Abstract Algebra
Mid-term Exam

Apr. 22, 2024
Exam Duration: 180 minutes
Each problem carries 10 marks.

Problem 1

Let σ, τ, γ be elements of S_{13} given by

$$\sigma = (1\ 4\ 6\ 8\ 10)\ (2\ 5)\ (3\ 6)\ (11\ 7\ 9), \tag{1}$$
$$\tau = (1\ 8)\ (2\ 3\ 6), \tag{2}$$
$$\gamma = (2\ 4)\ (3\ 9). \tag{3}$$

Compute $|\sigma|, \sigma^3, \sigma^{-1}\tau, \langle \tau\gamma \rangle$.

Problem 2

Let G be a group and N be a normal subgroup of G. Prove that for arbitrary subgroup H of G, $N \cap H$ is normal in H.

Problem 3

Let G be a group and H be a subgroup of index p where p is a prime. Prove that for any subgroup K of G at least one of the following holds: (i) $K \leq H$; (ii) $KH = G$ and $[K : K \cap H] = p$.

Problem 4

Let $p < q < r$ be distinct primes. Prove that any group of order pqr is solvable and has a normal Sylow r-subgroup.

Problem 5

Let p be a prime and G_p be an infinite group such that any non-trivial subgroup of it has order p. Such groups are known as Tarski monster group.

(a) Prove that any element of G_p, except for the identity, has order p.
(b) Prove that G_p is simple.
(c) Prove that G_p can be generated by two elements.

Problem 6

Let G be a group and H a subgroup of it with $[G : H] = 4$. It is known that for any normal subgroup K of G that is contained in H with $[H : K] \leq 100$, the quotient group H/K is never cyclic. Prove that there is some normal subgroup P of G contained in H such that $G/P \simeq S_4$.

Problem 7

Prove that there are exactly four homomorphisms from $\mathbb{Z}_2$ to $\mathrm{Aut}(\mathbb{Z}_8)$, and find one θ among them so that $\mathbb{Z}_8 \rtimes_\theta \mathbb{Z}_2 \simeq D_{16}$.

Problem 8

This problem does not require detailed reasons for answers.

Let the quasi-dihedral group of order 16, QD_{16}, be defined by

$$\left\langle \sigma, \tau \mid \sigma^8 = \tau^2 = \mathrm{id}, \sigma\tau = \tau\sigma^3 \right\rangle . \tag{4}$$

The following is a lattice of it. (The figure may be updated in future versions.)

(a) Fill in the blanks in the lattice, for each of which you should provide no more than 2 elements that generate the corresponding subgroup.

(b) Draw a lattice for $QD_{16}/\left\langle \sigma^4 \right\rangle$ and use it to explicitly construct an isomorphism $QD_{16}/\left\langle \sigma^4 \right\rangle \simeq D_8$.

Problem 9

This problem tries to apply group theory to combinatorics.

(a) Let G be a finite group such that every non-identity element has order 2. Prove that the order of G is a power of 2. (Partial marks may be awarded for those who successfully prove this result for Abelian groups, which is sufficient for the following of this problem.)

(b) There are n mathematicians attending a meeting. Some of them are friends while some are not. The organisers would like to divide the mathematicians into 2 dining halls, and each mathematician would like to have an even number of friends in his / her dining hall. Prove that the number of ways to divide these mathematicians into 2 dining halls is either 0 or a power of 2.

Problem 10

It is well known that conjugacy classes of S_n is determined by the lengths of the cycles formed. In this problem we work out the conjugacy classes of A_n. Let K be a conjugacy class in S_n whose elements are all in A_n.

(a) Find out all conjugacy classes in A_4.

(b) Prove that if an element of K consists of cycles of distinct odd lengths then K is not a conjugacy class in A_n.

(c) In the contrary to (b), prove that if an element of K contains a cycle of even length or two cycles of same odd length, then K is a conjugacy class in A_n.