The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole

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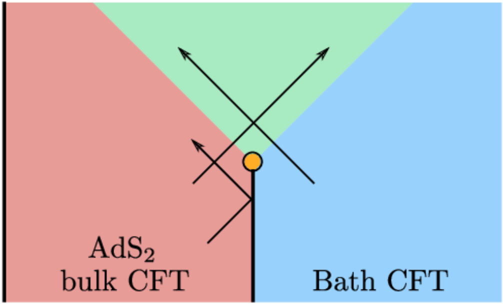
Contents I

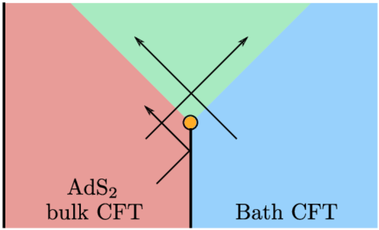
Bulk quantum fields are often said to contribute to the generalized entropy $S_{\text{gen}} = \frac{A}{4G_N} + S_{\text{bulk}}$ only at $O(1)$. But in the context of evaporating black holes, the quantum correction is important. The authors examine the effect of such bulk quantum on quantum extremal surfaces(QESs) and resulting entanglement wedge in a simple two-boundary 2 dimensional bulk system defined by Jackiw-Teitelboim gravity coupled to a 1+1 CFT.

This article is trying to answer the following questions:

- Can we calculate the generalized entropy more precisely in this specific setting? How important is the quantum correction?
- if we can do it, what's the physical intepretation of this generalized entropy?
- How the entropy is related to the quantum extremal surface(What's that?)? And we want to know the time evolution of the quantum extremal surface.



We consider a standard two-sided AdS_2 black hole in Jackiw-Teitelboim(JT) gravity coupled to a 1+1 CFT in the Hartle-Hawking state. With reflecting boundary conditions at AdS infinity. But at a finite time we couple the right boundary of our system to an auxiliary system B, which is a bath.



We assume that

- B is a copy of the same 1+1 CFT on the right half of Minkowski space
- B begins in its own vacuum.

The couling process is described as follows. At first the boundary is totally reflected. At some point the right boundary is fully transparent(the boundaries of these two system glues together). This leads to evaporation on the right of the two-sided AdS_2 black hole.

Model Setting

we will focus on some physical quantities. First is the generalized entropy and its evolution in this process. Second we introduce the quantum extremal surface, which is related to the entropy, to see the evolution of proxy QESs.

Holographic Hayden-Preskill

The information thrown into the black hole would be recoverable from the radiation in a relatively short time compared to the black hole lifetime. Hayden-Preskill protocol predicts an interesting timescale for when the message appears in the Hawking radiation, called the scrambling time(Hayden-Preskill time)

$$t_{HP} \sim \frac{1}{T} \log S_{BH} \tag{1}$$

where the T is the temperature and S_{BH} is the entropy of the black hole respectively. The scrambling time for near extremal black hole is controlled by the ADM energy E[1] above the ground state and the energy of the perturbation δE thrown into the black hole, assumed to be much smaller than E, as

$$t_{scr} = \alpha_S \frac{\beta}{2\pi} \log \frac{E}{\delta E} \tag{2}$$

[1] The ADM energy is a special way to define energy in GR which is computed as the strength of the gravitational field at infinity. We don't have to worry about it.

Holographic Hayden-Preskill

The result indicated the Hayden-Preskill time should be

$$t_{HP} = \alpha_{HP} \frac{\beta}{2\pi} \log(S - S_0) \tag{3}$$

for a small message with $\delta E \sim E/(S - S_0)$. One of the main achievements of this paper is to determine the factor α_{HP} for systems dual to JT gravity.

Jackiw-Teitelboim(JT) gravity is a model of 2 dimensional dilaton gravity. The dynamics of this theory are governed by the Lorentzian action $I = I_0 + I_G + I_M$ with

$$
\begin{aligned}
I_0 &= \frac{\phi_0}{16\pi G_N}\left[\int_M \mathrm{d}^2x\sqrt{-g}R + 2\int_{\partial M} K\right] \\
I_G &= \frac{1}{16\pi G_N}\left[\int_M \mathrm{d}^2x\sqrt{-g}\phi(R+2) + 2\int_{\partial M}\phi_b K\right] \\
I_M &= I_{CFT}[g]
\end{aligned}
\tag{4}
$$

where the I_0 is a pure topological term related to the Euler characteristic number χ. The remainling gravitational dynamics are governed by the action I_G. The metric g appears in the action, we have to claim what it is. The two-dimensional metric is locally AdS_2, In Poincaré coordinates this is

$$
ds^2 = \frac{-dt^2 + dz^2}{z^2} = \frac{-4dx^+dx^-}{(x^+ - x^-)^2},\ x^\pm = t \pm z \tag{5}
$$

In order to be more explicit, we write the metric in matrix form

$$g = \begin{pmatrix} g_{x^+x^+} & g_{x^+x^-} \\ g_{x^-x^+} & g_{x^-x^-} \end{pmatrix} = \begin{pmatrix} 0 & \frac{-2}{(x^+-x^-)^2} \\ \frac{-2}{(x^+-x^-)^2} & 0 \end{pmatrix} \tag{6}$$

The variation of the metric component yields the field equation

$$\begin{aligned} 2\partial_{x^+}\partial_{x^-}\phi + \frac{4}{(x^+-x^-)^2}\phi &= 16\pi G_N T_{x^+x^-} \\ -\frac{1}{(x^+-x^-)^2}\partial_{x^+}\left((x^+-x^-)^2\partial_{x^+}\phi\right) &= 8\pi G_N T_{x^+x^+} \\ -\frac{1}{(x^+-x^-)^2}\partial_{x^-}\left((x^+-x^-)^2\partial_{x^-}\phi\right) &= 8\pi G_N T_{x^-x^-} \end{aligned} \tag{7}$$

Now all these things are classical, we work in the limit where the gravitational sector can be treated semiclassically, so we may replace the stress tensors with their expectations values, $T_{ab} = \langle T_{ab} \rangle$.

There is a reparametrization between the bulk Poincaré time near the boundary t and the physical time u. The location of this physical boundary is specified by the boundary condition on the bulk fields

$$g_{uu}|_{bdy} = \frac{1}{\epsilon^2} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial u}\frac{\partial x^\beta}{\partial u} = \frac{1}{z^2}\left[-\left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}u}\right)^2\right] \tag{8}$$

and

$$\phi = \phi_b = \frac{\bar{\phi}_r}{\epsilon} \tag{9}$$

This equality comes from the assumption that we are just interested in $\phi_b \sim 1/\epsilon$ with a fixed constant $\bar{\phi}_r$. Now let us reconsider the action in (4),

$$I_G = \frac{1}{16\pi G_N}\left[\int_M \mathrm{d}^2x\sqrt{-g}\phi(R+2) + 2\int_{\partial M}\phi_b K\right] \tag{10}$$

Since the dilaton ϕ appears linealy in the first term and without derivatives, it is basically just a Largrange multiplier. Integrating it out leads to the constraint $R = -2$, which just tells us that the metric is AdS_2. The action is therefore a boundary term

$$S_G = \frac{1}{8\pi G_N} \int_{\partial M} \sqrt{-g_{bdy}} du \phi_b K = \frac{1}{8\pi G_N} \int_{\partial M} \frac{\mathrm{d}u}{\epsilon} \frac{\bar{\phi}_r}{\epsilon} K \tag{11}$$

The boundary terms involve the trace of the extrinsic curvature(second fundamental form). For codimension-one boundaries, it is given by

$$K = -\frac{h(T, \nabla_T n)}{h(T, T)} \tag{12}$$

And we could compute

$$T^a = (t', z'), n^a = \frac{z}{\sqrt{t'^2 + z'^2}}(-z', t') \tag{13}$$

where $z = \epsilon\sqrt{t'^2 + z'^2} = \epsilon t' + O(\epsilon^3)$

The result is

$$K = \frac{t'(t'^2 + z'^2 + z'z'') - zz't''}{(t'^2 + z'^2)^{3/2}} = 1 + \epsilon^2\{f(u), u\} + O(\epsilon^4) \tag{14}$$

In order to make the notification the same as the paper, we set $t(u) = f(u)$. The $\{f(u), u\}$ is Schwarzian derivatives, which is defined as

$$\{f(u), u\} = \frac{2f'f''' - 3f''^2}{2f'^2} \tag{15}$$

Now we could simplify the action

$$S_G = \frac{\bar{\phi}_r}{8\pi G_N} \int_{\partial M} \frac{\mathrm{d}u}{\epsilon^2} \left(1 + \epsilon^2\{f(u), u\}\right) \tag{16}$$

By generously neglecting the field independent divergent term, we arrive at the final result

$$S_G = \frac{\bar{\phi}_r}{8\pi G_N} \int \mathrm{d}u\{f(u), u\} \tag{17}$$

$t = f(u)$ is a diffeomorphism giving Poincaré time t in terms of boundary proper time u. This is the Schwarzian action, which is invariant under $SL(2,\mathbb{R})$(Its group is Diff$(\partial M)/SL(2,\mathbb{R})$). It is easy to see that the Noether charge under physical time translations $u \to u + \delta u$

$$E(u) = \frac{\partial \mathcal{L}}{\partial \dot{u}} \dot{u} - \mathcal{L} = -\frac{\bar{\phi}_r}{8\pi G_N} \{f(u), u\} \tag{18}$$

Using the diffeomorphism f, we could construct natural coordinates $y^{\pm}$ defined by $x^{\pm} = f(y^{\pm})$, in which the metric becomes

$$ds^2 = -\frac{f'(y^+) f'(y^-) \mathrm{d}y^+ \mathrm{d}y^-}{(f(y^+) - f(y^-))^2} \tag{19}$$

Solving the equation of motion (7) of dialtion, the vacuum solution(T = 0)[2] is

$$\phi = 2\bar{\phi}_r \frac{1 - (\pi T_0)^2 x^+ x^-}{x^+ - x^-} \tag{20}$$

[2] The proof is in another article arXiv:1402.6334.

The associated reparametrization is

$$f(u) = \frac{1}{\pi T_0} \tanh\left(\pi T_0 u\right) \tag{21}$$

Substitute it into the metric, dilaton and the Noether charge,

$$ds^2 = \frac{-4(\pi T_0)^2 \mathrm{d}y^+ \mathrm{d}y^-}{\sinh^2\left[\pi T_0 (y^+ - y^-)\right]} \tag{22}$$

$$\phi = 2\bar{\phi}_r \pi T_0 \coth\left[\pi T_0 (y^+ - y^-)\right] \tag{23}$$

$$E(u) = -\frac{\bar{\phi}_r}{8\pi G_N}\left\{\frac{1}{\pi T_0}\tanh\left(\pi T_0 u\right), u\right\} = \frac{\pi \bar{\phi}_r}{4 G_N} T_0^2 \equiv E_0 \tag{24}$$

We finish the calculation in the static case, before the coupling of the auxiliary system.

Actually we are ready to answer the questions raised at the beginning. But we want to give some consequences pertinent to the bulk dynamics, namely the resulting energy-momentum transfer into the black hole and give some general discussion.

- First, an important transient effect occurs when coupling the right boundary to this external sysetm is that there will be an initial injection of positive energy into the black hole. We denote this initial positive energy increase as E_S and find a lower bound on its value.
- After this initial 'shock' of energy, the energy of the black hole begins to be transferred into the bath via the Hawking radiation.
- In the later discussion, we would see that the ingoing stress-tensor expectation value vanishes in the flat metric, which via the conformal anomaly gives a flux of negative energy in the physical metric

$$\langle T_{x^- x^-}(x^-)\rangle = E_S \delta(x^-) - \frac{c}{24\pi}\{y^-, x^-\}\Theta(x^-) \tag{25}$$

$$\langle T_{x^-x^-}(x^-)\rangle = E_S\delta(x^-) - \frac{c}{24\pi}\{y^-, x^-\}\Theta(x^-) \tag{26}$$

We could build the balance equation of $E(u)$ by varying the action with respect to u,

$$\partial_u E(u) = f'(u)^2(T_{x^-x^-} - T_{x^+x^+}) \tag{27}$$

Recall we have already calculated the $E(u)$, then we have the following equation

$$-\frac{\bar{\phi}_r}{8\pi G_N}\partial_u\{f(u), u\} = \frac{c}{24\pi}\{f(u), u\} \Rightarrow f(u), u \propto e^{-ku} \tag{28}$$

where

$$k = \frac{c}{12}\frac{4G_N}{\bar{\phi}_r} \ll 1 \tag{29}$$

The whole expression of $E(u)$ is

$$E(u) = \Theta(-u)E_0 + \Theta(u)E_1 e^{-ku} \tag{30}$$

where $E_1 = E_0 + E_S$. For small positive time, we have a black hole with new temperature T_1 satisfying

$$E_1 \equiv \frac{\pi\bar{\phi}_r}{4G_N}T_1^2 \tag{31}$$

Where is similar to the expression in static case. The new horizon($u \to \infty$) is at $x^+ = t_\infty = \lim_{u\to\infty} f(u)$. We have already known the diffeomorphism for $f(u)$(21), but we don't know the diffeomorphism when $u > 0$. It is easy to see that $\lim_{u\to\infty} f(u) = 1/\pi T_0$, we require that the $t_\infty < \frac{1}{\pi T_0}$ and use this inequality to find a lower bound for E_S.

First we need to sovle the new diffeomorphism suitable for $u > 0$,

$$\{f(u), u\} = 2(\pi T_1)^2 e^{-ku}, \quad f(0) = 0, f'(0) = 1, f''(0) = 0, \tag{32}$$

The solution is

$$f(u) = \frac{1}{\pi T_1} \frac{-K_0\left[\frac{2\pi T_1}{k}\right] I_0\left[\frac{2\pi T_1}{k} e^{-ku/2}\right] + I_0\left[\frac{2\pi T_1}{k}\right] K_0\left[\frac{2\pi T_1}{k} e^{-ku/2}\right]}{K_1\left[\frac{2\pi T_1}{k}\right] I_0\left[\frac{2\pi T_1}{k} e^{-ku/2}\right] + I_1\left[\frac{2\pi T_1}{k}\right] K_0\left[\frac{2\pi T_1}{k} e^{-ku/2}\right]} \tag{33}$$

Take the limit $u \to \infty$

$$t_\infty = \lim_{u \to \infty} f(u) = \frac{1}{\pi T_1} \frac{I_0\left[\frac{2\pi T_1}{k}\right]}{I_1\left[\frac{2\pi T_1}{k}\right]} \xlongequal{Taylor} \frac{1}{\pi T_1} + \frac{k}{4(\pi T_1)^2} + O(k^2) \tag{34}$$

Using the inequality, we could see that

$$E_S > \frac{c}{24} T_0 + O(k) \tag{35}$$

Evaporation

We have the analytic expression of $f(u)$. Though it is complicated, we could use it under some specific cases. At the times of order k^{-1}, when the black hole has evaporated an order one fraction of its mass but remains sufficiently far from extremality, we can simplify further

$$\frac{K_n(z)}{\pi I_n(z)} \sim e^{-2z}\left(1+\frac{4n^2-1}{4z}+O(z^{-2})\right) \tag{36}$$

Fix the uk, and expand $f(u)$ at small k, then

$$\log\left(\frac{t_\infty - f(u)}{2t_\infty}\right) \sim -\frac{4\pi T_1}{k}\left(1-e^{-ku/2}\right)+O(ke^{ku/2}) \tag{37}$$

Take the derivative of u

$$\frac{1}{t_\infty - t}f'(u) \sim 2\pi T_1 e^{ku/2} + O(k^2 e^{ku/2}) \tag{38}$$

We could verify the following relation by taking higher order derivatives

$$\{u, t\} \sim \frac{1}{2(t_\infty - t)^2}\left(1 + O(k^2 e^{-ku/2})\right) \tag{39}$$

This is the expression we would use for later times($u \sim 1/k$). With a given stress tensor, the constraints and equation of motion for metric can be solved in terms of an integral of the stress tensor. To the furture of the shock $x^- > 0$, the solution is

$$\phi = 2\bar{\phi}_r \frac{1 - (\pi T)^2 x^+ x^- + \frac{1}{2} k I(x^+, x^-)}{x^+ - x^-} \tag{40}$$

where

$$I(x^+, x^-) = \int_0^{x^-} dt (x^+ - t)(x^- - t)\{u, t\} \tag{41}$$

Evaporation

We have to know an identity of this integral, which will be used in the calculation of the position of the quantum extremal surface

$$(x^+ - x^-)^2 \partial_\pm \left(\frac{I}{x^+ - x^-} \right) = \mp \int_0^{x^-} dt (x^\pm - t)^2 \{u, t\} \tag{42}$$

All the preparations have been done, we could move to discuss our specific model.

Contents I

Review of the notaion

In this section we want to compute the so called generalized entropy and see its evolution. But we still need to prepare something. Recalling the metric in AdS_2,

$$ds^2 = \frac{-dt^2 + dz^2}{z^2} = -\frac{4dx^+ dx^-}{(x^+ - x^-)^2} \tag{43}$$

In the following discussion we will use another notation, which will be more convenient,

$$ds^2 = \frac{4dx d\bar{x}}{(x + \bar{x}^2)}, \begin{cases} x^+ = t + z = \bar{x} \\ x^- = t - z = -x \end{cases} \tag{44}$$

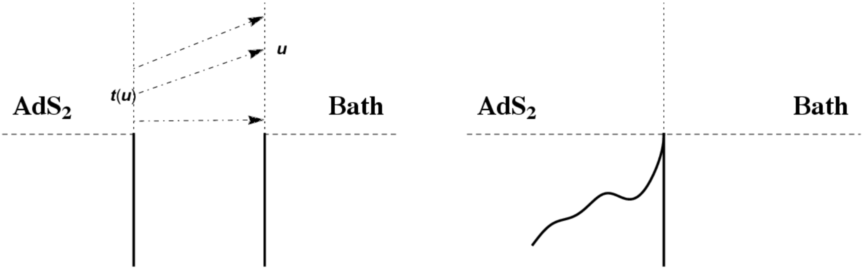
The logic of this part is

(1) Couple to the bath(The first conformal transformation)

(2) Map the whole space into the half plane(The second conformal transformation)

(3) Calculate the entropy of the half plane

Starting with the matter in Hartle-Hawking state at time $t = 0$, we want to allow the balck hole to evaporate via a coupling to an auxiliary system which acts as a bath to collect the Hawking radiation. For simplicity, we choose the bath to be another half-line supporting the same CFT as the bulk matter theory, also initially in the vacuum with the same AdS_2 and the heat bath, allowing matter to move freely between the two. But we have pay attention to the coupling because we use different coordinates in these two systems.



We have already solved the diffeomorphism before, which is $t = f(u)$. In the new coordiante$(x, \bar{x})$, we do the first conformal transformation $x = f(y)$ and $\bar{x} = f(\bar{y})$, then

$$ds^2 = \frac{4f'(y)f'(\bar{y})dyd\bar{y}}{(f(y)+f(\bar{y}))^2} = \Omega_y^{-2}dyd\bar{y} \tag{45}$$

The next step is to find a diffeomorphism taking the initial Cauchy surface(including both $t = 0$ slice of AdS_2 and the bath) to the half-line parameterised by a coordinate $w \in [0, \infty)$, which can map our state to the half-line vacuum[3]. In this new coordinate, if we could do the Weyl transformation again to make it flat, it would be convenient for the following calculations. To find this diffeomorphism, we need to use the condition

$$\langle T_{ww}(w)\rangle = 0 \tag{46}$$

[3] The authors often uses this name, I think they mean that there is no flux or matter in this line.

Mapping to the half plane

Besides, in AdS_2 at $t = 0$, the one-point function is zero in the physical metric(y), and also in the metric $dxd\bar{x}$ since the Weyl anomaly between AdS_2 and flat space in Poincaré coordinated vanishes.

$$\langle T_{xx}(x) \rangle = 0 \quad (x > 0) \tag{47}$$

In the bath, the one-point function is zero in the metric $dyd\bar{y}$,

$$\langle T_{yy}(y) \rangle = 0 \quad (y < 0) \tag{48}$$

Under the Weyl transformation, though the stress tensor is a tensor, it picks up an anomaly,

$$\left(\frac{dw}{dx} \right)^2 \langle T_{ww} \rangle = \langle T_{xx} \rangle + \frac{c}{24\pi} \{w, x\} \tag{49}$$

If we want to keep the equation exist, the anomaly must vanish. So we could find the diffeomorphism through the equation

$$\{w, x\} = 0 \tag{50}$$

The only possible map is Möbius map[4], so $w(x)$ is a Möbius map of $x = f(y)$ and for the part in the bath, w is a Möbius map of $y = f^{-1}(x)$. We can choose the AdS_2 region $x > 0$ to map to $w \in (0, w_0)$ and the bath region $y < 0$ to map to $(w_0, +\infty)$. Thus, the full diffeomorphism is

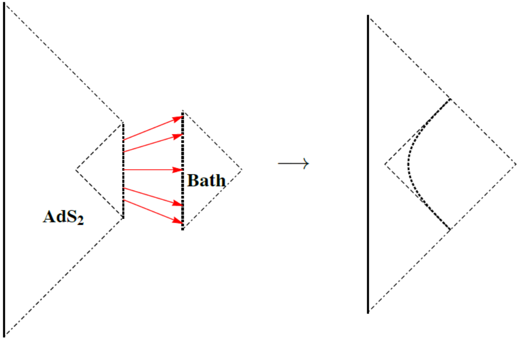
$$w(x) = \begin{cases} \dfrac{w_0}{w_0 + x} & x > 0 \\ w_0 + f^{-1}(-x) & x < 0 \end{cases} \tag{51}$$

In writing this, we have required that w and its first derivative are continuous at $x = y = 0$(This requires that $f'(0) = 1$).

[4] The map in such form $f(z) = \frac{az+b}{cz+d}$

Mapping to the half plane

The Penrose diagrams are shown as follows.



The resulting state is the half-line vacuum in the Minkowski half-space with auxiliary metric $dwd\bar{w}$. In the physically relevant limit $w_0 \to 0$, the worldline of the joined boundaries is pushed to the left in the w coordinates, becoming nearly null.

Recalling the equation (25), we can now use this map and the anomaly to compute stress tensor one-point function in x and y frames

$$\langle T_{xx}(x)\rangle = -\frac{c}{24\pi}\{w,x\} = E_S\delta(x) - \frac{c}{24\pi}\Theta(-x)\{y,x\} \tag{52}$$

$$\langle T_{yy}(y)\rangle = -\frac{c}{24\pi}\{w,y\} = E_S\delta(y) + \frac{c}{24\pi}\Theta(y)\{f(y),y\} \tag{53}$$

$$E_S = \frac{c}{24\pi}\left(\frac{2}{w_0} - f''(0)\right) \tag{54}$$

One tricky point is E_S has to be infinity($w_0 \to 0$), otherwise it is not compatible with the physics we described. Taking $w_0 \to 0$, the limiting map to the upper half-plane

$$w(x) \sim \begin{cases} \left(\frac{12\pi}{c}E_S\right)^{-2}\frac{1}{x} & x > 0 \\ f^{-1}(-x) & x < 0 \end{cases} \tag{55}$$

Entropy of the half-plane

Since we have mapped the state of the system to the half-plane, we could compute the entropy of a single internal now. A convenient way is to use the replica trick to use the Rényi entropies to compute the von-Neumann Entropy

$$S_{von} = -\lim_{n\to} \frac{1}{1-n} \operatorname{Tr} \rho^n = -\operatorname{Tr} \rho \log \rho. \tag{56}$$

We give a formula to calculate it[5]

$$S^{(n)} = -\frac{1}{n-1} \log \langle \sigma(x_1, \bar{x}_1) \tilde{\sigma}(x_2, \bar{x}_2) \rangle_{\mathcal{C}^n/\mathbb{Z}_n} \tag{57}$$

where σ and $\tilde{\sigma}$ are twist operators. The twist operators are local, primary operators in the orbifold theory, with dimension

$$\Delta_n = \frac{c}{12} \frac{(n-1)(n+1)}{n} \tag{58}$$

[5] The detial of this formula is in the article arXiv:0905.4013. And I think the authors have typo in the article, I use my version in this silde. This part we just admit the correctness and use the conclusion.

Under Weyl transforamtion

$$\langle\sigma(x_1,\bar{x}_1)\tilde{\sigma}(x_2,\bar{x}_2)\rangle_{\Omega^{-2}g} = \Omega(x_1,\bar{x}_1)^{\Delta_n}\Omega(x_2,\bar{x}_2)^{\Delta_n}\langle\sigma(x_1,\bar{x_1})\tilde{\sigma}(x_2,\bar{x_2})\rangle_g \tag{59}$$

The OPE of the twist operator is

$$\sigma(x_1)\tilde{\sigma}(x_2) \sim |x_1 - x_2|^{-2\Delta_n} \tag{60}$$

Then

$$S_{von} = \lim_{n\to 1}\frac{2\Delta_n \log|x_1 - x_2|}{n-1} = \lim_{n\to 1}\frac{c}{6}\frac{n+1}{n}\log l = \frac{c}{3}\log l \tag{61}$$

$$\begin{aligned} S_{\Omega^{-2}g} &= \lim_{n\to 1} -\frac{1}{n-1}\log\langle\sigma(x_1,\bar{x}_1)\tilde{\sigma}(x_2,\bar{x}_2)\rangle_{\Omega^{-2}g} \\ &= \lim_{n\to 1} -\frac{1}{n-1}\log\langle\sigma(x_1,\bar{x}_1)\tilde{\sigma}(x_2,\bar{x}_2)\rangle_g - \frac{c}{6}\sum_{\text{end points}}\log\Omega \\ &= S_g - \frac{c}{6}\sum_{\text{end points}}\log\Omega \end{aligned} \tag{62}$$

Next we constuct some simple cases and calculate the entropy, which will be used later. We begin with the case when the interval contains the boundary. We insert only one twist oprator in the bulk,

$$\langle\sigma(w,\bar{w})\rangle_{\mathsf{UHP}} = \frac{g_n}{(w+\bar{w})^{\Delta_n}} \tag{63}$$

So

$$\begin{aligned} S = \lim_{n\to 1} -\frac{1}{n-1}\log\langle\sigma(w,\bar{w})\rangle &= \lim_{n\to 1} \frac{-\log g_n}{n-1} + \frac{c}{6}\log(w+\bar{w}) \\ &= \frac{c}{6}\log(w+\bar{w}) + \log g \end{aligned} \tag{64}$$

where $\log g \equiv -\partial_n \log g_n|_{n=1}$.

Entropy of the half-plane

Next we take the interval to have both endpoints in the bulk of the system, away from the boundary. We could construct a conformally invariant cross ratio

$$\eta = \frac{(w_1 + \bar{w}_1)(w_2 + \bar{w}_2)}{(w_1 + \bar{w}_2)(w_2 + \bar{w}_1)} \tag{65}$$

The two point function of twise-operators does not have a fixed functional form, but contains an undetermined function $G_n(\eta)$, which can depend on the theory and boundary conditions

$$\langle \sigma(w_1, \bar{w}_1)\tilde{\sigma}(w_2, \tilde{w}_2)\rangle_{\mathsf{UHP}} = \frac{G_n(\eta)}{((w_1 - w_2)(\bar{w}_1 - \bar{w}_2)\eta)^{\Delta_n}} \tag{66}$$

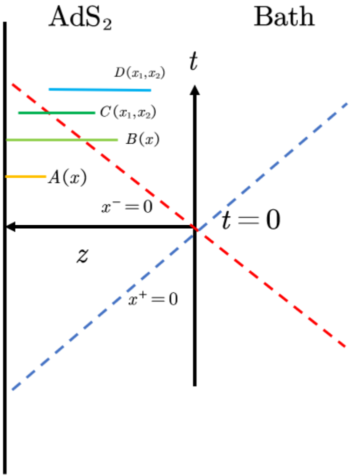
Thus,

$$S = \frac{c}{6}\log\left[(w_1 - w_2)(\bar{w}_1 - \bar{w}_2)\eta\right] + \log G(\eta) \tag{67}$$

where $\log G_n(\eta) = -\partial_n \log G_n(\eta)|_{\eta=1}$, satisfying $G(1) = 1, G(0) = g^2$.

Mapping to AdS_2

There are four different cases(four interval) we are interested in. The schematic is as follows, we would show how to compute the entropy of these four interval



Now we could calculate the entropy in the half-plane. When the boundary of the interval is spacelike separated from the coupling($x^+ > 0$ and $x^- < 0$), it is simplest

$$S = \frac{c}{6} \log 2 + \log g \tag{68}$$

Now we could calculate the entropy in the half-plane. When the boundary of the interval is spacelike separated from the coupling($x^+ > 0$ and $x^- < 0$), it is simplest

$$S = \frac{c}{6}\log 2 + \log g \tag{68}$$

Here are the Proof. Recalling the transformation $w(x)$ and $\bar{w}(\bar{x})$. And the calculation formula (64) and (62),

$$\begin{aligned} S &= \frac{c}{6}\left(\log(w+\bar{w}) - \log\Omega_w\right) = \frac{c}{6}\log\left(\frac{(12\pi E_S/c)^{-2}\left(\frac{1}{\bar{x}}+\frac{1}{x}\right)}{\frac{x+\bar{x}}{2}\sqrt{w'(x)\bar{w}'(\bar{x})}}\right) \\ &= \frac{c}{6}\log\left(\frac{(12\pi E_S/c)^{-2}\left(\frac{1}{\bar{x}}+\frac{1}{x}\right)}{\frac{x+\bar{x}}{2}(12\pi E_S/c)^{-2}\frac{1}{\bar{x}x}}\right) \\ &= \frac{c}{6}\log 2 + \log g \end{aligned} \tag{69}$$

If we take the endpoints to lie in the future of the shock

$$S = \frac{c}{6}\log\left(\frac{24\pi E_S}{\epsilon c}\frac{x^+y^-\sqrt{f'(y^-)}}{x^+-x^-}\right) \tag{70}$$

If we compurte the total entropy of the bath at physical time u,

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Finally, we compute the entropy of an interval in AdS_2. There are two mian cases of interest, with either one or both endpoints to the future of the shock. We begin with the case with one endpoint to the furture($x_1^{\pm} > 0$), and one endpoint to the past($x_2^+ > 0, x_2^- < 0$). In this case the cross ratio

$$\eta = \frac{x_1^+(x_2^+ - x_2^-)}{x_2^+(x_1^+ - x_2^-)} \tag{72}$$

Then the entropy is

$$S = \frac{c}{6}\log\left[\frac{48\pi E_S}{c}\frac{-y_1^- x_1^+ x_2^-(x_2^+ - x_1^+)\sqrt{f'(y_1^-)}}{x_2^+(x_1^+ - x_1^-)(x_1^+ - x_2^-)}\right] + \log G(\eta) \tag{73}$$

If two endpoints are both in the future of the shock$x_{\pm}^{(1,2)} > 0$, $\eta = 1$

$$S = \frac{c}{6}\log\left[\frac{4(y_1^- - y_2^-)(x_2^+ - x_1^+)\sqrt{f'(y_1^-)f'(y_2^-)}}{(x_1^+ - x_1^-)(x_2^+ - x_2^-}\right] \tag{74}$$

Finally we get the first main result in this paper. Take the limit of the above answers where one endpoint goes to the AdS_2 boundary at physical time u, Poincaré time $t = f(u)$, regulated to lie on the cutoff surface $z = \epsilon f'(u)$,

$$S = \begin{cases} \frac{c}{6}\log\left[\frac{24\pi E_S}{\epsilon c}\frac{-utx^-(x^+-t)}{x^+(t-x^-)\sqrt{f'(u)}}\right] + \log G\left(\frac{t(x^- - x^-)}{x^+(t-x^-)}\right) \\ (x^- < 0 < t < x^+) \\ \frac{c}{6}\log\left[\frac{2(u-y^-)(x^+-t)}{\epsilon(x^+-x^-)}\sqrt{\frac{f'(y^-)}{f'(u)}}\right] \\ (0 < x^- < t < x^+) \end{cases} \tag{75}$$

Contents I

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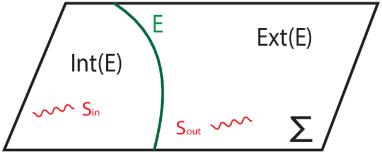
$$S \sim \min \left[\frac{\text{Area}}{4G_N} + S_{\text{outside}} \right] \tag{76}$$

The way to divide the region leads to the idea of quantum extremal surface.



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Definition 1

A codimension-2 surface X that minimizes the following expression is called the quantum extremal surface,

$$S = \min_X \left\{ \text{ext}\left[\frac{\text{Area}(X)}{4G_N} + S_{\text{semi-cl}}(\Sigma_X) \right] \right\} \tag{77}$$

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Naturally, we could also define the quantity in brackets as the generalized entropy.

$$S_{\text{gen}}(X) = \frac{\text{Area}(X)}{4G_N} + S_{\text{semi-cl}(\Sigma_X)} \tag{78}$$

But we define the generalized entropy in terms of renormalized quantities

$$S_{\text{gen}} = \frac{\phi_0^{(\text{Ren.})} + \phi^{(\text{Ren.})}}{4G_N} + S^{(\text{Ren.})} \tag{79}$$

First, we have already had the $S^{(\text{Ren})}$ part(75).

$$S^{(\text{Ren.})} = \begin{cases} \frac{c}{6}\log\left[\frac{24\pi E_S}{\epsilon c}\frac{-utx^-(x^+-t)}{x^+(t-x^-)\sqrt{f'(u)}}\right] + \log G\left(\frac{t(x^- - x^-)}{x^+(t-x^-)}\right) \\ (x^- < 0 < t < x^+) \\ \frac{c}{6}\log\left[\frac{2(u-y^-)(x^+-t)}{\epsilon(x^+-x^-)}\sqrt{\frac{f'(y^-)}{f'(u)}}\right] \\ (0 < x^- < t < x^+) \end{cases} \tag{80}$$

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The $\phi_0^{(\text{Ren.})}$ and $\phi^{(\text{Ren.})}$ are the dilaton's effect. Based on the talkathon we give before, we can discuss the entropy in chronological order. Before this, I want to explain the word 'shock' more explicitly. When we couple two boundaries of the system, there will be a positive energy injection into the black hole, this is so-called the 'shock' of energy. We regard this process is a transient process, which will take pretty short time.

Early times(before the shock)

Before the shock, for a surface located in the $x^- < 0$ region, the area is simply that of the original balck hole. We have already got the answer before the discussion.

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\begin{aligned}
S_{\text{gen}} &= \frac{\phi}{4G_N} + S, \quad \phi = \phi_0 + 2\bar{\phi}_r \frac{1 - (\pi T_0)^2 x^+ x^-}{x^+ - x^-}, \\
S &= 2k \frac{\bar{\phi}_r}{4G_N} \left[\log \left[\frac{24\pi E_S}{\epsilon c} \frac{-utx^-(x^+ - t)}{x^+(t - x^-)\sqrt{f'(u)}} \right] + \mathcal{G}\left(\frac{t(x^+ - x^-)}{x^+(t - x^-)} \right) \right]
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where $\mathcal{G} = \frac{6}{c}\log G(\eta)$. k is a very small parameter.We now simply make tha ansatz that the deviation from the horizon is small, which takes $x^\pm \mp \frac{1}{\pi T_0}$ of order k.

$$
\begin{aligned}
x^+ - \frac{1}{\pi T_0} &\sim \frac{k}{(\pi T_0)^2}\left(\eta - \eta(1-\eta)\mathcal{G}'(\eta)\right) \\
x^- + \frac{1}{\pi T_0} &\sim \frac{k}{(\pi T_0)^2}\left(\frac{\eta}{1-\eta} - \eta\mathcal{G}'(\eta)\right) \\
\eta &= \frac{2\pi T_0 t}{1+\pi T_0 t}
\end{aligned} \tag{82}
$$

The way to derive this is to let $\partial_+ S_{\text{gen}} = \partial_- S_{\text{gen}} = 0$. Expand the equation and let the position near the horizon.

Early times

If we use a specific physical time u, take a picture of the system, then we have to find the location of quantum extremal surface and calculate the entropy on it.

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For sufficiently early times, we have $1 - \eta \sim e^{-2\pi T_1 u}$, until η settles down to its maximum for $u \gtrsim -\frac{1}{T_1} \log\left(\frac{T_1 - T_0}{T_1 + T_0}\right)$.

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$$S_{\text{gen}} \sim \frac{\phi_0 + 2\pi T_0 \bar{\phi}_r}{4G_N} + \frac{c}{6} \log\left(\frac{2\pi E_S}{\epsilon c} \frac{u \sinh(\pi T_1 u)}{\pi T_1}(1 - \eta)\right) + \frac{c}{6}\mathcal{G}(\eta) \tag{84}$$

This is a quiet important result we get in this paper. The function is shown in the following.

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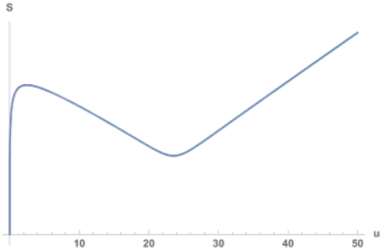
Remains to show

$$\frac{t}{\sqrt{f'(u)}} \sim \frac{\sinh(\pi T_1 u)}{\pi T_1} \tag{89}$$

Recalling we have $t = f(u)$ and we have analytic expression of $f(u)$(33), it is easy to prove.

Early times



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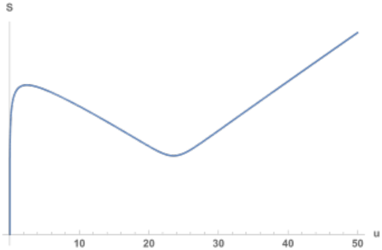
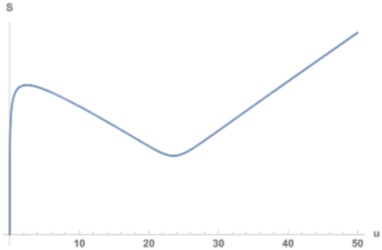


Intepretation: At very early times, we have a large logarithmic increase in the entropy, due to production of short-distance entanglement with the bath;



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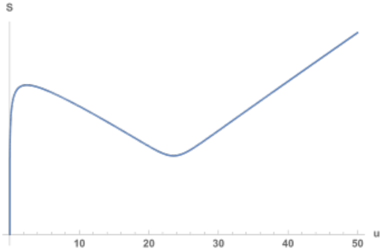

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$$t_{scr} = \frac{1}{2\pi T_1} \log\left(\frac{T_1}{T_1 - T_0}\right) = \frac{\beta_1}{2\pi} \log\left(\frac{E(T_1)}{E_S}\right) \tag{90}$$

Early times

for $E(T_1)$ the energy of a black hole at temperature T_1 and $T_1 - T_0 \ll T_0$, η settles down to its maximum, entropy reaches a local minimum, and then entropy begins to increase linearly. Here we have already defined $\alpha_S \equiv 1$, which answers a small question we raised before.

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Taylor expand the full generalized entropy and get

$$S_{\text{gen}} = \frac{\phi_0+\frac{2\bar{\phi}_r}{x^+}}{4G_N} + \frac{c}{6}\log\left[\frac{2u(x^+-t)}{\epsilon x^+\sqrt{f'(u)}}\right] + \frac{c}{6}\begin{cases}\log\left(-\frac{12\pi E_S}{c}x^-\right) & x^-<0\\ \left(\frac{1}{x^+}-\frac{1}{u}\right)x^- & x^->0\end{cases} \tag{92}$$

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For a quantum extremal surface to exist on the shock, we will also require that the generalized entropy is locally minimal under variations in the outgoing, x^- direction:

$$\frac{4G_N}{\bar{\phi}_r}\partial_- S_{\text{gen}} \sim \begin{cases} 2\frac{1}{(x^+)^2} - 2(\pi T_0)^2 + \frac{2k}{x^-} & x^- < 0 \\ 2\frac{1}{(x^+)^2} - 2(\pi T_1)^2 + \frac{2k}{x^+} - \frac{2k}{u} & x^- > 0 \end{cases} \tag{94}$$

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But the sigularity in the entropy will cause this to be decreasing just before the shock, but generically this will only be large enough to overcome the large dilaton increase with the microscopic scale of the shock E_S^{-1}, the result should not be trusted.

Soon after the shock

We now look for the QES in the region $x^- > 0$, to the future of the coupling to the bath, and the resulting shock of energy.

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$$\phi = 2\bar{\phi}_r \frac{1 - (\pi T_1)^2 x^+ x^-}{x^+ - x^-} \tag{95}$$

Thus the full entropy is

$$S_{\text{gen}} = \frac{\phi}{4G_N} + \frac{c}{6} \log\left[\frac{2(u - y^-)(x^+ - t)}{\epsilon(x^+ - x^-)}\sqrt{\frac{f'(y^-)}{f'(u)}}\right] \tag{96}$$

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First we extremize the entropy in x^+ direction,

$$\begin{aligned}\frac{4G_N}{\bar{\phi}_r}\partial_+ S_{\text{gen}} &= -2\frac{1-(\pi T_1 x^-)^2}{(x^+-x^-)^2} + \frac{2k}{x^+-t} + \frac{k}{x^+-x^-} \\ &\sim -2\frac{1-(\pi T_1 x^-)^2}{(x^+-x^-)^2} + \frac{2k}{x^+-t} = 0\end{aligned} \tag{97}$$

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If we let the position closed to the horizon, $x^+ \sim \frac{1}{\pi T_1}$, we could see that

$$x^+ - t \sim \frac{k}{(\pi T_1)^2}\frac{1-\pi T_1 x^-}{1+\pi T_1 x^+} \tag{98}$$

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Then we want to extremize in the x^- direction, along the horizon. Close to the horizon, the variation of the background dilaton is suppressed, so we must include the backreaction. Recall the identity of the backreaction(42),

$$\begin{aligned}(x^+-x^-)^2\partial_-\left(\frac{I}{x^+-x^-}\right) &\sim \int_0^{x^-} dt(x^+-t)^2\{\frac{\tanh^{-1}(\pi t T_1)}{\pi T_1}, t\} \\ &\sim \frac{2x^-}{1+\pi T_1 x^-}\end{aligned} \tag{99}$$

Soon after the shock

We calculate things near the horizon $x^+ \sim \frac{1}{\pi T_1}$

$$\partial_- \phi \sim \left(\frac{(2\pi T_1)^2 (1 - \pi T_1 x^+)}{(1 - \pi T_1 x^-)^2} + \frac{2(\pi T_1)^2 k x^-}{(1 + \pi T_1 x^-)(1 - \pi T_1 x^-)^2} \right) \tag{100}$$

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The rest part is $\partial_- S$, recalling the equation (75) and $x^- = f(y^-)$, and $f'(y^-) \sim 1 - (\pi T_1 x^-)^2$. For times $u \gg 1$, the variation of the $\log\big(u - y^-\big)$ term is negligible

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We now want to write the results using proper time u. In this setting(soon after the shock), $k e^{2\pi T_1 u} \sim 1$. Recalling $t = f(u)$, and two equations

$$\log \left(\frac{t_\infty - f(u)}{2 t_\infty} \right) \sim \frac{4\pi T_1}{k} \left(1 - e^{-ku/2} \right) \tag{103}$$

$$t_\infty = \frac{1}{\pi T_1} + \frac{k}{4(\pi T_1)^2} + O(k^2) \tag{104}$$

Soon after the shock

We have

$$t - \frac{1}{\pi T_1} \sim \frac{k}{(2\pi T_1)^2} - \frac{2}{\pi T_1} e^{-2\pi T_1 u} \tag{105}$$

The result is

$$\begin{aligned} x^- &\sim \frac{1}{\pi T_1} \frac{3ke^{2\pi T_1 u} - 8\pi T_1}{3ke^{2\pi T_1 u} + 8\pi T_1} \\ t_\infty - x^+ &\sim -\frac{2}{3} t_\infty e^{-2\pi T_1 u} \end{aligned} \tag{106}$$

So far all the result relies on the condition $e^{-2\pi T u} \ll k^2$. If we want to go to times of order k^{-1}, when a significant fraction of the black hole has evaporated, we require a different approach to treat the dilaton. Let us calculate term by term. First is the dilaton part. Its derivative along the $+$ direction is

$$(x^+ - x^-)^2 \partial_+ \phi = \bar{\phi}_r \left[2(\pi T_1 x^-)^2 - 2 - k \int_0^{x^-} dt (x^- - t)^2 \{u, t\} \right] \tag{107}$$

At $x^- = t_\infty$, it vanishes, so

$$I_\infty = \int_0^{t_\infty} dt (t_\infty - t)^2 \{u, t\} = \frac{2}{k} \left((\pi T_1 t_\infty)^2 - 1 \right) \tag{108}$$

Later times

Recalling the equation we got before(39)

$$
\begin{aligned}
\partial_- \int_0^{x^-} dt (x^- - t)^2 \{u,t\} &= 2\int_0^{x^-} dt (x^- - t)\{u,t\} \\
&\sim \int_0^{x^-} \frac{x^- - t}{(t_\infty - t)^2} dt = -\log\left(\frac{t_\infty - x^-}{t_\infty}\right) - \frac{x^-}{t_\infty} \\
&\sim -\log\left(\frac{t_\infty - x^-}{t_\infty}\right) + O(1)
\end{aligned}
\tag{109}
$$

Combining these two terms, the approximate form of the integral is

$$
\int_0^{x^-} dt (x^- - t)^2 \{u,t\} \sim \frac{2}{k}\left((\pi T_1 t_\infty)^2 - 1\right) + (t_\infty - x^-)\log\left(\frac{t_\infty - x^-}{t_\infty}\right) \tag{110}
$$

Substitute it into the derivative of the dilaton,

$$
\partial_+ \phi \sim -\bar{\phi}_r \left[(2\pi T_1)^2 \frac{t_\infty}{t_\infty - x^-} - \frac{k}{t_\infty - x^-}\log\left(\frac{t_\infty}{t_\infty - x^-}\right)\right] \tag{111}
$$

Later times

Here we put the details which the authors ignore in this article.

$$\frac{\bar{\phi}_r}{(x^+ - x^-)^2}\left[2(\pi T_1 x^-)^2 - 2(\pi T_1 t_\infty)^2 - k(t_\infty - x^-)\log\left(\frac{t_\infty - x^-}{t_\infty}\right)\right] \tag{112}$$

Then the factor

$$\frac{t_\infty - x^-}{(x^+ - x^-)^2} = \frac{t_\infty - x^-}{[(t_\infty - x^-) - (t_\infty - x^+)]^2} \sim \frac{1}{(t_\infty - x^-)} \tag{113}$$

And

$$\frac{(x^-)^2 - (t_\infty)^2}{(x^+ - x^-)^2} \sim \frac{(x^- + t_\infty)(x^- - t_\infty)}{(t_\infty - x^-)^2} \sim -\frac{2t_\infty}{t_\infty - x^-} \tag{114}$$

Notice that we have already prove the equation[6]

$$\log\left(\frac{t_\infty - f(u)}{2t_\infty}\right) \sim \frac{4\pi T_1}{k}\left(1 - e^{-ku/2}\right) \tag{115}$$

and $\pi T_1 t_\infty \sim 1$, the equation becomes

$$\sim -4\pi T_1 \bar{\phi}_r \left(\frac{\pi T_1 t_\infty}{t_\infty - x^-} - \frac{1 - e^{-ky^-/2}}{t_\infty - x^-}\right) \sim -4\pi T_1 \bar{\phi}_r \frac{e^{-ky^-/2}}{t_\infty - x^-} \tag{116}$$

[6] When we use this equation, we ignore the factor $\log 2$, which is not important.

The full entropy is still

$$S_{\text{gen}} = \frac{\phi}{4G_N} + \frac{c}{6}\log\left[\frac{2(u-y^-)(x^+-t)}{\epsilon(x^+-x^-)}\sqrt{\frac{f'(y^-)}{f'(u)}}\right] \tag{117}$$

And it derivative along $+$ direction is

$$-4\pi T_1 \frac{e^{-\frac{k}{2}y^-}}{t_\infty - x^-} + \frac{2k}{x^+-t} = 0 \tag{118}$$

Now we look at the derivative of dilaton along $-$ direction, again, using the equation (42)

$$(x^+-x^-)^2\partial_-\phi = \bar{\phi}_r\left[2(\pi T_1 x^-)^2 - 2 - k\int_0^{x^-} dt(x^+-t)^2\{u,t\}\right] \tag{119}$$

To analyse the integral, we expand in powers of $t_\infty - x^-$ and use the same approximation for $\{u,t\}$ as before:

$$\begin{aligned}&\int_0^{x^-} dt(x^+-t)^2\{u,t\} - \int_0^{t_\infty} dt(t_\infty-t)^2\{u,t\} = \\ &\int_0^{x^-} dt(x^+-t_\infty+t_\infty-t)^2\{u,t\} - \int_0^{t_\infty} dt(t_\infty-t)^2\{u,t\} =\end{aligned} \tag{120}$$

$$
\begin{aligned}
=&(x^+ - t_\infty)^2 \int_0^{x^-} dt\{u,t\} - 2(t_\infty - x^+)\int_0^{x^-} dt(t_\infty - t)\{u,t\} \\
&- \int_{x^-}^{t_\infty} dt(t_\infty - t)^2\{u,t\} \\
&\sim \frac{(t_\infty - x^+)^2}{2(t_\infty - x^-)} + (t_\infty - x^+)\log\left(\frac{t_\infty - x^-}{t_\infty}\right) - \frac{1}{2}(t_\infty - x^-)
\end{aligned} \tag{121}
$$

Again, due to $t_\infty - x^+ \ll t_\infty - x^-$, we could ignore the quadratic term. Then substitute to expression we have

$$
\begin{aligned}
\partial_- \phi &\sim \frac{\bar{\phi}_r}{(t_\infty - x^-)^2}\left[4\pi T_1(t_\infty - x^+) + k(t_\infty - x^+)\log\left(\frac{t_\infty - x^-}{t_\infty} - \frac{k}{2}(t_\infty - x^-)\right)\right] \\
&\sim \frac{\bar{\phi}_r}{(t_\infty - x^-)^2}\left[4\pi T_1(t_\infty - x^+)e^{-ky^-/2} - \frac{k}{2}(t_\infty - x^-)\right]
\end{aligned} \tag{122}
$$

Later times

The bulk entropy is

$$S = \frac{c}{6}\log\left[\frac{2(u-y^-)(x^+-t)}{\epsilon(x^+-x^-)}\sqrt{\frac{f'(y^-)}{f'(u)}}\right] \tag{123}$$

Only the $f'(y^-)$ term matters and $\partial_{x^-}\log f'(y^-) \sim -\frac{1}{t_\infty - x^-}$.

$$\partial_- S = \frac{c}{12}\frac{1}{t_\infty - x^-} \tag{124}$$

Combining the two parts of derivate, we have

$$\frac{4G_N}{\bar{\phi}_r}\partial_- S_{\text{gen}} = \frac{1}{(t_\infty - x^-)^2}\left[4\pi T_1(t_\infty - x^+)e^{-ky^-/2} + \frac{k}{2}(t_\infty - x^-)\right] = 0 \tag{125}$$

Hence,

$$x^+ - t_\infty \sim \frac{k}{8\pi T_1}(t_\infty - x^-)e^{ky^-/2} \tag{126}$$

Combinging the equation (118) and the equation (126), we can locate the quantum extremal surface

$$x^+ - t_\infty \sim \frac{t_\infty - t}{3} \tag{127}$$

$$t_\infty - t \sim \frac{3k}{8}t_\infty e^{\frac{k}{2}y^-}(t_\infty - x^-) \tag{128}$$

Later times

Expressing this entirely in terms of the boundary proper time u and corresponding ingoing coordinate y^-, we have our final location for the quanum extremal surface

$$u \sim y^- + \frac{e^{\frac{k}{2}y^-}}{2\pi T_1} \log\left(\frac{8\pi T_1 e^{-\frac{k}{2}y^-}}{3k}\right) \tag{129}$$

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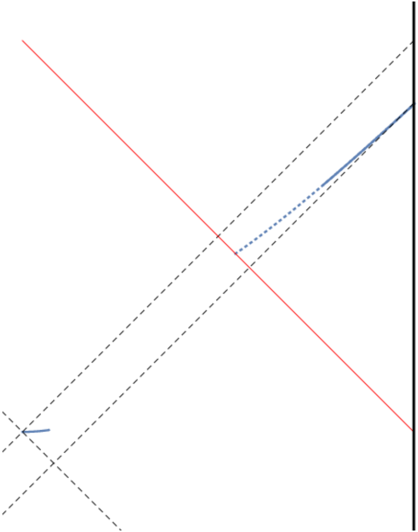
$$t_{HP} = \frac{\beta}{2\pi} \log\left[\frac{16}{c}(S - S_0)\right] \tag{130}$$

The result is precisely of the form we raised before with coefficient $\alpha_{HP} = 1$. This is a main conclusion in this paper.

Later times

The full evolution of the position of QES is shown as following



Summary

Now we briefly summarize what we have done in this paper.

- It is useful to take the viewpoint that begins with 3 systems L, R, B which at first do not interact. Here L, R are both AdS_2 black holes, and the black holes are highly entangled. The bath B begins in its ground state. This initial state is then perturbed by turning on a coupling between R and B, localized at the boundary of both systems.

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- Using the semiclassically calculation, we find a consistent but more precise expression of Hayden-Preskill time.