26-Nov 2023 Continuous-time MHE vs Discrete-time MHE

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Discrete-time MHE (DT-MHE)

$$\min_{\{x_k\}_0^N \{w_k\}_0^{N-1}} J_1 = \frac{1}{2}\|x_{(0)} - \hat{x}_0\|_P^2 + \frac{1}{2}\sum_{k=0}^{N}\|y_k - h(x_k)\|_R^2 + \frac{1}{2}\sum_{k=0}^{N-1}\|w_k\|_Q^2$$

$$\text{s.t.} \quad x_{k+1} = \underbrace{x_k + \Delta t \cdot \bar{f}(x_k, w_k)}_{:= f(x_k, w_k)}$$

$\{x_k\}_{k=0}^N$ $\{w_k\}_{k=0}^{N-1}$: sequences of state estimate & process noise

vs

Continuous-time MHE (CT-MHE)

$$\min_{x, w} J_2 = \frac{1}{2}\|x_{(0)} - \hat{x}_0\|_P^2 + \frac{1}{2}\int_0^T (\|y_t - h(x_t)\|_R^2 + \|w_t\|_Q^2)\, dt$$

$$\text{s.t.} \quad \dot{x}_t = \bar{f}(x_t, w_t)$$

$x_t \in \mathbb{R}^n$ $t \in [0, T]$, $w_t \in \mathbb{R}^m$ $t \in [0, T]$; trajectories of state estimate & process noise

* * *

Approximation of Continuous-time MHE (ACT-MHE)

$$\min_{\{x(k\epsilon)\}_0^N \{w(k\epsilon)\}_0^{N-1}} J_2 \approx \frac{1}{2}\|x_{(0)} - \hat{x}_0\|_P^2 + \frac{1}{2}\sum_{k=0}^{N-1}(\|y_k - h(x_k)\|_R^2 \cdot \epsilon + \|w_k\|_Q^2 \cdot \epsilon)$$

because $0 \cdots N-1$; totally N points $\rightarrow N \cdot \epsilon = T$.

$$\text{s.t.} \quad x_{k+1} = x_k + \epsilon \cdot \bar{f}(x_k, w_k)$$

where $N \cdot \epsilon = T$, ϵ: a very small time-step.

Derive KKT conditions for CT-MHE based on ACT-MHE

$$\mathcal{L}_{ACT\text{-}MHE} = \frac{1}{2}\|x_{(0)} - \hat{x}_0\|_P^2 + \frac{\epsilon}{2}\sum_{k=0}^{N-1}(\|y_k - h(x_k)\|_R^2 + \|w_k\|_Q^2) + \sum_{k=0}^{N-1}\lambda_k^T(x_{k+1} - x_k - \epsilon \cdot \bar{f}(x_k, w_k))$$

where $x_k := x(k \cdot \epsilon)$, $w_k := w(k \cdot \epsilon)$ so that $w_t = w(k \cdot \epsilon)$ $t = k \cdot \epsilon$

Initial Condition
$$\nabla_{x_0}\mathcal{L} = P(x_0 - \hat{x}_0) - H^T R \cdot \epsilon \cdot (y_0 - h(x_0)) - \lambda_0 - \epsilon \cdot \bar{F}_0^T \cdot \lambda_0 = 0$$

$$\nabla_{x_k}\mathcal{L} = -H^T R \cdot \epsilon (y_k - h(x_k)) + \lambda_{k-1} - \lambda_k - \epsilon \cdot \bar{F}_k^T \cdot \lambda_k = 0 \qquad (1)$$

$$\nabla_{w_k}\mathcal{L} = \epsilon Q \cdot w_k - \epsilon \bar{G}^T \lambda_k = 0$$

$$\nabla_{\lambda_k}\mathcal{L} = x_{k+1} - x_k - \epsilon \cdot \bar{f}(x_k, w_k) = 0$$

where $\bar{F} = \frac{\partial \bar{f}}{\partial x}$, $\bar{G} = \frac{\partial \bar{f}}{\partial w}$, $\lambda_N = 0$

If $\epsilon \rightarrow \Delta t$, Eq. (1) becomes the KKT conditions of DT-MHE

$$\nabla_{x_0}\mathcal{L} = P(x_0 - \hat{x}_0) - H^T R(y_0 - h(x_0)) - \underbrace{(\mathbb{I} + \Delta t \cdot \bar{F}_0^T)}_{:= F_0^T = \frac{\partial f}{\partial x_0}} \cdot \lambda_0 = 0$$

$$\nabla_{x_k}\mathcal{L} = -H^T R(y_k - h(x_k)) + \lambda_{k-1} - \underbrace{(\mathbb{I} + \Delta t \cdot \bar{F}_k^T)}_{F_k^T := \frac{\partial f}{\partial x_k}} \lambda_k = 0 \qquad (2)$$

$$\nabla_{w_k}\mathcal{L} = Q \cdot w_k - \underbrace{\Delta t \bar{G}_k^T}_{G_k^T} \lambda_k = 0 \qquad G_k := \frac{\partial f}{\partial w_k}$$

$$* \left(\frac{\partial[x_k + \Delta t \cdot \bar{f}(x_k, w_k)]}{\partial w_k} = \Delta t \cdot \frac{\partial \bar{f}(x_k, w_k)}{\partial w_k} = \Delta t \cdot \bar{G}_k\right)$$

$$\nabla_{\lambda_k}\mathcal{L} = x_{k+1} - \underbrace{(x_k + \Delta t \cdot \bar{f}(x_k, w_k))}_{f(x_k, w_k)} = 0$$

Eq. (2) is exactly the same as Eq. (10) in my paper!

2 mm Square 26 cm x 20 cm

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If $\epsilon \to 0$, Eq(1) will be like:

$$\nabla_{x_0} L = P(x_0 - \hat{x}_0) - \cancel{H^T R \cdot \underset{\Delta}{\epsilon}(y_0 - h(x_0))} - \lambda_0 - \cancel{\underset{\Delta}{\epsilon} \cdot \bar{F}_0^T} \lambda_0 = 0$$

→ Note: That is why the gradient of the running cost w.r.t x disappears in CT-MHE!

$$\Downarrow \quad \epsilon \to 0$$

$$P(x_0 - \hat{x}_0) - \lambda_0 = 0 \qquad \text{boundary condition!}$$

$$\nabla_{x_k} L = -H^T R \epsilon (y_k - h(x_k)) + \lambda_{k-1} - \lambda_k - \epsilon \cdot \bar{F}_k^T \lambda_k = 0$$

$$\Downarrow$$

$$\frac{\lambda_{k-1} - \lambda_k}{\epsilon} = H^T R (y_k - h(x_k)) + \bar{F}_k^T \lambda_k$$

$$\Downarrow \lim_{\epsilon \to 0}$$

$$-\dot{\lambda}_t = H^T R (y_k - h(x_t)) + \bar{F}_t^T \cdot \lambda_t$$

$$\nabla_{w_k} L = \cancel{\epsilon} Q w_k - \cancel{\epsilon} \cdot \bar{G}_k^T \lambda_k = 0$$

$$\Downarrow$$

$$w_k = Q^{-1} \cdot \bar{G}_k^T \lambda_k$$

$$\Downarrow$$

$$w_t = Q^{-1} \cdot \bar{G}_t^T \lambda_t$$

$$\nabla_{\lambda_k} L = x_{k+1} - x_k - \epsilon \cdot \bar{f}(x_k, w_k) = 0$$

$$\Downarrow$$

$$\frac{x_{k+1} - x_k}{\epsilon} = \bar{f}(x_k, w_k)$$

$$\Downarrow \lim_{\epsilon \to 0}$$

$$\dot{x}_t = \bar{f}(x_t, w_t)$$

In summary:

$$\begin{aligned} -\dot{\lambda}_t &= H^T R (y_k - h(x_t)) + \bar{F}_t^T \lambda_t \\ \dot{x}_t &= \bar{f}(x_t, w_t) \\ w_t &= Q^{-1} \bar{G}_t^T \cdot \lambda_t \\ \lambda_0 &= P(x_0 - \hat{x}_0) \\ \lambda_T &= 0 \end{aligned} \qquad (3)$$

λ_0 and λ_T: } boundary conditions

Let's check the optimality conditions (Pontryagin Principle) for CT-MHE using calculus of variation. Assume that there are small perturbations on x, w, and λ, denoted by δx, δw, $\delta\lambda$, respectively. So perturbed trajectories around the optimal trajectories will be:

$$x = x^* + \alpha \cdot \delta x, \quad w = w^* + \alpha \cdot \delta w, \quad \lambda = \lambda^* + \alpha \cdot \delta\lambda \qquad \text{where } \alpha \text{ is a small scalar.}$$

We have the variation of $\bar{J}_2$ of the following form (defined by the terms linear in $\delta x, \delta w, \delta\lambda$)

$$\delta \bar{J}_2 = \frac{d}{d\alpha}\bar{J}_2\Big|_{\alpha=0} = \left(P(x^*_{(0)} - \hat{x}_{(0)})\right)^T \delta x_{(0)} + \int_0^T \Big[-\left(H^T R (y_k - h(x_t^*))\right)^T \delta x_t + \left(Q \cdot w_t^*\right)^T \delta w_t + \left(\dot{x}_t - \bar{f}(x_t, w_t)\right)^T \delta\lambda_t$$
$$+ \lambda^T \delta \dot{x}_t - \left(\bar{F}_t^T \lambda_t\right)^T \delta x_t - \left(\bar{G}_t^T \lambda_t\right)^T \delta w_t \Big] dt$$

where $\bar{J}_2$ is ~~ag~~ J_2 augmented by the equality constraint $\bar{J}_2 := J_2 + \int_0^T \lambda_t^T (\dot{x}_t - \bar{f}(x_t, w_t)) dt$.

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Using integration by parts, we can rewrite the term $\int_0^T \lambda^T \cdot \delta \dot{x}_t \, dt$ as:

$$\int_0^T \lambda^T \cdot \delta \dot{x}_t \, dt = \lambda^T \delta x_t \Big|_0^T - \int_0^T \dot{\lambda}_t^T \delta x_t \, dt$$

$$= \lambda^T_{(T)} \cdot \delta x_T - \lambda^T_{(0)} \cdot \delta x_0 - \int_0^T \dot{\lambda}_t^T \delta x_t \, dt$$

Plugging the above equation into $\delta \bar{J}_2$ leads to:

$$\delta \bar{J}_2 = \left(P(x^*_{(0)} - \hat{x}_0)\right)^T \delta x_0 + \int_0^T \Big\{ \left[-\dot{\lambda}_t^T - (\bar{F}_t^T \lambda_t)^T - (H^T R (y_k - h(x_t^*)))^T \right] \delta x_t + \left[(Q \cdot w_t^*)^T - (\bar{G}_t^T \lambda_t)^T \right] \cdot \delta w_t$$

$$+ (\dot{x}_t - \bar{f}(x_t, w_t))^T \cdot \delta \lambda_t \Big\} dt + \lambda^T_{(T)} \cdot \delta x_T - \lambda^T_{(0)} \cdot \delta x_0 = 0$$

Since $\delta \bar{J}_2 = 0$ holds for any δx_t, δw_t, and $\delta \lambda_t$, the respective coefficients should be zero.

$$-\dot{\lambda}_t^T - (\bar{F}_t^T \lambda_t)^T - (H^T R (y_k - h(x_t^*)))^T = 0$$

$$\Downarrow$$

$$-\dot{\lambda}_t = H^T R (y_k - h(x_t^*)) + \bar{F}_t^T \lambda_t$$

$$(Q \cdot w_t^*)^T - (\bar{G}_t^T \lambda_t)^T = 0$$

$$\Downarrow$$

$$w_t^* = Q^{-1} \cdot \bar{G}_t^T \lambda_t$$

$$\dot{x}_t^* - \bar{f}(x_t^*, w_t^*) = 0$$

$$\Downarrow$$

$$\dot{x}_t^* = \bar{f}(x_t^*, w_t^*)$$

Note that $\delta x_{(0)} \neq 0$, $\delta x_{(T)} \neq 0$ hold true for MHE.

$$\left(P(x^*_{(0)} - \hat{x}_0)\right)^T - \lambda^T_{(0)} = 0$$

$$\Downarrow$$

$$\lambda_{(0)} = P(x^*_{(0)} - \hat{x}_0)$$

$$\lambda^T_{(T)} = 0$$

In summary:

$$\begin{aligned} -\dot{\lambda}_t &= H^T R (y_k - h(x_t^*)) + \bar{F}_t^T \lambda_t \\ w_t^* &= Q^{-1} \cdot \bar{G}_t^T \lambda_t \\ \dot{x}_t^* &= \bar{f}(x_t^*, w_t^*) \\ &\left.\begin{aligned} \lambda_0 &= P(x^*_{(0)} - \hat{x}_0) \\ \lambda_T &= 0 \end{aligned}\right\} \text{boundary conditions} \end{aligned} \qquad (4)$$

Eq. (4) is exactly the same as Eq. (3) obtained from ACT-MHE!

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