The Mathematician

Christmas Edition

Imaginary Numbers

Christmas–Advanced

Do you know about square roots? Well, you probably know that the square root of 4 is 2 and blah blah blah. That's easy. However, have you ever thought about the square root of -4? Think about this. Is it possible? For example,:

$(-2)^2 = -2\times-2 = 4$

$5^2 = 5\times5 = 25$

$(1.3)^2 = 1.3\times1.3 = 1.69$

No matter whether the squared number is negative or not, the result is never negative. So, we find that nothing square rooted can be negative.Then what number results in a negative square? It is an imaginary number. In math, we come across the square root of negative numbers many times, especially in the case of solving quadratic equations using the quadratic formula. In such cases, the usage of imaginary numbers is mandatory. However, how can we write imaginary numbers? We use i (iota), where $i = \sqrt{(-1)}$ In other words, $i^2 = -1$. To make other values, like $\sqrt{-3}$, we add a coefficient in front of it.

I know that sounds confusing, but I'll make it clearer, with an example. Take the expression 3i. $3i = 3 \times \sqrt{(-1)}$. We can also square this expression. $(3i)^2 = 9 \times i^2 = 9 \times -1 = -9$.

Try some other examples. Can you find 2i? How about 5i? What do you think -2i is?

What about $i\sqrt{3}$? Well, we can think about it this way: $\sqrt{3} = \sqrt{-1} * \sqrt{-3} = \sqrt{-3} * i$. So, $i\sqrt{3} = i * i \sqrt{-3} = -1 * \sqrt{-3}$. Hm. We can see that each of these numbers is a product of a non-zero real number and i. Thus, we can derive a rule for imaginary numbers which is:

- **$i\sqrt{x} = -1 * \sqrt{-x}$**

Try a couple more examples. Can you prove it? Try.

Let's try some more practices:

What is the simplest form of:

- $\sqrt{-16}$
- $\sqrt{-20}$
- $i\sqrt{4}$
- $i\sqrt{25}$
- $i\sqrt{120}$

This is only the tip of the iceberg of imaginary numbers. They are often used in much more complex ways, but this is enough for today. Keep on thinking, and have fun, mathematicians!

Infinity

Christmas—Intermediate

Have you ever wondered what the biggest number in the world is? A thousand? A million? A bazillion? A googol? Well, you'd be wrong. There actually isn't one number that is bigger than any other number! We'll prove that now. Think about the biggest number you know. You got your number? Ok., good. NOw add one to that number. Congratulations! You just made your big number bigger! Now add one again. You just made that number even bigger. You can add one to what you previously thought was the biggest number, and make it one bigger. This loop will occur endlessly. Soon, you will see that because of this, there is no actual 'biggest number'. Instead, mathematicians use the concept of infinity. There are many types of infinity, however, today we are focussing on the infinity reached by counting forever. There are other types of infinity, but that is for another day. Therefore, today we will think of infinity as a number, so we can add, subtract, multiply, divide, etc. Infinity is used as a way for mathematicians to represent something bigger than any number. We use the symbol ∞ when speaking and writing of the term. When talking about infinity, however, proves to be quite a dilemma. Infinity is not a number, so what happens if you add 1 to infinity? What if you add 2? What if you multiply by 5? Some people argue that because infinity is not a number, adding 1 would not make sense, as if we were adding one to an apple. However, most people say that it is possible. These people use an interesting example called Hilbert's Hotel to demonstrate this idea. It goes like this: Imagine Hilbert's Hotel is a hotel that has infinite rooms. So, every time someone new comes in, everybody in residency is moved one room down. So, if 1 new person comes,the manager asks the guest in room 1 to move to room 2, the guest in room 2 to move into room 3, the guest in room 3 to move into room 4, and so on. If there were only finitely many rooms, the guest in the last room would have nowhere to go, but since there are infinitely many, everybody will find a new abode. If 2 new people come, the person in room one will move to room 3, the person in room 2 will move to room 4, the person in room 3 will move to room 5, etc. This shows that infinity guests, + one guest, still equals infinity guests. So, $\infty + 1 = \infty$, $\infty + 200{,}543 = \infty$, and ∞ + any number still equals infinity. Now, what if that 'other number' is also infinity? What is infinity plus infinity? In Hilbert's Hotel, Infinity, we can also represent this. Assume that another hotel with infinite rooms and infinite guests has a plumbing issue, so they must move their residents to the Hilbert Hotel. If we use our usual way, we need everyone to move infinite rooms down. However, that is impossible. We need another method. Before I explain, I would like you to think about it.

Did you get it? Well, we can do it like this. We ask the person in room 1 to move to room 2, the person in room 2 to move to room 4, the person in room 3 to move to room 6, and the person in room 'x' to move to room 2x. By doing this, we make all the older residents occupy the even numbers, and free up all the odd numbers. Because there are infinite odd numbers, we free up a room for all the new guests. We see that 2 hotels worth of infinite people can fit into 1 hotel of infinite rooms. So, $\infty + \infty = \infty$. Hooray! All the guests fit into place, and word about this grand hotel has spread. Travelers from abroad come and stay at this magical hotel. Unfortunately, too many people come. One day, the hotel manager sees infinite buses, each full of infinite people waiting to get inside this hotel. This is an interesting dilemma, and I'm going to let you figure it out yourself. Have fun!

About the Authors

Thanks to Brooks Wang, Junxiao Wu, and Hady Jalloul for writing this edition, as well as everyone else in the math club for providing ideas and edits. If anyone wants to join the crew please contact any of us and we'll get you started right away. In addition, if you are curious or have any more questions about the topics covered in this edition, please contact us and we will be more than happy to help you with your questions. Thank you for reading, and keep on exploring math!