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For DT-LMHE

the relation between $\underbrace{\hat{x}_{k|k}}_{\text{Kalman filter}}$ and $\underbrace{\hat{x}_{k|N}}_{\text{MHE}}$ is inspired by the initial condition.

Similarly, why not figure out the general relation between $\hat{x}_{t|t}$ and $\hat{x}_{t|T}$ from the initial boundary condition of CT-MHE?

$t=0$, $$\hat{x}_{0|T} = \hat{x}_0 + \boxed{\overbrace{P^{-1}}^{P_0}}\lambda_0$$: initial boundary condition of CT-MHE.

Hypothesis: $$\hat{x}_{t|T} = \hat{x}_{t|t} + P_t \cdot \lambda_t \quad (9)$$ (general solution/relation between $\hat{x}_{t|T}$ and $\hat{x}_{t|t}$)

$$\dot{\hat{x}}_{t|T} = \bar{F}_t \cdot \hat{x}_{t|T} + \bar{G}_t \cdot w_t \quad (10) \qquad w_t = Q^{-1} \bar{G}_t^T \lambda_t$$

$*$ $\hat{x}_{t|T} = \hat{x}_{t|t} + P_t \lambda_t \Longrightarrow$ Kalman Bucy filter.

$$\dot{\lambda}_t = -[H^T R (y_t - H \cdot \hat{x}_{t|T}) + \bar{F}_t^T \lambda_t] \quad (11)$$

~~Substituting $\hat{x}_{t|T} = \hat{x}_{t|t} + P_t^{-1}\lambda_t$~~

$$\& \ (10) \rightarrow \quad \dot{\hat{x}}_{t|T} = \bar{F}_t \hat{x}_{t|T} + \bar{G}_t Q^{-1} \bar{G}_t^T \cdot \lambda_t \quad (12)$$

Differentiating (9) w.r.t time:

$$\dot{\hat{x}}_{t|T} = \dot{\hat{x}}_{t|t} + \dot{P}_t \cdot \lambda_t + P_t \cdot \dot{\lambda}_t \quad (13)$$

Substituting (11) into (13) yields:

$$\dot{\hat{x}}_{t|T} = \dot{\hat{x}}_{t|t} + \dot{P}_t \lambda_t - P_t [H^T R (y_t - H \cdot \hat{x}_{t|T}) + \bar{F}_t^T \lambda_t]$$

$$\overset{(9)}{=} \dot{\hat{x}}_{t|t} + \dot{P}_t \lambda_t - P_t [H^T R (y_t - H \hat{x}_{t|t} - H P_t \lambda_t) + \bar{F}_t^T \lambda_t]$$

$$= \dot{\hat{x}}_{t|t} + \dot{P}_t \lambda_t - P_t [H^T R (y_t - H \cdot \hat{x}_{t|t}) - H^T R H P_t \lambda_t + \bar{F}_t^T \lambda_t]$$

$$= \dot{\hat{x}}_{t|t} + \dot{P}_t \lambda_t - P_t H^T R (y_t - H \cdot \hat{x}_{t|t}) + P_t H^T R H P_t \lambda_t - P_t \bar{F}_t^T \lambda_t \quad (14)$$

~~Eliminating w_t from (10) using $w_t = Q^{-1} \bar{G}_t^T \lambda_t$ given~~

Substituting (9) into (12) gives:

$$\dot{\hat{x}}_{t|T} = \bar{F}_t \hat{x}_{t|t} + \bar{F}_t P_t \cdot \lambda_t + \bar{G}_t Q^{-1} \bar{G}_t^T \lambda_t \quad (15)$$

Comparing (14) & (15) yields:

$$\dot{\hat{x}}_{t|t} = \bar{F}_t \cdot \hat{x}_{t|t} + \overbrace{P_t \cdot H^T}^{K} R (y_t - H \cdot \hat{x}_{t|t}) \qquad \hat{x}_{0|0} = \hat{x}_0$$

$$\dot{P}_t = \bar{F}_t \cdot P_t + P_t \bar{F}_t^T + \bar{G}_t Q^{-1} \bar{G}_t^T - \underbrace{P_t H^T}_{K} R H P_t \quad (16) \qquad P_0 = P^{-1} \text{ given.}$$

Eq. (16) is exactly the same as Kalman-Bucy filter!

$$\dot{\lambda}_t = -[H^T R (y_t - H \cdot \hat{x}_{t|t} - H \cdot P_t \lambda_t) + \bar{F}_t^T \lambda_t] \qquad \lambda_T = 0. \quad (17)$$

Lastly, the general solution to CT-LMHE is given by

$$\hat{x}_{t|T} = \hat{x}_{t|t} + P_t \lambda_t$$

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CT-LMHE analytical solution

$$\underline{\hat{x}_{t|T} = \hat{x}_{t|t} + P_t \cdot \lambda_t} \Longleftarrow \begin{cases} \dot{\lambda}_t = -H^T R (y_t - H \cdot \hat{x}_{t|t} - H P_t \lambda_t) - \bar{F}_t^T \lambda_t \\ \text{Kalman Bucy filter} \begin{cases} \dot{\hat{x}}_{t|t} = \bar{F}_t \cdot \hat{x}_{t|t} + P_t H^T R (y_t - H \cdot \hat{x}_{t|t}) \\ \dot{P}_t = \bar{F}_t P_t + P_t \cdot \bar{F}_t^T + \bar{G}_t Q^{-1} \bar{G}_t^T - P_t H^T R H P_t \end{cases} \end{cases}$$

We are going to prove that $x = \hat{x}_{t|t} + P_t \cdot \lambda_t$ is the solution to CT-LMHE ($\hat{x}_{t|T}$)

$$\dot{x} = \dot{\hat{x}}_{t|t} + \dot{P}_t \lambda_t + P_t \cdot \dot{\lambda}_t$$

$$= \underbrace{\bar{F}_t \cdot \hat{x}_{t|t} + P_t H^T R (y_t - H \cdot \hat{x}_{t|t})}_{\dot{\hat{x}}_{t|t}} + (\underbrace{\bar{F}_t P_t + P_t \cdot \bar{F}_t^T + \bar{G}_t Q^{-1} \bar{G}_t^T - P_t H^T R H P_t}_{\dot{P}_t}) \lambda_t$$

$$+ P_t [\underbrace{-H^T R (y_t - H \hat{x}_{t|t} - H P_t \lambda_t) - \bar{F}_t^T \lambda_t}_{\dot{\lambda}_t}]$$

$$= \bar{F}_t \cdot \hat{x}_{t|t} + \bar{F}_t P_t \cdot \lambda_t + \cancel{P_t H^T R y_t} - \cancel{P_t H^T R H \hat{x}_{t|t}} + \cancel{P_t \bar{F}_t^T \lambda_t} + \bar{G}_t Q^{-1} \bar{G}_t^T \lambda_t - \cancel{P_t H^T R H P_t \lambda_t}$$

$$- \cancel{P_t H^T R y_t} + \cancel{P_t H^T R H \hat{x}_{t|t}} + \cancel{P_t H^T R H P_t \lambda_t} - \cancel{P_t \bar{F}_t^T \lambda_t}$$

$$= \bar{F}_t (\underbrace{\hat{x}_{t|t} + P_t \cdot \lambda_t}_{x}) + \bar{G}_t Q^{-1} \bar{G}_t^T \lambda_t$$

$$= \bar{F}_t \cdot x + \bar{G}_t Q^{-1} \bar{G}_t^T \cdot \lambda_t \qquad (18)$$

Comparing Eq. (18) with Eq. (12), we can conclude that $x = \hat{x}_{t|T}$. #

$$\dot{\hat{x}}_{t|T} = \bar{F}_t \cdot \hat{x}_{t|T} + \bar{G}_t Q^{-1} \bar{G}_t^T \lambda_t$$