元月6日 2024. Differential MHE using PDP developed for ~~OC~~ problem
optimal control.

$$\min_{X,W} \underbrace{\boxed{\tfrac{1}{2}\mathrm{Tr}\|X_0-\hat X_0\|^2_P} + \mathrm{Tr}\sum_{k=0}^{N-1}\left(\tfrac{1}{2}\begin{bmatrix}X_k\\W_k\end{bmatrix}^T\begin{bmatrix}L_k^{xx} & L_k^{xw}\\ L_k^{wx} & L_k^{ww}\end{bmatrix}\begin{bmatrix}X_k\\W_k\end{bmatrix} + \cancel{\tfrac{1}{2}\mathrm{Tr}\,X_N^T L_N^{xx} X_N} + \begin{bmatrix}L_k^{x\theta}\\L_k^{w\theta}\end{bmatrix}^T\begin{bmatrix}X_k\\W_k\end{bmatrix}\right) + \mathrm{Tr}\left(\tfrac{1}{2}X_N^T L_N^{xx} X_N + L_N^{x\theta T} X_N\right)}_{J}$$

s.t. $X_{k+1} = F_k X_k + G_k W_k$. $\quad k=0,\cdots,N-1$.

Compared to the initial state in control, this arrival cost can be viewed as a soft penalty on the initial state.

$$\mathcal{L} = J + \sum_{k=0}^{N-1} \underline{\Lambda_{k+1}^T}\,(F_k X_k + G_k W_k - X_{k+1})$$

$$\nabla_{X_0}\mathcal{L} = PX_0 - P\hat X_0 + L_0^{xx}X_0 + L_0^{xw}W_0 + L_0^{x\theta} + F_0^T\Lambda_1 = 0.$$ boundary 0.

$$\nabla_{X_k}\mathcal{L} = L_k^{xx}X_k + L_k^{xw}W_k + L_k^{x\theta} + F_k^T\Lambda_{k+1} - \Lambda_k = 0. \quad k=1,\cdots,N-1.$$

$$\nabla_{W_k}\mathcal{L} = L_k^{wx}X_k + L_k^{ww}W_k + L_k^{w\theta} + G_k^T\Lambda_{k+1} = 0 \quad k=0,\cdots,N-1.$$

$$\nabla_{\Lambda_k}\mathcal{L} = F_k X_k + G_k W_k - X_{k+1} = 0.$$

$$\nabla_{X_N}\mathcal{L} = L_N^{xx}X_N + L_N^{x\theta} - \Lambda_N = 0.$$ boundary N

$$\Lambda_N = \underbrace{L_N^{xx}}_{P_N}X_N + \underbrace{L_N^{x\theta}}_{O_N}. \qquad \Lambda_k = P_k X_k + O_k. \quad \underline{k=1,\cdots,N}$$

$$W_k = -(L_k^{ww})^{-1}(L_k^{wx}X_k + G_k^T\Lambda_{k+1} + L_k^{w\theta})$$

$$\Lambda_k = L_k^{xx}X_k - L_k^{xw}(L_k^{ww})^{-1}L_k^{wx}X_k - L_k^{xw}(L_k^{ww})^{-1}G_k^T\Lambda_{k+1} - L_k^{xw}(L_k^{ww})^{-1}L_k^{w\theta} + L_k^{x\theta} + F_k^T\Lambda_{k+1}$$

$$= \underbrace{\left(L_k^{xx} - L_k^{xw}(L_k^{ww})^{-1}L_k^{wx}\right)}_{S_k}X_k + \underbrace{\left(F_k^T - L_k^{xw}(L_k^{ww})^{-1}G_k^T\right)}_{\bar F_k^T}\Lambda_{k+1} + \underbrace{L_k^{x\theta} - L_k^{xw}(L_k^{ww})^{-1}L_k^{w\theta}}_{T_k}.$$

$$\Lambda_k = S_k X_k + \bar F_k^T\Lambda_{k+1} + T_k$$

$$X_{k+1} = F_k X_k - G_k(L_k^{ww})^{-1}L_k^{wx}X_k - \underbrace{G_k(L_k^{ww})^{-1}G_k^T}_{M_k}\Lambda_{k+1} - \underbrace{G_k(L_k^{ww})^{-1}L_k^{w\theta}}_{N_k}$$

$$= \bar F_k X_k - M_k\Lambda_{k+1} - N_k \qquad \Lambda_{k+1} = P_{k+1}X_{k+1} + O_{k+1}.$$

$$X_{k+1} = \bar F_k X_k - M_k P_{k+1}X_{k+1} - M_k O_{k+1} - N_k$$

$$(I + M_k P_{k+1})X_{k+1} = \bar F_k X_k - M_k O_{k+1} - N_k \qquad X_{k+1} = (I + M_k P_{k+1})^{-1}(\bar F_k X_k - M_k O_{k+1} - N_k) \quad k=0,\cdots,N-1.$$

$$W_k = -(L_k^{ww})^{-1}(L_k^{wx}X_k + G_k^T P_{k+1}X_{k+1} + G_k^T O_{k+1} + L_k^{w\theta})$$

$$= -(L_k^{ww})^{-1}\left(L_k^{wx}X_k + G_k^T P_{k+1}(I + M_k P_{k+1})^{-1}(\bar F_k X_k - M_k O_{k+1} - N_k) + G_k^T O_{k+1} + L_k^{w\theta}\right)$$

$$\Lambda_k = S_k X_k + \bar F_k^T P_{k+1}X_{k+1} + \bar F_k^T O_{k+1} + T_k \qquad k=1,\cdots,N-1.$$

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$$= S_k X_k + \bar F_k^T P_{k+1}(I + M_k P_{k+1})^{-1}\bar F_k X_k - \bar F_k^T P_{k+1}(M_k O_{k+1} + N_k) + \bar F_k^T O_{k+1} + T_k$$

$$= \underbrace{\left[S_k + \bar F_k^T P_{k+1}(I + M_k P_{k+1})^{-1}\bar F_k\right]}_{P_k}X_k + \underbrace{T_k + \bar F_k^T O_{k+1} - \bar F_k^T P_{k+1}(M_k O_{k+1} + N_k)}_{O_k}$$

$$\Lambda_1 = P_1 X_1 + O_1$$

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$$W_0 = -(L_0^{ww})^{-1}\left(L_0^{wx}X_0 + G_0^T P_1 (I+M_0P_1)^{-1}(\bar F_0 X_0 - M_0 O_1 - N_0) + G_0^T O_1 + L_0^{w\theta}\right)$$

$$= -(L_0^{ww})^{-1}\left[(L_0^{wx} + G_0^T P_1 (I+M_0P_1)^{-1}\bar F_0)X_0 - G_0^T P_1 (I+M_0P_1)^{-1}(M_0O_1+N_0) + G_0^T O_1 + L_0^{w\theta}\right]$$

$$= -(L_0^{ww})^{-1}(L_0^{wx} + G_0^T P_1 (I+M_0P_1)^{-1}\bar F_0)X_0 + (L_0^{ww})^{-1}G_0^T P_1 (I+M_0P_1)^{-1}(M_0O_1+N_0) - (L_0^{ww})^{-1}(G_0^T O_1 + L_0^{w\theta})$$

$\Lambda_1 = P_1 X_1 + O_1$ — X_0 is unknown and needs to be solved from $\nabla_{X_0} L$

($X_0 = 0$ holds for optimal control ~~since ...~~ because $x_0 = x(0) \Rightarrow \frac{\partial x_0}{\partial\theta} = 0$) constant

$= \cancel{P_1}\quad X_1 = (I+M_0P_1)^{-1}(\bar F_0 X_0 - M_0 O_1 - N_0)$

$\Lambda_1 = P_1(I+M_0P_1)^{-1}\bar F_0 X_0 - P_1(I+M_0P_1)^{-1}(M_0O_1+N_0) + O_1$

$X_0 \doteq \frac{\partial x_0}{\partial\theta}$

Below is the solution of X_0.

$$\nabla_{X_0} L = PX_0 - P\hat X_0 + \bar L_0^{xx} X_0 + L_0^{xw} W_0 + L_0^{x\theta} + F_0^T \Lambda_1 = 0.$$

$$= \underbrace{(P + \bar L_0^{xx})}_{L_0^{xx}} X_0 - P\hat X_0 + L_0^{xw} W_0 + L_0^{x\theta} + F_0^T \Lambda_1 = 0.$$

a function of X_0. W_0: a function of X_0.

X_1 is a function of X_0.

$$= L_0^{xx} X_0 - P\hat X_0 - L_0^{xw}(L_0^{ww})^{-1}(L_0^{wx} + G_0^T P_1(I+M_0P_1)^{-1}\bar F_0)X_0 + L_0^{xw}(L_0^{ww})^{-1}G_0^T P_1(I+M_0P_1)^{-1}(M_0O_1+N_0) - L_0^{xw}(L_0^{ww})^{-1}(G_0^TO_1 + L_0^{w\theta})$$

$$+ L_0^{x\theta} + F_0^T P_1(I+M_0P_1)^{-1}\bar F_0 X_0 - F_0^T P_1 (I+M_0P_1)^{-1}(M_0O_1+N_0) + F_0 O_1 = 0$$

We can solve X_0 from the above equation.

$$L_0^{xx}X_0 - L_0^{xw}(L_0^{ww})^{-1}L_0^{wx}X_0 - L_0^{xw}(L_0^{ww})^{-1}G_0^T P_1(I+M_0P_1)^{-1}\bar F_0 X_0 + F_0^T P_1 (I+M_0P_1)^{-1}\bar F_0 X_0 = \cdots$$

$$\underbrace{(L_0^{xx} - L_0^{xw}(L_0^{ww})^{-1}L_0^{wx})}_{S_0}X_0 + (F_0^T - L_0^{xw}(L_0^{ww})^{-1}G_0^T)P_1(I+M_0P_1)^{-1}\bar F_0 X_0 = \cdots$$

$$\bar F_0 = F_0 - \cancel{L_0^{xw}(L_0^{ww})^{-1}G_0}\; G_0 (L_0^{ww})^{-1}L_0^{wx}$$

$$S_0 X_0 + \bar F_0^T P_1 (I+M_0P_1)^{-1}\bar F_0 X_0 = \cdots$$

$$[S_0 + \bar F_0^T P_1 (I+M_0P_1)^{-1}\bar F_0]X_0 = \cdots$$

Summary: We can solve the auxiliary linear MHE using PDP-based method but with some modifications, e.g., solving for X_0!

Such a PDP-based method, however, is inconsistent with the continuous-time linear MHE where $\lambda_T = 0$.

Follow the PDP method

co-state: $\Lambda_k \quad k=1,\cdots,N \qquad \Lambda_{k+1}^T(F_kX_k + G_kW_k - X_{k+1})$

with $\Lambda_N = \underbrace{L_N^{xx}}_{P_N}X_N + L_N^{x\theta}$ from the terminal cost.

We need to solve for X_0, so direct application of PDP is impossible.

inconsistent with the continuous-time MHE

if $N=0$, then only Λ_1 remains, in this case, how can we determine its value?

There are two unknown variables, but only one equation!

$$\nabla_{X_0} L = PX_0 - P\hat X_0 + \bar L_0^{xx}X_0 + L_0^{x\theta} + F_0^T\Lambda_1 = 0.$$

Use Kalman-filter

co-state: $\Lambda_k \quad k=0,\cdots,N-1 \qquad \Lambda_k^T(X_{k+1} - F_kX_k - G_kW_k)$

with $\Lambda_N = 0$

P_0 is determined by the weight matrix in the arrival cost

(We also need to solve for X_0 using a Kalman-filter and Λ_0.)

Consistent with the continuous-time MHE, as both terminal co-states are zero!

Explore the solution structure.

When $N=0$, there is not such an issue, as $\Lambda_0 = 0$ by definition.

So X_0 can be easily obtained from its KKT condition:

$$\nabla_{X_0} L = PX_0 - P\hat X_0 + \bar L_0^{xx}X_0 + L_0^{x\theta} = 0.$$