Dynamic Programming

Chapter 5: Markov Decision Processes

Thomas J. Sargent and John Stachurski

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Topics

- Introduction to Markov decision processes
- Lifetime value
- Optimality
- Value function iteration
- Howard policy iteration
- Optimistic policy iteration
- Applications
- Refactoring dynamic programs

Markov Decision Processes (MDPs)

A class of dynamic programs

- broad enough to encompass many economic applications
- includes optimal stopping problems as a special case
- admits a clean, powerful theory

A foundation stone for

- reinforcement learning
- artificial intelligence
- operations research, etc.

States and Actions

We take as given

1. a finite set X called the **state space** and
2. a finite set A called the **action space**

We study a controller who, at each integer $t \geqslant 0$

1. observes the current state $X_t \in \mathsf{X}$
2. responds with an action $A_t \in \mathsf{A}$

Her aim is to maximize

$$\mathbb{E} \sum_{t \geqslant 0} \beta^t r(X_t, A_t) \quad \text{given } X_0 = x_0$$

Restrictions on actions / assumptions

Time travel is forbidden

- A_t cannot depend on X_{t+j} for any $j \geqslant 1$

While A_t can depend on the history

$$H_t := (A_0, X_0, \ldots, A_{t-1}, X_{t-1}, X_t),$$

for MDPs, conditioning on X_t is sufficient

In other words,

- A_t depends only on X_t

Below we write $A_t = \sigma(X_t)$ and call σ a **policy function**

Actions also restricted by a **feasible correspondence** Γ

- $\Gamma(x)$ is a nonempty subset of A for each $x \in \mathsf{X}$
- interpretation: $\Gamma(x) =$ actions available in state x

Given Γ, we set

$$\mathsf{G} = \{(x, a) \in \mathsf{X} \times \mathsf{A} : a \in \Gamma(x)\}$$

- called the set of **feasible state-action pairs**

Reward $r(x, a)$ is received at feasible state-action pair (x, a)

A **stochastic kernel** from G to X is a map $P \colon \mathsf{G} \times \mathsf{X} \to \mathbb{R}_+$ satisfying

$$\sum_{x' \in \mathsf{X}} P(x, a, x') = 1 \quad \text{for all } (x, a) \text{ in } \mathsf{G}$$

Interpretation

- next period state x' is drawn from distribution $P(x, a, \cdot)$

Now let's put it all together:

Given X and A, an **MDP** is a tuple $\mathscr{M} = (\Gamma, \beta, r, P)$ where

1. Γ is a nonempty correspondence from X $\to$ A
2. β is a constant in $(0, 1)$
3. r is a function from G to $\mathbb{R}$
4. P is a stochastic kernel from G to X

In what follows,

- β is called the **discount factor**
- r is called the **reward function**

MDP dynamics:

$t \leftarrow 0$
input X_0
while $t < \infty$ **do**
    observe X_t
    choose action A_t from $\Gamma(X_t)$
    receive reward $r(X_t, A_t)$
    draw X_{t+1} from $P(X_t, A_t, \cdot)$
    $t \leftarrow t + 1$
end

The **Bellman equation** is

$$v(x) = \max_{a \in \Gamma(x)} \left\{ r(x, a) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x, a, x') \right\}$$

Key idea: reduce infinite horizon problem to a two period problem

— "Bellman's principle of optimality"

In the two period problem, the controller trades off

1. current rewards and
2. expected discounted value from future states

Example. Rust (1987 ECMA) examined an engine replacement problem for a bus workshop

At each t, the supervisor decides whether or not to replace a bus engine

Replacement is costly but delaying risks unexpected failure

We consider a version of Rust's problem with binary action A_t

- $A_t = 1 \implies$ state resets to some fixed **renewal state** $\bar{x}$
    - e.g., mileage resets to zero when an engine is replaced
- $A_t = 0 \implies$ state updates according to $Q \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$
    - e.g., mileage increases stochastically when not replaced

Current reward is $r(x, a)$ at state x and action a

- $x \in \mathsf{X} =$ some finite set (e.g., possible milage values)

The discount factor is $\beta \in (0, 1)$

The Bellman equation is

$$v(x) = \max \left\{ r(x, 1) + \beta v(\bar{x}),\ r(x, 0) + \beta \sum_{x' \in \mathsf{X}} v(x') Q(x, x') \right\}$$

- first term is the value from action 1
- second is the value of action 0

Ex. Show that the renewal problem can be framed as an MDP

Proof: We set

- X = as above
- A $= \{0, 1\}$
- $\Gamma(x) =$ A for all $x \in$ X
- $r =$ as above

We define

$$P(x, a, x') := a \mathbb{1}\{x' = \bar{x}\} + (1 - a) Q(x, x')$$

You can check that P is a stochastic kernel from G to X

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You can check that P is a stochastic kernel from G to X

By construction, $\mathscr{M} = (\Gamma, \beta, r, P)$ is an MDP

Moreover,

$$v(x) = \max_{a \in \Gamma(x)} \left\{ r(x,a) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x,a,x') \right\}$$

$$= \max_{a \in \{0,1\}} \left\{ r(x,a) + \beta \left[a v(\bar{x}) + (1-a) \sum_{x' \in \mathsf{X}} v(x') Q(x,x') \right] \right\}$$

$$= \max \left\{ r(x,1) + \beta v(\bar{x}),\ r(x,0) + \beta \sum_{x' \in \mathsf{X}} v(x') Q(x,x') \right\}$$

We have recovered the renewal Bellman equation on slide 12

Example. Cake eating, where

$$W_{t+1} = R(W_t - C_t) \qquad (t = 0, 1, \ldots)$$

- Investing d dollars today returns Rd next period
- $C_t, W_t \geqslant 0$ are current consumption and wealth
- We restrict wealth to finite grid $\mathsf{W} \subset \mathbb{R}_+$

The agent seeks to maximize $\mathbb{E} \sum_{t \geqslant 0} \beta^t u(C_t)$ given $W_0 = w$

Bellman equation:

$$v(w) = \max_{0 \leqslant w' \leqslant Rw} \left\{ u\left(w - \frac{w'}{R} \right) + \beta v(w') \right\}$$

This model can be framed as an MDP with W as the state space

- The action is $A_t = W_{t+1} =$ savings
- Hence the action space is also W
- The feasible correspondence is

$$\Gamma(w) = \{a \in \mathsf{W} : a \leqslant Rw\}$$

- Since $A_t = R(W_t - C_t)$, current reward is

$$r(w, a) = u(w - a/R) \qquad (a \in \Gamma(w))$$

- The stochastic kernel is $P(w, a, w') = \mathbb{1}\{w' = a\}$

Example. All optimal stopping problems can be framed as MDPs

- See the text for details

This is important from a theoretical perspective

- illustrates the generality of MDPs

However, expressing optimal stopping problems as an MDP requires an extra state variable

- status $S_t = \mathbb{1}\{\text{already stopped}\}$

Hence treating optimal stopping problems separately is neater

Policies

As discussed above, actions are governed by policies

- respond to state X_t with action $A_t := \sigma(X_t)$ at **all** $t \geqslant 0$

A **feasible policy** is a

$$\sigma \in \mathsf{A}^{\mathsf{X}} \text{ such that } \sigma(x) \in \Gamma(x) \text{ for all } x \in \mathsf{X}$$

- Let $\Sigma :=$ the set of all feasible policies

If $A_t := \sigma(X_t)$ at all $t \geqslant 0$, then

$$X_{t+1} \sim P(X_t, \sigma(X_t), \cdot) \quad \text{for all } t \geqslant 0$$

Thus, X_t is P_σ-Markov for

$$P_\sigma(x, x') := P(x, \sigma(x), x') \qquad (x, x' \in \mathsf{X})$$

- Fixing a policy "closes the loop" in the state dynamics
- Solving an MDP means choosing a Markov chain!

Rewards

Under the policy σ, rewards at x given by $r(x, \sigma(x))$

Let

- $r_\sigma(x) := r(x, \sigma(x))$
- $\mathbb{E}_x := \mathbb{E}[\,\cdot \mid X_0 = x]$

Now

$$\mathbb{E}_x \, r(X_t, A_t) = \mathbb{E}_x \, r_\sigma(X_t) = \sum_{x'} r_\sigma(x') P_\sigma^t(x, x') = (P_\sigma^t \, r_\sigma)(x)$$

The **lifetime value of σ** starting from x is

$$v_\sigma(x) := \mathbb{E}_x \sum_{t \geqslant 0} \beta^t r_\sigma(X_t)$$

$$= \sum_{t \geqslant 0} \mathbb{E}_x \left[\beta^t r_\sigma(X_t)\right]$$

$$= \sum_{t \geqslant 0} \beta^t (P_\sigma^t \, r_\sigma)(x)$$

Since $\beta < 1$, the spectral radius of βP is < 1, so

$$v_\sigma = \sum_{t \geqslant 0} (\beta P_\sigma)^t \, r_\sigma = (I - \beta P_\sigma)^{-1} \, r_\sigma$$

Policy Operators

How should we compute v_σ given σ?

We saw above that

$$v_\sigma = (I - \beta P_\sigma)^{-1} \, r_\sigma$$

- Computationally helpful when X is small
- Problematic for large problems

Example. If $|\mathsf{X}| = 10^6$, then $I - \beta \, P_\sigma$ is $10^6 \times 10^6$

Matrices of this size are difficult invert—or even store in memory

Another way to compute v_σ: use the **policy operator**

$$(T_\sigma \, v)(x) = r(x, \sigma(x)) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x, \sigma(x), x')$$

- Defined at all $v \in \mathbb{R}^{\mathsf{X}}$
- Analogous to T_σ for the optimal stopping problem

In vector notation, we can write

$$T_\sigma \, v = r_\sigma + \beta P_\sigma v$$

- T_σ is order-preserving on $\mathbb{R}^{\mathsf{X}}$ — why?

Ex. Show that T_σ is a contraction of modulus β on $\mathbb{R}^{\mathsf{X}}$

For any v, w in $\mathbb{R}^{\mathsf{X}}$ we have

$$
\begin{aligned}
|T_\sigma v - T_\sigma w| &= \beta \, |P_\sigma v - P_\sigma w| \\
&= \beta \, |P_\sigma (v - w)| \\
&\leqslant \beta \, P_\sigma \, |v - w| \\
&\leqslant \beta \, P_\sigma \, \|v - w\|_\infty \mathbb{1} \\
&= \beta \, \|v - w\|_\infty \mathbb{1}
\end{aligned}
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Now use $|a| \leqslant |b|$ implies $\|a\|_\infty \leqslant \|b\|_\infty$

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Ex. Show that v_σ is the unique fixed point of T_σ in $\mathbb{R}^\mathsf{X}$

Proof: Since $\beta < 1$, we have

$$
\begin{aligned}
v = T_\sigma v \iff & v = r_\sigma + \beta P_\sigma v \\
\iff & v = (I - \beta P_\sigma)^{-1} r_\sigma \\
\iff & v = v_\sigma
\end{aligned}
$$

Hence

$$v \text{ is a fixed point of } T_\sigma \iff v = v_\sigma$$

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Hence

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Greedy Policies

Fix $v \in \mathbb{R}^{\mathsf{X}}$

A policy σ is called v-**greedy** if

$$\sigma(x) \in \operatorname*{argmax}_{a \in \Gamma(x)} \left\{ r(x,a) + \beta \sum_{x'} v(x') P(x,a,x') \right\}$$

for all $x \in \mathsf{X}$

Ex. Prove: at least one v-greedy policy exists in Σ

Proof: Immediate because $\Gamma(x)$ is finite and nonempty at all x

Greedy Policies

Fix $v \in \mathbb{R}^{\mathsf{X}}$

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The Bellman Operator

Recall: the Bellman equation is

$$v(x) = \max_{a \in \Gamma(x)} \left\{ r(x,a) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x,a,x') \right\}$$

The **Bellman operator** for $\mathscr{M} = (\Gamma, \beta, r, P)$ is the self-map on $\mathbb{R}^{\mathsf{X}}$ defined by

$$(Tv)(x) = \max_{a \in \Gamma(x)} \left\{ r(x,a) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x,a,x') \right\}$$

$$Tv = v \quad \iff \quad v \text{ satisfies the Bellman equation}$$

Ex. Prove: σ is v-greedy if and only if

$$T_\sigma v = Tv$$

Proof: σ is v-greedy if and only if

$$\sigma(x) \in \operatorname*{argmax}_{a \in \Gamma(x)} \left\{ r(x,a) + \beta \sum_{x'} v(x') P(x,a,x') \right\} \qquad \forall \, x \in \mathsf{X}$$

This is equivalent to

$$r(x, \sigma(x)) + \beta \sum_{x'} v(x') P(x, \sigma(x), x') = (Tv)(x) \qquad \forall \, x \in \mathsf{X}$$

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Ex. Prove that, for all $v \in \mathbb{R}^{\mathsf{X}}$,

$$Tv = \bigvee_{\sigma \in \Sigma} T_\sigma \, v$$

Proof:

Fixing $v \in \mathbb{R}^{\mathsf{X}}$, $\sigma \in \Sigma$ and $x \in \mathsf{X}$, we have

$$(T_\sigma \, v)(x) = r(x, \sigma(x)) + \beta \sum_{x'} v(x') P(x, \sigma(x), x') \leqslant (Tv)(x)$$

Conversely, for any v-greedy $\sigma \in \Sigma$, we have $T_\sigma \, v = Tv$

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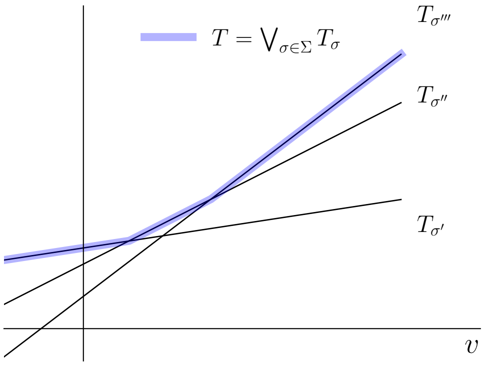

Figure: Visualization in one dimension

Ex. Prove: T is a contraction of modulus β on $\mathbb{R}^{\mathsf{X}}$

Proof: Recall that, for finite S and any $f, g \in \mathbb{R}^S$,

$$|\max_{s \in S} f(s) - \max_{s \in S} g(s)| \leqslant \max_{s \in S} |f(s) - g(s)|$$

Hence, for any v, w in $\mathbb{R}^{\mathsf{X}}$, we have

$$|(Tv)(x) - (Tw)(x)| = \left| \max_{\sigma \in \Sigma} (T_\sigma \, v)(x) - \max_{\sigma \in \Sigma} (T_\sigma \, w)(x) \right|$$

$$\leqslant \max_{\sigma \in \Sigma} |(T_\sigma \, v)(x) - (T_\sigma \, w)(x)|$$

$$\therefore \quad |Tv - Tw| \leqslant \max_{\sigma \in \Sigma} |T_\sigma \, v - T_\sigma \, w| \leqslant \beta \|v - w\|_\infty$$

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Optimality

The **value function** is defined by $v^* := \bigvee_{\sigma \in \Sigma} v_\sigma$

More explicitly,

$$v^*(x) := \max_{\sigma \in \Sigma} v_\sigma(x) \qquad (x \in \mathsf{X})$$

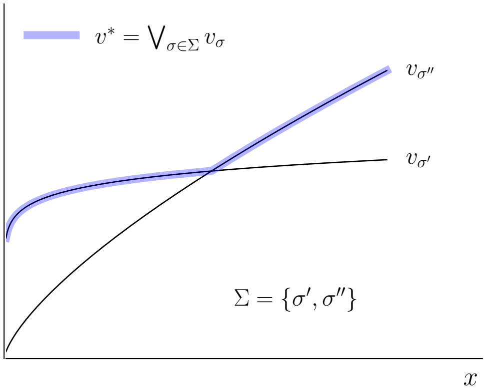
Thus, $v^*(x) = \underline{\text{maximal lifetime value}}$ from state x

A policy $\sigma \in \Sigma$ is called **optimal** if

$$v_\sigma = v^*$$

Thus, σ is optimal $\iff$ lifetime value is maximal at each state

Example. Consider



In this case there is no optimal policy

Indeed, v^* differs from both $v_{\sigma'}$ and $v_{\sigma''}$

Hence no σ satisfies $v^* = v_\sigma$

Below we show that such an outcome is **not** possible for MDPs

In other words, an optimal policy always exists

This leads to our next slide. . .

Theorem. If $\mathscr{M}$ is an MDP with Bellman operator T and value function v^*, then

1. v^* is the unique fixed point of T in $\mathbb{R}^{\mathsf{X}}$
2. A feasible policy is optimal if and only it is v^*-greedy
3. At least one optimal policy exists

Remark: Point (2) is called **Bellman's principle of optimality**

We prove the theorem below through some exercises

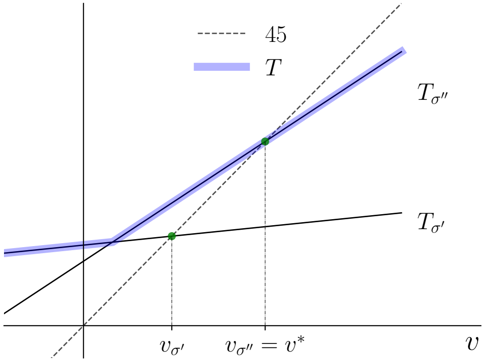


Figure: Illustration of optimality for MDPs

Ex. Show that Bellman's principle of optimality implies existence of an optimal policy

Proof:

Let Bellman's principle of optimality hold

For each $v \in \mathbb{R}^{\mathsf{X}}$, a v-greedy policy exists

Hence a v^*-greedy policy exists

Hence an optimal policy exists

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Proof:

Let Bellman's principle of optimality hold

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Hence a v^*-greedy policy exists

Hence an optimal policy exists

We know that T is a contraction mapping on $\mathbb{R}^\mathsf{X}$

Hence T is globally stable on $\mathbb{R}^\mathsf{X}$ with unique fixed point $\bar{v} \in \mathbb{R}^\mathsf{X}$

To prove (1) of the above theorem, we have to show $\bar{v} = v^*$

Ex. Show $\bar{v} \leqslant v^*$

Proof: Let $\sigma \in \Sigma$ be $\bar{v}$-greedy

Then $T_\sigma \bar{v} = T\bar{v} = \bar{v}$

Hence $\bar{v}$ is a fixed point of T_σ

But the only fixed point of T_σ in $\mathbb{R}^{\mathsf{X}}$ is v_σ

Hence $\bar{v} = v_\sigma$

But then $\bar{v} \leqslant v^*$, since $v^* = \sup_\sigma v_\sigma$

Ex. Show $v^* \leqslant \bar v$

Proof: Fix $\sigma \in \Sigma$ and note that $T_\sigma \, \bar v \leqslant T \bar v = \bar v$

Since T_σ is order-preserving and $T_\sigma \, \bar v \leqslant \bar v$, we have

$$T_\sigma^2 \, \bar v \leqslant T_\sigma \, \bar v \leqslant \bar v$$

Continuing in this way gives $T_\sigma^k \, \bar v \leqslant \bar v$ for all k

Taking the limit yields $v_\sigma \leqslant \bar v$

Hence $\bar v$ is an upper bound of $\{v_\sigma\}_{\sigma \in \Sigma}$

Therefore $v^* \leqslant \bar v$

Ex. Show $v^* \leqslant \bar{v}$

Proof: Fix $\sigma \in \Sigma$ and note that $T_\sigma \, \bar{v} \leqslant T\bar{v} = \bar{v}$

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Taking the limit yields $v_\sigma \leqslant \bar{v}$

Hence $\bar{v}$ is an upper bound of $\{v_\sigma\}_{\sigma \in \Sigma}$

Therefore $v^* \leqslant \bar{v}$

We have now shown that $v^* = \bar{v}$

Thus, v^* is a fixed point of T in $\mathbb{R}^{\mathsf{X}}$

But T is globally stable on $\mathbb{R}^{\mathsf{X}}$

Hence v^* is the only fixed point of T in $\mathbb{R}^{\mathsf{X}}$

In other words,

$$v^* = \text{the unique solution to the Bellman equation in } \mathbb{R}^{\mathsf{X}}$$

To prove Bellman's principle of optimality, we use $Tv^* = v^*$

We have

$$\sigma \text{ is } v^*\text{-greedy} \quad \iff \quad T_\sigma v^* = Tv^* \quad \iff \quad T_\sigma v^* = v^*$$

The last statement is equivalent to $v_\sigma = v^*$ (why?)

Hence

$$\sigma \text{ is } v^*\text{-greedy} \iff v^* = v_\sigma \iff \sigma \text{ is optimal}$$

In other words, Bellman's principle of optimality holds

Algorithms

Previously we used value function iteration (VFI) to solve optimal stopping problems

Here we

1. present a generalization suitable for arbitrary MDPs
2. introduce two other important methods

The two other methods are called

1. Howard policy iteration (HPI) and
2. Optimistic policy iteration (OPI)

Algorithm 1: VFI for MDPs

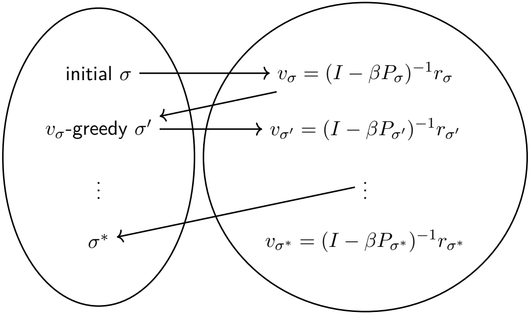
input $v_0 \in \mathbb{R}^{\mathsf{X}}$, an initial guess of v^*
input τ, a tolerance level for error
$\varepsilon \leftarrow \tau + 1$
$k \leftarrow 0$
while $\varepsilon > \tau$ **do**
    for $x \in \mathsf{X}$ **do**
        $v_{k+1}(x) \leftarrow (T v_k)(x)$
    end
    $\varepsilon \leftarrow \|v_k - v_{k+1}\|_\infty$
    $k \leftarrow k + 1$
end
Compute a v_k-greedy policy σ
return σ

VFI is

- globally convergent
- relatively robust
- easy to implement
- very popular

However, we can often find faster methods with a bit of effort

Howard Policy Iteration



Iterates between computing the value of a given policy and computing the greedy policy associated with that value

Algorithm 2: Howard policy iteration for MDPs

input $\sigma_0 \in \Sigma$, an initial guess of σ^*

$k \leftarrow 0$

$\varepsilon \leftarrow 1$

while $\varepsilon > 0$ **do**

- $v_k \leftarrow$ the σ_k-value function $(I - \beta P_{\sigma_k})^{-1} r_{\sigma_k}$
- $\sigma_{k+1} \leftarrow$ a v_k-greedy policy
- $\varepsilon \leftarrow \|\sigma_k - \sigma_{k+1}\|_\infty$
- $k \leftarrow k + 1$

end

return σ_k

Proposition. HPI returns an exact optimal policy in a finite number of steps

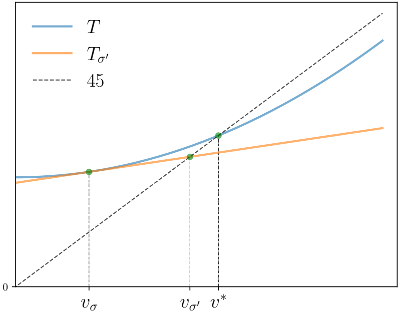
Also, rate of convergence is faster than VFI

In fact HPI is analogous to gradient-based Newton iteration on T

- Details are in the text

In general, for a given fixed point problem,

1. Newton iteration yields a quadratic rate of convergence
2. Successive approximation yields a linear rate of convergence



- σ' is v_σ-greedy if $T_{\sigma'} v_\sigma = T v_\sigma$
- $v_{\sigma'}$ is the fixed point of $T_{\sigma'}$

Optimistic Policy Iteration

OPI is a "convex combination" of VFI and HPI

Similar to HPI except that

- HPI takes current σ and obtains v_σ
- OPI takes current σ and iterates m times with T_σ

Recall that $T_\sigma^m \to v_\sigma$ as $m \to \infty$

Hence OPI replaces v_σ with an approximation

Algorithm 3: Optimistic policy iteration for MDPs

input $v_0 \in \mathbb{R}^{\mathsf{X}}$, an initial guess of v^*
input τ, a tolerance level for error
input $m \in \mathbb{N}$, a step size
$k \leftarrow 0$
$\varepsilon \leftarrow \tau + 1$
while $\varepsilon > \tau$ **do**
    $\sigma_k \leftarrow$ a v_k-greedy policy
    $v_{k+1} \leftarrow T_{\sigma_k}^m v_k$
    $\varepsilon \leftarrow \|v_k - v_{k+1}\|_\infty$
    $k \leftarrow k + 1$
end
return σ_k

Proposition. For all values of m we have $v_k \to v^*$

Ex. Show that OPI = VFI when $m = 1$

Proof. Suppose we are at step k with $v_k \in \mathbb{R}^{\mathsf{X}}$

Now we take σ_k to be v_k-greedy and set $v_{k+1} = T_{\sigma_k} v_k$

Since σ_k is v_k-greedy, we also have $T_{\sigma_k} v_k = T v_k$

Hence v_{k+1} is also the next step for VFI

Proposition. For all values of m we have $v_k \to v^*$

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Since σ_k is v_k-greedy, we also have $T_{\sigma_k} v_k = T v_k$

Hence v_{k+1} is also the next step for VFI

Intuitively, if $m = \infty$, OPI is identical to HPI

This is because $\lim_{m \to \infty} T_{\sigma_k}^m v_k = v_{\sigma_k}$

Without paralleization, an intermediate value of m is usually better than both

More rules of thumb:

- parallelization tends to favor HPI
- VFI works well when β is small and optimization is cheap

Application: Optimal Inventories

Previously we analyzed S-s inventory dynamics

- our aim was to understand Markov chains

But are such dynamics realistic?

We now investigate whether S-s behavior arises naturally in optimizing model

- firm chooses its inventory path to maximize firm value

We assume for now that the firm only sells one product

Given a demand process $(D_t)_{t \geqslant 0}$, inventory $(X_t)_{t \geqslant 0}$ obeys

$$X_{t+1} = f(X_t, A_t, D_{t+1})$$

where

- $f(x, a, d) = (x - d) \vee 0 + a$
- A_t is units of stock ordered this period
- The firm can store at most K items at one time

The state space is $\mathsf{X} := \{0, \ldots, K\}$

We assume $(D_t) \overset{\text{IID}}{\sim} \varphi \in \mathscr{D}(\mathbb{Z}_+)$

Profits are given by

$$\pi_t := X_t \wedge D_{t+1} - cA_t - \kappa \mathbb{1}\{A_t > 0\}$$

- output price $\equiv 1$
- orders in excess of inventory are lost
- c is unit product cost (and unit sales prices $= 1$)
- κ is a fixed cost of ordering inventory

With $\beta := 1/(1+r)$ and $r > 0$, the value of the firm is

$$V_0 = \mathbb{E} \sum_{t \geqslant 0} \beta^t \pi_t$$

Expected current profit is

$$r(x,a) := \sum_{d \geqslant 0} (x \wedge d)\varphi(d) - ca - \kappa \mathbb{1}\{a > 0\}$$

The set of feasible order quantities at x is

$$\Gamma(x) := \{0, \ldots, K - x\}$$

The Bellman equation is

$$v(x) = \max_{a \in \Gamma(x)} \left\{ r(x,a) + \beta \sum_{d \geqslant 0} v(f(x,a,d))\varphi(d) \right\}$$

The model is an MDP $\mathscr{M} = (\Gamma, \beta, r, P)$ with

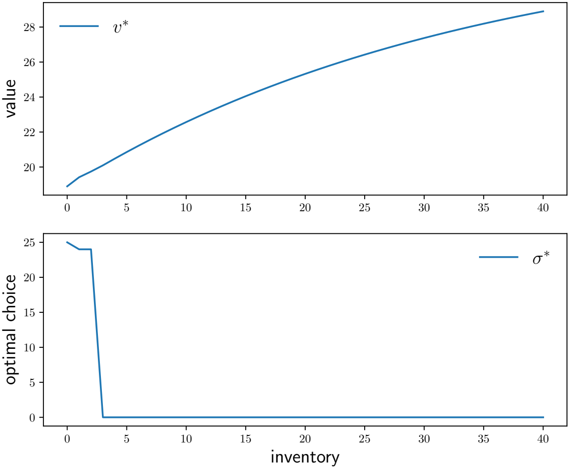
- state space X and action space A := X
- Γ, r and β as above
- stochastic kernel

$$P(x, a, x') := \mathbb{P}\{f(x, a, D) = x'\}$$
$$= \sum_{d \geqslant 0} \mathbb{1}\{f(x, a, d) = x'\} \varphi(d)$$

Since $\mathscr{M}$ is an MDP, all optimality results on slide 35 apply

```julia
using Distributions

f(x, a, d) = max(x - d, 0) + a  # Inventory update

function create_inventory_model(; β=0.98,     # discount factor
                                  K=40,        # maximum inventory
                                  c=0.2, κ=2, # cost paramters
                                  p=0.6)       # demand parameter
    ϕ(d) = (1 - p)^d * p          # demand pdf
    x_vals = collect(0:K)         # set of inventory levels
    return (; β, K, c, κ, p, ϕ, x_vals)
end

"The function B(x, a, v) = r(x, a) + β Σ_x′ v(x′) P(x, a, x′)."
function B(x, a, v, model; d_max=100)
    (; β, K, c, κ, p, ϕ, x_vals) = model
    revenue = sum(min(x, d) * ϕ(d) for d in 0:d_max)
    current_profit = revenue - c * a - κ * (a > 0)
    next_value = sum(v[f(x, a, d) + 1] * ϕ(d) for d in 0:d_max)
    return current_profit + β * next_value
end
```

```julia
"The Bellman operator."
function T(v, model)
    (; β, K, c, κ, p, ϕ, x_vals) = model
    new_v = similar(v)
    for (x_idx, x) in enumerate(x_vals)
        Γx = 0:(K - x)
        new_v[x_idx], _ = findmax(B(x, a, v, model) for a in Γx)
    end
    return new_v
end

"Get a v-greedy policy.  Returns a zero-based array."
function get_greedy(v, model)
    (; β, K, c, κ, p, ϕ, x_vals) = model
    σ_star = zero(x_vals)
    for (x_idx, x) in enumerate(x_vals)
        Γx = 0:(K - x)
        _, a_idx = findmax(B(x, a, v, model) for a in Γx)
        σ_star[x_idx] = Γx[a_idx]
    end
    return σ_star
end
```



value
v^*
28
26
24
22
20
0
5
10
15
20
25
30
35
40
optimal choice
σ^*
25
20
15
10
5
0
inventory

Ex. Try to replicate these plots

- Use the code given above (or at least the same parameters)
- Use value function iteration

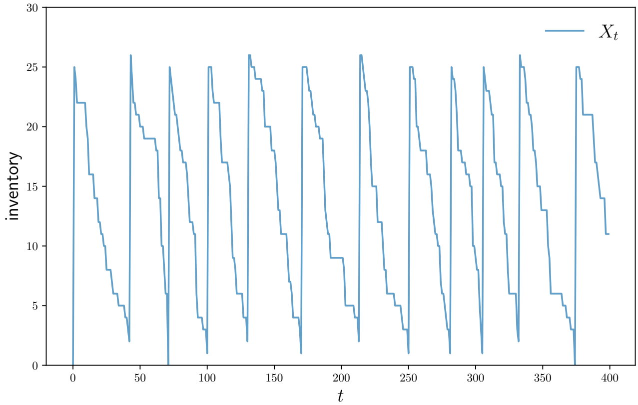


Figure: Optimal inventory dynamics

Optimal Savings with Labor Income

Wealth evolves according to

$$W_{t+1} = R(W_t + Y_t - C_t) \qquad (t = 0, 1, \ldots)$$

- (W_t) takes values in finite set $\mathsf{W} \subset \mathbb{R}_+$
- (Y_t) is Q-Markov chain on finite set Y
- $C_t, W_t \geqslant 0$

The household maximizes

$$\mathbb{E} \sum_{t \geqslant 0} \beta^t u(C_t)$$

The model is an MDP with state space $\mathsf{X} := \mathsf{W} \times \mathsf{Y}$

- action $=$ gross savings $=$ next period wealth
- The feasible correspondence is

$$\Gamma(w, y) = \{s \in \mathsf{W} : s \leqslant R(w + y)\}$$

- current reward is

$$r(w, y, s) = u(w + y - s/R) \qquad \text{(utility of consumption)}$$

- stochastic kernel is

$$P((w, y), s, (w', y')) = \mathbb{1}\{w' = s\} Q(y, y')$$

Bellman operator:

$$(Tv)(w, y) =$$

$$\max_{w' \in \Gamma(w,y)} \left\{ u(w + y - w'/R) + \beta \sum_{y' \in \mathsf{Y}} v(w', y') Q(y, y') \right\}$$

The policy operator for given $\sigma \in \Sigma$ is

$$(T_\sigma \, v)(w, y) =$$

$$u(w + y - \sigma(w, y)/R) + \beta \sum_{y' \in \mathsf{Y}} v(\sigma(w, y), y') Q(y, y')$$

How to solve $v_\sigma = (I - \beta\, P_\sigma)^{-1} r_\sigma$?

Set

$$P_\sigma((w,y),(w',y')) := \mathbb{1}\{\sigma(w,y) = w'\} Q(y,y')$$

and

$$r_\sigma(w,y) := u(w + y - \sigma(w,y)/R)$$

Either

- reshape for matrix algebra (compress (w,y) to single index)
- use linear operator routines

How to solve $v_\sigma = (I - \beta \, P_\sigma)^{-1} r_\sigma$?

Set

$$P_\sigma((w, y), (w', y')) := \mathbb{1}\{\sigma(w, y) = w'\} Q(y, y')$$

and

$$r_\sigma(w, y) := u(w + y - \sigma(w, y)/R)$$

Either

- reshape for matrix algebra (compress (w, y) to single index)
- use linear operator routines

```julia
using QuantEcon, LinearAlgebra, IterTools

function create_savings_model(; R=1.01, β=0.98, γ=2.5,
                                w_min=0.01, w_max=20.0, w_size=200,
                                ρ=0.9, ν=0.1, y_size=5)
    w_grid = LinRange(w_min, w_max, w_size)
    mc = tauchen(y_size, ρ, ν)
    y_grid, Q = exp.(mc.state_values), mc.p
    return (; β, R, γ, w_grid, y_grid, Q)
end

"B(w, y, w′, v) = u(R*w + y - w′) + β Σ_y′ v(w′, y′) Q(y, y′)."
function B(i, j, k, v, model)
    (; β, R, γ, w_grid, y_grid, Q) = model
    w, y, w′ = w_grid[i], y_grid[j], w_grid[k]
    u(c) = c^(1-γ) / (1-γ)
    c = w + y - (w′ / R)
    @views value = c > 0 ? u(c) + β * dot(v[k, :], Q[j, :]) : -Inf
    return value
end
```

```julia
"The Bellman operator."
function T(v, model)
    w_idx, y_idx = (eachindex(g) for g in (model.w_grid, model.y_grid))
    v_new = similar(v)
    for (i, j) in product(w_idx, y_idx)
        v_new[i, j] = maximum(B(i, j, k, v, model) for k in w_idx)
    end
    return v_new
end

"The policy operator."
function T_σ(v, σ, model)
    w_idx, y_idx = (eachindex(g) for g in (model.w_grid, model.y_grid))
    v_new = similar(v)
    for (i, j) in product(w_idx, y_idx)
        v_new[i, j] = B(i, j, σ[i, j], v, model)
    end
    return v_new
end
```

```julia
include("finite_opt_saving_0.jl")

"Compute a v-greedy policy."
function get_greedy(v, model)
    w_idx, y_idx = (eachindex(g) for g in (model.w_grid, model.y_grid))
    σ = Matrix{Int32}(undef, length(w_idx), length(y_idx))
    for (i, j) in product(w_idx, y_idx)
        _, σ[i, j] = findmax(B(i, j, k, v, model) for k in w_idx)
    end
    return σ
end
```

```julia
"Get the value v_σ of policy σ."
function get_value(σ, model)
    # Unpack and set up
    (; β, R, γ, w_grid, y_grid, Q) = model
    w_idx, y_idx = (eachindex(g) for g in (w_grid, y_grid))
    wn, yn = length(w_idx), length(y_idx)
    n = wn * yn
    u(c) = c^(1-γ) / (1-γ)
    # Build P_σ and r_σ as multi-index arrays
    P_σ = zeros(wn, yn, wn, yn)
    r_σ = zeros(wn, yn)
    for (i, j) in product(w_idx, y_idx)
            w, y, w′ = w_grid[i], y_grid[j], w_grid[σ[i, j]]
            r_σ[i, j] = u(w + y - w′/R)
        for (i′, j′) in product(w_idx, y_idx)
            if i′ == σ[i, j]
                P_σ[i, j, i′, j′] = Q[j, j′]
            end
        end
    end
    # Reshape for matrix algebra
    P_σ = reshape(P_σ, n, n)
    r_σ = reshape(r_σ, n)
    # Apply matrix operations --- solve for the value of σ
    v_σ = (I - β * P_σ) \ r_σ
    # Return as multi-index array
    return reshape(v_σ, wn, yn)
end
```

```julia
"Value function iteration routine."
function value_iteration(model, tol=1e-5)
    vz = zeros(length(model.w_grid), length(model.y_grid))
    v_star = successive_approx(v -> T(v, model), vz, tolerance=tol)
    return get_greedy(v_star, model)
end

"Howard policy iteration routine."
function policy_iteration(model)
    wn, yn = length(model.w_grid), length(model.y_grid)
    σ = ones(Int32, wn, yn)
    i, error = 0, 1.0
    while error > 0
        v_σ = get_value(σ, model)
        σ_new = get_greedy(v_σ, model)
        error = maximum(abs.(σ_new - σ))
        σ = σ_new
        i = i + 1
        println("Concluded loop $i with error $error.")
    end
    return σ
end
```

```julia
"Optimistic policy iteration routine."
function optimistic_policy_iteration(model; tolerance=1e-5, m=100)
    v = zeros(length(model.w_grid), length(model.y_grid))
    error = tolerance + 1
    while error > tolerance
        last_v = v
        σ = get_greedy(v, model)
        for i in 1:m
            v = T_σ(v, σ, model)
        end
        error = maximum(abs.(v - last_v))
    end
    return get_greedy(v, model)
end
```

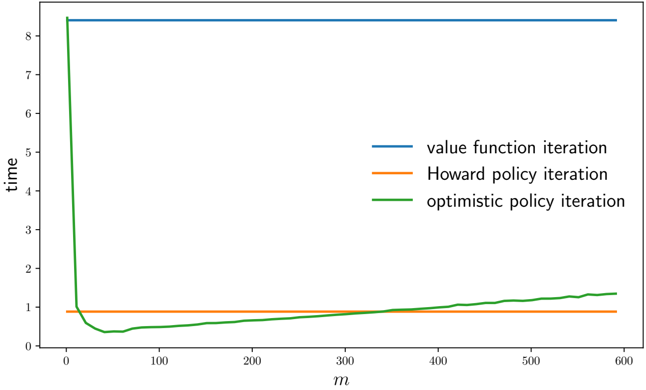


Figure: Timings for alternative algorithms

Investment with Adjustment Costs

A monopolist faces an inverse demand function of the form

$$P_t = a_0 - a_1 Y_t + Z_t$$

- a_0, a_1 are positive parameters
- Y_t is output
- P_t is price and
- the demand shock Z_t follows

$$Z_{t+1} = \rho Z_t + \sigma \eta_{t+1}, \qquad \{\eta_t\} \overset{\text{IID}}{\sim} N(0,1)$$

Current profits are given by

$$\pi_t := P_t Y_t - cY_t - \gamma(Y_{t+1} - Y_t)^2$$

- $\gamma(Y_{t+1} - Y_t)^2$ represents adjustment costs
- rapid changes to capacity are expensive when γ large

Objective: maximize value

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t \pi_t$$

- $\beta = 1/(1+r)$, where $r > 0$ is a fixed interest rate

Building intuition: If $\gamma = 0$ then no intertemporal trade-off

Therefore maximize current profit each period:

$$\max_{Y_t} \{P_t Y_t - c Y_t\} = \max_{Y_t} \{(a_0 - a_1 Y_t + Z_t) Y_t - c Y_t\}$$

Ex. Show that the maximizer is

$$\bar{Y}_t := \frac{a_0 - c + Z_t}{2a_1}$$

On the other hand, if γ is very large then $(Y_t)_{t \geqslant 0}$ should be almost constant

Thus, we expect the following:

- $\gamma \approx 0 \implies Y_t$ tracks $\bar{Y}_t$ closely
- γ large $\implies Y_t$ will be smoother than $\bar{Y}_t$

In short,

$$\text{more adjustment costs} \implies \text{smoother time path for } (Y_t)_{t \geqslant 0}$$

Implementation as an MDP

Let $\mathsf{Y} \subset \mathbb{R}_+$ be a grid containing output values

We discretize (Z_t) using Tauchen's method:

$$(Z_t) \text{ is } Q\text{-Markov on finite set } \mathsf{Z} \subset \mathbb{R}$$

The state space $\mathsf{X} := \mathsf{Y} \times \mathsf{Z}$

The action space is Y

The feasible correspondence is $\Gamma(x) = \mathsf{Y}$ for all x

- choice of output is not restricted by the state

The set Σ is all $\sigma \colon \mathsf{Y} \times \mathsf{Z} \to \mathsf{Y}$

The current reward function is current profits:

$$r(y, z, \hat{y}) = (a_0 - a_1 y + z - c) y - \gamma (\hat{y} - y)^2$$

- $\hat{y}$ = action = choice of next-period output

The stochastic kernel is

$$P((y, z), \hat{y}, (y', z')) = \mathbb{1}\{y' = \hat{y}\} Q(z, z')$$

Now the problem defines an MDP

- all of the optimality theory for MDPs applies

Optimal Investment

The Bellman and policy operators for this problem are

$$(Tv)(y,z) = \max_{y' \in \mathbb{R}} \left\{ r(y,z,y') + \beta \sum_{z' \in \mathsf{Z}} v(y',z') Q(z,z') \right\}$$

$$(T_\sigma\, v)(y,z) = r(y,z,\sigma(y,z)) + \beta \sum_{z' \in \mathsf{Z}} v(\sigma(y,z),z') Q(z,z')$$

On $\mathbb{R}^{\mathsf{X}}$, both are

- order-preserving
- contractions of modulus β

A v-greedy policy is a $\sigma \in \Sigma$ that obeys

$$\sigma(y, z) = \operatorname*{argmax}_{y' \in \mathsf{Y}} \left\{ r(y, z, y') + \beta \sum_{z' \in \mathsf{Z}} v(y', z') Q(z, z') \right\}$$

By our results for MDPs

- v^*-greedy policies = optimal policies
- OPI / HPI / VFI converge

Implications for output can be studied by

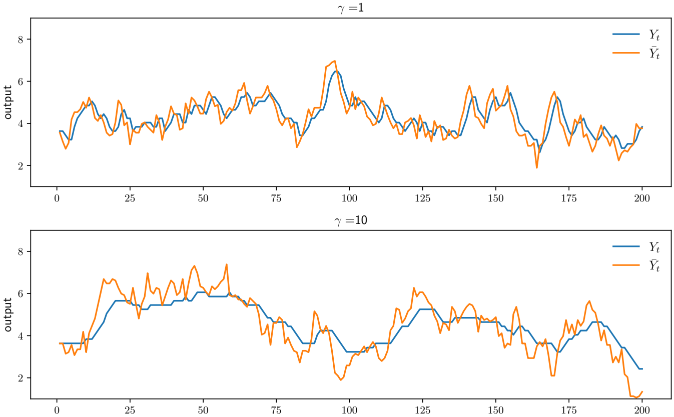
1. generating a Q-Markov chain $(Z_t)_{t=1}^T$
2. simulating optimal output via $Y_{t+1} = \sigma^*(Y_t, Z_t)$

```julia
using QuantEcon, LinearAlgebra, IterTools
include("s_approx.jl")

function create_investment_model(;
        r=0.04,                                 # Interest rate
        a_0=10.0, a_1=1.0,                      # Demand parameters
        γ=25.0, c=1.0,                          # Adjustment and unit cost
        y_min=0.0, y_max=20.0, y_size=100,      # Grid for output
        ρ=0.9, ν=1.0,                           # AR(1) parameters
        z_size=25)                              # Grid size for shock
    β = 1/(1+r)
    y_grid = LinRange(y_min, y_max, y_size)
    mc = tauchen(y_size, ρ, ν)
    z_grid, Q = mc.state_values, mc.p
    return (; β, a_0, a_1, γ, c, y_grid, z_grid, Q)
end
```

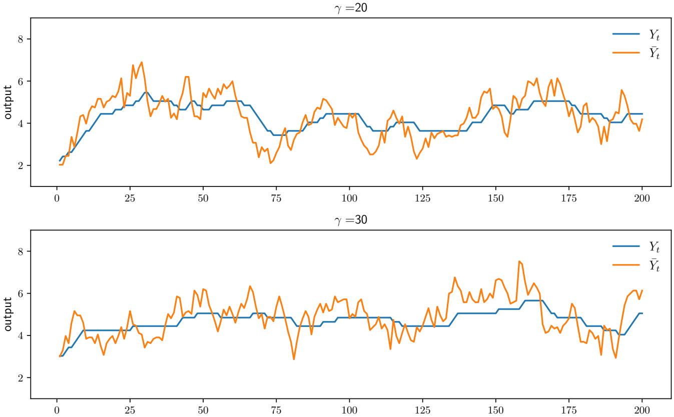
```julia
"""
The aggregator B is given by

    B(y, z, y′) = r(y, z, y′) + β Σ_z′ v(y′, z′) Q(z, z′)."

where

    r(y, z, y′) := (a_0 - a_1 * y + z - c) y - γ * (y′ - y)^2

"""
function B(i, j, k, v, model)
    (; β, a_0, a_1, γ, c, y_grid, z_grid, Q) = model
    y, z, y′ = y_grid[i], z_grid[j], y_grid[k]
    r = (a_0 - a_1 * y + z - c) * y - γ * (y′ - y)^2
    return @views r + β * dot(v[k, :], Q[j, :])
end
```

```julia
"The policy operator."
function T_σ(v, σ, model)
    y_idx, z_idx = (eachindex(g) for g in (model.y_grid, model.z_grid))
    v_new = similar(v)
    for (i, j) in product(y_idx, z_idx)
        v_new[i, j] = B(i, j, σ[i, j], v, model)
    end
    return v_new
end

"The Bellman operator."
function T(v, model)
    y_idx, z_idx = (eachindex(g) for g in (model.y_grid, model.z_grid))
    v_new = similar(v)
    for (i, j) in product(y_idx, z_idx)
        v_new[i, j] = maximum(B(i, j, k, v, model) for k in y_idx)
    end
    return v_new
end
```

```julia
"Compute a v-greedy policy."
function get_greedy(v, model)
    y_idx, z_idx = (eachindex(g) for g in (model.y_grid, model.z_grid))
    σ = Matrix{Int32}(undef, length(y_idx), length(z_idx))
    for (i, j) in product(y_idx, z_idx)
        _, σ[i, j] = findmax(B(i, j, k, v, model) for k in y_idx)
    end
    return σ
end

"Value function iteration routine."
function value_iteration(model; tol=1e-5)
    vz = zeros(length(model.y_grid), length(model.z_grid))
    v_star = successive_approx(v -> T(v, model), vz, tolerance=tol)
    return get_greedy(v_star, model)
end
```



γ =1
output
Y_t
$\bar{Y}_t$
γ =10
output
Y_t
$\bar{Y}_t$



γ =20
output
Y_t
$\bar{Y}_t$
γ =30
output
Y_t
$\bar{Y}_t$

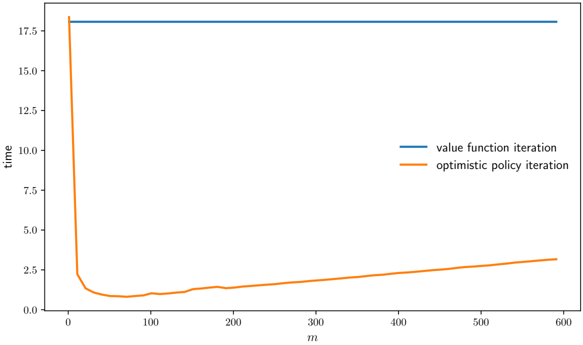


Figure: Timings for alternative algorithms, investment model

Notes on timing

- horizontal axis shows $m =$ step parameter in OPI
- Result for HPI not shown because time is 12x larger than VFI
- OPI dominates both VFI and HPI for almost all values of m

Why is VFI faster than HPI here?

Discount factor is relatively small, which tends to favor VFI

But note that well-parallelized HPI is typically faster than well-parallelized VFI

Fixed Costs in Hiring and Firing

Consider a firm that maximizes expected present value

Future profits are discounted at rate

$$\beta = \frac{1}{1+r} \qquad r > 0$$

The only production input is labor

Hiring and firing involves fixed costs

Letting ℓ_t be employment, current profits are

$$\pi_t = pZ_t\ell_t^\alpha - w\ell_t - \kappa\mathbb{1}\{\ell_{t+1} \neq \ell_t\}$$

- p is the output price
- w is the wage rate
- α is a production parameter
- productivity $(Z_t)_{t \geqslant 0}$ is Q-Markov on Z and
- κ is a fixed cost of hiring and firing

Let $\mathsf{L} \subset \mathbb{R}_+$ be a finite grid for labor stock

The model is an MDP with state space $\mathsf{L} \times \mathsf{Z}$ and action space L

The feasible correspondence is

$$\Gamma(\ell, z) = \mathsf{L}$$

The reward function is

$$r(\ell, z, \ell') := pz\ell^{\alpha} - w\ell_t - \kappa \mathbb{1}\{\ell' \neq \ell\}$$

The stochastic kernel is

$$P((\ell, z), \ell', (\ell', z')) = \mathbb{1}\{\ell = \ell'\} Q(z, z')$$

Bellman operator:

$$(Tv)(\ell, z) = \max_{\ell' \in \Gamma(\ell, z)} \left\{ r(\ell, z, \ell') + \beta \sum_{z' \in \mathsf{Y}} v(\ell', z') Q(z, z') \right\}$$

The policy operator for given $\sigma \in \Sigma$ is

$$(T_\sigma \, v)(\ell, z) = r(\ell, z, \sigma(\ell, z)) + \beta \sum_{z' \in \mathsf{Y}} v(\sigma(\ell, z), z') Q(z, y')$$

A policy σ is v-greedy if

$$\sigma(\ell, z) \in \operatorname*{argmax}_{\ell' \in \Gamma(\ell, z)} \left\{ r(\ell, z, \ell') + \beta \sum_{z' \in \mathsf{Y}} v(\ell', z') Q(z, z') \right\}$$

```julia
using QuantEcon, LinearAlgebra, IterTools

function create_hiring_model(;
        r=0.04,                              # Interest rate
        κ=1.0,                               # Adjustment cost
        α=0.4,                               # Production parameter
        p=1.0, w=1.0,                        # Price and wage
        l_min=0.0, l_max=30.0, l_size=100,   # Grid for labor
        ρ=0.9, ν=0.4, b=1.0,                 # AR(1) parameters
        z_size=100)                          # Grid size for shock
    β = 1/(1+r)
    l_grid = LinRange(l_min, l_max, l_size)
    mc = tauchen(z_size, ρ, ν, b, 6)
    z_grid, Q = mc.state_values, mc.p
    return (; β, κ, α, p, w, l_grid, z_grid, Q)
end
```

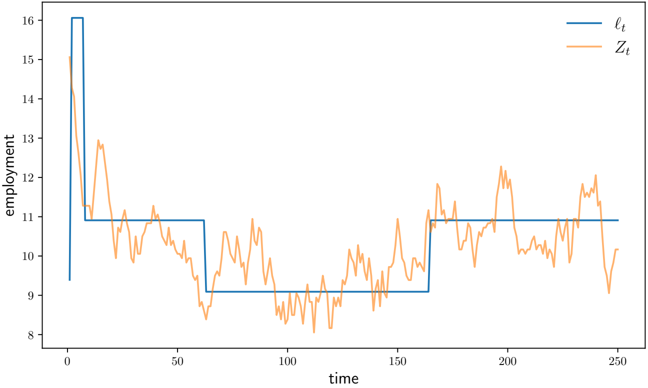


Figure: Fixed costs lead to jumps

Refactoring MDPs

Sometimes direct application of MDP theory is suboptimal

Example. We simplified job search by "refactoring" the Bellman equation

$$v^*(w) = \max \left\{ \frac{w}{1-\beta},\, c + \beta \sum_{w'} v^*(w') \varphi(w') \right\} \qquad (w \in \mathsf{W})$$

into a recursion on the continuation value

$$h^* = c + \beta \sum_{w'} \max \left\{ \frac{w'}{1-\beta},\, h^* \right\} \varphi(w')$$

- reduces dimensionality of iterative solution methods

Let's now examine this refactoring idea more systematically

Steps:

1. provide more examples
2. construct a theoretical foundation
3. connect with Bellman's principle of optimality
4. connect with VFI, OPI and HPI

Example: A Discrete Choice Problem

A structural estimation type the Bellman equation:

$$v(y, \varepsilon) = \max_{a \in \Gamma(y)} \left\{ r(y, \varepsilon, a) + \beta \sum_{y'} \int v(y', \varepsilon') P(y, a, y') \varphi(\varepsilon') \, \mathrm{d}\varepsilon' \right\}$$

for all $y \in \mathsf{Y}$ and $\varepsilon \in \mathsf{E}$

Here

- Y is a finite set — the **endogenous state** space
- ε is the **preference shock** — often vector-valued
- E, the outcome space for ε, is allowed to be continuous

Fix $v \in \mathbb{R}^{\mathsf{X}}$

We define the **expected value function** corresponding to v as

$$g(y,a) := \sum_{y'} \int v(y', \varepsilon') P(y, a, y') \varphi(\varepsilon') \, \mathrm{d}\varepsilon'$$

Key idea: work with expected value functions rather than value functions

Potential advantages:

- $|\mathsf{A}|$ can be much smaller than $|\mathsf{E}|$ (e.g., binary choice)
- integration provides smoothing to g

Using

$$g(y, a) = \sum_{y'} \int v(y', \varepsilon') P(y, a, y') \varphi(\varepsilon') \, \mathrm{d}\varepsilon'$$

We rewrite the Bellman equation as

$$v(y, \varepsilon) = \max_{a \in \Gamma(y)} \{ r(y, \varepsilon, a) + \beta g(y, a) \}$$

Taking expectations of both sides gives

$$g(y, a) = \sum_{y'} \int \max_{a' \in \Gamma(y')} \left\{ r(y', \varepsilon', a') + \beta g(y', a') \right\} P(y, a, y') \varphi(\varepsilon') \, \mathrm{d}\varepsilon'$$

Next step: solve this equation for g

To solve for g we introduce the **expected value Bellman operator** R via

$$(Rg)(y,a) :=$$

$$\sum_{y'} \int \max_{a' \in \Gamma(y')} \left\{ r(y', \varepsilon', a') + \beta g(y', a') \right\} P(y, a, y') \varphi(\varepsilon') \, \mathrm{d}\varepsilon'$$

Ex. Show that R is an order-preserving self-map on $\mathbb{R}^{\mathsf{G}}$

Ex. Prove that R is a contraction of modulus β on $\mathbb{R}^{\mathsf{G}}$

- let g^* be the fixed point of R in $\mathbb{R}^{\mathsf{G}}$
- note g^* can be computed by successive approximation

Knowing this fixed point is enough to solve the dynamic program:

Proposition. A policy $\sigma \in \Sigma$ is optimal if and only if

$$\sigma(y, \varepsilon) \in \operatorname*{argmax}_{a \in \Gamma(y)} \{r(y, \varepsilon, a) + \beta g^*(y, a)\} \quad \text{for all } (y, \varepsilon) \in \mathsf{Y} \times \mathsf{E}$$

Rather than prove this here, we show something more general below

Q-Learning

Q-learning is a branch of reinforcement learning

- a sub-field of control and artificial intelligence
- dynamic optimization when some aspects of the system are unknown to the controller

Example. Solve MDPs where the full specification of state dynamics and rewards is not known

Q-learning relies on the essential equivalence of value functions and the "Q-factor"

We omit discussion of learning and focus on this equivalence

Fix

- an MDP (Γ, β, r, P) with state space X and action space A
- $v \in \mathbb{R}^{\mathsf{X}}$

The **Q-factor** corresponding to v is the function

$$q(x, a) = r(x, a) + \beta \sum_{x'} v(x') P(x, a, x') \qquad ((x, a) \in \mathsf{G})$$

- some economists call q the "post-action value function"

Given such a q, the Bellman equation can be written as

$$v(x) = \max_{a \in \Gamma(x)} q(x, a)$$

Taking expectations and discounting on both sides of

$$v(x) = \max_{a \in \Gamma(x)} q(x, a)$$

gives

$$\beta \sum_{x'} v(x') P(x, a, x') = \beta \sum_{x'} \max_{a' \in \Gamma(x')} q(x', a') P(x, a, x')$$

Adding $r(x, a)$ and using the definition of q again gives

$$q(x, a) = r(x, a) + \beta \sum_{x'} \max_{a' \in \Gamma(x')} q(x', a') P(x, a, x')$$

This is the Bellman equation expressed in terms of Q-factors

To solve for q we introduce the **Q-factor Bellman operator**

$$(Sq)(x,a) = r(x,a) + \beta \sum_{x'} \max_{a' \in \Gamma(x')} q(x',a') P(x,a,x')$$

Ex. Prove: S is an order-preserving contraction map on $\mathbb{R}^{\mathsf{G}}$

Let q^* be the unique fixed point of S in $\mathbb{R}^{\mathsf{G}}$

Proposition. A policy $\sigma \in \Sigma$ is optimal if and only if

$$\sigma(x) \in \operatorname*{argmax}_{a \in \Gamma(x)} q^*(x,a) \qquad \text{for all } (x,a) \in \mathsf{G}$$

We prove a more general result below

Operator Factorizations

Fix an MDP (Γ, β, r, P) with state space X and action space A

We seek general framework for "refactoring" this MDP

Questions: If we refactor the dynamic program,

- is Bellman's principle of optimality still valid?
- do the resulting "Bellman" operators always converge?
- can we use versions of OPI / HPI?

Our first step is to decompose T into separate parts

First we define

- $E \colon \mathbb{R}^{\mathsf{X}} \to \mathbb{R}^{\mathsf{G}}$ by $(Ev)(x,a) = \sum_{x'} v(x')P(x,a,x')$
- $D \colon \mathbb{R}^{\mathsf{G}} \to \mathbb{R}^{\mathsf{G}}$ by $(Dg)(x,a) = r(x,a) + \beta g(x,a)$
- $M \colon \mathbb{R}^{\mathsf{G}} \to \mathbb{R}^{\mathsf{X}}$ by $(Mq)(x) = \max_{a \in \Gamma(x)} q(x,a)$

Since

$$(Tv)(x) = \max_{a \in \Gamma(x)} \left\{ r(x,a) + \beta \sum_{x' \in \mathsf{X}} v(x')P(x,a,x') \right\}$$

we have

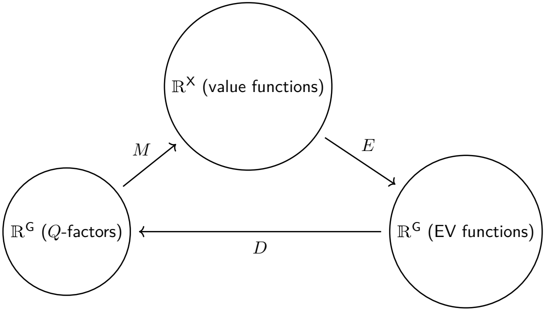
$$Tv = MDEv \qquad (v \in \mathbb{R}^{\mathsf{X}})$$

$$
\begin{tikzcd}
 & \mathbb{R}^{\mathsf{X}} \text{ (value functions)} \arrow[dr, "E"] & \\
\mathbb{R}^{\mathsf{G}} \text{ (Q-factors)} \arrow[ur, "M"] & & \mathbb{R}^{\mathsf{G}} \text{ (EV functions)} \arrow[ll, "D"]
\end{tikzcd}
$$

Figure: $T = MDE$

T is a round trip from the set of value functions

But notice that we have two other round trips



1. DME, from the expected value (EV) functions
2. MED, from the Q-factors

Thus we define

$$R := EMD, \quad S := DEM, \quad T := MDE$$

Ex. Show that

$$(Rg)(x,a) = \sum_{x'} \max_{a' \in \Gamma(x')} \left\{ r(x',a') + \beta g(x',a') \right\} P(x,a,x')$$

and

$$(Sq)(x,a) = r(x,a) + \beta \sum_{x'} \max_{a' \in \Gamma(x')} q(x',a') P(x,a,x')$$

We met these before!

- R is the **expected value Bellman operator**
- S is the **Q-factor Bellman operator**

Lemma. E and M are nonexpansive and D is a contraction. In particular, with $\| \cdot \| := \| \cdot \|_\infty$,

1. $\|Ev - Ev'\| \leqslant \|v - v'\|$ for all $v, v' \in \mathbb{R}^{\mathsf{X}}$
2. $\|Mq - Mq'\| \leqslant \|q - q'\|$ for all $g, g' \in \mathbb{R}^{\mathsf{G}}$
3. $\|Dg - Dg'\| \leqslant \beta\|g - g'\|$ for all $q, q' \in \mathbb{R}^{\mathsf{G}}$

Proof: For E we have

$$|(Ev)(x,a)| = \left| \sum_{x'} v(x') P(x,a,x') \right| \leqslant \sum_{x'} \left| v(x') \right| P(x,a,x') \leqslant \|v\|$$

Taking the sup gives $\|Ev\| \leqslant \|v\|$ and linearity now gives

$$\|Ev - Ev'\| = \|E(v - v')\| \leqslant \|v - v'\|$$

For M we can use the bound

$$|\max_{s \in S} f(s) - \max_{s \in S} g(s)| \leqslant \max_{s \in S} |f(s) - g(s)|$$

to obtain

$$\begin{aligned} |(Mq)(x) - (Mq')(x)| &= |\max_{a \in \Gamma(x)} q(x,a) - \max_{a \in \Gamma(x)} q'(x,a)| \\ &\leqslant \max_{a \in \Gamma(x)} |q(x,a) - q'(x,a)| \leqslant \|q - q'\| \end{aligned}$$

Now take the supremum over x to get

$$\|Mq - Mq'\| \leqslant \|q - q'\|$$

Regarding D, for fixed $g, g' \in \mathbb{R}^{\mathsf{G}}$, we have

$$\begin{aligned}
|(Dg)(x,a) - (Dg')(x,a)| &= |r(x,a) + \beta g(x,a) - r(x,a) - \beta g'(x,a)| \\
&= \beta |g(x,a) - g'(x,a)| \\
&\leqslant \beta \|g - g'\|
\end{aligned}$$

$$\therefore \quad \|Dg - Dg'\| \leqslant \beta \|g - g'\|$$

Proposition. R, S and T are all contractions of modulus β

Proof: Let's just prove this for $R = EMD$

Fixing $g, g' \in \mathbb{R}^{\mathsf{G}}$, we have

$$
\begin{aligned}
\|Rg - Rg'\| &= \|EMDg - EMDg'\| \\
&\leqslant \|MDg - MDg'\| \\
&\leqslant \|Dg - Dg'\| \\
&\leqslant \beta \|g - g'\|
\end{aligned}
$$

The proofs for S and T are very similar

It follows that R, S and T all have unique fixed points

We denote them by g^*, q^* and v^* respectively:

$$Rg^* = g^*, \quad Sq^* = q^*, \quad \text{and} \quad Tv^* = v^*$$

As suggested by notation, $v^* = \sup_\sigma v_\sigma$

- We proved above that v^* is the unique fixed point of T in $\mathbb{R}^\mathsf{X}$

How are these fixed points related to each other?

Proposition. The fixed points of R, S and T are connected via

1. $g^* = Ev^*$
2. $q^* = Dg^*$
3. $v^* = Mq^*$

Proof of 1:

We have $Ev^* = ETv^* = EMDEv^* = REv^*$

Hence Ev^* is a fixed point of R

But R has only one fixed point, which is g^*

Therefore, $g^* = Ev^*$

The proofs of 2–3 are analogous

The results in the last proposition can be written more explicitly as

$$g^*(x,a) = \sum_{x'} v^*(x') P(x,a,x') \quad ((x,a) \in \mathsf{G})$$

$$q^*(x,a) = r(x,a) + \beta g^*(x,a) \quad ((x,a) \in \mathsf{G})$$

and

$$v^*(x) = \max_{a \in \Gamma(x)} q^*(x,a) \quad (x \in \mathsf{X})$$

A policy σ is called v-**greedy** if

$$\sigma(x) \in \operatorname*{argmax}_{a \in \Gamma(x)} \left\{ r(x,a) + \beta \sum_{x'} v(x') P(x,a,x') \right\}$$

for all $x \in \mathsf{X}$

Fix $g, q \in \mathbb{R}^{\mathsf{G}}$

We call $\sigma \in \Sigma$ g-**greedy** if

$$\sigma(x) \in \operatorname*{argmax}_{a \in \Gamma(y)} \{ r(x,a) + \beta g(x,a) \} \qquad (x \in \mathsf{X})$$

and q-**greedy** if

$$\sigma(x) \in \operatorname*{argmax}_{a \in \Gamma(y)} q(x,a) \qquad (x \in \mathsf{X})$$

Proposition. For $\sigma \in \Sigma$, the following statements are equivalent:

1. σ is v^*-greedy
2. σ is g^*-greedy
3. σ is q^*-greedy

In particular,

$$\sigma \text{ is optimal} \iff \text{any one (and hence all) of 1–3 holds}$$

"Refactored" versions of Bellman's principle of optimality

One consequence: we can modify VFI to operate on either expected value functions or Q-factors

Example. Suppose we find it more convenient to iterate in expected value space

Then we can proceed as follows:

1. Fix $g \in \mathbb{R}^{\mathsf{G}}$
2. Iterate with R to obtain $g_k := R^k g \approx g^*$
3. Compute a g_k-greedy policy

Since $g_k \approx g^*$, the resulting policy will be approximately optimal

Refactored OPI

We saw above that VFI is often outperformed by HPI / OPI

Can we apply these methods to refactored MDPs?

Then we could combine

1. the speed gains from HPI/OPI
2. the potential efficiency gains obtained by refactoring

Below we provide an affirmative answer: build a version of OPI that can compute expected value functions

The same is true for OPI for Q-factors, HPI — details omitted

First, given $\sigma \in \Sigma$, we introduce

$$(M_\sigma \, q)(x) := q(x, \sigma(x)) \qquad (x \in \mathsf{X}, \; q \in \mathbb{R}^{\mathsf{G}})$$

Then we define

$$R_\sigma := E \, M_\sigma \, D \qquad S_\sigma := D \, E \, M_\sigma \qquad T_\sigma := M_\sigma \, D \, E$$

In fact T_σ is just the ordinary MDP policy operator:

$$(T_\sigma \, v)(x) = r(x, \sigma(x)) + \beta \sum_{x' \in \mathsf{X}} v(x') P(x, \sigma(x), x')$$

Let's call R_σ and S_σ the **expected-value policy operator** and **Q-factor policy operator** respectively

Here's an expected value version of the OPI algorithm

---

input $g_0 \in \mathbb{R}^{\mathsf{G}}$, an initial guess of g^*
input τ, a tolerance level for error
input $m \in \mathbb{N}$, a step size
$k \leftarrow 0$
$\varepsilon \leftarrow \tau + 1$
while $\varepsilon > \tau$ **do**
    $\sigma_k \leftarrow$ a g_k-greedy policy
    $g_{k+1} \leftarrow R^m_{\sigma_k} g_k$
    $\varepsilon \leftarrow \|g_k - g_{k+1}\|_\infty$
    $k \leftarrow k + 1$
end
return σ_k

Expected value OPI is globally convergent in the same sense as regular OPI

Indeed, suppose we

1. pick $v_0 \in \mathbb{R}^{\mathsf{X}}$,
2. apply regular OPI starting from v_0, and
3. apply expected value OPI applied to $g_0 := E v_0$,

then

- the sequences $(v_k)_{k \geqslant 0}$ and $(g_k)_{k \geqslant 0}$ generated by the two algorithms are connected via $g_k = E v_k$ for all $k \geqslant 0$
- the policy sequences generated by the two algorithms are identical in value

Proof: (assuming for convenience that greedy policies are unique)

Consider the claim that $g_k = Ev_k$ for all $k \geqslant 0$

True by assumption when $k = 0$

Suppose, as an induction hypothesis, that $g_k = Ev_k$ holds at arbitrary k

If σ is g_k-greedy, then

$$\sigma(x) = \operatorname*{argmax}_{a \in \Gamma(y)} \{r(x, a) + \beta(Ev_k)(x, a)\}$$

$$= \operatorname*{argmax}_{a \in \Gamma(y)} \left\{ r(x, a) + \beta \sum_{x'} v_k(x') P(x, a, x') \right\}$$

Hence σ is both g_k-greedy and v_k-greedy

Therefore σ is the next policy selected by both versions of OPI

Moreover, updating via expected value OPI,

$$\begin{aligned} g_{k+1} = R_\sigma^m g_k &= E T_\sigma^{m-1} M_\sigma D g_k \\ &= E T_\sigma^{m-1} M_\sigma D E v_k \\ &= E T_\sigma^m v_k \end{aligned}$$

Since σ is v_k-greedy, we have $v_{k+1} = T_\sigma^m v_k$

Hence $g_{k+1} = E v_{k+1}$

This completes the proof that $g_k = E v_k$ for all k

In fact we just showed that the policy functions generated by the algorithms are identical as well