

# Dynamic Programming

## Mini-Lecture: Asset Pricing

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## Pricing a single payoff

An asset pays a random amount  $G_{t+1}$  at time  $t + 1$  and nothing thereafter

One way to price this asset is **risk-neutral** pricing:

$$P_t = \beta \mathbb{E}_t G_{t+1}$$

Here

- $\beta$  is a constant discount factor
- $\mathbb{E}_t =$  expectation at time  $t$

Roughly speaking,

$$\text{cost} = \text{expected benefit}$$

This model fails empirically: same rate of return regardless of risk

Most investors demand higher rates of return for holding risky assets

Hence we modify to

$$P_t = \mathbb{E}_t M_{t+1} G_{t+1}$$

In this expression,  $M_{t+1}$

- replaces  $\beta$
- is called the **stochastic discount factor (SDF)** / **pricing kernel**

This model fits the data somewhat better when  $M_{t+1}$  is suitably chosen

# Pricing a cash flow

Now let's try to price an asset like a share

- delivers a cash flow  $D_t, D_{t+1}, \dots$

We will call these payoffs **dividends**

Ex-dividend contract: buy share, hold for one period sell it again  
yields payoff

$$G_{t+1} = D_{t+1} + P_{t+1}$$

Inserting into the previous equation gives

$$P_t = \mathbb{E}_t M_{t+1} [D_{t+1} + P_{t+1}]$$

## Growing prices

We could try to solve the last equation

$$P_t = \mathbb{E}_t M_{t+1} [D_{t+1} + P_{t+1}]$$

But prices grow over time, which complicates analysis

Easier to solve for the **price-dividend ratio**  $V_t := P_t/D_t$

Inserting this into the last equation and rearranging gives

$$V_t = \mathbb{E}_t \left[ M_{t+1} \frac{D_{t+1}}{D_t} (1 + V_{t+1}) \right]$$

We can also write this as

$$V_t = \mathbb{E}_t \left[ M_{t+1} \exp(G_{t+1}^d) (1 + V_{t+1}) \right]$$

where

$$G_{t+1}^d = \ln \frac{D_{t+1}}{D_t} = \text{growth rate of dividends}$$

To solve for  $V_t$  we need to specify

1. the stochastic discount factor  $M_{t+1}$  and
2. the growth rate of dividends  $G_{t+1}^d$

# Choosing the SDF

For simplicity we adopt the Lucas SDF

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}$$

where

- $u$  = utility
- $C_t$  is time  $t$  consumption (representative consumer)

We assume CRRA utility:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad \implies \quad u'(c) = c^{-\gamma}$$

With

$$G_{t+1}^c := \ln \frac{C_{t+1}}{C_t}$$

the Lucas SDF becomes

$$\begin{aligned} M_{t+1} &= \beta \frac{u'(C_{t+1})}{u'(C_t)} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \\ &= \beta \exp(G_{t+1}^c)^{-\gamma} = \beta \exp(-\gamma G_{t+1}^c) \end{aligned}$$



## Solving for the price-dividend ratio

Recall that our aim is to solve

$$V_t = \mathbb{E}_t \left[ M_{t+1} \exp(G_{t+1}^d) (1 + V_{t+1}) \right]$$

Substituting in the last expression gives

$$V_t = \beta \mathbb{E}_t \left[ \exp(G_{t+1}^d - \gamma G_{t+1}^c) (1 + V_{t+1}) \right]$$

We assume that  $(G_t^d, G_t^c)$  are driven by **state process**  $(X_t)$

- $P$ -Markov chain on finite state  $X = \{x_1, \dots, x_N\}$
- aggregate shock

More specifically, we assume

$$G_{t+1}^c = \mu_c + X_t + \sigma_c \varepsilon_{t+1}^c$$

$$G_{t+1}^d = \mu_d + X_t + \sigma_d \varepsilon_{t+1}^d$$

Here

- $(\varepsilon_t^c) \stackrel{\text{iid}}{\sim} N(0, 1)$
- $(\varepsilon_t^d) \stackrel{\text{iid}}{\sim} N(0, 1)$
- independent of each other

Recall that

$$V_t = \beta \mathbb{E}_t \left[ \exp(G_{t+1}^d - \gamma G_{t+1}^c) (1 + V_{t+1}) \right]$$

Guess:  $V_t = v(X_t)$  for some unknown  $v \in \mathbb{R}^X$

With  $a := \mu_d - \gamma \mu_c$  this means

$$v(X_t) =$$

$$\beta \mathbb{E}_t \left\{ \exp[a + (1 - \gamma)X_t + \sigma_d \varepsilon_{t+1}^d - \gamma \sigma_c \varepsilon_{t+1}^c] (1 + v(X_{t+1})) \right\}$$

Shocks  $\varepsilon_{t+1}^c$  and  $\varepsilon_{t+1}^d$  are IID so we integrate them out

$Y = \exp(c\varepsilon)$  for  $c \in \mathbb{R}$ ,  $\varepsilon \sim N(0, 1)$  implies  $\mathbb{E}Y = \exp(c^2/2)$

This yields

$$v(X_t) = \beta \mathbb{E}_t \left\{ \exp \left[ a + (1 - \gamma)X_t + \frac{\sigma_d^2 + \gamma^2 \sigma_c^2}{2} \right] (1 + v(X_{t+1})) \right\}$$

Conditioning on  $X_t = x$ , we can write this as

$$v(x) = \beta \sum_{y \in \mathcal{X}} \exp \left[ a + (1 - \gamma)x + \frac{\sigma_d^2 + \gamma^2 \sigma_c^2}{2} \right] (1 + v(y)) P(x, y)$$

Considering  $v, x$  as arrays we get

$$v[i] = \beta \sum_{j=1}^N \exp \left[ a + (1 - \gamma)x[i] + \frac{\sigma_d^2 + \gamma^2 \sigma_c^2}{2} \right] (1 + v[j])P[i, j]$$

Equivalently,

$$v[i] = \sum_{j=1}^N K[i, j](1 + v[j])$$

where

$$K[i, j] := \beta \exp \left[ a + (1 - \gamma)x[i] + \frac{\sigma_d^2 + \gamma^2 \sigma_c^2}{2} \right] P[i, j]$$

Thus we have

$$v = K(\mathbb{1} + v) = K\mathbb{1} + Kv$$

where

$$K[i, j] = \beta \exp \left[ a + (1 - \gamma)x[i] + \frac{\sigma_d^2 + \gamma^2 \sigma_c^2}{2} \right] P[i, j]$$

- When does this equation have a unique solution?
- How can we compute it?

Suppose  $r(K) := \text{spectral radius} < 1$

Then  $I - K$  is invertible and

$$v = (I - K)^{-1} K \mathbb{1}$$

Once we specify  $P$ , state space  $X$  and all the parameters, we can

1. obtain  $K$
2. verify  $r(K) < 1$
3. compute the solution

To specify  $P$  and the state space  $X$  we assume

$$X_{t+1} = \rho X_t + \sigma \eta_{t+1}$$

where

- $\rho, \sigma$  are parameters
- $(\eta_t)$  is IID and standard normal

Now discretize

- Eg., use QuantEcon.py's tauchen function



## An Extended Example

One problem: volatility is not constant over time

To accommodate this, we now suppose that

$$G_{t+1}^i = \mu_i + Z_t + \bar{\sigma} \exp(H_t^i) \varepsilon_{t+1}^i \quad i \in \{c, d\}$$

- $(Z_t)$  is  $R$ -Markov on  $Z$
- $(H_t^c)$  and  $(H_t^d)$  are volatility processes.

In particular,

$$H_{t+1}^i = \rho_i H_t^i + \sigma_i \eta_{t+1}^i \quad i \in \{c, d\}$$

Here  $(\eta_t^c)$  and  $(\eta_t^d)$  are IID standard normal

We set  $X := (H^c, H^d, Z)$  and consider again

$$V_t = \beta \mathbb{E}_t \left[ \exp(G_{t+1}^d - \gamma G_{t+1}^c) (1 + V_{t+1}) \right]$$

Guessing  $V_t = v(X_t)$  for some  $v \in \mathbb{R}^X$  yields

$$v(x) = \beta \mathbb{E}_x \exp[a + (1 - \gamma)z + \bar{\sigma} e^{h_d} \varepsilon_{t+1}^d - \gamma \bar{\sigma} e^{h_c} \varepsilon_{t+1}^c] (1 + v(X_{t+1}))$$

As before, we integrate out the independent shocks and use the rules for expectations of lognormals to obtain

$$v(x) = \beta \mathbb{E}_x \exp \left[ a + (1 - \gamma)z + \bar{\sigma}^2 \frac{e^{2h_d} + \gamma^2 e^{2h_c}}{2} \right] (1 + v(X_{t+1}))$$

## Setting

$$A(h_c, h_d, z, h'_c, h'_d, z') =$$

$$\beta \exp \left[ a + (1 - \gamma)z + \bar{\sigma}^2 \frac{\exp(2h_d) + \gamma^2 \exp(2h_c)}{2} \right]$$

$$P(h_c, h'_c)Q(h_d, h'_d)R(z, z')$$

we can rewrite the equation as

$$v(h_c, h_d, z) = \sum_{h'_c, h'_d, z'} (1 + v(h'_c, h'_d, z')) A(h_c, h_d, z, h'_c, h'_d, z')$$

Using indices  $(i, j, k)$  for  $(h_c, h_d, z)$ , we can express as

$$v[i, j, k] = \sum_{i', j', k'} A[i, j, k, i', j', k'] (1 + v[i', j', k'])$$

where

$$A[i, j, k, i', j', k'] =$$

$$\beta \exp \left[ a + (1 - \gamma)z[k] + \bar{\sigma}^2 \frac{\exp(2h_d[j]) + \gamma^2 \exp(2h_c[i])}{2} \right]$$

$$P[i, i']Q[j, j']R[k, k']$$

One option to solve

$$v[i, j, k] = \sum_{i', j', k'} A[i, j, k, i', j', k'] (1 + v[i', j', k'])$$

Reshape

- $v$  into an  $N$ -vector, where  $N = I \times J \times K$
- $A$  into an  $N \times N$  matrix

Then we write as

$$v = A(\mathbb{1} + v)$$

Provided that  $r(A) < 1$ , the solution is given by

$$v = (I - A)^{-1} A \mathbb{1}$$

See [https://jax.quantecon.org/markov\\_asset.html](https://jax.quantecon.org/markov_asset.html)

- NumPy code
- JAX code
- JAX code with linear operators

The code assumes that  $(Z_t)$  is a discretization of

$$Z_{t+1} = \rho_z Z_t + \sigma_z \xi_{t+1}$$