Algorithms, 2023 Fall, Homework 6
(Due: Nov 3)

October 24, 2023

You can discuss homeworks with your classmates. But you will have to write down the solution on your own. You need to acknowledge other students who help you in the homework. If you read some other source that is helpful, you should list all of them.

Problem 1 :(Minimum Dominating Set) Given a undirected graph $G(V, E)$, a subset S of vertices is a dominating set if for every $v \in V \setminus S$, there exists a vertex $u \in S$ which is a neighbor of v. The decision version of the minimum dominating set problem asks whether there is a dominating set of size at most k for a given undirected graph $G(V, E)$ and integer k. Prove the problem is NP-Complete.

Problem 2 : Prove the Hamitonian path problem (with specified starting point s and ending point t) in undirected graph is NP-Complete. (A Hamitonian path is a path that visit every node of the graph exactly once). You may assume that the Hamitonian cycle problem in either directed or undirected graphs is NP-Complete.

Problem 3 : Consider the following: Suppose we can solve an NP-Complete problem in quasi-polynomial time (e.g., $O(n^{\log n})$). Show that we can solve all NP problems in this time.

In fact, many TCS researchers believe that an NP-Complete problem requires at least exponential time i.e., $O(2^n)$, and anything substantially smaller (e.g., $2^{\sqrt{n}}$) is unlikely (this is called Exponential Time Hypothesis, which is a stronger conjecture than $P \neq NP$).

Problem 4 : (Max-clique and Densest k-Subgraph) A clique in a graph G is a complete subgraph of G. The decision version of the maximum clique problem is as follows: Given a undirected graph G and an positive integer k, is there a clique in G of size at least k. Prove the problem is NP-Complete.

Problem 5 : Consider the densest k subgraph problem. We are given an undirected graph G and positive integers k and Z. The densest k-subgraph problem asks whether there is a subgraph of G with k vertices such that the total number of edges in the subgraph is at least Z. Prove this problem is NP-Complete.

Problem 6 : You are given a graph $G(V, E)$ with distance measure d (i.e., the distance between node u and v is $d(u, v)$). Each node s has a facility opening cost f_s. You are asked to open a set A of facilities to provide services to every node in the graph. Given the set A of opened facilities, the service cost of a node v is defined to be $d(v, A) = \min_{s \in A} d(v, s)$. The goal is to select a subset A such that the total cost (facility cost plus service cost)

$$\sum_{s \in A} f_s + \sum_{v \in V} d(v, A)$$

is minimized. Prove the decision version of the facility location problem is NP-Complete.

Problem 7 : Consider a nonnegative integral submodular function $f : 2^{[m]} \to [n]$, i.e.,

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T), \ \ \forall S, T \subseteq [m].$$

In fact, submodular functions are discrete analogues of the concave functions in the continuous case.

1. To see this, you are ask to prove the following property: f is submodular if and only if

   $$f(S \cup \{x\}) - f(S) \geq f(T \cup \{x\}) - f(T), \ \ \forall S \subset T \subseteq [m] \text{ and } x \notin T.$$

   The above inequality says the marginal profit of an element is smaller if we already have more. Compare this with a concave function $g(x)$ for which the derivative of g gets smaller when x is getting bigger.

2. We are given a ground set U of n element and a collection of subsets $S_1, \ldots, S_m$. Show that $f(S) = |\cup_{i \in S} S_i|$ is submodular.

3. Recall the definition of matroid $\mathcal{M}(E, \mathcal{I})$ from previous homework. Define the rank function as follows: $r(S) = \max\{|T| : T \in \mathcal{I}, T \subseteq S\}$ for $S \subset E$ (i.e., $r(S)$ is the maximum cardinality of an independent set in S) (Note that the concept of independent set has nothing to do with independent sets in graphs). Show that $r(S)$ is a submodular function.

Bonus problems: Bonus problems are problems that may be (may be not) more challenging than other problems. You do not lose points if you do not answer bonus problems. But you will get extra points if you get correct answers for such problems. **To make your life easier, we give you two weeks for the bonus problems)**.

All reductions you learn in our class are not very complicated. If you are interested, you can take a look at the following more difficult examples.

(1) The paper shows that Tetris is NP-Hard

`http://publications.csail.mit.edu/publications/pubs/pdf/MIT-LCS-TR-865.pdf`

(2) This paper shows that the minimum triangulation problem is NP-hard. This was an open problem for a long time and the problem was solved with computer assistance (some gadgets are designed with the help of computer).

`https://arxiv.org/pdf/cs/0601002.pdf`