Dynamic Programming

Chapter 3: Markov Dynamics

Thomas J. Sargent and John Stachurski

2023

Topics

1. Markov chains
2. Stationarity and ergodicity
3. Approximation
4. Expectations
5. Job search revisited

Markov Models

In this chapter we review Markov models

- An essential workhorse
  - economics
  - finance
  - operations research
  - etc., etc.
- General
- Beautiful theory
- Natural fit for dynamic programming (Markov decisions)

Markov Chains

Let X be a finite set (the state space) with typical members x, x'

Consider a process $(X_t) = (X_0, X_1, \ldots)$ that jumps from state x to state x' according to given probabilities $P(x, x')$

Since $P(x, \cdot)$ is a distribution, we require that

$$P(x, x') \geqslant 0 \text{ for all } x' \text{ and } \sum_{x' \in \mathsf{X}} P(x, x') = 1$$

In operator notation,

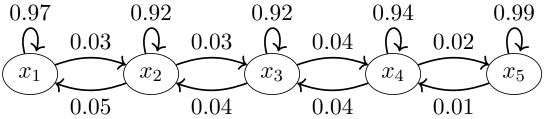
$$P \in \mathcal{L}(\mathbb{R}^{\mathsf{X}}) \text{ with } P \geqslant 0 \text{ and } P\mathbb{1} = \mathbb{1}$$

$$\iff P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$$

Example.

$$P = \begin{pmatrix} 0.97 & 0.03 & 0.00 & 0.00 & 0.00 \\ 0.05 & 0.92 & 0.03 & 0.00 & 0.00 \\ 0.00 & 0.04 & 0.92 & 0.04 & 0.00 \\ 0.00 & 0.00 & 0.04 & 0.94 & 0.02 \\ 0.00 & 0.00 & 0.00 & 0.01 & 0.99 \end{pmatrix}$$

Transition probabilities:



More formally, let

- $(X_t)_{t \geqslant 0}$ be a sequence of X-valued random variables
- P be in $\mathcal{M}(\mathbb{R}^{\mathsf{X}})$

Def. We call $(X_t)_{t \geqslant 0}$ P-**Markov** if

$$\mathbb{P}\{X_{t+1} = x' \mid X_0, \ldots, X_t\} = P(X_t, x') \quad \text{for all} \quad t \geqslant 0, \; x' \in \mathsf{X}$$

Terminology:

- $(X_t)_{t \geqslant 0}$ is a **Markov chain**
- P is the **transition matrix** of $(X_t)_{t \geqslant 0}$
- X_0 and/or its distribution ψ_0 is the **initial condition**

This algorithm yields a P-Markov chain with initial condition ψ_0

$t \leftarrow 0$

$X_t \leftarrow$ a draw from ψ_0

while $t < \infty$ **do**

  $X_{t+1} \leftarrow$ a draw from the distribution $P(X_t, \cdot)$

  $t \leftarrow t + 1$

end

Application: Day Laborer

A worker is either

- unemployed $(X_t = 1)$ or
- employed $(X_t = 2)$ each day

Transitions:

- In state 1 he is hired with probability $\alpha \in (0, 1)$
- In state 2 he is fired with probability $\beta \in (0, 1)$

The corresponding state space and transition matrix are

$$\mathsf{X} = \{1, 2\} \quad \text{and} \quad P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

```julia
function create_laborer_model(; α=0.3, β=0.2)
    return (; α, β)
end

function laborer_update(x, model)  # update X from t to t+1
    (; α, β) = model
    if x == 1
        x′ = rand() < α ? 2 : 1
    else
        x′ = rand() < β ? 1 : 2
    end
    return x′
end
```

Application: S-s Dynamics

Consider a firm whose inventory behavior follows S-s dynamics

Meaning:

- firm waits until its inventory falls below some level $s > 0$
- then replenishes by buying some fixed amount S

Reasonable if ordering inventory involves a fixed cost

(We will see this behavior later in a DP problem with fixed costs)

Inventory $(X_t)_{t \geqslant 0}$ obeys

$$X_{t+1} = \max\{X_t - D_{t+1}, 0\} + S\mathbb{1}\{X_t \leqslant s\}$$

where

- demand $(D_t)_{t \geqslant 1}$ is $\overset{\text{IID}}{\sim} \varphi \in \mathscr{D}(\mathbb{Z}_+)$
- $S =$ amount of stock ordered when inventory $\leqslant s$

We assume φ obeys the geometric distribution:

$$\varphi(d) = \mathbb{P}\{D_t = d\} = p(1-p)^d \text{ for } d \in \mathbb{Z}_+$$

We take $\mathsf{X} := \{0, \ldots, S + s\}$ to be the state space

Ex. Show that X satisfies

$$X_t \in \mathsf{X} \implies \mathbb{P}\{X_{t+1} \in \mathsf{X}\} = 1$$

Proof: Let $X_t = x \in S$, so that

$$X_{t+1} = \max\{x - D_{t+1}, 0\} + S\mathbb{1}\{x \leqslant s\}$$

Evidently $X_{t+1} \in \mathbb{Z}_+$. Also,

$$x \leqslant s \quad \implies \quad X_{t+1} = \max\{x - D_{t+1}, 0\} + S \leqslant s + S$$

and

$$s < x \leqslant S + s \quad \implies \quad X_{t+1} = \max\{x - D_{t+1}, 0\} \leqslant x \leqslant s + S$$

Ex. Show that X satisfies

$$X_t \in \mathsf{X} \implies \mathbb{P}\{X_{t+1} \in \mathsf{X}\} = 1$$

<u>Proof</u>: Let $X_t = x \in S$, so that

$$X_{t+1} = \max\{x - D_{t+1}, 0\} + S\mathbb{1}\{x \leqslant s\}$$

Evidently $X_{t+1} \in \mathbb{Z}_+$. Also,

$$x \leqslant s \quad \implies \quad X_{t+1} = \max\{x - D_{t+1}, 0\} + S \leqslant s + S$$

and

$$s < x \leqslant S + s \quad \implies \quad X_{t+1} = \max\{x - D_{t+1}, 0\} \leqslant x \leqslant s + S$$

If

$$h(x, d) = \max\{x - d, 0\} + S\mathbb{1}\{x \leqslant s\}$$

then

$$X_{t+1} = h(X_t, D_{t+1}) \quad \text{for all } t \geqslant 0$$

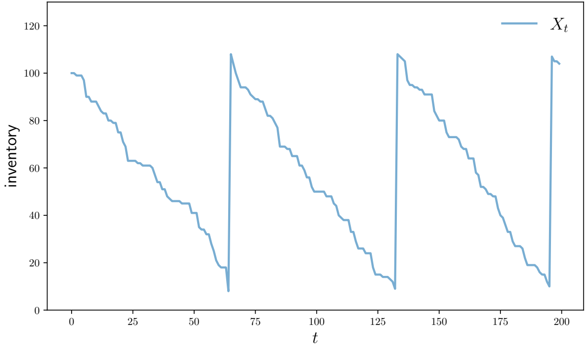
The transition matrix can be expressed as

$$P(x, x') = \mathbb{P}\{h(x, D_{t+1}) = x'\}$$

$$= \sum_{d \geqslant 0} \mathbb{1}\{h(x, d) = x'\} \varphi(d)$$

(In calculations we truncate the sum)

```
using Distributions, QuantEcon, IterTools

function create_inventory_model(; S=100,  # Order size
                                  s=10,   # Order threshold
                                  p=0.4)  # Demand parameter
    ϕ = Geometric(p)
    h(x, d) = max(x - d, 0) + S * (x <= s)
    return (; S, s, ϕ, h)
end

"Simulate the inventory process."
function sim_inventories(model; ts_length=200)
    (; S, s, ϕ, h) = model
    X = Vector{Int32}(undef, ts_length)
    X[1] = S  # Initial condition
    for t in 1:(ts_length-1)
        X[t+1] = h(X[t], rand(ϕ))
    end
    return X
end
```


X_t
inventory
0
20
40
60
80
100
120
0
25
50
75
100
125
150
175
200
t

Multistep transitions

Fix finite X and $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$

- P^k (k-th power) is called the **k-step transition matrix**
- $P^k(x, x') :=$ the (x, x')-th element of P^k

Claim:

$$P^k(x, x') = \mathbb{P}\{X_{t+k} = x' \mid X_t = x\} \quad \text{for any } P\text{-chain } (X_t)_{t \geqslant 0}$$

Proof: Fix $x, x' \in \mathsf{X}$ and $t \geqslant 0$

True by definition when $k = 1$

Multistep transitions

Fix finite X and $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$

- P^k (k-th power) is called the **k-step transition matrix**
- $P^k(x, x') :=$ the (x, x')-th element of P^k

Claim:

$$P^k(x, x') = \mathbb{P}\{X_{t+k} = x' \mid X_t = x\} \quad \text{for any } P\text{-chain } (X_t)_{t \geqslant 0}$$

Proof: Fix $x, x' \in \mathsf{X}$ and $t \geqslant 0$

True by definition when $k = 1$

Now suppose true at k, so that

$$P^k(x, x') = \mathbb{P}\{X_{t+k} = x' \mid X_t = x\}$$

By the law of total probability, we have

$$\mathbb{P}\{X_{t+k+1} = x' \mid X_t = x\}$$

$$= \sum_z \mathbb{P}\{X_{t+k+1} = x' \mid X_{t+k} = z\} \mathbb{P}\{X_{t+k} = z \mid X_t = x\}$$

Therefore

$$\mathbb{P}\{X_{t+k+1} = x' \mid X_t = x\} = \sum_z P^k(x, z) P(z, x') = P^{k+1}(x, x')$$

Irreducible Markov chains

Lemma. The following statements are equivalent:

1. $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ is irreducible
2. For any P-chain (X_t) and any $x, x' \in \mathsf{X}$, $\exists\, k \geqslant 0$ s.t.

$$\mathbb{P}\{X_k = x' \,|\, X_0 = x\} > 0$$

Proof:

$$\sum_{k \geqslant 0} P^k \gg 0 \iff \forall\, x, x' \in \mathsf{X},\ \exists\, k \geqslant 0 \text{ s.t. } P^k(x, x') > 0$$

$$\iff \quad \text{statement 2}$$

Marginals

Fix $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ and P-chain (X_t), let $\psi_t \stackrel{d}{:=} X_t$ for all t

By the law of total probability, for all $x, x' \in \mathsf{X}$,

$$\mathbb{P}\{X_{t+1} = x'\} = \sum_{x \in \mathsf{X}} \mathbb{P}\{X_{t+1} = x' \,|\, X_t = x\} \mathbb{P}\{X_t = x\}$$

Equivalently,

$$\psi_{t+1}(x') = \sum_{x \in \mathsf{X}} P(x, x') \psi_t(x) \quad \text{for all } x \in \mathsf{X}$$

Treating each ψ_t as a <u>row</u> vector, we get $\psi_{t+1} = \psi_t P$

Stationarity

Distributions update via $\psi_{t+1} = \psi_t P$

Recall also that ψ^* is called **stationary** for P if $\psi^* = \psi^* P$

Now we can interpret this expression

If ψ^* is stationary for P, then

$$X_t \stackrel{d}{=} \psi^* \implies X_{t+1} \stackrel{d}{=} \psi^*$$

We recall that

- each $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ has at least one stationary distribution, and
- uniqueness in $\mathscr{D}(\mathsf{X})$ holds whenever P is irreducible

Ergodicity

Theorem. Let P be irreducible with stationary distribution ψ^*

For any P-Markov chain (X_t) and any $x \in \mathsf{X}$, we have

$$\mathbb{P}\left\{\lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}\{X_t = x\} = \psi^*(x)\right\} = 1$$

Meaning: For (almost) every P-Markov chain that we generate,

fraction of time chain in state $x \approx \psi^*(x)$

Markov chains with this property are said to be **ergodic**

Example. Recall the model

$$\mathsf{X} = \{1, 2\}, \quad P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

Ex. Assuming $0 < \alpha, \beta \leqslant 1$, show that

$$\psi^* := \frac{1}{\alpha+\beta} \begin{pmatrix} \beta & \alpha \end{pmatrix} = \text{ the unique stationary distribution}$$

Since P is irreducible, ergodicity holds:

$$\mathbb{P}\left\{ \lim_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}\{X_t = x\} = \psi^*(x) \right\} = 1$$

Ex. Using simulation, confirm that, for large k,

$$\frac{1}{k}\sum_{t=x}^{k} \mathbb{1}\{X_t = x\} \approx \psi^*(x) \qquad x = 1, 2$$

Check: this convergence does not depend on distribution of X_0

```julia
function sim_chain(k, p, model)
    X = Array{Int32}(undef, k)
    X[1] = rand() < p ? 1 : 2
    for t in 1:(k-1)
        X[t+1] = laborer_update(X[t], model)
    end
    return X
end

function test_convergence(; k=10_000_000, p=0.5)
    model = create_laborer_model()
    (; α, β) = model
    ψ_star = (1/(α + β)) * [β α]

    X = sim_chain(k, p, model)
    ψ_e = (1/k) * [sum(X .== 1)  sum(X .== 2)]
    error = maximum(abs.(ψ_star - ψ_e))
    approx_equal = isapprox(ψ_star, ψ_e, rtol=0.01)
    println("Sup norm deviation is $error")
    println("Approximate equality is $approx_equal")
 end
```

And now in Python

```python
import numpy as np
from collections import namedtuple
from numba import njit, int32

Model = namedtuple("Model", ("α", "β"), defaults=(0.3, 0.2))

@njit
def laborer_update(x, model):  # update X from t to t+1
    α, β = model
    if x == 1:
        y = 2 if np.random.rand() < α else 1
    else:
        y = 1 if np.random.rand() < β else 2
    return y
```

```python
@njit
def sim_chain(k, p, model):
    X = np.empty(k, dtype=int32)
    X[0] = 1 if np.random.rand() < p else 2
    for t in range(k-1):
        X[t+1] = laborer_update(X[t], model)
    return X

@njit
def test_convergence(model, k=10_000_000, p=0.5):
    α, β = model
    ψ_star = (1/(α + β)) * np.array((β,  α))

    X = sim_chain(k, p, model)
    ψ_e = (1/k) * np.array((sum(X == 1), sum(X == 2)))
    error = np.max(np.abs(ψ_star - ψ_e))
    return error

model = Model()
error = test_convergence(model)
print(f"Sup norm deviation is {error}")
```

The next slide shows the results of an analogous simulation for the inventory model (slide 11)

The bar plot shows mean occupation time

$$M(x) := \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}\{X_t = x\}$$

for each inventory level x when k is large

The black line shows the stationary distribution ψ^*

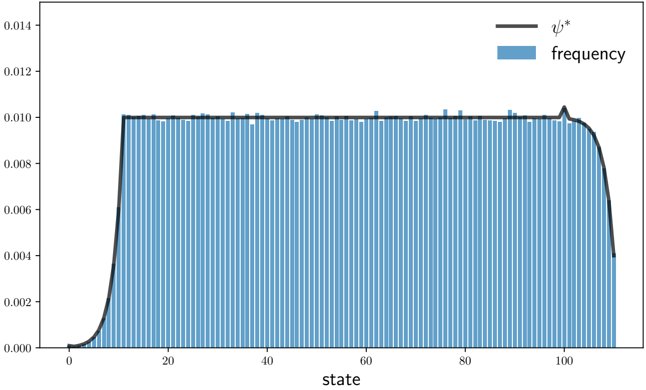


Figure: Ergodicity in the inventory model

Fix $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ with $P \gg 0$ and let ψ^* obey $\psi^* = \psi^* P$

Ex. Prove: $\psi P^t \to \psi^*$ as $t \to \infty$ for any $\psi \in \mathscr{D}(\mathsf{X})$

Proof: Since P is positive and $\rho(P) = 1$, PF theorem implies

$$P^t \to e \, \varepsilon \text{ as } t \to \infty,$$

where

- e is the dominant right eigenvector
- ε is the dominant left eigenvector
- the vectors are normalized s.t. $\langle e, \varepsilon \rangle = 1$

Fix $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ with $P \gg 0$ and let ψ^* obey $\psi^* = \psi^* P$

Ex. Prove: $\psi P^t \to \psi^*$ as $t \to \infty$ for any $\psi \in \mathscr{D}(\mathsf{X})$

Proof: Since P is positive and $\rho(P) = 1$, PF theorem implies

$$P^t \to e\,\varepsilon \text{ as } t \to \infty,$$

where

- e is the dominant right eigenvector
- ε is the dominant left eigenvector
- the vectors are normalized s.t. $\langle e, \varepsilon \rangle = 1$

In the current setting,

- $\mathbb{1}$ is the dominant right eigenvector and
- ψ^* is the dominant left eigenvector

Hence

$$P^t \to \mathbb{1}\,\psi^* \quad \text{as} \quad t \to \infty$$

Thus, for any $\psi \in \mathscr{D}(\mathsf{X})$,

$$\psi P^t \to \psi \mathbb{1}\,\psi^* = \psi^*$$

Recall the model

$$\mathsf{X} = \{1, 2\}, \quad P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \quad \text{and} \quad \psi^* := \frac{1}{\alpha+\beta} \begin{pmatrix} \beta & \alpha \end{pmatrix}$$

Ex. Fix $\alpha = 0.3$ and $\beta = 0.2$

Compute the sequence (ψP^t) for different choices of ψ

Confirm that your results are consistent with the claim that

$$\psi P^t \to \psi^* \text{ as } t \to \infty \text{ for any } \psi \in \mathscr{D}(\mathsf{X})$$

Approximation

It can be helpful to reduce continuous state Markov models to finite state models

For example, suppose that $(X_t)_{t \geqslant 0}$ evolves in $\mathbb{R}$ according to

$$X_{t+1} = \rho X_t + b + \nu \varepsilon_{t+1}, \quad (\varepsilon_t) \overset{\text{IID}}{\sim} N(0,1). \tag{1}$$

This is a **linear Gaussian AR(1)** model

To approximate it we use **Tauchen's method**

We assume throughout that $|\rho| < 1$

Under this assumption, (1) has a unique **stationary distribution** ψ^* given by

$$\psi^* = N(\mu_x, \sigma_x^2) \quad \text{with} \quad \mu_x := \frac{b}{1-\rho} \quad \text{and} \quad \sigma_x^2 := \frac{\nu^2}{1-\rho^2}$$

This means that ψ^* has the following property:

$$X_t \stackrel{d}{=} \psi^* \text{ and } X_{t+1} = \rho X_t + b + \nu \varepsilon_{t+1} \text{ implies } X_{t+1} \stackrel{d}{=} \psi^*$$

Ex. Prove this. Hints: When $X_t \stackrel{d}{=} \psi^*$,

- is X_{t+1} normally distributed?
- what is its mean and variance?

Tauchen's discretization method

We start with the case $b = 0$

Fix $m, n \in \mathbb{N}$

Create state space $\mathsf{X} := \{x_1, \ldots, x_n\}$ via

- set $x_1 = -m\,\sigma_x$,
- set $x_n = m\,\sigma_x$ and
- set $x_{i+1} = x_i + s$ for $i \in \{1, \ldots, n-1\}$ where

$$s = \frac{x_n - x_1}{n - 1}$$

A grid that brackets the stationary mean on both sides by m standard deviations:

Create an $n \times n$ matrix P such that, For $i, j \in [n]$,

1. if $j = 1$, then set $P(x_i, x_j) = F(x_1 - \rho x_i + s/2)$
2. If $j = n$, then set $P(x_i, x_j) = 1 - F(x_n - \rho x_i - s/2)$
3. Otherwise, set

$$P(x_i, x_j) = F(x_j - \rho x_i + s/2) - F(x_j - \rho x_i - s/2)$$

Finally, if $b \neq 0$, then replace x_i with $x_i + \mu_x$ for each i

- shift the grid X to be centered on the stationary mean

```julia
using QuantEcon

ρ, b, ν = 0.9, 0.0, 1.0
μ_x = b/(1-ρ)
σ_x = sqrt(ν^2 / (1-ρ^2))

n = 15
mc = tauchen(n, ρ, ν)
approx_sd = stationary_distributions(mc)[1]

function psi_star(y)
    c = 1/(sqrt(2 * pi) * σ_x)
    return c * exp(-(y - μ_x)^2 / (2 * σ_x^2))
end
```

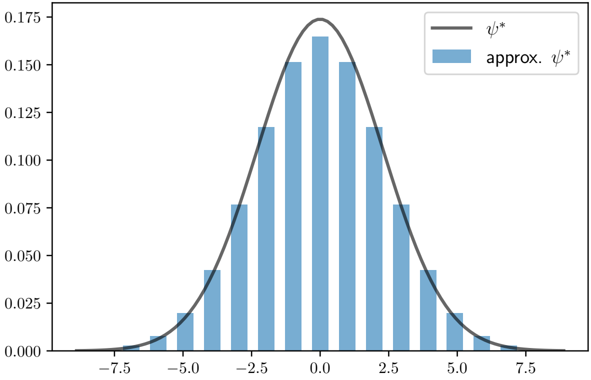


Figure: Comparison of $\psi^* = N(\mu_x, \sigma_x^2)$ and its discrete approximant

Conditional Expectations

Fix $P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$

For each $h \in \mathbb{R}^{\mathsf{X}}$ and $x \in \mathsf{X}$, we define

$$(Ph)(x) := \sum_{x' \in \mathsf{X}} h(x') P(x, x')$$

Equivalently

$$(Ph)(x) = \mathbb{E}[h(X_{t+1}) \,|\, X_t = x] \quad \text{when } (X_t) \text{ is } P\text{-Markov}$$

This interpretation extends to powers:

$$(P^k h)(x) = \sum_{x' \in \mathsf{X}} h(x') P^k(x, x') = \mathbb{E}[h(X_{t+k}) \,|\, X_t = x]$$

Note on conventions

When updating distributions we use row vectors:

$$\psi_{t+1}(x') = (\psi_t P)(x') = \sum_{x \in \mathsf{X}} P(x, x') \psi_t(x)$$

When taking conditional expectations we use column vectors:

$$(Ph)(x) := \sum_{x' \in \mathsf{X}} h(x') P(x, x')$$

The Law of Iterated Expectations

We now prove a version of the **law of iterated expectations**

Let (X_t) be P-Markov with $X_0 \stackrel{d}{=} \psi_0$

Fix $t, k \in \mathbb{N}$ and set $\mathbb{E}_t := \mathbb{E}[\cdot \mid X_t]$

We claim that

$$\mathbb{E}[\mathbb{E}_t[h(X_{t+k})]] = \mathbb{E}[h(X_{t+k})] \quad \text{for any } h \in \mathbb{R}^{\mathsf{X}}$$

(A special case of the general rule $\mathbb{E}[\mathbb{E}[Y \mid X]] = \mathbb{E}[Y]$)

To see this, recall that $\mathbb{E}[h(X_{t+k}) \mid X_t = x] = (P^k h)(x)$

Hence $\mathbb{E}[h(X_{t+k}) \mid X_t] = (P^k h)(X_t)$

Therefore,

$$\mathbb{E}[\mathbb{E}_t[h(X_{t+k})]] = \mathbb{E}[(P^k h)(X_t)]$$

$$= \sum_{x'} (P^k h)(x') \psi_t(x') = \sum_{x'} (P^k h)(x') (\psi_0 P^t)(x')$$

Since $\psi_0 P^t$ is a row vector, we can write the last expression as

$$\psi_0 P^t P^k h = \psi_0 P^{t+k} h = \psi_{t+k} h = \mathbb{E} h(X_{t+k})$$

(First Order) Stochastic Dominance

Some Markov chains have useful monotonicity properties

To use state these results, we need to order distributions

That is, we need a partial order over distributions

The most important of these partial orders is called "first order stochastic dominance"

In this section we define it

To start, let's consider ordering distributions in a special case

Example. $X \sim B(n, 0.5)$ counts heads in n flips of a fair coin

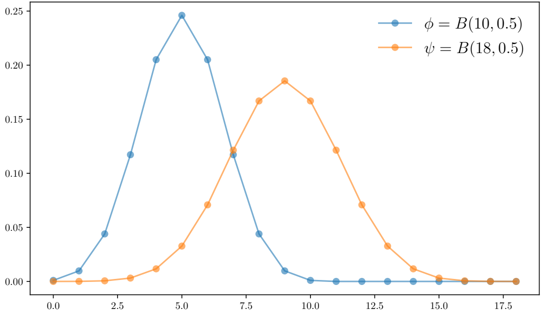
Suppose

- $\varphi \stackrel{d}{=} X \sim B(10, 0.5)$ and
- $\psi \stackrel{d}{=} Y \sim B(18, 0.5)$

- Y counts over more flips, so it should be "larger"

Hence we expect φ is "$\preceq$" ψ in some sense

Distribution ψ seems "larger than" φ — more mass on higher draws



But how can we make this idea precise?

Let X be a finite set partially ordered by $\preceq$

Fix $\varphi, \psi \in \mathscr{D}(\mathsf{X})$

Write $\langle u, \varphi \rangle$ for $\sum_x u(x)\varphi(x)$, etc.

We say that ψ **stochastically dominates** φ and write $\varphi \preceq_{\mathbf{F}} \psi$ if

$$u \in i\mathbb{R}^{\mathsf{X}} \quad \implies \quad \langle u, \varphi \rangle \leqslant \langle u, \psi \rangle$$

Example. If

- $\varphi \stackrel{d}{=} X \sim B(10, 0.5)$ and
- $\psi \stackrel{d}{=} Y \sim B(18, 0.5)$,

then $\varphi \preceq_{\mathrm{F}} \psi$

Proof: Fix $u \in i\mathbb{R}^{\mathsf{X}}$ and let

- $\mathsf{X} = \{0, \ldots, 18\}$ and
- $W_1, \ldots, W_{18}$ be IID Bernoulli with $\mathbb{P}\{W_i = 1\} = 0.5$ for all i

Then $X := \sum_{i=1}^{10} W_i \stackrel{d}{=} \varphi$ and $Y := \sum_{i=1}^{18} W_i \stackrel{d}{=} \psi$

Clearly $X \leqslant Y$ with probability one (i.e., for any draw of $\{W_i\}_{i=1}^{18}$)

Hence $u(X) \leqslant u(Y)$

Hence $\mathbb{E}u(X) \leqslant \mathbb{E}u(Y)$

In other words,

$$\langle u, \varphi \rangle \leqslant \langle u, \psi \rangle$$

Example. An agent has preferences over outcomes in X

Preferences are determined by a utility function $u \in \mathbb{R}^{\mathsf{X}}$

The agent prefers more to less, so $u \in i\mathbb{R}^{\mathsf{X}}$

Suppose that the agent ranks lotteries over X according to expected utility

- evaluates $\varphi \in \mathscr{D}(\mathsf{X})$ according to $\sum_x u(x)\varphi(x)$

Then the agent (weakly) prefers ψ to φ whenever $\varphi \preceq_{\mathrm{F}} \psi$

Another Perspective

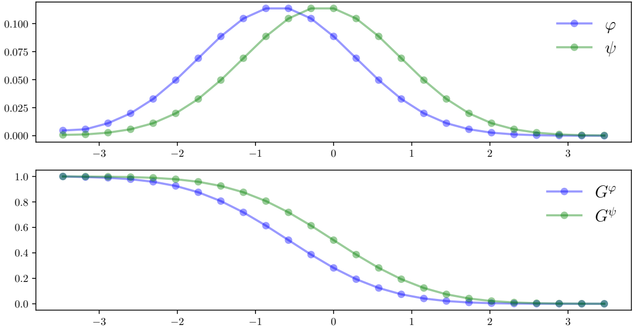
Given $\varphi \in \mathscr{D}(\mathsf{X})$, let

$$G^{\varphi}(y) := \sum_{x \in \mathsf{X}} \mathbb{1}\{y \preceq x\}\varphi(x) \qquad (y \in \mathsf{X})$$

This is the **counter** CDF of φ

Lemma. For each $\varphi, \psi \in \mathscr{D}(\mathsf{X})$, the following statements hold:

1. $\varphi \preceq_{\mathrm{F}} \psi \implies G^{\varphi} \leqslant G^{\psi}$
2. If X is totally ordered by $\preceq$, then $G^{\varphi} \leqslant G^{\psi} \implies \varphi \preceq_{\mathrm{F}} \psi$



0.100
0.075
0.050
0.025
0.000
−3
−2
−1
0
1
2
3
φ
ψ
1.0
0.8
0.6
0.4
0.2
0.0
−3
−2
−1
0
1
2
3
G^{φ}
G^{ψ}

Lemma. $\preceq_{\mathrm{F}}$ is a partial order on $\mathscr{D}(\mathsf{X})$

Proof:

Let's just prove transitivity

Suppose $f, g, h \in \mathscr{D}(\mathsf{X})$ with $f \preceq_{\mathrm{F}} g$ and $g \preceq_{\mathrm{F}} h$

Fixing $u \in i\mathbb{R}^{\mathsf{X}}$, we have

$$\langle u, f \rangle \leqslant \langle u, g \rangle \quad \text{and} \quad \langle u, g \rangle \leqslant \langle u, h \rangle$$

Hence $\langle u, f \rangle \leqslant \langle u, h \rangle$

Since u was arbitrary in $i\mathbb{R}^{\mathsf{X}}$, we are done

Monotone Markov Chains

$P \in \mathcal{M}(\mathbb{R}^{\mathsf{X}})$ is called **monotone increasing** if

$$x, y \in \mathsf{X} \text{ and } x \preceq y \quad \implies \quad P(x, \cdot) \preceq_{\mathrm{F}} P(y, \cdot)$$

Example. Consider the AR(1) model $X_{t+1} = \rho X_t + \sigma \varepsilon_{t+1}$

Apply Tauchen discretization, mapping to

- $n \times n$ stochastic matrix P on
- state space $\mathsf{X} = \{x_1, \ldots, x_n\} \subset \mathbb{R}$

Lemma. If $\rho \geqslant 0$ (+ve autocorrelation), then P is monotone increasing

Ex. Prove that P is monotone increasing if and only if P is invariant on $i\mathbb{R}^{\mathsf{X}}$

Proof of $\implies$

Suppose P is monotone increasing and fix $u \in i\mathbb{R}^{\mathsf{X}}$

We claim that $Pu \in i\mathbb{R}^{\mathsf{X}}$

To see this, pick any $x, y \in \mathsf{X}$ with $x \preceq y$

Since $P(x, \cdot) \preceq_{\mathrm{F}} P(y, \cdot)$, we have

$$(Pu)(x) := \sum_{x'} u(x')P(x, x') \leqslant \sum_{x'} u(x')P(y, x') =: (Pu)(y)$$

Hence $Pu \in i\mathbb{R}^{\mathsf{X}}$, as was to be shown

Ex. Prove: If P is monotone increasing then so is P^t for all $t \in \mathbb{N}$

Proof: P is a self-map on $i\mathbb{R}^{\mathsf{X}}$

Hence P^t is a self-map on $i\mathbb{R}^{\mathsf{X}}$

Ex. Prove: If P is monotone increasing then so is P^t for all $t \in \mathbb{N}$

Proof: P is a self-map on $i\mathbb{R}^{\mathsf{X}}$

Hence P^t is a self-map on $i\mathbb{R}^{\mathsf{X}}$

Valuation

Solve optimization problems requires defining objectives clearly

Objective of dynamic programs = max **lifetime rewards**

Examples.

- lifetime wages for a worker
- lifetime utility for a consumer
- net present value for a firm

Let's start to look at computation of lifetime rewards

Fixed Discount Rates

Task: compute $\mathbb{E}$ of discounted future rewards

These sums take the form

$$v(x) := \mathbb{E}_x \sum_{t \geqslant 0} \beta^t h(X_t) := \mathbb{E} \left[\sum_{t \geqslant 0} \beta^t h(X_t) \,|\, X_0 = x \right]$$

Here

- $\beta \in \mathbb{R}_+$ and $h \in \mathbb{R}^{\mathsf{X}}$
- (X_t) is P-Markov on finite set X
- $\mathbb{E}_x$ indicates we are conditioning on $X_0 = x$

Example. Computing expected present value of a cash flow

Lemma. If $\beta \in (0, 1)$, then

$$v(x) := \mathbb{E}_x \sum_{t \geqslant 0} \beta^t h(X_t)$$

is finite for all $x \in \mathsf{X}$, $I - \beta P$ is invertible and

$$v = \sum_{t \geqslant 0} (\beta P)^t h = (I - \beta P)^{-1} h \tag{2}$$

Proof: Observe that

$$\mathbb{E}_x \sum_{t \geqslant 0} \beta^t h(X_t) = \sum_{t \geqslant 0} \beta^t \mathbb{E}_x h(X_t) = \sum_{t \geqslant 0} \beta^t (P^t h)(x)$$

Result (2) holds because $\rho(\beta P) < 1$ — why?

Application: A firm valuation problem

A firm receives profit stream $(\pi_t)_{t \geqslant 0}$ — a stochastic process

Valuation = expected present value of its profit stream:

$$V_0 = \mathbb{E} \sum_{t=0}^{\infty} \beta^t \pi_t \quad \text{with} \quad \beta := \frac{1}{1+r}$$

Assume $\pi_t = \pi(X_t)$ where $(X_t)_{t \geqslant 0}$ is P-Markov, set

$$v(x) := \mathbb{E}_x \sum_{t=0}^{\infty} \beta^t \pi_t$$

Now $r > 0$ implies $v = (I - \beta P)^{-1} \pi$

Ex. Suppose

- X is partially ordered
- $\pi \in i\mathbb{R}^{\mathsf{X}}$ and P is monotone increasing

Prove that, under these conditions, v **is increasing on** X

Proof 1: Let π and P satisfy the stated conditions

Given $t \in \mathbb{N}$

- P monotone increasing implies P^t monotone increasing
- Since $\pi \in i\mathbb{R}^{\mathsf{X}}$, we see that $P^t\pi \in i\mathbb{R}^{\mathsf{X}}$

Hence $v = \sum_{t \geqslant 0} \beta^t P^t \pi$ is also increasing

Ex. Suppose

- X is partially ordered
- $\pi \in i\mathbb{R}^{\mathsf{X}}$ and P is monotone increasing

Prove that, under these conditions, v **is increasing on** X

Proof 1: Let π and P satisfy the stated conditions

Given $t \in \mathbb{N}$

- P monotone increasing implies P^t monotone increasing
- Since $\pi \in i\mathbb{R}^{\mathsf{X}}$, we see that $P^t \pi \in i\mathbb{R}^{\mathsf{X}}$

Hence $v = \sum_{t \geqslant 0} \beta^t P^t \pi$ is also increasing

Proof 2: Let π and P satisfy the stated conditions

Rearranging $v = (I - \beta P)^{-1}\pi$ gives $v = \pi + \beta P v$

Hence v is the fixed point in $\mathbb{R}^{\mathsf{X}}$ of $Tv = \pi + \beta P v$

Since $\pi \in i\mathbb{R}^{\mathsf{X}}$ and P is monotone, T is invariant on $i\mathbb{R}^{\mathsf{X}}$

The operator T is a contraction of modulus β (Ex: check it)

Since $i\mathbb{R}^{\mathsf{X}}$ is closed, nonempty and invariant for T, the fixed point of T must lie in this set

In other words, v is increasing on X

Job Search Revisited

Now we return to the job search problem

Aims:

1. drop some of the restrictive assumptions we made earlier
2. analyze optimality

First extension: wage draws are correlated

- (W_t) is P-Markov on finite set $\mathsf{W} \subset \mathbb{R}_+$

The value function is denoted v^*

- $v^*(w) =$ max. lifetime value given current wage offer w

The value function satisfies the Bellman equation

$$v^*(w) = \max \left\{ \frac{w}{1-\beta}, \, c + \beta \sum_{w' \in \mathsf{W}} v^*(w') P(w, w') \right\}$$

The corresponding Bellman operator is

$$(Tv)(w) = \max \left\{ \frac{w}{1-\beta}, \, c + \beta \sum_{w' \in \mathsf{W}} v(w') P(w, w') \right\}$$

Ex. Prove that T is an order-preserving self-map on $V := \mathbb{R}_+^{\mathsf{W}}$

Proof of the order-preserving property

Given $f, g \in V$ with $f \leqslant g$, we claim that $Tf \leqslant Tg$

Indeed, if $w \in \mathsf{W}$, then

$$\sum_{w'} f(w') P(w, w') \leqslant \sum_{w'} g(w') P(w, w')$$

It easily follows that $(Tf)(w) \leqslant (Tg)(w)$

Since w was arbitrary, we have $Tf \leqslant Tg$

Ex. Prove that T is an order-preserving self-map on $V := \mathbb{R}_+^{\mathsf{W}}$

Proof of the order-preserving property

Given $f, g \in V$ with $f \leqslant g$, we claim that $Tf \leqslant Tg$

Indeed, if $w \in \mathsf{W}$, then

$$\sum_{w'} f(w')P(w, w') \leqslant \sum_{w'} g(w')P(w, w')$$

It easily follows that $(Tf)(w) \leqslant (Tg)(w)$

Since w was arbitrary, we have $Tf \leqslant Tg$

Set

$$\|f - g\|_\infty = \max_{w \in \mathsf{W}} |f(w) - g(w)|$$

Ex. Prove: T is a contraction of modulus β on V w.r.t. $\|\cdot\|_\infty$

Proof:

- Similar to the IID case
- Please complete as an exercise

Ex. Show that v^* is increasing on W whenever P is monotone increasing

Proof: Let $iV =$ all increasing functions in V

Since iV is closed, suffices to show that T is invariant on iV

Fix $v \in iV$

Then

- $h(w) := c + \beta(Pv)(w)$ is in iV and
- $e(w) := w/(1-\beta)$ is in iV

It follows that $Tv = e \vee h$ is in iV

Ex. Show that v^* is increasing on W whenever P is monotone increasing

Proof: Let $iV =$ all increasing functions in V

Since iV is closed, suffices to show that T is invariant on iV

Fix $v \in iV$

Then

- $h(w) := c + \beta(Pv)(w)$ is in iV and
- $e(w) := w/(1 - \beta)$ is in iV

It follows that $Tv = e \vee h$ is in iV

Fix $v \in V$

A policy $\sigma \colon \mathsf{W} \to \{0, 1\}$ is called v-**greedy** if

$$\sigma(w) = \mathbb{1}\left\{\frac{w}{1-\beta} \geqslant c + \beta \sum_{w'} v(w') P(w, w')\right\}$$

- treat v as the value function, choose optimally

We use value function iteration (VFI) to solve the model

- Iterate from arbitrary $v \in V$ to get $v_k := T^k v$
- Compute the v_k-greedy policy

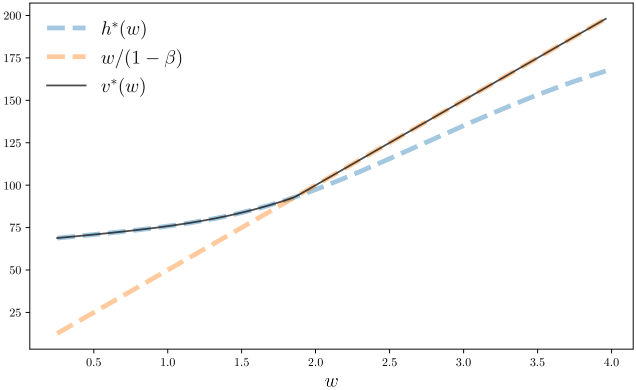
```julia
using QuantEcon, LinearAlgebra
include("s_approx.jl")

"Creates an instance of the job search model with Markov wages."
function create_markov_js_model(;
        n=200,       # wage grid size
        ρ=0.9,       # wage persistence
        ν=0.2,       # wage volatility
        β=0.98,      # discount factor
        c=1.0        # unemployment compensation
    )
    mc = tauchen(n, ρ, ν)
    w_vals, P = exp.(mc.state_values), mc.p
    return (; n, w_vals, P, β, c)
end
```

```julia
" The Bellman operator Tv = max{e, c + β P v} with e(w) = w / (1-β)."
function T(v, model)
    (; n, w_vals, P, β, c) = model
    h = c .+ β * P * v
    e = w_vals ./ (1 - β)
    return max.(e, h)
end

" Get a v-greedy policy."
function get_greedy(v, model)
    (; n, w_vals, P, β, c) = model
    σ = w_vals / (1 - β) .>= c .+ β * P * v
    return σ
end

"Solve the infinite-horizon Markov job search model by VFI."
function vfi(model)
    v_init = zero(model.w_vals)
    v_star = successive_approx(v -> T(v, model), v_init)
    σ_star = get_greedy(v_star, model)
    return v_star, σ_star
end
```

The **continuation value function** is given by

$$h^*(w) := c + \beta \sum_{w' \in \mathsf{W}} v^*(w') P(w, w')$$

- depends on w due to correlated wages



$h^*(w)$
$w/(1-\beta)$
$v^*(w)$
200
175
150
125
100
75
50
25
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
w

Ex. Explain why h^* is increasing in the last figure

Answer Since $\rho > 0$, P is monotone increasing

Hence $v^* \in iV$

Since $h^* = c + \beta P v^*$, it follows that $h^* \in iV$

Positive autocorrelation in wages means that

- high current wages predict high future wages
- value of waiting rises with current wages

Ex. Explain why h^* is increasing in the last figure

Answer Since $\rho > 0$, P is monotone increasing

Hence $v^* \in iV$

Since $h^* = c + \beta P v^*$, it follows that $h^* \in iV$

Positive autocorrelation in wages means that

- high current wages predict high future wages
- value of waiting rises with current wages

Job Search with Separation

Let's now allow for separation

- matches between workers and firms terminate with probability α every period

Other aspects of the problem are unchanged

Conditional on current offer w, let

- $v_u^*(w) =$ max lifetime value for unemployed worker
- $v_e^*(w) =$ max lifetime value for employed worker

We have

$$v_u^*(w) = \max\left\{v_e^*(w),\; c + \beta \sum_{w'} v_u^*(w') P(w, w')\right\}$$

and

$$v_e^*(w) = w + \beta\left[\alpha \sum_{w'} v_u^*(w') P(w, w') + (1 - \alpha) v_e^*(w)\right]$$

Proposition If $0 < \alpha, \beta < 1$, then this system has a unique solution (v_u^*, v_e^*) in $V \times V$

Step one: solve for v_e^* as

$$v_e^*(w) = \frac{1}{1 - \beta(1-\alpha)} \left(w + \alpha\beta (P v_u^*)(w)\right)$$

Substitute to get

$$v_u^*(w) = \max \left\{ \frac{1}{1 - \beta(1-\alpha)} \left(w + \alpha\beta (P v_u^*)(w)\right), \, c + \beta \, (P v_u^*)(w) \right\}$$

Ex.

- Prove that $\exists$ a unique $v_u^* \in V$ that solves this equation
- Propose a convergent method for solving for both v_u^* and v_e^*

When unemployed, the stopping and continuation values are

$$s^*(w) := \frac{1}{1 - \beta(1-\alpha)} \left(w + \alpha\beta(Pv_u^*)(w)\right)$$

and

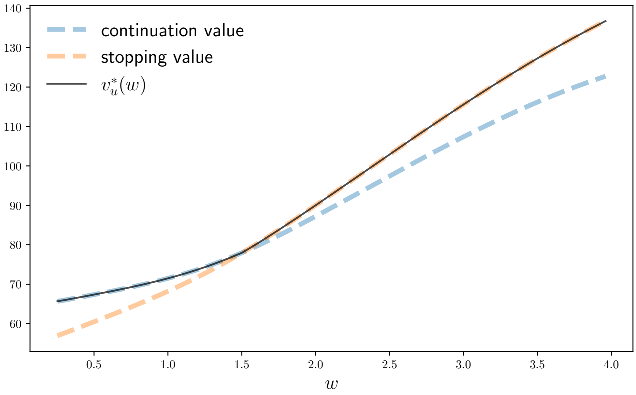
$$h^*(w) := c + \beta \, (Pv_u^*)(w)$$

Note $v_u^* = s^* \vee h^*$

Unemployed agent's optimal policy:

$$\sigma^*(w) := \mathbb{1}\{s^*(w) \geqslant h^*(w)\}$$

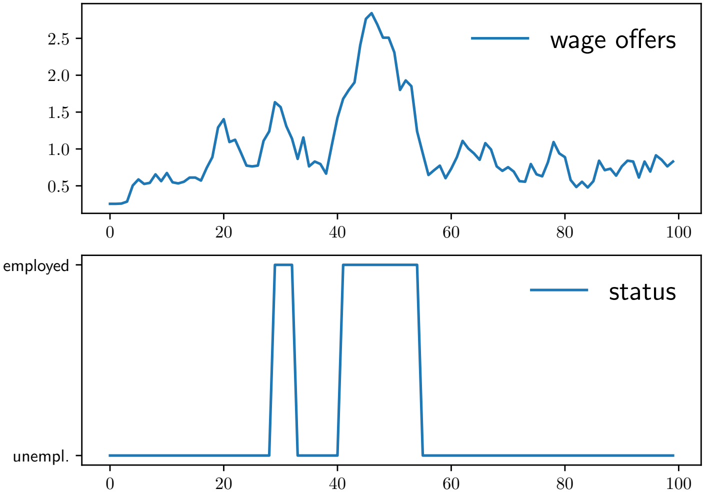
Reservation wage $w^* := \min\{w \in \mathsf{W} : s^*(w) \geqslant h^*(w)\}$


continuation value
stopping value
$v_u^*(w)$
140
130
120
110
100
90
80
70
60
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
w

```julia
include("markov_js_with_sep.jl")   # Code to solve model
using Distributions

# Create and solve model
model = create_js_with_sep_model()
(; n, w_vals, P, β, c, α) = model
v_star, σ_star = vfi(model)

# Create Markov distributions to draw from
P_dists = [DiscreteRV(P[i, :]) for i in 1:n]

function update_wages_idx(w_idx)
    return rand(P_dists[w_idx])
end
```

```julia
function sim_wages(ts_length=100)
    w_idx = rand(DiscreteUniform(1, n))
    W = zeros(ts_length)
    for t in 1:ts_length
        W[t] = w_vals[w_idx]
        w_idx = update_wages_idx(w_idx)
    end
    return W
end
```

```julia
function sim_outcomes(; ts_length=100)
    status = 0
    E, W = [], []
    w_idx = rand(DiscreteUniform(1, n))
    ts_length = 100
    for t in 1:ts_length
        if status == 0
            status = σ_star[w_idx] ? 1 : 0
        else
            status = rand() < α ? 0 : 1
        end
        push!(W, w_vals[w_idx])
        push!(E, status)
        w_idx = update_wages_idx(w_idx)
    end
    return W, E
end
```


wage offers
2.5
2.0
1.5
1.0
0.5
0
20
40
60
80
100
employed
unempl.
status
0
20
40
60
80
100

Ex. Here's an open-ended optional exercise

Let $E_t :=$ employment status

- Show that $X_t := (W_t, E_t)$ is a Markov chain
- Write down the state space and prove irreducibility

Let ψ^* be the unique stationary distribution

Ergodicity: fraction of time a worker spends unemployed should be equal to prob of unemployment under ψ^*

- Check it

Prob of unemployment under ψ^* equals unemployment rate

Adjust model parameters to match current unemployment rate