Dynamic Programming

Chapter 1: Introduction

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Introduction

Summary of this lecture:

- Finite horizon job search
- Linear equations
- Fixed point theory
- Infinite horizon job search

Dynamic program

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an initial state X_0 is given
$t \leftarrow 0$
while $t < T$ **do**
  observe current state X_t
  choose action A_t
  receive reward R_t based on (X_t, A_t)
  state updates to X_{t+1}
  $t \leftarrow t + 1$
end

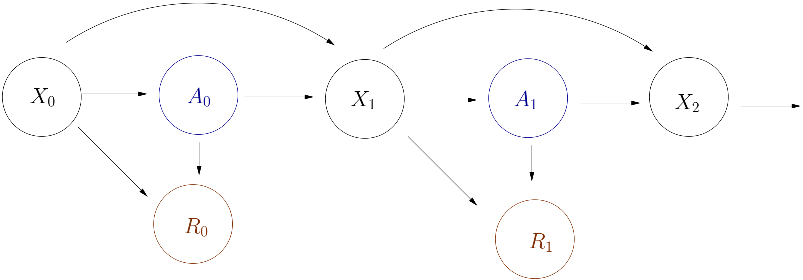


Figure: A dynamic program

Comments:

- Objective: maximize **lifetime rewards**
    - Some aggregation of $R_0, R_1, \ldots$
    - Example. $\mathbb{E}[R_0 + \beta R_1 + \beta^2 R_2 + \cdots]$ for some $\beta \in (0, 1)$
- If $T < \infty$, then **finite horizon**
- Otherwise it is called an **infinite horizon** problem
- The update rule can depend on random elements:

$$X_{t+1} = F(X_t, A_t, \xi_{t+1})$$

Example. A retailer sets prices and manages inventories to maximize profits

- X_t measures
  - current business environment
  - the size of the inventories
  - prices set by competitors, etc.
- A_t specifies current prices and orders of new stock
- R_t is current profit π_t
- Lifetime reward is

$$\mathbb{E}\left[\pi_0 + \frac{1}{1+r}\pi_1 + \left(\frac{1}{1+r}\right)^2 \pi_2 + \cdots\right] = \text{EPV}$$

Flow of this lecture:

1. Begin with simple finite-horizon dynamic program
2. Introduce the recursive structure of dynamic programming
3. Shift to an infinite-horizon version
4. Show how the problem produces a system of nonlinear equations
5. Discuss how we can solve nonlinear equations
    - Fixed point theory

Finite-Horizon Job Search

A model of job search created by **John J. McCall**

We model the decision problem of an unemployed worker

Job search depends on

- current and likely future wage offers
- impatience, and
- the availability of unemployment compensation

We begin with a very simple version of the McCall model

(Later we consider extensions)

Set Up

An agent begins working life at time $t = 1$ without employment

Receives a new job offer paying wage W_t at each date t

She has two choices:

1. **accept** the offer and work permanently at W_t or
2. **reject** the offer, receive unemployment compensation c, and reconsider next period

Assume $\{W_t\}$ is $\overset{\text{IID}}{\sim}$ $\varphi \in \mathscr{D}(\mathsf{W})$, where

- $\mathsf{W} \subset \mathbb{R}_+$ is a finite set of wage outcomes and
- the agent knows φ (but not future draws W_{t+j})

The agent cares about the future but is **impatient**

Impatience is parameterized by a **time discount factor** $\beta \in (0, 1)$

- Present value of a next-period payoff of y dollars is βy

Trade off:

- $\beta < 1$ indicating some impatience
- hence the agent will be tempted to accept reasonable offers, rather than always waiting for a better one
- The key question is how long to wait

The Two Period Problem

Suppose that the working life is just two periods ($t = 1, 2$)

Let's start at $t = 2$, when W_2 is observed

backward induction – start at the end, work back

- If already employed, continue working at W_1
- If unemployed, take the max of c and W_2

Set $v_2(w) = \max\{c, w\}$ = the **time 2 value function**

- max value available for unemployed worker at $t = 2$ given w

Now we shift to $t = 1$

At $t = 1$, given W_1, the unemployed worker's options are

1. **accept** W_1 and receive it at $t = 1, 2$
2. **reject** it, receive compensation c, and then, at $t = 2$, get $v_2(W_2) = \max\{c, W_2\}$

Expected present value (**EPV**) of option 1 is $W_1 + \beta W_1$

- sometimes called the **stopping value**

EPV of option 2 is

$$h_1 := c + \beta \sum_{w' \in \mathsf{W}} v_2(w')\varphi(w'), \tag{1}$$

- sometimes called the **continuation value**

Decision at $t = 1$

1. Look at EPV of two choices (accept, reject)
2. Choose the one with highest EPV

Let's label the actions

$$0 := \text{ reject}$$

$$1 := \text{ accept}$$

Then optimal choice is

$$\mathbb{1}\{W_1 + \beta W_1 \geqslant h_1\}$$

$$=: \mathbb{1}\{\text{stopping value} \geqslant \text{continuation value}\}$$

Let

$$w_1^* := \frac{h_1}{1+\beta} := \textbf{reservation wage}$$

We have

$$\begin{aligned} W_1 \geqslant w^* \quad &\iff \quad W_1 \geqslant \frac{h_1}{1+\beta} \\ &\iff \quad W_1 + \beta W_1 \geqslant h_1 \\ &\iff \quad \text{stopping value} \geqslant \text{continuation value} \end{aligned}$$

Hence

$$\text{accept} \quad \iff \quad W_1 \geqslant w_1^*$$

The **time 1 value function** v_1 is

$$v_1(w_1) = \max \left\{ w_1 + \beta w_1, \, c + \beta \sum_{w' \in \mathsf{W}} v_2(w') \varphi(w') \right\}$$

$$= \max \{ w_1 + \beta w_1, \, h_1 \}$$

$$= \max \{ \text{stopping value}, \, \text{continuation value} \}$$

The maximum lifetime value available at $t = 1$ given

- currently unemployed
- current offer w_1

```julia
using Distributions

"Creates an instance of the job search model, stored as a NamedTuple."
function create_job_search_model(;
        n=50,        # wage grid size
        w_min=10.0,  # lowest wage
        w_max=60.0,  # highest wage
        a=200,       # wage distribution parameter
        b=100,       # wage distribution parameter
        β=0.96,      # discount factor
        c=10.0       # unemployment compensation
    )
    w_vals = collect(LinRange(w_min, w_max, n+1))
    ϕ = pdf(BetaBinomial(n, a, b))
    return (; n, w_vals, ϕ, β, c)
end

" Computes lifetime value at t=1 given current wage w_1 = w. "
function v_1(w, model)
    (; n, w_vals, ϕ, β, c) = model
    h_1 = c + β * max.(c, w_vals)'ϕ
    return max(w + β * w, h_1)
end

" Computes reservation wage at t=1. "
function res_wage(model)
    (; n, w_vals, ϕ, β, c) = model
    h_1 = c + β * max.(c, w_vals)'ϕ
    return h_1 / (1 + β)
end
```

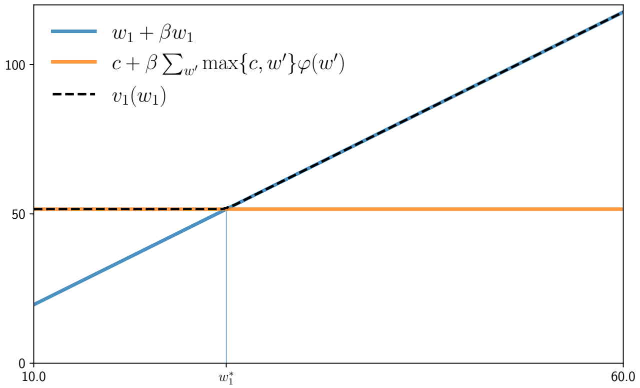


Figure: The value function v_1 and the reservation wage

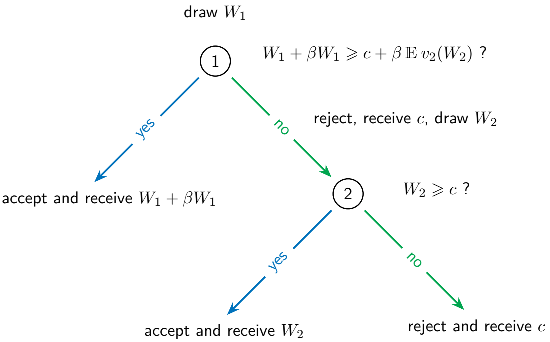


Figure: Decision tree for the two period problem

Three Period Problem

Now suppose we extend to three periods, with $t = 0, 1, 2$

At $t = 0$, the EPV of accepting W_0 is $W_0 + \beta W_0 + \beta^2 W_0$

Maximal EPV of rejecting is

1. unemployment compensation plus
2. max value we can expect from $t = 1$ when unemployed

That is,

$$\text{continuation value} = h_0 := c + \beta \sum_{w'} v_1(w') \varphi(w')$$

Putting it together,

$$v_2(w_2) = \max\{c, w_2\}$$

$$v_1(w_1) = \max\left\{w_1 + \beta w_1,\; c + \beta \sum_{w' \in \mathsf{W}} v_2(w')\varphi(w')\right\}$$

$$v_0(w_0) = \max\left\{w_0 + \beta\, w_0 + \beta^2\, w_0,\; c + \beta \sum_{w' \in \mathsf{W}} v_1(w')\varphi(w')\right\}$$

- **solve for all v_t** by **backward induction** (start from top)

From Values to Choices

Now we know the optimal values we can make optimal choices

At time $t = 0$

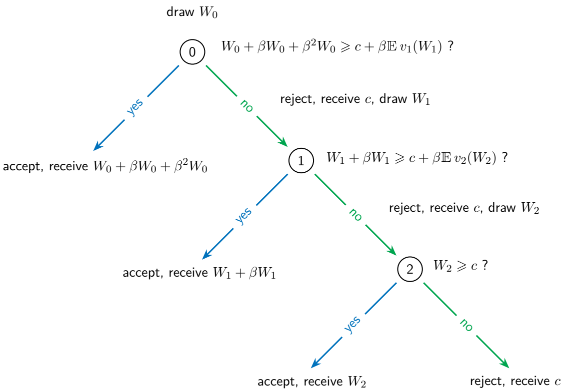
$$\text{action} \; = \mathbb{1}\left\{w_0 + \beta\, w_0 + \beta^2\, w_0 \geqslant c + \beta \sum_{w' \in \mathsf{W}} v_1(w')\varphi(w')\right\}$$

At time $t = 1$, if still unemployed

$$\text{action} \; = \mathbb{1}\left\{w_1 + \beta w_1 \geqslant c + \beta \sum_{w' \in \mathsf{W}} v_2(w')\varphi(w')\right\}$$

At time $t = 2$, if still unemployed

$$\text{action} \; = \mathbb{1}\{w_2 \geqslant c\}$$



draw W_0
0
$W_0 + \beta W_0 + \beta^2 W_0 \geqslant c + \beta \mathbb{E}\, v_1(W_1)$?
yes
no
reject, receive c, draw W_1
accept, receive $W_0 + \beta W_0 + \beta^2 W_0$
1
$W_1 + \beta W_1 \geqslant c + \beta \mathbb{E}\, v_2(W_2)$?
yes
no
reject, receive c, draw W_2
accept, receive $W_1 + \beta W_1$
2
$W_2 \geqslant c$?
yes
no
accept, receive W_2
reject, receive c

Summary

We reduced the multi-stage problem to two period problems

- the **key idea** of dynamic programming!

The equation

$$v_0(w_0) = \max \left\{ w_0 + \beta\, w_0 + \beta^2\, w_0,\ c + \beta \sum_{w' \in \mathsf{W}} v_1(w')\varphi(w') \right\}$$

is an example of the **Bellman equation**

Similar ideas easily extend to time T:

$$v_T(w_T) = \max\{c, w_T\}$$

and

$$v_t(w_t) =$$
$$\max\left\{w_t + \beta\, w_t + \cdots + \beta^{T-t}\, w_t,\; c + \beta \sum_{w' \in \mathsf{W}} v_{t+1}(w')\varphi(w')\right\}$$

for $t = 0, \ldots, T-1$

Infinite Horizons

Now let us consider a worker who aims to maximize

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t Y_t \qquad (Y_t \text{ is income at time } t)$$

- offer process $\{W_t\} \overset{\text{IID}}{\sim} \varphi$ for $\varphi \in \mathscr{D}(\mathsf{W})$
- $\mathsf{W} \subset \mathbb{R}_+$ with $|\mathsf{W}| < \infty$
- $c, \beta > 0$ and $\beta < 1$
- jobs are permanent
- φ is known but future draws are not

What is max EPV of each option when lifetime is infinite?

What if we **accept** $w \in \mathsf{W}$ now?

$$\text{max EPV} \;= \text{stopping value} \;= w + \beta w + \beta^2 w + \cdots = \frac{w}{1-\beta}$$

What if we **reject**?

$$\begin{aligned} \text{max EPV} \;&= \text{continuation value} \\ &= c + \text{EPV of optimal choice at } t+1, t+2, \ldots \end{aligned}$$

But what are optimal choices?!

Calculating optimal choice requires knowing optimal choices!

The Value Function

Let $v^*(w) :=$ max lifetime EPV given wage offer w

We call v^* the **value function**

Suppose that we know v^*

Then the (maximum) **continuation value** is

$$h^* := c + \beta \sum_{w' \in \mathsf{W}} v^*(w') \varphi(w')$$

The optimal choice is then

$$\mathbb{1}\left\{\text{stopping value} \geqslant \text{continuation value}\right\} = \mathbb{1}\left\{\frac{w}{1-\beta} \geqslant h^*\right\}$$

But how can we calculate v^*?

Key idea: Use the Bellman equation to solve for v^*

Theorem. The value function v^* satisfies the **Bellman equation**

$$v^*(w) = \max \left\{ \frac{w}{1-\beta}, \, c + \beta \sum_{w' \in \mathsf{W}} v^*(w') \varphi(w') \right\} \qquad (w \in \mathsf{W})$$

Intuition: max value today is max of

- accept, get $w/(1-\beta)$
- reject, get c plus EPV of optimal future choices

(Full proof coming later)

So how can we use the Bellman equation

$$v^*(w) = \max \left\{ \frac{w}{1-\beta},\, c + \beta \sum_{w' \in \mathsf{W}} v^*(w') \varphi(w') \right\} \qquad (w \in \mathsf{W})$$

to solve for v^*?

For this we need **fixed point theory**

So now let's back up and cover some necessary mathematics. . .

Vector Space

Let's recall some basic facts about finite-dimensional vector space, such as

- different kinds of norms
- equivalence of norms
- matrix operations and eigenvalues
- Neumann series (geometric series of matrices)

Norms in Vector Space

Let $\|\cdot\|$ be a function from $\mathbb{R}^n \to \mathbb{R}$

The function $\|\cdot\|$ is called a **norm** on $\mathbb{R}^n$ if, for any $\alpha \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$,

(a) $\|u\| \geqslant 0$

(b) $\|u\| = 0 \iff u = 0$

(c) $\|\alpha u\| = |\alpha| \|u\|$ and

(d) $\|u + v\| \leqslant \|u\| + \|v\|$

Example. The **Euclidean norm** $\|u\| := \sqrt{\langle u, u \rangle}$ obeys (a)–(d)

Example. The ℓ_1 **norm** of a vector $u \in \mathbb{R}^n$ is defined by

$$u = (u_1, \ldots, u_n) \mapsto \|u\|_1 := \sum_{i=1}^{n} |u_i|$$

Example. The **supremum norm** of u is defined by

$$\|u\|_\infty := \max_i |u_i|$$

Ex. Verify that

1. the ℓ_1 norm on $\mathbb{R}^n$ satisfies (a)–(d) above
2. the supremum norm on $\mathbb{R}^n$ satisfies (a)–(d) above

Equivalence of Norms

Fix $u \in \mathbb{R}^n$ and $(u_m) := (u_m)_{m \in \mathbb{N}} \subset \mathbb{R}^n$

We say that (u_m) **converges** to u and write $u_m \to u$ if

$$\|u_m - u\| \to 0 \text{ as } m \to \infty \text{ for some norm } \|\cdot\| \text{ on } \mathbb{R}^n$$

Don't we need to say "convergence with respect to $\|\cdot\|$"?

No because any two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on $\mathbb{R}^n$ are **equivalent**

That is, for any such pair, $\exists\, M, N$ such that

$$M\|u\|_a \leqslant \|u\|_b \leqslant N\|u\|_a \quad \text{for all } u \in \mathbb{R}^n$$

- See, e.g., Kreyszig (1978)

Hence convergence is independent of the norm

Ex. Let $\| \cdot \|_a$ and $\| \cdot \|_b$ be any two norms on $\mathbb{R}^n$

Given u in $\mathbb{R}^n$ and a sequence (u_m) in $\mathbb{R}^n$, confirm that

$$\|u_m - u\|_a \to 0 \text{ implies } \|u_m - u\|_b \to 0 \text{ as } m \to \infty$$

Proof: Let $\| \cdot \|_a$ and $\| \cdot \|_b$ be as stated

Equivalence implies $\exists\, M \in \mathbb{R}$ with

$$0 \leqslant \|u_m - u\|_b \leqslant M \|u_m - u\|_a \text{ for all } m \in \mathbb{N}$$

Since $\|u_m - u\|_a \to 0$, we also have $\|u_m - u\|_b \to 0$

Real operations such as $|\cdot|$ and $\vee$ are applied to vectors <u>element-by-element</u>

Example.

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \Longrightarrow \quad |a| = \begin{pmatrix} |a_1| \\ \vdots \\ |a_n| \end{pmatrix}$$

$$a \vee b = \begin{pmatrix} a_1 \vee b_1 \\ \vdots \\ a_n \vee b_n \end{pmatrix} \quad \text{and} \quad a \wedge b = \begin{pmatrix} a_1 \wedge b_1 \\ \vdots \\ a_n \wedge b_n \end{pmatrix}$$

etc.

Topology in $\mathbb{R}^n$

Some quick reminders

A set $C \subset \mathbb{R}^n$ is called **closed** in $\mathbb{R}^n$ if

$$(u_m) \subset C \text{ and } u_m \to u \implies u \in C$$

A set G is called **open** if G^c is closed

$T \colon U \to V$ is called **continuous at** $u \in U$ if

$$(u_m) \subset U \text{ and } u_m \to u \implies Tu_m \to Tu$$

We call T **continuous on** U if T is continuous at every $u \in U$

Matrices

Let $A = (a_{ij})$ be an $n \times n$ matrix

Fix $u \in \mathbb{R}^n$

Recall:

$$\text{the } i\text{-th element of } Au \text{ is } \sum_{j=1}^{n} a_{ij} u_j$$

and

$$\text{the } j\text{-th element of } u^\top A \text{ is } \sum_{i=1}^{n} a_{ij} u_i$$

Think of $u \mapsto Au$ and $u \mapsto u^\top A$ as two maps

Each takes an n-vector and produces a new n-vector

The **operator norm** of $n \times n$ matrix B is

$$\|B\| := \max_{\|u\|=1} \|Bu\| \tag{2}$$

Here $u \in \mathbb{R}^n$ and the norm on the r.h.s. is the Euclidean norm

For any $\alpha \in \mathbb{R}$ and $n \times n$ matrices A, B

(a) $\|A\| \geqslant 0$

(b) $\|A\| = 0 \iff A = 0$

(c) $\|\alpha A\| = |\alpha| \|A\|$

(d) $\|A + B\| \leqslant \|A\| + \|B\|$

(e) $\|AB\| \leqslant \|A\| \|B\|$ (submultiplicative property)

Let $\mathbb{C}^n$ be the set of complex n-vectors

If A is $n \times n$, then

- $\lambda \in \mathbb{C}$ is called an **eigenvalue** of A if

$$\exists \text{ nonzero } e \in \mathbb{C}^n \text{ such that } Ae = \lambda e \tag{3}$$

- The vector e satisfying (3) is called an **eigenvector** of A

Together, (λ, e) is called an **eigenpair**

Geometric Series

In one-dimension we have

$$|a| < 1 \quad \Longrightarrow \quad \sum_{k \geqslant 0} a^k = \frac{1}{1-a}$$

Is there an equivalent result when $a = A$ is $n \times n$?

Let A be $n \times n$ and let I be the $n \times n$ identity

We define the **spectral radius** of A as

$$\rho(A) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$$

Neumann Series Lemma. If $\rho(A) < 1$, then

1. $I - A$ is nonsingular
2. $\sum_{k \geqslant 0} A^k$ converges and
3. $\sum_{k \geqslant 0} A^k = (I - A)^{-1}$

Intuitive idea: with $S := \sum_{k \geqslant 0} A^k$, we have

$$I + AS = I + A(I + A + \cdots) = I + A + A^2 + \cdots = S$$

Rearranging $I + AS = S$ gives $S = (I - A)^{-1}$

To make this rigorous, we need to show that

- $\sum_{k \geqslant 0} A^k$ converges
- the matrix $I - A$ is invertible

To complete the proof, we use properties of the matrix norm

Lemma. If B is any square matrix, then

1. $\rho(B)^k \leqslant \|B^k\|$ for all $k \in \mathbb{N}$ and
2. $\|B^k\|^{1/k} \to \rho(B)$ as $k \to \infty$

The last expression is often called **Gelfand's formula**

Ex. Prove: $\rho(A) < 1 \implies \sum_{k \geqslant 0} A^k$ converges

- Hint 1: Suffices to show $\lim_{N \to \infty} \| \sum_{k \geqslant 0}^{N} A^k \| < \infty$
- Hint 2: Use triangle inequality and Cauchy's root test

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Final step: Show that $(I - A)^{-1}$ exists:

- Suffices to show existence of a right inverse
    - See, e.g., §6.1.4.5 of networks.quantecon.org
- That is, we need an S such that $(I - A)S = I$
- Let $S = \sum_{k \geqslant 0} A^k$

We have

$$(I - A)S = I \sum_{k \geqslant 0} A^k - A \sum_{k \geqslant 0} A^k = \sum_{k \geqslant 0} A^k - \sum_{k \geqslant 1} A^k = I$$

Hence $(I - A)^{-1}$ exists and equals $\sum_{k \geqslant 0} A^k$

Fixed Points

To solve more complex equations we use **fixed point theory**

Recall that, if U is any set then

- T is a **self-map** on U if T maps U into itself
- $u^* \in U$ is called a **fixed point** of T in U if $Tu^* = u^*$

Conventions: Given self-map T on U, common to

- write Tu instead of $T(u)$ and
- If U is multi-dimensional, we typically call T an **operator** rather than a function

Fixed point theory tells us how to solve equations

$$\text{solving equation } u = Tu \iff \text{finding fixed points of } T$$

Example. If $U = \mathbb{R}^n$ and $Tu = Au + b$, then

$$u^* \text{ solves } u = Au + b \iff u^* \text{ is a fixed point of } T$$

Example. Every u in arbirary set U is fixed under the **identity map**

$$I : u \mapsto u$$

Example. If $U = \mathbb{N}$ and $Tu = u + 1$, then T has no fixed point

Example. If $U \subset \mathbb{R}$, then

$$Tu = u \iff T \text{ meets the 45 degree line}$$

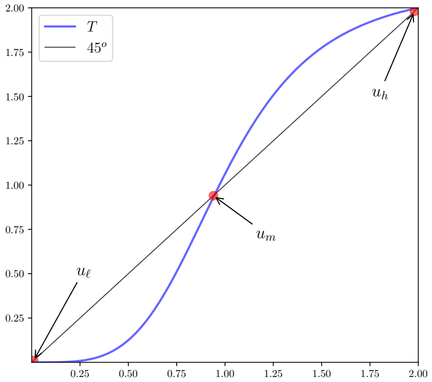

Figure: Graph and fixed points of $T: u \mapsto 2.125/(1 + u^{-4})$

Example. Letting

- $U = \mathbb{R}^n$ and
- $Tu = Au + b$ with $\rho(A) < 1$, we have

$$
\begin{aligned}
u \text{ is a fixed point of } T &\iff u = Au + b \\
&\iff (I - A)u = b \\
&\iff u = (I - A)^{-1} b
\end{aligned}
$$

$\therefore \quad u^* := (I - A)^{-1} b$ is the unique fixed point of T in $\mathbb{R}^n$

Point on notation:

- $T^2 = T \circ T$
- $T^3 = T \circ T \circ T$
- etc.

Example. If $Tu = Au + b$, then

- $T^2 u = A(Au + b) + b = A^2 u + Ab + b$
- $T^3 u = A(A^2 u + Ab + b) + b = A^3 u + A^2 b + Ab + A + b$
- etc.

Let U be any set and let T be a self-map on U

Lemma If

$$\exists\, \bar{u} \in U,\ m \in \mathbb{N} \ \text{ s.t. } \ T^k u = \bar{u} \text{ for all } u \in U \text{ and } k \geqslant m$$

then $\bar{u}$ is the unique fixed point of T in U

Proof of uniqueness:

Let u, v be fixed points of T in U

Since $T^m u = \bar{u}$ and $T^m v = \bar{u}$, we have $T^m u = T^m v$

But u and v are fixed points, so

$$u = T^m u \text{ and } v = T^m v$$

Hence $u = v$

Proof of existence:

We claim that $\bar{u}$ is a fixed point

To see this, recall that

$$T^k u = \bar{u} \text{ for } k \geqslant m \text{ and all } u \in U$$

Hence $T^m \bar{u} = \bar{u}$ and $T^{m+1} \bar{u} = \bar{u}$

But then

$$T\bar{u} = T(T^m \bar{u}) = T^{m+1} \bar{u} = \bar{u}$$

That is, $\bar{u}$ is a fixed point of T

Let T be a self-map on $U \subset \mathbb{R}^d$

Ex. Prove the following: If

1. $T^m u \to u^*$ as $m \to \infty$ for some pair $u, u^* \in U$ and
2. T is continuous at u^*

then u^* is a fixed point of T

Proof: Assume hypotheses and let $u_m := T^m u$ for all $m \in \mathbb{N}$

By continuity and $u_m \to u^*$ we have $Tu_m \to Tu^*$

But $(Tu_m)_{m \geqslant 1}$ is just $(u_2, u_3, \ldots)$

Since $u_m \to u^*$, we just have $Tu_m \to u^*$

Limits are unique, so $u^* = Tu^*$

Let T be a self-map on $U \subset \mathbb{R}^d$

Ex. Prove the following: If

1. $T^m u \to u^*$ as $m \to \infty$ for some pair $u, u^* \in U$ and
2. T is continuous at u^*

then u^* is a fixed point of T

<u>Proof:</u> Assume hypotheses and let $u_m := T^m u$ for all $m \in \mathbb{N}$

By continuity and $u_m \to u^*$ we have $Tu_m \to Tu^*$

But $(Tu_m)_{m \geqslant 1}$ is just $(u_2, u_3, \ldots)$

Since $u_m \to u^*$, we just have $Tu_m \to u^*$

Limits are unique, so $u^* = Tu^*$

Self-map T is called **globally stable** on $U \subset \mathbb{R}^n$ if

1. T has a unique fixed point u^* in U and
2. $T^k u \to u^*$ as $k \to \infty$ for all $u \in U$

Example. Consider Solow–Swan growth dynamics

$$k_{t+1} = g(k_t) := sAk_t^{\alpha} + (1 - \delta)k_t, \qquad t = 0, 1, \ldots,$$

where

- k_t is capital stock per worker,
- $A, \alpha > 0$ are production parameters, $\alpha < 1$
- $s > 0$ is a savings rate, and
- $\delta \in (0, 1)$ is a rate of depreciation

Iterating with g from k_0 generates a time path for capital stock

The map g is globally stable on $(0, \infty)$

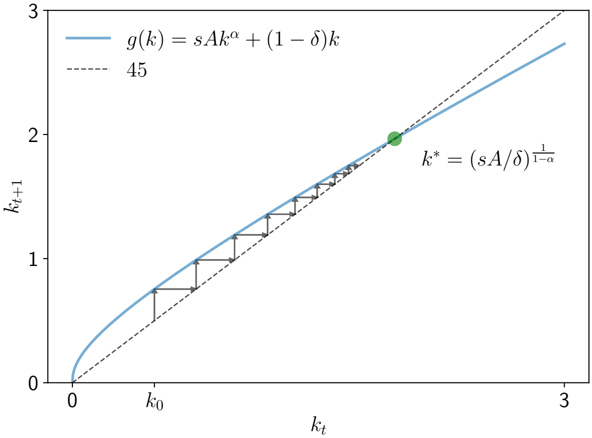

Figure: Global stability for the Solow–Swan model

Note from last slide

- If g is flat near k^*, then $g(k) \approx k^*$ for k near k^*
- A flat function near the fixed point $\implies$ fast convergence

Conversely

- If g is close to the 45 degree line near k^*, then $g(k) \approx k$
- Close to 45 degree line means high persistence, slow convergence

Fix $n \times n$ matrix A and $b = \mathbb{R}^n$

Let $Tu = Au + b$

Ex. Show that T is globally stable on $\mathbb{R}^n$ whenever $\rho(A) < 1$

Proof: Suppose $\rho(A) < 1$

We previously showed that $I - A$ is nonsingular and that the unique fixed point of T in U is

$$u^* = (I - A)^{-1} b$$

It remains to prove convergence

Fix $n \times n$ matrix A and $b = \mathbb{R}^n$

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Ex. Show that T is globally stable on $\mathbb{R}^n$ whenever $\rho(A) < 1$

Proof: Suppose $\rho(A) < 1$

We previously showed that $I - A$ is nonsingular and that the unique fixed point of T in U is

$$u^* = (I - A)^{-1} b$$

It remains to prove convergence

Fix $u \in \mathbb{R}^n$

For $k \in \mathbb{N}$ we have

$$T^k u = A^k u + A^{k-1} b + A^{k-2} b + \cdots + Ab + b$$

Since $\rho(A) < 1$, we have $A^k u \to 0$ and $\sum_{i=0}^{k-1} A^i \to (I - A)^{-1}$

Therefore

$$\lim_{k \to \infty} T^k u = \lim_{k \to \infty} \left[A^k u + \sum_{i=0}^{k} A^{i-1} b \right] = (I - A)^{-1} b = u^*$$

Let T be a self-map on $U \subset \mathbb{R}^n$

We call $C \subset U$ **invariant** for T if

$$u \in C \implies Tu \in C$$

(Equivalently, T is a self-map on C as well as U)

Lemma. If

1. T is globally stable on $U \subset \mathbb{R}^n$ with fixed point u^* and
2. C is nonempty, closed and invariant for T,

then $u^* \in C$

Proof: Let the stated hypotheses hold

Fix $u \in C$ (exist because C is nonempty)

By global stability we have

$$T^k u \to u^* \qquad (k \to \infty)$$

Since T is invariant on C we have $(T^k u)_{k \in \mathbb{N}} \subset C$

Since C is closed, the any limit point of this sequence is in C

Hence $u^* \in C$, as claimed

Contractions

Let

- U be a nonempty subset of $\mathbb{R}^n$ and T be a self-map on U
- $\| \cdot \|$ be a norm on $\mathbb{R}^n$

T is called a **contraction** on U with respect to $\| \cdot \|$ if

$$\exists \, \lambda < 1 \text{ such that } \|Tu - Tv\| \leqslant \lambda \|u - v\| \quad \text{for all} \quad u, v \in U$$

Example. For $Tx = ax + b$ we have

$$|Tx - Ty| = |ax + b - ay - b| = |a||x - y|$$

Hence T is a contraction on $\mathbb{R}$ with respect to $| \cdot |$ iff $|a| < 1$

Ex. Prove: If T is a contraction on U with respect to any norm, then

1. T is continuous on U and
2. T has at most one fixed point in U

Proof: Regarding part 1, if $u_k \to u$, then

$$\|Tu_k - Tu\| \leqslant \lambda\|u_k - u\| \to 0 \quad k \to \infty$$

Regarding part 2, if u, v are fixed points of T in U, then

$$\|u - v\| = \|Tu - Tv\| \leqslant \lambda\|u - v\| \qquad \text{with } \lambda < 1$$

$$\therefore \quad \|u - v\| = 0$$

$$\therefore \quad u = v$$

Ex. Prove: If T is a contraction on U with respect to any norm, then

1. T is continuous on U and
2. T has at most one fixed point in U

<u>Proof</u>: Regarding part 1, if $u_k \to u$, then

$$\|Tu_k - Tu\| \leqslant \lambda \|u_k - u\| \to 0 \quad k \to \infty$$

Regarding part 2, if u, v are fixed points of T in U, then

$$\|u - v\| = \|Tu - Tv\| \leqslant \lambda \|u - v\| \quad \text{with } \lambda < 1$$

$$\therefore \quad \|u - v\| = 0$$

$$\therefore \quad u = v$$

Banach's Contraction Mapping Theorem

Let $\| \cdot \|$ be a norm on $U \subset \mathbb{R}^n$

Theorem. If

1. U is closed in $\mathbb{R}^n$ and
2. T is a contraction of modulus λ on U w.r.t. $\| \cdot \|$

then T has a unique fixed point u^* in U and

$$\|T^k u - u^*\| \leqslant \lambda^k \|u - u^*\| \quad \text{for all } k \in \mathbb{N} \text{ and } u \in U$$

In particular, T is globally stable on U

Proof: See book

Successive Approximation

A common algorithm for computing a fixed point of T in $U \subset \mathbb{R}^n$ is **successive approximation**:

---

fix $u_0 \in U$ and $k = 0$
while *some stopping condition fails* **do**
    $u_{k+1} \leftarrow T u_k$
    $k \leftarrow k + 1$
end
return u_k

Example. If T is globally stable on U with fixed point u^*, then

$$u_k = T^k u_0 \to u^* \qquad (k \to \infty)$$

```julia
function successive_approx(T,                    # operator (callable)
                           u_0;                  # initial condition
                           tolerance=1e-6,       # error tolerance
                           max_iter=10_000,      # max iteration bound
                           print_step=25)        # print at multiples
    u = u_0
    error = Inf
    k = 1

    while (error > tolerance) & (k <= max_iter)

        u_new = T(u)
        error = maximum(abs.(u_new - u))

        if k % print_step == 0
            println("Completed iteration $k with error $error.")
        end

        u = u_new
        k += 1
    end

    if error <= tolerance
        println("Terminated successfully in $k iterations.")
    else
        println("Warning: hit iteration bound.")
    end

    return u
end
```

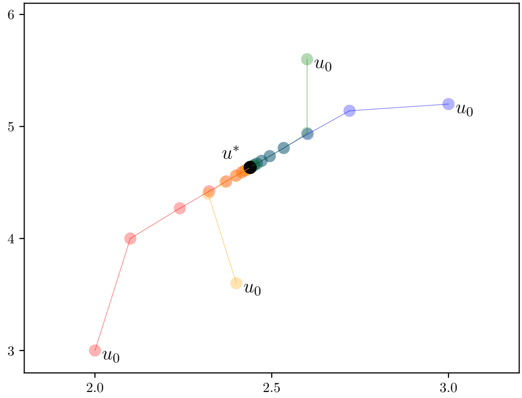


Figure: Successive approximation from different initial conditions

Finite-Dimensional Function Space

Let X be any set

If $u \colon \mathsf{X} \to \mathbb{R}$, then we call u a **real-valued function** on X

Let $\mathbb{R}^{\mathsf{X}}$ be the **set of all real-valued functions on** X

If $u, v \in \mathbb{R}^{\mathsf{X}}$ and $\alpha, \beta \in \mathbb{R}$, then

- $(\alpha u + \beta v)(x) := \alpha u(x) + \beta v(x)$
- $(uv)(x) := u(x)v(x)$
- $(u \vee v)(x) := u(x) \vee v(x)$
- $(u \wedge v)(x) := u(x) \wedge v(x)$
- etc.

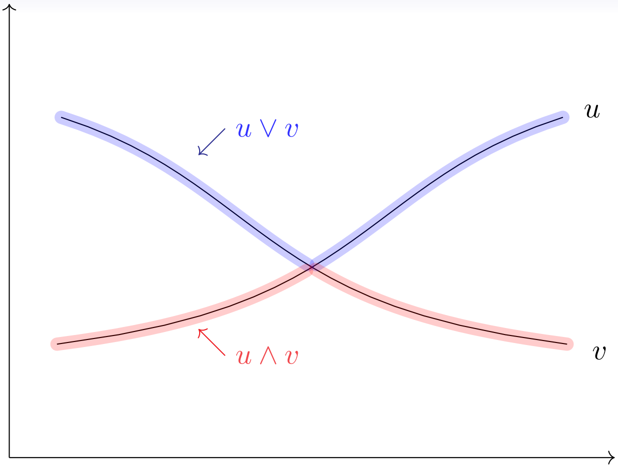


Figure: Functions $u \vee v$ and $u \wedge v$ when $\mathsf{X} \subset \mathbb{R}$

Functions on Finite Sets are Vectors!

If $\mathsf{X} = \{x_1, \ldots, x_n\}$ = some finite set, then

$$\mathbb{R}^{\mathsf{X}} \text{ and } \mathbb{R}^n \text{ are the "same" set}$$

Indeed, $u \in \mathbb{R}^{\mathsf{X}}$ is defined by its values $u(x_1), \ldots, u(x_n)$

In other words, there is an exact a one-to-one correspondence between $\mathbb{R}^{\mathsf{X}}$ and $\mathbb{R}^n$:

$$\mathbb{R}^{\mathsf{X}} \ni u \quad \longleftrightarrow \quad (u(x_1), \ldots, u(x_n)) \in \mathbb{R}^n$$

All properties of $\mathbb{R}^n$ map directly to $\mathbb{R}^{\mathsf{X}}$

Examples.

- The Euclidean norm of $u \in \mathbb{R}^{\mathsf{X}}$ is

$$\left(\sum_{i=1}^{n} u(x_i)^2\right)^{1/2} =: \left(\sum_{x \in \mathsf{X}} u(x)^2\right)^{1/2}$$

- $C \subset \mathbb{R}^{\mathsf{X}}$ closed if C is closed when regarded as a subset of $\mathbb{R}^n$

- $O \subset \mathbb{R}^{\mathsf{X}}$ open if O is open when regarded as a subset of $\mathbb{R}^n$

- etc., etc.

Our notation emphasizes functions ($\mathbb{R}^{\mathsf{X}}$ rather than $\mathbb{R}^n$)

Example. $v^*(w)$ instead of v_i^* for value function at wage $w = w_i$

- Generally clearer
- Accustom ourselves to the language of functional analysis

When we shift to general state / action spaces, value functions are functions, not vectors

Given finite X, the set of **distributions** on X is

$$\mathscr{D}(\mathsf{X}) := \left\{ \varphi \in \mathbb{R}^{\mathsf{X}} \mid \varphi \geqslant 0 \text{ and } \sum_{x \in \mathsf{X}} \varphi(x) = 1 \right\}$$

We say that X-valued random element X has distribution $\varphi \in \mathscr{D}(\mathsf{X})$ (and write $X \sim \varphi$) if

$$\mathbb{P}\{X = x\} = \varphi(x) \quad \text{for all } x \in \mathsf{X}$$

For $h \in \mathbb{R}^{\mathsf{X}}$ and $X \sim \varphi$, the **expectation** of $h(X)$ is

$$\mathbb{E} h(X) := \sum_{x \in \mathsf{X}} h(x)\varphi(x) = \langle h, \varphi \rangle$$

If $\mathsf{X} \subset \mathbb{R}$, then

$$\Phi(x) := \mathbb{P}\{X \leqslant x\} = \sum_{x' \in \mathsf{X}} \mathbb{1}\{x' \leqslant x\} \varphi(x')$$

is called the **cumulative distribution function** (CDF)

If $\tau \in [0, 1]$, then the τ-th **quantile** of X is

$$Q_\tau \, X := \min\{x \in \mathsf{X} : \Phi(x) \geqslant \tau\}$$

In particular, if $\tau = 1/2$, then $Q_\tau \, X$ is called the **median** of X

Let X be a random variable on finite set X

Given $\alpha \in \mathbb{R}$, we have

$$\mathbb{E}[X + \alpha] = \mathbb{E}X + \alpha$$

The same property holds for quantiles:

Ex. Prove that, for any $\tau \in [0, 1]$, and $\alpha \in \mathbb{R}$,

$$Q_\tau(X + \alpha) = Q_\tau(X) + \alpha$$

Let X be a finite set

Lemma. If f and g are elements of $\mathbb{R}^{\mathsf{X}}$, then

$$|\max_{x \in \mathsf{X}} f(x) - \max_{x \in \mathsf{X}} g(x)| \leqslant \max_{x \in \mathsf{X}} |f(x) - g(x)| \tag{4}$$

Proof Fixing $f, g \in \mathbb{R}^{\mathsf{X}}$, we have

$$f = f - g + g \leqslant |f - g| + g$$

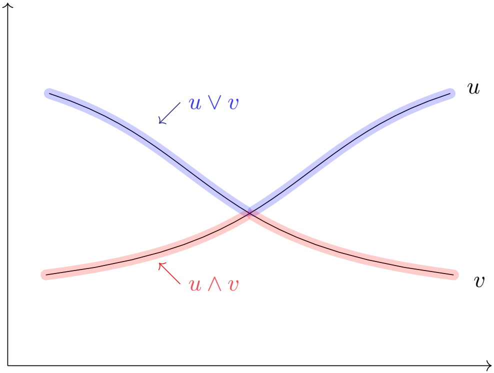
$$\therefore \quad \max f \leqslant \max(|f - g| + g) \leqslant \max |f - g| + \max g$$

$$\therefore \quad \max f - \max g \leqslant \max |f - g|$$

Reversing the roles of f and g proves the claim

Pointwise Operations

Given $u, v \in \mathbb{R}^{\mathsf{X}}$, recall that



Let $\{v_\sigma\}_{\sigma\in\Sigma}$ be a finite subset of $\mathbb{R}^{\mathsf{X}}$

Generalizing, we set

$$\left(\bigwedge_{\sigma} v_\sigma\right)(x) := \min_{\sigma\in\Sigma} v_\sigma(x)$$

and

$$\left(\bigwedge_{\sigma} v_\sigma\right)(x) := \min_{\sigma\in\Sigma} v_\sigma(x)$$

Lattice Properties

Let X be a finite set

A subset V of $\mathbb{R}^{\mathsf{X}}$ is called a **sublattice** of $\mathbb{R}^{\mathsf{X}}$ if

$$u, v \in V \text{ implies } u \vee v \in V \text{ and } u \wedge v \in V$$

Example. The following are sublattices of $\mathbb{R}^{\mathsf{X}}$

- $\mathbb{R}^{\mathsf{X}}$
- $\mathbb{R}^{\mathsf{X}}_{+} := \{f \in \mathbb{R}^{\mathsf{X}} : f \geqslant 0\}$
- $(0, \infty)^{\mathsf{X}} := \{f \in \mathbb{R}^{\mathsf{X}} : f \gg 0\}$

Let

- V be sublattice of $\mathbb{R}^{\mathsf{X}}$
- $\{T_\sigma\}_{\sigma \in \Sigma}$ be a finite family of self-maps on V

Set

$$Tv = \bigvee_{\sigma \in \Sigma} T_\sigma \, v \qquad (v \in V)$$

By the sublattice property, T is a self-map on V

Lemma. If T_σ is a contraction of modulus λ_σ w.r.t. $\| \cdot \|_\infty$ for each $\sigma \in \Sigma$, then T is a contraction of modulus $\max_\sigma \lambda_\sigma$ under the same norm

Proof Let the stated conditions hold and fix $u, v \in V$

By the max inequality on slide 76,

$$\|Tu - Tv\|_\infty = \max_x |\max_\sigma (T_\sigma\, u)(x) - \max_\sigma\, (T_\sigma\, v)(x)|$$

$$\leqslant \max_x \max_\sigma\, |(T_\sigma\, u)(x) - (T_\sigma\, v)(x)|$$

$$= \max_\sigma\, \max_x\, |(T_\sigma\, u)(x) - (T_\sigma\, v)(x)|$$

$$\therefore \quad \|Tu - Tv\|_\infty \leqslant \max_\sigma\, \|T_\sigma\, u - T_\sigma\, v\|_\infty \leqslant \max_\sigma\, \lambda_\sigma\, \|u - v\|_\infty$$

Back to infinite-horizon job search

Recall from slide 28 that the value function v^* solves the Bellman equation:

$$v^*(w) = \max \left\{ \frac{w}{1-\beta}, \; c + \beta \sum_{w' \in \mathsf{W}} v^*(w')\varphi(w') \right\} \qquad (w \in \mathsf{W})$$

The infinite-horizon **continuation value** is defined as

$$h^* := c + \beta \sum_{w'} v^*(w')\varphi(w')$$

Key question: <u>how to solve for v^*?</u>

We introduce the **Bellman operator**, defined at $v \in \mathbb{R}^{\mathsf{W}}$ by

$$(Tv)(w) = \max \left\{ \frac{w}{1-\beta}, \, c + \beta \sum_{w' \in \mathsf{W}} v(w') \varphi(w') \right\} \qquad (w \in \mathsf{W})$$

By construction, $Tv = v$ iff v solves the Bellman equation

Proposition. T is a contraction on $\mathbb{R}^{\mathsf{W}}$ with respect to $\| \cdot \|_\infty$

In the proof, we use the elementary bound

$$|\alpha \vee x - \alpha \vee y| \leqslant |x - y| \qquad (\alpha, x, y \in \mathbb{R})$$

Fixing f, g in $\mathbb{R}^{\mathsf{W}}$ fix any $w \in \mathsf{W}$, we have

$$|(Tf)(w) - (Tg)(w)| \leqslant \left| \beta \sum_{w'} f(w')\varphi(w') - \beta \sum_{w'} g(w')\varphi(w') \right|$$

$$= \beta \left| \sum_{w'} [f(w') - g(w')]\varphi(w') \right|$$

Applying the triangle inequality,

$$|(Tf)(w) - (Tg)(w)| \leqslant \beta \sum_{w'} |f(w') - g(w')|\varphi(w') \leqslant \beta \|f - g\|_\infty$$

$$\therefore \quad \|Tf - Tg\|_\infty \leqslant \beta \|f - g\|_\infty$$

As a consequence

1. T has a unique fixed point $\bar{v}$ in $\mathbb{R}^{\mathsf{W}}$
2. $T^k v \to \bar{v}$ as $k \to \infty$ for all $v \in \mathbb{R}^{\mathsf{W}}$

Moreover, we know that $v^* \in \mathbb{R}^{\mathsf{W}}$ and v^* solves the Bellman equation

Hence $\bar{v} = v^*$

Summary: We can compute v^* by successive approximation:

1. choose any initial $v \in \mathbb{R}^{\mathsf{W}}$
2. iterate with T to obtain $T^k v \approx v^*$ (k large)

Optimal Policies

Recall: The optimal decision facing current offer w is

$$\mathbb{1}\left\{\frac{w}{1-\beta} \geqslant h^*\right\} \quad \text{where } h^* := c + \beta \sum_{w'} v^*(w')\varphi(w')$$

Let's try to write this in the language of dynamic programming

Dynamic programming centers around the problem of finding optimal **policies**

In general, for a dynamic program, choices $=$ sequence $(A_t)_{t \geqslant 0}$

- specifies how the agent acts at each t

Agents are not clairvoyant: A_t cannot depend on future events

Hence, for some function σ_t,

$$A_t = \sigma_t(X_t, A_{t-1}, X_{t-1}, A_{t-2}, X_{t-2}, \dots A_0, X_0)$$

In dynamic programming, σ_t is called a **policy function**

Key idea Design the state such that X_t is

- sufficient to determine the optimal current action
- but not so large as to be unmanageable

Finding the state is an art!

Example. Recall retailer who chooses stock orders and prices in each period

What to include in the current state?

- level of current inventories
- interest rates and inflation?
- competitors prices?

So suppose state X_t determines the current action A_t

Then we can write $A_t = \sigma(X_t)$ for some function σ

Note that we dropped the time subscript on σ

No loss of generality: can include time in the current state

- i.e., expand X_t to $\hat{X}_t = (t, X_t)$

Depends on the problem at hand

- For the job search model with finite horizon, the date matters
- For the infinite horizon version of the problem, however, the agent always looks forward toward an infinite horizon

For job search model,

- state $=$ current wage offer and
- possible actions are accept (1) or reject (0)

A policy is a map σ from W to $\{0, 1\}$

Let Σ be the set of all such maps

For each $v \in \mathbb{R}^{\mathsf{W}}$, let us define a **$v$-greedy policy** to be a $\sigma \in \Sigma$ satisfying

$$\sigma(w) = \mathbb{1}\left\{\frac{w}{1-\beta} \geqslant c + \beta \sum_{w' \in \mathsf{W}} v(w')\varphi(w')\right\} \quad \text{for all } w \in \mathsf{W}$$

Accepts iff $w/(1-\beta) \geqslant$ continuation value computed using v

Optimal choice:

- agent should adopt a v^*-greedy policy
- Sometimes called **Bellman's principle of optimality**

We can also express a v^*-greedy policy via

$$\sigma^*(w) = \mathbb{1}\{w \geqslant w^*\} \quad \text{where} \quad w^* := (1-\beta)h^* \tag{5}$$

The term w^* in (5) is called the **reservation wage**

- Same ideas as before, different language
- We prove optimality more carefully later

Computation

Since T is globally stable on $\mathbb{R}^{\mathsf{W}}$, we can compute an approximate optimal policy by

1. applying successive approximation on T to compute v^*
2. calculate a v^*-greedy policy

In dynamic programming, this approach is called **value function iteration**

input $v_0 \in \mathbb{R}^{\mathsf{W}}$, an initial guess of v^*
input τ, a tolerance level for error
$\varepsilon \leftarrow \tau + 1$
$k \leftarrow 0$
while $\varepsilon > \tau$ **do**
    for $w \in \mathsf{W}$ **do**
        $v_{k+1}(w) \leftarrow (Tv_k)(w)$
    end
    $\varepsilon \leftarrow \|v_k - v_{k+1}\|_\infty$
    $k \leftarrow k + 1$
end
Compute a v_k-greedy policy σ
return σ

```julia
include("two_period_job_search.jl")
include("s_approx.jl")

" The Bellman operator. "
function T(v, model)
    (; n, w_vals, ϕ, β, c) = model
    return [max(w / (1 - β), c + β * v'ϕ) for w in w_vals]
end

" Get a v-greedy policy. "
function get_greedy(v, model)
    (; n, w_vals, ϕ, β, c) = model
    σ = w_vals ./ (1 - β) .>= c .+ β * v'ϕ  # Boolean policy vector
    return σ
end

" Solve the infinite-horizon IID job search model by VFI. "
function vfi(model=default_model)
    (; n, w_vals, ϕ, β, c) = model
    v_init = zero(model.w_vals)
    v_star = successive_approx(v -> T(v, model), v_init)
    σ_star = get_greedy(v_star, model)
    return v_star, σ_star
end
```

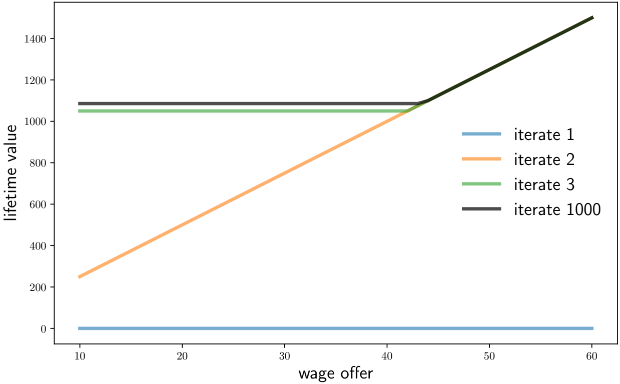


Figure: A sequence of iterates of the Bellman operator

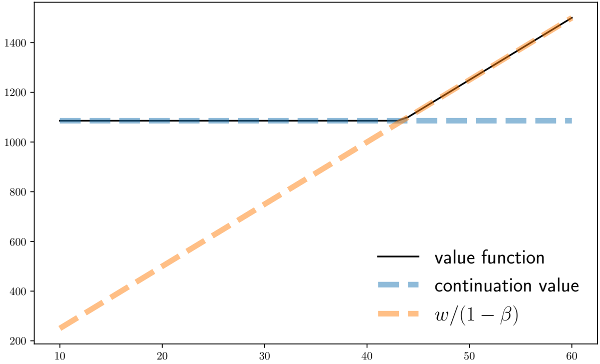


Figure: The approximate value function for job search

Computing the Continuation Value Directly

We used a programming approach to solve this problem

For the infinite horizon job search problem, a more efficient way exists

The idea is to compute the continuation value h^* directly

This shifts the problem from n-dimensional to one-dimensional

- large efficiency gain

Method: Recall that

$$v^*(w) = \max\left\{\frac{w}{1-\beta},\, c + \beta \sum_{w'} v^*(w')\varphi(w')\right\} v^*(w')$$

$$= \max\left\{\frac{w'}{1-\beta},\, h^*\right\}$$

$$\therefore \quad \sum_{w'} v^*(w')\varphi(w') = \sum_{w'} \max\left\{\frac{w'}{1-\beta},\, h^*\right\}\varphi(w')$$

$$\therefore \quad \beta \sum_{w'} v^*(w')\varphi(w') = \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, \, h^* \right\} \varphi(w')$$

$$\therefore \quad h^* = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, \, h^* \right\} \varphi(w')$$

We can find h^* by solving this equation

To find h^* from

$$h^* = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, \, h^* \right\} \varphi(w') \tag{6}$$

we introduce the map $g \colon \mathbb{R}_+ \to \mathbb{R}_+$,

$$g(h) = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta}, \, h \right\} \varphi(w')$$

By construction,

$$h^* \text{ solves (6)} \iff h^* \text{ is a fixed point of } g$$

Ex. Show that g is a contraction map on $\mathbb{R}_+$

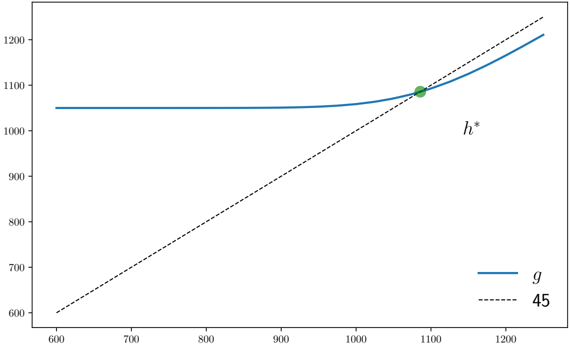


Figure: Computing the continuation value as the fixed point of g

New algorithm:

1. Compute h^* via successive approximation on g
    - Iteration in $\mathbb{R}$, not $\mathbb{R}^n$
2. Optimal policy is

$$\sigma^*(w) = \mathbb{1}\left\{\frac{w}{1-\beta} \geqslant h^*\right\}$$