Notes for CIE 6002 Matrix Analysis

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Contents

Lecture 2. Fundamental and Basic Operations

Subspace

Definition

The set $S \subseteq \mathbb{R}^m$ is a subspace, if for $x, y \in S$
$$\alpha x + \beta y = S, \quad \alpha, \beta \in \mathbb{R}^m$$
This indicates

$$x_i \in S, \quad i = 1, ..., N \implies \sum_{i=1}^{N} \alpha_i x_i \in S$$

Some important subspaces

1. Span: a collection of vectors

$$span\{a_1, ..., a_n\} = \{y \in \mathbb{R}^m | y = \sum_{i=1}^{n} \alpha_i a, \alpha_i \in \mathbb{R}\}$$

2. Orthogonal complement: given a subspace $S \subseteq \mathbb{R}^m$

$$S_\perp = \{y \in \mathbb{R}^m | y^T x = 0, \forall x \in S\}$$

3. Range space of A (linear comb. of col. vectors)
$$\begin{aligned} R(A) &= span\{a_1, a_2, ..., a_n\} \\ &= \{y \in \mathbb{R}^m | y = Ax, x \in \mathbb{R}^n\} \end{aligned}$$

4. Null space of A (linear comb. of col. vectors)
$$Null(A) = \{x \in \mathbb{R}^n | Ax = 0\}$$

Linear independence

Given a set $\{a_1, a_2, ..., a_n\}$, we say they are linear independent if

$$\sum_{i=1}^{n} \alpha_i a_i = 0 \implies \alpha_i = 0, \forall i = 1, ..., n$$

Basis

$\{b_1, b_2, ..., b_k\} \subseteq \mathbb{R}^m$ is a basis for the subspace $S \subseteq \mathbb{R}^m$ if $S = span\{b_1, b_2, ..., b_k\}$ and $\{b_1, b_2, ..., b_k\}$ is linear independent.
The number of the vectors of the basis for S
$$k = dim(S)$$

1. k: max number of linear indep. vectors in S
2. k: min number of vectors that can span S

Rank

For a matrix $A \in \mathbb{R}^{m \times n}$
$$dim(R(A)) \leq min\{m, n\}$$
Rank of a matrix (?)
$$\begin{aligned} rank(A) &= dim(R(A)) \\ &= \text{max number of linearly indep.} \\ &\quad \text{columns(rows) of } A \\ &= rank(A^T) \end{aligned}$$
And

$$Null(A) = (R(A^T))_\perp$$

Nonsingularity/Invertibility

A symmetric matrix $A \in \mathbb{R}^{m \times m}$ is non-singular if

$$Ax = 0 \implies x = 0$$

$\implies$ columns of A are linearly indep.
$\implies \exists A^{-1} \in \mathbb{R}^{m \times m}$ that $AA^{-1} = A^{-1}A = I_m$
Some properties

1. $(AB)^{-1} = B^{-1}A^{-1}$
2. $(A^{-1})^T = (A^T)^{-1}$

Determinant

For a matrix $A \in \mathbb{R}^{m \times m}$,
Inductive

1. If $m = 1$, $det(A) = A$
2. If $m > 1$, Define: $A_{ij} \in \mathbb{R}^{(m-1) \times (m-1)}$ is a submatrix that removes ith row and jth column of A.
   Define: $c_i j = (-1)^{i+j} det(A_{ij})$.
   $$\begin{aligned} det(A) &= \sum_{j=1}^{m} a_{ij} c_{ij} \quad \forall i = 1, ..., m \\ &= \sum_{i=1}^{m} a_{ij} c_{ij} \quad \forall j = 1, ..., m \end{aligned}$$

For $A, B \in \mathbb{R}^{m \times m}$

- $det(AB) = det(A)det(B)$
- $det(A) = det(A^T)$
- $det(\alpha A) = \alpha^m det(A)$
- $det(A) = 0 \iff A$ is singular
- $det(A^{-1}) = 1/det(A)$ if A is invertible

Vector Norm

A mapping $f : \mathbb{R}^> \to \mathbb{R}$ is a norm if it satisfies that $\forall x, y \in \mathbb{R}^m$

- $f(x) \geq 0$
- $f(x) = 0$ *iff.* $x = 0$
- $f(x + y) \leq f(x) + f(y)$
- $f(\alpha x) = |\alpha| f(x)$

Examples

- $\|x\|_2 = \sqrt{\sum_{i=1}^{m} |x_i|^2}$ (Euclidean norm)
- $\|x\|_1 = \sum_{i=1}^{m} |x_i|$
- $\|x\|_\infty = \max_{i=1,...,m} |x_i|$
- $\|x\|_p = \sqrt[p]{\sum_{i=1}^{m} |x_i|^p}$

Inner Product

$$< x, y >= x^T y$$

$$< x, x >= \|x\|_2^2$$

Cauchy-Schwarz Inequality

$$|x^T y| \leq \|x\|_2 \cdot \|y\|_2$$

Holder Inequality

$$|x^T y| \leq \|x\|_p \cdot \|y\|_q$$

where

$$1/p + 1/q = 1 \quad , \quad p \cdot q \geq 1$$

Projection onto a Subspace

Projection

Given a set $S \subseteq \mathbb{R}^m$ and a point $y \in \mathbb{R}^m$, the projection of y onto S is

$$y_S = \arg\min_{z \in S} \|z - y\|_2^2$$

Theorem 1
Suppose S is a subspace,

1. y_S exists and is unique
2. Let $y_S = \Pi_S(y)$ iff.
   $z^T(y_S - y) = 0 \ \forall z \in S$

Orthogonal complement

$S_\perp = \{y \in \mathbb{R}^m | y^T x = 0, \forall x \in S\}$ is the orthogonal complement of subspace S.

- Orthogonal complement is a subspace
- $S \cap S_\perp$

Theorem 2
Let $S \subseteq \mathbb{R}^m$ be a subspace. For any $y \in \mathbb{R}^m$

$$y = \Pi_S(y) + \Pi_{S_\perp}(y)$$

Proof

$$y = \Pi_S(y) + \Pi_{S_\perp}(y)$$

$$\begin{aligned}
\iff & \quad y - \Pi_S(y) = \Pi_{S_\perp}(y) \\
\iff & \quad z^T(y - y + \Pi_S(y)) = 0 \quad \forall z \in S_\perp \\
\iff & \quad z^T \Pi_S(y) = 0 \quad \forall z \in S_\perp
\end{aligned}$$

$$\because \quad \Pi_S(y)) \in S$$

$$\therefore z^T \Pi_S(y)) = 0 \quad \forall z \in S_\perp$$

$\square$

Sum of subspaces

For two sets $X, Y \subseteq \mathbb{R}^m$,

$$X + Y = \{x + y \in \mathbb{R}^m\}$$

For a subspace $S \subseteq \mathbb{R}^m$

1. $S + S_\perp = \mathbb{R}^m$
2. $dim(S) + dim(S_\perp) = m$
3. $(S_\perp)_\perp = S$

Proof

1. Comes from Theorem 2
2. Let $\{a_1, a_2, ..., a_k\}$ be a basis for S $(dim(S) = k)$ and $\{b_1, b_2, ..., b_\ell\}$ be a basis for $S_\perp$ $(dim(S_\perp) = \ell)$.
   $\{a_1, a_2, ..., a_k\} \cup \{b_1, b_2, ..., b_\ell\}$ is
   (a) linear independent
   (b) Span $S + S_\perp = \mathbb{R}^m$
   Therefore, $dim(S) + dim(S_\perp) = m$
3. Obvious

$\square$

Orthogonal Set

$\{a_1, a_2, ..., a_n\}$ is orthogonal/orthonormal set if

1. $\|a_i\|_2 = 1 \quad \forall i = 1, ..., n$
2. $a_i^T a_j = 0 \quad \forall i \neq j$

The orthogonal set is linear independent.

$$y = \sum_{i=1}^{n} \alpha_i a_i \implies \alpha_i = a_i^T y = \alpha \|a_i\|_2^2$$

Given a linear independent set $\{a_1, a_2, ..., a_n\}$. The procedure of finding an orthogonal set $\{q_1, q_2, ..., 1_n\}$ so that $span\{a_1, a_2, ..., a_n\} = span\{q_1, q_2, ..., q_n\}$ is called "Gram-Schimit" procedure.

Orthogonal/Unitary Matrix

$Q = [q_1, q_2, ..., q_m] \in \mathbb{R}^{n \times m}$, $q_i, i = 1, ..., m$ are orthogonal.

- If $m = n$,
  Q is orthogonal matrix $\implies Q^T Q = I_m = QQ^T$.
- If $n > m$,
  semi-orthogonal $\implies Q^T Q = I_m \neq QQ^T$.

$Q = [q_1, q_2, ..., q_m] \in \mathbb{C}^{n \times m}$, $q_i, i = 1, ..., m$ are orthogonal.

- If $m = n$,
  Q is unitary matrix $\implies Q^H Q = I_m = QQ^H$.
- If $n > m$,
  semi-unitary $\implies Q^H Q = I_m \neq QQ^H$.

If $Q \in \mathbb{R}^{n \times m}$ $(n > m)$ is semi-orthogonal, $\exists \tilde{Q} \in \mathbb{R}^{n \times (n-m)}$ so that $\begin{bmatrix} Q & \tilde{Q} \end{bmatrix}$ is orthogonal.

Dimensional Theorem

For a matrix $A \in \mathbb{R}^{m \times n}$, it holds

$$dim(R(A)) + dim(Null(A)) = n$$

Proof

$$dim(R(A^T)) + dim(R(A^T)_\perp) = n$$
$$dim(R(A^T)) = rank(A^T) = rank(A) = dim(R(A))$$
$$dim(R(A^T)_\perp) = Null(A) = n$$

$\square$

Complexity of Matrix Computation

Flops

Arithmetic operations including addition, substraction, multiplication, division.

Big O notation

Big O notation: "order" of the complexity.

$$f(n) = O(g(n))$$

for some $g(n)$ if $\exists$ a constant $C > 0$ and n_0 so that for $n > n_0$

$$|f(n)| \leq C \cdot |g(n)|$$

Examples: for $x \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times p}$

$x + y$	m flops	$O(m)$
$x^T y$	$2m - 1$ flops	$O(m)$
Ax	$n \cdot (2m - 1)$ flops	$O(nm)$
AB	$p \cdot n \cdot (2m - 1)$ flops	$O(pnm)$

Lecture 3. Least Square Problem (LSP)

Formulation

LS problem

Given $y \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$

$$\min_{x \in \mathbb{R}^n} \|y - Ax\|_2^2$$

Linear representation of signals

$$y = Ax$$

In practice,

$$y = Ax + v$$

where

- y is the observation (Given)
- A is the model (Given)
- x is the parameter to be estimated/determined
- v is the noice follow multi-variate Gaussian distribution

$$\mathrm{pdf}(v) \propto e^{-\|v\|^2}$$

$\implies$ Maximum likelihood estimator

$$\max_x P(y|x) \propto \min_x \|y - Ax\|_2^2$$

AR and Polynomial Models

Autoregressive (AR) model

Given a time series y_t, $t = 0, 1, 2, ...$,

$$\underbrace{y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \cdots + \alpha_p y_{t-p}}_{\text{AR model}} + \underbrace{v_t}_{\text{noise}}$$

If $\alpha_1, \alpha_2, ..., \alpha_p$ are known, estimate y_t by

$$\hat{y}_t = \underbrace{\alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \cdots + \alpha_p y_{t-p}}_{\text{historical observation}}$$

Collect

$$\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_T \end{bmatrix}}_{y \in \mathbb{R}^T} = \underbrace{\begin{bmatrix} y_0 & & & \\ y_1 & y_0 & & \\ \vdots & \ddots & \ddots & \\ y_{p-1} & \cdots & \cdots & y_0 \\ \vdots & \ddots & \ddots & \vdots \\ y_{T-1} & \cdots & \cdots & y_T \end{bmatrix}}_{A \in \mathbb{R}^{T \times p}} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix}}_{x \in \mathbb{R}^p} + \underbrace{\begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_T \end{bmatrix}}_{v \in \mathbb{R}^T}$$

$\implies$ Solve LS problem to obtain α_i, $i = 1, ..., p$

Polynomial model

$$\underbrace{y_t = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_p t^p}_{\text{polynomial model}} + \underbrace{v_t}_{\text{noise}}$$

Collect

$$\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_T \end{bmatrix}}_{y \in \mathbb{R}^T} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ & \vdots & & \\ 1 & t & \cdots & t^p \\ & \vdots & & \\ 1 & T-1 & \cdots & (T-1)^p \end{bmatrix}}_{A \in \mathbb{R}^{T \times p}} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix}}_{x \in \mathbb{R}^p} + \underbrace{\begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_T \end{bmatrix}}_{v \in \mathbb{R}^T}$$

$\implies$ Solve LS problem to obtain α_i, $i = 1, ..., p$

Basis Representation

Basis matrix

For $y \in \mathbb{R}^m$, given a set of basis vectors (can be given or learned)

$$\Phi = \{\phi_1, \phi_2, ..., \phi_n\} \in \mathbb{R}^{m \times n}$$

We model

$$y = \sum_{i=1}^{n} \phi_i x_i = \Phi x$$

- $m \gg n$, assume data lies in a low-dimensional subspace
- $m \ll n$, dictionary $\implies x$ is sparse

Fourier basis

Discrete Fourier Transform (DFT)
Inverse Discrete Fourier Transform (IDFT)

$$\Phi = [\phi_1, ..., \phi_n] \in \mathbb{C}^{n \times n}$$

$$\phi_i = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ e^{j \frac{2\pi}{n}(i-1)} \\ e^{j \frac{2\pi}{n}(i-1) \times 2} \\ \vdots \\ e^{j \frac{2\pi}{n}(i-1) \times n-1} \end{bmatrix} \in \mathbb{R}^n$$

- Φ is IDFT matrix, Φ^H is DFT matrix.
- Φ is unitary matrix.

LTI System

Lienar time invariant system

- Linearity
$$\begin{cases} x_t^1 \to \boxed{\text{LTI}} \to y_t^1 \\ x_t^2 \to \boxed{\text{LTI}} \to y_t^2 \\ \alpha x_t^1 + \beta x_t^2 \to \boxed{\text{LTI}} \to \alpha y_t^1 + \beta y_t^2 \end{cases}$$

- Time Invariant $\quad x_{t-d} \to \boxed{\text{LTI}} \to y_{t-d}$

Impulse $\delta(t)$
$$\begin{cases} \delta(t) = 0, t \neq 0 \\ \int_{-\infty}^{+\infty} \delta(t)\, dt \end{cases}$$

LTI system is characterized by an impulse response
$$h_t, \quad t = 0, ..., p-1$$

$\implies x_t$ and y_t is related through "convolution"
$$\begin{aligned} y_t &= \sum_{i=1}^{p} h_i x_{t-i} \\ &= h_t * x_t \end{aligned}$$

$$x_t \to \overbrace{\boxed{h_t}}^{\text{LTI}} \to y_t$$

System identification

Given $\{y_t\}_{t=0}^{T-1}$ and $\{x_t\}_{t=0}^{T-1}$, find the impulse response $\{h_t\}_{t=0}^{p}$

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_p \\ \vdots \\ y_T \end{bmatrix} = \begin{bmatrix} x_0 & & & \\ x_1 & x_0 & & \\ \vdots & \ddots & \ddots & \\ x_p & \cdots & \cdots & x_0 \\ \vdots & \ddots & \ddots & \vdots \\ x_{T-1} & \cdots & \cdots & x_{T-p} \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_p \end{bmatrix} + \text{noise}$$

$\implies$ Solve LS problem to obtain $\{h_t\}_{t=0}^{p}$

Deconvolution

Given $\{y_t\}_{t=0}^{T-1}$ and $h_{t=0}^{p}$, find out $\{x_t\}_{t=0}^{T-1}$
Put them into a linear system of equations

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_p \\ \vdots \\ y_T \end{bmatrix} = \underbrace{\begin{bmatrix} h_0 & 0 & & \cdots & & 0 \\ h_1 & h_0 & 0 & \cdots & & \ddots \\ \vdots & \vdots & & \ddots & & \vdots \\ h_p & h_{p-1} & & \cdots & h_1 & h_0 \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & \cdots & h_p & h_{p-1} & \cdots & h_1 & h_0 \end{bmatrix}}_{A \in \mathbb{R}^{T \times T}} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_p \\ \vdots \\ x_T \end{bmatrix} + \text{noise}$$

$\implies$ Solve LS problem to obtain $\{x_t\}_{t=0}^{T-1}$

Toeplitz matrix

$$A = \begin{bmatrix} a_0 & a_{-1} & \cdots & \cdots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-n+2} \\ a_2 & a_1 & a_0 & a_{-1} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-1} & \cdots & a_2 & a_1 & a_0 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Circulant Matrix

Circulant matrix is a special case of Toeplitz matrix

$$H = \begin{bmatrix} h_0 & h_{n-1} & \cdots & \cdots & h_1 \\ h_1 & h_0 & h_{n-1} & \cdots & h_2 \\ h_2 & h_1 & h_0 & h_{n-1} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ h_{n-1} & \cdots & h_2 & h_1 & h_0 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$H \overset{O(n \log n)}{\Longleftrightarrow} H^{-1}$$

The key property:
$$H \cdot \underbrace{\phi_i}_{\text{IDFT}} = d_i \cdot \phi_i \quad , d_i \in \mathbb{R}$$

Let
$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

Then
$$\begin{aligned} H \cdot \Phi &= H \cdot [\phi_1, ..., \phi_n] \\ &= [d_1 \phi_1, ..., d_n \phi_n] \\ &= \Phi \cdot D \end{aligned}$$

$\implies$

$$\underbrace{H = \Phi \cdot D \cdot \Phi^H}_{\text{eigen-decomposition of } H} \iff \Phi^H H \Phi = D$$

Lecture 4. Eigenvalues and Eigenvectors

Definition (Eigenvalues and Eigenvector)
Let $A \in \mathbb{R}^{m \times m}$, if a scalar $\lambda \in \mathbb{C}$ and a nonzero vector $v \in \mathbb{C}^m$ satisfy the equation

$$Av = \lambda v$$

then λ is called the eigenvalue of A and v is called the eigenvector of A associated with λ.

$\Longrightarrow$

$$\underbrace{(A - \lambda I)}_{\text{singular}} v = 0$$

$P_A(\lambda) \triangleq det(A - \lambda I) = \prod_{i=1}^{m}(\lambda - \lambda_i)$ is a polynomial of λ with degree m. The equation $P_A(\lambda) \triangleq det(A - \lambda I) = 0$ is called the characteristic euqation of A.

For each λ_i find v_i such that
$$(A - \lambda_i I)v_i = 0$$
$v_i \in Null(A - \lambda_i I)$ and $Null(A - \lambda_i I)$ is the eigenspace for λ_i

- Even if $A \in R^{m \times m}$ is real-valued, λ_i and v_i could be complex. *e.g.*
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies \lambda = \pm j$$
- But if $\lambda_i \in \mathbb{R}$ is real-valued, $v_i \in \mathbb{R}^m$ should be in $\mathbb{R}^m$.

Diagonalizability

Similar matrices

Definition (Similar matrices)
$A \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{R}^{m \times m}$ are similar to each other if there exists a nonsingular $S \in \mathbb{R}^{m \times m}$ such that
$$B = SAS^{-1}$$

If A and B are similar to each other, then they have the same eigenvalues.
Proof
$$\begin{aligned} P_B(\lambda) &= det(B - \lambda I) \\ &= det(SAS^{-1} - \lambda I) \\ &= det(S(A - \lambda I)S^{-1}) \\ &= det(S)det(A - \lambda I)det(S^{-1}) \\ &= det(A - \lambda I) \\ &= P_A(\lambda) \end{aligned}$$
$\square$

Diagonalizable matrix

Definition (Diagonalizable)
A matrix $A \in \mathbb{R}^{m \times m}$ is diagonalizable if A is similar to a diagonal matrix.

$\Longrightarrow$ (**Eigenvalue decomposition**)
There exists an invertible matrix $V \in \mathbb{R}^{m \times m}$ and a diagonal matrix $D \in \mathbb{R}^{m \times m}$ s.t.
$$\underbrace{A = VDV^{-1}}_{\text{eigenvalue decomposition (EVD)}}$$

$\Longrightarrow$

$$AV = VD$$
$$V = [v_1, v_2, ..., v_m] \quad , \quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}$$

$\Longleftrightarrow$

$$Av_i = \lambda_i v_i, \quad \forall i = 1, ..., m$$

$\Longrightarrow$

A is diagonalizable *iff.* there is a set of linear indep. vectors which are engenvectors of A. **Properties**
If a matrix $A \in \mathbb{R}^{m \times m}$ is diagonalizable

1. $det(A) = \prod_{i=1}^{m} \lambda_i$
2. $Tr(A) = \sum_{i=1}^{m} \lambda_i$
3. $rank(A) =$ the number of non-zero eigenvalues

Proof

1. $det(A) = det(VDV^{-1}) = det(V)det(D)det(V^{-1}) = det(D) = \prod_{i=1}^{m} \lambda_i$
2. $Tr(A) = Tr(VDV^{-1}) = Tr(DV^{-1}V) = Tr(D) = \sum_{i=1}^{m} \lambda_i$

$\square$

When Diagonalizable

Distinct eigenvalues

Let $\{\lambda_1, \lambda_2, ..., \lambda_k\}_{k \in M}$ $(k \geq 2)$ be a subset of eigenvalues of A and $\lambda_i \neq \lambda_j$ $\forall i \neq j$. Then the corresponding eigenvectors, denoted with $v_1, ..., v_k$ respectively, are linearly independent.

If A has $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_k$, then the corresponding eigenvectors $v_1, ..., v_k$ are linear independent.
Proof

Suppose $v_1, ..., v_k$ are linearly **dependent**. Then there exists $\alpha_1, \alpha_2, ..., \alpha_k$ not all zero (without losing generality, assume $\alpha_1 \neq 0$) s.t.

$$\sum_{i=1}^{k} \alpha_i v_i = 0$$

Then
$$A \cdot \left(\sum_{i=1}^{k} \alpha_i v_i \right) = \sum_{i=1}^{k} \alpha_i Av_i = \sum_{i=1}^{k} \alpha_i \lambda_i v_i = 0$$

$$\sum_{i=1}^{k-1} \alpha_i \lambda_i v_i + \alpha_k \lambda_k v_k = 0 \qquad (1)$$

On the other hand,

$$\lambda_k \cdot \left(\sum_{i=1}^{k} \alpha_i v_i \right) = \sum_{i=1}^{k} \alpha_i \lambda_k v_i = 0$$

$$\sum_{i=1}^{k-1} \alpha_i \lambda_k v_i + \alpha_k \lambda_k v_k = 0 \qquad (2)$$

$(1) - (2)$:

$$\sum_{i=1}^{k-1} \alpha_i (\lambda_i - \lambda_k) v_i = 0$$

Then

$$A \cdot \left(\sum_{i=1}^{k-1} \alpha_i (\lambda_i - \lambda_k) v_i \right) = \sum_{i=1}^{k-1} \alpha_i (\lambda_i - \lambda_k) A v_i$$
$$= \sum_{i=1}^{k-1} \alpha_i (\lambda_i - \lambda_k) \lambda_i v_i$$
$$= 0$$

$$\sum_{i=1}^{k-2} \alpha_i (\lambda_i - \lambda_k) \lambda_i v_i + \alpha_{k-1} (\lambda_{k-1} 0 \lambda_k) \lambda_{k-1} v_{k-1} = 0 \quad (3)$$

On the other hand

$$\lambda_{k-1} \cdot \left(\sum_{i=1}^{k-1} \alpha_i (\lambda_i - \lambda_k) v_i \right) = \sum_{i=1}^{k-1} \alpha_i (\lambda_i - \lambda_k) \lambda_{k-1} v_i = 0$$

$$\sum_{i=1}^{k-2} \alpha_i (\lambda_i - \lambda_k) \lambda_{k-1} v_i + \alpha_{k-1} (\lambda_{k-1} 0 \lambda_k) \lambda_{k-1} v_{k-1} = 0$$
$$(4)$$

$(3) - (4)$:

$$\sum_{i=1}^{k-2} \alpha_i (\lambda_i - \lambda_k)(\lambda_i - \lambda_{k-1}) v_i = 0$$

Repeat until $i = 1$

$$\alpha_1 \left(\prod_{i=1}^{k} (\lambda_1 - \lambda_i) \right) v_1 = 0$$

Since $\alpha_1 \neq 0$ and $\prod_{i=1}^{k} (\lambda_1 - \lambda_i) \neq 0$, so $v_1 = 0$ (Contradict!)

Therefore, $v_1, ..., v_k$ are linearly **independent** □

If $A \in \mathbb{R}^{m \times m}$ has m distinct eigenvalues, then A is diagonalizable.

Repeated eigenvalues

Let λ be an eigenvalue of A repeated r times. Define

- Algebra multiplicity: r.
- Geometric multiplicity: $s = dim(Null(A - \lambda I))$

Then

$$s \leq r$$

Proof
Let $dim(Null(A - \bar{\lambda} I)) = s$ and $\{q_1, ..., q_s\}$ be an orthogonal basis for $Null(A - \bar{\lambda} I)$.
Then $Q_1 = [q_1, ..., q_s] \in \mathbb{R}^{m \times s}$ is semi-unitary and $\exists Q_2 \in \mathbb{R}^{m \times (m-s)}$ s.t. $Q \triangleq [Q_1, Q_2]$ is unitary.
Consider

$$Q^H A Q = \begin{bmatrix} Q_1^H A Q_1 & Q_1^H A Q_2 \\ Q_2^H A Q_1 & Q_2^H A Q_2 \end{bmatrix} = \begin{bmatrix} \bar{\lambda} I_s & Q_1^H A Q_2 \\ 0 & Q_2^H A Q_2 \end{bmatrix}$$

Then

$$P_A(\lambda) = P_{Q^H A Q}(\lambda)$$
$$= det(Q^H A Q - \lambda I)$$
$$= det \left(\begin{bmatrix} (\bar{\lambda} - \lambda) I_s & Q_1^H A Q_2 \\ 0 & Q_2^H A Q_2 - \lambda I_{m-s} \end{bmatrix} \right)$$
$$= \underbrace{(\bar{\lambda} - \lambda)^s}_{\bar{\lambda} \text{ repeats } s \text{ times}} det(Q_2^H A Q_2 - \lambda I_{m-s})$$
$$= 0$$

So

$$r \geq s$$
□

If $s = r$, A is diagonalizable.

Symmetric/Hermitian Matrices

- **Symmetric** $A = A^T$ $A \in \mathbb{R}^{m \times n}$

- **Hermitian** $A = A^H$ $A \in \mathbb{C}^{m \times n}$

Property 1: The eigenvalues of a Hermitian matrix $A \in \mathbb{C}^{m \times n}$ are real-valued.
Proof
Let λ be the eigenvalue of A, v be the eigenvector of A and $\|v\|_2 = 1$.
$\implies Av = \lambda v$
$\implies v^H A v = \lambda v^H v = \lambda$
$\overset{\text{take} H}{\implies} v^H A^H v = \lambda^*$
Since A is Hermitian, $A = A^H$ and hence $\lambda = \lambda^*$
So λ is real-valued □

Property 2: If $A \in \mathbb{C}^{m \times n}$ is Hermitian and that

- $\lambda_i \neq \lambda_j$ be two eigenvalues,
- v_i, v_j be the corresponding eigenvectors.

Then the eigenvectors are mutually orthogonal.
Proof
$$\begin{cases} v_i^H A v_j = v_i^H (\lambda_j v_j) = \lambda_j v_i^H v_j \\ v_j^H A v_i = v_j^H (\lambda_i v_i) = \lambda_i v_j^H v_i \overset{\text{take} H}{\implies} v_i^H A^H v_j = \lambda_i v_i^H v_j \end{cases}$$
Since $v_i^H A^H v_j = v_i^H A v_j$, so we have
$$(\lambda_i - \lambda_j) v_i^H v_j = 0 \implies v_i^H v_j = 0$$
□

Unitarily diagonalizable

If Hermitian/symmetric A is diagonalizable

$$A = VDV^{-1}$$

where

- V contains orthogonal eigenvectors and is unitary matrix,
- D real-valued diagonal eigenvalues.

Therefore, $VV^H = I$ and $V^{-1} = V^H$. Thus,

$$A = VDV^H$$

And we say A is unitarily diagonalizable.

Schur triangularization

Any $A \in \mathbb{C}^{m \times m}$ can be decomposed as an upper triangular matrix.

> **Theorem 3 (Schur triangularization)**
> Given $A \in \mathbb{C}^{m \times m}$ with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_m \in \mathbb{C}$ there exists a unitary matrix $U \in \mathbb{C}^{m \times m}$ such that
> $$U^H A U = T \iff A = UTU^H$$
> where T is an upper triangular matrix with $[T]_{ii} = \lambda_i$, $i = 1, 2, ..., m$

If $A \in \mathbb{C}^{m \times m}$, by Schur triangularization, there exists unitary $U \in \mathbb{C}^{m \times m}$ s.t.

$$U^H A U = T$$

Then by taking H, we have

$$\underset{(=U^H A U)}{U^H A^H U} = T^H$$

$\implies$

$$T = T^H$$

Since T is an upper triangular matrix, so T^H is a lower triangular matrix. Therefore, T is diagonal. $\implies$

$$U^H A U = T$$

Thus, A is diagonalizable.

Proof   for Schur Triangularization (**Theorem 3**)
This proof is also an algorithm.
Let $v_1 \in \mathbb{R}^m$ be the eigenvalue of A associated with λ_1, i.e., $A v_1 = \lambda_1 v_1$. Without losing generality, we can assume $v_1^H v_1 = 1$ and $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m$

Construct a unitary matrix U_1

$$U_1 = \begin{bmatrix} v_1 & V_1 \end{bmatrix} \in \mathbb{C}^{m \times m}$$

where $V_1 \in \mathbb{C}^{m \times (m-1)}$. U_1 being unitary indicates

$$v_1^H V_1 = \overrightarrow{0}^T$$

Then we have

$$U_1^H A U_1 = \begin{bmatrix} v_1^H \\ V_1^H \end{bmatrix} A \begin{bmatrix} v_1 & V_1 \end{bmatrix} = \begin{bmatrix} v_1^H A v_1 & v_1^H A V_1 \\ V_1^H A v_1 & V_1^H A V_1 \end{bmatrix}$$

where

$$\begin{cases} v_1^H A v_1 = \lambda_1 \\ v_1^H A V_1 \neq \overrightarrow{0} \\ V_1^H A v_1 = \overrightarrow{0} \end{cases}$$

So,

$$U_1^H A U_1 = \begin{bmatrix} \lambda_1 & \times \\ 0 & V_1^H A V_1 \end{bmatrix}$$

Define

$$A_1 \triangleq V_1^H A V_1 \in \mathbb{C}^{(m-1) \times (m-1)}$$

Then

$$\begin{aligned} P_A(\lambda) &= det(A - \lambda I) \\ &= det(U_1^H A U_1 - \lambda I)^\dagger \\ &= det\left(\begin{bmatrix} \lambda_1 & \times \\ 0 & A_1 \end{bmatrix} - \lambda I \right) \\ &= det\left(\begin{bmatrix} \lambda_1 - \lambda & \times \\ 0 & A_1 - \lambda I \end{bmatrix} \right) \\ &= (\lambda_1 - \lambda) \underbrace{det(A_1 - \lambda I)}_{=P_{A_1}(\lambda)} \end{aligned}$$

$\dagger$: $A = U_1(U_1^H A U_1)U_1^{-1}$ so A is similar to $U_1^H A U_1$ and hence A and $U_1^H A U_1$ have same eigenvalues.

Therefore, A_1 has eigenvalues $\lambda_2, \lambda_3, ..., \lambda_m$
Let $v_2 \in \mathbb{C}^{m-1}$ be the eigenvector of A_1 associated with λ_2.
Construct a unitary matrix U_2
$$U_2 = \begin{bmatrix} v_2 & V_2 \end{bmatrix} \in \mathbb{C}^{(m-1) \times (m-1)}$$
where $V_2 \in \mathbb{C}^{(m-1) \times (m-2)}$. U_2 being unitary indicates
$$v_2^H V_2 = \overrightarrow{0}^T$$

$\implies$

$$U_2^H A_1 U_2 = \begin{bmatrix} \lambda_2 & \times \\ 0 & V_2^H A_1 V_2 \end{bmatrix}$$

Let $\tilde{U}_2 = \begin{bmatrix} 1 & 0^T \\ 0 & U_2 \end{bmatrix}$, then $\tilde{U}_2$ is unitary.

So,

$$\begin{aligned} \tilde{U}_2^H [U_1^H A U_1] \tilde{U}_2 &= \tilde{U}_2^H \begin{bmatrix} \lambda_1 & \times \\ 0 & A_1 \end{bmatrix} \tilde{U}_2 \\ &= \begin{bmatrix} \lambda_1 & \times \\ 0 & U_2^H A_1 U_2 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \times \\ 0 & \begin{bmatrix} \lambda_2 & \times \\ 0 & V_2^H A_1 V_2 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \times & \times \\ 0 & \lambda_2 & \times \\ 0 & 0 & V_2^H A_1 V_2 \end{bmatrix} \end{aligned}$$

Repeat this operations and we can finally get
$$\underbrace{\tilde{U}_m^H \cdots \tilde{U}_3^H \tilde{U}_2^H (U_1 A U_1) \tilde{U}_2 \tilde{U}_3 \cdots \tilde{U}_m}_{\triangleq U \text{ unitary}}$$

$$= \begin{bmatrix} \lambda_1 & \times & \times & \times \\ & \lambda_2 & \times & \times \\ & & \ddots & \times \\ & & & \lambda_m \end{bmatrix} \triangleq T$$

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Therefore,
$$U^H A U = T$$
$\square$

Variational Characterization of Eigenvalues

Rayleigh quotient

> **Definition (Rayleigh quotient)**
>
> $$R(A,x) = \frac{x^H A x}{\|x\|_2^2} = \left(\frac{x}{\|x\|_2}\right)^H A \left(\frac{x}{\|x\|_2}\right)$$

> **Theorem 4 (Rayleigh-Ritz theorem)**
> Let $A \in \mathbb{C}^{m \times m}$ be Hermitian, then
>
> $$\begin{aligned}
\lambda_{\max}(A) &= \max_{x \in \mathbb{C}^m} \frac{x^H A x}{\|x\|_2^2} \\
&= \max_{x \in \mathbb{C}^m} x^H A x \quad (\|x\|_2 = 1) \\
\lambda_{\min}(A) &= \min_{x \in \mathbb{C}^m} \frac{x^H A x}{\|x\|_2^2}
\end{aligned}$$
>
> and
>
> $$\lambda_{\min}(A) \cdot \|x\|_2^2 \le x^H A x \le \lambda_{\max}(A) \cdot \|x\|_2^2$$

Proof
According to the EVD of A,
$$A = V \Lambda V^H = \sum_{i=1}^{m} (\lambda_i \cdot v_i v_i^H)$$
$\implies$
$$\begin{aligned}
x^H A x &= x^H V \Lambda V^H x \\
&= \sum_{i=1}^{m} (\lambda_i \cdot x^H v_i v_i^H x) \\
&= \sum_{i=1}^{m} (\lambda_i \cdot |v_i^H x|^2) \\
&\le \lambda_{\max}(A) \cdot \sum_{i=1}^{m} |v_i^H x|^2
\end{aligned}$$
Since
$$\sum_{i=1}^{m} |v_i^H x|^2 = x^H V V^H x = \|x\|_2^2$$
So,
$$x^H A x \le \lambda_{\max}(A) \cdot \|x\|_2^2$$
$\implies$
$$\frac{x^H A x}{\|x\|_2^2} \le \lambda_{(\max)}(A)$$
The equality holds only when $x = v_1$, which is an eigenvector associated with $\lambda_{\max}(A)$ and is called the principal eigenvector. $\square$

Courant-Fischer theorem

> **Theorem 5 (Courant-Fischer theorem)**
> Let $A \in \mathbb{C}^{m \times m}$ be Hermitian and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m$ are ordered eigenvalues. Then
>
> $$\begin{aligned}
\lambda_k &= \min_{\omega_1, \omega_2, \ldots, \omega_{k-1}} \left\{ \max_{x \in \mathbb{C}^m, x \perp \omega_1, \omega_2, \ldots, \omega_{k-1}} \frac{x^H A x}{\|x\|_2^2} \right\} \\
&= \max_{\omega_{k+1}, \omega_{k+2}, \ldots, \omega_m} \left\{ \min_{x \in \mathbb{C}^m, x \perp \omega_{k+1}, \omega_{k+2}, \ldots, \omega_m} \frac{x^H A x}{\|x\|_2^2} \right\}
\end{aligned}$$

Proof ideas:
1.
$$\max_{x \in \mathbb{C}^m, x \perp v_1, v_2, \ldots, v_{k-1}} \frac{x^H A x}{\|x\|_2^2} = \lambda_k$$
where v_k is the eigenvector associated with λ_k of A
2.
$$\max_{x \in \mathbb{C}^m, x \perp \omega_1, \omega_2, \ldots, \omega_{k-1}} \frac{x^H A x}{\|x\|_2^2} \ge \lambda_k$$
$1+2 \implies$
$$\min_{\omega_1, \omega_2, \ldots, \omega_{k-1}} \left\{ \max_{x \in \mathbb{C}^m, x \perp \omega_1, \omega_2, \ldots, \omega_{k-1}} \frac{x^H A x}{\|x\|_2^2} \right\} = \lambda_k$$
attained when $\omega_i = v_i$ $(i = 1, \ldots, k-1)$

Proof
1.
Since $A = V \Lambda V^H = \sum_{i=1}^{m} \lambda_i v_i v_i^H$,

$$\begin{aligned}
&\max_{x \in \mathbb{C}^m, x \perp v_1, v_2, \ldots, v_{k-1}} \frac{x^H A x}{\|x\|_2^2} \\
&= \max_{x \in \mathbb{C}^m, x \perp v_1, v_2, \ldots, v_{k-1}} \frac{\sum_{i=1}^{m} \lambda_i |v_i^H x|^2}{\|x\|_2^2} \\
&= \max_{x \in \mathbb{C}^m, x \perp v_1, v_2, \ldots, v_{k-1}} \frac{\sum_{i=k}^{m} \lambda_i |v_i^H x|^2}{\|x\|_2^2} \\
&\le \max_{x \in \mathbb{C}^m} \frac{\sum_{i=k}^{m} \lambda_i |v_i^H x|^2}{\|x\|_2^2} \quad \text{(Relax constraint)} \\
&\le \max_{x \in \mathbb{C}^m} \frac{\lambda_k \sum_{i=k}^{m} |v_i^H x|^2}{\|x\|_2^2} {}^{\dagger} \\
&\le \lambda_k \quad \text{(equality attained if } x = v_k)
\end{aligned}$$

†: $\sum_{i=k}^{m} |v_i^H x|^2 = x^H V V^H x = \|x\|_2^2$.

Therefore,
$$\max_{x \in \mathbb{C}^m, x \perp v_1, v_2, \ldots, v_{k-1}} \frac{x^H A x}{\|x\|_2^2} = \lambda_k$$

2.
Let $y = V^H x$
$$\begin{aligned}
&\max_{x \perp \omega_1, \ldots, \omega_{k-1}} \frac{x^H A x}{\|x\|_2^2} \\
&= \max_{x \perp \omega_1, \ldots, \omega_{k-1}} \frac{x^H V \Lambda V^H x}{\|x\|_2^2} \\
&= \max_{y \perp V^H \omega_1, \ldots, V^H \omega_{k-1}} \frac{y^H \Lambda y}{\|y\|_2^2} \\
&\ge \max_{y \perp V^H \omega_1, \ldots, V^H \omega_{k-1}, y_{k+1} = y_{k+2} = \cdots = y_m = 0, \|y\|_2 = 1} y^H \Lambda y \\
&= \max_{y \perp V^H \omega_1, \ldots, V^H \omega_{k-1}, |y_1|^2 + |y_2|^2 + \cdots + |y_k|^2 = 1} \sum_{i=1}^{k} \lambda_i |y_i|^2 \\
&\ge \lambda_k
\end{aligned}$$

$\square$

Weyl's inequality

For $A, B \in \mathbb{C}^{m \times m}$ be two Hermitian matrices. Then

1. $\lambda_k(A) + \lambda_m(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda_1(B)$
2. $\lambda_{k+1}(A) \leq \lambda_k(X \pm zz^H) \leq \lambda_{k-1}(A)$
3. $\lambda_{k+r}(A) \leq \lambda_k(A + B) \leq \lambda_{k-r}(A)$ where $rank(B) \leq r$
4. $\lambda_{j+k-1}(A + B) \leq \lambda_j(A) + \lambda_k(B) \leq \lambda_{j+k-m}(A + B)$
5. Let A_I be the principal submatrix of A associated with subset $I = \{i_1, i_2, ..., i_r\} \subseteq \{1, 2, ..., m\}$
$$\lambda_{k+m-r}(A) \leq \lambda_k(A_I) \leq \lambda_k(A)$$
6. Let $U \in \mathbb{C}^{m \times r]}$ be semiunitary
$$\lambda_{k+m-r} \leq \lambda_k(U^H A U) \leq \lambda_k(A)$$

Proof 1.

$$\lambda_k(A+B) = \min_{\omega_1,...,\omega_{k-1}} \left\{ \max_{x \in \mathbb{C}^m, x \perp \omega_1,...,\omega_{k-1}} \frac{x^H(A+B)x}{\|x\|_2^2} \right\}$$

where

$$\frac{x^H(A+B)x}{\|x\|_2^2} = \frac{x^H A x}{\|x\|_2^2} + \underbrace{\frac{x^H B x}{\|x\|_2^2}}_{\leq \lambda_1(B)}$$

So,

$$\lambda_k(A + B)$$
$$\leq \underbrace{\min_{\omega_1,...,\omega_{k-1}} \left\{ \max_{x \in \mathbb{C}^m, x \perp \omega_1,...,\omega_{k-1}} \frac{x^H A x}{\|x\|_2^2} \right\}}_{=\lambda_k(A)} + \lambda_1(B)$$

Therefore,

$$\lambda_k(A + B) \leq \lambda_k(A) + \lambda_1(B)$$

$\square$

Proof 2.

$$\lambda_k(A \pm zz^H)$$
$$= \min_{\omega_1,...,\omega_{k-1}} \left\{ \max_{x \in \mathbb{C}^m, x \perp \omega_1,...,\omega_{k-1}} \frac{x^H(A \pm zz^H)x}{\|x\|_2^2} \right\}$$
$$\geq \min_{\omega_1,...,\omega_{k-1}} \left\{ \max_{x \in \mathbb{C}^m, x \perp \omega_1,...,\omega_{k-1}, \boxed{z}} \frac{x^H(A \pm zz^H)x}{\|x\|_2^2} \right\}$$
$$= \min_{\omega_1,...,\omega_{k-1}, \omega_k = z} \left\{ \max_{x \in \mathbb{C}^m, x \perp \omega_1,...,\omega_{k-1}, \omega_k} \frac{x^H A x}{\|x\|_2^2} \right\}$$
$$\geq \min_{\omega_1,...,\omega_{k-1}, \omega_k} \left\{ \max_{x \in \mathbb{C}^m, x \perp \omega_1,...,\omega_{k-1}, \omega_k} \frac{x^H A x}{\|x\|_2^2} \right\}$$
$$= \lambda_{k+1}(A)$$

$\square$

Proof 3.

$$\because B = \sum_{i=1}^{r} \lambda_i(B) u_i u_i^H$$

$$\therefore A + B = A + \sum_{i=1}^{r-1} \lambda_i(B) u_i u_i^H + \lambda_r(B) u_r u_r^H$$

Define $B^{r-k} \triangleq \sum_{i=1}^{r-k} \lambda_i(B) u_i u_i^H$ $k = 0, 1, ..., r$
$$\lambda_{k+1}(A+B^{r-1}) \leq \lambda_k(A+B^{r-1}+\lambda_r(B)u_r u_r^H) \leq \lambda_{k-1}(A+B^{r-1})$$
Do again and we get
$$\lambda_{k+2}(A + B^{r-2}) \leq \lambda_{k+1}(A + B^{r-2} + \lambda_{r-1}(B)u_{r-1}u_{r-1}^H)$$
$$= \lambda_{k+1}(A + B^{r-1})$$
Repeat r times, we can get
$$\lambda_{k+r}(A) \leq \lambda_k(A + B) \leq \lambda_{k-r}(A)$$
where $rank(B) \leq r$ $\square$

Proof 4.
Define $A_{j-1} = \sum_{i=1}^{j-1} \lambda_i(A) v_i v_i^H$ (if $A = V \Lambda V^H$).
Define $B_{k-1} = \sum_{i=1}^{k-1} \lambda_i(B) u_i u_i^H$ (if $B = U \Lambda U^H$).
Then
$$\lambda_j(A) = \lambda_1(A - A_{j-1})$$
$$\lambda_k(B) = \lambda_1(B - B_{k-1})$$
$\implies$
$$\lambda_j(A) + \lambda_k(B) = \lambda_1(A - A_{j-1}) + \lambda_1(B - B_{k-1})$$
$$\geq \lambda_1(A + B - (A_{j-1} + B_{k-1}))$$
$$\geq \lambda_{j+k-1}(A + B)^\dagger$$
$\dagger$: $rank((A_{j-1} + B_{k-1}) \leq j + k - 2$ and from 3. $\square$

Proof 5.
$\lambda_k(A)$
$$= \min_{\omega_1,...,\omega_{k-1}} \left\{ \max_{x \perp \omega_1,...,\omega_{k-1}, \|x\|_2=1} x^H A x \right\}$$
$$\geq \min_{\omega_1,...,\omega_{k-1}} \left\{ \max_{x \perp \omega_1,...,\omega_{k-1}, \|x\|_2=1, x \in span\{e_{i_1},...,e_{i_r}\}} x^H A x \right\}^\dagger$$
$$= \min_{v_1,...,v_{k-1}} \left\{ \max_{y \perp v_1,...,v_{k-1}, \|y\|_2=1} y^H A y \right\}^\ddagger$$
$$= \lambda_k(A_I)$$
$\dagger$: $e_i = \begin{bmatrix} \mathbf{0} \\ 1 \\ \mathbf{0} \end{bmatrix}$, only the ith entry equal to 1.
$\ddagger$: Define $y \in \mathbb{C}^r$, $y_\ell = x_{i_\ell}$ ($\ell = 1, ..., r$) and take $v_i \in \mathbb{C}^r$, $(v_i)_\ell = (\omega_i)_{i_\ell}$ $\square$

Proof 6.
Let $\tilde{U} = [\underbrace{U}_{\in \mathbb{C}^{m \times r}}, \underbrace{\bar{U}}_{\in \mathbb{C}^{m \times r}}]$ be unitary. Then
$$\tilde{U}^H A \tilde{U} = \begin{bmatrix} U^H A U & \times \\ \times & \times \end{bmatrix}$$
where $U^H A U$ is the principal submatrix of $\tilde{U}^H A \tilde{U}$ with $I = 1, 2, ..., r$
Therefore,
$$\lambda_k(A) = \lambda_k(\tilde{U}^H A \tilde{U}) \geq \lambda_k(U^H A U)$$

$\square$

The Power Method

The power method is a simple method to obtain eigenvector/eigenvalue of a matrix, particularly for large scale problem.

Algorithm 1 (Power Method)
Given a matrix $A \in \mathbb{C}^{m \times m}$ and $v^0 \in \mathbb{C}^m$

$$v^{(k+1)} \leftarrow \frac{A \cdot v^{(k)}}{\|A \cdot v^{(k)}\|_2}$$

until a stopping condition is met.

$\implies v^{(k+1)}$ takes as an estimate of the **principle eigenvector** of A.

Assumption:

- A is diagonalizable (in particular, A is Hermitian (symmetric)). $A = V\Lambda V^H$.
- $|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_m|$ and $|\lambda_1| > |\lambda_2|$

$v^{(k+1)} = \frac{A \cdot v^{(k)}}{\|A \cdot v^{(k)}\|_2} = \frac{A^{k+1} \cdot v^{(0)}}{\|A^{k+1} \cdot v^{(0)}\|_2}$

Consider $A^k x$, x satisfies $[V^H x]_1 \neq 0$. (x is $v^{(0)}$)
Let $\alpha = V^H x$

$$A^k \cdot x = (V\Lambda^k V^H) \cdot x = V\Lambda^k \alpha$$

Let $V = [v_1, v_2, ..., v_m]$

$$
\begin{aligned}
A^k \cdot x &= V\Lambda^k \alpha \\
&= \sum_{i=1}^m v_i \cdot \lambda_i^k \cdot \alpha_i \\
&= v_1(\lambda_1 \alpha_1) + \sum_{i=2}^m \alpha_i \lambda_i^k v_i \\
&= \lambda_1^k \alpha_1 (v_1 + \underbrace{\sum_{i=2}^m \frac{\alpha_i}{\alpha_1} \left(\frac{\lambda_i}{\lambda_1}\right)^k v_i}_{\triangleq r_k \text{(residule)}})
\end{aligned}
$$

$\implies$

$$
\begin{aligned}
\|r_k\|_2 &= \|\sum_{i=2}^m \frac{\alpha_i}{\alpha_1} \left(\frac{\lambda_i}{\lambda_1}\right)^k v_i\|_2 \\
&\leq \sum_{i=1}^m \frac{\alpha_i}{\alpha_1} \cdot \left(\frac{\lambda_i}{\lambda_1}\right)^k \cdot \underbrace{\|v_i\|_2}_{=1} \\
&\leq \underbrace{|\frac{\lambda_2}{\lambda_1}|^k}_{<1} \cdot \sum_{i=1}^m |\frac{\alpha_i}{\alpha_1}| \to 0 \text{ as } k \to \infty
\end{aligned}
$$

Consider $c_k = \frac{|\alpha_1||\lambda_1^k|}{\alpha_1 \lambda_1^k}$ and $\lim_{k \to \infty} c_k \cdot \frac{A^k \cdot x}{\|A^k \cdot x\|_2}$

$$
\begin{aligned}
&c_k \cdot \frac{A^k \cdot x}{\|A^k \cdot x\|_2} \\
&= \frac{|\alpha_1||\lambda_1^k|}{\alpha_1 \lambda_1^k} \cdot \frac{1}{\|A^k \cdot x\|_2^{\dagger}} \cdot \lambda_1^k \alpha_1 \left(v_1 + \sum_{i=2}^m \frac{\alpha_i}{\alpha_1} \left(\frac{\lambda_i}{\lambda_1}\right)^k v_i\right) \\
&= \frac{|\alpha_1||\lambda_1^k|}{\sqrt{\sum_{i=1}^m (\lambda_i^k)^2 \alpha_i^2}}^{\ddagger} \cdot \left(v_1 + \underbrace{\sum_{i=2}^m \frac{\alpha_i}{\alpha_1} \left(\frac{\lambda_i}{\lambda_1}\right)^k v_i}_{\to 0 \text{ as } k \to \infty}\right)
\end{aligned}
$$

$\dagger$: $\|A^k x\|_2 = \|V\Lambda^k V^H x\|_2 = \|V\Lambda^k \alpha\|_2 = \|\Lambda^k \alpha\|_2 = \sqrt{\sum_{i=1}^m (\lambda_i^k \alpha_i)^2}$

$\ddagger$:
$$
\begin{aligned}
\frac{|\alpha_1||\lambda_1^k|}{\sqrt{\sum_{i=1}^m (\lambda_i^k)^2 \alpha_i^2}} &= \frac{|\alpha_1||\lambda_1^k|}{\sqrt{(\lambda_1^k)^2 \alpha_1^2 + \sum_{i=2}^m (\lambda_i^k)^2 \alpha_i^2}} \\
&= \frac{1}{\sqrt{1 + \sum_{i=2}^m \left(\frac{\lambda_i}{\lambda_1}\right)^{2k} \left(\frac{\alpha_i}{\alpha_1}\right)^2}}
\end{aligned}
$$

Since $\sum_{i=2}^m \left(\frac{\lambda_i}{\lambda_1}\right)^{2k} \left(\frac{\alpha_i}{\alpha_1}\right)^2 \to 0$ as $k \to \infty$, so $\frac{|\alpha_1||\lambda_1^k|}{\sqrt{\sum_{i=1}^m (\lambda_i^k)^2 \alpha_i^2}} \to 1$ as $k \to \infty$.

Therefore, $\lim_{k \to \infty} c_k \cdot \frac{A^k \cdot x}{\|A^k \cdot x\|_2} \to v_1$.

Deflation:
$$
\begin{aligned}
A &= \sum_{i=1}^m \lambda_i v_i v_i^H \\
&= \lambda_1 v_1 v_1^H + \sum_{i=2}^m \lambda_i v_i v_i^H
\end{aligned}
$$

Obtain $A^{(1)} = A - \lambda_1 v_1 v_1^H$ and apply power method to obtain v_2
Obtain $A^{(2)} = A - \lambda_2 v_2 v_2^H$ and apply power method to obtain v_3
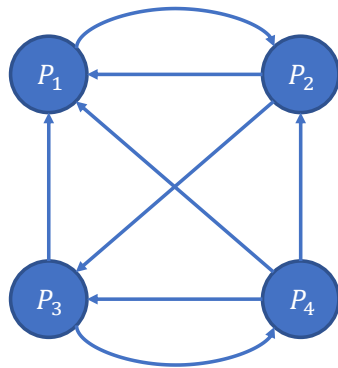...

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Lecture 5. PageRank

PageRank Model

Importance score

- v_i: importance score of page i
- c_j: number of outgoing links of page j
- $\mathcal{L}_i$: set of pages that refers to page i

$$v_i = |\mathcal{L}_i|$$
$$\Downarrow$$
$$v_i = \sum_{j \in \mathcal{L}_i} v_j$$
$$\Downarrow$$
$$v_i = \sum_{j \in \mathcal{L}_i} \frac{v_j}{c_j}$$

The model

e.g.

$v_1 = 1 \cdot v_2 + \frac{1}{2}v_3 + \frac{1}{3}v_4$
$v_2 = 1 \cdot v_1 + \frac{1}{3}v_4$
...
$\implies$

$$\underbrace{\begin{bmatrix} 0 & 1/2 & 1/2 & 1/3 \\ 1 & 0 & 0 & 1/3 \\ 0 & 1/2 & 0 & 1/3 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}}_{\text{link matrix}} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$\implies$

$$Av = v \quad , \quad v \geq 0$$

PageRank Problem

Find solution $v \geq 0$ satisfying $Av = v$
The foure questions:

1. Does A admit a solution of $Av = v$?
   (Does A has eigenvalue equal to 1?)
2. Does $Av = v$ admit a non-negative solution?
   ($v \geq 0$ or $v \leq 0$)

3. Is the solution unique?
   (unique ranking)
4. Can we obtain the solution efficiently?
   (Power method applicable?)

Answer to Question 1

Does A admit a solution of $Av = v$? (Does A has eigenvalue equal to 1?)

Note that $A \geq 0$ and column-sum eqaul to 1 (column-stochastic matrix).
$\implies \mathbf{1}^T A = \mathbf{1}^T$
$\implies A^T \cdot \mathbf{1} = \mathbf{1}$
$\implies A^T$ has an eigenvalue equal to 1
Since A and A^T have the same set of eigenvalues
$\implies A$ has an eigenvlue equal to 1

> **Theorem 6**
> Let A be non-negative and column(row)-stochastic, then
>
> 1. $\lambda = 1$ is an eigenvlue of A
> 2. $|\lambda| \leq 1$ for all eigenvalues of A

Proof   Let $Av = \lambda v$ for some $|\lambda| \neq 1$.
Let $|v_k| = \max\{|v_1|, |v_2|, ..., |v_m|\}$.

$$\lambda v_k = \sum_{i=1}^{m} [A]_{ki} v_i$$

$\implies$

$$\begin{aligned} |\lambda||v_k| &= |\sum_{i=1}^{m} [A]_{ki} v_i| \\ &\leq \sum_{i=1}^{m} |[A]_{ki}| \cdot |v_i| \\ &\leq |v_k| \cdot \sum_{i=1}^{m} |[A]_{ki}| \\ &= |v_k| \end{aligned}$$

$\implies$

$$|\lambda| \leq 1$$

$\square$

Disjoint page networks

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

$$A_1 v_1 = v_1$$
$$A_2 v_2 = v_2$$

$$A\left(\alpha_1 \begin{pmatrix} v_1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} v_1 \\ 0 \end{pmatrix}\right) = \left(\alpha_1 \begin{pmatrix} v_1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} v_1 \\ 0 \end{pmatrix}\right)$$

Modified model

$$M \triangleq (1-\beta)A + \beta\left(\frac{11^T}{m}\right) > 0$$

This is positive column-stochastic

$$1^T M = (1-\beta)1^T A + \beta 1^T \left(\frac{11^T}{m}\right) = 1^T$$

Answer to Question 2

Does $Av = v$ admit a non-negative solution? ($v \geq 0$ or $v \leq 0$)

> **Theorem 7**
> For positive and column-stochastic matrix M, any eigenvector associated with λ_1 has either all positive or all negative components.

Proof   Note that for $y \in \mathbb{R}^m$, $|\sum_{i=1}^m y_i| < \sum_{i=1}^m |y_i|$ if $\{y_i\}$ have mixed signs.

Suppose the above theorem is not true. Then we have

$$|v_i| = |\sum_{j=1}^m [M]_{ij} v_j| < \sum_{j=1}^m [M]_{ij} |v_j|$$

$$\Longrightarrow$$

$$\sum_{i=1}^m |v_i| < \sum_{i=1}^m \sum_{j=1}^m [M]_{ij} v_j = \sum_{j=1}^m \sum_{i=1}^m [M]_{ij} v_j = \sum_{j=1}^m |v_j|$$

Contradict! So the above theorem is true. □

Answer to Question 3

Is the solution unique? (unique ranking)
Since $Mv = v \Longrightarrow (M - I)v = 0$, show

$$dim(Null(M - I)) = 1$$

i.e., there is a unique ranking.

Proof

Simple lemma: Let a and b be linearly independent, then $\exists x \in span\{a, b\}$ which has mixed sign.
Proof for this lemma:
If $1^T q = 0 \Longrightarrow q$ has mixed sign.
If $1^T q \neq 0$, set $\alpha_1 = \frac{-1^T b}{1^T q}$, $\alpha_2 = 1$ and let $x = \alpha_1 \cdot a + \alpha_2 \cdot b$
$\Longrightarrow x$ satisfies $1^T x = 0$, so x has mixed sign.

Suppose $dim(Null(M - I)) > 1$, then for $q_1, q_2 \in Null(M - I)$ that are linear independent. $\Longrightarrow \exists v = \alpha_1 q_1 + \alpha_2 q_2 \in Null(M - I)$
$\Longrightarrow Mv = v$ which has mixed sign. (Contradict!)
$\Longrightarrow dim(Null(M - I)) = 1$ □

Answer to Question 4

Can we obtain the solution efficiently? (Power method applicable?)
Property: Let M be positive and column-stochastic.
Let $S = \{v \in \mathbb{R}^m | 1^T v = 0\}$.
Then,

1. $Mv \in S$ if $v \in S$
2. $\|Mv\|_1 \leq c\|v\|_1$ with $c < 1$ $\forall$ any $v \in S$
3. the eigenvector associated with $\lambda = 1$, denoted v_1, can be obtainable by the power method

Proof
1)
$v \in S \iff 1^T v = 0$
2)
$w \triangleq Mv$
$$\begin{aligned}
\|w\|_1 &= \|Mv\|_1 \\
&= \sum_{i=1}^m |w_i| = \sum_{i=1}^m sign(w_i) \cdot w_i \\
&= \sum_{i=1}^m \left[sign(w_i) \cdot \sum_{j=1}^m M_{ij} v_j \right] \\
&= \sum_{j=1}^m \left[\left(\sum_{j=1}^m sign(w_i) M_{ij} \right) \cdot v_j \right]
\end{aligned}$$

$a_j \triangleq \left(\sum_{j=1}^m sign(w_i) M_{ij} \right)$

$$\begin{aligned}
|a_j| &= \left| \sum_{j=1}^m sign(w_i) M_{ij} \right| \\
&< \sum_{j=1}^m |sign(w_i) M_{ij}| \\
&= \sum_{j=1}^m M_{ij} = 1
\end{aligned}$$

$c \triangleq \max_{j=1,\ldots,m} |a_j| < 1$

$$\begin{aligned}
\|Mv\_1\| = \|w\|_1 &= \sum_{j=1}^m a_j v_j \\
&\leq \sum_{j=1}^m |a_j| |v_j| \\
&\leq c \sum_{j=1}^m |v_j| \\
&= c \cdot \|v\|_1
\end{aligned}$$

$\Longrightarrow \|Mv\|_1 \leq c\|v\|_1 \ \forall v \in S$ □

Lecture 6. Positive Semidefinite Matrix

For $A \in \mathbb{C}^{m \times m}$, if

- A is symmetric, we denote $A \in S^m$
- A is Hermitian, we denote $A \in H^m$

Definition
We say $A \in H^m$ is positive semidefinite (PSD) if
$$x^H A x \geq 0 \quad \forall x \in \mathbb{C}^m$$
We say $A \in H^m$ is positive definite (PD) if
$$x^H A x > 0 \quad \forall x \in \mathbb{C}^m, x \neq 0$$
We say $A \in H^m$ is indefinite if A is not PSD.

Properties of PSD Matrices

Propeties

Property 1: If A is PSD then any principal submatrix of A (A_I) is PSD.
Proof
Let $x \in \mathbb{C}^m$ with $x_{i_k} \neq 0$ for all $i_k \notin I$.
Let $(x_I)_k = x_{i_k}$ for $i_k \in I$.
Then
$$x^H A x = x_I^H A_I x_I \geq 0, \quad \forall x_I \in \mathbb{C}^{|I|}$$
$\square$

By Property 1, we know
1)
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$
$$A_{11} \in H^{k \times k}, A_{22} \in H^{(m-k) \times (m-k)}$$
If A is PSD(PD) $\implies A_{11}, A_{22}$ are PSD(PD). 2)
If A is PSD(PD), then $A_{ii} \geq 0(> 0)$

Eigenvalues of PSD matrices

Theorem 1
$A \in H^m$ is PSD(PD) if and only if $\lambda_1, \lambda_2, ..., \lambda_3 \geq 0(> 0)$

Proof
Consider EVD of A as
$$A = V \Lambda V^H, \quad \Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_m)$$
Consider $x^H A x = x^H V \Lambda V^H x = z^H \Lambda z$.

A is PSD $\iff x^H A x \geq 0 \ \forall x \in \mathbb{C}^m \iff z^H \Lambda z \geq 0$
$\forall z \in \mathbb{C}^m$.

$z^H \Lambda z = \sum_{i=1}^{m} \lambda_i |z_i|^2 \geq 0 \ \forall z \in \mathbb{C}^m \iff \lambda_i \geq 0$
$\forall i = 1, ..., m$ $\square$

If A is PSD(PD),

- $Tr(A) \geq 0(> 0)$

- $det(A) \geq 0(> 0)$

If A is PD, then A is invertible.
$$A = V \Lambda V^H$$
$$A^{-1} = V \Lambda^{-1} V^H$$

Property 2: $A \in H^m$ can be factorized as $A = BB^H$ for some matrix B if and only if A is PSD.
Proof
$\implies$:
$$x^H A x = x^H BB^H x = \|B^H x\|^2 \geq 0 \implies A \text{ is PSD}$$
$\impliedby$:
If A is PSD, define $\Lambda^{1/2} = diag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, ..., \sqrt{\lambda_m})$
$$\begin{aligned} A &= V \lambda V^H \\ &= V \Lambda^{1/2} \Lambda^{1/2} V^H \\ &= V \Lambda^{1/2} Q^H Q \Lambda^{1/2} V^H \end{aligned}$$
where Q is a Hermitian matrix.
Define $B \triangleq V \Lambda^{1/2} Q^H$. Then, $A = BB^H$. $\square$

Theorem 2
Consider $A \in H^m$, $B \in \mathbb{C}^{m \times n}$ and $C = B^H A B$

- If A is PSD, then C is PSD.
- If A is PD, then C is PD if and only if $rank(B) = n$
- If B is square and invertible, then A is PSD(PD) if and only if C is PSD(PD)

Proof of 2)
($\implies$) Show C is PSD $\implies rank(B) = n$
Assume B is not full column rank.
Consider $x^H C x = x^H B^H A B x$, then there exists $x \neq 0$
s.t. $Bx = 0 \implies x^H C x = 0$ (Contradict!) ($\impliedby$) Show
$rank(B) = n \implies$ Show C is PSD
$rank(B) = n \implies Bx = 0$ iff $x = 0$. So $\forall x \neq 0 \implies Bx \neq 0$.
$\implies x^H C x = x^H B^H A B x > 0$ as A is PD. So C is PD.
$\square$

Property 3: Let $A \in \mathbb{C}^{m \times k}$ and $B \in \mathbb{C}^k$ and B has full row rank ($rank(B) = k$). Then
$$Range(A) = Rnage(AB)$$
Proof
$Range(B) = \mathbb{C}^k$ as B is full row rank.
$$\begin{aligned} Range(AB) &= \{y | y = ABx, x \in \mathbb{C}^n\} \\ &= \{y | y = Az, \underbrace{z = Bx, x \in \mathbb{C}^n}_{z \in Range(B) = \mathbb{C}^k}\} \\ &= \{y | y = Az, z \in \mathbb{C}^k\} \\ &= Range(A) \end{aligned}$$
$\square$

Property 4: Let $B \in \mathbb{C}^{m \times k}$ and $C \in \mathbb{C}^{m \times k}$ be two matrices with full column rank. Then

$$BB^H = CC^H$$

if and only if $C = BQ$ for some unitary Q. *Proof*
($\Longleftarrow$): obvious.
($\Longrightarrow$): $(C^H)^+ \triangleq C(C^HC)^{-1}$. Since $rank(C^HC) = rank(C)$, C^HC is invertible †.
†: As C has full column rank, then $Cx = 0 \Longleftrightarrow x = 0$. $Cx = 0 \Longrightarrow C^HCx = 0 \Longrightarrow x^HC^HCx = 0 \Longleftrightarrow \|Cx\|^2 = 0 \Longrightarrow x = 0$. Therefore, $C^HCx = 0 \Longrightarrow x = 0$ and hence $rank(C^HC) = rank(C)$.

$$
\begin{aligned}
BB^H(C^H)^+ &= CC^H(C^H)^+ \\
&= CC^HC(C^HC)^{-1} \\
&= C
\end{aligned}
$$

$$\Longrightarrow$$

$$
\begin{aligned}
C &= BB^H(C^H)^+ \\
&= B\underbrace{B^HC(C^HC)^{-1}}_{\triangleq Q(\text{square})}
\end{aligned}
$$

$$
\begin{aligned}
Q^HQ &= (C^HC)^{-H}C^HBB^HC(C^HC)^{-1} \\
&= \underbrace{(C^HC)^{-H}C^HC}_{=I}\underbrace{C^HC(C^HC)^{-1}}_{=I} \\
&= I
\end{aligned}
$$

Therefore, Q is unitary. $\qquad\square$

Matrix Inequality

For $A, B \in H^{m \times m}$, we write $A \succeq (\succ)B$ if $A - B$ is a PSD(PD) matrix.
So $A \succeq 0$ if A is PSD.
Basic Properties:

- If $A \succeq 0$, then $\alpha A \succeq 0 \ \forall \alpha \geq 0$
- If $A \succeq 0$, $A \succeq 0$, then $\alpha A + B \succeq 0$
- If $A \succeq B$, $B \succeq C$, then $\alpha A \succeq C$
- If $A \nsucceq B$, it doesn't imply $B \succ A$

Let $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_m(A)$ for $A \in H^{m \times m}$, then

1. $A \succeq I \Longleftrightarrow \lambda_i(A) \geq 1 \ \forall i = 1, ..., m$
2. $I \succeq A \Longleftrightarrow \lambda_i(A) \leq 1 \ \forall i = 1, ..., m$
3. Suppose $A, B \succ 0$, then $A \succeq B \Longleftrightarrow B^{-1} \succeq A^{-1}$
4. If $A \succeq B$, then $\lambda_i(A) \geq \lambda_i(B) \ \forall i = 1, ..., m$

Proof
1)
$A \succeq I \Longleftrightarrow A - I \succ 0 \Longleftrightarrow V\Lambda V^H - I \succeq 0 \Longleftrightarrow V(\Lambda - I)V^H \succeq 0 \Longleftrightarrow \Lambda - I \succeq 0 \Longleftrightarrow \lambda_i(A) \geq 1 \ \forall i = 1, ..., m$

2) similar to 1)

3)

$$
\begin{aligned}
A \succeq B &\Longleftrightarrow A - B \succeq 0 \\
&\Longleftrightarrow A^{\frac{1}{2}}A^{\frac{1}{2}} - B \succeq 0 \\
&\Longleftrightarrow A^{\frac{1}{2}}(I - A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \succeq 0 \\
&\Longleftrightarrow I - A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \succeq 0 \\
&\Longleftrightarrow \lambda_i(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \leq 1 \quad \forall i = 1, ..., m \\
&\Longleftrightarrow \frac{1}{\lambda_i(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})} \leq 1 \quad \forall i = 1, ..., m^\dagger \\
&\Longleftrightarrow \lambda_i(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}) \geq 1 \quad \forall i = 1, ..., m \\
&\Longleftrightarrow A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}} - I \succeq 0 \\
&\Longleftrightarrow A^{-\frac{1}{2}}(I - A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})A^{-\frac{1}{2}} \succeq 0 \\
&\Longleftrightarrow B^{-1} - A^{-1} \succeq 0 \\
&\Longleftrightarrow B^{-1} \succeq A^{-1}
\end{aligned}
$$

†: $\lambda_i(A^{-1}) = \frac{1}{\lambda_i(A)}$

4)

$$
\begin{aligned}
\lambda_i(A) &= \lambda_i(A - B + B) \\
&\geq \lambda_i(B) + \underbrace{\lambda_m(A - B)}_{\geq 0 \text{ since } A - B \succeq 0} \\
&\geq \lambda_i(B) \quad \forall i = 1, ..., m
\end{aligned}
$$
$\qquad\square$

From 4), if $A \succeq B \Longrightarrow Tr(A) \geq Tr(B)$, if additionally $A, B \succeq 0 \Longrightarrow det(A) \geq det(B)$, and if $A, B \succ 0$ and $A \succeq B$ then $Tr(B^{-1}) \geq Tr(A^{-1})$

> **Definition (Schur Complement)**
> Consider a matrix $X \in H^{(m+n) \times (m+n)}$ to be partitioned
> $$X = \begin{bmatrix} A & B \\ B^H & C \end{bmatrix}$$
> where $A \in H^{m \times m}$, $B \in \mathbb{C}^{m \times n}$, $C \in H^{n \times n} \succ 0$. The term
> $$S \triangleq A - BC^{-1}B^H \in H^{m \times m}$$
> is called *Schur Complement* of X.

Then $X \succeq (\succ)0 \Longleftrightarrow S \succeq (\succ)0$ *Proof*
Let $Y = \begin{bmatrix} I_m & 0 \\ -C^{-1}B^H & I_n \end{bmatrix} \in \mathbb{C}^{(m+n) \times (m+n)}$.
Then Y is a lower triangular matrix and $det(Y) = 1 \Longrightarrow Y$ is invertible.
Then

$$Y^H X Y = \begin{bmatrix} A - BC^{-1}B^H & 0 \\ 0 & C \end{bmatrix}$$

$Y^H X Y \succeq (\succ)0 \Longleftrightarrow x \succeq (\succ)0 \Longrightarrow X \succeq (\succ)0 \Longleftrightarrow X \succeq (\succ)0$ $\qquad\square$

$$
\begin{aligned}
det(YX^HY) &= det(X) \\
&= det(S)det(C)
\end{aligned}
$$

Lecture 7. Singular Value Decomposition

Singular Value Decomposition (SVD)

Theorem of SVD

> **Theorem 1**
> Any $A \in \mathbb{C}^{m \times n}$ can be decomposed as
> $$\underbrace{A}_{\mathbb{R}^{m \times n}} = \underbrace{V}_{\mathbb{R}^{m \times m}} \underbrace{\Sigma}_{\mathbb{R}^{m \times n}} \underbrace{U^H}_{\mathbb{R}^{m \times n}}$$
> where V and U are unitary,
> $$[\Sigma]_{ij} = \begin{cases} \sigma_i & i = j \\ 0 & i \neq j \end{cases}$$
> and
> $$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0 \quad , \quad p = \min\{m, n\}$$
> $V = [v_1, v_2, ..., v_m]$: left singular matrix.
> $U = [u_1, u_2, ..., u_m]$: left singular matrix.
> $\sigma_i \geq 0$: singular matrix.
>
> If A is real-valued, then V, U can be orthogonal real-valued.

Proof $\quad r \triangleq rank(A)$
Consider
$$\underbrace{AA^H}_{rank(AA^H)=r,PSD} = V \Lambda V^H = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \tilde{\Lambda} & \\ & 0 \end{bmatrix} \begin{bmatrix} V_1^H \\ V_2^H \end{bmatrix}$$
where

- $V_1 \in \mathbb{C}^{m \times r}$
- $V_2 \in \mathbb{C}^{m \times (m-r)}$
- $\tilde{\Lambda} = diag(\lambda_1, ..., \lambda_r) \succ 0 \in \mathbb{C}^{r \times r}$

First note that $V_2^H A = 0$ as
$$V_2^H A A^H V_2 = V_2^H (V_1 \tilde{Lambda} V_1^H) V_2 = 0$$
Second, define
$$\tilde{\Sigma} = \tilde{\Lambda}^{\frac{1}{2}} = diag(\sqrt{\lambda_1}, ..., \sqrt{\lambda_r})$$
$$U_1 = A^H V_1 \tilde{\Sigma}^{-1}$$
Then U_1 is semi-unitary $(U_1^H U_1 = Ir)$.
Since
$$U_1^H U_1 = \tilde{\Sigma}^{-1} V_1^H \underbrace{A A^H}_{=V_1 \tilde{\Lambda} V_1^H} V_1 \tilde{\Sigma}^{-1} = \tilde{\Sigma}^{-1} \tilde{\Lambda} \tilde{\Sigma}^{-1} = I_r$$
Then we have $U_2 \in \mathbb{C}^{n \times (n-r)}$ and $U \triangleq \begin{bmatrix} U_1 & U_2 \end{bmatrix} \in \mathbb{C}^{n \times n}$ is unitary.
Consider
$$V^H A U = \begin{bmatrix} V_1^H \\ V_2^H \end{bmatrix} A \begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} V_1^H A U_1 & V_1^H A V_2 \\ V_2^H A U_1 & V_2^H A U_2 \end{bmatrix}$$
Since $V_2^H A = 0$ and $V_2^H A V_1 = u_2^H (U_1 \tilde{\Sigma}) = 0$, so
$$V^H A U = \begin{bmatrix} V_1^H A U_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore,
$$A = V \Sigma U^H = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \tilde{\Sigma}_{r \times r} & \\ & 0 \end{bmatrix} \begin{bmatrix} U_1^H \\ U_1^H \end{bmatrix} = V_1 \tilde{\Sigma} U_1^H \text{(thin SVD)}$$
or
$$A = \sum_{i=1}^{r} \sigma_i v_i u_i^H$$
$\square$

Mention that

- $AA^H = V \Sigma^2 V^H$, $\lambda_i(AA^H) = \sigma_i^2(A)$
- $A^H A = U \Sigma^2 U^H$, $\lambda_i(AA^H) = \sigma_i^2(A)$
- or $\sigma_i(A) = \sqrt{\lambda_i(AA^H)} = \sqrt{\lambda_i(A^H A)}$

Properties

$$A = V \Sigma U^H = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \tilde{\Sigma}_{r \times r} & \\ & 0 \end{bmatrix} \begin{bmatrix} U_1^H \\ U_1^H \end{bmatrix} = V_1 \tilde{\Sigma} U_1^H$$
$$A^H = U_1 \tilde{\Sigma} V_1^H$$

- $Range(A) = Range(V_1)$
- $Range(A)_\perp = Range(V_2)$
- $Range(A^H) = Range(V_1)$
- $Range(A^H)_\perp = Null(A) = Range(V_2)$

System of Linear Equations

SVD as a solution of a linear system

For $y \in \mathbb{R}^{m \times 1}$, $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^{n \times 1}$
$$y = Ax$$
has a solution iff.
$$y \in Range(A)$$
Let A SVD be $A = V \Sigma U^H$
$$y = Ax = (V \Sigma U^H) x$$
$$V^H y = \Sigma U^H x$$
Define $\tilde{y} = V^H y$, $\tilde{x} = U^H x$.
$$\tilde{y} = \Sigma \tilde{x}$$

If A is invertible, then $y = Ax \implies x = A^{-1} y$.
If A is general, not necessarily be square
$$y = Ax \implies x = A^+ y$$
where
$$A^+ \triangleq U \begin{bmatrix} \tilde{\Sigma}^{-1} & \\ & 0 \end{bmatrix} V^H$$
if SVD of A is $A = V \begin{bmatrix} \tilde{\Sigma} & \\ & 0 \end{bmatrix} U^H$

Case 1: $rank(A) = n$ (full column rank)
$$\Sigma = \begin{bmatrix} \tilde{\Sigma} \\ 0 \end{bmatrix} \quad ; \quad \tilde{y} = \Sigma \tilde{x}$$

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$$\Longrightarrow$$

$$\tilde{y} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} \tilde{\Sigma} \\ 0 \end{bmatrix} \tilde{x} = \begin{bmatrix} \tilde{\Sigma}\tilde{x} \\ 0 \end{bmatrix}$$

$$\tilde{x} = \tilde{\Sigma}^{-1}\tilde{y}_1 \quad ; \quad \tilde{y}_2 = 0$$

Then

$$\begin{aligned}
x \quad &= U\tilde{x} \\
&= U\tilde{\Sigma}^{-1}\tilde{y}_1 \\
&= U \begin{bmatrix} \tilde{\Sigma}^{-1} & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} \\
&= U \begin{bmatrix} \tilde{\Sigma}^{-1} & 0 \end{bmatrix} V^H y
\end{aligned}$$

Define $A^+ = U \begin{bmatrix} \tilde{\Sigma}^{-1} & 0 \end{bmatrix} V^H$ given $A = V \begin{bmatrix} \tilde{\Sigma} & \\ & 0 \end{bmatrix} U^H$.

Then

$$x = A^+ y$$

If $y \in Range(A)$, given y and $rank(A) = n$, x is unique.
In fact, $A^+ = (A^H A)^{-1} A^H = U \begin{bmatrix} \tilde{\Sigma}^{-1} & 0 \end{bmatrix} V^H$

Case 1: $rank(A) = m$ (full row rank) Then

$$\Sigma = \begin{bmatrix} \tilde{\Sigma} & 0 \end{bmatrix}$$

$$\Longrightarrow$$

$$\tilde{y} = \Sigma\tilde{x} = \begin{bmatrix} \tilde{\Sigma} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \tilde{\Sigma}\tilde{x}_1$$

$$\tilde{x}_1 = \tilde{\Sigma}^{-1}\tilde{y} \quad , \quad \tilde{x}_2 \text{ arbitrary}$$

$$x = U\tilde{x} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = U_1\tilde{x}_1 + U_2\tilde{x}_2 = U_1\tilde{\Sigma}^{-1}\tilde{y} + U_1\tilde{x}_2$$