Quantum Computer Science, Final Exam, Spring 2022

Please explicitly write down the derivation process for each question, otherwise you will lose some of your marks.

I. CONCEPTS (24 PTS)

Please briefly explain the following conceptual questions (No need of calculations):

(a) What are the basic requirements for universal quantum computing? (4 pts)

(b) An unknown quantum state can be faithfully teleported from Alice to Bob. Does the quantum teleportation violate the quantum no-cloning theorem? Does this process allow superluminal (faster than light) transfer of information? Briefly explain your reasons. (6 pts)

(c) Give a description of the CHSH inequality, and show how the quantum mechanics predictions that based on Bell states maximally violate the inequality. (8 pts)

(d) Briefly summarize the main idea of the initial state preparation and final state detection for trapped ion qubits. (6 pts)

II. STABILIZER FORMALISM AND THE GHZ EXPERIMENT (14 PTS)

In class we have demonstrated the contradictions between the local hidden-variable theory and quantum mechanical theory through the Bell's inequality test. Actually, in order to generate more contrasting predictions, we can also consider the following Greenberger-Horne-Zeilinger experiment. For the three-qubit GHZ state,

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle), \tag{II.1}$$

(a) Show that $|GHZ\rangle$ is the co-eigenstate of the following four operators, and calculate the corresponding eigenvalues. Namely, the four operators are stabilizers of the GHZ state. (X_i, Y_i are Pauli matrices of the qubit i.) (7 pts)

$$\begin{aligned} &X_1 \otimes Y_2 \otimes Y_3 \\ &Y_1 \otimes X_2 \otimes Y_3 \\ &Y_1 \otimes Y_2 \otimes X_3 \\ &X_1 \otimes X_2 \otimes X_3 \end{aligned} \tag{II.2}$$

(b) If we measure the above four operators on the GHZ state, what are the measurement expectation values? (2 pts)

(c) Next we consider the prediction of the local hidden-variable theory. If we have pre-assigned the measurement values of the observables X_i, Y_i as m_i^x, m_i^y ($m_i^{x(y)}$ takes the values ± 1), the measurement values of the above four operators are

$$\begin{aligned} &m_1^x m_2^y m_3^y \\ &m_1^y m_2^x m_3^y \\ &m_1^y m_2^y m_3^x \\ &m_1^x m_2^x m_3^x. \end{aligned} \tag{II.3}$$

When you multiply the four measurement values in Eq. (II.3) together, compared with the multiplied value predicted by quantum mechanics in (b), can you find a contradiction? (5 pts)

III. SWAP GATE (14 PTS)

The SWAP gate can be written in the matrix form

$$\text{SWAP} = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}$$

(a) Explain that the SWAP gate, assisted by any single qubit gates, is not universal for quantum computation. You need to give explicit contruction of an operation that these gates cannot reach. (4 pts)

(b) Construct a gate R that satisfies $R^2 = \text{SWAP}$ (so we may also call it $\sqrt{\text{SWAP}}$ gate). (3 pts)
Hint: $\sqrt{X} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$ satisfies $\left(\sqrt{X}\right)^2 = X = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

(c) Part of the transformation $R \circ Z_1 \circ R$ has been given in the matrix form below, where Z_i is the Puali Z matrix on the i-th qubit.

$$R \circ Z_1 \circ R = \begin{pmatrix} ? & ? & & \\ ? & ? & & \\ ? & ? & -i & \\ ? & ? & & -1 \end{pmatrix}$$

(I) Fill in the rest of the matrix. (4 pts)

(II) Briefly explain why the $\sqrt{\text{SWAP}}$ gate, together with single qubit gates, is universal. (3 pts)

Hint: The phase shift gate on the i-th qubit is defined as $P_i(\alpha) = \begin{pmatrix} 1 & \\ & e^{i\alpha} \end{pmatrix}$. Appending $P_1(\alpha_1), P_2(\alpha_2)$ is already sufficient to transform $R \circ Z_1 \circ R$ into the controlled phase gate, which is known to be universal when assisted by single qubit gates.

IV. GROVER ALGORITHM(17 PTS)

Suppose we wish to search through a search space of $N = 2^{10}$ elements, and we concentrate on the index to those elements, which is just a number in the range 0 to $N-1$. The search problem has only **one** solution (marked element), with index ω $(0 \leq \omega \leq N-1)$. The Grover algorithm consists of repeated application of a quantum subroutine known as the **Grover iteration**, which we denote G. Note that $G = DU_\omega$, where $D = 2|\psi\rangle\langle\psi| - I, U_\omega = 2|\omega\rangle\langle\omega| - I, |\psi\rangle = \frac{1}{\sqrt{N}} \sum\limits_{x=0}^{N-1} |x\rangle$.

(a) When using a classical algorithm, what is the possibility to find the solution within **3** queries? And what is the possibility when using Grover search algorithm with **3** iterations? (4 pts)

(b) (5pts)

(I) Estimate how many **Grover iteration** shall we use to find the solution?

(II) Assume we keep the Grover algorithm process running, estimate how many **Grover iteration** shall we use to get the solution for the second time?

(c) Consider a search problem with **3** solutions denoted by their indexs S_1, S_2, S_3. The state of the solutions is $|S\rangle = \frac{1}{\sqrt{3}}(|S_1\rangle + |S_2\rangle + |S_3\rangle)$. Grover search algorithm still applies by just replacing $|\omega\rangle$ with $|S\rangle$. (8 pts)

(I) Re-express the initial state $|\psi\rangle$ in the space spanned by $|S\rangle$ and $|S_\perp\rangle$.

(II) What is the minimum number of solutions in a search problem where the possibility to find the solutions is larger than $\frac{1}{2}$ **before** applying the first G.

(III) Is it possible to find all the solutions correctly by doing the final measurement only **once**? If not, please explian why.

V. QFT AND TRANSLATIONAL INVARIANCE (14 PTS)

In class, we have introduced the n-qubit Quantum Fourier Transformation (QFT)

$$|x\rangle \to \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i x y/2^n} |y\rangle$$

Here we extend 2^n to an arbitrary integer $N \in \mathbb{Z}^+$. We define the QFT modulo N as

$$\hat{F}_N |x\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i x y/N} |y\rangle$$

where the state space is $\mathbb{C}^N$ with orthonormal basis $\{|0\rangle, |1\rangle, ..., |N-1\rangle\}$ and $0 \le x, y \le N-1$. Let $\hat{P}$ denote the unitary operation that adds 1 modulo N: for any $0 \le x \le N-1$

$$\hat{P}|x\rangle = |(x+1) \mod N\rangle$$

a Show that $\hat{F}_N$ is a unitary transformation. (6 pts)

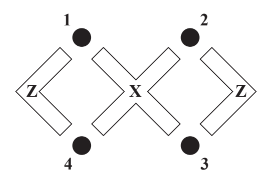
b Show that the Fourier basis states ($\hat{F}_N|x\rangle$) are eigenvectors of $\hat{P}$. What are their eigenvalues? (Equivalently you could also show that $\hat{F}_N^{-1}\hat{P}\hat{F}_N$ is diagonal, and find its diagonal entries.) (4 pts)

c Let $|\psi\rangle \in \mathbb{C}^N$. Show that if $\hat{P}|\psi\rangle$ is measured in the Fourier basis (or equivalently, if we apply the inverse Fourier Transformation and then measure in the computational basis), the probability of all measurement outcomes are the same as if the state had been $|\psi\rangle$. (4 pts)

VI. QUANTUM ERROR CORRECTION (17 PTS)

A 4-qubit "error detection" code is defined by stabilizers $G = \{g_i\}$ as follows

$$\begin{aligned} g_1 &= X_1X_2X_3X_4 \\ g_2 &= Z_1Z_4 \\ g_3 &= Z_2Z_3 \end{aligned}$$



(a) Verify that $|\phi\rangle = \frac{1}{2}(|0000\rangle + |1111\rangle) + \frac{1}{2}(|1001\rangle + |0110\rangle)$ is "stabilized" by G, that is, $g_i|\phi\rangle = |\phi\rangle \quad \forall i$. (3 pts)

(b) Write down the code space in the form $|\psi\rangle = a|\psi_1\rangle + b|\psi_2\rangle$. (4 pts)

(c) Suppose we initialize the system as $|\phi\rangle$. After some time, there is an X_1 error. What is the measurement outcome of the three stabilizers in G? Show that an X flip on another qubit can also produce the same measurement outcome. (4 pts)

(d) For $|\phi\rangle$, show that from the measurement outcome of G we can

(I) fully recover the state if there is no more than one X_i error;

(II) detect that an error occurs (we may not be able to recover the state so we will just throw it away) if there is no more than one Z_j error.

Note that all qubits are exposed to noise equally so i, j are not known. (6 pts)