Dynamic Programming

Mini-Lecture: Asset Pricing

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Pricing a single payoff

An asset pays

1. a random amount G_{t+1} at time $t+1$
2. nothing thereafter

Risk-neutral pricing implies

$$P_t = \beta\, \mathbb{E}_t G_{t+1}$$

- β is a constant discount factor
- $\mathbb{E}_t =$ expectation at time t

Roughly speaking,

$$\text{cost} = \text{expected benefit}$$

This model fails empirically: same rate of return regardless of risk

Most investors demand higher rates of return for holding risky assets

Hence we modify to

$$P_t = \mathbb{E}_t M_{t+1} G_{t+1}$$

In this expression, M_{t+1}

- replaces β
- is called the **stochastic discount factor (SDF)** / **pricing kernel**

This model fits the data somewhat better when M_{t+1} is suitably chosen

Pricing a cash flow

Now let's try to price an asset like a share

- delivers a cash flow $D_t, D_{t+1}, \ldots$

We will call these payoffs **dividends**

Ex-dividend contract: buy share, hold for one period, sell

$$\implies \quad \text{payoff } = D_{t+1} + P_{t+1}$$

Inserting into the previous equation gives

$$P_t = \mathbb{E}_t \, M_{t+1}[D_{t+1} + P_{t+1}]$$

Growing prices

We could try to solve the last equation

$$P_t = \mathbb{E}_t M_{t+1}[D_{t+1} + P_{t+1}]$$

But dividends grow over time, so prices grow, which complicates analysis

Easier to solve for the **price-dividend ratio** $V_t := P_t / D_t$

Inserting this into the last equation and rearranging gives

$$V_t = \mathbb{E}_t \left[M_{t+1} \frac{D_{t+1}}{D_t} (1 + V_{t+1}) \right]$$

We can also write this as

$$V_t = \mathbb{E}_t \left[M_{t+1} \exp(G^d_{t+1})(1 + V_{t+1}) \right]$$

where

$$G^d_{t+1} = \ln \frac{D_{t+1}}{D_t} = \text{ growth rate of dividends}$$

To solve for V_t we need to specify

1. the stochastic discount factor M_{t+1} and
2. the growth rate of dividends G^d_{t+1}

Choosing the SDF

For simplicity we adopt the Lucas SDF

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}$$

where

- $u =$ utility
- C_t is time t consumption (representative consumer)

We assume CRRA utility:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad \Longrightarrow \quad u'(c) = c^{-\gamma}$$

With

$$G^c_{t+1} := \ln \frac{C_{t+1}}{C_t}$$

the Lucas SDF becomes

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

$$= \beta \exp(G^c_{t+1})^{-\gamma} = \beta \exp(-\gamma G^c_{t+1})$$

Solving for the price-dividend ratio

Recall that our aim is to solve

$$V_t = \mathbb{E}_t \left[M_{t+1} \exp(G^d_{t+1})(1 + V_{t+1}) \right]$$

Substituting in the last expression gives

$$V_t = \beta \mathbb{E}_t \left[\exp(G^d_{t+1} - \gamma G^c_{t+1})(1 + V_{t+1}) \right]$$

We assume that (G^d_t, G^c_t) are driven by **state process** (X_t)

- P-Markov chain on finite state $\mathsf{X} = \{x_1, \ldots, x_N\}$
- aggregate shock

More specifically, we assume

$$G^c_{t+1} = \mu_c + X_t + \sigma_c \varepsilon^c_{t+1}$$

$$G^d_{t+1} = \mu_d + X_t + \sigma_d \varepsilon^d_{t+1}$$

Here

- $(\varepsilon^c_t) \overset{\text{IID}}{\sim} N(0,1)$
- $(\varepsilon^d_t) \overset{\text{IID}}{\sim} N(0,1)$
- independent of each other

Recall that

$$V_t = \beta \mathbb{E}_t \left[\exp(G^d_{t+1} - \gamma G^c_{t+1})(1 + V_{t+1}) \right]$$

Guess: $V_t = v(X_t)$ for some unknown $v \in \mathbb{R}^{\mathsf{X}}$

With $a := \mu_d - \gamma \mu_c$ this means

$$v(X_t) =$$

$$\beta \mathbb{E}_t \left\{ \exp[a + (1 - \gamma) X_t + \sigma_d \varepsilon^d_{t+1} - \gamma \sigma_c \varepsilon^c_{t+1}](1 + v(X_{t+1})) \right\}$$

Shocks ε^c_{t+1} and ε^d_{t+1} are IID so we integrate them out

$Y = \exp(c\varepsilon)$ for $c \in \mathbb{R}, \ \varepsilon \sim N(0,1)$ implies $\mathbb{E} Y = \exp(c^2/2)$

This yields

$$v(X_t) = \beta \mathbb{E}_t \left\{ \exp\left[a + (1-\gamma) X_t + \frac{\sigma_d^2 + \gamma^2 \sigma_c^2}{2} \right] (1 + v(X_{t+1})) \right\}$$

Conditioning on $X_t = x$, we can write this as

$$v(x) = \beta \sum_{y \in \mathsf{X}} \exp\left[a + (1-\gamma) x + \frac{\sigma_d^2 + \gamma^2 \sigma_c^2}{2} \right] (1 + v(y)) P(x, y)$$

Considering v, x as arrays we get

$$v[i] = \beta \sum_{j=1}^{N} \exp\left[a + (1-\gamma)x[i] + \frac{\sigma_d^2 + \gamma^2\sigma_c^2}{2}\right](1 + v[j])P[i,j]$$

Equivalently,

$$v[i] = \sum_{j=1}^{N} K[i,j](1 + v[j])$$

where

$$K[i,j] := \beta \exp\left[a + (1-\gamma)x[i] + \frac{\sigma_d^2 + \gamma^2\sigma_c^2}{2}\right]P[i,j]$$

Thus we have

$$v = K(\mathbb{1} + v) \tag{1}$$

where

$$K[i,j] = \beta \exp\left[a + (1-\gamma)x[i] + \frac{\sigma_d^2 + \gamma^2\sigma_c^2}{2}\right] P[i,j]$$

- When does (1) have a unique solution?
- How can we compute it?

Suppose $r(K) :=$ spectral radius < 1

Then $I - K$ is invertible and the unique solution to $v = K(\mathbb{1} + v)$ is

$$v = (I - K)^{-1} K \mathbb{1}$$

Once we specify P, state space X and all the parameters, we can

1. obtain K
2. verify $r(K) < 1$
3. compute the solution

To specify P and the state space X we assume

$$X_{t+1} = \rho X_t + \sigma \eta_{t+1}$$

where

- ρ, σ are parameters
- (η_t) is IID and standard normal

Now discretize

- Eg., use QuantEcon.py's `tauchen` function

An Extended Example

One problem: volatility is not constant over time

To accommodate this, we now suppose that

$$G_{t+1}^{i} = \mu_i + Z_t + \bar{\sigma} \exp(H_t^i) \varepsilon_{t+1}^i \qquad i \in \{c, d\}$$

- (Z_t) is R-Markov on Z
- (H_t^c) and (H_t^d) are volatility processes.

In particular,

$$H_{t+1}^{i} = \rho_i H_t^i + \sigma_i \eta_{t+1}^i \qquad i \in \{c, d\}$$

Here (η_t^c) and (η_t^d) are IID standard normal

We set $X := (H^c, H^d, Z)$ and consider again

$$V_t = \beta \mathbb{E}_t \left[\exp(G^d_{t+1} - \gamma G^c_{t+1})(1 + V_{t+1}) \right]$$

Guessing $V_t = v(X_t)$ for some $v \in \mathbb{R}^{\mathsf{X}}$ yields

$$v(x) = \beta \mathbb{E}_x \exp[a + (1-\gamma)z + \bar{\sigma} \mathrm{e}^{h_d} \varepsilon^d_{t+1} - \gamma \bar{\sigma} \mathrm{e}^{h_c} \varepsilon^c_{t+1}](1 + v(X_{t+1}))$$

As before, we integrate out the independent shocks and use the rules for expectations of lognormals to obtain

$$v(x) = \beta \mathbb{E}_x \exp \left[a + (1-\gamma)z + \bar{\sigma}^2 \frac{\mathrm{e}^{2h_d} + \gamma^2 \mathrm{e}^{2h_c}}{2} \right] (1 + v(X_{t+1}))$$

Setting

$$A(h_c, h_d, z, h'_c, h'_d, z') =$$

$$\beta \exp \left[a + (1-\gamma)z + \bar{\sigma}^2 \frac{\exp(2h_d) + \gamma^2 \exp(2h_c)}{2} \right]$$

$$P(h_c, h'_c) Q(h_d, h'_d) R(z, z')$$

we can rewrite the equation as

$$v(h_c, h_d, z) = \sum_{h'_c, h'_d, z'} (1 + v(h'_c, h'_d, z')) A(h_c, h_d, z, h'_c, h'_d, z')$$

Using indices (i, j, k) for (h_c, h_d, z), we can express as

$$v[i, j, k] = \sum_{i', j', k'} A[i, j, k, i', j', k'](1 + v[i', j', k'])$$

where

$$A[i, j, k, i', j', k'] =$$

$$\beta \exp\left[a + (1-\gamma)z[k] + \bar{\sigma}^2 \frac{\exp(2h_d[j]) + \gamma^2 \exp(2h_c[i])}{2}\right]$$

$$P[i, i']Q[j, j']R[k, k']$$

One option to solve

$$v[i,j,k] = \sum_{i',j',k'} A[i,j,k,i',j',k'](1 + v[i',j',k'])$$

Reshape

- v into an N-vector, where $N = I \times J \times K$
- A into an $N \times N$ matrix

Then we write as

$$v = A(\mathbb{1} + v)$$

Provided that $r(A) < 1$, the solution is given by

$$v = (I - A)^{-1} A \mathbb{1}$$

See https://jax.quantecon.org/markov_asset.html

- NumPy code
- JAX code
- JAX code with linear operators

The code assumes that (Z_t) is a discretization of

$$Z_{t+1} = \rho_z Z_t + \sigma_z \xi_{t+1}$$