The Pattern Recogition 's homework

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October 20, 2015

1 Ex 1

The log-likelihood function is:

$$L(\mu) = \ln p(\mathscr{K}|\mu) = \sum_{i=1}^{N} lnp(x_i|\mu) = \frac{1}{2}\sum_{i=1}^{N}(x_i - \mu)^2 + C \tag{1}$$

The derivation of $L(\mu)$,we could get that :

$$\frac{\partial L(\mu)}{\partial \mu} = \sum_{i=1}^{N} x_i - N\mu = 0$$

So the maximum likelihood estimation of $\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$

The bayesian estimation :

First we get the posterior:

$$\begin{aligned}
p(\mu|\mathscr{K}) &= \frac{p(\mathscr{K}|\mu)p(\mu)}{\int p(\mathscr{K}|\mu)p(\mu)du} \\
&= \alpha \prod_{i=1}^{N} p(x_i|u)p(u) \\
&= \alpha \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} exp[-\frac{(x_i - u)^2}{2\sigma^2}] \cdot \sqrt{2\pi}\sigma_0 exp[-\frac{(u-u_0)^2}{2\sigma_0^2}] \\
&= \alpha^{'}[-\frac{1}{2}[(\frac{N}{\sigma^2} + \frac{1}{\sigma_0^2})\mu^2 - 2(\frac{1}{\sigma^2}\sum_{i=1}^{N} x_i + \frac{\sigma_0}{\sigma_0^2})\mu]]
\end{aligned}$$

According to the formula ,apply the method of undetermined coefficients ,we can get that :

$$\hat{u} = \int \mu p(\mu|\mathscr{K})du = \int \mu \frac{1}{\sqrt{2\pi}\sigma_N} exp[-\frac{1}{2}(\frac{u-u_N}{\sigma_N})^2]du = u_N$$

Combine with $\sigma = 1, u_0 = 0, \sigma_0 = 1$,we can get the bayesian estimation :

$$\hat{\mu} = \frac{1}{N+1}\sum_{i=1}^{N} x_i$$

2 Ex 2

According to the theory of maximum likelihood estimation and derivation of EX1,

$$l(\theta) = P(x|\theta) = \prod_{k=1}^{n} p(x_k|\theta)$$

$$H(\theta) = \ln l(\theta) = \sum_{k=1}^{n} \ln p(x_k|\theta) = \sum_{k=1}^{n}\sum_{i=1}^{d}[x_{ki}ln(\theta_i) + (1 - x_{ki}\ln(1-\theta_i))]$$

In terms of $0 = \dfrac{\partial H(\theta)}{\partial \theta_i} = \sum_{k=1}^{n}[\dfrac{x_{ki}}{\theta_i} - \dfrac{1-x_{ki}}{1-\theta_i}]$

We get that:

$$\hat{\theta} = \frac{1}{n}\sum_{k=1}^{n} x_{ki}, i = 1, \cdots, d;$$

So,$\hat{\theta} = \dfrac{1}{n}\sum_{k=1}^{n} x_k$

3 Ex3

According to the theory of bayes estimation,and the derivation of Ex1,the results of multivariate normal distribution is :

$$\begin{cases} \Sigma_N^{-1} = \Sigma_0^{-1} + N\Sigma^{-1} \\ u_N^T = (\Sigma^{-1}\sum_{i=1}^{N} x_i^T + \mu_0^T\Sigma_0^{-1})\Sigma_N \end{cases} \tag{2}$$

In terms of the equation of the matrix ,we have:

$$\begin{aligned} \Sigma_N &= (\Sigma_0^{-1} + (\frac{1}{N}\Sigma)^{-1})^{-1} \\ &= \Sigma_0(\Sigma_0 + \frac{1}{N}\Sigma)^{-1}\frac{1}{N}\Sigma \\ &= \Sigma_0(N\Sigma_0 + \Sigma)^{-1}\Sigma \end{aligned}$$

$$\begin{aligned} \mu_N^T &= (\Sigma^{-1}\sum_{i=1}^{N} x_i^T + u_0^T\Sigma_0^{-1}) \cdot \Sigma_0(N\Sigma_0 + \Sigma)^{-1}\Sigma \\ &= \sum_{i=1}^{N} x_i^T(N\Sigma_0 + \Sigma)^{-1}\Sigma_0 + \mu_0^T(N\Sigma_0 + \Sigma)^{-1}\Sigma \\ &= \Sigma_0(N\Sigma_0 + \Sigma)^{-1}\sum_{i=1}^{N} x_i + \Sigma(N\Sigma_0 + \Sigma)^{-1}\mu_0 \end{aligned}$$

4 Ex4

4.1 (1)

The maximum likelihood estimation of the mean of the normal distribution μ is $\hat{\mu} = \dfrac{1}{N}\sum_{i=1}^{N} x_i$.So,we have:

$$E[\hat{\mu}] = E[\frac{1}{N}\sum_{i=1}^{N}] = \frac{1}{N}\sum_{i=1}^{N}\mu = \mu$$

So the $\hat{\mu}$ is ubbiased estimation.

4.2 (2)

The maximum likelihood estimation of the variance of the normal distribution σ^2 is $\hat{\sigma}^2 = \dfrac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu})^2$. So:

$$\begin{aligned}
E[\hat{\sigma}^2] &= E[\frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu})^2] \\
&= E[\frac{1}{N}\sum_{i=1}^{N}x_i^2 - \hat{\mu}^2] \\
&= \frac{1}{N}\sum_{i=1}^{N}(\sigma^2 + \mu^2) - (\frac{\sigma^2}{N} + \mu^2) \\
&= \frac{N-1}{N}\sigma^2 \neq \sigma^2
\end{aligned}$$

So $\hat{\sigma}^2$ is not a ubbiased estimation.

5 Ex5

5.1 (1)

$$\begin{aligned}
\lim_{N\to\infty} E[\hat{p}_N(x)] &= \lim_{N\to\infty} E[\frac{1}{NV_N}\sum_{i=1}^{N}\phi(\frac{x - x_i}{h_N})] \\
&= E[\lim_{N\to\infty}\sigma_N(x - x_i)] \\
&= \int \delta(x - V)p(V)dv \\
&= p(x)
\end{aligned}$$

5.2 (2)

$$
\begin{aligned}
\lim_{N\to\infty} Var[\hat{p}_N(x)] &= \lim_{N\to\infty} E\hat{p}_N^2(x) - (E\hat{p}_N(x))^2 \\
&= \lim_{N\to\infty} E[\frac{1}{NV_N}\sum_{i=1}^{N}\phi(\frac{x-x_i}{h_N})\frac{1}{NV_N}\sum_{j=1}^{N}\phi(\frac{x-x_j}{h_N})] - [\lim_{N\to\infty} E\hat{p}_N(x)]^2 \\
&= \frac{1}{N^2}E[\lim_{N\to\infty}\frac{1}{V_N}\sum_{i=1}^{N}\phi(\frac{x-x_i}{h_N})\lim_{N\to\infty}\frac{1}{V_N}\sum_{j=1}^{N}\phi(\frac{x-x_j}{h_N})] - p^2(x) \\
&= E[\lim_{N\to\infty}\delta_N(x-V_1)\delta(x-V_2)] - p^2(x) \\
&= E(\delta(x-V_1)\delta(x-V_2) - p^2(x) \\
&= \int \delta^2(x-V)^2p^2(V)dv - p^2(x) \\
&= p^2(x) - p^2(x) \\
&= 0
\end{aligned}
$$