# Algorithms, 2023 Fall, Homework 3
# (Due: Oct 18)

October 7, 2023

You can discuss homeworks with your classmates. But you will have to write down the solution on your own. You need to acknowledge other students who help you in the homework. If you read some other source that is helpful, you should list all of them.

Problems marked with (@) can be skipped by those who have extensive experience in OI

**Problem 1 :** (@) KT book. CH 4. Problem 4. (pp. 190)

**Problem 2 :** (@) KT book. CH 4. Problem 9. (pp. 192)

**Problem 3 :** (@) KT book. CH 4. Problem 10. (pp. 192)

**Problem 4 :** (@) Write down the pseudocode for the implementation of priority queue using heap. (you need to use an array for storing the heap. you need to specify how to access visit the children of a node or the parent of a node).

**Problem 5 :** (@) In the class, I assume that all weights are distinct in the minimum spanning tree problem. Please show how to remove the assumption. In particular, you should prove that Kruskal's algorithm is still correct even if some edges have the same weights.

**Problem 6 :** Graph Laplacian is a fundamental notion in algebraic/spectral graph theory. For undirected graph, the graph Laplacian is defined as

$$L = D - A$$

where $D$ is the diagnal matrix with diagnal being the vertex degrees and $A$ is the adjacency matrix. Since $L$ is a matrix, it can be viewed as an operator (which maps a vector to a vector).

Recall in the calculus class, the laplacian operator $\Delta$ (which operators on a smooth function) is defined as

$$\Delta f = div \nabla f = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}$$

Intuitively argue that why the graph Laplacian can be also viewed as the discrete version of the Laplacian operator you learn from the calculus (or PDE) class. (Hint:

Recall that the physical interpretation for $\Delta$: $\Delta f(x)$ as the extent to which a point $x$ represents a source or sink of the density $f$ (recall Gauss theorem).

sidenote: There is a relation between the (undirected) graph Laplacian and the incidence matrix (the $|E| \times |V|$ 0/1 matrix which indicates the incidence relation of vertices and edges). This is in fact the specialized definition of Laplacian in algebraic topology (in particular Homology theory for simplicial complex). It is the product of the boundary operator (which is the incidence matrix) and the coboundary operator (which is the transpose of the incidence matrix). So many results about Laplacians in Riemannian geometry have discrete analogue for graph Laplacians, and vice versa. For those who are interested in this topic, you can read Fan Chung's book "spectral graph theory". Or Gary Miller's lectures: http://www.cs.cmu.edu/ 15859n/schedule.html

**Problem 7 :** For undirected graph, the graph Laplacian is defined as

$$L = D - A$$

where $D$ is the diagnal matrix with diagnal being the vertex degrees and $A$ is the adjacency matrix. Try to compute the rank of the Laplacian of the following undirected graphs. (1) A clique of size 3. (2) A path of length 3. (3) An unconnected graph consisting two cliques of size 3. (4) An unconnected graph consisting two paths of length 3. (5) An unconnected graph consisting three cliques of size 3.

From the above results, try to derive a relation between the rank of $L$ and the number of connected components. Formally prove your finding. (don't look up the internet. it is not difficult. you can do this!)