Dynamic Programming

Chapter 7: Valuation

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Topics

- The Knaster–Tarski fixed point theorem
- Du's theorem
- Risk-sensitive preferences
- Epstein-Zin preferences
- Koopmans operators

Nonlinear valuation

In DP, the objective to be optimized is always lifetime value

Lifetime value is an aggregation of a flow of rewards $(R_t)_{t \geqslant 0}$

Key issue: how to combine $(R_t)_{t \geqslant 0}$ into lifetime value?

What we have seen so far:

- Standard approach: $\mathbb{E} \sum_{t \geqslant 0} \beta^t R_t$
- State-dependent discounting: $\mathbb{E} \sum_{t \geqslant 0} \left[\prod_{i=0}^{t} \beta_i \right] R_t$

Now we go further

In particular, we drop the linearity assumption

- lifetime value computed from a (nonlinear) recursion
- closely related to "recursive preferences"

To solve these problems we need more fixed point theory

We focus on fixed point results for order-preserving maps

- existence results
- global stability results

Knaster–Tarski Fixed Point Theorem

We start with a simple version of the famous KT theorem, where

- X is a finite set
- $I = [v_1, v_2]$ is a nonempty order interval in $(\mathbb{R}^{\mathsf{X}}, \leqslant)$
- T is a self-map on I
- $\text{fix}(T) =$ the set of fixed points of T in I

Theorem (Knaster–Tarski). If T is order-preserving on I, then $\text{fix}(T)$ is nonempty and contains least and greatest elements $a \leqslant b$

Moreover,

$$T^k v_1 \leqslant a \leqslant b \leqslant T^k v_2 \quad \text{for all } k \geqslant 0$$

Ex. Sketch the one-dimensional case $I = [0, 1]$ and convince yourself that a fixed point must exist

The KT fixed point theorem has a huge range of applications

However, it <u>only yields existence</u>

- Uniqueness and stability can fail

Ex. Continuing your sketch, provide a counterexample to uniqueness when $I = [0, 1]$

Next we seek additional conditions that guarantee uniqueness and stability

Concavity, Convexity and Stability

A self-map T on a convex subset D of $\mathbb{R}^n$ is called **convex** if

$$T(\lambda u + (1-\lambda)v) \leqslant \lambda Tu + (1-\lambda)Tv$$

whenever $u, v \in D$ and $\lambda \in [0, 1]$

T is called **concave** if

$$\lambda Tu + (1-\lambda)Tv \leqslant T(\lambda u + (1-\lambda)v)$$

whenever $u, v \in D$ and $\lambda \in [0, 1]$

- Here $\leqslant$ is, as usual, the pointwise order
- Note that the definitions look identical to the scalar case

Du's Theorem

Theorem. Let

- X be a finite set
- $I := [v_1, v_2]$ be a nonempty order interval in $(\mathbb{R}^\mathsf{X}, \leqslant)$
- T be a self-map on I

If T is order-preserving, then T is globally stable on I under any one of

1. T is concave and $Tv_1 \gg v_1$
2. T is concave and $\exists$ a $\delta > 0$ such that $Tv_1 \geqslant v_1 + \delta(v_2 - v_1)$
3. T is convex and $Tv_2 \ll v_2$
4. T is convex and $\exists$ a $\delta > 0$ such that $Tv_2 \leqslant v_2 - \delta(v_2 - v_1)$

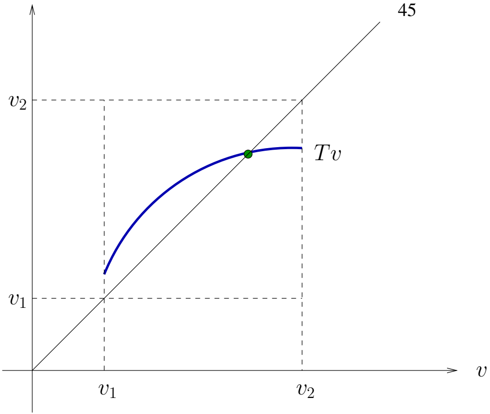

Figure: Concave case (one-dimensional)

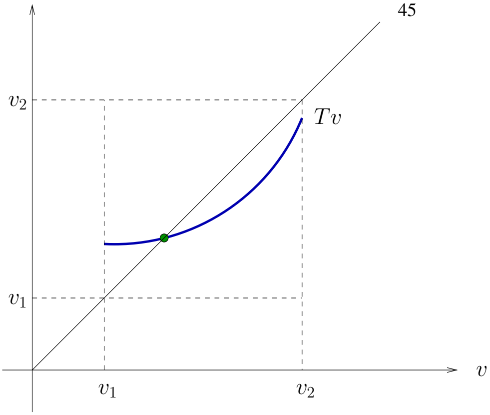

Figure: Convex case (one-dimensional)

Power-transformed affine equations

Now let's add more structure

- extends global stability to unbounded domains
- necessary and sufficient conditions

To begin, let X be finite and consider the vector equation

$$v = [h + (Av)^{1/\theta}]^{\theta} \qquad (v \in V)$$

where

- θ is a nonzero parameter
- $A \in \mathcal{L}(\mathbb{R}^{\mathsf{X}})$
- $V = (0, \infty)^{\mathsf{X}}$ and $h \in V$

To analyze the equation we introduce the self-map

$$Gv = [h + (Av)^{1/\theta}]^{\theta} \qquad (v \in V)$$

By extending Du's theorem, we can obtain

Theorem. If A is irreducible, then the following statements are equivalent

1. $\rho(A)^{1/\theta} < 1$
2. G is globally stable on V

In the case $\rho(A)^{1/\theta} \geqslant 1$, the map G has no fixed point in V

The one-dimensional case (so $\rho(A) = A$)

- globally stable iff $A^{1/\theta} < 1$

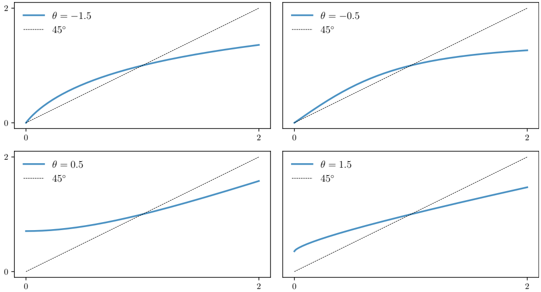

Figure: Shape properties of $Gv = [h + (Av)^{1/\theta}]^\theta$ on $(0, \infty)$

Example. Kleinman et al. (ECMA 2023) study a model of migration with capital accumulation

Optimal consumption for landlords is $c_t = \sigma_t R_t k_t$ where

- k_t is capital and R_t is the gross rate of return
- σ_t is a state-dependent process obeying

$$\sigma_t^{-1} = 1 + \beta^\psi \left[\mathbb{E}_t R_{t+1}^{(\psi-1)/\psi} \sigma_{t+1}^{-1/\psi} \right]^\psi \tag{1}$$

Assume $R_t = f(X_t)$ where

- X is finite
- (X_t) is P-Markov

Setting $v_t = \sigma_t^{-1/\psi}$, we can write (1) as

$$v_t = \left\{ 1 + \beta^\psi \left[\mathbb{E}_t R_{t+1}^{(\psi-1)/\psi} v_{t+1} \right]^\psi \right\}^{1/\psi} \tag{2}$$

Define $A \in \mathcal{L}(\mathbb{R}^{\mathsf{X}})$ by

$$(Av)(x) = \beta \sum_{x'} f(x')^{(\psi-1)/\psi} v(x') P(x, x')$$

Ex. Show the following are equivalent

1. $\exists$ a unique $v \in V := (0, \infty)^{\mathsf{X}}$ s.t. $v_t = v(X_t)$ solves (2)
2. $\rho(A)^\psi < 1$

Proof: Consider a $v \in V$ such that $v_t = v(X_t)$ solves (2)

This v must satisfy

$$v(x) = \left\{ 1 + \beta^{\psi} \left[\sum_{x'} f(x')^{(\psi - 1)/\psi} v(x') P(x, x') \right]^{\psi} \right\}^{1/\psi}$$

Equivalently,

$$v = [1 + (Av)^{\psi}]^{1/\psi}$$

By the result on slide 12,

$$\text{a unique solution exists in } V \quad \iff \quad \rho(A)^{\psi} < 1$$

Motivation: Limitations of time additive preferences

Why do we need to go beyond standard specifications such as

$$v(x) = \mathbb{E}_x \sum_{t \geqslant 0} \beta^t u(C_t)$$

Well documented problems

- failure of discounted expected utility model in experiments
- failure of constant positive discount rate in surveys / experiments
- does not accommodate ambiguity
- cannot separate risk aversion and elasticity of intertemporal substitution

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- cannot separate risk aversion and elasticity of intertemporal substitution

Another issue: agent is indifferent to variations in the joint distribution of rewards that leaves marginal distributions unchanged

Example. Suppose you accept a new job

- duration is for life
- daily consumption = daily wage

Your boss offers you two options:

(A) Your boss will flip a coin on day 1 only and set

$$\text{daily wage } = \mathbb{1}\{\text{heads}\} \times 10000 + \mathbb{1}\{\text{tails}\} \times 1$$

(B) Your boss will flip a coin every day and set

$$\text{daily wage } = \mathbb{1}\{\text{heads}\} \times 10000 + \mathbb{1}\{\text{tails}\} \times 1$$

Strict preference between A and B implies choice cannot be rationalized with time additive preferences

To see why, let $\varphi(10000) = \varphi(1) = 1/2$

- Option A $\implies$ $C_1 \sim \varphi$ and $C_t = C_1$ for $t \geqslant 2$
- Option B $\implies$ $(C_t)_{t \geqslant 1} \overset{\text{IID}}{\sim} \varphi$

Either way,

$$\mathbb{E} \sum_{t \geqslant 1} \beta^t u(C_t) = \sum_{t \geqslant 1} \beta^t \mathbb{E} u(C_t) = \frac{\beta}{1-\beta} \left[\frac{u(1) + u(10000)}{2} \right]$$

A recursive view of time additive preferences

As a warm up for recursive preferences, let's investigate traditional preferences from a recursive point of view

Consider the recursion

$$V_t = u(C_t) + \beta \, \mathbb{E}_t V_{t+1}$$

where

- $(V_t)_{t \geqslant 0}$ is to be determined
- $\mathbb{E}_t$ is expectation conditional on $X_0, \ldots, X_t$
- (X_t) is P-Markov on X
- $C_t = c(X_t)$ is a consumption path and $u \in \mathbb{R}^{\mathsf{X}}$

How to solve

$$V_t = u(C_t) + \beta \, \mathbb{E}_t V_{t+1} \tag{3}$$

Since

- consumption is a function of $(X_t)_{t \geqslant 0}$
- knowledge of X_t is sufficient to forecast $(C_{t+j})_{j \geqslant 1}$

we guess that $V_t = v(X_t)$ for some $v \in \mathbb{R}^{\mathsf{X}}$

- Called a **stationary Markov** solution to (3)

Under this conjecture, (3) can be rewritten as

$$v(X_t) = u(c(X_t)) + \beta \mathbb{E}_t v(X_{t+1})$$

Conditioning on $X_t = x$ and setting $r := u \circ c$, this becomes

$$\begin{aligned} v(x) &= r(x) + \beta\, \mathbb{E}_x\, v(X_{t+1}) \\ &= r(x) + \beta \sum_{x'} v(x') P(x, x') \end{aligned}$$

In vector form, $v = r + \beta P v$

From the NSL, the unique solution is

$$v^* = (I - \beta P)^{-1} r = \sum_{t \geqslant 0} \beta^t P^t r$$

As proved earlier, the r.h.s. is equal to

$$v^*(x) = \mathbb{E}_x \sum_{t \geqslant 0} \beta^t r(X_t) \quad \text{when } (X_t) \text{ is } P\text{-Markov}$$

In summary:

the **recursive representation**

$$V_t = u(C_t) + \beta \, \mathbb{E}_t V_{t+1}$$

and the **sequential representation**

$$V_0 = \mathbb{E}_0 \sum_{t \geqslant 0} \beta^t r(X_t)$$

specify the same lifetime value

So is the recursive formulation redundant?

No because it gives us a formula to build on!

In summary:

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specify the same lifetime value

So is the recursive formulation redundant?

No because it gives us a formula to build on!

Pursuing this idea leads to lifetime valuations without any sequential representations

- Occurs when V_t is nonlinear in current rewards and continuation values
- Such specifications are called **recursive preferences**

Remark.

The term "recursive preferences" is confusing, since traditional time additive preferences also admit a recursive specification

Economists use the term "recursive preferences" when lifetime utility can only be expressed recursively

We follow this convention

Risk-Sensitive Preferences

Let's turn to our first nonlinear example

For the consumption problem, **risk-sensitive preferences** means replacing recursion

$$v(x) = r(x) + \beta \sum_{x'} v(x') P(x, x')$$

with

$$v(x) = r(x) + \beta \frac{1}{\theta} \ln \left\{ \sum_{x'} \exp(\theta v(x')) P(x, x') \right\}$$

- $r(x) = u(c(x)) =$ utility of consumption at current state
- θ is a nonzero constant in $\mathbb{R}$

We understand

$$v(x) = r(x) + \beta \frac{1}{\theta} \ln \left\{ \sum_{x'} \exp(\theta v(x')) P(x, x') \right\}$$

as "defining" lifetime utility under risk-sensitive preferences

- solution $v \in \mathbb{R}^{\mathsf{X}}$ gives lifetime value from any state

But we can't be sure we have a definition at this point

- What if there is no v that solves this recursion?
- What there are many?

Key task: find conditions under which a unique solution exists

As a preliminary step, let's try to understand the nonlinear expectation term

In general, the **entropic risk-adjusted expectation** of random variable ξ is

$$\mathcal{E}_\theta[\xi] = \frac{1}{\theta} \ln \mathbb{E}[\exp(\theta \xi)]$$

- θ is a nonzero parameter

Ex. Prove that, if ξ is normally distributed, then

$$\mathcal{E}_\theta[\xi] = \mathbb{E}[\xi] + \theta \frac{\mathrm{Var}[\xi]}{2}$$

Notice that $\theta \downarrow$ lowers appetite for risk and $\theta \uparrow$ does the opposite

More generally,

For any random variable ξ taking values in X,

1. $\mathcal{E}_\theta[\xi] \leqslant \mathbb{E}[\xi]$ for all $\theta < 0$
2. $\mathcal{E}_\theta[\xi] \geqslant \mathbb{E}[\xi]$ for all $\theta > 0$

Moreover, these inequalities are strict if and only if $\mathrm{Var}[\xi] > 0$

Proof: Apply Jensen's lemma

(Strict case uses a strict version of Jensen's lemma)

Existence and uniqueness

Let's return to existence and uniqueness of lifetime utility under risk-sensitive preferences

Consider the **risk-sensitive Koopmans operator** $K_\theta \colon \mathbb{R}^\mathsf{X} \to \mathbb{R}^\mathsf{X}$

$$(K_\theta \, v)(x) = r(x) + \beta \frac{1}{\theta} \ln \left\{ \sum_{x' \in \mathsf{X}} \exp(\theta v(x')) P(x, x') \right\}$$

Equivalent:

1. $v \in \mathbb{R}^\mathsf{X}$ solves the risk-sensitive preference specification
2. v is a fixed point of K_θ

We continue to assume that

- $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ and $r \in \mathbb{R}^{\mathsf{X}}$
- θ is nonzero and $\beta \geqslant 0$

Proposition. If $\beta < 1$, then K_θ is globally stable on $\mathbb{R}^{\mathsf{X}}$

We prove a more general result below

Implications

1. For all nonzero θ, lifetime utility is uniquely defined
2. The unique solution v^* can be computed by successive approximation using K_θ

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Example. Suppose $r(x) = x$ and $X_{t+1} = \rho X_t + \sigma W_{t+1}$ where

- $(W_t) \stackrel{\text{IID}}{\sim} N(0,1)$
- $|\rho| < 1$

In this setting, we need to solve for v satisfying

$$v(x) = x + \beta \mathcal{E}_\theta [v(\rho x + \sigma W_1)] \tag{4}$$

Ex. Verify that $v(x) = ax + b$ solves (4) when

$$a := \frac{1}{1 - \rho\beta} \quad \text{and} \quad b := \theta \frac{\beta}{1-\beta} \frac{(a\sigma)^2}{2}$$

We can also compute an approximation to v by discretizing the state process via Tauchen's method and iterating on K_θ

```julia
using LinearAlgebra, QuantEcon

function create_rs_utility_model(;
        n=180,      # size of state space
        β=0.95,     # time discount factor
        ρ=0.96,     # correlation coef in AR(1)
        σ=0.1,      # volatility
        θ=-1.0)     # risk aversion
    mc = tauchen(n, ρ, σ, 0, 10)  # n_std = 10
    x_vals, P = mc.state_values, mc.p
    r = x_vals      # special case u(c(x)) = x
    return (; β, θ, ρ, σ, r, x_vals, P)
end
```

Listing 1: Risk sensitive utility model parameters (rs_utility.jl)

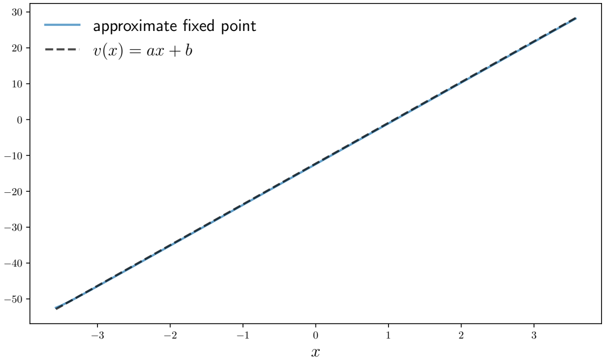

Figure: Approximate and true solutions in the Gaussian case

Ex. Replicate the figure

Epstein–Zin preferences

One of the most popular recursion preferences specifications

Has been used to study

- asset pricing
- business cycles
- monetary policy
- fiscal policy
- optimal taxation
- climate policy, etc., etc.

With **Epstein–Zin** preferences, lifetime value obeys

$$V_t = \left\{ (1-\beta) C_t^{\alpha} + \beta [\mathbb{E}_t V_{t+1}^{\gamma}]^{\alpha/\gamma} \right\}^{1/\alpha}$$

where

- γ, α are nonzero parameters
- $\beta \in (0, 1)$

Assume

- $C_t = c(X_t)$ where $c \in \mathbb{R}_+^{\mathsf{X}}$
- $(X_t)_{t \geqslant 0}$ is P-Markov on finite set X

We conjecture a solution of the form $V_t = v(X_t)$

Now condition on x to get

$$v(x) = \left\{ (1-\beta) c(x)^\alpha + \beta \left[\sum_{x'} v(x')^\gamma P(x, x') \right]^{\alpha/\gamma} \right\}^{1/\alpha}$$

Equivalently, v is a fixed point of the **Epstein–Zin Koopmans operator**

$$(Kv)(x) = \left\{ (1-\beta) c(x)^\alpha + \beta \left[\sum_{x'} v(x')^\gamma P(x, x') \right]^{\alpha/\gamma} \right\}^{1/\alpha}$$

Let's rewrite more generally as

$$(Kv)(x) = \left\{ h(x) + \beta \left[\sum_{x'} v(x')^\gamma P(x, x') \right]^{\alpha/\gamma} \right\}^{1/\alpha}$$

Pointwise on X this is

$$Kv = \left\{ h + \beta [P v^\gamma]^{\alpha/\gamma} \right\}^{1/\alpha}$$

- all operations are pointwise on vectors/functions
- we assume $h \gg 0$ to make sure Kv is real-valued

Ex. Prove that K is a self-map on $V := (0, \infty)^{\mathsf{X}}$

Epstein–Zin stability

Proposition. If P is irreducible and $h \gg 0$, then K is globally stable on V

A proof is provided below

Ex. Compute the fixed point of K in V starting with this code

```julia
include("s_approx.jl")
using LinearAlgebra, QuantEcon

function create_ez_utility_model(;
        n=200,      # size of state space
        ρ=0.96,     # correlation coef in AR(1)
        σ=0.1,      # volatility
        β=0.99,     # time discount factor
        α=0.75,     # EIS parameter
        γ=-2.0)     # risk aversion parameter

    mc = tauchen(n, ρ, σ, 0, 5)
    x_vals, P = mc.state_values, mc.p
    c = exp.(x_vals)

    return (; β, ρ, σ, α, γ, c, x_vals, P)
end
```

The figure plots every 10th iterate, repeated 100 times.

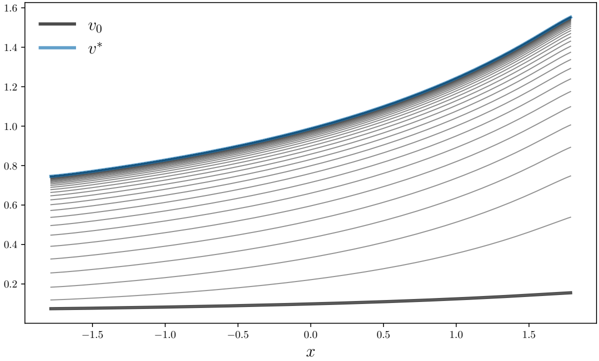

Figure: Convergence of Koopmans iterates for Epstein–Zin utility

Proof of global stability

K is difficult to work with because

- linear and nonlinear transformations are intertwined
- there are several cases for the parameters that we need to handle

To tackle these issues we

1. introduce an operator $\hat{K}$ obtained from K via a smooth transformation
2. prove that $(\hat{V}, \hat{K})$ and (V, K) are topologically conjugate
3. obtain conditions under which $\hat{K}$ is globally stable on V

We define $\hat{K}$ via

$$\hat{K}v = \left\{h + \beta(Pv)^{1/\theta}\right\}^{\theta} \qquad \text{where} \quad \theta := \frac{\gamma}{\alpha}$$

The operator $\hat{K}$ is simpler than K because

- one parameter θ rather than α, γ
- separates linear and nonlinear parts of the mapping

We continue to assume that P is irreducible and $h \gg 0$

Ex. Prove that $\hat{K}$ is a self-map on $V := (0, \infty)^{\mathsf{X}}$

Let Φ be defined by $\Phi v = v^\gamma$

Ex. Show that

1. Φ is a homeomorphism from V to itself and
2. (V, K) and $(V, \hat{K})$ are topologically conjugate under Φ

Proof: Evidently Φ is a homeomorphism from V to itself

In addition, for $v \in V$,

$$\hat{K}\Phi v = \left\{h + \beta(P\Phi v)^{1/\theta}\right\}^\theta = \left\{h + \beta(Pv^\gamma)^{\alpha/\gamma}\right\}^{\gamma/\alpha} = \Phi K v$$

Hence (V, K) and $(V, \hat{K})$ are topologically conjugate, as claimed

Now set $A = \beta^\theta P$, so that

$$\hat K v = [h + (Av)^{1/\theta}]^\theta$$

By the result on slide 12,

$$\hat K \text{ is globally stable on } V \iff \rho(A)^{1/\theta} < 1$$

In our case

$$\rho(A)^{1/\theta} = \rho(\beta^\theta P)^{1/\theta} = \beta$$

It follows that $\hat K$ is globally stable on V whenever $\beta < 1$

Since (V, K) and $(V, \hat K)$ are topologically conjugate, K has the same properties on V

A general representations

We have discussed two examples of recursive preferences

Now let's build a general representation

While various constructions can be found in the decision theory literature, many are challenging for applied researchers hoping to do quantitative work

Here we give a relatively parsimonious definition based

- a Markov environment
- fixed point theory

We will build our theory from two components

1. an “aggregation function” and
2. a “certainty equivalent operator”

Below we define these objects and give examples

Then we put them together to get recursive preferences

Certainty equivalent operators

Given $V \subset \mathbb{R}^{\mathsf{X}}$, a **certainty equivalent operator** on V is a self-map R on V such that

1. R is order-preserving on V and
2. $R(\lambda \mathbb{1}) = \lambda \mathbb{1}$ for all $\lambda \in \mathbb{R}$ with $\lambda \mathbb{1} \in V$

Example. Conditional expectations is a certainty equivalent operator

To see this, set $V = \mathbb{R}^{\mathsf{X}}$, fix $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$

- $u \leqslant v$ implies $Pu \leqslant Pv$, so P preserves order on V and
- $P(\lambda \mathbb{1}) = \lambda P \mathbb{1} = \lambda \mathbb{1}$

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Example. Let $V = \mathbb{R}^{\mathsf{X}}$ and fix

- $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ and
- nonzero θ

The operator

$$(R_\theta \, v)(x) = \frac{1}{\theta} \ln \left\{ \sum_{x'} \exp(\theta v(x')) P(x, x') \right\}$$

is called the **entropic certainty equivalent operator**

Ex. Prove that R_θ is a certainty equivalent operator on V

Example. Let $V = (0, \infty)^{\mathsf{X}}$ and fix

- $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ and
- nonzero γ

The operator

$$(R_\gamma \, v)(x) = \left\{ \sum_{x'} v(x')^\gamma P(x, x') \right\}^{1/\gamma}$$

is called the **Kreps-Porteus certainty equivalent operator**

Ex. Confirm that R_γ is a certainty equivalent operator on V

Let $V = \mathbb{R}^{\mathsf{X}}$ and fix $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ and $\tau \in [0, 1]$

Let R_τ be defined by $(R_\tau \, v)(x) = Q_\tau \, v(X)$ where

- $X \sim P(x, \cdot)$
- Q_τ is the quantile functional

More specifically,

$$(R_\tau \, v)(x) = \min \left\{ y \in \mathbb{R} \;\middle|\; \sum_{x'} \mathbb{1}\{v(x') \leqslant y\} P(x, x') \geqslant \tau \right\}$$

R_τ is called the **quantile certainty equivalent**

Confirm that R_τ defines a certainty equivalent operator on V

A certainty equivalent operator R on V is called

- **positive homogeneous** on V if

$$R(\lambda v) = \lambda R v \text{ for all } v \in V \text{ and } \lambda \geqslant 0 \text{ with } \lambda v \in V$$

- **superadditive** on V if

$$R(v + w) \geqslant Rv + Rw \text{ for all } v, w \in V \text{ with } v + w \in V$$

- **subadditive** on V if

$$R(v + w) \leqslant Rv + Rw \text{ for all } v, w \in V \text{ with } v + w \in V$$

- **constant-subadditive** on V if

$$R(v + \lambda \mathbb{1}) \leqslant Rv + \lambda \mathbb{1} \text{ for all } v \in V,\ \lambda \geqslant 0 \text{ with } v + \lambda \mathbb{1} \in V$$

Example. Let $V = (0, \infty)^{\mathsf{X}}$ and fix $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$

In this setting, the Kreps–Porteus certainty equivalent operator

$$(R_\gamma \, v)(x) = \left\{ \sum_{x'} v(x')^\gamma P(x, x') \right\}^{1/\gamma}$$

is

1. subadditive on V when $\gamma \geqslant 1$
2. superadditive on V when $\gamma \leqslant 1$

Proof: See the book

Ex. Prove that the quantile certainty equivalent operator R_τ is constant-subadditive

Ex. Show that the entropic certainty equivalent operator R_θ is constant-subadditive

Proof: Fix $v \in V$, $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ and $\lambda \in \mathbb{R}_+$

Let X be a draw from $P(x, \cdot)$

We have

$$(R_\theta(v + \lambda))(x) = \frac{1}{\theta} \ln \{\mathbb{E} \exp[\theta(v(X) + \lambda)]\}$$

$$= \frac{1}{\theta} \ln \{\mathbb{E} \exp[\theta v(X)] \cdot \exp(\theta\lambda)\}$$

$$= \frac{1}{\theta} \ln \{\mathbb{E} \exp[\theta v(X)]\} + \lambda$$

Hence constant-subadditivity holds

Let V be convex and let R be a positive homogeneous certainty equivalent operator on V

Lemma

1. R is subadditive on $V \implies R$ is convex on V
2. R is superadditive on $V \implies R$ is concave on V

Proof: Regarding (i), fix $\lambda \in [0,1]$ and $v, w \in V$

Using subadditivity and positive homogeneity, we have

$$R(\lambda v + (1-\lambda)w) \leqslant R(\lambda v) + R((1-\lambda)w) = \lambda Rv + (1-\lambda)Rw$$

This proves that R is convex on V. The proof of (ii) is similar.

Lemma The Kreps–Porteus certainty equivalent operator R_γ is

1. convex on V when $\gamma \geqslant 1$ and
2. concave on V when $\gamma \leqslant 1$

Ex. Check it

This result will be useful later, when we introduce ambiguity

- Convexity / concavity connect to Du's theorem

Monotonicity

Let X be partially ordered and let $i\mathbb{R}^{\mathsf{X}}$ be the set of increasing functions in $\mathbb{R}^{\mathsf{X}}$

Let V be such that $i\mathbb{R}^{\mathsf{X}} \subset V \subset \mathbb{R}^{\mathsf{X}}$ and let R be a certainty equivalent on V

We call R **monotone increasing** if R is invariant on $i\mathbb{R}^{\mathsf{X}}$

Ex. Prove: If $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ is monotone increasing, then

1. the entropic certainty equivalent operator R_θ is monotone increasing on $V = \mathbb{R}^{\mathsf{X}}$
2. the Kreps–Porteus certainty equivalent operator R_γ is monotone increasing on $V = (0, \infty)^{\mathsf{X}}$

Aggregation

Koopmans operators are typically constructed by combining

1. a certainty equivalent operator and
2. an aggregation function

Given $V \subset \mathbb{R}^{\mathsf{X}}$, an **aggregator** on V is a map $A \colon \mathsf{X} \times \mathbb{R} \to \mathbb{R}$ such that

1. $w(x) = A(x, v(x))$ is in V whenever $v \in V$ and
2. $y \mapsto A(x, y)$ is increasing for all $x \in \mathsf{X}$

Intuitively, an aggregator combines current state and continuation values to measure lifetime value

Common types of aggregators include the

- **Leontief aggregator**

$$A_{\mathrm{MIN}}(x,y) = \min\{r(x), \beta y\} \text{ with } r \in \mathbb{R}^{\mathsf{X}} \text{ and } \beta \geqslant 0$$

- **Uzawa aggregator**

$$A_{\mathrm{UZAWA}}(x,y) = r(x) + b(x)y \text{ with } r \in \mathbb{R}^{\mathsf{X}} \text{ and } b \in \mathbb{R}^{\mathsf{X}}_{+}$$

- **CES aggregator**

$$A_{\mathrm{CES}}(x,y) = \{r(x)^{\alpha} + \beta y^{\alpha}\}^{1/\alpha} \text{ with } r \in \mathbb{R}^{\mathsf{X}}_{+},\ \beta \geqslant 0,\ \alpha \neq 0$$

An important special case of the CES & Uzawa aggregators is the

- **additive aggregator**

$$A_{\mathrm{ADD}}(x,y) = r(x) + \beta y \text{ with } r \in \mathbb{R}^{\mathsf{X}} \text{ and } \beta \geqslant 0$$

Given $V \subset \mathbb{R}^{\mathsf{X}}$, we call K a **Koopmans operator** on V if

$$(Kv)(x) = A(x, (Rv)(x)) \tag{5}$$

for

1. some aggregator A on V and
2. a certainty equivalent operator R on V

Ex. Show that every Koopmans operator on V is an order-preserving self-map on V

In what follows we write (5) as

$$K = A \circ R$$

We have already met the following special cases

1. The risk-sensitive Koopmans operator

$$K_\theta = A_{\text{ADD}} \circ R_\theta$$

where R_θ is the entropic certainty equivalent operator

2. The Epstein–Zin Koopmans operator

$$K = A_{\text{CES}} \circ R_\gamma$$

where R_γ is the Kreps–Porteus expectations operator

Another special case is

$$K = A_{\text{ADD}} \circ P$$

where $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ is ordinary conditional expectations

In this case

- K is called **time additive**
- corresponds to traditional discounted expectations model

The **lifetime value** generated by Koopmans operator K on $V \subset \mathbb{R}^{\mathsf{X}}$ is the unique fixed point v of K in V, whenever it exists

- $v(x) =$ lifetime value given initial state x

Example. In the case of time additive preferences

- $Kv = r + \beta P v$ with $\beta \in (0, 1)$
- unique fixed point is $v^* = (I - \beta P)^{-1} r$

We know that

$$v^*(x) = \mathbb{E} \sum_{t \geqslant 0} \beta^t r(X_t) \quad \text{when } (X_t) \text{ is } P\text{-Markov and } X_0 = x$$

$$= \text{lifetime value}$$

Example. Let K_θ be the risk sensitive Koopmans operator

K_θ is globally stable on $V = \mathbb{R}^{\mathsf{X}}$ when $\beta \in (0, 1)$

- see the result on slide 30

The unique fixed point of K in V is interpreted as lifetime value under risk-sensitive preferences

- for one instance, see the fig on slide 33

Finite horizons

Given arbitrary Koopmans operator K, we identify $K^m w$ with total m-period utility given terminal condition w

Example. For $Kv = r + \beta P v$, we have

$$K^m w = \sum_{t=0}^{m-1} (\beta P)^t r + (\beta P)^m w$$

If (X_t) is P-Markov, we can write this as

$$(K^m w)(x) = \mathbb{E}_x \left[\sum_{t=0}^{m-1} \beta^t r(X_t) + \beta^m w(X_m) \right]$$

$$= \text{lifetime utility up to date } m$$

Let

- $\mathsf{X} = (\mathsf{X}, \preceq)$ be partially ordered
- $i\mathbb{R}^{\mathsf{X}}$ be the set of increasing functions in $\mathbb{R}^{\mathsf{X}}$
- V be such that $i\mathbb{R}^{\mathsf{X}} \subset V \subset \mathbb{R}^{\mathsf{X}}$

Let $K = A \circ R$ be a globally stable Koopmans operator on V

Ex. Show that the unique fixed point v^* is increasing on X whenever

1. $A(x, v) \leqslant A(x', v)$ whenever $v \in V$ and $x \preceq x'$, and
2. R is monotone increasing on V

We call aggregator A on V a **Blackwell aggregator** if $\exists$ a $\beta \in (0, 1)$ such that

$$A(x, y + \lambda) \leqslant A(x, y) + \beta\lambda \text{ for all } x \in \mathsf{X},\ y \in \mathbb{R} \text{ and } \lambda \in \mathbb{R}_+$$

Ex. Show that $A_{\text{MIN}}(x, y) = \min\{r(x), \beta y\}$ is a Blackwell aggregator when $\beta < 1$

Proof: Fixing $x \in \mathsf{X}$, $y \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$, we have

$$\begin{aligned}\min\{r(x), \beta y + \beta\lambda\} &\leqslant \min\{r(x) + \beta\lambda, \beta y + \beta\lambda\} \\ &= \min\{r(x), \beta y\} + \beta\lambda\end{aligned}$$

That is, $A(x, y + \lambda) \leqslant A(x, y) + \beta\lambda$

We call aggregator A on V a **Blackwell aggregator** if $\exists$ a $\beta \in (0, 1)$ such that

$$A(x, y + \lambda) \leqslant A(x, y) + \beta\lambda \text{ for all } x \in \mathsf{X},\ y \in \mathbb{R} \text{ and } \lambda \in \mathbb{R}_+$$

Ex. Show that $A_{\text{MIN}}(x, y) = \min\{r(x), \beta y\}$ is a Blackwell aggregator when $\beta < 1$

Proof: Fixing $x \in \mathsf{X}$, $y \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$, we have

$$\begin{aligned} \min\{r(x), \beta y + \beta\lambda\} &\leqslant \min\{r(x) + \beta\lambda, \beta y + \beta\lambda\} \\ &= \min\{r(x), \beta y\} + \beta\lambda \end{aligned}$$

That is, $A(x, y + \lambda) \leqslant A(x, y) + \beta\lambda$

Proposition. If

1. A is a Blackwell aggregator and
2. R is constant-subadditive

then $K = A \circ R$ is a contraction on V with respect to $\| \cdot \|_\infty$

Proof: Fixing $v \in V$ and $\lambda \in \mathbb{R}_+$, we have

$$K(v + \lambda) = A(\cdot, R(v + \lambda)) \leqslant A(\cdot, Rv + \lambda)$$

Since A is a Blackwell aggregator, the last bound yields

$$K(v + \lambda) \leqslant A(\cdot, Rv) + \beta\lambda = Kv + \beta\lambda$$

Hence K satisfies Blackwell's condition for a contraction

Let's now return to the global stability claim for the risk-sensitive Koopmans operator K_θ on slide 30

Pick any nonzero θ

Recall that $K_\theta = A_{\text{ADD}} \circ R_\theta$ when

- A_{ADD} is the additive aggregator and
- R_θ is the entropic certainty equivalent

Note that

1. A_{ADD} is a Blackwell aggregator and
2. R_θ is constant-subadditive

Hence K_θ is globally stable on $\mathbb{R}^{\mathsf{X}}$

Suppose $V = \mathbb{R}^{\mathsf{X}}$ and

$$(K_\tau v)(x) = r(x) + \beta (R_\tau v)(x) \qquad (x \in \mathsf{X}) \tag{6}$$

for $\beta \in (0,1)$, $r \in \mathbb{R}^{\mathsf{X}}$ and quantile certainty equivalent R_τ

- represents **quantile preferences** (de Castro and Galvao 2019)
- τ parameterizes attitude to risk

Since R_τ is constant-subadditive and A_{ADD} is Blackwell, K_τ is globally stable

Ex. Suppose K_τ in (6) is replaced by with $K = A_{\mathrm{MIN}} \circ R_\tau$. Under analogous assumptions, prove that K is globally stable.

Uzawa aggregation and stability

Consider the Koopmans operator $K = A_{\text{UZAWA}} \circ R$

- V is some subset of $\mathbb{R}^{\mathsf{X}}$ and
- R is a certainty equivalent operator on V

Example. Suppose $V = \mathbb{R}^{\mathsf{X}}$ and $R = P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$

Then

$$Kv = r + Lv \text{ where } L(x, x') = b(x) P(x, x')$$

Ex. Show that K is globally stable on V whenever $\rho(L) < 1$

Example. In Krussell and Smith 1998, Toda 2019, Cao 2020, etc.,

$$v(x) = \mathbb{E}_x \sum_{t=0}^{\infty} \left[\prod_{i=0}^{t} \beta_i \right] u(C_t) \quad \text{with} \quad \beta_0 := 1 \tag{7}$$

Suppose $C_t = c(X_t)$ and $\beta_t = b(X_t)$ where

- $b \geqslant 0$ and
- (X_t) is P-Markov for some $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$

Set $L(x, x') := b(x) P(x, x')$ and suppose $\rho(L) < 1$

Then v in (7) is the unique f.p. of

$$Kv = u \circ c + bPv = A_{\text{UZAWA}} Pv$$

Stability via concavity

Now consider $Kv = r + bRv$ when R is not in $\mathcal{M}(\mathbb{R}^{\mathsf{X}})$

When does global stability hold (allowing $b > 1$ in some states)?

Suppose

(a) $bRv \leqslant c + Lv$ for $c \in \mathbb{R}^{\mathsf{X}}$ and $L \in \mathcal{L}(\mathbb{R}^{\mathsf{X}})$ with $\rho(L) < 1$

(b) $r \gg 0$ and R is concave on $\mathbb{R}^{\mathsf{X}}_{+}$

Let $V = [0, \bar{v}]$ where $\bar{v} := (I - L)^{-1}(r + c)$

Proposition. If (a)–(b) hold, then K is globally stable on V

Proof: Under (a)–(b), K is a self-map on V

Indeed, we have

$$0 \ll r = r + 0 = r + bR0 = K0$$

and

$$K\bar{v} = r + bR\bar{v} \leqslant r + c + L\bar{v} = r + c - (I - L)\bar{v} + \bar{v} = \bar{v}$$

In addition, K is concave and order-preserving on V

The claim now follows from Du's theorem

EZ preferences with state-dependent discounting

CES-Uzawa aggregator + Kreps–Porteus certainty equivalent operator leads to

$$Kv = \left\{ h + b \left[P v^{\gamma} \right]^{\alpha / \gamma} \right\}^{1/\alpha}$$

In what follows we

- take $V = (0, \infty)^{\mathsf{X}}$
- assume $h, b \in V$ and $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ is irreducible

Ex. Show that K is self-map on V

To discuss stability of K we introduce the operator $A \in \mathcal{L}(\mathbb{R}^{\mathsf{X}})$ defined by

$$(Av)(x) := b(x)^\theta \sum_{x'} v(x') P(x, x') \quad \text{where} \ \ \theta := \frac{\gamma}{\alpha}$$

Note that we can now write K as follows

$$Kv = \left\{ h + [Av^\gamma]^{\alpha/\gamma} \right\}^{1/\alpha}$$

Proposition. K is globally stable on V if and only if $\rho(A)^{\alpha/\gamma} < 1$

Proof: Let

$$\hat{K}v = \left\{ h + (Av)^{1/\theta} \right\}^{\theta}$$

Let Φ be defined by $\Phi v = v^{\gamma}$

Ex. Show that $\hat{K}$ is globally stable on V if and only if $\rho(A)^{\alpha/\gamma} < 1$

- Hint: See the result on slide 12

Ex. Show that (V, K) and $(V, \hat{K})$ are topologically conjugate under Φ

The claim in the proposition follows