) + Pr(twin_1 \mid B)Pr(B)}$$

$$= \frac{.2 \times .5}{.2 \times .5 + .1 \times .5} = \frac{2}{3}$$

$$Pr(twin_2 \mid twin_1) = Pr(A \mid twin_1)Pr(twin_2 \mid A) + Pr(B \mid twin_1)Pr(twin_2 \mid B)$$

$$= \frac{1}{3}.1 + \frac{2}{3}.2 = \frac{1}{6}$$

```
twins_sim <- function(){
  # simulate 10000 individuals from either species with prob = .5 A = 1, B = 0
  species <- rbinom(1e5, 1, .5)
  # simulate probability of first set being twins (1) given the species
  first_twins <- rbinom(1e5, 1, prob = ifelse(species==1, .1, .2))
  # simulate second twins
  second_twins <- rbinom(1e5, 1, ifelse(species == 1, .1, .2))
  # calculate the probability of second twins if the first set was twins
  sum(first_twins==1 & second_twins==1)/sum(first_twins==1)
}

hist(replicate(100, twins_sim()), main = "Simulations of second twins given first
twins",
     xlab = "Proportion Second Twins given First Twins")
abline(v = 1/6, col = "white", lwd = 6)
abline(v = 1/6, col = "blue", lwd = 3)
```

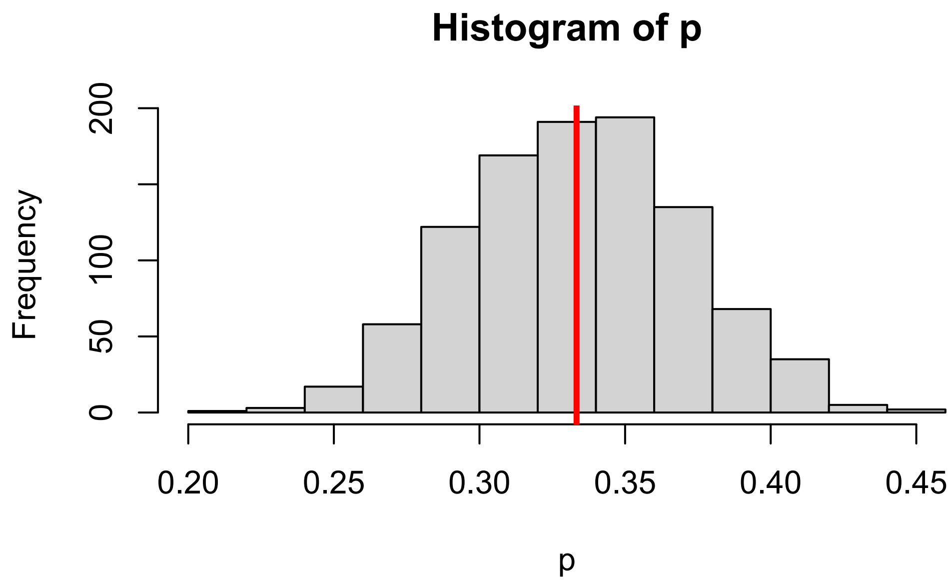
H2

$$Pr(A \mid twin_1) = \frac{Pr(twin_1 \mid A)Pr(A)}{Pr(twin_1 \mid A)Pr(A) + Pr(twin_1 \mid B)Pr(B)}$$

$$= \frac{.1 \times .5}{.1 \times .5 + .2 \times .5} = \frac{1}{3}$$

Check with a simulation

```
p <- c()
for(i in 1:1000){
  species <- rbinom(1000, 1, .5)
  twins <- rbinom(1000, 1, ifelse(species==0, .1, .2))
  p[i] <- sum(species==0&twins==1)/sum(twins==1)
}
hist(p)
abline(v = 1/3, col = "red", lwd = 3)
```



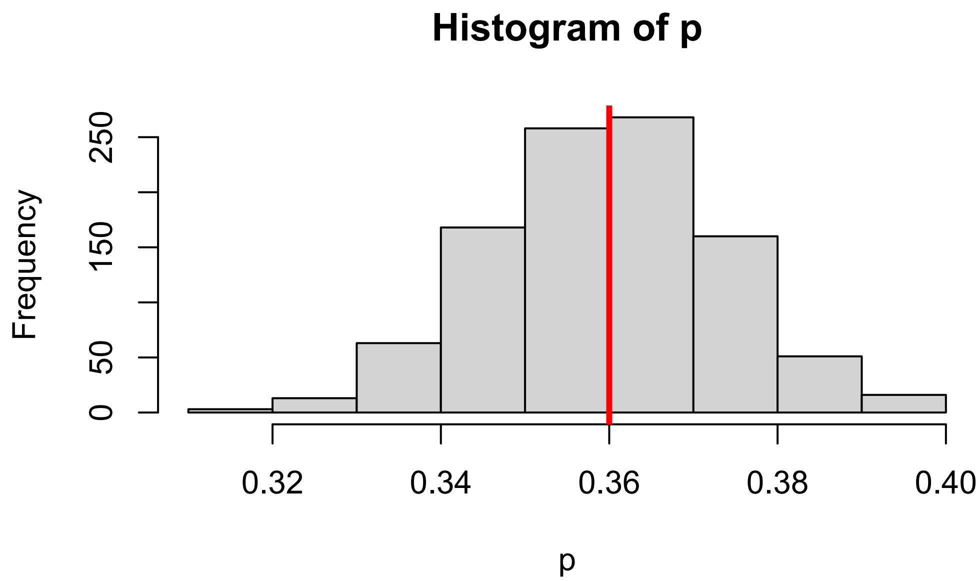
H3

$$Pr(A \mid sing.) = \frac{Pr(sing \mid A)Pr(A)}{Pr(sing \mid A)Pr(A) + Pr(sing \mid B)Pr(B)}$$

$$= \frac{.9 \times \frac{1}{3}}{.9 \times \frac{1}{3} + .8 \times \frac{2}{3}}$$

Simulate to check

```
expected <- (.9/3)/(.9/3 + .8*2/3)
p <- c()
for(i in 1:1000){
  species <- rbinom(1e4, 1, .5)
  first_twins <- rbinom(1e4, 1, ifelse(species==1, .1, .2))
  second_twins <- rbinom(1e4, 1, ifelse(species==1, .1, .2))
  p[i] <- sum(first_twins==1 & second_twins==0 & species==1)/sum(first_twins==1
& second_twins==0)
}

hist(p)
abline(v = expected, col = "red", lwd = 3)
```



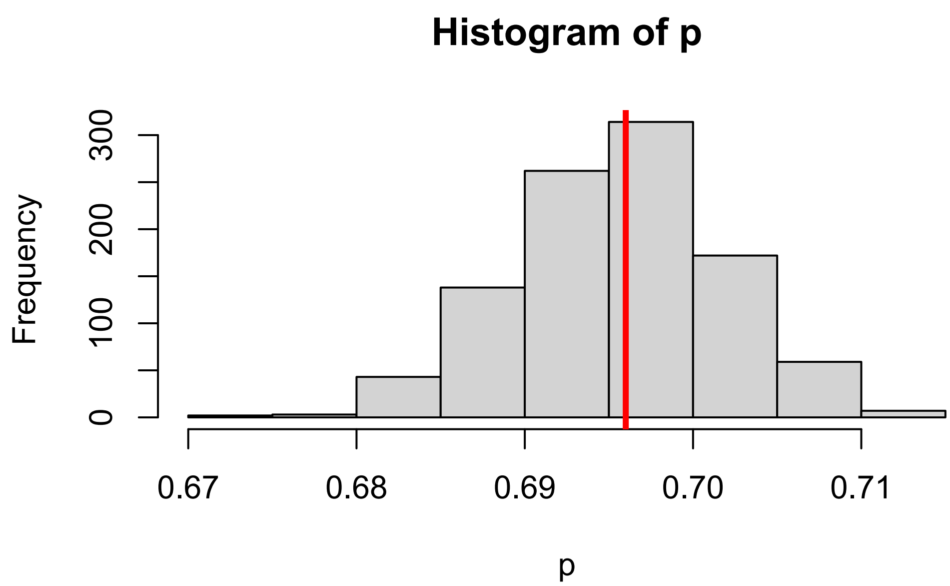
H4

Calculation without the previous knowledge.

$$
\begin{aligned}
Pr(A \mid +A) &= \frac{Pr(+A \mid A)Pr(A)}{Pr(+A \mid A)Pr(A) + Pr(+A \mid B)Pr(B)} \\
&= \frac{.8 \times \frac{1}{2}}{.8 \times \frac{1}{2} + .35 \times \frac{1}{2}} \\
&\approx .696
\end{aligned}
$$

Simulate this:

```r
p <- c()
for(i in 1:1000){
  species <- rbinom(1e4,1, .5)
  test_a <- rbinom(1e4, 1, ifelse(species==1, .8, .35))
  p[i] <- sum(species==1&test_a==1)/sum(test_a==1)
}

hist(p)
abline(v = .696, col = "red", lwd = 3)
```

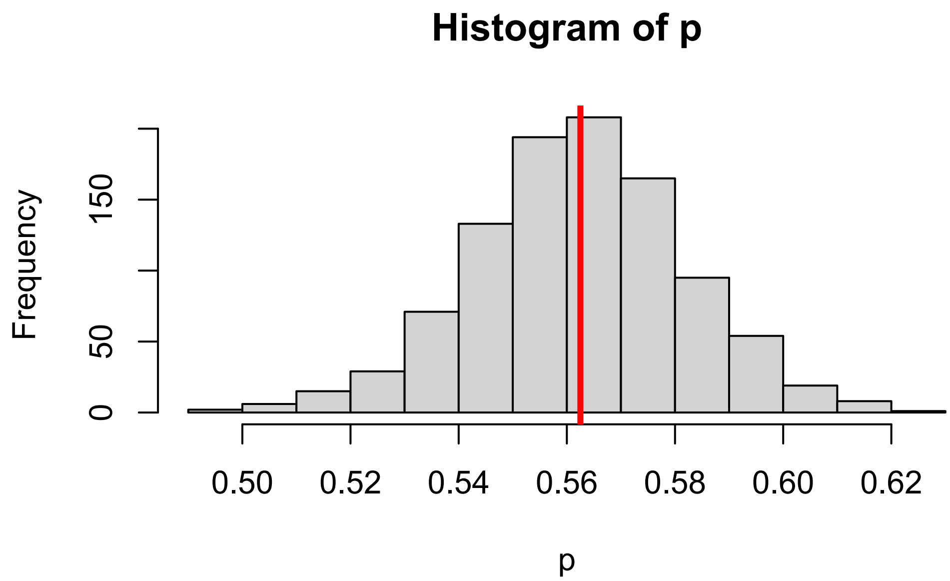


Calculate taking into account prior knowledge.

$$
\begin{aligned}
Pr(A \mid +A) &= \frac{Pr(+A \mid A)Pr(A)}{Pr(A+ \mid A)Pr(A) + Pr(+A \mid B)Pr(B)} \\
&= \frac{.8 \times .36}{.8 \times .36 + .35 \times .64} \\
&= .5625
\end{aligned}
$$

Simulate this

```
p <- c()
for(i in 1:1000){
  species <- rbinom(1e4, 1, .5)
  first_twins <- rbinom(1e4, 1, ifelse(species==1, .1, .2))
  second_twins <- rbinom(1e4, 1, ifelse(species==1, .1, .2))
  test_a <- rbinom(1e4, 1, ifelse(species==1, .8, .35))
  p[i] <- sum(species==1&test_a==1&first_twins==1&second_twins==0)/
sum(test_a==1&first_twins==1&second_twins==0)
}

hist(p)
abline(v = .5625, col = "red", lwd = 3)
```


Histogram of p
Frequency
0
50
150
0.50
0.52
0.54
0.56
0.58
0.60
0.62
p