Algorithms, 2023 Fall, Homework 14
(Due: Jan 6)

December 25, 2023

Required reading: Section 12.4

Problem 1 : Consecutive 1 Matrix. It is a matrix satisfying the following two properties: 1) Each entry is either 0 or 1. 2) In every row, all the 1s are consecutive. This matrix is used in many scheduling problems. For example, interval packing problem (We want to maximize the size of the set of disjoint intervals of integrals) can be solved by linear programming, while its corresponding matrix is a consecutive 1 matrix. Prove such a matrix is TUM.

Problem 2 : Let $G = (V, E)$ be a bipartite graph and let k be an integer. Let S_k be the set of indicator vectors of the matchings in G of size at most k. Show that the following polytope is the convex hull of the vectors in S_k.

$$\sum_{e \in E} x_e \leq k, \forall e \in E; \quad \sum_{e \in \delta(v)} x_e \leq 1, \forall v \in E; \quad 0 \leq x_e \leq 1, \forall e \in E.$$

$\delta(v)$ denotes the set of edges incident on v.

Problem 3 :

Definition 1 ***(Network Matrix)*** *For a directed tree $T = (V, E)$ and a set of ordered pairs of vertices $P \subseteq V \times V$ ($|P| = k$), with the edges of T labeled from 1 to m, the network matrix $M_{m \times k}$ is defined as follows: The rows of a network matrix correspond to arcs E. Each arc has an arbitrary orientation (it is not necessary that there exists a root vertex r such that the arcs are oriented towards r or out of r). Each column corresponds to an ordered pair in P. To compute the entry at row e_t and column (v_i, v_j), look at the v_i-to-v_j path p in T; then the entry is:*

1. *+1 if arc R appears in the forward direction in P,*
2. *-1 if arc R appears in the backwards direction in P,*
3. *0 if arc R does not appear in P.*

(1) Prove that a consecutive 1 matrix is a network matrix. (2) Prove the matrix in problem 2 is a network matrix.

Problem 4 : Prove a network matrix is TUM. (do it by yourself. don't look up the internet.)

Problem 5 : Using the theory of TUM, show the following result: For a min-cost circulation instance (with cost, demands and lower bounds), if the demands and capacity (lower and upper) bounds are integers, there is an integer min-cost circulation.

Problem 6 : KT book pp704 problem 4.