The Mathematician

March Edition

Mental Roots

If you have read some of our newspapers, you probably know about square and cube roots. This strategy is only shown for these roots, but feel free to try to extend it to a larger root, like the 10th root if you want to try.

Let's start with the square roots of 961. First, 1 is in the ones place and it is a 3 or 4 digit number, and 1^1 has a value of 1 in the ones place, so our answer is _1. Now, pay attention to 9, not 6. 9 is 3^2 or at least close to that or greater, or equal like in this case. This means that 3 is in the tens place so the square root of 961 is 31. Now try the square root of 2704. If you have problems trying to find the numbers, just square a digit.

For solving, 4 is 2^2 so 2 is in the ones place. 27 is greater than 5^2, but we are okay with that so our answer is 52. This trick only works for perfect squares.

For cube rooting 4-6 digit numbers, do a similar thing. So the cube root of 2197 is solved like this. 7 is in the ones place for 3^3, so 3 is in the ones place, and 2 is between 1^3 and 2^3, and 1 is less, meaning it is more accurate. So our answer is 13. For 5 digit numbers, split it into 2 and 3 digits in either way depending on the situation. For 6 digits, just split into every 3 hundreds. So the cube root of 74088 should have a _ in the ones place and _ in the tens place. That means your answer is __. Fill the blanks.

If your answer had the first _ as 2, and the 2nd _ was 4, and the __ was 42, you have probably mastered mental square and cube rooting. Now a challenge. What is the 4th root of 14641? If you are smart, you can tell it is square rooting twice, but after the first attempt, you will get a 3 digit number(Hint: think about divisibility rules). If you found 11 working (1-4+6-4+1=0) then you know this answer is a multiple of 11. Since 1 is in the ones place, then the answer is 11. As a check, put 11*11*11*11 into a calculator or by hand. You should have gotten 14641!

Mental Squares

Before, we talked about mentally rooting numbers, now we will do the opposite! How do you mentally square numbers?

Let's start with two digit numbers. Every two digit number can be split into the tens digit and the ones digit. If the tens part is x and the ones part is y (ex. In 47, x = 40, y = 7), then the number is x + y. The square of this is $(x + y)^2$, which can be expanded to $x^2 + 2xy + y^2$. To square a number, just do each of these and add them together. Of course, this can still be quite mentally tormenting and slow, so we can use these following tricks.

Firstly, if this number is close to a multiple of ten, like 49, we can rewrite this as 50 - 1 instead. $(50 - 1)^2 = 50^2 + 1^2 - 2 \times 50 \times 1$. This is much easier to calculate. 50^2 is just 5^2 with 2 zeros at the end, 1^2 is always 1, and $2 \times 50 \times 1 = 100$. At the end, we get 2500 + 1 - 100 = 2401. This also works if the multiple is a little above a multiple of ten. Try using this method to calculate 42^2, then look at our solution: $42^2 = (40 + 2)^2 = 40^2 + 2^2 + 2 \times 40 \times 2 = 1600 + 4 + 160 = 1764$.

But what if this number is close to the middle, like 75? If the number ends in 5, we multiply the tens digit by the number one above it, and that forms the first two digits. Then, we add 25 to the end to finish this square. For example, $75^2 = 5625$, where $56 = 7 \times 8$. But why is this true? Well, any number ending in five can be represented by 10a + 5 ('a' is the tens digit value. Why do we have to multiply ten?) Using the squaring methods above, we can get that $(10a + 5)^2 = 10a \times 10a + 2 \times 10a \times 5 + 5^2$. Simplifying, we get $10a \times 10a + 10 \times 10a + 25$. Combining the first two terms, we can get $(10a + 10) \times 10a + 25$. $(10a + 10) \times 10a$, in other words, means the tens digit multiplied by the tens term above it. Since we multiply two tens in the product, the result always ends in 00. This means that once we add 25, the one and tens digit will remain 25. If you don't understand this proof, feel free to come to us in recess or lunch.

But we still haven't covered the case where the number we need is 64 or 66. Well, we can use 65 how we used the multiples of 10 in the above methods. 64 = 65 - 1, so $64^2 = (65 - 1)^2$. Try to figure out the rest of the steps before looking at our steps. Okay, time to tell the solution: $(65 - 1)^2 = 65^2 - 2 \times 65 \times 1 + 1^2 = 4225 - 130 + 1 = 4096$. Of course, this is a bit harder to calculate than it would be using multiples of 10, but after practice, it will be quite easy to do mentally.

We have covered all these methods and tricks, but we are limiting ourselves to two digit numbers. Why stop there? Let's try three digit numbers now. Again, the fives trick works. Except now, the hundreds and tens digit count as one. For example, $125 \times 125 = 15625$, where $156 = 12 \times 13$. All our other tricks work as well, they just take more practice. Truthfully, all these tricks need much practice before the answer appears to you instantly. After much practicing, many people, named mental calculators, can square four digit, perhaps five digit numbers in their head, as well as multiply random four digit numbers together instantly, such as Arthur Benjamin and so much more (if you are interested, you can check out his TED talk on 'Mathemagic').

Mental Multiplication

We have talked about squares and roots, but why stop there? How impressive would it be to be able to mentally multiply two-digit numbers? Guess what? That's what we are studying here!

Firstly, there is some algebra we need to go over. $(x + y)(x - y) = x^2 - y^2$. If we expand this, we can see why. This is equal to $x^2 - xy + xy - y^2 = x^2 - y^2$. This can help with many problems. If we can turn two hard to multiply numbers into two easy squares, the multiplication will be a lot simpler. For example, $38 \times 42 = 40^2 - 2^2 = 1600 - 4 = 1596$. This is easy enough to do inside your head. Try to use this method on 72×78. $72 \times 78 = 75^2 - 3^2$. If you recall from previous chapters, $75^2 = 5625$, so the answer is $5625 - 9 = 5616$. This method can be used to prove a couple other tricks.

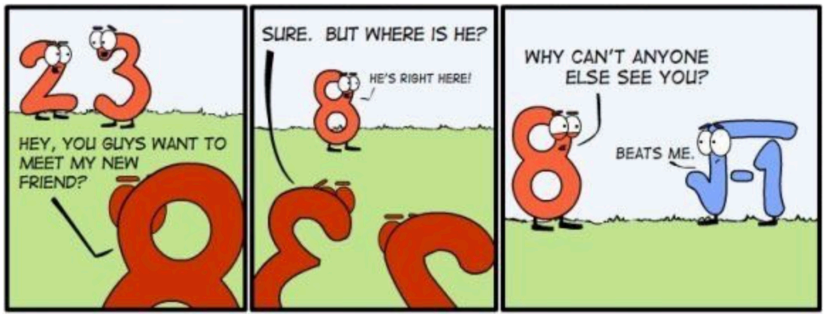
The first trick is that if two numbers have the same tens digit and ones digit that sum to 10, then the first two digits of the product are the product of the tens digit and the number one more than that, while the last two digits are the product of the two ones digits. For example, $43 \times 47 = 2021$ because $4 \times 5 = 20$, $3 \times 7 = 21$. Why is this true? Well, the average of these two numbers is always something ending with 5. Remember, $(x - y)(x + y) = x^2 - y^2$. These two numbers can be expressed as this format, with x equal to something ending in 5, while y is a small one digit number. After some simplification, we can get the formula for these types of numbers. If you don't understand, feel free to ask our members.

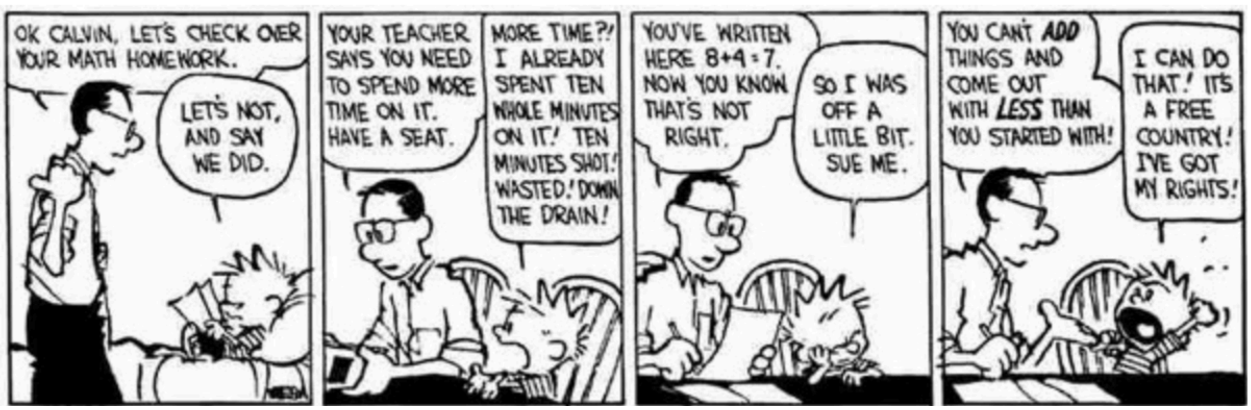
The second trick is that if two numbers have the same ones digit numbers and two tens digit numbers that sum up to 10, the product of them is the two tens digit numbers multiplied with the ones digit added to the product, followed by the square of the ones digit. For example, if we want to calculate 26×86, the first two digits will be $2 \times 8 + 6$, and the last two will be 6^2, so the result will be 2236. Why is this true? Try to prove this yourself. If you want the answer, contact any of us in the math club.

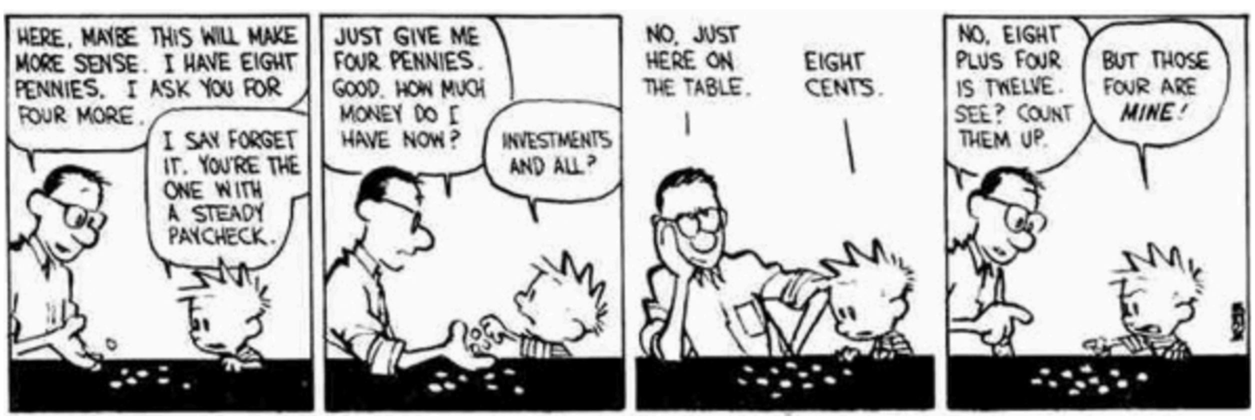
There are also many more tricks. We will share two more. Firstly, if you want to multiply two numbers a little more than 100, we can first write one, then write the sum of the last two digits, then the product of the last two digits. For example, $122 \times 103 = 12566$, with $22 + 3 = 25$, $22 \times 3 = 66$. Why is this true? First try to prove it yourself, then look at our solution. Firstly, the two numbers can be represented as $100 + 10x + y$ and $100 + 10a + b$. Each variable is one digit. The product, after being expanded, is $(100 + 10x + y)(100 + 10a + b) = 10000 + 1000(a + x) + 100(b + y) + 10(bx + ay) + by$. How does this turn into our formula? Try to figure it out yourself, and if you need help, feel free to find us for help. Does this formula have any restrictions? How much bigger can these numbers be until it doesn't work?

Our last trick is used to multiply numbers a little smaller than 100. This trick is quite hard to describe, so I will demonstrate using numbers. Say, 98×93. The difference between each number and 100 is 2 and 7. The first two digits are $100 - 2 - 7$, while the second two digits are 2×7. Then, the product is 9114. See the trick? Try to prove this. Feel free to explore more tricks that make multiplication easier. Have fun!

Math Comics


HEY, YOU GUYS WANT TO MEET MY NEW FRIEND?
SURE. BUT WHERE IS HE?
HE'S RIGHT HERE!
WHY CAN'T ANYONE ELSE SEE YOU?
BEATS ME.


OK CALVIN, LET'S CHECK OVER YOUR MATH HOMEWORK.
LET'S NOT, AND SAY WE DID.
YOUR TEACHER SAYS YOU NEED TO SPEND MORE TIME ON IT. HAVE A SEAT.
MORE TIME?! I ALREADY SPENT TEN WHOLE MINUTES ON IT! TEN MINUTES SHOT! WASTED! DOWN THE DRAIN!
YOU'VE WRITTEN HERE 8+4=7. NOW YOU KNOW THAT'S NOT RIGHT.
SO I WAS OFF A LITTLE BIT. SUE ME.
YOU CAN'T ADD THINGS AND COME OUT WITH LESS THAN YOU STARTED WITH!
I CAN DO THAT! IT'S A FREE COUNTRY! I'VE GOT MY RIGHTS!


HERE, MAYBE THIS WILL MAKE MORE SENSE. I HAVE EIGHT PENNIES. I ASK YOU FOR FOUR MORE.
I SAY FORGET IT. YOU'RE THE ONE WITH A STEADY PAYCHECK.
JUST GIVE ME FOUR PENNIES. GOOD. HOW MUCH MONEY DO I HAVE NOW?
INVESTMENTS AND ALL?
NO, JUST HERE ON THE TABLE.
EIGHT CENTS.
NO, EIGHT PLUS FOUR IS TWELVE. SEE? COUNT THEM UP.
BUT THOSE FOUR ARE MINE!

Game Time:

Today's game time is not exactly a game. But it's still quite interesting. So you might have come across a problem while trying our methods for mental calculation: The bigger these numbers get, the harder it is to remember one number while trying to calculate the other. This issue is quite difficult to solve. However, after digging into the methods of the best mental calculators in the world, Their method is to use a mnemonic device. Basically, each number can be turned into a consonant sound, and vowels are added in the middle to make words. The mnemonic device that Arthur Benjamin uses is:

1 = t or d	5 = l	9 = p or b
2 = n	6 = j, ch, or sh	0 = z or s
3 = m	7 = k or g	
4 = r	8 = f or v	

This takes a bit of practice to memorize and use well, so try to turn the numbers 1 - 100 into words. Then, try to turn them back randomly. This memorization trick can even help you memorize the digits of pi! We did the first couple digits:

My turtle Pancho will, my love, pick up my new mover, Ginger
3 .1 415 9 26 5 3 5 8 9 7 9 3 2 3 8 4 6 26 4

Notice that because we assign each number as a sound, the G in Ginger is the j sound, and hence a 6, not a 7. In addition, since the two l's in will make one l sound, it counts as one 5.

Try to make more digits!

About The Authors

There were many people that contributed to this amazing Mathematician article. Firstly, to Rohan Hallur for doing the first article, about mental roots. Brooks Wang wrote Mental Squares, Mental Multiplication, and Game Time. Thanks for every Mathematician Member who edited and revised this amazing newspaper, including:

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If you want to join the staff, again, feel free to contact any of us, and as always, thanks for reading, keep on learning!