Algorithms, 2023 Fall, Homework 1
(Due: Sep 27)

September 18, 2023

Readings: Our algorithm class assumes basic knowledge about basic algorithms knowledge (like what is a graph, BFS, DFS) and basic data structures (like array, link list, queue and stack), and college mathematics (basic calculus, linear algebra and probability). In case you have not learnt much before, you are required to read the following chapters.

1. Kleinberg-Tardos book Chapter 2.1-2.4 for big O notations, basic data structures. (You need to finish reading this chapter this week in order to do the homework)
2. Chapter 3 for BSF and DFS, also Chapter 22.1,22.2,22.3 in CLRS. (You need to finish reading this chapter this week in order to do the homework.)

You can discuss homeworks with your classmates. But you will have to write down the solution on your own. You need to acknowledge other students who help you in the homework. If you read some other source that is helpful, you should list all of them.

Problems marked with @ can be skipped by those who have OI medals (show TA your certificate).

Problem 1 : @ Write down the pseudocode of BSF (using queues).

Problem 2 : @ KT book pp. 67 problem 4.

Problem 3 : @ KT book pp. 107 problem 3.

Problem 4 : @ KT book pp. 108 problem 5.

Problem 5 : @ KT book pp. 108 problem 6.

Problem 6 : @ KT book pp.110, problem 10) (Note: We only need the number, not the actually paths. Please learn BFS first.)

Problem 7 : We are given n balls and n bins. Each ball is thrown into a random bin (each bin is chosen with probability $1/n$). Prove that:

$$\lim_{n \to +\infty} \mathbb{E}\left[\frac{\text{the number of empty bins}}{n}\right] = \frac{1}{e}$$

where e is the base of the natural logarithm.

Problem 8 : For two women w and w', we write $w <_m w'$ to denote that w is worse than w' in the preference list of man m. Given two stable matchings f and f' (You can easily construct an example in which there are multiple stable matchings), define a mapping $g = f \vee f'$ as follows:

- for each man m, assign him more preferred partner

$$g(m) = f(m) \text{ if } f(m) \geq_m f'(m)$$

$$g(m) = f'(m) \text{ if } f'(m) >_m f(m)$$

- for each woman w, assign her less preferred partner

$$g(w) = f(w) \text{ if } f(w) \leq_w f'(w)$$

$$g(w) = f'(w) \text{ if } f'(w) <_w f(w)$$

Show that if both f and f' are stable matchings, so is g. (note: We can similarly define $f \wedge f'$. Then, all stable matchings form a distributive lattice, an abstract yet very popular object studied in combinatorics.)

Problem 9 : As we mentioned, there could be multiple stable matchings. The Gale-Shapley algorithm only finds one such stable matching. For any man m, let $best(m)$ be the best woman matched to m in all possible stable matchings. Show that Gale-Shapley algorithm is man-optimal, in the sense that it returns a stable matching where for any man m, m is matched to $best(m)$. (If we let women propose, the resulting matching is women-optimal).