The Z-Transform and Its Application to the Analysis of LTI Systems

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Why we need the Z transform ?

The Direct Z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) \equiv Z\{x(n)\}$$

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

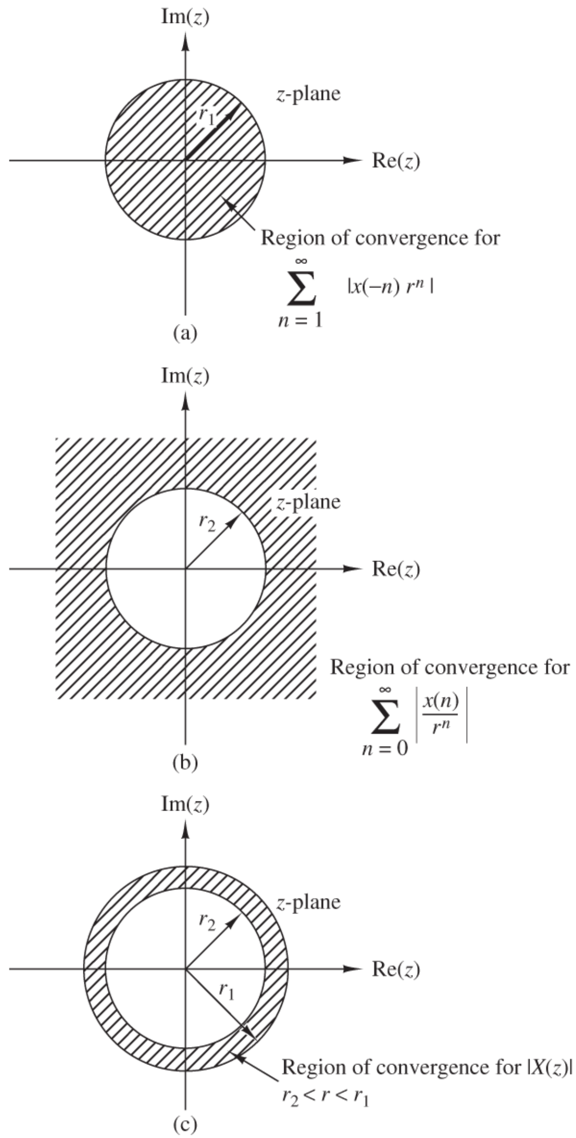

Figure 3.1.1 Region of convergence for $X(z)$ and its corresponding causal and anticausal components.

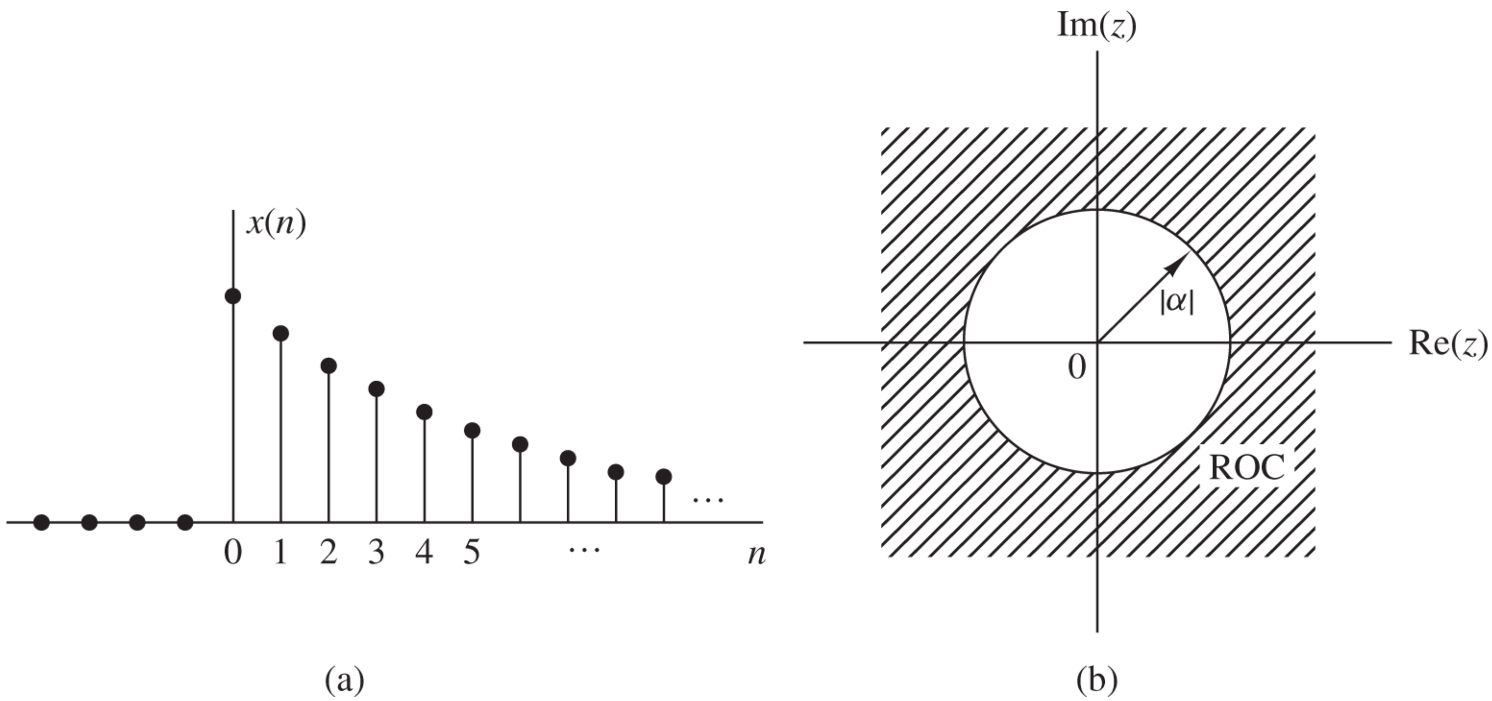

Figure 3.1.2 The exponential signal $x(n) = \alpha^n u(n)$ (a), and the ROC of its z-transform (b).

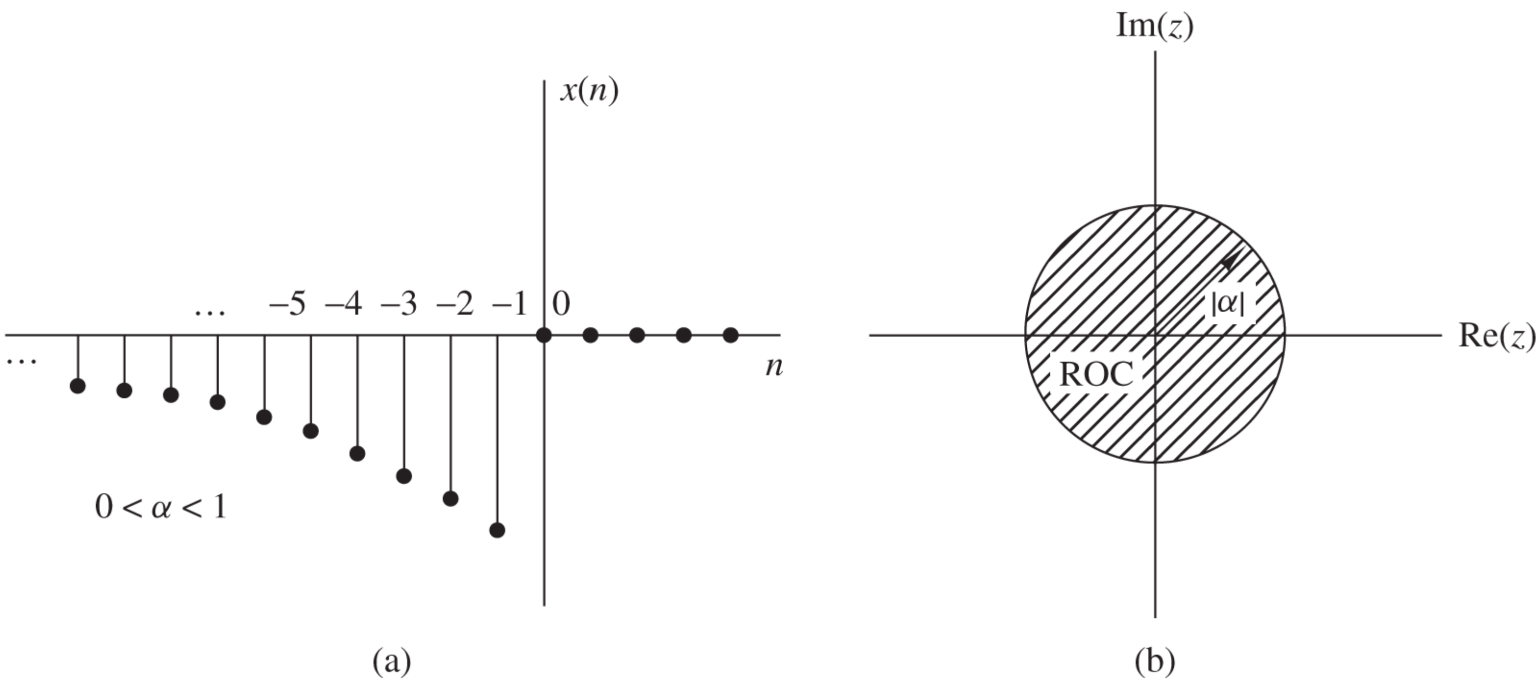

Figure 3.1.3 Anticausal signal $x(n) = -\alpha^n u(-n-1)$ (a), and the ROC of its z-transform (b).

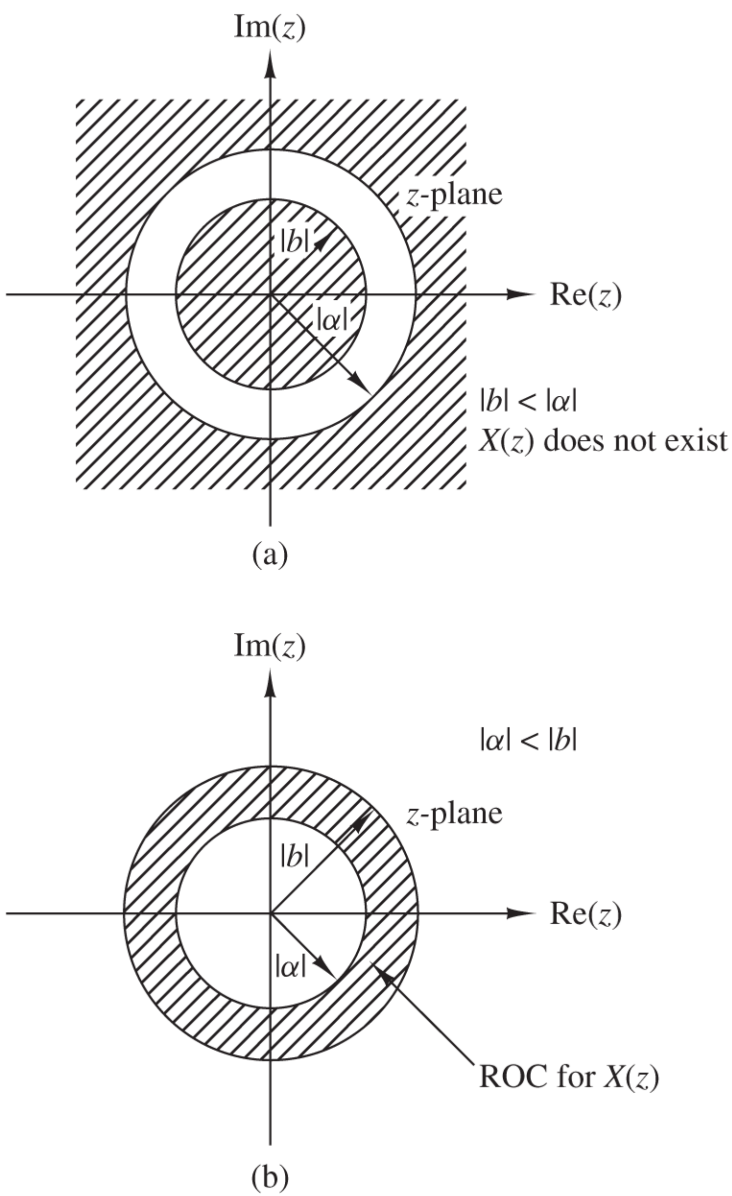

Figure 3.1.4 ROC for z-transform in Example 3.1.5.

The Inverse z-Transform

$$x(n)=\frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

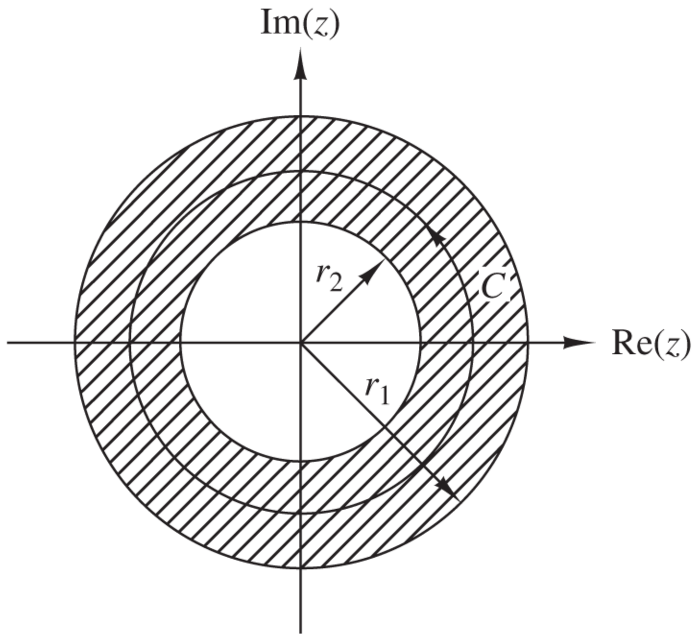

Figure 3.1.5 Contour C for integral in (3.1.13).

Properties of the z-Transform

- Linear
- Time shifting
- Scaling in the z-domain
- Time reversal
- Differentiation in the z-domain
- Convolution of two sequences
- Correlation of two sequences
- Multiplication of two sequences
- Parseval’s relation
- The Initial Value Theorem

Poles and Zeros

$$X(z)=\frac{B(z)}{A(z)}=\frac{b_0+b_1z^{-1}+\ldots+b_Mz^{-M}}{a_0+a_1z^{-1}+\ldots+a_Nz^{-N}}$$

$$=\frac{b_0}{a_0}z^{-M+N}\frac{(z-z_1)(z-z_2)\ldots(z-z_M)}{(z-p_1)(z-p_2)\ldots(z-p_N)}$$

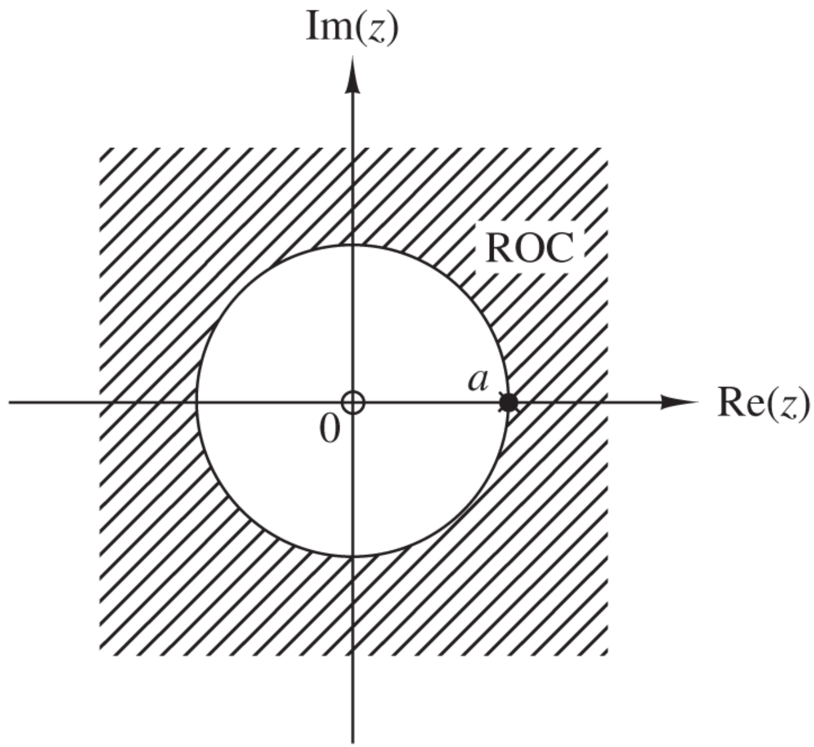

Figure 3.3.1 Pole–zero plot for the causal exponential signal $x(n)=a^nu(n)$.

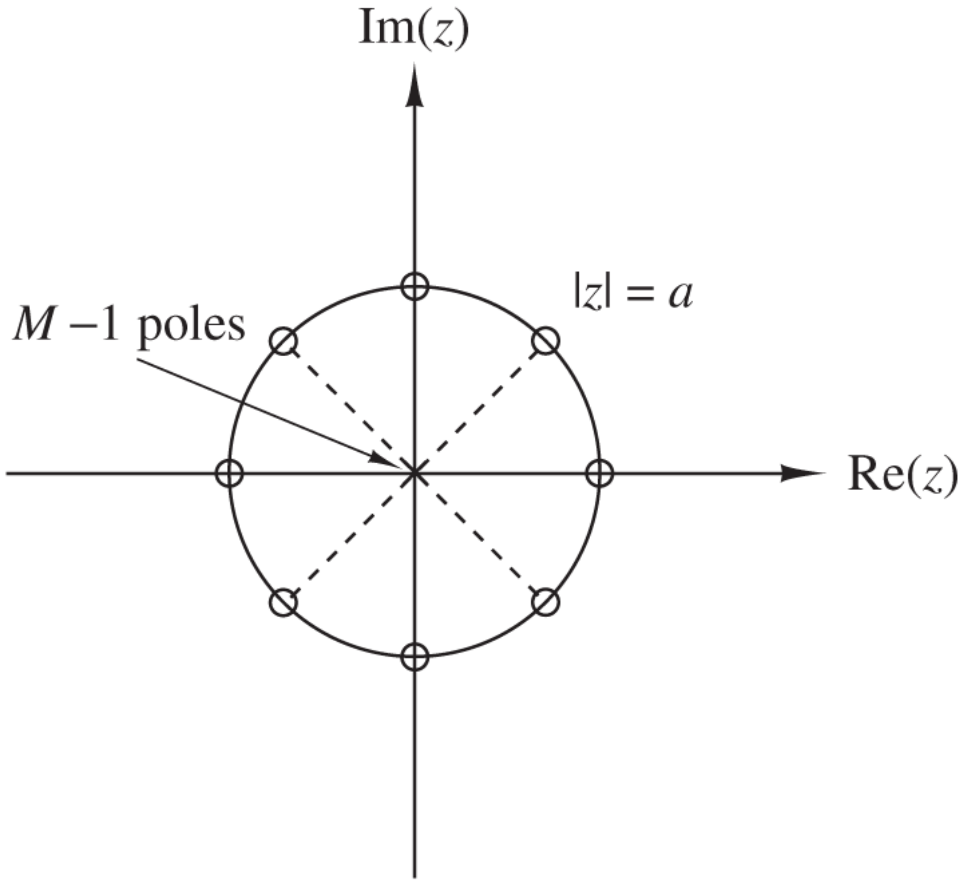

Figure 3.3.2 Pole–zero pattern for the finite-duration signal $x(n) = a^n$, $0 \le n \le M - 1 (a > 0)$, for $M = 8$.

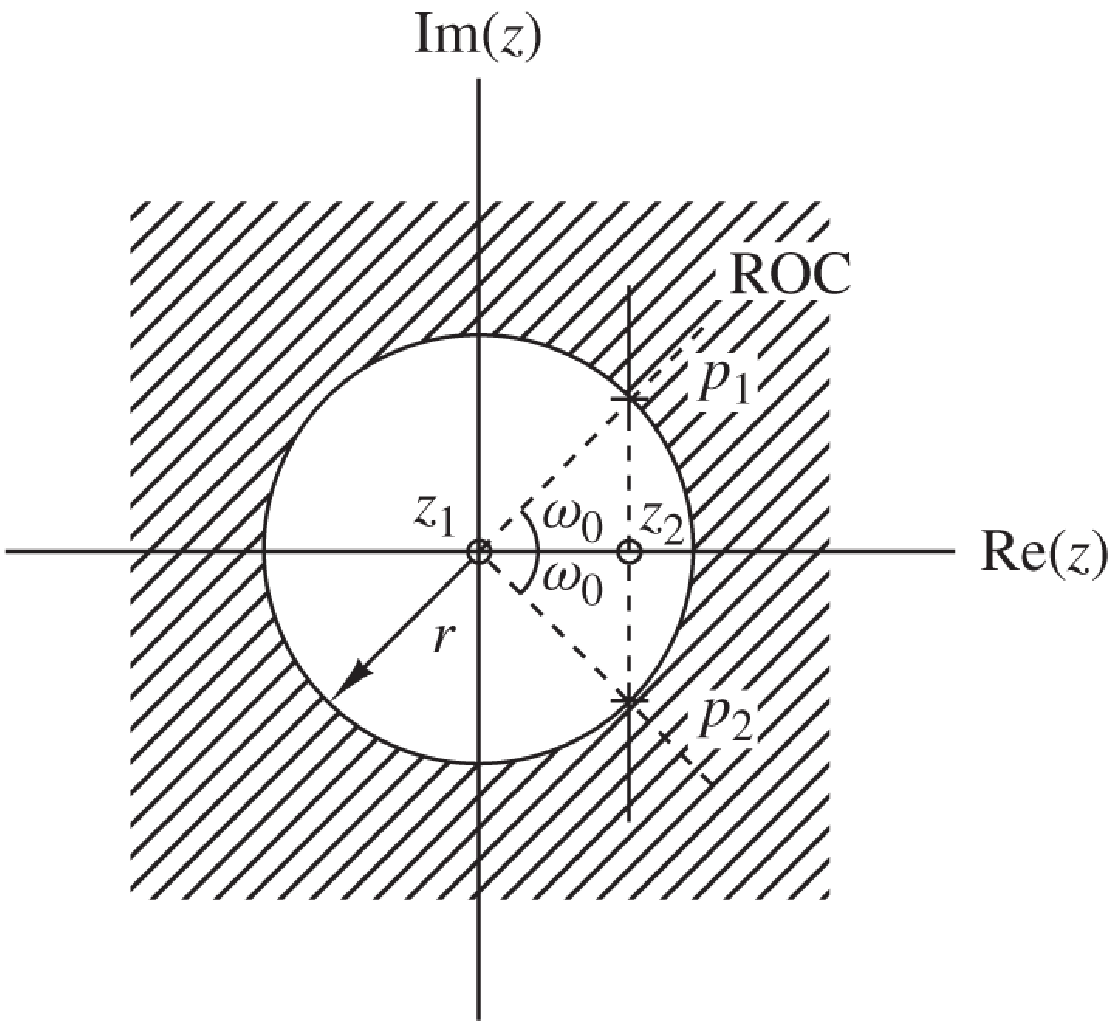

Figure 3.3.3 Pole-zero pattern for Example 3.3.3.

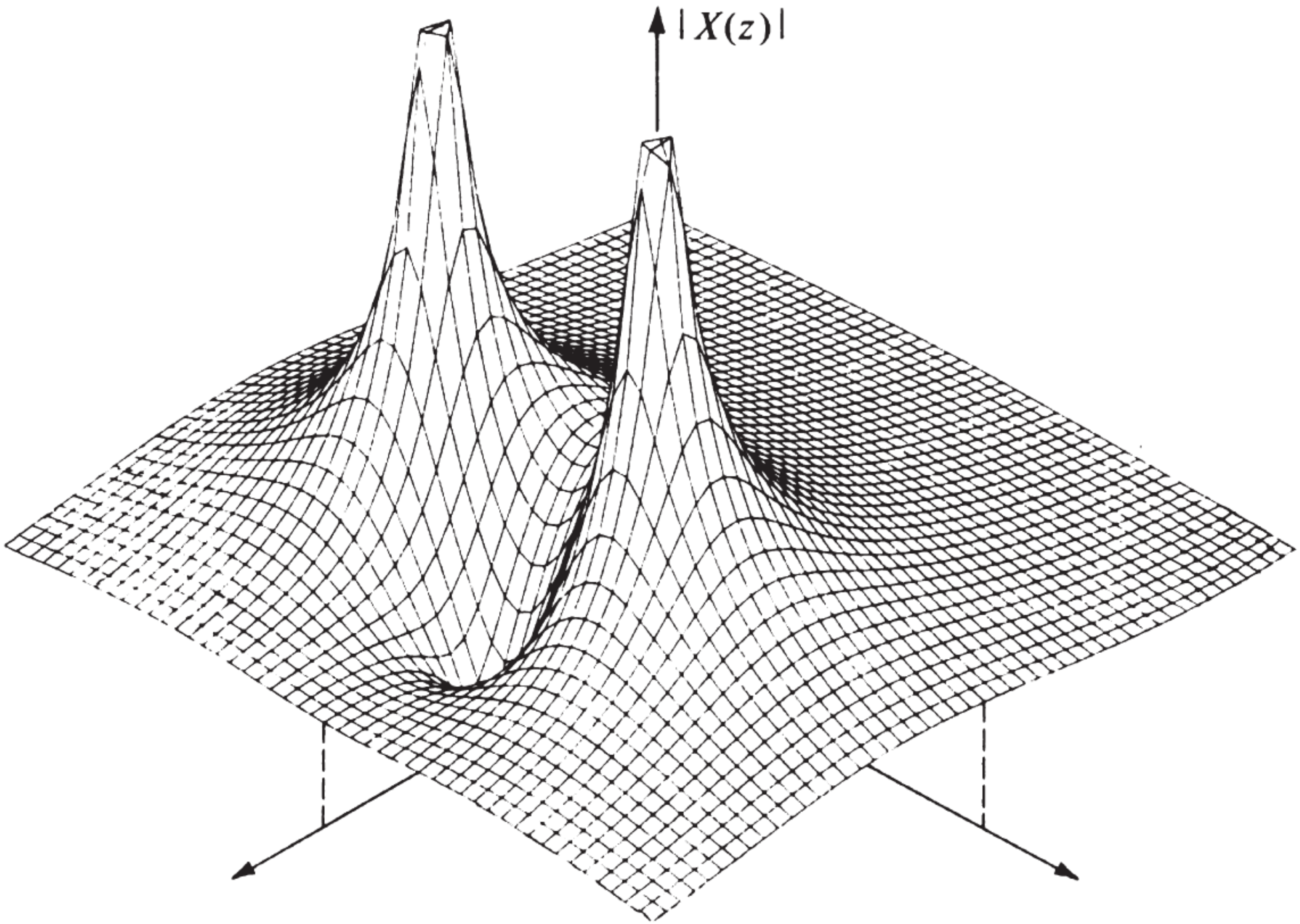

Figure 3.3.4 Graph of $|X(z)|$ for the z-transform in (3.3.3).

Pole Location and Time-Domain Behavior for Causal Signals

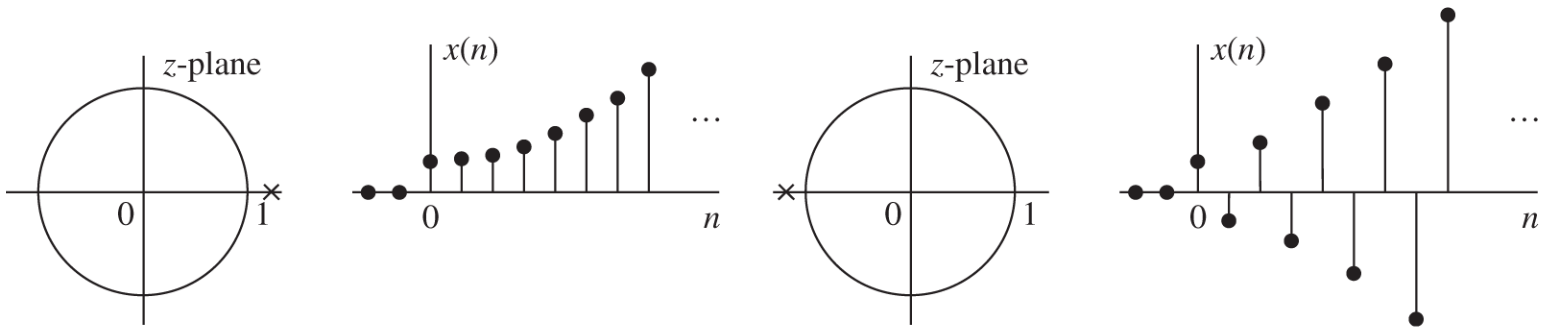

Figure 3.3.5 Time-domain behavior of a single-real-pole causal signal as a function of the location of the pole with respect to the unit circle.

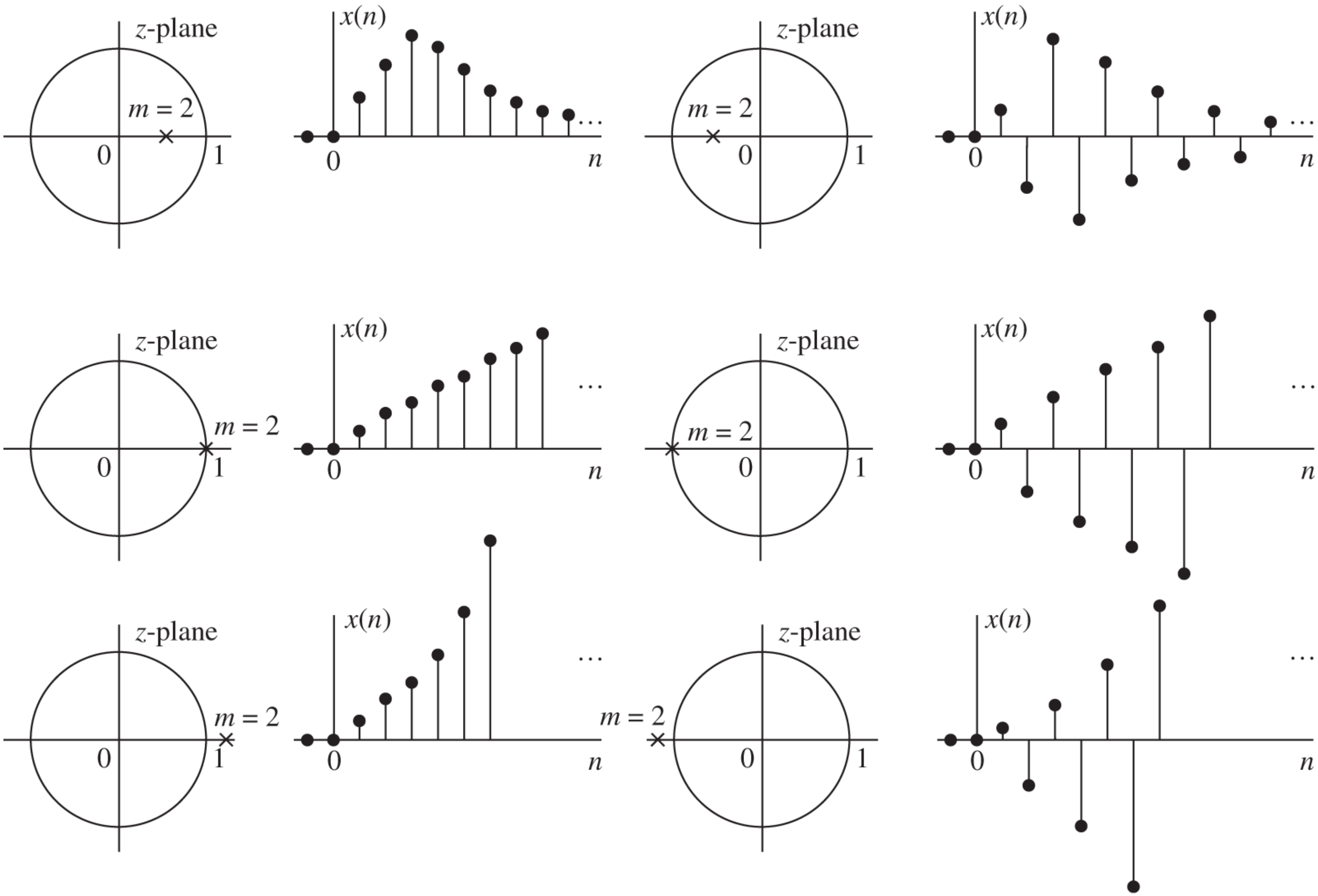

Figure 3.3.6 Time-domain behavior of causal signals corresponding to a double ($m = 2$) real pole, as a function of the pole location.

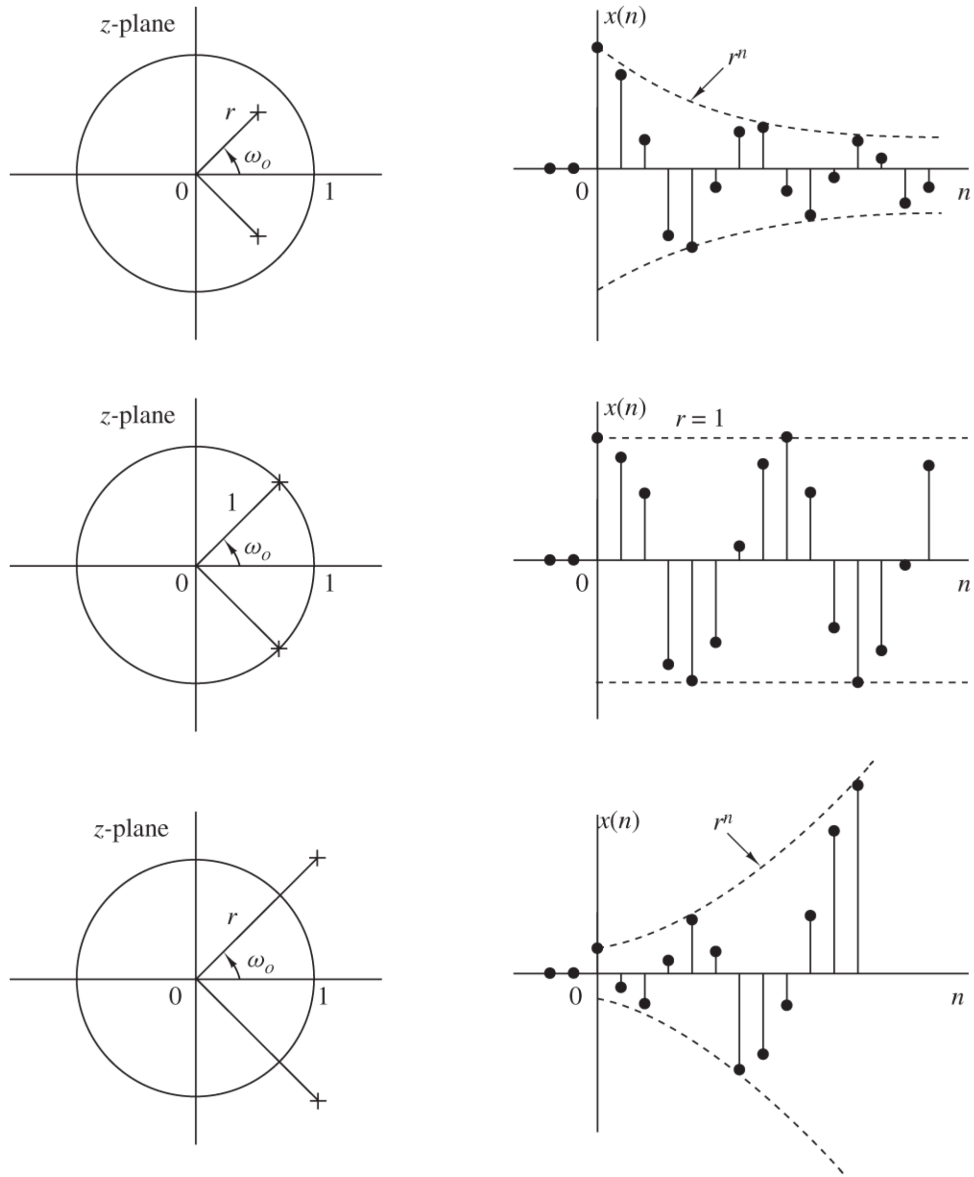

Figure 3.3.7 A pair of complex-conjugate poles corresponds to causal signals with oscillatory behavior.

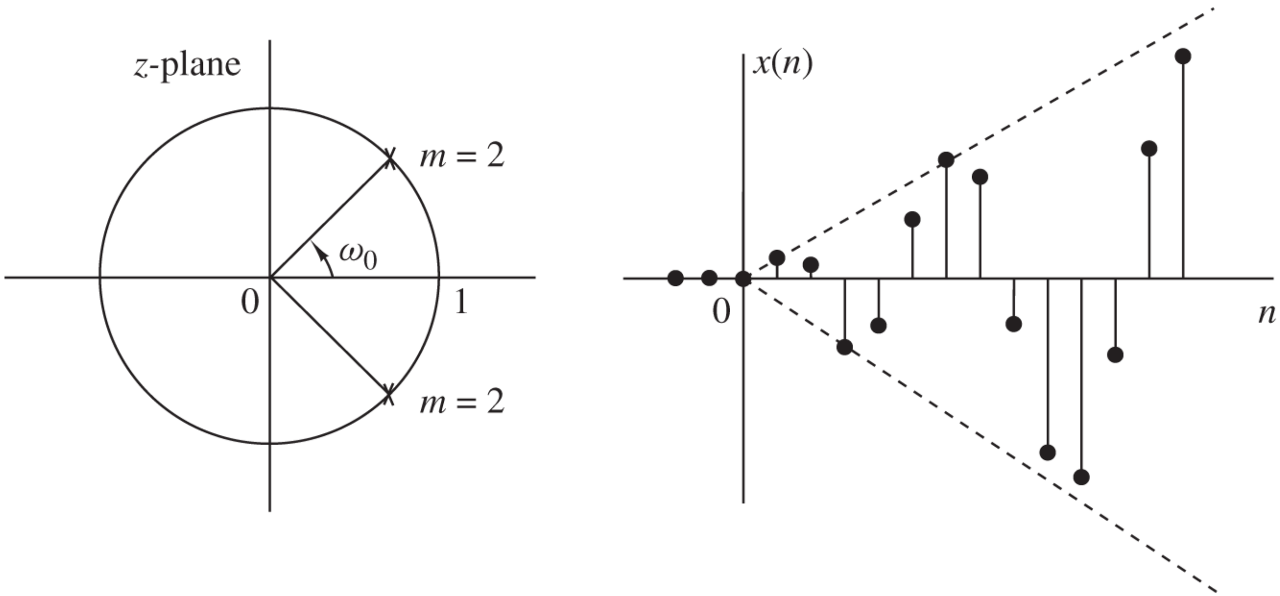

Figure 3.3.8 Causal signal corresponding to a double pair of complex-conjugate poles on the unit circle.

The Inverse z-Transform

- Power Series Expansion
- Partial-Fraction Expansion
- Decomposition of Rational z-Transforms

Analysis of Linear Time-Invariant Systems in the z-Domain

- Response of Systems with Rational System Functions
- Transient and Steady-State Responses
- Causality and Stability
- Pole-Zero Cancellations
- Multiple-Order Poles and Stability
- Stability of Second-Order Systems

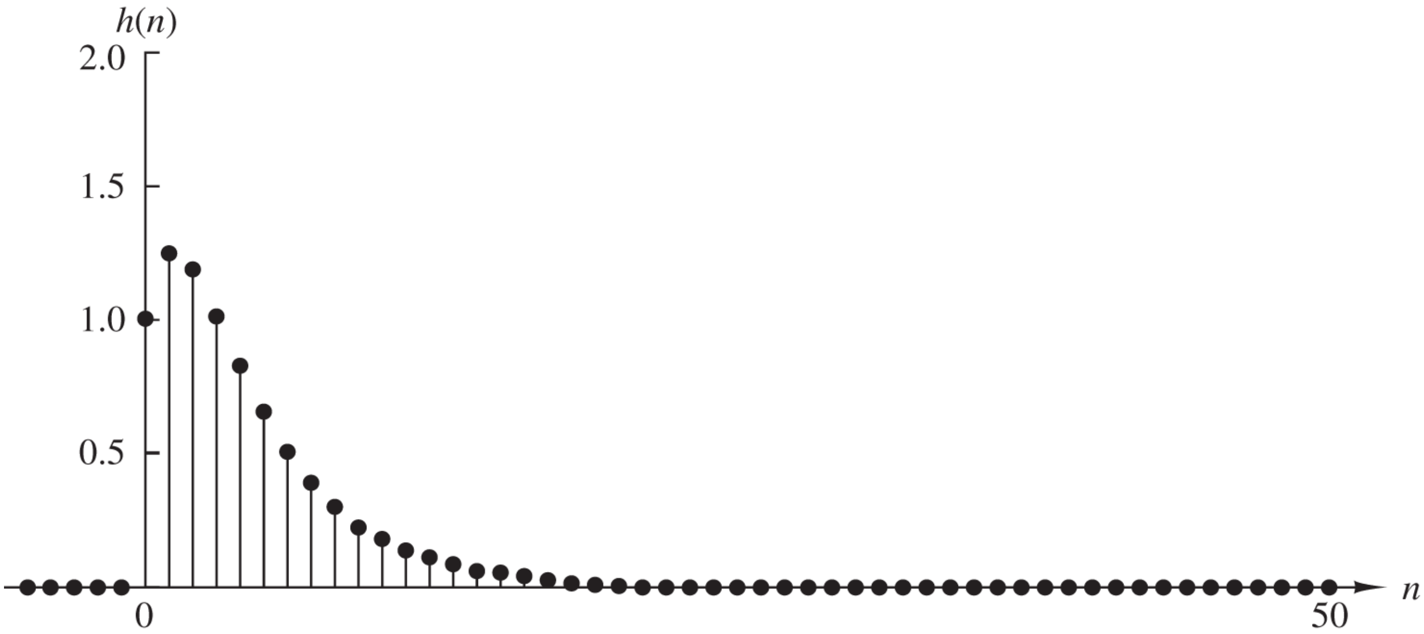

Figure 3.5.2 Plot of $h(n)$ given by (3.5.16) with $p_1 = 0.5,\ p_2 = 0.75$; $h(n) = [1/(p_1 - p_2)](p_1^{n+1} - p_2^{n+1})u(n)$.

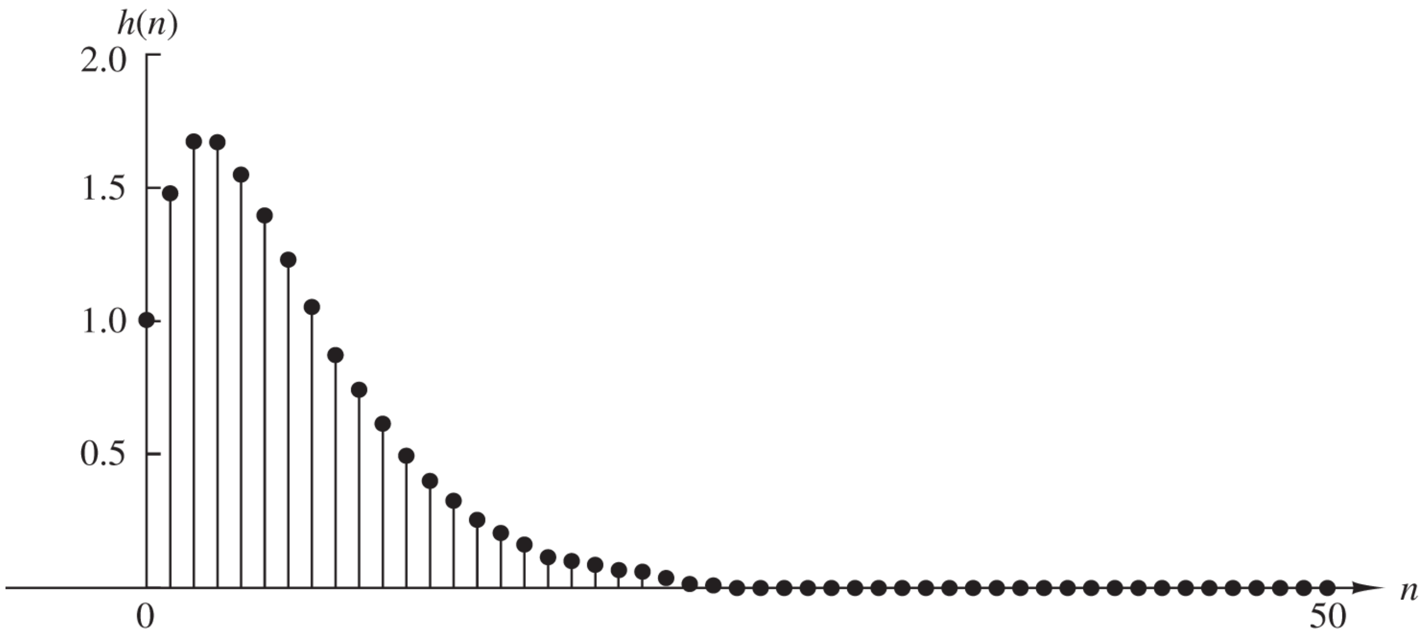

Figure 3.5.3 Plot of $h(n)$ given by (3.5.18) with $p = \frac{3}{4}$; $h(n) = (n+1)p^n u(n)$.

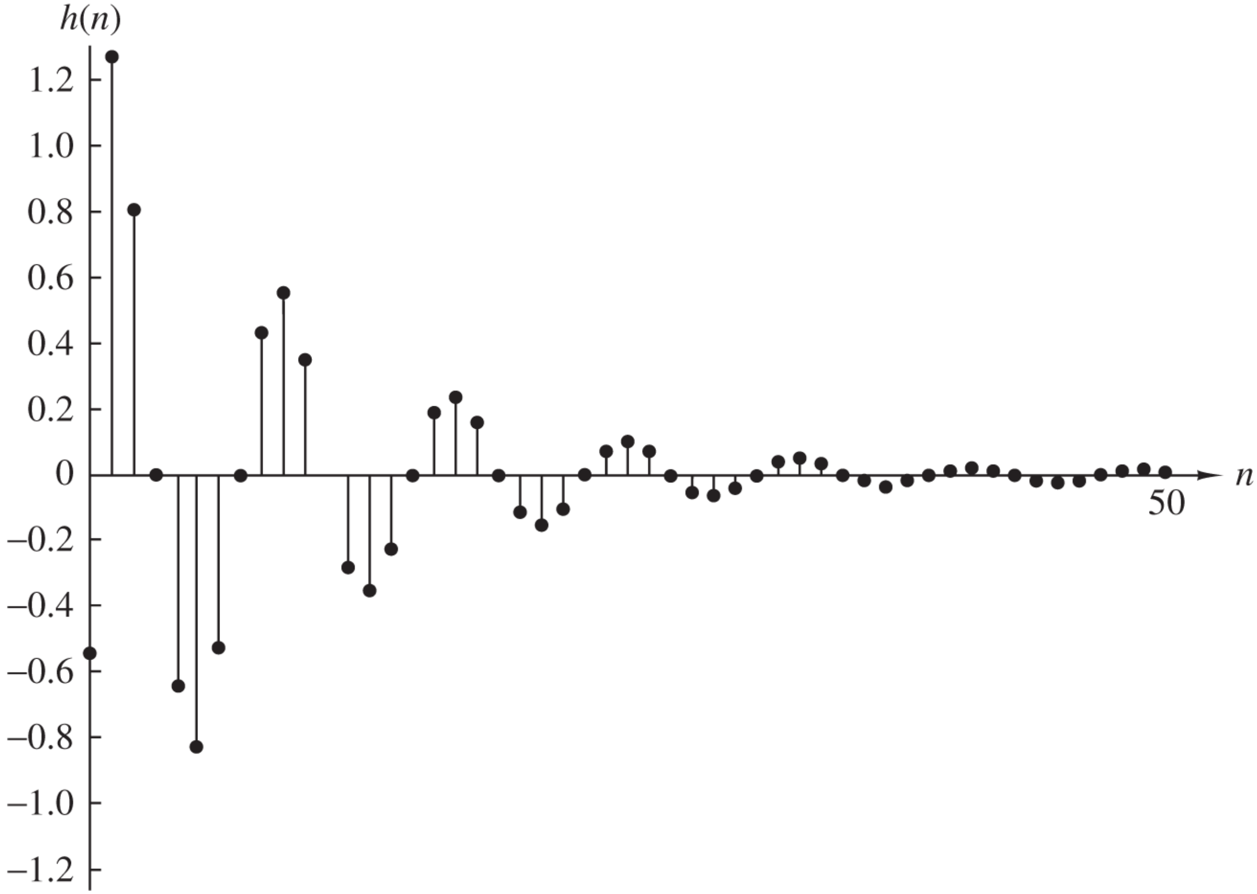

Figure 3.5.4 Plot of $h(n)$ given by (3.5.22) with $b_0 = 1$, $\omega_0 = \pi/4$, $r = 0.9$; $h(n) = [b_0 r^n/(\sin\omega_0)] \sin[(n+1)\omega_0]u(n)$.