CIE 6022 Dynamic Programming

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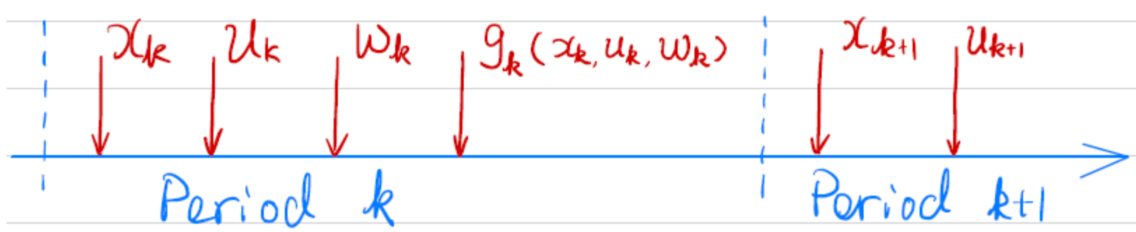
Contents

Lecture 1. Introduction and Basic DP Algorithm

Basic Strucutre of Stochastic DP

- 2 principle features of DP
  - Discrete-time dynamic system
  - Cost function is additive over time
- 6 key elements of DP formulation
  - **Stage**: k
    - Finite horizon. Total number of horizon: N.
  - **State**: x_k
  - **Control**: $u_k \in U_k(x_k)$ (or called **Decision, Action**)
  - **Disturbance**: w_k
  - **"Memoryless" System Dynamics**: $x_{k+1} = f_k(x_k, u_k, w_k)$
    - Alternative system equation: $P(x_{k+1}|x_k, u_k)$
  - **"Additive" Stage Cost**: $g_k(x_k, u_k, w_k)$
    - Terminal Stage Cost: $g_N(x_N)$
    - Total cost: $\mathbb{E}_{w_0, w_{k-1}}\{g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k)\}$
- Additional Assumptions
  - The feasible set of u_k, $U_k(x_k)$, depends at most x_k and not on prior x or u.
  - The probability distribution of w_k may depends on x_k, u_k but not past values $w_0, ..., w_{k-1}$.
    - $P(w_k|x_k, u_k)$

    This two assumptions ensures the optimization of u_k only requires the information of x_k and u_k, which is "Memoryless". Past values of x or w should be useless for future optimization.
- Sequence of events envisioned in period k:



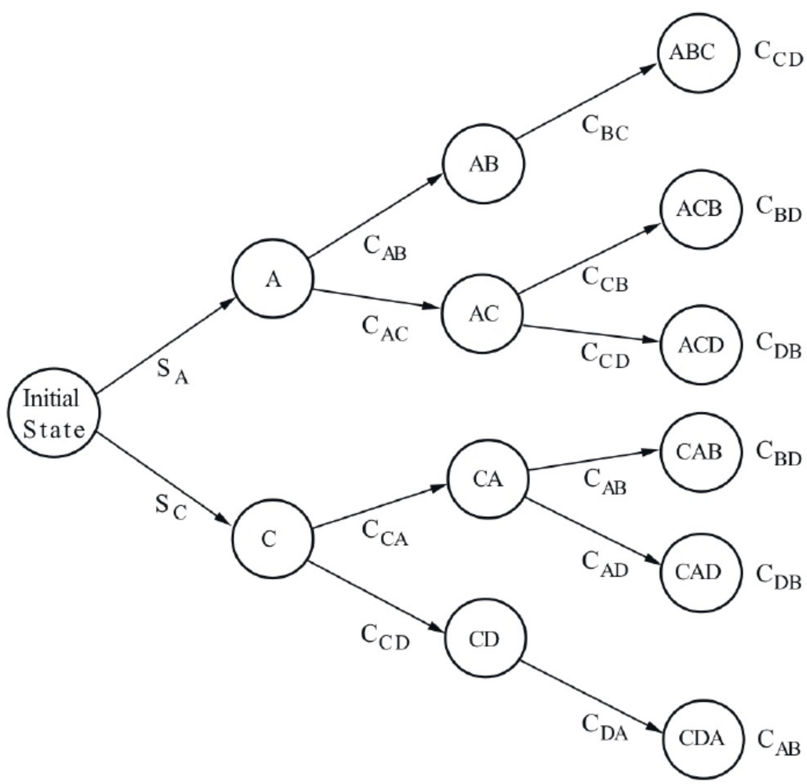
    - x_k occurs according to system dynamics
        * $x_k = f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1})$
    - u_k is selected with knowledge of x_k
        * $u_k \in U_k(x_k)$
    - w_k is random and generated according to a distribution
        * $w_k \; P(w_k | x_k, u_k)$
    - $g_k(x_k, u_k, w_k)$ is incurred and added to previous costs
- Basic problem
    - **Policies** $\pi = \{\mu_0, ..., \mu_{N-1}\}$
        * μ_k maps states x_k into controls $u_k = \mu_k(x_k)$
        * $\mu_k(x_k) \in U_k(x_k)$ for all x_k
    - **Expected cost** of policy π starting at x_0 is
        * $J_\pi(x_0) = \mathbb{E}\{g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k(x_k), w_k)\}$
    - **Optimal cost function**
        * $J^*(x_0) = \min_\pi J_\pi(x_0)$
    - **Optimal policy** π^*
        * $J_{\pi^*}(x_0) = J^*(x_0)$
- 2 aims
    - **Optimal policy**
        * $u_k = \mu_k^*(x_k)$ for all $k = 1, ..., N-1$
        * $\pi^* = (\mu_0^*, \mu_1^*, ..., \mu_{N-1}^*)$
    - **Value function**
        * $J_N(x_N) = g_N(x_N)$
        * $J_k(x_k) = \min\limits_{u_k \in U_k(x_k)} \mathbb{E}_{w_k}\{g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k))\}, \forall k = 0, 1, ..., N-1$

Example 1. Inventory Control

- **Stage**: Ordering period k
- **State**: Inventory level at period k, x_k
- **Control**: Sotck ordered at period k, $u_k \geq 0$
- **Disturbance**: Demand at period k, w_k
- **System Dynamics**: $x_{k+1} = f_k(x_k, u_k, w_k) = x_k + u_k - w_k$
- **Stage Cost**: $g_k(x_k, u_k, w_k) = \underbrace{k \cdot \mathbb{1}_{[u_k>0]}}_{\text{Set up}} + \underbrace{c \cdot u_k}_{\text{Ordering fee}} + \underbrace{h(x_k + u_k - w_k)^+}_{\text{Holding cost}} + \underbrace{p(x_k + u_k - w_k)^-}_{\text{Shortage penalty}}$
- **Terminal Cost**: $g_N(x_N) = 0$

Example 2. Scheduling

- Find the optimal sequence of operations A, B, C, D
- A must precede B, C must precede D
- Given startup cost S_A, S_C and setup transition cost C_{nm} from operation m to n
- This is a **Deterministic Finite-State Problem**.



Example 3. Two-game Chess

- Find two-game chess match strategy
- **State**: Match score
- **Control**: Timid play *or* Bold play
- **System Dynamics**:
  - Timid play: draws with $P_d > 0$ and loses with $1 - P_d$
  - Bold play: wins with $P_w \leq \frac{1}{2}$ and loses with $1 - P_w$
- **Value of information** $= J^*_{\text{closed-loop}} - J^*_{\text{open-loop}}$
  - Open-loop policy: play always Bold
  - Close-loop policy: Play Timid if and only if you are ahead

  See the example in Apendix.1

Principle of Optimality

- Consider the **"tail subproblem"**. Minimize the "cost-to-go" from i to N
  - $\mathbb{E}\{g_N(x_N) + \sum_{k=1}^{N-1} g_k(x_k, \mu_k(x_k), w_k)\}$
- Consider the **"tail policy"**
  - $\{\mu_i^*, \mu_{i+1}^*, ..., \mu_{N-1}^*\}$
- **Principle of optimality**: The tail policy is optimal for the tail subproblem.
- DP first solve ALL tail subproblems of final stage
- At the generic step, DP solves ALL tail subproblems of a given time length, using the solution of the subproblems of shorter length

Demonstration for the Principle of Optimality (Inventory Control)

- $J_N(x_N) = 0$
- Tail subproblem of length 1 ($u_{N-1} = \mu^*_{N-1}(x_{N-1})$)

$$\begin{aligned} J_{N-1}(x_{N-1}) \quad &= \min\nolimits_{u_{N-1} \geq 0} \mathbb{E}_{w_{N-1}}\{cu_{N-1} + r(x_{N-1}) + u_{N-1} - w_{N-1} + J_N(x_{N-1} + u_{N-1} - w_{N-1})\} \\ &= \min\nolimits_{u_{N-1} \geq 0} \mathbb{E}_{w_{N-1}}\{cu_{N-1} + r(x_{N-1}) + u_{N-1} - w_{N-1})\} \end{aligned}$$

- Tail subproblem of length $N - k$ ($u_k = \mu_k^*(x_k)$)

$$J_k(x_k) = \min_{u_k \geq 0} \mathbb{E}_{w_k}\{cu_k + r(x_k) + u_k - w_k + J_{k+1}(x_k + u_k - w_k)\}$$

- $J_0(x_0)$ is the optimal cost of initial state x_0

DP Algorithm

Original Problem:

$$\begin{aligned} J^*(x_0) = \ & \min_{x\in\pi} J_\pi(x_0) = \min_{x\in\pi} \mathbb{E}\left\{g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k(x_k), w_k)\right\} \\ & \text{s.t. } x_{k+1} = f_k(x_k, \mu_k(x_k), w_k) \\ & \text{where } \pi = \{\mu_0, \mu_1, ..., \mu_{N-1}\} \end{aligned}$$

DP algorithm:

- Start with $J_N(x_N) = g_N(x_N)$ and go backwards using

  $$J_k(x_k) = \min_{u_k \in U_k(x_k)} \mathbb{E}_{w_k}\{g_k(x_k, u_k, w_k) + J_{k+1}\left(f_k(x_k, u_k, w_k)\right)\} \quad , k = 0, 1, ..., N-1$$

  This means that $u_k = \mu_k^*(x_k)$.

- Then $J_0(x_0)$, generated at the last setp, is equal to the optimal cost $J^*(x_0)$.
  Also, the policy $\pi^* = \{\mu_0^*, ..., \mu_{N-1}^*\}$.

Some remark:

- Ideally, the DP algorithm can be used to obtain closed-form expressions for J_k, the cost-to-go function at k, or an optimal policy
- Even if models may rely on over-simplified assumption, they may provide valuable insights about the structures of the optimal solution of more complex models and form the basis for suboptimal control scheme
- Unfortunately, an analytical solution is not possible in many practical cases, and one has to resort to numerical execution of the DP algorithm
- DP is the only general approach for sequential optimization under uncertainty, and even when it is computationally prohibitive, it can serve as the basis for more practicval suboptimal approaches

[Example, Proof ...]

State Augmentation

- When assumptions of the basic problem are violated (e.g., disturbances are correlated, cost is nonadditive, etc) reformulate/augment the state
- DP algorithm still applies, but the problem gets bigger
- **Example**:
  Time lags

  $$x_{k+1} = f_k(x_k, x_{k-1}, u_k, w_k)$$

  $$x_1 = f_0(x_0, u_0, w_0)$$

– Introduce additional state variable $y_k = x_{k-1}$. New system takes the form

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} f_k(x_k, y_k, u_k, w_k) \\ x_k \end{pmatrix}$$

View $\tilde{x}_k = (x_k, y_k)$ as the new state.

– DP algorithm for the formulated problem:

$$J_k(x_k, x_{k-1}) = \min_{u_k \in U_k(x_k)} \mathbb{E}_{w_k}\{g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, x_{k-1}, u_k, w_k), x_k)\}$$

- **Example**:
  Correlated disturbances

$$w_k = \lambda w_k + \xi_k$$

– Let $y_k = w_{k-1}$

$$\tilde{x}_{k+1} = \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} f_k(x_k, u_k, \lambda y_k + \xi_k) \\ \lambda y_k + \xi_k \end{pmatrix}$$

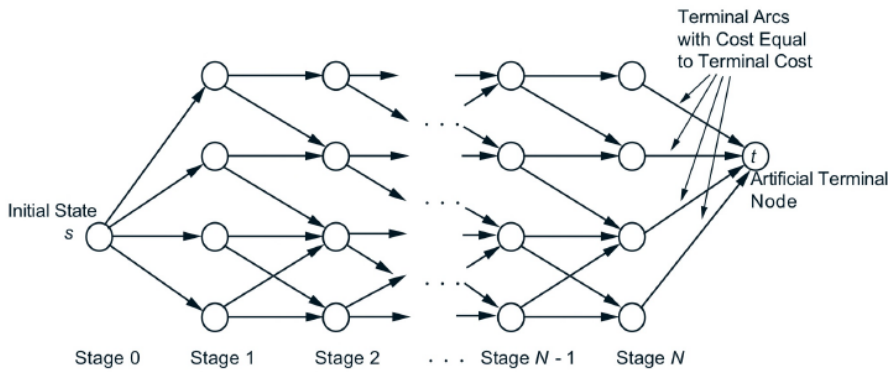
View $\tilde{x}_k = (x_k, y_k)$ as the new state.

– DP algorithm for the formulated problem:

$$J_k(x_k, y_k) = \min_{u_k \in U_k(x_k)} \mathbb{E}_{\xi_k}\{g_k(x_k, u_k, \lambda y_k + \xi_k) + J_{k+1}(f_k(x_k, u_k, \lambda y_k + \xi_k), \lambda y_k + \xi_k)\}$$

Lecture 2. Deterministic Discrete-Time System and Shortest Path Problems

Deterministic Finite-State DP Problems



- Two conditions required for the transformation to the shortest path problem
  - Deterministic
  - Finite-state

  A deterministic finite-state problem is equivalent to finding a shortest path from s (initial state) to t (terminal state)
- States: Nodes
- Controls: Arcs
- Control sequences (Open-loop): paths from initial state to terminal state
- a_{ij}^k: cost of transition from state $i \in S_k$ to state $j \in S_{k+1}$ at stage k ("length" of the arc)
- a_{it}^N: terminal cost of state $i \in S_N$
- Cost of control sequence: cost of the corresponding path ("length" of the path)

Backward and Forward DP Algorithm

- DP algorithm
  - $J_N(i) = a_{it}^N, \quad i \in S_N$
  - $J_k(i) = \min_{j \in S_{k+1}} [a_{ij}^k + J_{k+1}(j)], \quad i \in S_k, \quad k = 0, ..., N-1$

  The optimal cost is $J_0(s)$ and is equal to the length of the shortest path from s to t
- Observation: An optimal path $s \to t$ is also an optimal path $t \to s$ in a "reverse" shortest path problem where the direction of each arc is reversed and its length is left unchanged
- In this situation: forward DP algorithm = backward DP algorithm

- For the reverse problem:
  - $\tilde{J}_N(i) = a_{it}^0, \quad i \in S_1$
  - $\tilde{J}_k(i) = \min_{j \in S_{N-k}}[a_{ij}^{N-k} + \tilde{J}_{k+1}(i)], \quad i \in S_{N-k+1}, \quad k = 1, ..., N$

  The optimal cost is $\tilde{J}_0(t) = \min_{i \in S_N}[a_{it}^N + \tilde{J}_1(i)]$
- View $\tilde{J}_k(j)$ as optimal-to-arrive to state j from initial state s

Note on Forward DP algorithms

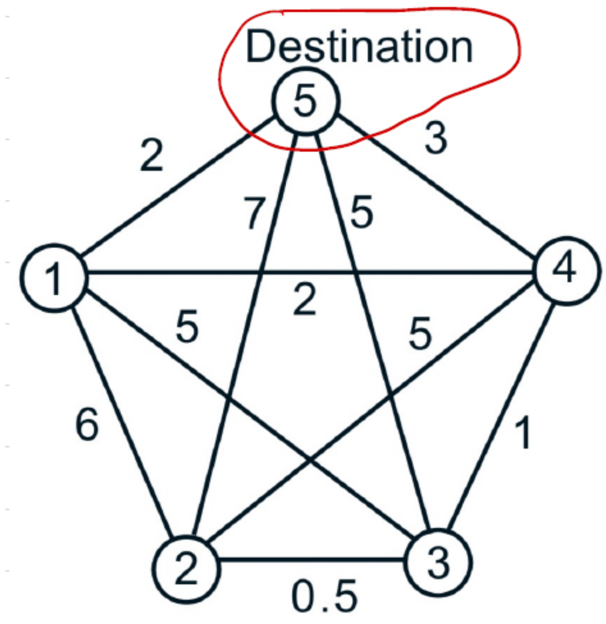
- There is no Forward DP Algorithm for **stochastic** problems
- Mathmatically, for stochastic problems, we cannot restrict ourselves to open loop sequences, so the shortest path viewpoint fails
- Conceptually, in the presence of uncertainty the concept of "optimal cost-to-arrive" at a state s_k does not make sence. It may be impossible to guarantee (with prob. 1) that any given state can be reached.
- By contrast, even in stochastic problems, the concept "optimal cost-to-go" from any state s_k makes clear sense.

Generic Shortest Path Problems

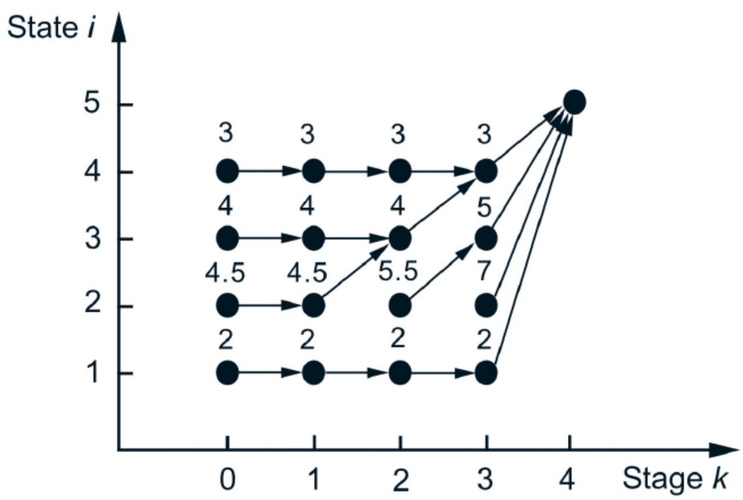
- $\{1, 2, ..., N, t\}$: Nodes of a graph (t: the destination)
- a_{ij}: cost of moving from node i to node j
- Find a shortest (minimum cost) path from each node i to node t
- Assumption: all cycles have nonnegative length. Then an optimal path need not take more than N moves.
  Some arcs are allowed to have nagative length
- We formulate the problem as one where we require exactly N moves but allows degenerate moves from a node i to itself with cost $a_{ii} = 0$
  - $J_k(i)$: optimal cost of getting from i to t in $N - k$ moves
  - $J_0(i)$: cost of the optimal path from i to t
- DP algorithm:
  - $J_k(i) = \min_{j=1,...,N}[a_{ij} + J_{k+1}(j)], \quad k = 0, ..., N - 2$
  - $J_{N-1}(i) = a_{it}, \quad i = 1, 2, ..., N$

Example for Generic Shortest Path Problem

At most 4 moves to the destination



Convert to equivalent staged-shortest-path-problem



Bellman-Ford Algorithm

Critical Path Analysis

[To be completed ...]

Estimation/Hidden Markov Models

- Markov chain with transition probabilities p_{ij}

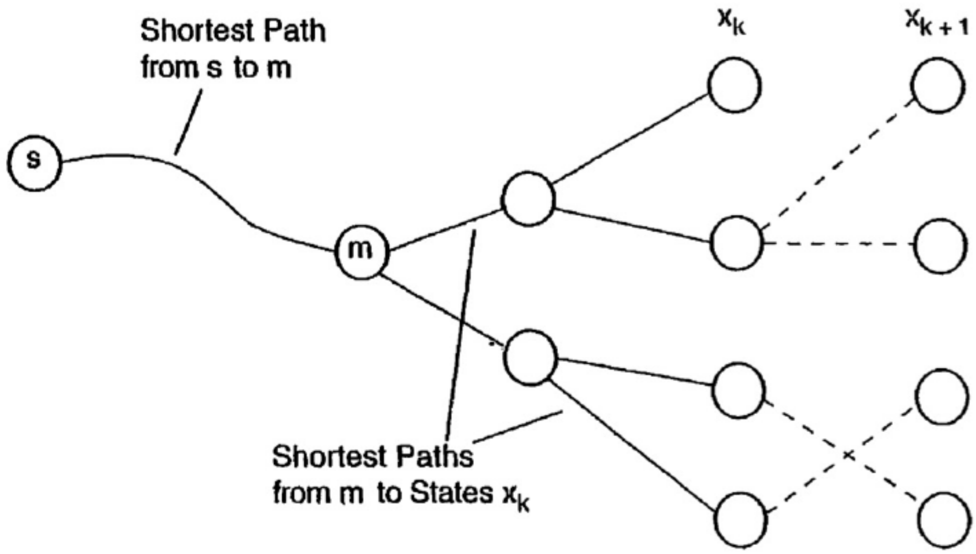
- State transitions are hidden from view
- For each trasition, we get an (independent) observation
- $r(z; i, j)$: Prob. the observation take value z when the state transition is from i to j
- **Trajectory estimation problem**: Given the observation sequence $Z_N = \{z_1, z_2, ..., z_N\}$, what is the "most likely" state transition sequence $\hat{X}_N = \{\hat{x}_0, \hat{x}_1, ..., \hat{x}_N\}$ [one that maximizes $p(X_N|Z_N)$ over all $X_N = \{x_0, x_1, ..., x_N\}$]

Viterbi Algorithm

Assumption of independent observation: an observation only depends on its corresponding transition

- We have
$$p(X_N|Z_N) = \frac{p(X_N, Z_N)}{p(Z_N)}$$
where $p(X_N, Z_N)$ and $p(Z_N)$ are the unconditional probabilities of occurrance of (X_N, Z_N) and Z_N
- Maximizing $p(X_N|Z_N)$ is equivalent with maximizing $\ln(p(X_N, Z_N))$
- We have
$$\begin{aligned} p(X_N, Z_N) &= p(x_0, x_1, ..., x_N, z_1, z_2, ..., z_N) \\ &= \pi_{x_0} p(x_1, ..., x_N, z_1, z_2, ..., z_N|x_0) \\ &= \pi_{x_0} p(x_1, z_1|x_0) p(x_2, ..., x_N, z_2, ..., z_N|x_0, x_1, z_1) \\ &= \pi_{x_0} p_{x_0 x_1} r(z_1; x_0, x_1) p(x_2, ..., x_N, z_2, ..., z_N|x_0, x_1, z_1)^{\dagger} \\ &= \cdots \\ &= \pi_{x_0} \textstyle\prod_{k=1}^{N} p_{x_{k-1}x_k} r(z_k; x_{k-1}, x_k) \end{aligned}$$
so the problem is equivalent to
$$\min -\ln(\pi_{x_0}) - \sum_{k=1}^{N} \ln(p_{x_{k-1}x_k} r(z_k; x_{k-1}, x_k))$$
over all possible sequences $\{x_0, x_1, ..., x_N\}$
†: The calculation can be continued by writing
$$\begin{aligned} p(x_2, ..., x_N, z_2, ..., z_N|x_0, x_1, z_1) &= p(x_2, z_2|x_0, x_1, z_1) p(x_3, ..., x_N, z_3, ..., z_N|x_0, x_1, z_1, x_2, z_2) \\ &= p_{x_1 x_2} r(z_2; x_1, x_2) p(x_3, ..., x_N, z_3, ..., z_N|x_0, x_1, z_1, x_2, z_2)^{\ddagger} \end{aligned}$$
‡:
  - $P(x_2|x_0, x_1, z_1) = P(x_2, x_1)$: Markov Property
  - $P(z_2|x_1, x_2) = P(z_2|x_0, x_1, x_2, z_1)$: Independenc pf observation
- This is a shortest path problem

Advantage of Forward Alogrithm (for Viterbi Algorithm)



Suppose that the shortest paths from s to all states x_k pass through a single node m. If an additional observation is received, the shortest paths from s to all states x_{k+1} will continue pass through m. Therefore, the portion of the state sequence up to node m can be safely estimated because additional observations will not change the initial portion of the shortest paths from s up to m.

Thereofre, we can estimate a portion of state sequence **without waiting** to receive the entire observation sequence Z_N for a number of practical schemes.

This is useful is Z_N is a long sequence.

General Shortest Path Algorithm

- There are many nonDP shortest path algorithms. They ca all be used to solve deterministic finite-state problems
- They may be preferable than DP if they avoid calculating the optimal cost-to-go of **EVERY** state
- This is essential for problems with **HUGE** state spaces. Such problems arise for example in combinatorial optimization
- Example: Traveling Salesman Problem (TSP)

Traveling Salesman Problem

: Find the shortest route that starts from Home City o, visits every other city $1, 2, ..., N$ exactly once and returns to Home City o

- State: (i, S). Now at city i, and still have to visit cities in S and return to home city o
- Control: j. Which city should be visited next.
- System Dynamic: $(i, s) \overset{\text{Control: } j}{\longrightarrow} (k, S\backslash\{j\})$
- Stage cost: d_{ij}. Distance from i to j.
  - $J_N(i, \emptyset) = d_{i,o}$
  - $J_k(i, S) = \min_{i\in S}\{d_{i,j} + J_{k+1}(j, S\backslash\{j\})\}^{\dagger}$
  - $J_0(0, \{1, 2, ..., N\})$ gives the shortest tour.
- Mehotds comparison
  - Brute Force: $N!$
  - DP: $N^2 \cdot 2^N$. much less than $N!$

†: Take N evaluations for each pair of (i, S). There are N cities and 2^N subsets S and N stages. Total: $N \cdot N \cdot 2^N$

Knapsack Problem

$$\begin{array}{rl} \max & \sum_{i=1}^{N} v_i \cdot u_i \\ \text{s.t.} & \sum_{i=1}^{N} w_i \cdot u_i \leq W \text{ (Integer)} \\ & u_i \in \{0, 1\}, \quad \forall i = 1, ..., N \end{array}$$

- Stage k: Items $1, ..., k$ have been considered
- Stage x_k: The remaining capacity after considering items $1, ..., k-1$. $x_k \in \{0, 1, 2, ..., W\}$
- Control u_k: Whether to take item k. $u_k \in \{0, 1\}$
- Stage Cost: $v_k \cdot u_k$
- System Dynamics: $x_{k+1} = x_k - w_k \cdot u_k$
- DP Algorithm:
  - $J_k(x_k) = \begin{cases} J_{k+1}(x_k) & \text{if } x_k < w_k \\ \max\{J_{k+1}(x_k), v_k + J_{k+1}(x_k - w_k)\} & \text{if } x_k \geq w_k \end{cases}, \quad 1 \leq k \leq N-1$
  - $J_N(x_N) = \begin{cases} 0 & \text{if } x_k < w_k \\ v_N & \text{if } x_k \geq w_k \end{cases}$
- Computational Complexity: $O(N \cdot W)$, Pseudo-Polynomial.
  For each stage k and each state x_k, at most 2 operations. There are N stages and $W + 1$ possible states for every stage.
  $\Longrightarrow$ Total complexity: $O(N \cdot W)$

Label Correcting Methods

- Given: origin s, destination t, lengths $a_{ij} \geq 0$
- Idea is to progressively discover shorter paths from the origin s to every other node i
- Notation
  - d_i (label of i): length of the shortest path found (initially $d_s = 0$, $d_i = \infty$ for $i \neq s$)
  - UPPER: the label d_t of the destination
  - OPEN list: contains nodes that are currently active in the sense that they are candidates for further examination (initially OPEN=$\{s\}$)

Lable Correcting Algorithm

- Step 1 (Node Removal): Remove a node i from OPEN and for each child j of i, do Step 2
- Step 2 (Node Insertion Test): If $d_i + a_{ij} < \min\{d_j, \text{UPPER}\}$, set $d_j = d_i + a_{ij}$ and set i to be the parent of j. In addition, if $j \neq t$, place j in OPEN if it is not already in OPEN, while if $j = t$, set UPPER to the new value $d_i + a_{it}$ of d_t
- Step 3 (Termination Test): If OPEN is empty, terminate; else go to Step 1

Specific Label Correcting Methods

- Breadth-first search
  - Bellman-Ford Algorithm. Works for the cases with negative weights.
  - Computational complexity: $O(|E| \cdot |V|) = O(|V|^3)$
- Depth-first search
- Best-first search
-
  - Dijkstra's Algorithm. Requires the assumption of nonnegative weights.
  - Computational complexity: $O(|E| + |V|^2) = O(|V|^2)$

Label Correcting Variations

- A* Algorithm can speed up the computational substantially by placing a node j in OPEN in Step 2 only when
$$d_i + a_{ij} + h_j < \text{UPPER}$$
where h_j is an underestimate of the true shortest distance from j to the destination. In this way, fewer nodes will potentially placed in OPEN before termination.
Since h_j is an underestimate, if $d_i + a_{ij} + h_j \geq \text{UPPER}$, node j need not enter OPEN.

- Another way to sharpen the test $d_i + a_{ij} <$ UPPER for admission of node j into the OPEN list is to try to reduce UPPER by obtaining an upper bound m_j of the shortest distance from j to t. Then, if $d_j + m_j <$ UPPER, we can reduce UPPER to $d_j + m_j$, thereby making the test in Step 2 more stringent.

Lecture 4. Problems with Perfect State Information

Linear System and Quadratic Cost

Definition

- State: $x_k \in \mathbb{R}^{n\times 1}$
- Control: $u_k \in \mathbb{R}^{m\times 1}$
- Disturbance: $w_k \in \mathbb{R}^{n\times 1}$
  - independent and zero mean $\mathbb{E}(w_k) = 0$
- Linear Dynamics: $x_{k+1} = A_k x_k + B_k u_k + w_k$
  - $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$
- Quadratic Stage Cost: $u_k^T R_k u_k + x_k^T Q_k x_k$
  - $R_k \in \mathbb{R}^{m\times m}$ positive definite symmetric matrix
  - $Q_k \in \mathbb{R}^{n\times n}$ positive semidefinite symmetric matrix
- Terminal Cost: $J_N(x_N) = x_N^T Q_N x_N$

Derivation

DP algorithm

- $J_N(x_N) = x_N^T Q_N x_N$
- $J_k(x_k) = \min_{u_k} \mathbb{E}\{x_k^T Q_k x_k + u_k^T R_k u_k + J_{k+1}(A_k x_k + B_k u_k + w_k)\}$
- $J_{N-1}(x_{N-1}) = \min_{u_{N-1}}\{x_{N-1}^T Q_{N-1} x_{N-1} + u_{N-1}^T R_{N-1} u_{N-1} + (A_{N-1}x_{N-1} + B_{N-1}u_{N-1} + w_{N-1})^T Q_N (A_{N-1}x_{N-1} + B_{N-1}u_{N-1} + w_{N-1})\}$
- $(A_{N-1}x_{N-1} + B_{N-1}u_{N-1} + w_{N-1})^T Q_N (A_{N-1}x_{N-1} + B_{N-1}u_{N-1} + w_{N-1})$
  $= x_{N-1}^T A_{N-1}^T Q_N A_{N-1} x_{N-1} + u_{N-1}^T B_{N-1}^T Q_N B_{N-1} u_{N-1} + \mathbb{E}(w_{N-1}^T Q_N w_{N-1}) + 2x_{N-1}^T A_{N-1}^T Q_N B_{N-1} u_{N-1}$

- Define
  - $C = \underbrace{R_{N-1}}_{>0} + \underbrace{B_{N-1}^T Q_N B_{N-1}}_{\geq 0} > 0$ (invertible), and $C \in \mathbb{R}^{m\times m}$
  - $D = x_{N-1}^T A_{N-1}^T Q_N B_{N-1}$, and $D \in \mathbb{R}^{1\times m}$
  - $E = x_{N-1}^T Q_{N-1} x_{N-1} + x_{N-1}^T A_{N-1}^T Q_N A_{N-1} x_{N-1} + w_{N-1}^T Q_N w_{N-1}$ independent of u_{N-1}, and $E \in \mathbb{R}$
- $J_{N-1}(x_{N-1}) = \min_{u_{N-1}}\{u_{N-1}^T C u_{N-1} + 2Du_{N-1} + E\}$
- Since $J_{N-1}(x_{N-1})$ is quadratic in u_{N-1}, so $Cu_{N-1}^* = -D^T$ and $u_{N-1}^* = -C^{-1}D^T$

- $u^*_{N-1} = -(R_{N-1} + B^T_{N-1} Q_N B_{N-1})^{-1} B^T_{N-1} Q_N A_{N-1} \cdot x_{N-1}$ (Linear in state x_{N-1})
- $J_{N-1}(x_{N-1}) = x^T_{N-1} K_{N-1} x_{N-1} + w^T_{N-1} Q_N w_{N-1}$
  where $K_{N-1} = A^T_{N-1}(Q_N - Q_N B_{N-1}(B^T_{N-1} Q_N B_{N-1} + R_{N-1})^{-1} B^T_{N-1} Q_N) A_{N-1} + Q_{N-1}$
  and K_{N-1} is symmetric positive semidefinite
- $J_{N-2}(x_{N-2}) = \min_{u_{N-2}} \{x^T_{N-2} Q_{N-2} x_{N-2} + u^T_{N-2} R_{N-2} u_{N-2} + (A_{N-2} x_{N-2} + B_{N-2} u_{N-2} + w_{N-2})^T K_{N-1} (A_{N-2} x_{N-2} + B_{N-2} u_{N-2} + w_{N-2})\}$

Conclusion

- Optimal policy: $u^* = L_k x_k$
- Optimal cost: $J_k(x_k) = x_k^T K_k x_k + \text{constant}$
  where $L_k = -(B_k^T K_{k+1} B_k + R_k)^{-1} B_k^T K_{k+1} A_k$
  and K_k are symmetric positive semidefinite matrices given by

$$K_N = Q_N$$

$$K_k = A_k^T (K_{k+1} - K_{k+1} B_k (B_k^T K_{k+1} B_k + R_k)^{-1} B_k^T K_{k+1}) A_k + Q_k$$

- This is called the *discrete-time Riccati equation*
- Certainty equivalence holds
  optimal policy is the same as when w_k is replaced by $\mathbb{E}\{w_k\} = 0$

Asymptotic Behavior of Riccati Equation

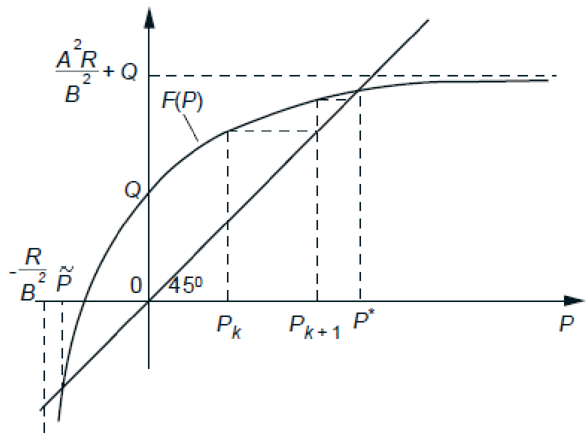
What if finite horizon $\to$ infinite horizon ($N \to \infty$)?

- $u^* = L_k x_k$
  where
  - $L_k = -(B_k^T K_{k+1} B_k + R_k)^{-1} B_k^T K_{k+1} A_k$
  - $K_k = A_k^T (K_{k+1} - K_{k+1} B_k (B_k^T K_{k+1} B_k + R_k)^{-1} B_k^T K_{k+1}) A_k + Q_k$
- As $k \to \infty$, $K_{k+1} = K_k$
- **Assume** stationary case holds
  - $A_k = A$, $B_k = B$, $R_k = R$, $Q_k = Q$
- So the Riccati Equation converges $\lim_{k\to\infty} K_k = K$
  where K is PD and is the unique solution of the *algebraic Riccati equation*

$$K = A^T (K - KB(B^T KB + R)^{-1} B^T K) A + Q$$

- We can plot K in simple case
  - A, B, R and Q are scalar

$K = A^2 \left[K - \frac{K^2 B^2}{B^2 K + R} \right] + Q$
Define $P_k = K_{N-k}$
$P_{k+1} = F(P) = A^2 \left[P_k - \frac{P_k^2 B^2}{B^2 P_k + R} \right] + Q$



- Gnerally, as $k \to \infty$, the corresponding steady-state controller $\mu^*(x) = Lx$, where $L = -(B^T K B + R)^{-1} B^T K A$
  L is stable in the sense that the matrix $(A + BL)$ of the closed-loop system $x_{k+1} = (A + BL)x_k + w_k$ satisfies $\lim_{k \to \infty} (A + BL)^k = 0$

Random System Dynamics

- Suppose that $\{A_0, B_0\}, ..., \{A_{N-1}, B_{N-1}\}$ are not known but rather are independent random matrices that are also independent of the w_k
- $J_k(x_k) = \min_{u_k} \mathbb{E}_{w_k, A_k, B_k} \{x_k^T Q_k x_k + u_k^T R_k u_k + J_{k+1}(A_k x_k + B_k u_k, w_k)\}$
- $L_k = -(R_k + \mathbb{E}\{B_k^T K_{k+1} B_k\})^{-1} \cdot \mathbb{E}\{B_k^T K_{k+1} A_k\}$
- $K_k = \mathbb{E}\{A_k^T K_{k+1} A_k\} - \mathbb{E}\{A_k^T K_{k+1} B_k\} \cdot (R_k + \mathbb{E}\{B_k^T K_{k+1} B_k\})^{-1} \cdot \mathbb{E}\{A_k K_{k+1} B_k^T\} + Q_k$
- Properties
  - Certainty equivalence may not hold.
  - Riccati equation may not converge to a steady-state.
  - In simple case, we have $P_{k+1} = \tilde{F}(P_k)$, where
    $\tilde{F}(P) = \frac{\mathbb{E}\{A^2\} R P}{\mathbb{E}\{B^2\} P + R} + Q + \frac{T P^2}{\mathbb{E}\{B^2\} P + R}$
    $T = \mathbb{E}\{A^2\} \mathbb{E}\{B^2\} - (\mathbb{E}\{A\})^2 (\mathbb{E}\{B\})^2$

Inventory Control

- State: Inventory level x_k
- Control: Number of orders u_k
- Disturbance: Demand w_k
- Dynamics: $x_{k+1} = x_k + u_k - w_k$
- Stage Cost: Ordering cost + maintenance cost (hold cost, penalty cost)
  - $c \cdot u_k + r(x_k + u_k - w_k)$
  - $r(x_k + u_k - w_k) = h \cdot \max\{0, x_k + u_k - w_k\} + p \cdot \max\{0, -(x_k + u_k - w_k)\}$
- Terminal Cost: $J_N(x_N) = 0$
- Cost-to-go: $J_k(x + k) = \min_{u_k \geq 0}\{cu_k + \mathbb{E}_{w_k}\{r(x_k + u_k - w_k)\} + \mathbb{E}_{w_k}\{J_{k+1}(x_k + u_k - w_k)\}\}$

Appendix.1 Value of Information (Two-game Chess)

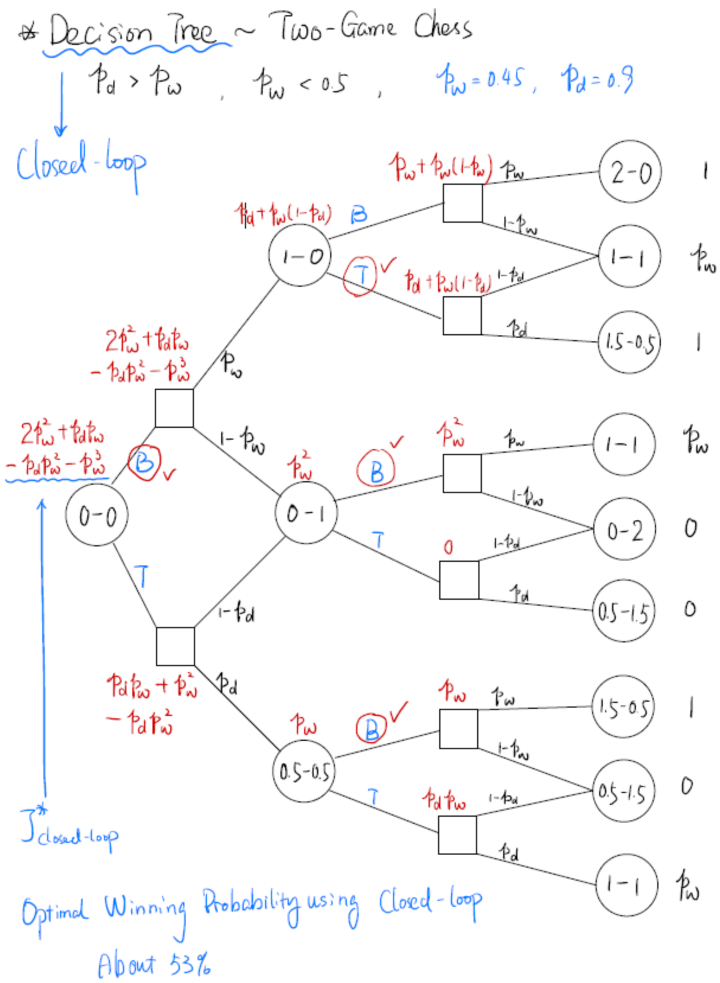
Set $P_w = 0.45,\ P_d = 0.9$

Open-loop
There are 4 possible choices

$$\begin{cases} \begin{array}{ccc} \text{Policy} & \text{Possible cases to win} & \text{Possibility to win} \\ \{B, B\} & \{w, w\}, \{l, w, w\}, \{w, l, w\} & P_w^2 + 2 \cdot P_w^2(1 - P_w) \\ \{B, T\} & \{w, d\}, \{w, l, w\} & P_w P_d + P_w^2(1 - P_d) \\ \{T, B\} & \{d, w\}, \{l, w, w\} & P_d P_w + (1 - P_d) \cdot P_w^2 \\ \{T, T\} & \{d, d, w\} & P_d^2 P_w \end{array} \end{cases}$$

$J^*_{\text{open-loop}} = \max\{P_w^2 + 2 \cdot P_w^2(1 - P_w), P_w P_d + P_w^2(1 - P_d), P_d^2 P_w\} = 42.5\%$

Closed-loop



$J^*_{\text{closed-loop}} = 53\%$

Value of information
$J^*_{\text{closed-loop}} - J^*_{\text{open-loop}} = 10.5\%$