# Dynamic Programming

## Chapter 2: Operators and Fixed Points

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# Summary

- Conjugate maps
- Convergence rates
- Newton's method
- Partial orders
- Pointwise orders
- Order-preserving maps
- Fixed points and order
- Linear operators

## Conjugate Maps

Suppose we are concerned with the dynamics induced by a self-map $T$ on $\mathbb{R}^n$

- does a unique fixed point of $T$ exist?
- do iterates of $T$ always converge to a fixed point?

Option A: Apply fixed point theory to $T$

Option B:

1. transform $T$ into a "simpler" operator $\hat{T}$
2. apply fixed point theory to $\hat{T}$
3. translate properties we discover about $\hat{T}$ back to $T$

To implement Option B we need the following definitions:

A **dynamical system** is a pair $(U, T)$, where

- $U$ is a subset of $\mathbb{R}^n$ and
- $T$ is a self-map on $U$

Dynamical systems $(U, T)$ and $(\hat{U}, \hat{T})$ are called **conjugate** under $\Phi$ if

1. $\Phi$ is a bijection from $U$ into $\hat{U}$ and
2. $T = \Phi^{-1} \circ \hat{T} \circ \Phi$ on $U$

The condition $T = \Phi^{-1} \circ \hat{T} \circ \Phi$ can be understood as follows:

Shifting a point $u \in U$ to $Tu$ via $T$ is equivalent to

1. shifting $u$ from $U$ to $\hat{U}$ via $\hat{u} = \Phi u$
2. applying $\hat{T}$, and
3. shifting the result back to $U$ via $\Phi^{-1}$:

$$\begin{tikzcd} u \arrow[r, "T"] \arrow[d, "\Phi"] & Tu \\ \hat{u} \arrow[r, "\hat{T}"] & \hat{T}\hat{u} \arrow[u, "\Phi^{-1}"'] \end{tikzcd}$$

**Ex.** (Log-linearization) Fix $A > 0$, $\alpha \in \mathbb{R}$ and suppose

- $U := (0, \infty)$ and $Tu = Au^\alpha$
- $\hat{U} := \mathbb{R}$ and $\hat{T}\hat{u} = \ln A + \alpha\hat{u}$

Show that $(U, T)$ and $(\hat{U}, \hat{T})$ are conjugate under $\Phi := \ln$

Proof: The condition $T = \Phi^{-1} \circ \hat{T} \circ \Phi$ is equivalent to

$$\Phi \circ T = \hat{T} \circ \Phi$$

This holds because, for $u \in \mathbb{R}$,

- $\Phi T u = \ln A + \alpha \ln u$
- $\hat{T} \Phi u = \ln A + \alpha \ln u$

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- $\Phi\, T\, u = \ln A + \alpha \ln u$
- $\hat{T}\, \Phi\, u = \ln A + \alpha \ln u$

**Ex.** Let $(U, T)$ and $(\hat{U}, \hat{T})$ be conjugate under $\Phi$

Show that the following statements are equivalent

1. $u \in U$ is a fixed point of $T$ on $U$
2. $\Phi u \in \hat{U}$ is a fixed point of $\hat{T}$ on $\hat{U}$

Proof: Let $(U, T)$ and $(\hat{U}, \hat{T})$ be as stated

- then $\hat{T} \circ \Phi = \Phi \circ T$

The claimed equivalence holds because

$$Tu = u \iff \Phi T u = \Phi u \iff \hat{T} \Phi u = \Phi u$$

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# Topological Conjugacy

Let $U$ and $\hat{U}$ be two subsets of $\mathbb{R}^n$

$\Phi\colon U \to \hat{U}$ is called a **homeomorphism** if it is a continuous bijection and $\Phi^{-1}$ is also continuous

Example. $\Phi = \ln$ is a homeomorphism from $(0, \infty) \to \mathbb{R}$ with

$$\Phi^{-1} = \exp$$

Example. Let $A$ be an $n \times n$ matrix understood as a map $u \mapsto Au$

$$A\colon \mathbb{R}^n \to \mathbb{R}^n \text{ is a homeomorphism} \iff A \text{ is nonsingular}$$

Let

- $(U, T)$ and $(\hat{U}, \hat{T})$ be two dynamical systems
- $\Phi$ be a map from $U$ to $\hat{U}$

We call $(U, T)$ and $(\hat{U}, \hat{T})$ **topologically conjugate** under $\Phi$ if

1. $(U, T)$ and $(\hat{U}, \hat{T})$ are conjugate under $\Phi$ and
2. $\Phi$ is a homeomorphism

Example. An $n \times n$ matrix $A$ is called **diagonalizable** if $\exists$ a diagonal matrix $D$ and a nonsingular matrix $P$ such that

$$A = P^{-1}DP$$

- $D$ and $P$ can be complex-valued
- we view $A$ as a self-map on $\mathbb{R}^n$
- and $D$ and a self-map on $\mathbb{C}^n$

Note that $P$ and $P^{-1}$ are continuous bijections

- linear maps between finite-dimensional spaces are continuous

Hence $(A, \mathbb{R}^n)$ and $(D, \mathbb{C}^n)$ are topologically conjugate

**Ex.** Let $(U, T)$ and $(\hat{U}, \hat{T})$ be topologically conjugate under $\Phi$

Given $u, u^* \in U$, show that

$$T^k u \to u^* \quad \iff \quad \hat{T}^k \Phi u \to \Phi u^*$$

<u>Proof</u>: From $\hat{T} = \Phi \circ T \circ \Phi^{-1}$ we can show that

$$\hat{T}^k = \Phi \circ T^k \circ \Phi^{-1} \quad \text{for all } k \in \mathbb{N}$$

Hence, using continuity of $\Phi$ and $\Phi^{-1}$,

$$T^k u \to u^* \iff \Phi T^k u \to \Phi u^* \iff \hat{T}^k \Phi u \to \Phi u^*$$

**Ex.** Let $(U, T)$ and $(\hat{U}, \hat{T})$ be topologically conjugate under $\Phi$

Given $u, u^* \in U$, show that

$$T^k u \to u^* \quad \Longleftrightarrow \quad \hat{T}^k \Phi u \to \Phi u^*$$

Proof: From $\hat{T} = \Phi \circ T \circ \Phi^{-1}$ we can show that

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Hence, using continuity of $\Phi$ and $\Phi^{-1}$,

$$T^k u \to u^* \iff \Phi T^k u \to \Phi u^* \iff \hat{T}^k \Phi u \to \Phi u^*$$

Let $(U, T)$ and $(\hat{U}, \hat{T})$ be two dynamical systems

**Proposition**. If $(U, T)$ and $(\hat{U}, \hat{T})$ are topologically conjugate, then

$$(U, T) \text{ is globally stable } \iff (\hat{U}, \hat{T}) \text{ is globally stable}$$

Moreover, if one and hence both are globally stable, then the unique fixed points $u^* \in U$ and $\hat{u}^* \in \hat{U}$ satisfy

$$\hat{u}^* = \Phi u^*$$

**Ex.** Prove this

# Local Stability

Let $U$ be a subset of $\mathbb{R}^n$ and let $T$ be a self-map on $U$

A fixed point $u^*$ of $T$ in $U$ is called **locally stable** for $T$ if $\exists$ an open set $O \subset U$ such that

$$u^* \in O \text{ and } T^k u \to u^* \text{ as } k \to \infty \text{ for every } u \in O$$

**Ex.** Let $(U, T)$ and $(\hat{U}, \hat{T})$ be topologically conjugate and let $u^*$ be a fixed point of $T$ in $U$

Show that

$$u^* \text{ is locally stable for } T \iff \Phi u^* \text{ is locally stable for } \hat{T}$$

# Derivative tests

One way to verify local stability of a one-dimensional map $g$ at fixed point $x^*$ is to show that $|g'(x^*)| < 1$

Intuition: The first-order linear approximation of $g$ near $x^*$ is

$$\begin{aligned}\hat{g}(x) &:= g(x^*) + g'(x^*)(x - x^*) \\ &= x^* + g'(x^*)(x - x^*)\end{aligned}$$

When $|g'(x^*)| < 1$, the map $\hat{g}$ is a contraction of modulus $|g'(x^*)|$

Moreover, $\hat{g}$ and $g$ have "the same" dynamics near $x^*$

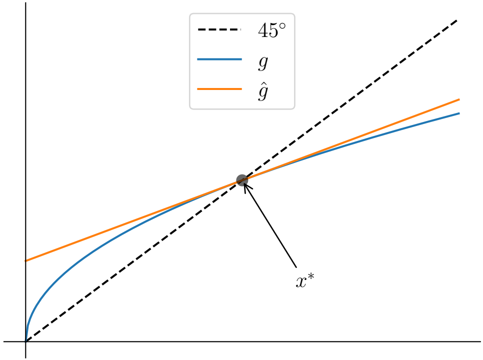



Figure: Local stability when $|g'(x^*)| < 1$

Let's formalize this argument and generalize to vector space

Take $T$ to be a continuously differentiable self-map on $U \subset \mathbb{R}^n$ with fixed point $u^*$

Recall that the **Jacobian** of $T$ at $u \in U$ is

$$J_T(u) := \begin{pmatrix} \frac{\partial T_1}{\partial u_1}(u) & \cdots & \frac{\partial T_1}{\partial u_n}(u) \\ & \cdots & \\ \frac{\partial T_n}{\partial u_1}(u) & \cdots & \frac{\partial T_n}{\partial u_n}(u) \end{pmatrix} \quad \text{where} \quad Tu = \begin{pmatrix} T_1 u \\ \vdots \\ T_n u \end{pmatrix}$$

We set $\hat{T}$ to be the first-order approximation to $T$ at $u^*$:

$$\hat{T}u = u^* + J_T(u^*)(u - u^*) \qquad (u \in U)$$

**Theorem**. (Hartman–Grobman) If $J_T(u^*)$

1. is nonsingular and
2. has no eigenvalues on the unit circle in $\mathbb{C}$,

then $\exists$ an open neighborhood $O$ of $u^*$ such that $(O, T)$ and $(O, \hat{T})$ are topologically conjugate

In particular,

$$u^* \text{ locally stable for } T \text{ whenever } \hat{T} \text{ globally stable on } \mathbb{R}^n$$

Recall that

$$\hat{T}u = u^* + J_T(u^*)(u - u^*) \qquad (u \in U)$$

By the Neumann series lemma,

$$\rho(J_T(u^*)) < 1 \implies \hat{T} \text{ is globally stable on } \mathbb{R}^n$$

**Corollary**. Under the conditions of the Hartman–Grobman theorem,

$$\rho(J_T(u^*)) < 1 \quad \implies \quad u^* \text{ is locally stable for } T$$

# Convergence Rates

Fix norm $\|\cdot\|$ on $\mathbb{R}^n$

In what follows

1. $(u_k)_{k\geqslant 0} \subset \mathbb{R}^n$ converges to $u^* \in \mathbb{R}^n$
2. $e_k := \|u_k - u\|$

In particular, we have

$$e_k \to 0 \qquad (k \to \infty)$$

We wish to quantify the rate of convergence

We say $(u_k)_{k\geqslant 0}$ converges to $u^*$ **at rate at least $q$** if

1. $q \geqslant 1$
2. for some $\beta \in (0,\infty)$ and $N \in \mathbb{N}$,

$$e_{k+1} \leqslant \beta e_k^q \quad \text{for all } k \geqslant N$$

We say that convergence occurs **at rate $q$** if, in addition,

$$\limsup_{k\to\infty} \frac{e_{k+1}}{e_k^q} = \beta$$

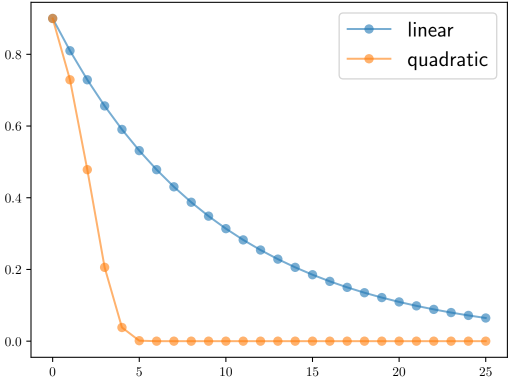
- If $q = 2$ we say that convergence is (at least) **quadratic**
- If $q = 1$ and $\beta < 1$, we say convergence is (at least) **linear**

Example. With $|\beta| \in (0,1)$ and $q \geqslant 1$, the real sequence

$$x_{k+1} = \beta x_k^q, \qquad |x_0| < 1$$

converges to $0$ at rate $q$ because

$$\begin{aligned}
\limsup_{k\to\infty} \frac{e_{k+1}}{e_k^q} &= \limsup_{k\to\infty} \frac{|x_{k+1}|}{|x_k^q|} \\
&= \limsup_{k\to\infty} \frac{|\beta||x_k^q|}{|x_k^q|} \\
&= |\beta|
\end{aligned}$$


linear
quadratic
0.0
0.2
0.4
0.6
0.8
0
5
10
15
20
25

Example. Suppose $u_k$ converges to $u^*$ quadratically, so that

$$\limsup_{k\to\infty} \frac{e_{k+1}}{e_k^2} = \beta$$

Suppose also that $\beta$ is not large

If, say, $e_k = 10^{-5}$, then

$$e_{k+1} \approx \beta 10^{-10} \approx 10^{-10}$$

Number of accurate digits roughly doubles at each step

Example. Let

- $T$ be a contraction of modulus $\lambda$ on closed set $U \subset \mathbb{R}^n$
- $u^*$ be the unique fixed point of $T$ in $U$
- $u$ be fixed and $u_k := T^k u$

Then $(u_k)$ converges at least linearly to $u^*$, since

$$\begin{aligned} e_{k+1} &= \|u_{k+1} - u^*\| \\ &= \|Tu_k - Tu^*\| \\ &\leqslant \lambda \|u_k - u^*\| \\ &= \lambda e_k \end{aligned}$$

Rule of thumb: Successive approximation often converges linearly

The next exercise provides some evidence

**Ex.** Let $T \colon U \to U$ be smooth on open interval $U$ in $\mathbb{R}$

Suppose that

- $T$ has a fixed point $u^* \in U$ and
- $u_k := T^k u_0$ converges to $u^*$ as $k \to \infty$

Prove that the rate of convergence is linear whenever

$$0 < |T' u^*| < 1$$

Hint: Taylor expansion yields a $v_k \in (u_k, u^*)$ such that

$$Tu_k = u^* + T'u^*(u_k - u^*) + \frac{T''v_k}{2}(u_k - u^*)^2$$

Proof: Since $u_{k+1} = Tu_k$, we have

$$\frac{u_{k+1} - u^*}{u_k - u^*} = T'u^* + \frac{T''v_k}{2}(u_k - u^*)$$

$$\therefore \quad \limsup_{k\to\infty} \frac{e_{k+1}}{e_k} = |T'u^*|$$

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$$\therefore \quad \limsup_{k\to\infty} \frac{e_{k+1}}{e_k} = |T'u^*|$$

# Newton's Method

Successive approximation is typically linear

Faster algorithms can often be obtained for smooth functions

Strategy: leverage the extra information provided by gradients

One important gradient-based technique is Newton's method

Suppose that

- $T$ is a differentiable self-map on an open set $U \subset \mathbb{R}^n$
- our aim is to find a fixed point of $T$

Our plan: start with a guess $u_0$ and then update it to $u_1$

To do this we

1. construct the first-order approximation $\hat{T}$ of $T$ around $u_0$
2. obtain the fixed point of $\hat{T}$ (exactly, since $\hat{T}$ is linear)

We take this new point $u_1$ and repeat

The first order approximation is at $u_0$

$$\hat{T}u := Tu_0 + J_T(u_0)(u_1 - u_0)$$

where $J_T(u_0)$ is the Jacobian of $T$ at $u_0$

Now solve $\hat{T}u_1 = u_1$ for $u_1$, which gives

$$u_1 = (I - J_T(u_0))^{-1}(Tu_0 - J_T(u_0)u_0) \qquad (I = \text{identity})$$

Then repeat the process starting at $u_1$, etc.

More generally, define

$$u_{k+1} = Qu_k \qquad (k \geqslant 0)$$

where

$$Qu := (I - J_T(u))^{-1}(Tu - J_T(u)u) \qquad k = 0, 1, \ldots$$

Newton's fixed point method: apply successive approximation to $Q$

The next figure shows $u_0, u_1$ when $n = 1$ and $Tu = 1 + u/(u+1)$

- $u_0 = 0.5$
- $u_1$ is the fixed point of $\hat{T}$
- $u_1$ is closer to the fixed point of $T$ than $u_0$, as desired

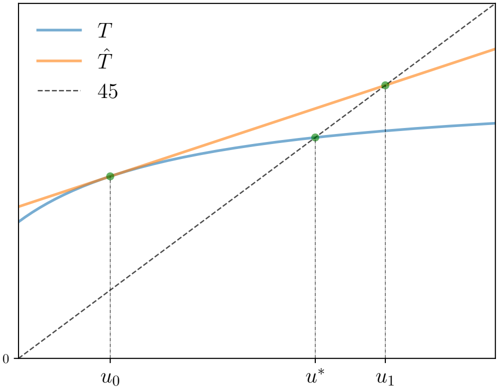


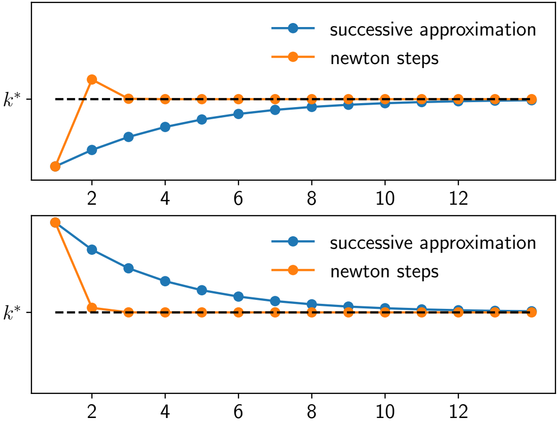

Figure: The first step of Newton's method applied to $T$

The next figure shows an application to computing the fixed point of the Solow–Swan model

We use

- both Newton's method and successive approximation
- same initial conditions

Both sequences converge but the Newton sequences converge faster


successive approximation
newton steps
$k^*$
2
4
6
8
10
12
successive approximation
newton steps
$k^*$
2
4
6
8
10
12

Fast rates of convergence can be confirmed theoretically for the Newton scheme

Under mild conditions, $\exists$ a neighbourhood of the fixed point within which the Newton iterates converge quadratically

- See the text for references

This fast rate of convergence will be significant when we study dynamic programming algorithms

- An algorithm called Howard policy iteration is a version of Newton's method

# Parallelization

Successive approx. is highly serial: cannot compute $T^{k+1}u$ until $T^k u$ is available

- slow convergence
- many sequential steps

Newton's method is also serial — we are just iterating with a different map — but

- fewer steps
- each one is more computationally intensive

Hence well suited to parallelization

# Automatic differentiation

Many software platforms now offer automatic differentiation

- simular to symbolic (exact) differentiation
- but more numerically efficient
- also more efficient/stable than numerical differentiation

Automatic differentiation can be used for computing Jacobians in the Newton step

Combines well with parallelization

# Order

Let's review the fundamentals of **order theory**

One of the foundational subjects of maths, on par with

- algebra
- geometry
- topology, etc.

But not commonly taught in foundational math courses

Why?

Rarely used in

- physics
- chemistry
- biology, etc.

Math courses are biased toward these subjects!

But not commonly taught in foundational math courses

Why?

Rarely used in

- physics
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Math courses are biased toward these subjects!

But very important for econ / OR / finance

Examples.

- Does consumer X prefer good A or good B?
- Is welfare greater under policy A or policy B?
- Does R & D increase profits?
- How can firm Y minimize costs?

In these lectures, we need order for

- studying optimality
- fixed point results

# Partial orders

Let $P$ be a nonempty set

A **partial order** on a $P$ is a binary relation $\preceq$ on $P \times P$ satisfying, for any $p, q, r$ in $P$,

$p \preceq p$,

$p \preceq q$ and $q \preceq p$ implies $p = q$ and

$p \preceq q$ and $q \preceq r$ implies $p \preceq r$

(Reflexivity, antisymmetry, transitivity)

We call $(P, \preceq)$ a **partially ordered set**

- $P := (P, \preceq)$ when $\preceq$ understood

**Ex.**

1. Show that the usual order $\leqslant$ on $\mathbb{R}$ is a partial order on $\mathbb{R}$
2. Given set $M$, show that $\subset$ is a partial order on $\wp(M)$

Proof for 2: Clearly, for all $A, B, C \subset M$,

- $A \subset A$ holds
- $A \subset B$ and $B \subset A$ implies $A = B$
- $A \subset B$ and $B \subset C$ implies $A \subset C$

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A partial order $\preceq$ on $P$ is called a **total order** if

$$\text{either } p \preceq q \text{ or } q \preceq p \text{ for all } p, q \in P$$

Example. $\leqslant$ is a total order on $\mathbb{R}$

**Ex.** Prove: $\subset$ is not a total order on $\wp(M)$ when $|M| > 1$

Proof: If $|M| \geqslant 2$, then $\exists$ nonempty $A, B \subset M$ with $A \cup B = \emptyset$

But then $A \subset B$ and $B \subset A$ both fail

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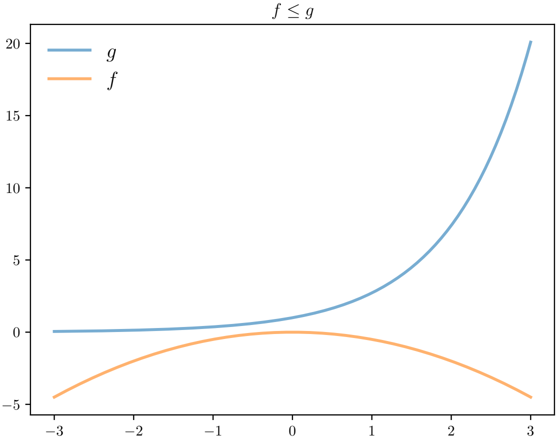
# Pointwise Orders

Let

- $\mathsf{X}$ be any set
- $\mathbb{R}^{\mathsf{X}}$ be all $f\colon \mathsf{X} \to \mathbb{R}$

The **pointwise order** over $\mathbb{R}^{\mathsf{X}}$ is written as $\leqslant$ and defined via

$$f \leqslant g \iff f(x) \leqslant g(x) \text{ for all } x \in \mathsf{X}$$


$f \leq g$
$g$
$f$
20
15
10
5
0
−5
−3
−2
−1
0
1
2
3

**Ex.** Show $\leqslant$ is a partial order on $\mathbb{R}^{\mathsf{X}}$

Proof:

Let's just check antisymmetry

Fix $f, g \in \mathbb{R}^{\mathsf{X}}$ and suppose $f \leqslant g$ and $g \leqslant f$

Pick any $x \in \mathsf{X}$

By definition, $f(x) \leqslant g(x)$ and $g(x) \leqslant f(x)$

Therefore, $f(x) = g(x)$

Since $x$ was arbitrary, we have $f = g$

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Let's define the **pointwise order for matrices**

Let $\mathbb{M}^{n\times k} :=$ all $n \times k$ matrices

For $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{M}^{n\times k}$, we set

$$A \leqslant B \iff a_{ij} \leqslant b_{ij} \text{ for all } i, j$$

Example.

$$\begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix} \leqslant \begin{pmatrix} 10 & 20 \\ 0 & 10 \end{pmatrix}$$

**Ex.** Show that $\leqslant$ is a partial order on $\mathbb{M}^{n\times k}$

Special case: **pointwise order for vectors**

Recall $[n] := \{1, \ldots, n\}$

For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, we write

$$x \leqslant y \quad \iff \quad x_i \leqslant y_i \text{ for all } i \in [n]$$

Pointwise order $\leqslant$ on $\mathbb{R}^2$:

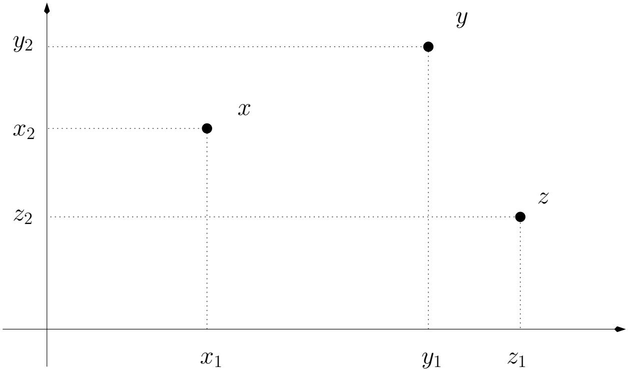


Figure: $x \leqslant y$ but neither $x, z$ nor $y, z$ are comparable

**Ex.** Prove: for $a, b \in \mathbb{R}^n$ and sequence $(x_k)$ in $\mathbb{R}^n$, we have

$$a \leqslant x_k \leqslant b \text{ for all } k \in \mathbb{N} \text{ and } x_k \to x \text{ implies } a \leqslant x \leqslant b$$

Proof: Fix $i \in [n]$

Let $a^i$ be the $i$-th element of $a$, etc.

It suffices to show that

$$a^i \leqslant x^i \leqslant b^i \tag{1}$$

Note $x_k \to x$ implies $x_k^i \to x^i$

Moreover, $a^i \leqslant x_k^i \leqslant b^i$ for all $k$

Weak inequalities in $\mathbb{R}$ are preserved under limits, so (1) holds

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Weak inequalities in $\mathbb{R}$ are preserved under limits, so (1) holds

In other words, the pointwise order $\leqslant$ is preserved under limits

As a result, these sets are **closed**

- $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : 0 \leqslant x\}$
- $[a, b] := \{x \in \mathbb{R}^n : a \leqslant x \leqslant b\}$
- etc.

A key connection between order and topology!

**Ex.** Prove: If $B$ is $m \times k$ and $B \geqslant 0$, then

$$|Bx| \leqslant B|x| \text{ for all } k \times 1 \text{ column vectors } x$$

Proof: Fix $B \in \mathbb{M}^{m\times k}$ with $b_{ij} \geqslant 0$ for all $i, j$

Fix $i \in [m]$ and $x \in \mathbb{R}^k$

By the triangle inequality, we have $|\sum_j b_{ij}x_j| \leqslant \sum_j b_{ij}|x_j|$

Stacking these inequalities yields

$$|Bx| \leqslant B|x|$$

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Stacking these inequalities yields

$$|Bx| \leqslant B|x|$$

**Lemma.** If X is finite and $f, g, h \in \mathbb{R}^{\mathsf{X}}$, then

1. $|f+g| \leqslant |f| + |g|$
2. $(f \wedge g) + h = (f+h) \wedge (g+h)$
3. $(f \vee g) + h = (f+h) \vee (g+h)$
4. $(f \vee g) \wedge h = (f \wedge h) \vee (g \wedge h)$
5. $(f \wedge g) \vee h = (f \vee h) \wedge (g \vee h)$
6. $|f \wedge h - g \wedge h| \leqslant |f - g|$
7. $|f \vee h - g \vee h| \leqslant |f - g|$

Also, if $f, g, h \in \mathbb{R}_+^{\mathsf{X}}$, then

$$(f+g) \wedge h \leqslant (f \wedge h) + (g \wedge h) \tag{2}$$

**Ex.** Prove: If $a, b, c \in \mathbb{R}_+$, then $|a \wedge c - b \wedge c| \leqslant |a - b| \wedge c$

Proof: Fix $a, b \in \mathbb{R}_+$ and $c \in \mathbb{R}_+$

By (2), we have

$$a \wedge c = (a - b + b) \wedge c \leqslant (|a - b| + b) \wedge c \leqslant |a - b| \wedge c + b \wedge c$$

Thus, $a \wedge c - b \wedge c \leqslant |a - b| \wedge c$

Reversing the roles of $a$ and $b$ gives $b \wedge c - a \wedge c \leqslant |a - b| \wedge c$

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Reversing the roles of $a$ and $b$ gives $b \wedge c - a \wedge c \leqslant |a-b| \wedge c$

# Least and Greatest Elements

In dynamic programming, our aim is to maximize lifetime value

This is a function over the state space — not a number

Thus, the objective takes values in a partially ordered set (a set of functions over the state space)

This leads us to consider "least" and "greatest" elements, rather than traditional maxima and minima

Definitions follow...

Let $(P, \preceq)$ be a partially ordered set

Given $A \subset P$, we say that

- $\ell \in P$ is a **least element** of $A$ if
$$\ell \in A \text{ and } a \succeq \ell \text{ for all } a \in A, \text{ and}$$
- $g \in P$ is a **greatest element** of $A$ if
$$g \in A \text{ and } a \preceq g \text{ for all } a \in A.$$

**Ex.** Let $P$ be any partially ordered set and fix $A \subset P$. Prove that $A$ has at most one greatest element and at most one least element.

Example. The pointwise case

Let X be a nonempty set and $V$ be a subset of $\mathbb{R}^{\mathsf{X}}$

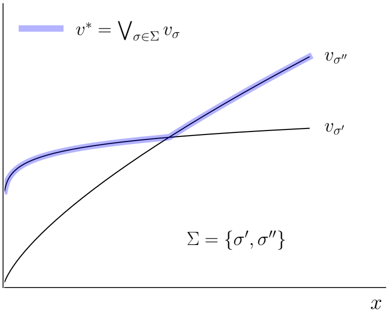
- $V$ as a partially ordered set under the pointwise order $\leqslant$

Let $\{v_\sigma\} := \{v_\sigma\}_{\sigma \in \Sigma}$ be a finite collection of functions in $V$

Let $\vee_\sigma v_\sigma$ be the pointwise maximum

If $\vee_\sigma v_\sigma \in \{v_\sigma\}$, then $\vee_\sigma v_\sigma$ is the greatest element of $\{v_\sigma\}$

Example. The set $\{v_\sigma\}$ has no greatest element: neither $v_{\sigma'} \leqslant v_{\sigma''}$ nor $v_{\sigma''} \leqslant v_{\sigma'}$



# Order-preserving maps

Let

- $(P, \preceq)$ and $(Q, \trianglelefteq)$ be partially ordered sets
- $T \colon P \to Q$

$T$ is called **order-preserving** if, for all $x, y \in P$,

$$x \preceq y \quad \implies \quad Tx \trianglelefteq Ty$$

- Meaning: If $x$ goes up then $Tx$ goes up
- Very important concept for dynamic programming

Example. Let $(P, \preceq) = (\mathscr{C}, \leqslant)$ where

- $\mathscr{C}$ is all continuous functions from $[a, b]$ to $\mathbb{R}$
- $\leqslant$ is the pointwise order

If $I : \mathscr{C} \to \mathbb{R}$ is defined by

$$Ig := \int_a^b g(x)dx \qquad (g \in \mathscr{C})$$

then $I$ is order-preserving on $\mathscr{C}$

(Larger functions have larger integrals)

Example. Let $\leqslant$ denote the pointwise order on $\mathbb{R}^n$

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Tx = Ax + b$

If $A \geqslant 0$, then $T$ is order preserving on $\mathbb{R}^n$

Proof: Fix $x \leqslant y$

Then $0 \leqslant y - x$

$$\therefore \quad 0 \leqslant A(y - x) \leqslant Ay - Ax$$

$$\therefore \quad Ax \leqslant Ay$$

$$\therefore \quad Tx \leqslant Ty$$

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# Special Case: Real-Valued Functions

Special case: maps from $(P, \preceq)$ into $(\mathbb{R}, \leqslant)$

Then "order-preserving" $=$ "increasing"

In particular, we also call $h \in \mathbb{R}^P$

- **increasing** if $x \preceq y$ implies $h(x) \leqslant h(y)$ and
- **decreasing** if $x \preceq y$ implies $h(x) \geqslant h(y)$

Let $P$ be partially ordered by $\preceq$

We write $i\mathbb{R}^P$ **for the increasing functions in** $\mathbb{R}^P$

Thus,

$$h \in i\mathbb{R}^P \quad \iff \quad x, y \in P \text{ and } x \preceq y \text{ implies } h(x) \leqslant h(y)$$

Example. Let $P = \{1, \ldots, n\}$ and let $\preceq$ be the usual order $\leqslant$ on $\mathbb{R}$

Then

- $x \mapsto 2x$ and $x \mapsto \mathbb{1}\{2 \leqslant x\}$ are in $i\mathbb{R}^P$
- $x \mapsto -x$ and $x \mapsto \mathbb{1}\{x \leqslant 2\}$ are not

**Ex.** Prove the following:

If $f, g \in i\mathbb{R}^P$, then

- $\alpha f + \beta g \in i\mathbb{R}^P$ when $\alpha, \beta \geqslant 0$
- $f \vee g \in i\mathbb{R}^P$
- $f \wedge g \in i\mathbb{R}^P$

**Ex.** Given finite $P$, show that $i\mathbb{R}^P$ is closed in $\mathbb{R}^P$

Proof: Take $(f_k)_{k \geqslant 1}$ in $i\mathbb{R}^P$ and $f \in \mathbb{R}^P$ with $f_k \to f$

Since $f_k \to f$ we have $f_k(z) \to f(z)$ for all $z \in P$

- norm convergence implies pointwise convergence

Fix $x, y \in P$ with $x \preceq y$

From $(f_k) \subset i\mathbb{R}^P$ we have $f_k(x) \leqslant f_k(y)$ for all $k$

Since weak inequalities are preserved under limits, $f(x) \leqslant f(y)$

Hence $f \in i\mathbb{R}^P$

## Strict inequalities

We write

- $f \ll g$ if $f(x) < g(x)$ for all $x \in$ some given set $M$
- $x \ll y$ if $x_i < y_i$ for all $i \in [n]$
- $A \ll B$ if $a_{ij} < b_{ij}$ for all $i, j$

These are not partial orders

**Ex.** Why is $f \ll g$ not a partial order on $\mathbb{R}^M$?

# Order Isomorphisms

Let $P$ and $\hat{P}$ be two partially ordered sets and let $\Phi$ be a map from $P$ to $\hat{P}$

The map $\Phi$ is called

- an **order isomorphism** if $\Phi$ is bijective and order-preserving, and
- an **order anti-isomorphism** if $\Phi$ is bijective and order-reversing.

Example. If $P = \hat{P} = \mathbb{R}^n_+$, then $p \mapsto p^2$ is an order isomorphism

**Ex.** Check: If $\Phi$ is an order isomorphism from $P$ to $\hat{P}$, then $\Phi^{-1}$ is order-preserving.

**Ex.** Suppose there exists an order isomorphism from $P$ to $\hat{P}$. Prove: If $P$ is totally ordered, then so is $\hat{P}$.

# Blackwell's Condition

Fix $U \subset \mathbb{R}^{\mathsf{X}}$ with X finite

Assume $u \in U$ and $c \in \mathbb{R}_+$ implies $u + c \in U$

Let $T$ be an order preserving self-map on $U$

**Lemma.** If there exists a constant $\beta \in (0, 1)$ such that

$$T(u + c) \leqslant Tu + \beta c \quad \text{for all } u \in U \text{ and } c \in \mathbb{R}_+$$

then $T$ is a contraction of modulus $\beta$ on $U$ w.r.t. $\| \cdot \|_\infty$

Proof:

Let $U, T$ have the stated properties and fix $u, v \in U$

We have

$$\begin{aligned} Tu &= T(v + u - v) \\ &\leqslant T(v + \|u - v\|_\infty) \\ &\leqslant Tv + \beta\|u - v\|_\infty \end{aligned}$$

Hence

$$Tu - Tv \leqslant \beta\|u - v\|_\infty$$

Reversing the roles of $u$ and $v$ proves the claim

# Parametric Monotonicity

Let $(P, \preceq)$ be a partially ordered set

Given two self-maps $S$ and $T$ on $P$, we set

$$S \preceq T \iff Sx \preceq Tx \text{ for every } x \in P$$

We say that $T$ **dominates** $S$ on $P$

**Ex.** Show that $\preceq$ is a partial order on

$$\mathscr{S}_P := P^P := \text{ set of all self-maps on } P$$

Proof of antisymmetry of $\preceq$ on $\mathscr{S}_P$:

Let $(P, \preceq)$ and $S, T \in \mathscr{S}_P$ be as defined above

Suppose $S \preceq T$ and $T \preceq S$

Fix any $x \in P$

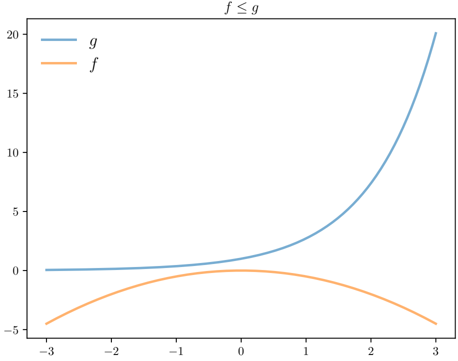
We have $Sx \preceq Tx$ and $Tx \preceq Sx$

Since $\preceq$ is antisymmetric on $P$, we have $Sx = Tx$

Since $p$ was arbitrary, $S = T$

Hence $\preceq$ is antisymmetric on $\mathscr{S}_P$

Example. If $(\preceq, P) = (\leqslant, \mathbb{R})$, then $\leqslant$ is the pointwise order over functions



Example. Consider $\mathbb{R}^n_+$ with the pointwise order $\leqslant$

- Called the **positive cone** in $\mathbb{R}^n$

Let

- $Sx = Ax + b$
- $Tx = Bx + b$

**Ex.** Show that $0 \leqslant A \leqslant B \implies T$ dominates $S$ on $\mathbb{R}^n_+$

Proof: Fixing $x \in \mathbb{R}^n_+$, suffices to show that $Sx \leqslant Tx$

Since $A \leqslant B$ and $x \geqslant 0$, we have $Ax \leqslant Bx$

Hence $Sx \leqslant Tx$

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Since $A \leqslant B$ and $x \geqslant 0$, we have $Ax \leqslant Bx$

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**Conjecture**: If $S \leqslant T$, then the fixed points of $T$ will be larger

This is not true in general...

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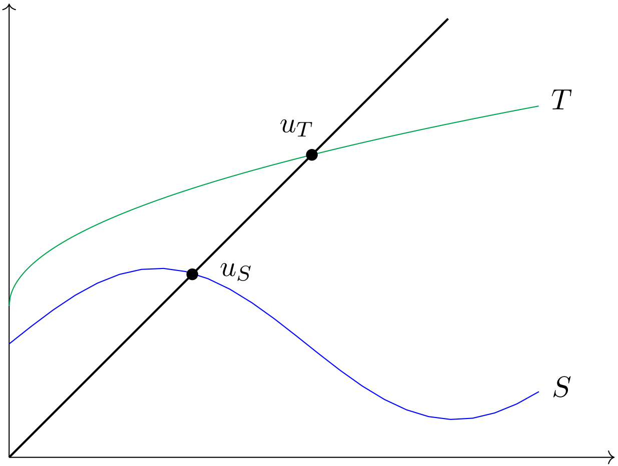
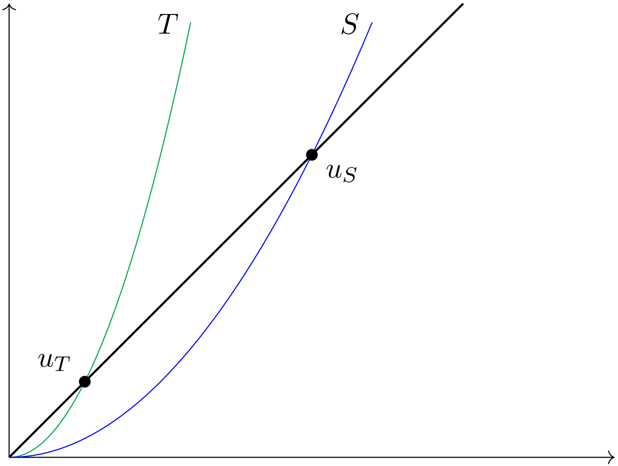
Sometimes true:



And sometimes false:



One difference: in the first case, $T$ is globally stable

This leads us to our next result

**Proposition.** Let

- $S$ and $T$ be self-maps on $M \subset \mathbb{R}^n$
- $\leqslant$ be the pointwise order on $M$

If

1. $T$ dominates $S$ on $M$ and
2. $T$ is order-preserving and globally stable on $M$,

then the unique fixed point of $T$ dominates any fixed point of $S$

Proof: Assume the conditions

Let

- $u_T$ be the unique fixed point of $T$ and
- $u_S$ be any fixed point of $S$

Since $S \leqslant T$, we have $u_S = Su_S \leqslant Tu_S$

Applying $T$ to both sides of $u_S \leqslant Tu_S$ gives

$$u_S \leqslant Tu_S \leqslant T^2 u_S$$

Continuing in this fashion yields $u_S \leqslant T^k u_S$ for all $k \in \mathbb{N}$

Since $\leqslant$ is preserved under limits and $T$ is globally stable,

$$u_S \leqslant \lim_k T^k u_S = u_T$$

Example. Recall that, in the job search model,

$$h^* = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, \, h^* \right\} \varphi(w')$$

We found $h^*$ as the fixed point of $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$g(h) = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, \, h \right\} \varphi(w')$$

In the exercise, you showed that $g$ is a contraction map on $\mathbb{R}_+$

**Ex.** Prove that the optimal continuation value $h^*$ is increasing in $\beta$

Proof: Fix $\beta_1 \leqslant \beta_2$ and let

- $h_i^* :=$ fixed point corresponding to $\beta_i$
- $g_i :=$ fixed point map corresponding to $\beta_i$

Since $\beta_1 \leqslant \beta_2$, we have $g_1(h) \leqslant g_2(h)$ for all $h \in \mathbb{R}_+$

In addition, $g_2$ is

1. a contraction (so globally stable) and
2. increasing (order-preserving)

Hence $h_1^* \leqslant h_2^*$

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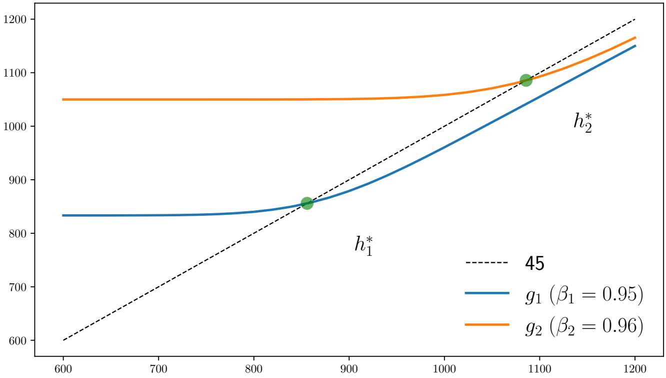
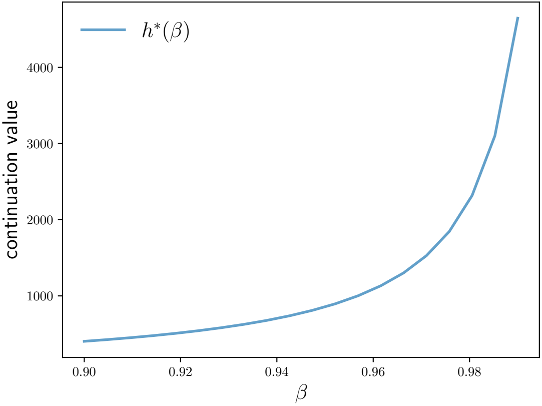

1200
1100
1000
900
800
700
600
600
700
800
900
1000
1100
1200
$h_1^*$
$h_2^*$
45
$g_1$ ($\beta_1 = 0.95$)
$g_2$ ($\beta_2 = 0.96$)

**Ex.** Replicate this figure



# Reminders

Def. $(\lambda, v) \in \mathbb{C} \times \mathbb{C}^n$ is an **eigenpair** of $n \times n$ matrix $A$ if

$$v \neq 0 \quad \text{and} \quad Av = \lambda v$$

The **eigenspace** of eigenvalue $\lambda$ is

$$E_\lambda := \{w \in \mathbb{C}^n : w = 0 \text{ or } (\lambda, w) \text{ is an eigenpair of } A\}$$

**Ex.** Show that $E_\lambda$ is a linear subspace of $\mathbb{C}^n$

Proof: If $v, w \in E_\lambda$ and $\alpha, \beta \in \mathbb{C}$, then

$$A(\alpha w + \beta v) = \alpha A w + \beta A v = \alpha \lambda w + \beta \lambda v = \lambda(\alpha w + \beta v)$$

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$$A(\alpha w + \beta v) = \alpha A w + \beta A v = \alpha \lambda w + \beta \lambda v = \lambda(\alpha w + \beta v)$$

Implication: exists a continuum of eigenvectors paired with $\lambda$

So what can we say about uniqueness?

Let $(\lambda, v)$ be an eigenpair for $A$

Def. $v$ has (geometric) **multiplicity one** if $\dim E_\lambda = 1$

In other words,

$$w \in E_\lambda \implies w = \alpha v \text{ for some } \alpha \in \mathbb{C}$$

In a sense, there is "just one" eigenvector corresponding to $\lambda$, since any other is a scalar multiple

# Nonnegative Matrices

Def. Matrix $A$ is called

- **nonnegative**, and we write $A \geqslant 0$, if all elements of $A$ are nonnegative
- **positive**, and we write $A \gg 0$, if every element of $A$ is strictly positive
- **irreducible** if it is square, nonnegative and

$$\sum_{k \in \mathbb{N}} A^k \gg 0$$

Note: positive $\implies$ irreducible $\implies$ nonnegative

Let $A$ be $n \times n$

It is <u>not</u> always true that $\rho(A)$ is an eigenvalue of $A$

Example. Let

$$A := \begin{pmatrix} -1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Set of eigenvalues (the **spectrum**) of $A$ is $\sigma(A) = \{-1, 1/2\}$

Hence $\rho(A) = |-1| = 1 \notin \sigma(A)$

However, when $A \geqslant 0$, we have the following result

**Theorem.** (Perron–Frobenius) If $A \geqslant 0$, then $\rho(A)$ is an eigenvalue of $A$ with nonnegative, real-valued right and left eigenvectors

In particular, there exists

- a nonnegative, nonzero column vector $e$ s.t. $Ae = \rho(A)e$
- a nonnegative, nonzero row vector $\varepsilon$ s.t. $\varepsilon A = \rho(A)\varepsilon$

If $A$ **is irreducible**, then these eigenvectors are everywhere positive and have multiplicity of one

If $A$ **is positive**, then with $e$ and $\varepsilon$ such that $\langle \varepsilon, e \rangle = 1$, we have

$$\rho(A)^{-t} A^t \to e\,\varepsilon \qquad (t \to \infty)$$

In this setting,

- $\rho(A)$ is also called the **dominant eigenvalue**
- $e$ is called the **dominant right eigenvector**
- $\varepsilon$ is called the **dominant left eigenvector**

Note also

$$\varepsilon A = \rho(A)\varepsilon \quad \iff \quad A^\top \varepsilon^\top = \rho(A)\varepsilon^\top$$

Hence $\varepsilon^\top$ is the dominant right eigenvector of $A^\top$

Since the dominant eigenvectors are only defined up to constant multiples, we often normalize so that $\langle \varepsilon, e \rangle = 1$

Let's check these results for arbitrary positive $A$

```
julia> using LinearAlgebra

julia> A = [0.3 0.9;
            1.0 0.1];

julia> λ_1, λ_2 = eigvals(A)
2-element Vector{Float64}:
 -0.7539392014169456
  1.1539392014169458

julia> rA = λ_2    # r(A) is the positive eigenvalue
1.1539392014169458
```

```
julia> right_evecs = eigvecs(A)
2×2 Matrix{Float64}:
 -0.649386  0.725426
  0.760459  0.6883

julia> e = right_evecs[:, 2]   # dominant right eigenvector
2-element Vector{Float64}:
 0.7254262498099013
 0.6882999027217298

julia> left_evs = eigvecs(A')  # transpose to get left eigenvector
2×2 Matrix{Float64}:
 -0.6883    0.760459
  0.725426  0.649386

julia> ϵ = left_evs[:, 2]'     # dominant left eigenvector
1×2 adjoint(::Vector{Float64}) with eltype Float64:
 0.760459  0.649386
```

```
#  Checking the eigenpair relations

 julia> A * e
2-element Vector{Float64}:
 0.8370977873925273
 0.7942562400820743

julia> rA * e
2-element Vector{Float64}:
 0.8370977873925274
 0.7942562400820744


julia> ϵ * A
1×2 adjoint(::Vector{Float64}) with eltype Float64:
 0.877524  0.749352

julia> rA * ϵ
1×2 adjoint(::Vector{Float64}) with eltype Float64:
 0.877524  0.749352
```

```
#  The matrix A is everywhere positive
#
#  Hence we expect, for large k,
#
#       r(A)^(-k) * A^k ≈ e ϵ


julia> k = 1000
1000

julia> rA^(-k) * A^k
2×2 Matrix{Float64}:
 0.552414  0.471728
 0.524142  0.447586

julia> e * ϵ
2×2 Matrix{Float64}:
 0.551657  0.471082
 0.523424  0.446972
```

# Bounds on the spectral radius

Fix $n \times n$ matrix $A$ and set

- $\mathrm{rs}_i(A) :=$ the $i$-th row sum of $A$ and
- $\mathrm{cs}_j(A) :=$ the $j$-th column sum of $A$

**Corollary**. If $A \geqslant 0$, then

1. $\min_i \mathrm{rs}_i(A) \leqslant \rho(A) \leqslant \max_i \mathrm{rs}_i(A)$ and
2. $\min_j \mathrm{cs}_j(A) \leqslant \rho(A) \leqslant \max_j \mathrm{cs}_j(A)$

**Ex.** Prove this via the PF theorem

Proof for the column sum case

Fix $A \geqslant 0$ and let $e$ be the dominant right eigenvector

We normalize $e$ by setting $\mathbb{1}^\top e = \sum_j e_j = 1$

From $\rho(A)e = Ae$ we have

$$\rho(A) = \rho(A)\mathbb{1}^\top e = \mathbb{1}^\top(\rho(A)e) = \mathbb{1}^\top Ae = \sum_j \mathrm{cs}_j(A)e_j$$

Therefore, $\rho(A)$ is a weighted average of the column sums

Hence $\min_j \mathrm{cs}_j(A) \leqslant \rho(A) \leqslant \max_j \mathrm{cs}_j(A)$

# Stochastic Matrices

Let $P$ be a square matrix

Def. $P$ is called **stochastic** if $P \geqslant 0$ and $P\mathbb{1} = \mathbb{1}$

**Ex.** Show that $P$ is stochastic $\implies \rho(P) = 1$

Row vector $\psi$ is called a **stationary distribution** of $P$ if

$$\psi \geqslant 0, \quad \psi\mathbb{1} = 1 \quad \text{and} \quad \psi P = \psi$$

Stationary distributions very important for Markov dynamics...

# Existence of Stationary Distributions

Let $P$ be a stochastic matrix

**Ex.** Prove: $P$ has at least one stationary distribution

Proof: By the PF theorem,

$\exists$ a nonzero, nonnegative row vector $\varphi$ satisfying $\varphi P = \varphi$

Since $\varphi$ is nonzero, $\varphi \mathbb{1} > 0$

Setting $\psi := \varphi / (\varphi \mathbb{1})$ gives the desired vector

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**Ex.** Prove: $P$ has at least one stationary distribution

Proof: By the PF theorem,

$$\exists \text{ a nonzero, nonnegative row vector } \varphi \text{ satisfying } \varphi P = \varphi$$

Since $\varphi$ is nonzero, $\varphi \mathbb{1} > 0$

Setting $\psi := \varphi / (\varphi \mathbb{1})$ gives the desired vector

# Uniqueness of Stationary Distributions

**Ex.** Prove: If $P$ is also irreducible, then the stationary vector $\psi$ is everywhere positive and unique

Proof of Positivity: See Perron–Frobenius theorem

Proof of Uniqueness: Let $\varphi \geqslant 0$ satisfy $\varphi \mathbb{1} = 1$ and $\varphi P = \varphi$

By the Perron–Frobenius theorem, $\varphi = \alpha \psi$ for some $\alpha > 0$

But then $1 = \varphi \mathbb{1} = \alpha \psi \mathbb{1} = \alpha$

Hence $\varphi = \psi$

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```julia
julia> P = [0.2 0.8;
            0.1 0.9]
2×2 Matrix{Float64}:
    0.2  0.8
    0.1  0.9

julia> using QuantEcon

julia> mc = MarkovChain(P)
    Discrete Markov Chain
    stochastic matrix of type Matrix{Float64}:
    [0.2 0.8; 0.1 0.9]

julia> is_irreducible(mc)
true

julia> stationary_distributions(mc)
    1-element Vector{Vector{Float64}}:
    [0.1111111111111111, 0.8888888888888888]
```

# Lake Model of Employment

An illustration of the Perron–Frobenius theorem

We analyze a model of employment and unemployment flows in a large population

The model is sometimes called a "lake model"

Two "pools" of workers:

- those who are currently employed and
- those who are currently unemployed but still seeking work

FP theorem helps us analyze dynamics

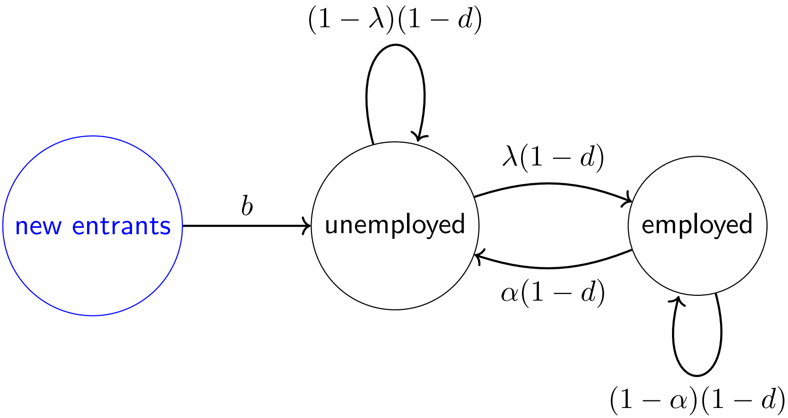
Workers

- **exit** the workforce at rate $d$
- **enter** the workforce at rate $b$
- **separate** from their jobs at rate $\alpha$
- **find jobs** at rate $\lambda$

Assumptions:

- All parameters lie in $(0, 1)$
- New workers are initially unemployed

Transition rates:



Let

- $u_t :=$ number of **unemployed workers** at time $t$
- $e_t :=$ number of **employed workers**
- $n_t := e_t + u_t :=$ total **population** of workers

Dynamics are

$$u_{t+1} = (1-d)\alpha e_t + (1-d)(1-\lambda)u_t + bn_t$$

$$e_{t+1} = (1-d)(1-\alpha)e_t + (1-d)\lambda u_t$$

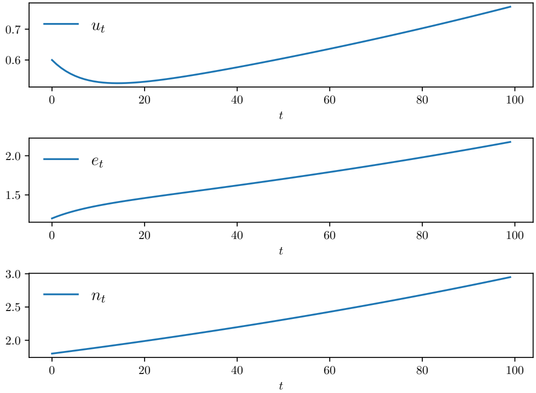



Figure: Example simulation when $b > d$ (population growth)

Can we say more about the dynamics of this system?

For example,

- what long run unemployment rate should we expect?
- do outcomes depend on the initial conditions $u_0$ and $e_0$?
- Or are there general statements we can make?

We define

$$x_t := \begin{pmatrix} u_t \\ e_t \end{pmatrix}$$

and

$$A := \begin{pmatrix} (1-d)(1-\lambda)+b & (1-d)\alpha+b \\ (1-d)\lambda & (1-d)(1-\alpha) \end{pmatrix}$$

Dynamics can now be written

$$x_{t+1} = Ax_t$$

Hence

$$x_t = A^t x_0 \quad \text{where} \quad x_0 = \begin{pmatrix} u_0 \\ e_0 \end{pmatrix}$$

**Ex.** With $g := b - d$, show that $n_{t+1} = (1+g)n_t$ for all $t$

Proof: The column sums of $A$ are

$$(1-d)(1-\lambda) + b + (1-d)\lambda = 1 + g$$

and

$$(1-d)\alpha + b + (1-d)(1-\alpha) = 1 + g$$

From $x_{t+1} = Ax_t$ and $n_t = u_t + e_t$ we have

$$n_{t+1} = \mathbb{1}^\top x_{t+1} = \mathbb{1}^\top A x_t = (1+g)\mathbb{1}^\top x_t = (1+g)n_t$$

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**Ex.** Prove that $\rho(A) = 1 + g$

Proof: We know that

$$\min_j \mathrm{cs}_j(A) \leqslant \rho(A) \leqslant \max_j \mathrm{cs}_j(A)$$

Hence $1 + g \leqslant \rho(A) \leqslant 1 + g$

PF theorem $\implies$ $1 + g$ is the dominant eigenvalue of $A$

**Ex.** Show that $\mathbb{1}^\top := (1\ 1)$ is the dominant left eigenvector of $A$

Proof:

$$\mathbb{1}^\top A = \begin{pmatrix} 1+g & 1+g \end{pmatrix} = \rho(A)\mathbb{1}^\top$$

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**Ex.** Prove that

$$\bar{x} := \begin{pmatrix} \bar{u} \\ \bar{e} \end{pmatrix}$$

with

$$\bar{u} := \frac{1 + g - (1-d)(1-\alpha)}{1 + g - (1-d)(1-\alpha) + (1-d)\lambda} \quad \text{and} \quad \bar{e} := 1 - \bar{u}$$

is the dominant right eigenvector of $A$

Proof: Just show $A\bar{x} = (1+g)\bar{x}$

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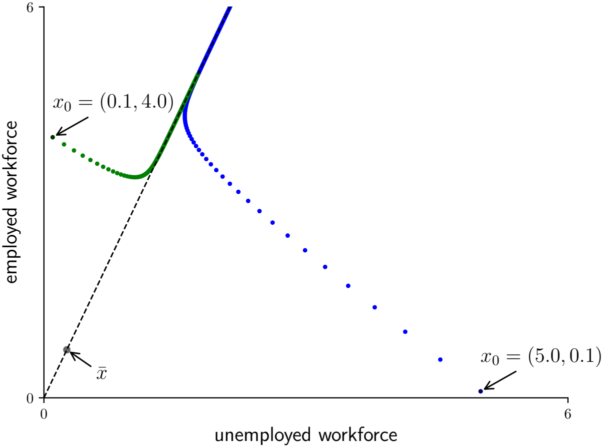
```julia
using LinearAlgebra

α, λ, d, b = 0.01, 0.1, 0.02, 0.025
g = b - d
A = [(1 - d) * (1 - λ) + b     (1 - d) * α + b;
     (1 - d) * λ               (1 - d) * (1 - α)]

ū = (1 + g - (1 - d) * (1 - α)) /
        (1 + g - (1 - d) * (1 - α) + (1 - d) * λ)

ē = 1 - ū
x̄ = [ū; ē]

println(isapprox(A * x̄, (1 + g) * x̄))   # prints true
```


6
0
0
6
employed workforce
unemployed workforce
$x_0 = (0.1, 4.0)$
$x_0 = (5.0, 0.1)$
$\bar{x}$

Let

$$D := \{x \in \mathbb{R}^2 : x = \alpha \bar{x} \text{ for some } \alpha > 0\}$$

- Shown as a dashed black line in the last figure
- The two time paths are of the form $(x_t)_{t \geqslant 0} = (A^t x_0)_{t \geqslant 0}$
- In both cases, the paths converge to $D$ over time

Suggests all paths are "eventually almost" multiples of $\bar{x}$

How can we explain this strong regularity?

From the Perron–Frobenius theorem, since $A \gg 0$, we have

$$A^t \approx \rho(A)^t \cdot \bar{x}\mathbb{1}^\top = (1+g)^t \begin{pmatrix} \bar{u} & \bar{u} \\ \bar{e} & \bar{e} \end{pmatrix} \quad \text{for large } t$$

Hence, $\forall\, x_0 = (u_0 \; e_0)^\top$,

$$A^t x_0 \approx (1+g)^t \begin{pmatrix} \bar{u} & \bar{u} \\ \bar{e} & \bar{e} \end{pmatrix} \begin{pmatrix} u_0 \\ e_0 \end{pmatrix}$$

$$= (1+g)^t (u_0 + e_0) \begin{pmatrix} \bar{u} \\ \bar{e} \end{pmatrix} = n_t \bar{x},$$

where $n_t = (1+g)^t n_0$ and $n_0 = u_0 + e_0$

Regardless of $x_0$, state scales along $\bar{x}$ at rate of population growth

# Rates

Unemployment rate $= u_t / n_t$

For large $t$, we have $u_t \approx n_t \bar{u}$

Hence unemployment rate $\approx (n_t \bar{u}) / n_t = \bar{u}$

Hence $\underline{\bar{u} \text{ is the long run rate of unemployment}}$

Similarly, $\bar{e}$ is the long run employment rate

$$\Longrightarrow \text{ dominant eigenvector gives unemployment rates}$$

# Extensions

Further analysis: how are $\alpha$, $\lambda$, $b$ and $d$ determined?

For the hiring rate $\lambda$, we could use the job search model

In particular, with $w^*$ as the reservation wage, we could set

$$\lambda = \mathbb{P}\{w_t \geqslant w^*\} = \sum_{w \in \mathsf{W}} \varphi(\mathbb{1}\{w \geqslant w^*\})$$

Doing so would allow us to study the crucial rate $\lambda$ in terms of fundamental primitives, such as

- unemployment compensation
- impatience of individual agents, etc.

# Linear Operators

An $n \times n$ matrix $A = (a_{ij})$ is

1. an $n \times n$ array of (real) numbers $a_{ij}$
2. a linear operator from $\mathbb{R}^k$ to $\mathbb{R}^n$ mapping $u \mapsto Au$

The matrix representation is important

But the linear operator representation is more fundamental

Let's clarify these ideas

A **linear operator** on $\mathbb{R}^n$ is a map $L$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

$$L(\alpha u + \beta v) = \alpha L u + \beta L v \quad \text{for all } u, v \in \mathbb{R}^n \text{ and } \alpha, \beta \in \mathbb{R}$$

**Ex.** Prove: if $A$ is an $n \times n$ matrix, then $u \mapsto Au$ defines a linear operator

In fact the converse is also true!

**Theorem.** If $L$ is a linear operator on $\mathbb{R}^n$, then there exists an $n \times n$ matrix $A = (a_{ij})$ such that $Lu = Au$ for all $u \in \mathbb{R}^n$

Proof: See, e.g., this link

# Linear operators on function space

Let $\mathsf{X} = \{x_1, \ldots, x_n\}$

A **linear operator** on $\mathbb{R}^{\mathsf{X}}$ is a self-map $L$ on $\mathbb{R}^{\mathsf{X}}$ such that, for all $u, v \in \mathbb{R}^{\mathsf{X}}$ and $\alpha, \beta \in \mathbb{R}$, we have

$$L(\alpha u + \beta v) = \alpha L u + \beta L v$$

- acts as a linear operator on $\mathbb{R}^n$ when elements of $\mathbb{R}^{\mathsf{X}}$ are viewed as vectors

Below,

$$\mathcal{L}(\mathbb{R}^{\mathsf{X}}) := \text{all linear operators on } \mathbb{R}^{\mathsf{X}}$$

**Ex.** Let $\ell$ be a map from $\mathsf{X} \times \mathsf{X}$ to $\mathbb{R}$

Show that $L$ defined by

$$(Lu)(x) = \sum_{x' \in \mathsf{X}} \ell(x, x') u(x') \qquad \left(u \in \mathbb{R}^{\mathsf{X}}\right)$$

is in $\mathcal{L}(\mathbb{R}^{\mathsf{X}})$

In fact every $L \in \mathcal{L}(\mathbb{R}^{\mathsf{X}})$ takes the form above

The proof is as follows:

1. The "kernel" $\ell$ is just a matrix
2. In finite dimensions, linear operator $\leftrightarrow$ matrix

Let's summarize (assuming $\mathsf{X} = \{x_1, \dots, x_n\}$)

The following sets are in one-to-one correspondence

(a) The set of all $n \times n$ real matrices

(b) The set of all linear operators on $\mathbb{R}^n$

(c) The set $\mathcal{L}(\mathbb{R}^{\mathsf{X}})$ of linear operators on $\mathbb{R}^{\mathsf{X}}$

(d) The set of all functions from $\mathsf{X} \times \mathsf{X}$ to $\mathbb{R}$

That said, linear operators are more general

- extend to infinite dims
- extend to abstract vector space

Also, the matrix representation can be

1. tedious to construct and
2. difficult to instantiate in memory in large problems

Example. Suppose $\mathsf{X} = \mathsf{Y} \times \mathsf{Z}$ and let $n = |\mathsf{Y}| \times |\mathsf{Z}|$

Fix $Q \colon \mathsf{Z} \times \mathsf{Z} \to \mathbb{R}$ and consider

$$(Lu)(x) = (Lu)(y, z) = \sum_{z'} u(y, z') Q(z, z')$$

**Ex.** Show that $L \in \mathcal{L}(\mathbb{R}^{\mathsf{X}})$

Hence $L$ can be represented as an $n \times n$ matrix

Choose between

1. **column-major** order (Julia, Fortran, Matlab, etc.)
2. **row-major** order (Python, C, etc.)

## But

- an $n \times n$ matrix has to be instantiated in memory, even though the operation is only an inner product in Z
- construction is tedious / error-prone
- confusion when swapping between column- and row-major orderings

Fortunately, some modern scientific computing environments support linear operators directly

- defining linear operators
- providing linear algebra routines (inversion, etc.)

In what follows we take an operator-centric approach

## Positive operators

Let $\mathbb{R}^{\mathsf{X}}_+ :=$ all $u \in \mathbb{R}^{\mathsf{X}}$ with $u \geqslant 0$

- called the **positive cone** of $\mathbb{R}^{\mathsf{X}}$

An operator $L \in \mathcal{L}(\mathbb{R}^{\mathsf{X}})$ is called **positive** if

$$u \geqslant 0 \implies Lu \geqslant 0$$

**Lemma.** $L \in \mathcal{L}(\mathbb{R}^{\mathsf{X}})$ is positive if and only if its matrix representation is nonnegative

**Ex.** Prove $L \in \mathcal{L}(\mathbb{R}^{\mathsf{X}})$ is positive if and only if $L$ is order-preserving on $(\mathbb{R}^{\mathsf{X}}, \leqslant)$

$P \in \mathcal{L}(\mathbb{R}^{\mathsf{X}})$ is called a **Markov operator** on $\mathbb{R}^{\mathsf{X}}$ if $P$ is positive and $P\mathbb{1} = \mathbb{1}$

We let

$$\mathcal{M}(\mathbb{R}^{\mathsf{X}}) := \text{ the set of all Markov operators on } \mathbb{R}^{\mathsf{X}}$$

**Ex.** Prove: If $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ and $v \in \mathbb{R}^{\mathsf{X}}$ with $v \gg 0$, then $Pv \gg 0$.

**Ex.** Given $P \in \mathcal{L}(\mathbb{R}^{\mathsf{X}})$, prove that $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ if and only if $\varphi P$ defined by

$$(\varphi P)(x') = \sum_{x \in \mathsf{X}} P(x, x') \varphi(x) \qquad (x' \in \mathsf{X})$$

is in $\mathscr{D}(\mathsf{X})$ whenever $\varphi \in \mathscr{D}(\mathsf{X})$