Review of 3 Dimensional Gravity

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Contents

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1 Preface

This is a review of quantum gravity in three dimensions based on many materials, in order to collect and reconsider the 3 dimensional gravity of 2+1 D gravity. There are already tons of notes to do this, but here I give my personal version. Of course I don't create most of the statements by myself. In this note I just rewrite them and try to organise them in a more formal and mathematical way.

2 Introduction

2.1 Motivation, Advantages and Drawbacks

There is a long history of studying 3 dimensional gravity, or 2+1 D gravity with 2 spatial dimension and 1 time dimension. And there are many papers and lecture notes related to these topics. Why people are interested in it is that it is in low dimension and it seems that it is solvable. The 2+1 dimensional theory provides us with one useful approach to more complicated 3+1 dimensional classical gravity. The vacuum Einstein equation

$$R_{\mu\nu} - \left(\frac{1}{2}R - \Lambda\right) g_{\mu\nu} = 0 \tag{1}$$

implies that the space-time is locally flat, corresponding to the absence of the gravitational radiation (Weyl tensor) in three dimensions. But the locally flat property doesn't means that this theory is trivial. Many papers prove that there exist black holes solutions in 2+1 dimensional gravity. A typical examples is the BTZ black hole.

Definition 1. *BTZ black hole is a black hole solution for (2+1)-dimensional topological gravity with a negative cosmological constant. It carries the following properties.*

Properties 1. - *Properties of BTZ black hole:*

1 *It admits a no hair theorem, fully characterizing the solution by its ADM-mass, angular momentum and charge.*

2 *It has the same thermodynamical properties as traditional black hole solutions such as Schwarzschild or Kerr black holes, e.g. its entropy is captured by a law directly analogous to the Bekenstein bound in (3+1)-dimensions.*

3 A rotating BTZ black hole contains an inner and an outer horizon

Another interesting things deserves noticing that all the solutions exist only if the equation includes a negative cosmological constant. This is proved by Daisuke [6].

Theorem 1. *Let (M, g) be a three dimensional space-time manifold subject to the Einstein equation with $\Lambda > 0$. If the stress-energy tensor T satisfies the dominant energy condition, then (M, g) contains no apparent horizons.*

Through this, we can see that 3 dimensional gravity is not a very trivial model. It might provide some ways to figure out how to quantize the gravity in 3+1 dimension.

But the drawback of this theory is obvious–it is a theory in 3 dimension instead of 4 dimension, which is the real physical world. Besides, the gravity of these models has many ways to quantize gravity, we cannot tell which one is the correct way. Last but not least, the partition function in this model is not satisfying, which will be discussed later this note. It is not a satisfying theory,

Remark 1. *What is dimension? Physicists never claim or define what is dimension. They just discuss it based on experience and intuition. People say that the space-time of our universe is 4 dimensional because we have to use 4 coordinates to describe the exact position of objects. But is that correct? Is there another more intrinsic but more abstract way to define dimension? It there another way to tell the natural differences between 4 dimensional gravity or 3 dimensional gravity?*

3 Classical Three Dimensional Gravity

Let's see the classical theory first[3].

3.1 Special thing in three dimension

First let us see how the lower dimension simplifies the general relativity. In 2+1 dimensions, the Weyl tensor $C_{\mu\nu\rho}{}^{\sigma}$ vanishes automatically, and the full curvature tensor is determined algebraically by the curvature scalar and the Ricci tensor

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}R\right) \tag{2}$$

The another formalism of Einstein field equation is

$$R_{\mu\nu} - \frac{2}{D-2}\Lambda g_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{D-2}T g_{\mu\nu}\right) \tag{3}$$

Specially for $D = 3$ and the vacuum, the equation field equation becomes

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu} \tag{4}$$

Which means that the scalar curvature is a constant. So the case of $\Lambda < 0$ corresponds to the AdS space. There are no gravitational waves in the classical theory, and no gravitons in the quantum theory. Based on the simple discussion above, we now move to study the 2+1 dimensional gravity with negative cosmological constant. There is no gravitational wave in 3 dimensional gravity model. Naively, there are no perturbative excitations at all above the AdS_3 vacuum.

3.2 Alternative Formulation

We now give several formulations to study this theory. Of course this is not a important point so we won't pay too much attention to this.

3.2.1 Classical Action

As the usual formalism of the study of GR, we wrote the classical action

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R + \frac{2}{l^2}\right), \tag{5}$$

where $\Lambda = -2/l^2$. Starting from this action we could derive the Einstein field equation and try to solve the metric in 2+1 dimension.

3.2.2 Chen-Simons Formulation

Here we use the explanation in [1]. Again, staring form the classical action (5), similar to the bosonic string theory, we introduce the fundamental variables ${e_\mu}^a$,

$$g_{\mu\nu} = {e_\mu}^a \eta_{ab} {e_\nu}^b \tag{6}$$

and spin connection ${\omega_\mu}^a{}_b$,

$$\Gamma^\sigma{}_{\mu\nu} = e^\sigma{}_a \partial_\nu e^a{}_\mu + e^\sigma{}_a {\omega_b}^a{}_\nu {e_\mu}^b \tag{7}$$

And the corresponding covariant derivative in the spin connection is D_μ and $D_\mu {e_\nu}^a = \Gamma^\rho{}_{\nu\mu} {e_\rho}^a$. The next object we want to write in the new coordinate is the curvature tensor. The curvature tensor is a tensorial 2-form, for that reason we only transform two of its four indices as,

$$R^{\lambda\sigma}{}_{\mu\nu} = {e_a}^\lambda {e_b}^\sigma R^{ab}{}_{\mu\nu}, \tag{8}$$

where $R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$, $\omega^a = {\omega_\mu}^{bc} dx^\mu$. The first term in the equation (5) can be rewritten as

$$\begin{aligned} \int \sqrt{g}R &= \frac{1}{2}\int \epsilon^{\mu\nu\lambda}\epsilon_{\alpha\beta\lambda} e R^{\alpha\beta}{}_{\mu\nu} = \frac{1}{2}\int \epsilon^{\mu\nu\lambda}\epsilon_{abc} R^{\alpha\beta}{}_{\mu\nu} {e_\alpha}^a {e_\beta}^b {e_\lambda}^c \\ &= \int \epsilon^{\mu\nu\lambda}\epsilon_{abc}\left(\frac{1}{2}R^{ab}{}_{\mu\nu}\right){e_\lambda}^c = \int \epsilon_{abc} R^{ab}\wedge e^c \end{aligned} \tag{9}$$

Collecting all formulae together we get a new version of the action

$$I[e^a,\omega^{ab}] = \frac{1}{16\pi G}\int \epsilon_{abc}\left(R^{ab} + \frac{1}{3l^2} e^a \wedge e^b\right)\wedge e^c. \tag{10}$$

Define the dual form

$$\omega^a = -\frac{1}{2}\epsilon^a{}_{bc}\omega^{bc}, \quad R^a = -\frac{1}{2}\epsilon^a{}_{bc}R^{bc} \tag{11}$$

Explicitly,

$$R^a = d\omega^a + \frac{1}{2}\omega^a{}_{bc}\omega^b \wedge \omega^c \tag{12}$$

We are now ready to connect to Chern-Simons or gauge theory. Let x be a complex number and let A^a and $\bar{A}^a$ to fields related to e and w by,

$$A^a = \omega^a + xe^a, \quad \bar{A}^a = \omega^a - xe^a \tag{13}$$

The term in the integral is

$$2e_a R^a + \frac{x^2}{3}\epsilon_{abc}e^a e^b e^c = \frac{1}{2x}\left(A_a dA^a + \frac{1}{3}\epsilon_{abc}A^a A^b A^c\right) - \frac{1}{2x}\left(\bar{A}_a d\bar{A}^a + \frac{1}{3}\epsilon_{abc}\bar{A}^a \bar{A}^b \bar{A}^c\right) + dB \tag{14}$$

where we ignore $\wedge$ symbol here, but the product of two forms is defined by wedge product. dB is a total derivative term and B is an auxiliary field. Thus we can rewrite the action in Chern-Simons form

$$\begin{aligned} I[g,\Gamma] &= iI[A] - iI[\bar{A}] \\ I[A] &= \frac{k}{4\pi}\int \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right) \end{aligned} \tag{15}$$

where $x = i/l$. The 1-form A^a is a $SL(2,\mathbb{C})$ Yang-Mills gauge field. For Minkowskian gravity $x = 1/l$ is real and the relevant group is $SO(2,1) \times SO(2,1)$. In [8], Witten wrote more things to explain the connection between 2+1 dimensional gravity to the gauge theory.

3.2.3 ADM Formulation

Let us consider an Arnowitt-Deser-Misner decomposition of the metric $g_{\mu\nu}$,

$$ds^2 = N^2 dt^2 - g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right) \tag{16}$$

for which the action takes the usual form

$$I_{\text{grav}} = \int d^3x\sqrt{-g}(R - 2\Lambda) = \int dt \int_\Sigma d^2x \left(\pi^{ij}\dot{g}_{ij} - N^i \mathscr{H}_i - N\mathscr{H}\right) \tag{17}$$

Here the canonical momentum π^{ij} is given by

$$\pi^{ij} = \sqrt{g}\left(K^{ij} - g^{ij}K\right), \tag{18}$$

where K^{ij} is the extrinsic curvature of the surface $t =$ const, and the momentum and Hamiltonian constraints in 2+1 dimesnions are

$$\mathscr{H}_i = -2\nabla_j \pi^j{}_i, \quad \mathscr{H} = \frac{1}{\sqrt{g}}g_{ij}g_{kl}\left(\pi^{ik}\pi^{jl} - \pi^{ij}\pi^{kl}\right) - \sqrt{g}\,(R - 2\Lambda) \tag{19}$$

In this formalism, we could also solve the 2+1 dimensional gravity.

4 Geometry Background

What geometry structure can admit (2+1)D or 3D gravity? This is actually a very interesting and fundamental problem. In [7], Maloney and Witten somehow solved this problem. When we assume that M is a smooth and complete three dimensional manifold with Σ for its conformal boundary, which is a Riemann surface of genus 1. And Σ is the only "end" of M. They claim the possible choices of M and discuss their contributions to partition function. In order to be clear what we are talking about, we decide to give a clear definition of the conformal boundary first[5],

Definition 2. *(Envelopment.) A (conformal) envelopment of M is an open conformal embedding $i : M \hookrightarrow M_0$ in some strongly causal spacetime M_0.*

Then, the conformal completion of M with respect to i is the closure $\bar{M}_i := i(\bar{M}) \subset M_0$, and the conformal boundary is the topological one $\partial_i M := i(\bar{M}) \backslash i(M)$.

4.1 Possible Choice of M

The automorphism group of AdS_3 is $SO(3,1)$, which is the same as $SL(2,\mathbb{C})/\mathbb{Z}_2$. The metric on a dense open subset of AdS_3 as

$$ds^2 = \frac{|dz|^2 + du^2}{u^2}, u > 0, z \in \mathbb{C} \tag{20}$$

If we combine the (z, u) coordinates into a single quaternion $y = z + ju$, the action of an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ can be written succinctly as

$$y \to (ay + b)(cy + d)^{-1} \tag{21}$$

In this expression the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially. In general, any classical solution M of three-dimensional gravity with negative cosmological constant takes the form AdS_3'/Γ, wher Γ is a discrete subgroup of $SO(3,1)$ and AdS_3' is the part of AdS_3 on which Γ acts discretely.

Then we could construct the conformal boundary of M. First of all, the conformal boundary of AdS_3 is a two-sphere, which one can think of as $\mathbb{CP}^1$, acted on by $SL(2,\mathbb{C})$ in the usual way. This $\mathbb{CP}^1$ can be regarded as the complex z-plane in (20) at $u \to 0$ plus a point at infinity. So $SL(2,\mathbb{C})$ acts on this $\mathbb{CP}^1$ in the familiar fashion

$$z \to \frac{az + b}{cz + d} \tag{22}$$

To construct the conformal boundary of $M = AdS_3'/\Gamma$, we use the following procedure

1. Throws away a certain subset of $\mathbb{CP}^1$ in a neighborhood of which the discrete group Γ badly, which is a closed subset, so its complement is an open subset $U \subset \mathbb{CP}^1$.

   Remark 2. *Here we don't define the topology clearly. I think it is just the usual topology we used.*

2. The discrete group acts freely on U, and the conformal boundary of M is the quotient $\Sigma = U/\Gamma$.

3.

   Lemma 1. *if $x_0 \in X$, $y_0 \in Y$, then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0) \tag{23}$$

If we assume that Σ is of genus 1. Σ is topologically a torus, so its fundamental group is $\pi_1(\Sigma) = \mathbb{Z}\oplus\mathbb{Z}$.. If follows from the fact that $\Sigma = U/\Gamma$ the fundamental group of U is a subgroup of the fundamental group of Σ. The subgroup of $\mathbb{Z} \times \mathbb{Z}$ are of three types.

(i) $\pi_1(U)$ may be a subgroup of $\pi_1(\Sigma)$ of finite index, isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. A special case is $\pi_1(U) \cong \pi_1(\Sigma)$.

(ii) $\pi_1(U) \cong \mathbb{Z}$.

(iii) $\pi_1(U)$ may be trivial.

4. Different setting will give different geometry of U. But one can be sure is that the case (i) is impossible. In case (i), U is a finite cover of Σ, and therefore is itself a Riemann surface of genus 1. However, a Riemann surface of genus 1 is not isomorphic to an open subset of $\mathbb{CP}^1$. So we just need to discuss the two cases (ii) and (iii).

4.2 Case (iii) Geometry

It seems that case (iii) is simpler so we talk about it first. U is the universal cover of Σ, and so is isomorphic to $\mathbb{C}$(or $\mathbb{R}^2$). U is two dimensional. The holomorphic structure of $\mathbb{C}$ is unique up to isomorphism. $\mathbb{C}$ is isomorphic holomorphically to an open subset of $\mathbb{CP}^1$ in essentially only one way: it is the complement of one point, say the point at $z = \infty$. The subgroup of $SL(2, \mathbb{C})$ that leaves fixed the point at infinity consists of the triangular matrices

$$\begin{pmatrix} \lambda & w \\ 0 & \lambda^{-1} \end{pmatrix} \tag{24}$$

Proof 1. *Noticed that the action of the element of $SL(2, \mathbb{C})$ is*

$$z \to \frac{az + b}{cz + d} \tag{25}$$

In order to keep that $\lim_{z\to\infty}(az + b)/(cz + d) = \infty$, we can see that c must be 0. And the condition for the element is $ad - bc = ad = 1$, so $a = \lambda$ and $d = \lambda^{-1}$.

The point z is transform as $z \to \lambda^2 z + \lambda w$. In order to keep the fundamental group of Σ is $\mathbb{Z} \oplus \mathbb{Z}$, and U is simply connected. Γ must be isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Any discrete group of triangular matrices that is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ is generated by two strictly triangular matrices,

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \tag{26}$$

where a, b are complex number linearly independent over $\mathbb{R}$. Up to conjugacy by a diagonal matrix, the only invariant of such a subgroup is the ration b/a. Therefore, we can reduce to the case $a = 1$ and $b = \tau$. The group action is

$$z \to z + m + n\tau \quad m, n \in \mathbb{Z}, \tag{27}$$

where we can set $Im\tau > 0$ because we can take $b = -b$ which does not affect the group generated by these two matrices. In these group action, the quotient space is a genus 1 surface Σ with an arbitrary τ-parameter. Now the three-manifold is $M = \mathrm{AdS}_3'/\Gamma$ is given by the metric 20 and the identification 27.

Unfortunately though this case looks nice, but that's not what we want. Because we have two "end" of the manifold, instead of one "end". In addition to the "end" at $u = 0$, which is the one required by our boundary condition, M also has a second "end" at $u = \infty$. The second end of M is one at which the metric collapses to zero (rather than blowing up, as it does for $u \to 0$). An end of this kind is known as a "cusp".

Remark 3. *Actually we want to give a strict definition to the "end" and the "cusp geometry", but we found it is not very easy to give a general and satisfying definition. We suppose that the "end" should be defined by the casual curve or time-like curve.*

Remark 4. *Acutually there's no reasons to omit the cusp geometry, physictists just choose them by intuition. Because they found it is difficult the compute the partition function near the cusp, so they ignore it.*

4.3 Case (ii) Geometry(Semiclassical Geometry)

The last case we left is that the fundamental group of U is $\mathbb{Z}$. This means that topologically U is $\mathbb{R} \times S^1$. The holomorphic structure of U is then uniquely determined to be that of the z-plane minus a point, which we may as well take to be the point to be $z = 0$. The subgroup of triangular matrices that preserve the point $z = 0$ is simply the group of diagonal matrices, which means that Γ is a discrete subgroup of the group of diagonal matrices.

Theorem 2. Γ *cannot be a finite group.*

Proof 2. *If Γ is a finite group, the first Betti number of $\Sigma = U/\Gamma$ would be 1, while the first Betti number of torus is 2, which doesn't make sense.*

So Γ has the following possible choice:

(a) Γ is isomorphic to $\mathbb{Z}$, generated by a matrix of the form

$$W = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \in SL(2, \mathbb{C}) \tag{28}$$

because we could exchange the two eigenvalues so we can assume that $|q| < 1$.

(b) Γ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_n$, generated by W together with

$$Y = \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{-\frac{2\pi i}{n}} \end{pmatrix} \tag{29}$$

where n is an integer.

4.3.1 $\Gamma \cong \mathbb{Z}$

Let us first consider the case (a) when $\Gamma \cong \mathbb{Z}$. Then $\Sigma = U/\Gamma$ is obtained from the complex z-plane by throwing away the point $z = 0$ and dividing by the group generated by W. It is convenient to write $z = \exp(2\pi i w)$, and $w \in \mathbb{C}$. So that $w \sim w + 1$ and W acts by

$$w \to w + \frac{\log q}{2\pi i} \tag{30}$$

The quotient of the w-plane is a Riemann surface of genus 1, as required. The complex modulus of this surface is $\tau = \frac{\log q}{2\pi i}$, i.e. it is given by $q = e^{2\pi i\tau}$. But the Riemann surface should be defined up to $\tau \to (a\tau + b)/(c\tau + d)$ with integers a, b, c, d obeying $ad - bc = 1$. Therefore,

$$q = \exp(2\pi i(a\tau + b)/(c\tau + d)) \tag{31}$$

for such a, b, c, d. But the degrees of freedom is not the choice of a, b, c, d. First,, an overall sign change of a, b, c, d does not affect q or the associated three-manifold. Second, once c and d are given, a and b are uniquely determined by $ad - bc = 1$ up to shifts of the form $(a, b) \to (a, b) + t(c, d), t \in \mathbb{Z}$. So the possible three-manifold really only depend on the choice of the pair c, d of relatively prime integers, up to sign.

Remark 5. *I think the logic here is that c, d can determine q, then determine τ, then determine the modulus radius of the Riemann surface. Different complex modulus can give different geometry.*

Definition 3. *$M_{c,d}$ is a three-manifold which is defined in the way above.*

The simplest case is $M_{0,1}$. If we identity the real one-parameter subgroup $\mathrm{diag}(e^b, e^{-b})$ as the group of time translation, the AdS_3 metric is

$$ds^2 = \cosh^2 r dt^2 + dr^2 + \sinh^2 r d\phi^2 \tag{32}$$

with $-\infty < t < \infty$, $0 \le r < \infty$, and $0 \le \phi < 2\pi$. Described in this coordinate system is the subset $\mathrm{AdS}_3' \subset \mathrm{AdS}_3$ on which Γ acts nicely; its topology is $D \times \mathbb{R}$, where D is a two dimensional disk. Conformal infinity is at $r = \infty$. The element $\mathrm{diag}(e^b, e^{-b})$ will give $t \to t + b$. The group of spatial rotations is the one-parameter group $\mathrm{diag}(e^{i\theta}, e^{-i\theta})$, acting by $\phi \to \phi + \theta$.

The group element W therefore generates a combined time-translation and spatial rotation. To explicitly divide by W, we "cut" AdS_3' at times $t = 0$ and $t = 2\pi\mathrm{Im}\tau$. Then we glue together the top and bottom of the region $0 \le t \le 2\pi\mathrm{Im}\tau$ after making a spatial rotation by an angle $2\pi\mathrm{Re}\tau$. The resulting spacetime is indeed topologically $D \times S^1$. The other manifold $M_{c,d}$ are obtained from $M_{0,1}$ by modular transformations, that is, by diffeomorphisms that act non-trivially on the homology of Σ.

4.3.2 $\Gamma \cong \mathbb{Z} \times \mathbb{Z}_n$

Now let us consider Γ that has two generators. We may assume that $n = 2m$ is even, since the element $\mathrm{diag}(-1, -1)$ acts trivially on $\mathbb{CP}^1$. We may also assuume $m > 1$ because $m = 1$ is trivial. What we get is simply a three dimensional space of the form $M_{c,d}/\mathbb{Z}_m$.

Remark 6. *I think the logic here is as follows. What we got form this quotient is*

$$AdS_3'/\Gamma \sim AdS_3'/(\mathbb{Z} \times \mathbb{Z}_n) \sim M_{c,d}/Z_n \tag{33}$$

and it is easy to see that if n is odd, because $M_{c,d}$ is already $\mathbb{Z}_2$ graded, so

$$M_{c,d}/\mathbb{Z}_{2m+1} \sim M_{c,d}/\mathbb{Z}_{2m} \tag{34}$$

so we can assume that n is even.

The conformal boundary is still a Riemann surface of genus 1, and by changing q we can adjust its modular parameter as we wish. The conformal boundary is still of genus 1, but the group element acts on AdS_3' with fixed points, meaning that $M_{c,d}/\mathbb{Z}_m$ has orbifold singularities. The fixed points are of codimension 2.

So far we have already figure out the geometry of 2+1 dimensional gravity. Of course it is not quite satisfying. Based on this, we continue to discuss several topics.

5 Partition Function

5.1 Preliminaries

Before we start to discuss the partition function, we need to be clear what we are talking about and some basic properties.

5.1.1 Definition of partition function

There is also a conserved angular momentum J which generates a rotation at infinity of the asymptotically AdS_3 spacetime and commutes with H. Consequently, one can introduce an additional parameter θ and try to compute a more general partition function

$$Z(\beta,\theta) = \mathrm{Tr}\exp(-\beta H - i\theta J) \tag{35}$$

The text before we don't discuss the exact meaning of τ. According to the standard recipe, the integral is taken over Euclidean three-geometries that are conformal at infinity to a two-torus Σ with modular parameter $\tau = \theta/2\pi + i\beta$.

5.1.2 Properties of partition function

1. Because the manifolds $M_{c,d}$ are all diffeomorphic to each other, the functions $Z_{c,d}(\tau)$ can all be expressed in terms of any one of them, say $Z_{0,1}(\tau)$,by a modular transformation. The formula is simply
$$Z_{c,d}(\tau) = Z_{0,1}\left(\frac{a\tau+b}{c\tau+d}\right) \tag{36}$$
where a,b are integers and $ad-bc=1$. The whole sum of the partition function is to
$$Z(\tau) = \sum_{c,d} Z_{c,d}(\tau) = \sum_{c,d} Z_{0,1}\left(\frac{a\tau+b}{c\tau+d}\right) \tag{37}$$

2. This partition function is naturally computed via a Euclidean path integral. The path integral in 2+1 dimensional spacetime($M_{c,d}$) has a simple semiclassical meaning, since it may be interpreted in terms of Hamiltonian time evolution. A state is prepared at time zero and propagates a distance $\beta = 2\pi\mathrm{Im}\tau$ forward in Euclidean time. In this process, the state vector is multiplied by the time evolution operator $\exp(-\beta H)$, where H is the Hamiltonian.. Then, after a spatial rotation by an angle $\theta = 2\pi\mathrm{Re}(\tau)$, which acts on the state by $\exp(-i\theta J)$, we glue the top and bottom of the figure, which results in taking the inner product of the final state with the initial state. The whole operation

gives the trace $\mathrm{Tr}\exp(-\beta H - i\theta J)$ defined in the Hilbert space of pertubative fluctuations around AdS_3.

In the most naive semiclassical approximation, $Z_{0,1}(\tau)$ is just $\exp(-I)$, where I is the classical action.

Remark 7. *Here we just use the Euclidean path integral instead of Feynman path integral.*

In computing this action, one cannot just evaluate the action

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R + \frac{2}{l^2}\right) \tag{38}$$

for the solution

$$ds^2 = \cosh^2 r dt^2 + dr^2 + \sinh^2 r d\phi^2 \tag{39}$$

. such a computation would give an infinite answer, coming from the boundary $r \to \infty$. The full action includes the Gibbons-Hawking boundary term, which has the opposite sign of the Einstein-Hilbert term. This extra term removes the divergence, and one arrives at a finite (negative) answer for the action of $M_{0,1}$[4].

$$I = -4\pi k \mathrm{Im}\tau \tag{40}$$

where $k = l/16G$. Therefore, in this approximation, with $q = e^{2\pi i\tau}$

$$Z_{0,1}(\tau) = \exp(-4\pi k \mathrm{Im}\tau) = \exp\left(-4\pi k\frac{\tau - \bar{\tau}}{2i}\right) = (\exp(2\pi i\tau)\exp(-2\pi i\bar{\tau}))^{-k} = |q\bar{q}|^{-k} \tag{41}$$

Remark 8. *These results has a simple interpretation. Three dimensional pure gravity with the Einstein-Hilbert action is dual to a conformal field theory with central charge $c_L = c_R = 3l/2G = 24k$. Depending on how the theory is regularized, there may be quantum corrections to this formula, but they preserve $c_L = c_R$ because the Einstein-Hilbert theory is parity symmetric. So if we choose to parametrize the k to be $k = c_L/24 = c_R/24$. It is independent of the choice of formalism, ignoring the scheme for regularization. Let L_0 and $\tilde{L}_0$ be the Hamiltonians for left- and right- moving modes of the CFT. They are related to waht we have called H and J by*

$$\begin{aligned} H &= L_0 + \tilde{L}_0 \\ J &= L_0 - \tilde{L}_0 \end{aligned} \tag{42}$$

The CFT ground state has $L_0 = -c_L/24$, $\tilde{L}_0 = -c_R/24$, or in the present context $L_0 = \tilde{L}_0 = k$ so $H = -2k$ and $J = 0$. The ground state contribution tho the partition function is just $|q\bar{q}|^{-k}$. It does have good classical explanation.

5.2 Partition function compuation

5.2.1 $Z_{0,1}(\tau)$

Again, because we are solving the 3D model, naively we expect there the result is exact because there are no perturbative excitations. But this is not true. Brown and Henneaux said there must at least be states that correspond to Virasoro descendants[2] of the identity. If L_n and $\tilde{L}_n$ are the left- and right-moving modes of the Virasoro algebra, then a general such state is

$$\prod_{n=2}^{\infty} L_{-n}^{u_n} \prod_{m=2}^{\infty} \tilde{L}_{-m}^{v_m} |\Omega\rangle \tag{43}$$

with u_n and v_m are positive integers. Counting these states, we have the final partition function

$$Z_{0,1}(\tau) = |\bar{q}q|^{-k} \frac{1}{\prod_{n=2}^{\infty} |1-q^n|^2} \tag{44}$$

If we introduce Dedekind η, defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n) \tag{45}$$

Then we got the first important result of this part

$$Z_{0,1}(\tau) = \frac{1}{|\eta(\tau)|^2} |\bar{q}q|^{-(k-1/24)} |1-q|^2 \tag{46}$$

5.2.2 Computing the Sum over Geometries

The known contributions to the partition function of pure gravity in a spacetime asymptotic to AdS_3 come from smooth geometries $M_{c,d}$, where c and d are a pair of relatively prime integers. Their contribution from Brown-Henneaux excitations, is

$$Z(\tau) = \sum_{c,d} Z_{0,1}(\gamma\tau) \tag{47}$$

where

$$\gamma\tau := \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \tag{48}$$

The sum is over all relatively prime c and d with $c \geq 0$. Since $Z_{0,1}(\tau)$ is invariant under $\tau \to \tau + 1$ (Because $Z_{0,1}(\tau)$ is only relevant to $q = e^{2\pi i\tau}$.), so the summand is independent of the choice of a and b. And the sum over c, d should be thought of as a sum over the coset $PSL(2, \mathbb{Z})/\mathbb{Z}$ where $\mathbb{Z}$ is the subgroup of $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\pm 1\}$ that acts by $\tau \to \tau + n, n \in \mathbb{Z}$. Similar to String theory, the function $\sqrt{\mathrm{Im}\tau}|\eta(\tau)|^2$ is modular-invariant. We can therefore write $Z(\tau)$ as a much simple-looking Poincaré series.

$$Z(\tau) = \frac{1}{\sqrt{\mathrm{Im}\tau}|\eta(\tau)|^2} \sum_{c,d} \left(\sqrt{\mathrm{Im}\tau} |\bar{q}q|^{-k+1/24} |1-q^2| \right)_\gamma \tag{49}$$

where the notation $(\ldots)_\gamma$ represents the action of γ. If we expand $|1-q^2|$ as $1 - q - \bar{q} + \bar{q}q$, we see that we really need a sum of four Poincaré series, each of the form

$$E(\tau; n, m) = \sum_{c,d} \left(\sqrt{\mathrm{Im}\tau} q^{-n} \bar{q}^{-m} \right)_\gamma \tag{50}$$

with $n-m$ equal to 0 or ± 1. If we define $\kappa = n+m$ and $\mu = m-n$, and we use the fact that $\operatorname{Im}\gamma\tau = \operatorname{Im}\tau/|c\tau+d|^2$, then the basic Poincaré series can be written

$$E(\tau;\kappa,\mu) = \sqrt{\operatorname{Im}\tau}\sum_{c,d}|c\tau+d|^{-1}\exp\{2\pi\kappa\operatorname{Im}\gamma\tau + 2\pi i\mu\operatorname{Re}\gamma\tau\} \tag{51}$$

When $\kappa = 0$ and $\mu = 0$, this sum is a non-holomorphic Eiseinstein series of weight 1/2. If we omit the label τ, explicitly,

$$\begin{aligned} Z(\tau) = \frac{1}{\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2} & (E(2k-1/12,0) + E(2k+2-1/2,0) \\ & -E(2k+1-1/12,1) - E(2k+1-1/12,-1)) \end{aligned} \tag{52}$$

So now our task is to find a suitable way to compute such an Poincaré sum. But unfortunatly, the sum in 50 is divergent. It is not difficult to see that. First

$$\frac{a\tau+b}{c\tau+b} = \frac{a}{c} - \frac{1}{c(c\tau+d)} \tag{53}$$

If we write $\tau = x+iy$, explicitly we have

$$\begin{aligned} \operatorname{Im}\gamma\tau &= \frac{y}{(cx+d)^2+c^2y^2} \\ \operatorname{Re}\gamma\tau &= \frac{a}{c} - \frac{cx+d}{c((cx+d)^2+c^2y^2)} \end{aligned} \tag{54}$$

For example, if we assume that $\mu = 0$, the factor in the sum is

$$\lim_{c,d\to\infty}\exp\left\{2\pi\kappa\frac{y}{(cx+d)^2+c^2y^2}\right\} \sim 1 \tag{55}$$

So the behaviour of the sum is simply the

$$\sum_{c,d}|c\tau+d|^{-1} \to \infty \tag{56}$$

Luckily, this divergence is not too bad.

Remark 9. *I think the regularization scheme here is similar to the renormalization scheme in the field theory for the behaviour divergence is like the logarithm function.*

5.2.3 Regularization

We will show two possible schemes for the regularization and show the final results

- Using the Δ operator.

  Definition 4. *Delta operator on the upper half plane, which we call $\mathscr{H}$, there is a natural $SL(2,\mathbb{R})$-invariant Laplacian:*

  $$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{57}$$

And it is easy to verify that such an operator satisfies:

$$\Delta\sqrt{\mathrm{Im}(\gamma\tau)} = \frac{1}{4}\sqrt{\mathrm{Im}(\gamma\tau)} \tag{58}$$

Using this property, we can easily verify that

$$\left(\Delta - \frac{1}{4}\right) E(\tau;\kappa,\mu) \tag{59}$$

is convergent though $E(\tau,\kappa,\mu)$ itself is divergent. So $(\Delta - 1/4)E$ requires no regularization. Similarly, if we get $F = \sqrt{\mathrm{Im}\,\tau}|\eta|^2$, then $(\Delta - 1/4)(FZ)$ requires no regularization, where Z is what we need. The Laplacian Δ, acting on the Hilbert space of square-integrable $SL(2,\mathbb{Z})$-invariant functions on $\mathscr{H}$, has a continuous spectrum starting at $1/4$. It also has a discrete spectrum, but there are no discrete modes with eigenvalue $1/4$. So the operator $\Delta - 1/4$ is invertible acting on square-integrable functions. So once $(\Delta - 1/4)(FZ)$ is known, FZ is uniquely determined. Though this way sounds good(?), it does not give a convenient way to determine Z in practice. Now the second scheme is more achievable.

- ζ-function regularization scheme: In the present context, the analog of ζ-function regularization is to replace the Poincaré series by a more general one depending on a parameter s:

$$E(\tau;s,n,m) = \sum_{c,d}\left((\mathrm{Im}\,\tau)^s q^{-n}\bar{q}^{-m}\right)_\gamma \tag{60}$$

Since $(\mathrm{Im}\,\gamma\tau)^s = y^s/|c\tau+d|^{2s}$, the series $E(\tau;s,n,m)$ converges for $\mathrm{Re}\,s > 1$. If we can extend it to $s = 1/2$, we finish the regularization.

Remark 10. *The only problem with this approach is that the physical meaning of the parameter s in three-dimensional gravity is unclear; hence, it is not clear a priori that the analytic continuation will give the right answer.*

In terms of $\kappa = m+n$, $\mu = m-n$, the regularized Poincaré series is

$$E(s,\kappa,\mu) = \sum_{c,d}\frac{y^s}{|c\tau+d|^{2s}}\exp\{2\pi\kappa\,\mathrm{Im}\,\gamma\tau + 2\pi i\mu\,\mathrm{Re}\,\gamma\tau\} \tag{61}$$

5.2.4 Final Result

The computation is ugly, long and difficult so we don't want to show the derivation here. The detailed steps are all in [7],

$$\begin{aligned} Z = Z_{0,1} + \frac{1}{|\eta|^2}\Bigg(&-6 + \frac{(\pi^3-6\pi)(11+24k)}{9\zeta(3)}y^{-1} \\ &+\frac{5(53\pi^6-882\pi^2)+528(\pi^6-90\pi^2)k+576(\pi^6-90\pi^2)k^2}{2430\zeta(5)}y^{-2}+\mathscr{O}(y^{-3})\Bigg) \end{aligned} \tag{62}$$

The additional contributions to the partition function in this expression have two notable features. First, and most importantly, they are not zero. Thus, as described above, the partition function truly cannot be represented as a sum of exponential. Second, they differ qualitatively from the leading y^0 term: the additional coefficients appearing here are positive and irrational, rather than negative and integer.

5.3 Possible Interpretation

So far we have analyzed the sum of known contributions to the partition function of pure three-dimensional gravity. The resulting partition function (62) cannot be interpreted as $\operatorname{Tr}\exp(-\beta H)$ for any Hilbert space operator H. The most straight forward interpretation is to take the result as face value. Three-dimensional pure gravity may not exist as a quantum theory; to get a consistent theory, it may be necessary to complete it by adding additional degrees of freedom, and there may be no canonical way to do this. In other words, this theory has inherent difficulties, we failed to solve them.

The other possibility is that some unknown contributions to the partition function should be added to the terms that we have evaluated. Here all sorts of speculations are possible. All of them are just qualitatively explain what happens.

5.3.1 Cosmic Strings

Known consistent models of 2+1 dimensional graivty with negative cosmological constant arise from string theory. For example, a famous class of models comes form Type IIB superstring theory on $\mathrm{AdS}_3 \times S^3 \times X$, where X is either K3 surface or a four-torus,

Definition 5. *K3 surface is a compact connected complex manifold of dimension 2 with trivial canonical bundle and irregularity zero. Together with two-dimensional compact complex tori, K3 surfaces are the Calabi–Yau manifolds of codimension 2.*

It is always possible to have domain walls across which the fluxes jump. The domain walls are constructed from suitably wrapped branes. In 2+1 dimensions, a domain wall has a 1+1-dimensional world-volume and so can be viewed as a cosmic string. The existence of these comic strings makes the models much more unified, as regions with different fluxes can appear as different domains in a single spacetime.

In the usual $\mathrm{AdS}_3 \times S^3 \times X$ models with Type IIB supersymmetry, the existence of "long string" is possible. These are strings that can expand to an arbitrarily size at only a finite cost of energy. When long strings exist, the energy spectrum is continuous above a certain minimum energy, and the partition function $\operatorname{Tr}\exp(-\beta H)$ therefore diverges for all β. Since well-established models of three-dimensional gravity have such cosmic strings, perhaps they also present in minimal three dimensional gravity.

5.3.2 Doubled Sum over Geometries

We will now describe a quite different scenario. To motivate it, we return to the formula for the classical action of the basic spacetime $M_{0,1}$:

$$I = -4\pi k \operatorname{Im}\tau = 2\pi i k(\tau - \bar{\tau}) \tag{63}$$

So the classical approximation can be written as $\exp(-I) = q^{-k}\bar{q}^{-k}$. We notice that this is locally the product of a holomorphic function of k and an antiholomorphic function, and is globally such a product if k is an integer(ensuring q^{-k} is single-valued). The one-loop correction preserves this factorized form, and therefore the formula for the exact partition function $Z_{0,1}$ associated with $M_{0,1}$ has the same properties, in fact $Z_{0,1} = F_k(q)F_k(\bar{q})$, with

$$F_k(q) = q^{-k} \prod_{n=2}^{+\infty} (1-q^n)^{-1} \tag{64}$$

Whether the whole partition function has the same holomorphic factorized meaning? The answer seems yes. The gauge theory description of three-dimensional graivty gives a natural explanation of this. With negative cosmological, in Lorentz signature, the gauge group is $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$; the theory is a product of two decoupled $SL(2,\mathbb{R})$ theories, associated respectively with left- and right- moving modes in the boundary CFT, and this corresponds to holomorphic factorization in the Euclidean form of the theory. So we suggest that the exact partition function of pure three-dimensional gravity is holomorphically factorized. Now let us discuss holomorphic factorization in view of the sum over geometries. Associated to an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $SL(2,\mathbb{Z})$ is a classical spacetime $M_{c,d}$. Its action is obtained by applying a modular transformation to

$$I_\gamma(\tau) = 2\pi i k(\gamma\tau - \gamma\bar{\tau}) \tag{65}$$

As usual, $\gamma\tau = (a\tau + b)/(c\tau + d)$, $\gamma\bar{\tau} = (a\bar{\tau} + b)/(c\bar{\tau} + d)$. The partition function of the manifold $M_{c,d}$ is

$$Z_{c,d} = F_k(q)|_\gamma F_k(\bar{q})|_\gamma \tag{66}$$

So the sum over geometries is

$$Z = \sum_{\gamma \in \mathscr{W}} F_k(q)|_\gamma F_k(\bar{q})|_\gamma \tag{67}$$

Where the holomorphic factorization is lost. Here $\mathscr{W}$ represents the possible modular transformation. What could we do to restore holomorphic factorization? We could slightly change the partition function. If we formally introduce separate topological sums for holomorphic and antiholomorphic variables, defining an extended partition function

$$\hat{Z} = \sum_{\gamma,\gamma' \in \mathscr{W}} F_k(q)|_\gamma F_k(\bar{q})|'_\gamma \tag{68}$$

the holomorphic factorization is restored.

$$\hat{Z} = \left(\sum_{\gamma \in \mathscr{W}} F_k(q)|_\gamma \right) \left(\sum_{\gamma' \in \mathscr{W}} F_k(\bar{q})|_{\gamma'} \right) \tag{69}$$

Remark 11. *Could we define a new mathematical object similar to matrix, but the dimension is infinite and the index is the transformation, then the real partition function is similar to be the trace of such an matrix.*

What is the classical action corresponding to a given term in the above sum? The answer is easy

$$I_{\gamma.\gamma'} = 2\pi i(\gamma\tau - \gamma'\bar{\tau}) \tag{70}$$

For $\gamma \neq \gamma'$, this action is not real. The most easiest way to interpret it is to see this action as the action of a complex-valued solution of Einstein equations. Our first step is to find the possible manifold that can admit the three-dimensional gravity, does the manifold which doesn't admit the three-dimensional graivty that has contribution to the partition function? What happen if the metric is an symmetric complex-valued matrix?

6 Gravity Quantization

7 2+1 D Black Hole

8 Relation to the Gauge theory

9 The Hawking-Page Phase Transition

10 2+1 D Supergravity

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