# Algorithms, 2023 Fall, Homework 7
# (Due: Nov 12)

October 30, 2023

Required Reading:
1. page 493,494,495 in the textbook.

You can discuss homeworks with your classmates. But you will have to write down the solution on your own. You need to acknowledge other students who help you in the homework. If you read some other source that is helpful, you should list all of them.

**Problem 1 :**(Path Selection Problem) (KT pp.508, Problem 9) We are given a directed graph $G(V, E)$ and $c$ paths $P_1, \ldots, P_c$ in $G$. We are also given an integer $k$. We are asked whether it is possible to select $k$ of the paths so that no two selected paths share any nodes. Prove the problem is NP-Complete.

**Problem 2 :**(Minimum Leaf Spanning Tree) The decision version of the minimum leaf spanning tree problem asks whether there is a spanning tree of $G$ with at most $k$ leaves for a given undirected graph $G(V, E)$ and integer $k$. Prove the problem is NP-Complete.

**Problem 3 :** Prove the following special case of SUBSET SUM is also NPC. We are given $n$ positive integer numbers $w_1, w_2, \ldots, w_n$. Suppose $W = \sum_{i=1}^{n} w_i$ and $W$ is even. The problem asks whether there exists a subset of total sum $W/2$.

**Problem 4 :** You are given a metric graph $G(V, E, d)$. You need to assign each node one of $k$ given colors. Define the total weight of a color $c$ be

$$\sum_{(u,v):color(u)=color(v)=c} d(u, v).$$

Show that it is NP-complete to assign colors such that the total weight of each color is at most $t$.

**Problem 5 :** You are given an undirected graph $G(V, E)$. The goal is to find a partition of $V$ into two sets $S$ and $V - S$, such that the number of edges across the cut $(S, V - S)$ is maximized. The problem is NP-hard, unlike the min-cut problem (which we will learn shortly). Show that the following greedy algorithm is a 2-approximation: Start with two empty bin $S$ and $\bar{S}$. Process the vertices in an arbitrary order. For each vertex $v$ we are processing, add $v$ to the bin that maximizes the number of edges across $S$ and $\bar{S}$.

**Problem 6 :** When we talk about approximation factors, we usually mean multiplicative factors. In this question, we will see why additive approximation does not make much sense for many problems (sometimes it makes sense for certain special problems). Let us consider the vertex cover problem. Show that for any constant $\alpha > 0$, it is not possible to obtain a solution $SOL$ in poly-time (assuming $P \neq NP$ of course) such that

$$SOL \leq OPT + \alpha.$$

**Problem 7 :** Try to prove the following algorithm is a 2-approximation for $k$-center. Let $C$ be the set of centers. Initially $C = \emptyset$. In each iteration, we choose the vertex $v$ which is farthest from $C$ ($v = \arg\max d(v, S)$), and add it to $C$. The algorithm terminates until $|C| = k$.

**Problem 8 :** Show that it is NP-hard to approximate the (vertex) coloring problem within a factor of $4/3 - \epsilon$ for any constant $\epsilon > 0$.

**Bonus problems:** Bonus problems are problems that may be (may be not) more challenging than other problems. You do not lose points if you do not answer bonus problems. But you will get extra points if you get correct answers for such problems. **To make your life easier, we give you two weeks for the bonus problems**).

**Problem 1 :**(Max 2-SAT) Given a 2CNF with $n$ clauses and a integer $k < n$, we would like to ask whether there is an assignment such that at least $k$ clauses are satisfied. Show that the problem is NP-Complete.