

CSC 2515: Introduction to Machine Learning

Lecture 6: Neural Networks

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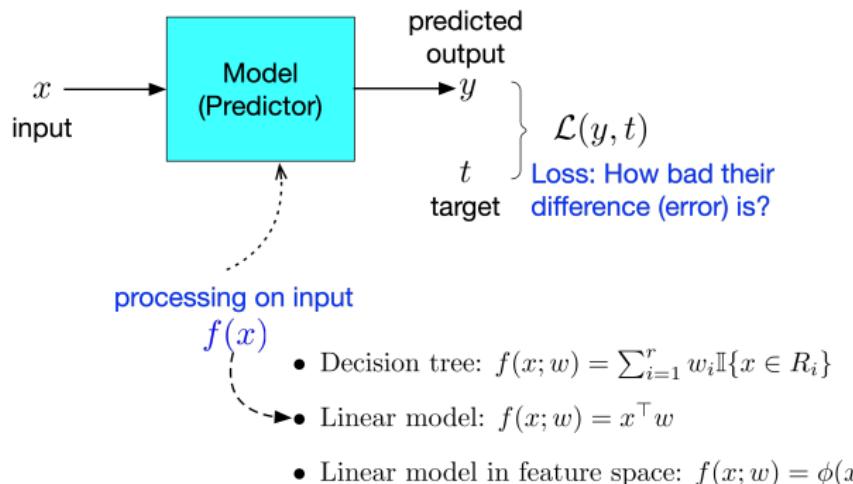
University of Toronto and Vector Institute

¹ Credit for slides goes to many members of the ML Group at the U of T, and beyond, including (recent past): Roger Grosse, Murat Erdogdu, Richard Zemel, Juan Felipe Carrasquilla, Emad Andrews, and myself.

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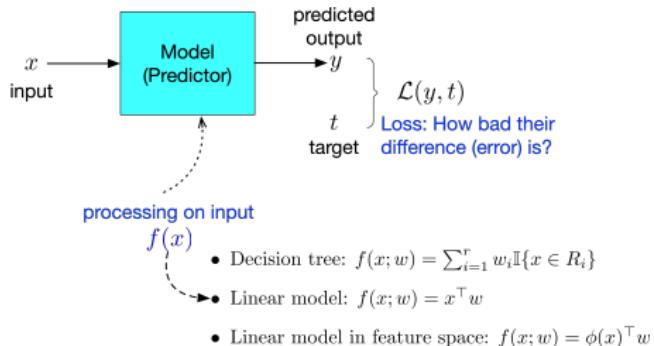
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Today



- We have considered a modular framework to ML.
- We considered several loss functions for regression and classifications
- We have “mostly” focused on linear models.

Today



- Feature mapping can make linear models much more powerful.
- Coming up with feature mapping can be challenging.
- Kernel-based approach is a way to partially address it.
- (Artificial) Neural Networks is a general approach to represent more complex models.
- The predictor can be seen as a **computer program** that processes the input in order to generate the output. Some programs are simpler, some are more complex.

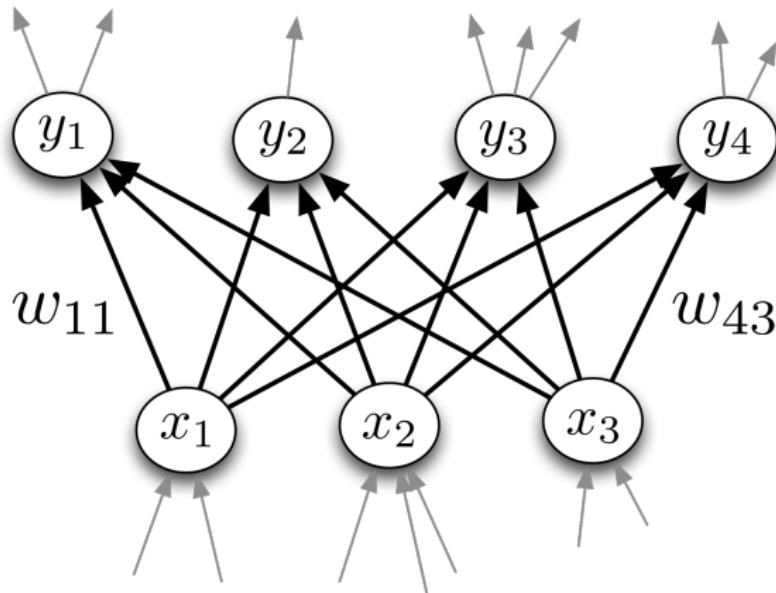
Today

Skills to Learn

- Multi-layer feedforward neural networks
- Backpropagation for training NN

Neural Networks

...



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Inspiration: The Brain

- Our brain has $\sim 10^{11}$ neurons, each of which communicates (is connected) to $\sim 10^4$ other neurons

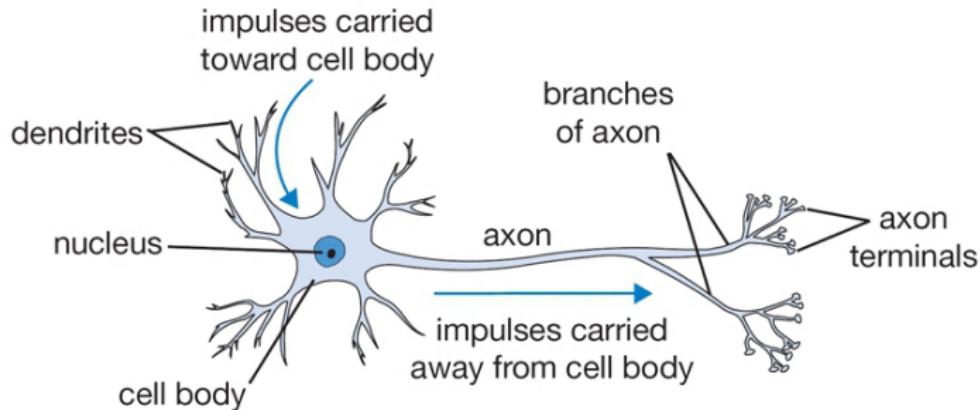
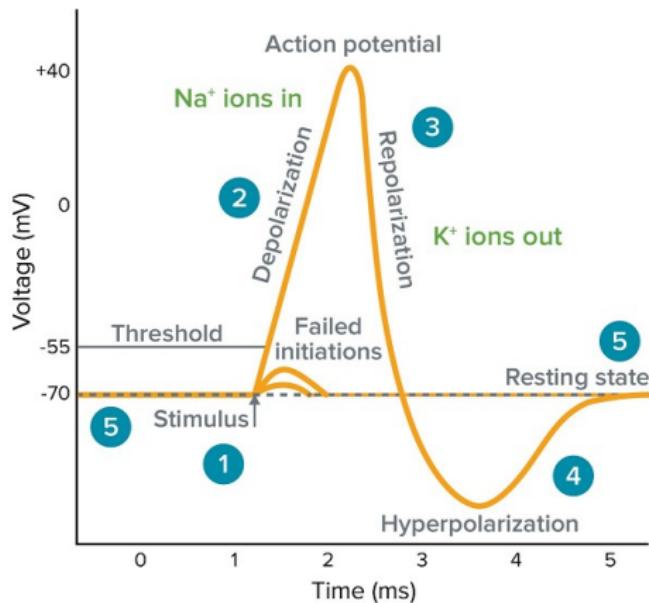


Figure: The basic computational unit of the brain: Neuron

[Pic credit: <http://cs231n.github.io/neural-networks-1/>]

Inspiration: The Brain

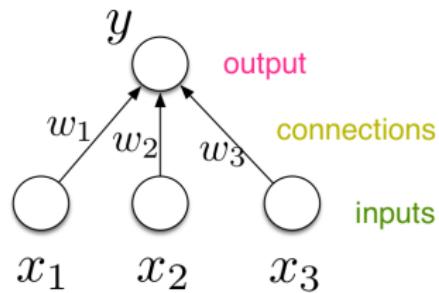
- Neurons receive input signals and accumulate voltage. After some threshold they will fire spiking responses.



[Pic credit: www.moleculardevices.com]

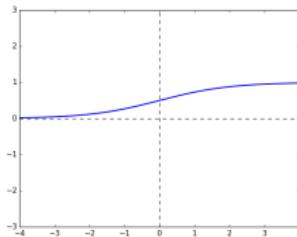
Inspiration: The Brain

- For (artificial) neural nets, we use a much simpler model neuron, or **unit**:



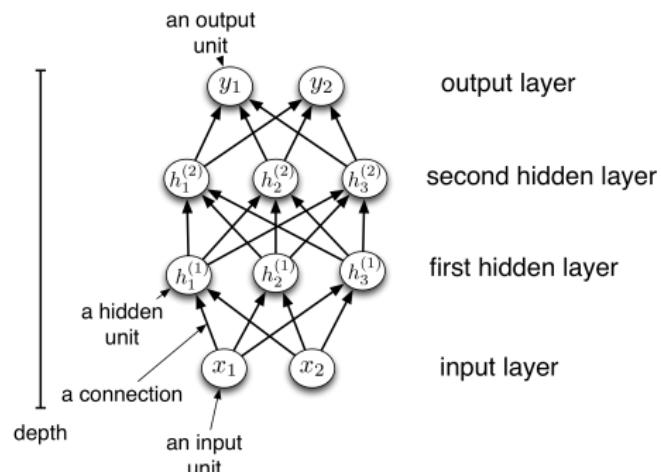
A diagram showing the mathematical equation for a neural unit. The equation is $y = \phi(\mathbf{w}^\top \mathbf{x} + b)$. To the left of the equation, a pink arrow labeled "output" points down to y . To the right, a blue arrow labeled "weights" points down to \mathbf{w}^\top . Another blue arrow labeled "bias" points down to b . Below the equation, a red arrow labeled "activation function" points up to ϕ . To the right of the equation, a green arrow labeled "inputs" points up to \mathbf{x} .

- Compare with logistic activation function used in LR:
 $y = \sigma(\mathbf{w}^\top \mathbf{x} + b)$

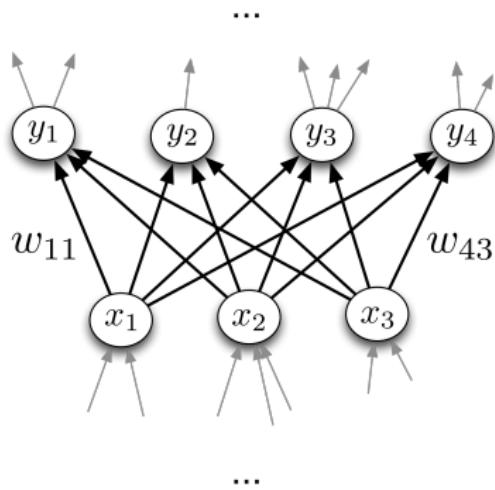


Multilayer Perceptrons (Feedforward Neural Networks)

- We can connect lots of units together into a directed acyclic graph.
- Typically, units are grouped together into layers.
- This gives a feed-forward neural network.
- That is in contrast to recurrent neural networks, which can have cycles.

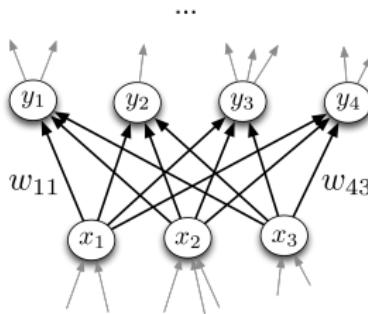


Multilayer Perceptrons (Feedforward Neural Networks)



- Each hidden layer i connects N_{i-1} input units to N_i output units.
- In the simplest case, all input units are connected to all output units. We call this a **fully connected layer**. We will consider other layer types later.
 - ▶ The inputs and outputs for a layer are distinct from the inputs and outputs to the network.

Multilayer Perceptrons (Feedforward Neural Networks)



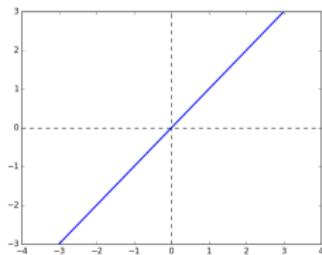
- If we need to compute $M [= N_i]$ outputs from $N = [N_{i-1}]$ inputs, we can do so in parallel using matrix multiplication. This means we will be using a $M \times N$ weight matrix.
- The output units are a function of the input units:

$$\mathbf{y} = f(\mathbf{x}) = \phi(\mathbf{Wx} + \mathbf{b})$$

- A multilayer network consisting of fully connected layers is called a **multilayer perceptron**. Despite the name, it has nothing to do with the Perceptron algorithm.

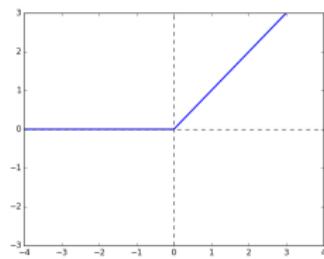
Activation Functions

Some activation functions:



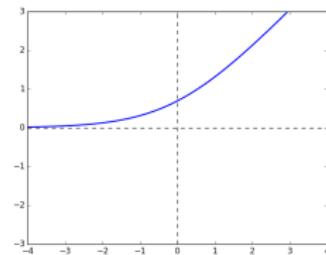
Identity

$$y = z$$



**Rectified Linear
Unit
(ReLU)**

$$y = \max(0, z)$$

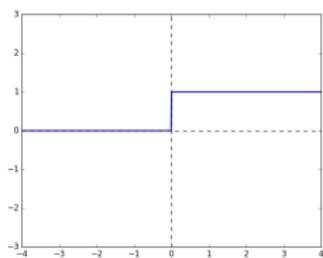


Soft ReLU

$$y = \log 1 + e^z$$

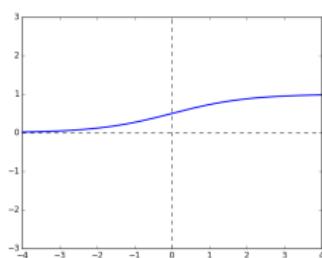
Activation Functions

Some activation functions:



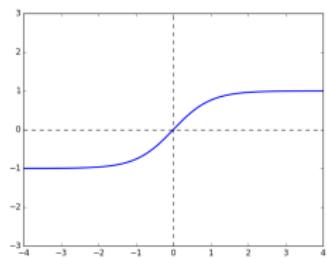
Hard Threshold

$$y = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z \leq 0 \end{cases}$$



Logistic

$$y = \frac{1}{1 + e^{-z}}$$



Hyperbolic Tangent
(tanh)

$$y = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

Multilayer Perceptrons (Feedforward Neural Networks)

- Each layer computes a function, so the network computes a composition of functions:

$$\mathbf{h}^{(1)} = f^{(1)}(\mathbf{x}) = \phi(\mathbf{W}^{(1)}\mathbf{x} + \mathbf{b}^{(1)})$$

$$\mathbf{h}^{(2)} = f^{(2)}(\mathbf{h}^{(1)}) = \phi(\mathbf{W}^{(2)}\mathbf{h}^{(1)} + \mathbf{b}^{(2)})$$

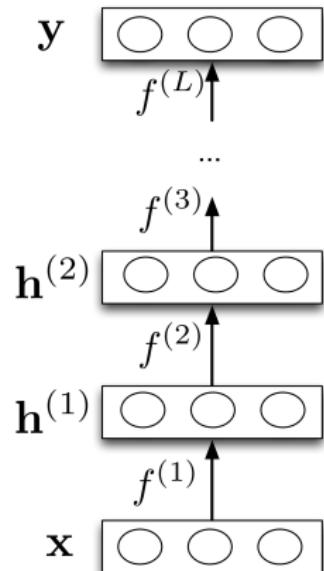
⋮

$$\mathbf{y} = f^{(L)}(\mathbf{h}^{(L-1)})$$

- Or more compactly:

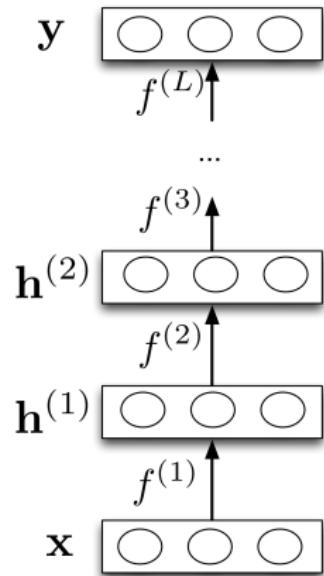
$$\mathbf{y} = f^{(L)} \circ \dots \circ f^{(1)}(\mathbf{x}).$$

- Neural nets provide modularity: we can implement each layer's computations as a black box.



Multilayer Perceptrons (Feedforward Neural Networks)

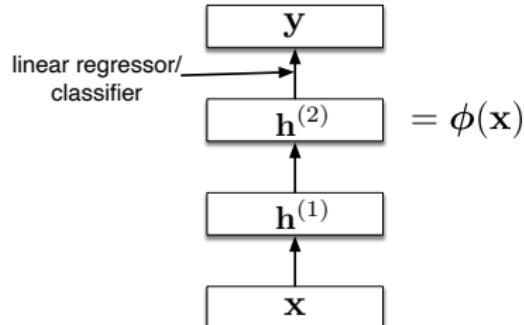
- Q: Write down the equations of a two layer NN (one hidden, one output), two hidden units, ϕ as the activation function of the hidden layer, and a linear one dimensional output layer.



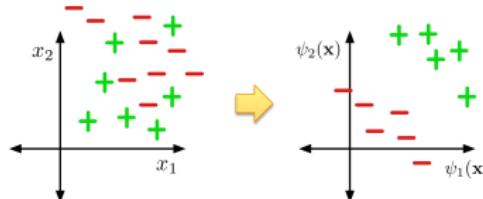
Feature Learning

Last layer:

- If task is regression: choose
 $\mathbf{y} = f^{(L)}(\mathbf{h}^{(L-1)}) = (\mathbf{w}^{(L)})^T \mathbf{h}^{(L-1)} + b^{(L)}$
- If task is binary classification: choose
 $\mathbf{y} = f^{(L)}(\mathbf{h}^{(L-1)}) = \sigma((\mathbf{w}^{(L)})^T \mathbf{h}^{(L-1)} + b^{(L)})$
- Neural nets can be viewed as a way of learning features:

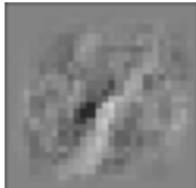


- The goal:



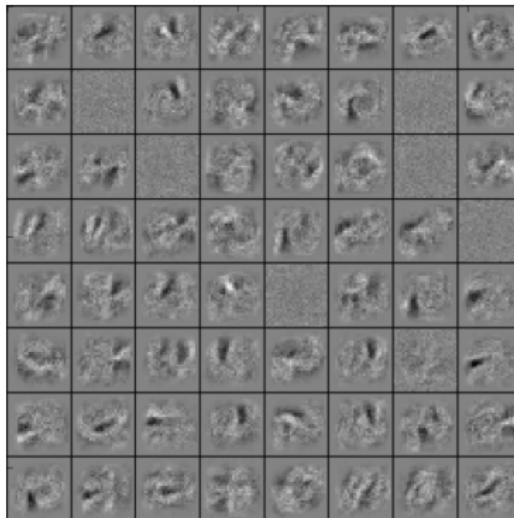
Feature Learning

- Suppose that we are trying to classify images of handwritten digits. Each image is represented as a vector of $28 \times 28 = 784$ pixel values.
- Each first-layer hidden unit computes $\phi(\mathbf{w}_i^T \mathbf{x})$. It acts as a **feature detector**.
- We can visualize \mathbf{w} by reshaping it into an image. Here is an example that responds to a diagonal stroke.



Feature Learning

Here are some of the features learned by the first hidden layer of a handwritten digit classifier:



Expressive Power

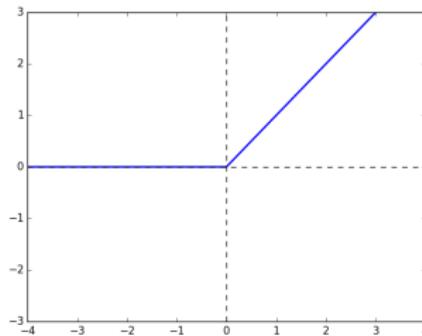
- We have seen that there are some functions that linear classifiers cannot represent. Are deep networks any better?
- Suppose a layer's activation function is the identity function, so the layer just computes an affine transformation of the input
 - ▶ We call this a linear layer
- Any sequence of *linear* layers can be equivalently represented with a single linear layer.

$$\mathbf{y} = \underbrace{\mathbf{W}^{(3)} \mathbf{W}^{(2)} \mathbf{W}^{(1)}}_{\triangleq \mathbf{W}'} \mathbf{x}$$

- ▶ Deep *linear* networks are no more expressive than linear regression.

Expressive Power

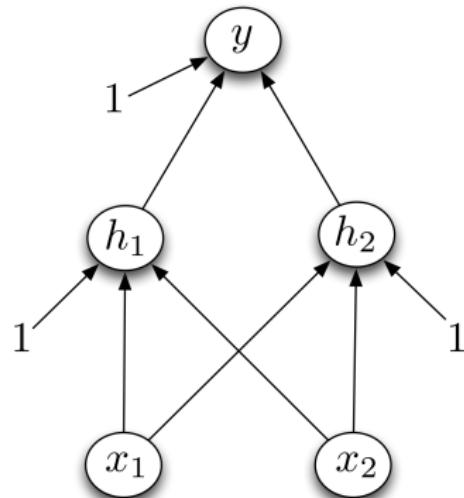
- Multilayer feed-forward neural nets with *nonlinear* activation functions are **universal function approximators**: they can approximate any function arbitrarily well.
- This has been shown for various activation functions (thresholds, logistic, ReLU, etc.)
 - ▶ Even though ReLU is “almost” linear, it is nonlinear enough.



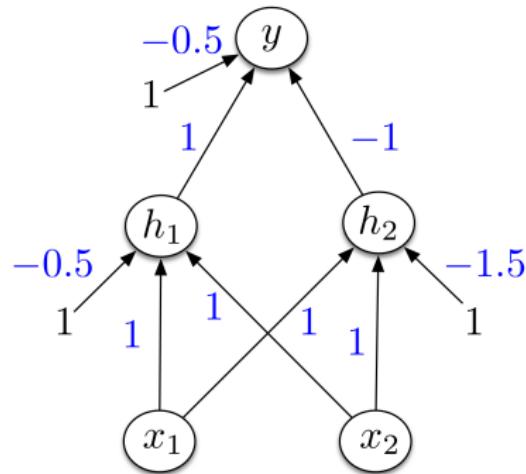
Multilayer Perceptrons

Designing a network to classify XOR:

Assume hard threshold activation function



Multilayer Perceptrons



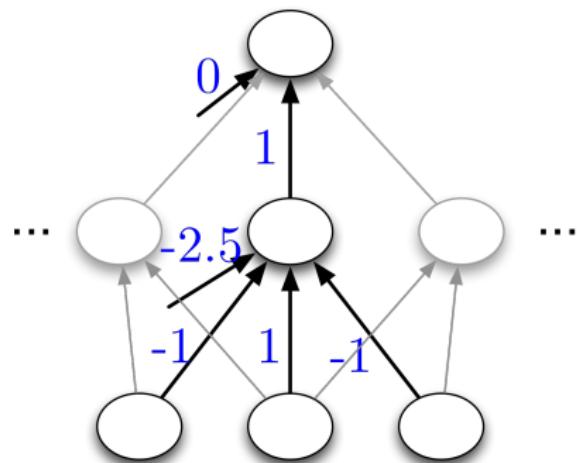
- h_1 computes $\mathbb{I}[x_1 + x_2 - 0.5 > 0]$
 - ▶ i.e. x_1 OR x_2
- h_2 computes $\mathbb{I}[x_1 + x_2 - 1.5 > 0]$
 - ▶ i.e. x_1 AND x_2
- y computes $\mathbb{I}[h_1 - h_2 - 0.5 > 0] \equiv \mathbb{I}[h_1 + (1 - h_2) - 1.5 > 0]$
 - ▶ i.e. h_1 AND (NOT h_2) = x_1 XOR x_2

Expressive Power

Universality for binary inputs and targets:

- Hard threshold hidden units, linear output
- Strategy: 2^D hidden units, each of which responds to one particular input configuration

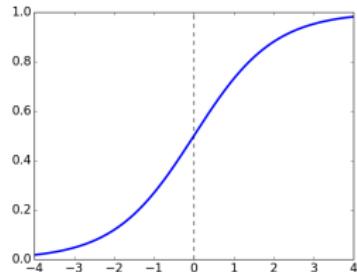
x_1	x_2	x_3	t
:	:	:	:
-1	-1	1	-1
-1	1	-1	1
-1	1	1	1
:		:	:



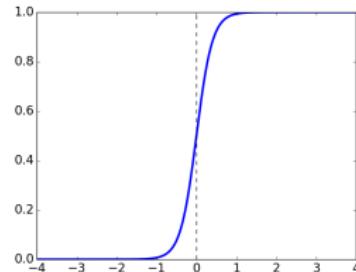
- Only requires one hidden layer, though it needs to be extremely wide.

Expressive Power

- What about the logistic activation function?
- You can approximate a hard threshold by scaling up the weights and biases:



$$y = \sigma(x)$$



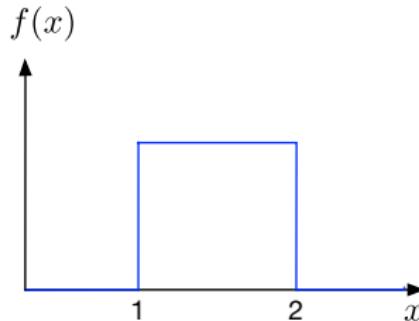
$$y = \sigma(5x)$$

- This is good: logistic units are differentiable, so we can train them with gradient descent.

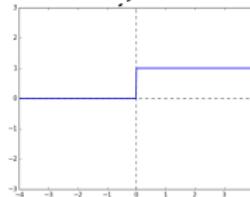
Expressive Power

Let us do some exercises ...

- Q: How can we represent the function that takes value of +1 in $x \in [1, 2]$ and 0 elsewhere using a simple NN with **hard threshold** activation function?



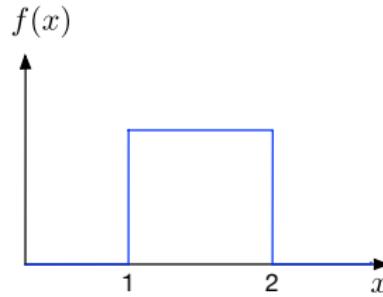
$$f(x) = w_1\phi(x - b_1) + w_2\phi(x - b_2)$$



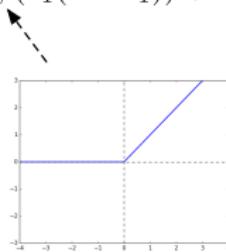
Expressive Power

Let us do some exercises ...

- Q: How can we **approximately** represent the function that takes value of +1 in $x \in [1, 2]$ and 0 elsewhere using a simple NN with ReLU activation function?



$$f(x) \approx w_1\phi(v_1(x - b_1)) + w_2\phi(v_2(x - b_2)) + \dots$$



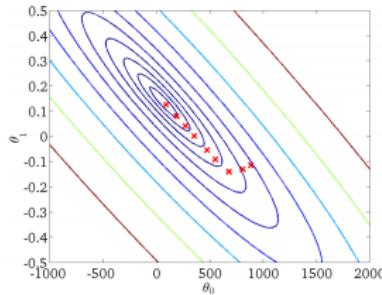
Expressive Power

- Limits of universality
 - ▶ You may need to represent an exponentially large network.
 - ▶ How can you find the appropriate weights to represent a given function?
 - ▶ If you can learn any function, you'll just overfit.
 - ▶ We desire a *compact* representation.

Training Neural Networks with Backpropagation

Recap: Gradient Descent

- **Recall:** gradient descent moves in the opposite of the gradient



- Weight space for a multilayer neural net: one coordinate for each weight or bias of the network, in *all* the layers
- Conceptually, not any different from what we have seen so far — just higher dimensional and harder to visualize!
- We want to define a loss \mathcal{L} and compute the gradient of the cost $d\mathcal{J}/d\mathbf{w}$, which is the vector of partial derivatives.
 - ▶ This is the average of $d\mathcal{L}/d\mathbf{w}$ over all the training examples, so in this lecture we focus on computing $d\mathcal{L}/d\mathbf{w}$.

Univariate Chain Rule

- We have already been using the univariate Chain Rule.
- Recall: if $f(x)$ and $x(t)$ are univariate functions, then

$$\frac{d}{dt} f(x(t)) = \frac{df}{dx} \frac{dx}{dt}.$$

Univariate Chain Rule

Recall: Univariate logistic least squares model

$$z = wx + b$$

$$y = \sigma(z)$$

$$\mathcal{L} = \frac{1}{2}(y - t)^2$$

Let's compute the loss derivatives $\frac{\partial \mathcal{L}}{\partial w}$, $\frac{\partial \mathcal{L}}{\partial b}$.

Univariate Chain Rule

How you would have done it in calculus class:

$$\mathcal{L} = \frac{1}{2}(\sigma(wx + b) - t)^2$$

$$\frac{\partial \mathcal{L}}{\partial w} = \frac{\partial}{\partial w} \left[\frac{1}{2}(\sigma(wx + b) - t)^2 \right]$$

$$= \frac{1}{2} \frac{\partial}{\partial w} (\sigma(wx + b) - t)^2$$

$$= (\sigma(wx + b) - t) \frac{\partial}{\partial w} (\sigma(wx + b) - t)$$

$$= (\sigma(wx + b) - t) \sigma'(wx + b) \frac{\partial}{\partial w} (wx + b)$$

$$= (\sigma(wx + b) - t) \sigma'(wx + b) x$$

$$\frac{\partial \mathcal{L}}{\partial b} = \frac{\partial}{\partial b} \left[\frac{1}{2}(\sigma(wx + b) - t)^2 \right]$$

=? (Exercise!)

What are the disadvantages of this approach?

Univariate Chain Rule

A more structured way to do it:

Computing the derivatives:

Computing the loss:

$$z = wx + b$$

$$y = \sigma(z)$$

$$\mathcal{L} = \frac{1}{2}(y - t)^2$$

$$\frac{d\mathcal{L}}{dy} = y - t$$

$$\frac{d\mathcal{L}}{dz} = \frac{d\mathcal{L}}{dy} \frac{dy}{dz} = \frac{d\mathcal{L}}{dy} \sigma'(z)$$

$$\frac{\partial \mathcal{L}}{\partial w} = \frac{d\mathcal{L}}{dz} \frac{dz}{dw} = \frac{d\mathcal{L}}{dz} x$$

$$\frac{\partial \mathcal{L}}{\partial b} = \frac{d\mathcal{L}}{dz} \frac{dz}{db} = \frac{d\mathcal{L}}{dz}$$

Remember: The goal is not to obtain closed-form solutions, but to be able to write a program that efficiently computes the derivatives.

Univariate Chain Rule

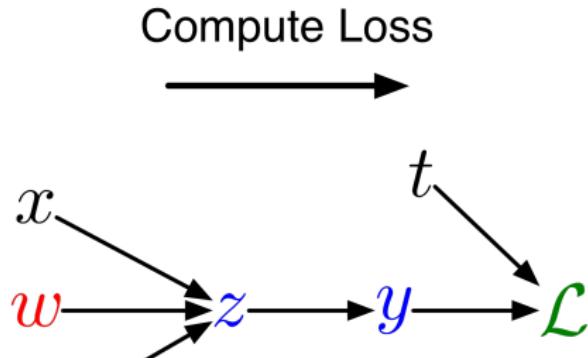
- We can diagram out the computations using a computation graph.
- The nodes represent all the inputs and computed quantities, and the edges represent which nodes are computed directly as a function of which other nodes.

Computing the loss:

$$z = wx + b$$

$$y = \sigma(z)$$

$$\mathcal{L} = \frac{1}{2}(y - t)^2$$



Compute Derivatives

Univariate Chain Rule

A slightly more convenient notation:

- Use \bar{y} to denote the derivative of the loss w.r.t. y (i.e., $d\mathcal{L}/dy$), sometimes called the **error signal**.
- This emphasizes that the error signals are just values our program is computing (rather than a mathematical operation).

Computing the loss:

$$z = wx + b$$

$$y = \sigma(z)$$

$$\mathcal{L} = \frac{1}{2}(y - t)^2$$

Computing the derivatives:

$$\bar{y} = y - t$$

$$\bar{z} = \bar{y} \sigma'(z)$$

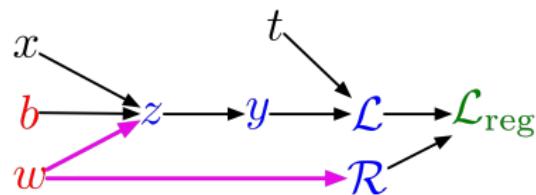
$$\bar{w} = \bar{z} x$$

$$\bar{b} = \bar{z}$$

Multivariate Chain Rule

Problem: what if the computation graph has fan-out > 1?
This requires the Multivariate Chain Rule!

L_2 -Regularized regression



$$z = wx + b$$

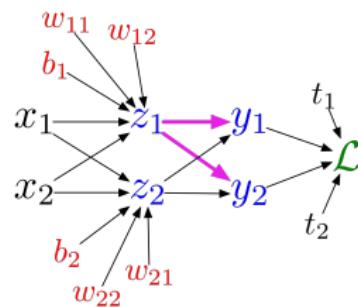
$$y = \sigma(z)$$

$$\mathcal{L} = \frac{1}{2}(y - t)^2$$

$$\mathcal{R} = \frac{1}{2}w^2$$

$$\mathcal{L}_{\text{reg}} = \mathcal{L} + \lambda \mathcal{R}$$

Softmax classifier with the cross-entropy loss



$$z_\ell = \sum_j w_{\ell j} x_j + b_\ell$$

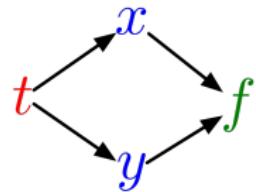
$$y_k = \frac{e^{z_k}}{\sum_\ell e^{z_\ell}}$$

$$\mathcal{L} = - \sum_k t_k \log y_k$$

Multivariate Chain Rule

- Suppose that we have a function $f(x, y)$ and functions $x(t)$ and $y(t)$. (All the variables here are scalar-valued). Then

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$



- Example:

$$f(x, y) = y + e^{xy}$$

$$x(t) = \cos t$$

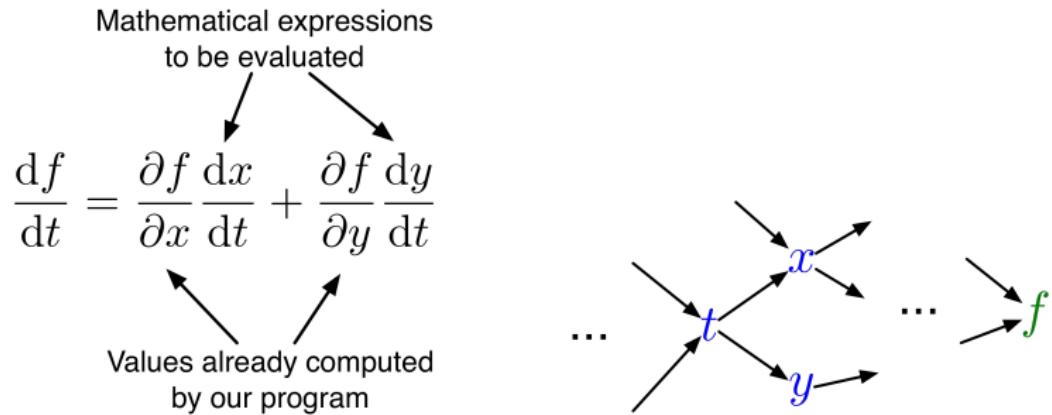
$$y(t) = t^2$$

- Plug in to Chain Rule:

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= (ye^{xy}) \cdot (-\sin t) + (1 + xe^{xy}) \cdot 2t\end{aligned}$$

Multivariate Chain Rule

- In the context of backpropagation:



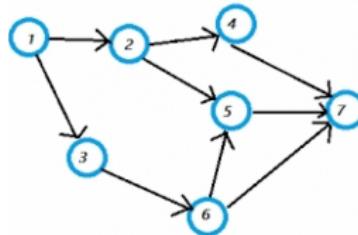
- In our notation:

$$\bar{t} = \bar{x} \frac{dx}{dt} + \bar{y} \frac{dy}{dt}$$

Backpropagation

Full backpropagation algorithm:

Let v_1, \dots, v_N be a topological ordering of the computation graph
(i.e. parents come before children.)

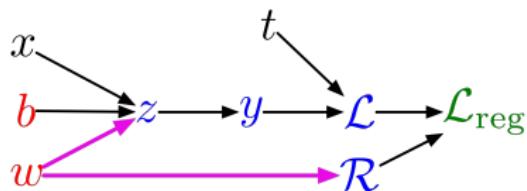


v_N denotes the variable we're trying to compute derivatives of (e.g. loss).

- forward pass
- backward pass
- | | |
|---|---|
| I | For $i = 1, \dots, N$ |
| | Compute v_i as a function of $\text{Pa}(v_i)$ |
| I | $\overline{v_N} = 1$ |
| | For $i = N - 1, \dots, 1$ |
| | $\overline{v_i} = \sum_{j \in \text{Ch}(v_i)} \overline{v_j} \frac{\partial v_j}{\partial v_i}$ |

Backpropagation

Example: univariate logistic least squares regression



Forward pass:

$$z = wx + b$$

$$y = \sigma(z)$$

$$\mathcal{L} = \frac{1}{2}(y - t)^2$$

$$\mathcal{R} = \frac{1}{2}w^2$$

$$\mathcal{L}_{\text{reg}} = \mathcal{L} + \lambda \mathcal{R}$$

Backward pass:

$$\overline{\mathcal{L}_{\text{reg}}} = 1$$

$$\overline{\mathcal{R}} = \overline{\mathcal{L}_{\text{reg}}} \frac{d\mathcal{L}_{\text{reg}}}{d\mathcal{R}}$$

$$= \overline{\mathcal{L}_{\text{reg}}} \lambda$$

$$\overline{\mathcal{L}} = \overline{\mathcal{L}_{\text{reg}}} \frac{d\mathcal{L}_{\text{reg}}}{d\mathcal{L}}$$

$$= \overline{\mathcal{L}_{\text{reg}}}$$

$$\overline{y} = \overline{\mathcal{L}} \frac{d\mathcal{L}}{dy}$$

$$= \overline{\mathcal{L}}(y - t)$$

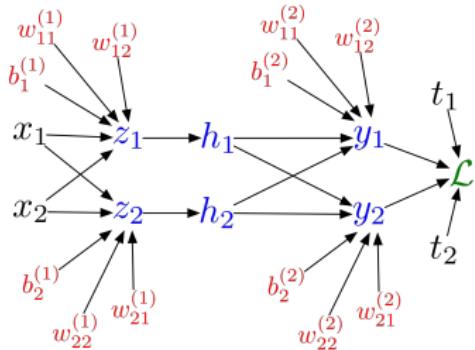
$$\begin{aligned}\overline{z} &= \overline{y} \frac{dy}{dz} \\ &= \overline{y} \sigma'(z)\end{aligned}$$

$$\begin{aligned}\overline{w} &= \overline{z} \frac{\partial z}{\partial w} + \overline{\mathcal{R}} \frac{d\mathcal{R}}{dw} \\ &= \overline{z} x + \overline{\mathcal{R}} w\end{aligned}$$

$$\begin{aligned}\overline{b} &= \overline{z} \frac{\partial z}{\partial b} \\ &= \overline{z}\end{aligned}$$

Backpropagation

Multilayer Perceptron (multiple outputs):



Forward pass:

$$z_i = \sum_j w_{ij}^{(1)} x_j + b_i^{(1)}$$

$$h_i = \sigma(z_i)$$

$$y_k = \sum_i w_{ki}^{(2)} h_i + b_k^{(2)}$$

$$\mathcal{L} = \frac{1}{2} \sum_k (y_k - t_k)^2$$

Backward pass:

$$\bar{\mathcal{L}} = 1$$

$$\bar{y}_k = \bar{\mathcal{L}} (y_k - t_k)$$

$$\bar{w}_{ki}^{(2)} = \bar{y}_k h_i$$

$$\bar{b}_k^{(2)} = \bar{y}_k$$

$$\bar{h}_i = \sum_k \bar{y}_k w_{ki}^{(2)}$$

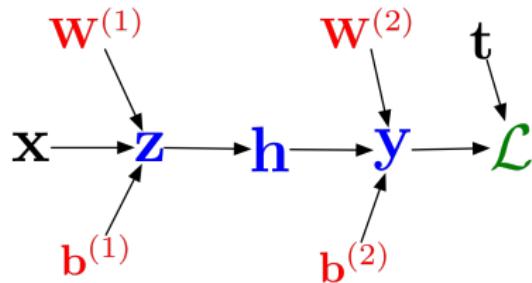
$$\bar{z}_i = \bar{h}_i \sigma'(z_i)$$

$$\bar{w}_{ij}^{(1)} = \bar{z}_i x_j$$

$$\bar{b}_i^{(1)} = \bar{z}_i$$

Backpropagation

In vectorized form:



Forward pass:

$$\mathbf{z} = \mathbf{W}^{(1)}\mathbf{x} + \mathbf{b}^{(1)}$$

$$\mathbf{h} = \sigma(\mathbf{z})$$

$$\mathbf{y} = \mathbf{W}^{(2)}\mathbf{h} + \mathbf{b}^{(2)}$$

$$\mathcal{L} = \frac{1}{2}\|\mathbf{y} - \mathbf{t}\|^2$$

Backward pass:

$$\bar{\mathcal{L}} = 1$$

$$\bar{\mathbf{y}} = \bar{\mathcal{L}}(\mathbf{y} - \mathbf{t})$$

$$\bar{\mathbf{W}^{(2)}} = \bar{\mathbf{y}}\mathbf{h}^\top$$

$$\bar{\mathbf{b}^{(2)}} = \bar{\mathbf{y}}$$

$$\bar{\mathbf{h}} = \mathbf{W}^{(2)\top}\bar{\mathbf{y}}$$

$$\bar{\mathbf{z}} = \bar{\mathbf{h}} \circ \sigma'(\mathbf{z})$$

$$\bar{\mathbf{W}^{(1)}} = \bar{\mathbf{z}}\mathbf{x}^\top$$

$$\bar{\mathbf{b}^{(1)}} = \bar{\mathbf{z}}$$

Computational Cost

- Computational cost of forward pass: one add-multiply operation per weight

$$z_i = \sum_j w_{ij}^{(1)} x_j + b_i^{(1)}$$

- Computational cost of backward pass: two add-multiply operations per weight

$$\overline{w_{ki}^{(2)}} = \overline{y_k} h_i$$

$$\overline{h_i} = \sum_k \overline{y_k} w_{ki}^{(2)}$$

- Rule of thumb: the backward pass is about as expensive as two forward passes.
- For a multilayer perceptron, this means the cost is linear in the number of layers, quadratic in the number of units per layer.

Backpropagation

- Backprop is used to train the overwhelming majority of neural nets today.
 - ▶ Even optimization algorithms much fancier than gradient descent (e.g. second-order methods) use backprop to compute the gradients.
- Despite its practical success, backprop is believed to be neurally implausible.

Conclusion

- Multi-layer feedforward NN addressed the feature learning problem
- Backpropagation as a method to learn NN

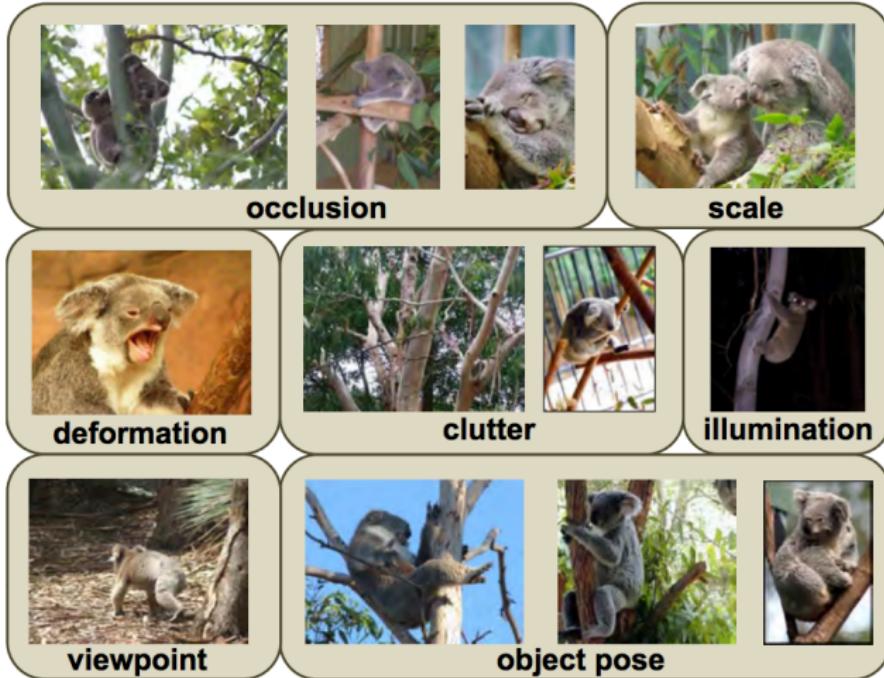
Convolutional Networks (Optional)

Neural Nets for Visual Object Recognition

- People are very good at recognizing shapes
 - ▶ Intrinsically difficult, computers are bad at it
- Why is it difficult?

Why is it a Problem?

- Difficult scene conditions



[From: Grauman & Leibe]

Why is it a Problem?

- Huge within-class variations. Recognition is mainly about modeling variation.



Why is it a Problem?

- Tons of classes



[Biederman]

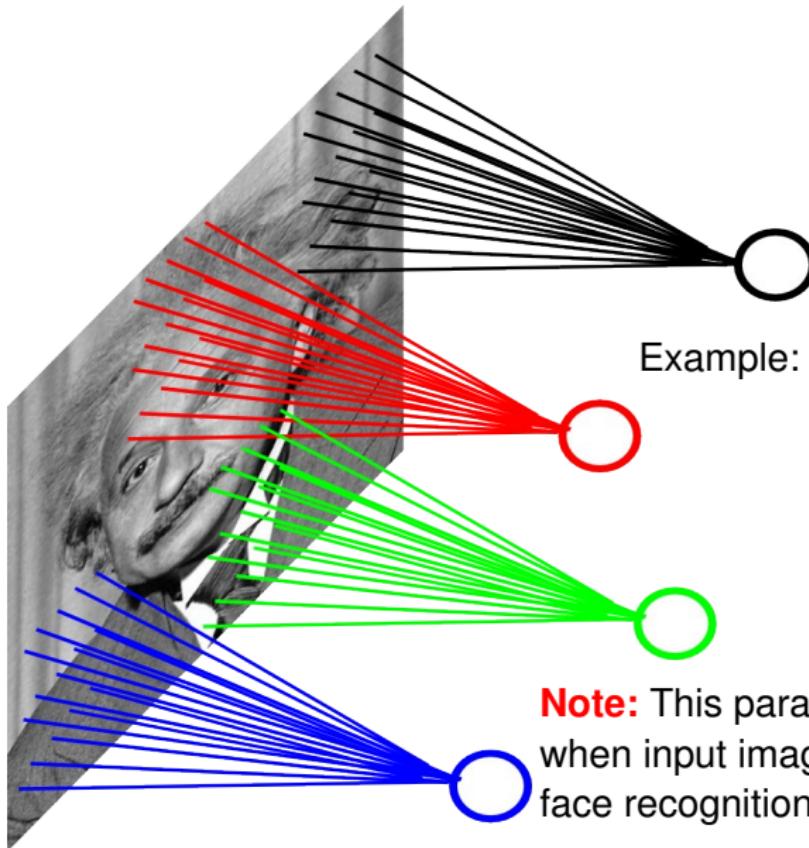
Neural Nets for Object Recognition

- People are very good at recognizing object
 - ▶ Intrinsically difficult, computers are bad at it
- Some reasons why it is difficult:
 - ▶ **Segmentation:** Real scenes are cluttered
 - ▶ **Invariances:** We are very good at ignoring all sorts of variations that do not affect class
 - ▶ **Deformations:** Natural object classes allow variations (faces, letters, chairs)
 - ▶ A huge amount of computation is required

How to Deal with Large Input Spaces

- How can we apply neural nets to images?
- Images can have millions of pixels, i.e., \mathbf{x} is very high dimensional
- How many parameters do we have?
- Prohibitive to have fully-connected layers
- What can we do?
- We can use a **locally connected layer**

Locally Connected Layer



Example:
200x200 image
40K hidden units
Filter size: 10x10
4M parameters

Note: This parameterization is good
when input image is registered (e.g.,
face recognition).

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When Will this Work?

When Will this Work?

- This is good when the **input** is (roughly) registered



General Images

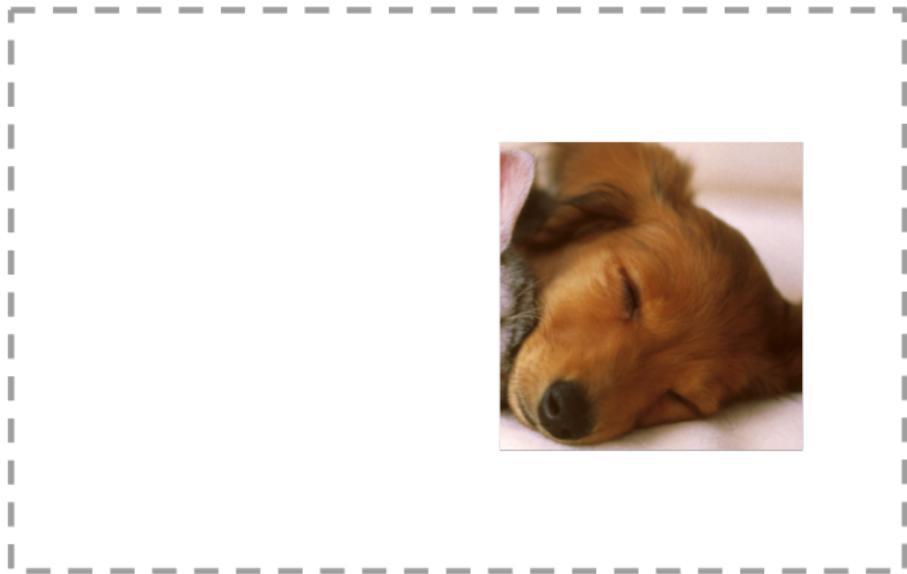
- The object can be anywhere



[Slide: Y. Zhu]

General Images

- The object can be anywhere



[Slide: Y. Zhu]

General Images

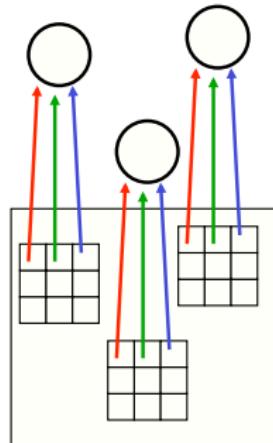
- The object can be anywhere



[Slide: Y. Zhu]

The replicated feature approach

The red connections all have the same weight.

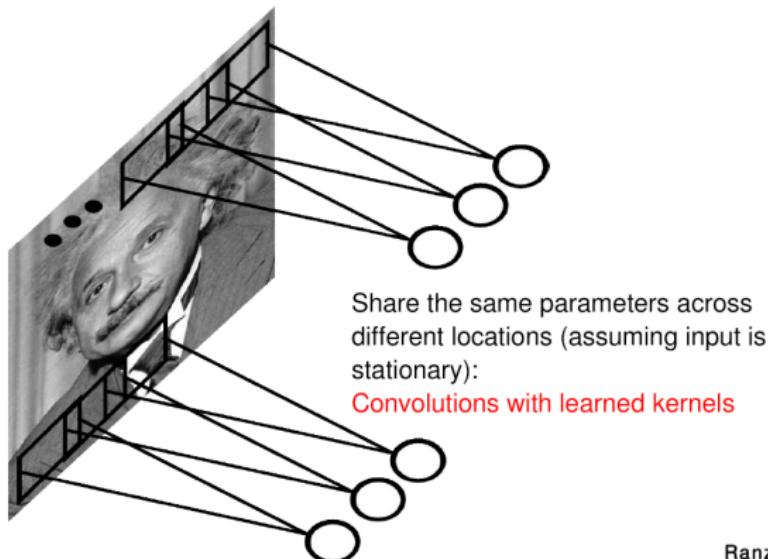


5

- Adopt approach apparently used in monkey visual systems
- Use many different copies of the same feature detector.
 - ▶ Copies have slightly different positions.
 - ▶ Could also replicate across scale and orientation.
 - ▶ Tricky and expensive
 - ▶ Replication **reduces the number of free parameters** to be learned.
- Use several **different feature types**, each with its own replicated pool of detectors.
 - ▶ Allows each patch of image to be represented in several ways.

Convolutional Neural Net

- Idea: Statistics are similar at different locations (Lecun 1998)
- Connect each hidden unit to a small input patch and share the weight across space
- This is called a **convolution layer** and the network is a **convolutional network**



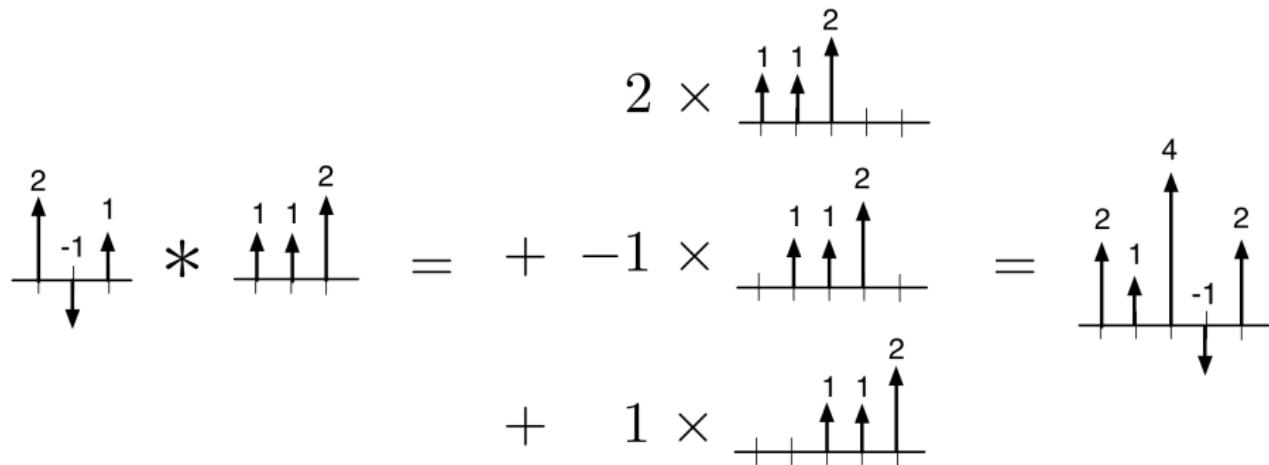
Convolution Operator

- Convolution layers are named after the **convolution** operator.
- If a and b are two arrays (or vector or signal), the convolution $a * b$ between them is defined as a new array (or vector or signal) with its t -th component being

$$(a * b)_t = \sum_{\tau} a_{\tau} b_{t-\tau}.$$

Convolution Operator

Method 1: translate-and-scale

$$\begin{array}{c} 2 \times \\ \begin{array}{c} 1 \\ 1 \\ 2 \end{array} \end{array} * \begin{array}{c} 2 \times \\ \begin{array}{c} 1 \\ 1 \\ 2 \end{array} \end{array} = + -1 \times \begin{array}{c} 2 \times \\ \begin{array}{c} 1 \\ 1 \\ 2 \end{array} \end{array} + 1 \times \begin{array}{c} 2 \times \\ \begin{array}{c} 1 \\ 1 \\ 2 \end{array} \end{array} = \begin{array}{c} 4 \\ 2 \\ 1 \\ -1 \\ 2 \end{array}$$


Convolution Operator

Method 2: flip-and-filter

$$\begin{array}{c} 2 \\ \uparrow \\ -1 \\ \uparrow \\ 1 \\ \downarrow \end{array} * \begin{array}{c} 1 \\ \uparrow \\ 1 \\ \uparrow \\ 2 \\ \uparrow \end{array} =$$

$$\begin{array}{c} 2 \\ \uparrow \\ -1 \\ \uparrow \\ 1 \\ \downarrow \end{array}$$

$$\begin{array}{c} 2 \\ \uparrow \\ 1 \\ \uparrow \\ 1 \\ \downarrow \end{array}$$

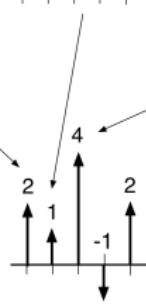
$$\begin{array}{c} 2 \\ \uparrow \\ -1 \\ \uparrow \\ 1 \\ \downarrow \end{array}$$

$$\begin{array}{c} 2 \\ \uparrow \\ 1 \\ \uparrow \\ 1 \\ \downarrow \end{array}$$

$$\begin{array}{c} 2 \\ \uparrow \\ -1 \\ \uparrow \\ 1 \\ \downarrow \end{array}$$

$$\begin{array}{c} 2 \\ \uparrow \\ 1 \\ \uparrow \\ 1 \\ \downarrow \end{array}$$

...


$$\begin{array}{c} 2 \\ \uparrow \\ 1 \\ \uparrow \\ 1 \\ \uparrow \\ 2 \\ \uparrow \\ -1 \\ \downarrow \end{array}$$

Convolution Operator

Convolution can also be viewed as matrix multiplication:

$$(2, -1, 1) * (1, 1, 2) = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 2 & 1 & 1 \\ & 2 & 1 \\ & & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Note: This is how convolution is typically implemented. It is more efficient than the fast Fourier transform (FFT) for modern conv nets on GPUs.

Convolution Operator

Some properties of convolution:

- Commutativity

$$a * b = b * a$$

- Distributivity

$$a * (\lambda_1 b + \lambda_2 c) = \lambda_1 a * b + \lambda_2 a * c$$

2-D Convolution Operator

2-D convolution is defined analogously to 1-D convolution.

If A and B are two 2-D arrays, then:

$$(A * B)_{ij} = \sum_s \sum_t A_{st} B_{i-s, j-t}.$$

2-D Convolution Operator

Method 1: Translate-and-Scale

$$\begin{array}{c} 1 \times \\ \begin{array}{|c|c|c|c|} \hline 1 & 3 & 1 & \\ \hline 0 & -1 & 1 & \\ \hline 2 & 2 & -1 & \\ \hline \end{array} \end{array} \\ \begin{array}{c} * \\ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 0 & -1 \\ \hline \end{array} \end{array} = + 2 \times \begin{array}{|c|c|c|c|} \hline & 1 & 3 & 1 \\ \hline & 0 & -1 & 1 \\ \hline & 2 & 2 & -1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 5 & 7 & 2 \\ \hline 0 & -2 & -4 & 1 \\ \hline 2 & 6 & 4 & -3 \\ \hline 0 & -2 & -2 & 1 \\ \hline \end{array}$$

$$+ -1 \times \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & 1 & 3 & 1 \\ \hline & 0 & -1 & 1 \\ \hline & 2 & 2 & -1 \\ \hline \end{array}$$

2-D Convolution Operator

Method 2: Flip-and-Filter

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 1 \\ \hline 0 & -1 & 1 \\ \hline 2 & 2 & -1 \\ \hline \end{array} \quad * \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 0 & -1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 1 \\ \hline 0 & -1 & 1 \\ \hline 2 & 2 & -1 \\ \hline \end{array} \quad \times \quad \begin{array}{|c|c|} \hline -1 & 0 \\ \hline 2 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 5 & 7 & 2 \\ \hline 0 & -2 & -4 & 1 \\ \hline 2 & 6 & 4 & -3 \\ \hline 0 & -2 & -2 & 1 \\ \hline \end{array}$$

The diagram illustrates the convolution process. A blue box highlights the top-left 3x3 receptive field of the output unit at position (0,0) in the result matrix. A red box highlights the bottom-right 3x3 receptive field of the input unit at position (2,2). Blue arrows show the weights from the blue box being multiplied by the corresponding input values (1, 3, 1, 0, -1, 1), resulting in the value -2 in the output matrix. A red arrow shows the weight 1 being multiplied by the value -1 from the red box, resulting in the value -2 in the output matrix.

2-D Convolution Operator

We convolve an input by a **kernel**, or **filter**.

- The term Filter is used due to the original of convolutional in **signal processing**, in which the convolution operator is used to compute the effect of a linear filter on an input.
- Do not confuse this kernel with the kernels in an RKHS.

What does this filter do?



*

0	1	0
1	4	1
0	1	0



2-D Convolution Operator

What does this filter do?



*

0	-1	0
-1	8	-1
0	-1	0



2-D Convolution Operator

What does this filter do?



*

0	-1	0
-1	4	-1
0	-1	0



2-D Convolution Operator

What does this filter do?

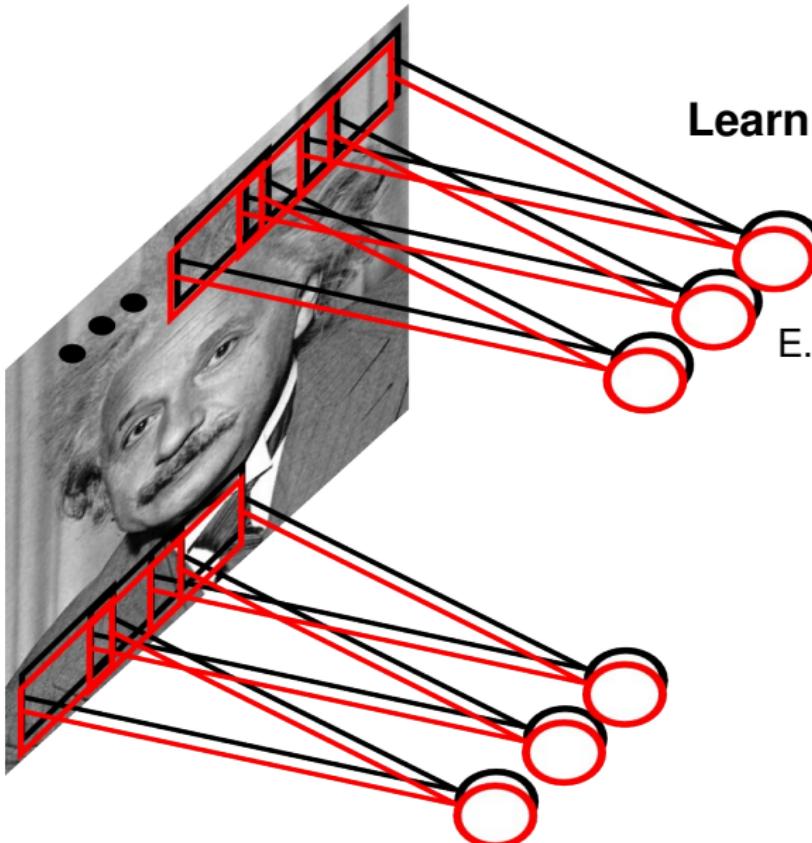


*

1	0	-1
2	0	-2
1	0	-1



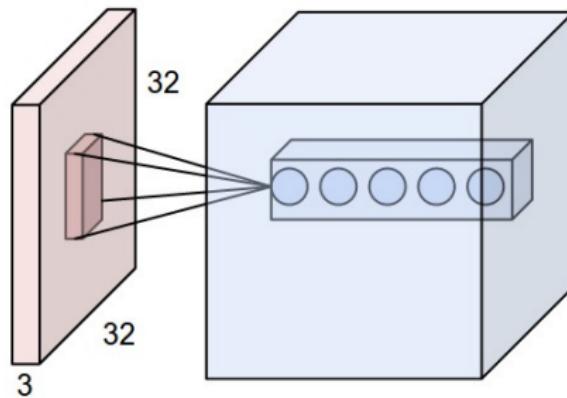
Convolutional Layer



Learn multiple filters.

E.g.: 200x200 image
100 Filters
Filter size: 10x10
10K parameters

Convolutional Layer

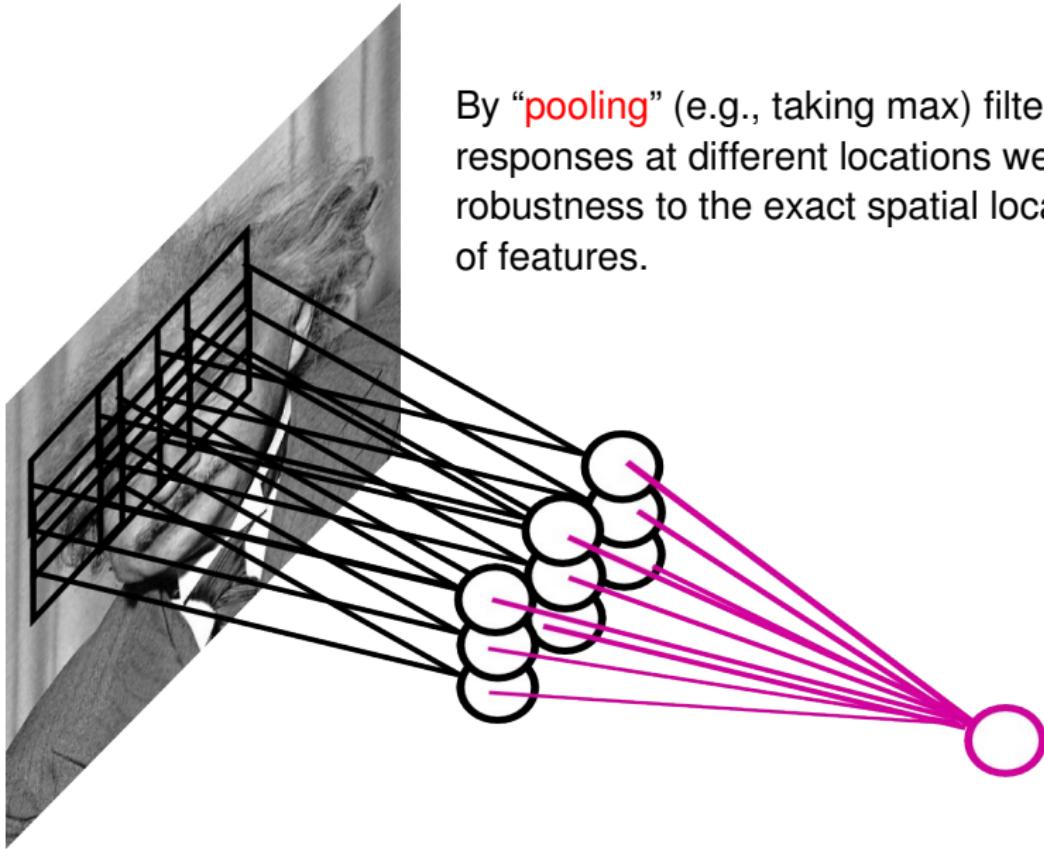


Hyperparameters of a convolutional layer:

- The number of filters (controls the **depth** of the output volume)
- The **stride**: how many units apart do we apply a filter spatially (this controls the spatial size of the output volume)
- The size $w \times h$ of the filters

[\[http://cs231n.github.io/convolutional-networks/\]](http://cs231n.github.io/convolutional-networks/)

Pooling Layer



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Ranzato

Pooling Options

- Max Pooling: return the maximal argument
- Average Pooling: return the average of the arguments
- Other types of pooling exist too

Pooling

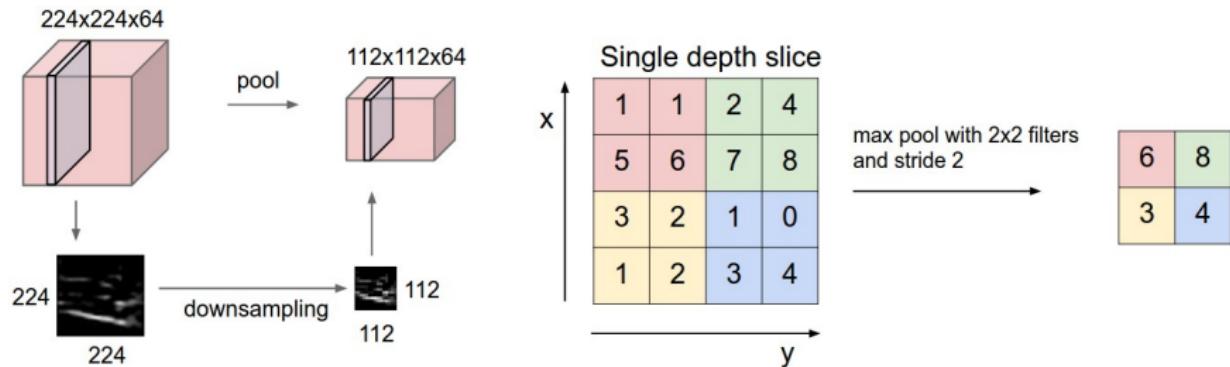


Figure: **Left:** Pooling, **right:** max pooling example

Hyperparameters of a pooling layer:

- The spatial extent F
- The stride

[<http://cs231n.github.io/convolutional-networks/>]

Backpropagation with Weight Constraints

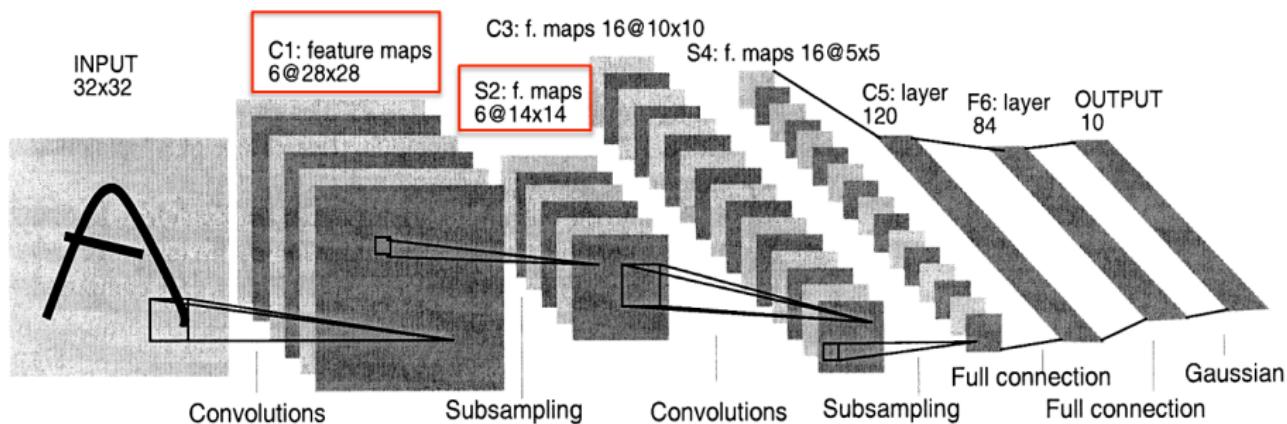
- The backpropagation algorithm from earlier can be applied directly to ConvNets
- This is covered in CSC2516.
- As a user, you do not need to worry about the details, since they are handled by automatic differentiation packages.

MNIST Dataset

- MNIST dataset of handwritten digits
 - ▶ **Categories:** 10 digit classes
 - ▶ **Source:** Scans of handwritten zip codes from envelopes
 - ▶ **Size:** 60,000 training images and 10,000 test images, grayscale, of size 28×28
 - ▶ **Normalization:** centered within in the image, scaled to a consistent size
 - ▶ The assumption is that the digit recognizer would be part of a larger pipeline that segments and normalizes images.
- In 1998, Yann LeCun and colleagues built a conv net called [LeNet](#) which was able to classify digits with 98.9% test accuracy.
 - ▶ It was good enough to be used in a system for automatically reading numbers on checks.

LeNet

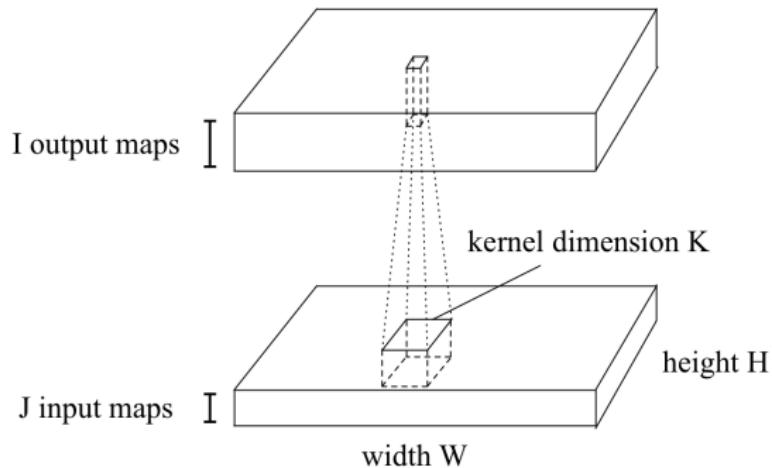
Here is the LeNet architecture, which was applied to handwritten digit recognition on MNIST in 1998:



Size of a Conv Net

- Ways to measure the size of a network:
 - ▶ **Number of units.** This is important because the activations need to be stored in memory during training (i.e. backprop).
 - ▶ **Number of weights.** This is important because the weights need to be stored in memory, and because the number of parameters affects the overfitting.
 - ▶ **Number of connections.** This is important because there are approximately 3 add-multiply operations per connection (1 for the forward pass, 2 for the backward pass).
- We saw that a fully connected layer with M input units and N output units has MN connections and MN weights.
- The story for conv nets is more complicated.

Size of a Conv Net



	fully connected layer	convolution layer
# output units	WHI	WHI
# weights	W^2H^2IJ	K^2IJ
# connections	W^2H^2IJ	WHK^2IJ

Size of a Conv Net

Sizes of layers in LeNet:

Layer	Type	# units	# connections	# weights
C1	convolution	4704	117,600	150
S2	pooling	1176	4704	0
C3	convolution	1600	240,000	2400
S4	pooling	400	1600	0
F5	fully connected	120	48,000	48,000
F6	fully connected	84	10,080	10,080
output	fully connected	10	840	840

Conclusions?

Size of a Conv Net

- Rules of thumb:
 - ▶ Most of the units and connections are in the convolution layers.
 - ▶ Most of the weights are in the fully connected layers.
- If you try to make layers larger, you'll run up against various resource limitations (i.e. computation time, memory)
- You'll repeat this exercise for AlexNet for homework.
 - ▶ Conv nets have gotten a LOT larger since 1998!

ImageNet

ImageNet is the modern object recognition benchmark dataset. It was introduced in 2009, and has led to amazing progress in object recognition since then.

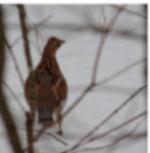
ILSVRC



flamingo



cock



ruffed grouse



quail



partridge

...



Egyptian cat



Persian cat



Siamese cat

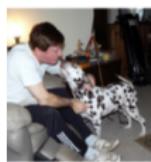


tabby



lynx

...



dalmatian



keeshond



miniature schnauzer



standard schnauzer



giant schnauzer

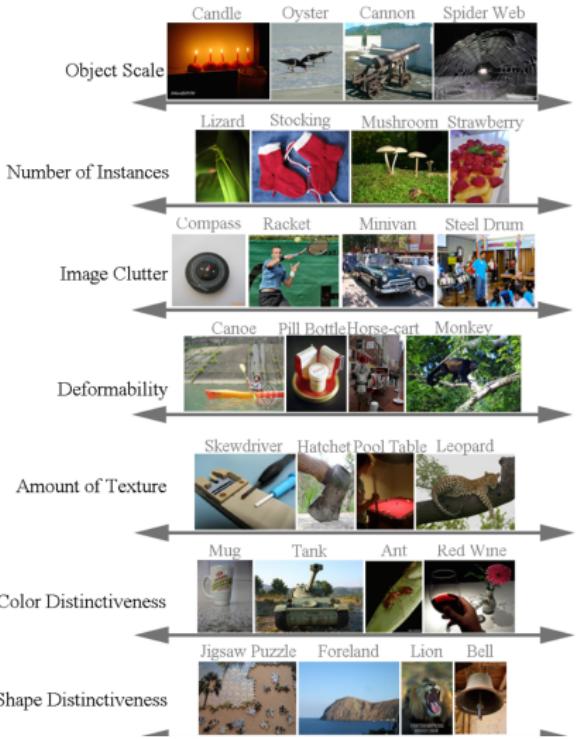
...

ImageNet

- Used for the ImageNet Large Scale Visual Recognition Challenge (ILSVRC), an annual benchmark competition for object recognition algorithms
- Design decisions
 - ▶ **Categories:** Taken from a lexical database called WordNet
 - ▶ WordNet consists of “synsets”, or sets of synonymous words
 - ▶ They tried to use as many of these as possible; almost 22,000 as of 2010
 - ▶ Of these, they chose the 1000 most common for the ILSVRC
 - ▶ The categories are really specific, e.g. hundreds of kinds of dogs
 - ▶ **Size:** 1.2 million full-sized images for the ILSVRC
 - ▶ **Source:** Results from image search engines, hand-labeled by Mechanical Turkers
 - ▶ Labeling such specific categories was challenging; annotators had to be given the WordNet hierarchy, Wikipedia, etc.
 - ▶ **Normalization:** none, although the contestants are free to do preprocessing

ImageNet

Images and object categories vary on a lot of dimensions



Russakovsky et al.

ImageNet

Size on disk:

MNIST
60 MB

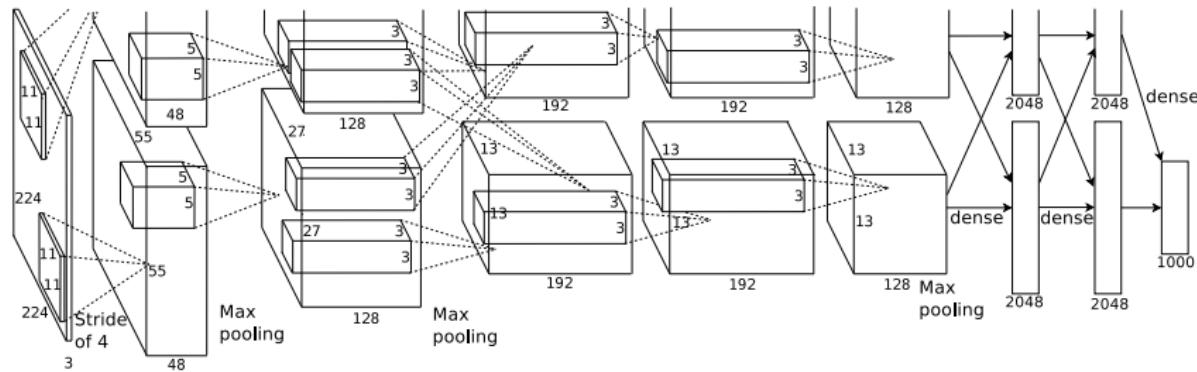


ImageNet
50 GB



AlexNet

- AlexNet, 2012. 8 weight layers. 16.4% top-5 error (i.e. the network gets 5 tries to guess the right category).



(Krizhevsky et al., 2012)

- The two processing pathways correspond to 2 GPUs. (At the time, the network couldn't fit on one GPU.)
- AlexNet's stunning performance on the ILSVRC is what set off the deep learning boom of the last 8-9 years.

Inception

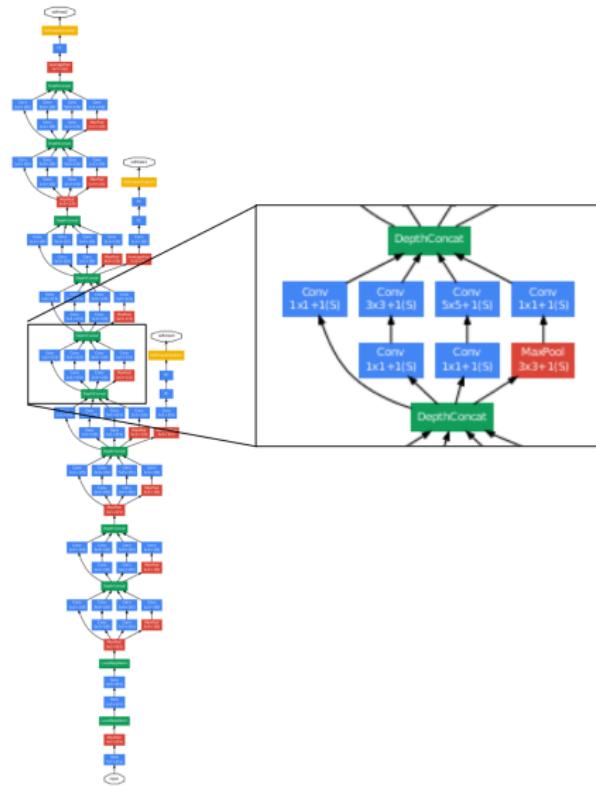
Inception, 2014. (“We need to go deeper!”)

22 weight layers

Fully convolutional (no fully connected layers)

Convolutions are broken down into a bunch of smaller convolutions

6.6% test error on ImageNet



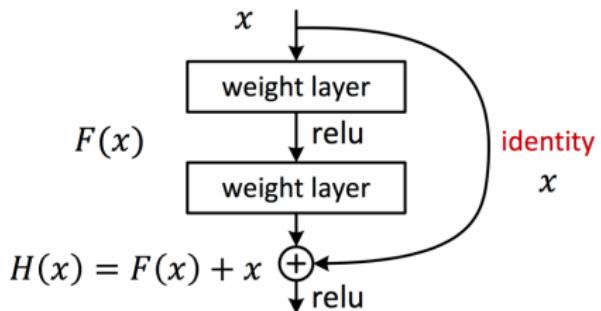
(Szegedy et al., 2014)

Inception

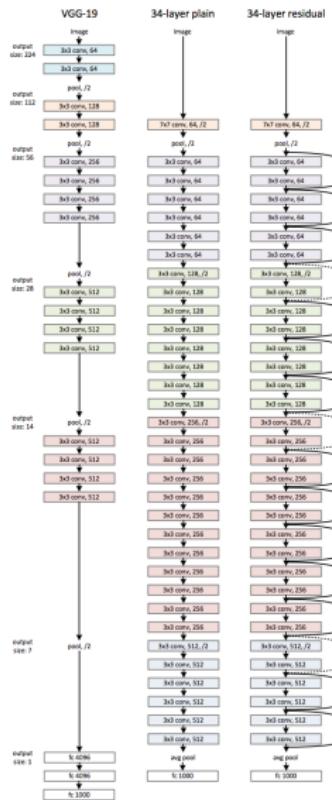
- They were really aggressive about cutting the number of parameters.
 - ▶ Motivation: train the network on a large cluster, run it on a cell phone
 - ▶ Memory at test time is the big constraint.
 - ▶ Having lots of units is OK, since the activations only need to be stored at training time (for backpropagation).
 - ▶ Parameters need to be stored both at training and test time, so these are the memory bottleneck.
 - ▶ How they did it
 - ▶ No fully connected layers (remember, these have most of the weights)
 - ▶ Break down convolutions into multiple smaller convolutions (since this requires fewer parameters total)
 - ▶ Inception has “only” 2 million parameters, compared with 60 million for AlexNet
 - ▶ This turned out to improve generalization as well. (Overfitting can still be a problem, even with over a million images!)

150 Layers!

- Networks are now at 150 layers
- They use a skip connections with special form
- In fact, they don't fit on this screen
- Amazing performance!
- A lot of “mistakes” are due to wrong ground-truth

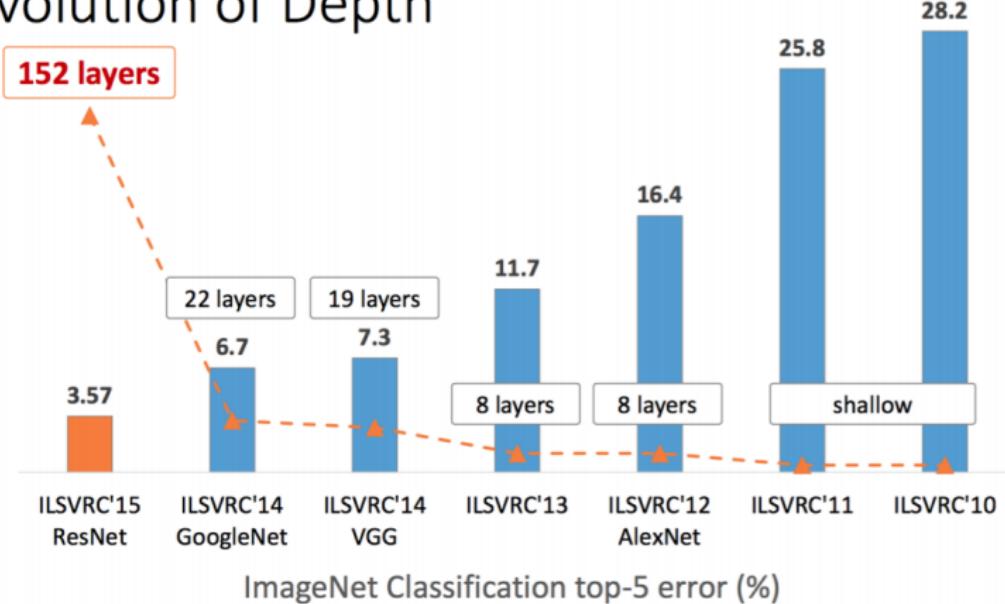


[He, K., Zhang, X., Ren, S. and Sun, J., 2015. Deep Residual Learning for Image Recognition. arXiv:1512.03385, 2016]



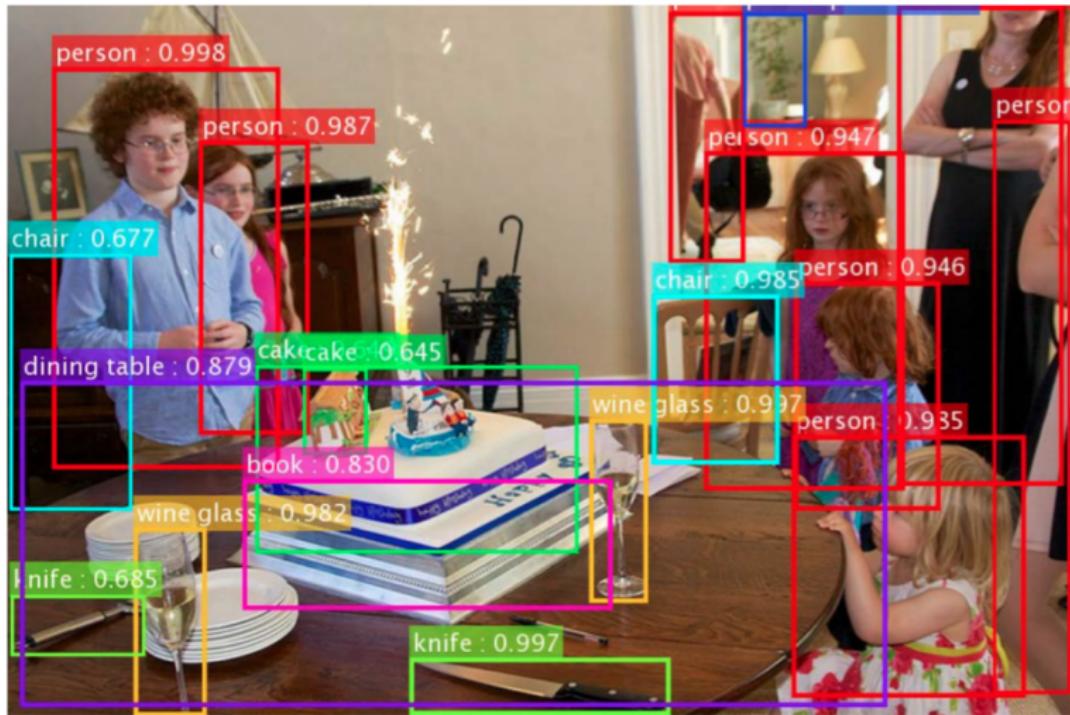
Results: Object Classification

Revolution of Depth



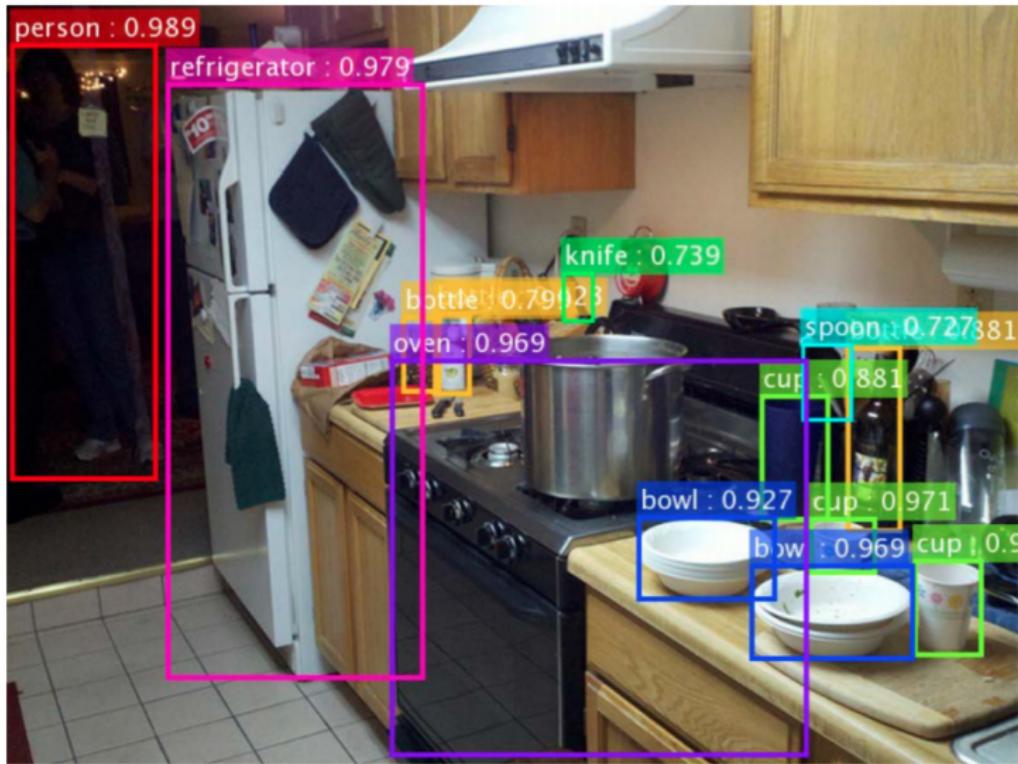
Slide: R. Liao, Paper: [He, K., Zhang, X., Ren, S. and Sun, J., 2015. Deep Residual Learning for Image Recognition. arXiv:1512.03385, 2016]

Results: Object Detection

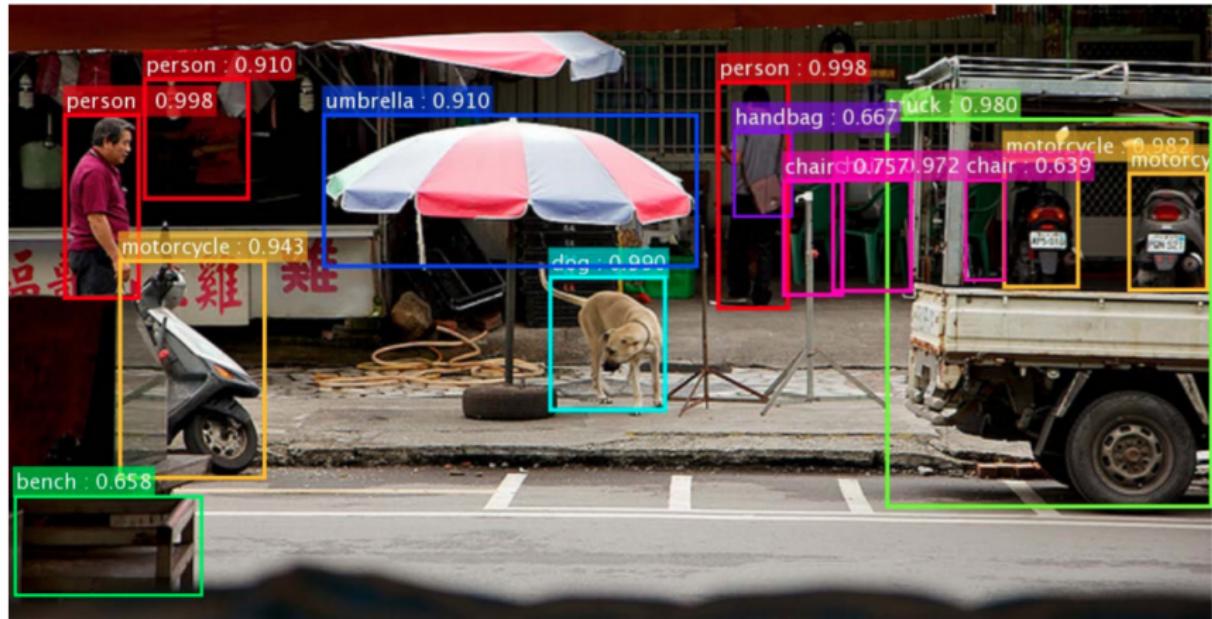


Slide: R. Liao, Paper: [He, K., Zhang, X., Ren, S. and Sun, J., 2015. Deep Residual Learning for Image Recognition. arXiv:1512.03385, 2016]

Results: Object Detection



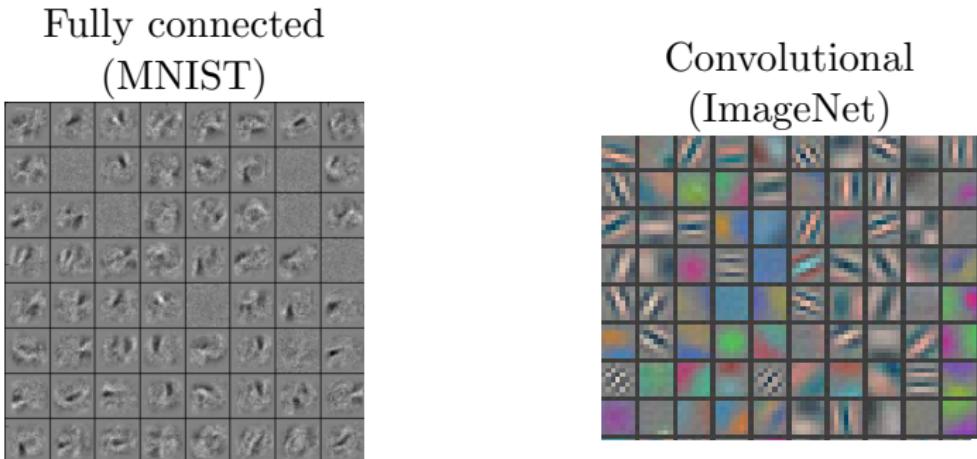
Results: Object Detection



Slide: R. Liao, Paper: [He, K., Zhang, X., Ren, S. and Sun, J., 2015. Deep Residual Learning for Image Recognition. arXiv:1512.03385, 2016]

What Do Networks Learn?

- Recall: we can understand what first-layer features are doing by visualizing the weight matrices.



- Higher-level weight matrices are hard to interpret.
- The better the input matches these weights, the more the feature activates.
 - ▶ Obvious generalization: visualize higher-level features by seeing what inputs activate them.