Image processing
- Fourier transforms

Objectives

- What is Fourier transform?
- Fourier transform of two- dimentional
- Properties of Discrete Fourier transform
- How using the Fast Fourier Transform
- Learn how linear filters work and their implementation
- Applications

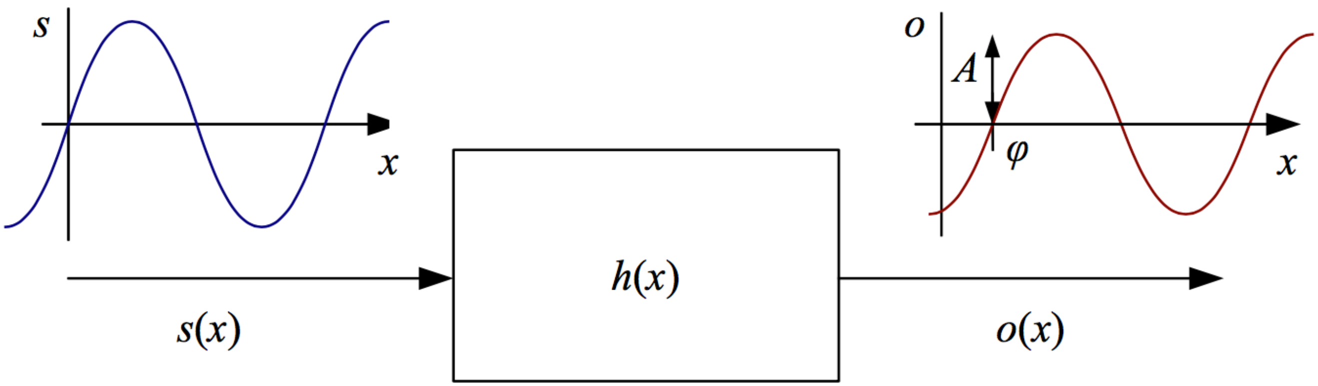
Fourier Transform

- The Fourier Transform is a tool that breaks a waveform into an alternate representation, characterized by sine and cosines.
- The Fourier Transform shows that any waveform can be re-written as the sum of sinusoidal functions.
- The Fourier Transform gives us a unique and powerful way of viewing these waveforms.

All waveforms, no matter what you scribble or observe in the universe, are actually just the sum of simple sinusoids of different frequencies.

Fourier Transform

- The Fourier transform is simply a tabulation of the magnitude and phase response at each frequency



$$o(x) = h(x) * s(x) = Ae^{j\omega x+\phi}$$

Fourier Transform

- The Fourier transform pair

$$h(x) \overset{\mathcal{F}}{\leftrightarrow} H(\omega)$$

- In the continuous domain

$$H(\omega) = \int_{-\infty}^{\infty} h(x) e^{-j\omega x} dx$$

- In the discrete domain

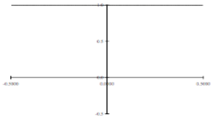
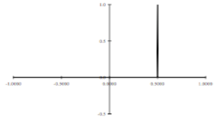
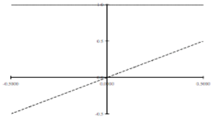
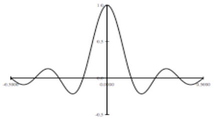
$$H(k) = \frac{1}{N} \sum_{x=0}^{N-1} h(x) e^{-j\frac{2\pi kx}{N}}$$

Fourier Transform's Properties

Property	Signal		Transform
superposition	$f_1(x) + f_2(x)$		$F_1(\omega) + F_2(\omega)$
shift	$f(x - x_0)$		$F(\omega)e^{-j\omega x_0}$
reversal	$f(-x)$		$F^*(\omega)$
convolution	$f(x) * h(x)$		$F(\omega)H(\omega)$
correlation	$f(x) \otimes h(x)$		$F(\omega)H^*(\omega)$
multiplication	$f(x)h(x)$		$F(\omega) * H(\omega)$
differentiation	$f'(x)$		$j\omega F(\omega)$
domain scaling	$f(ax)$		$1/aF(\omega/a)$
real images	$f(x) = f^*(x)$	$\Leftrightarrow$	$F(\omega) = F(-\omega)$
Parseval's Theorem	$\sum_x [f(x)]^2$	$=$	$\sum_\omega [F(\omega)]^2$

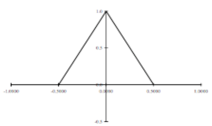
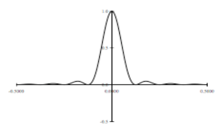
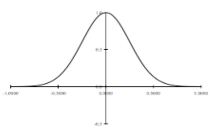
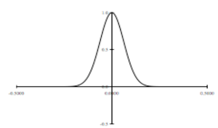
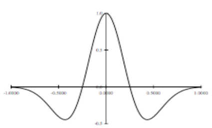
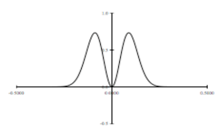
Fourier Transform pairs

- Impulse: The impulse response has a constant (all frequency) transform.
- Shifted impulse: The shifted impulse has unit magnitude and linear phase.
- Box filter: The box (moving average) filter

Name	Signal			Transform	
impulse		$\delta(x)$	$\Leftrightarrow$	1	
shifted impulse		$\delta(x-u)$	$\Leftrightarrow$	$e^{-j\omega u}$	
box filter		$\mathrm{box}(x/a)$	$\Leftrightarrow$	$a\mathrm{sinc}(a\omega)$	

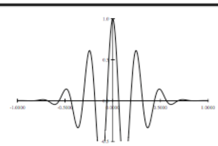
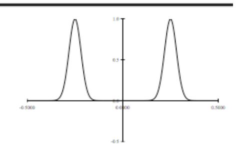
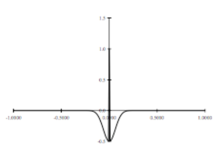
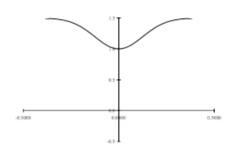
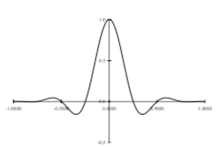
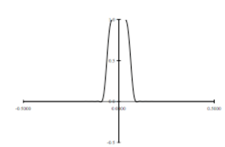
Fourier Transform pairs

- Tent: The piecewise linear tent function
- Gaussian: The (unit area) Gaussian of width σ
- Laplacian of Gaussian: The second derivative of a Gaussian of width σ

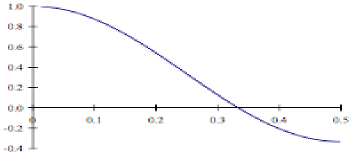
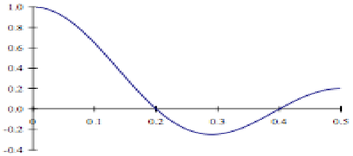
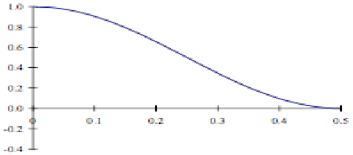
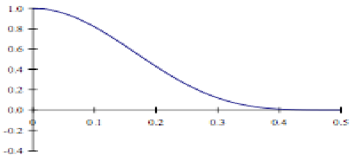
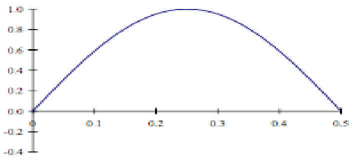
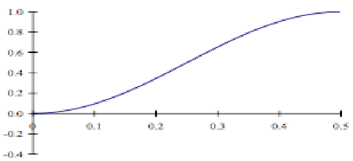
Name	Signal			Transform	
tent		$\mathrm{tent}(x/a)$	$\Leftrightarrow$	$a\,\mathrm{sinc}^2(a\omega)$	
Gaussian		$G(x;\sigma)$	$\Leftrightarrow$	$\frac{\sqrt{2\pi}}{\sigma}G(\omega;\sigma^{-1})$	
Laplacian of Gaussian		$(\frac{x^2}{\sigma^4}-\frac{1}{\sigma^2})G(x;\sigma)$	$\Leftrightarrow$	$-\frac{\sqrt{2\pi}}{\sigma}\omega^2 G(\omega;\sigma^{-1})$	

Fourier Transform pairs

- Gabor: The even Gabor function
- Unsharp mask: The unsharp mask
- Windowed sinc: The windowed (masked) sinc function

Name		Signal		Transform	
Gabor		$\cos(\omega_0 x) G(x; \sigma)$	$\Leftrightarrow$	$\frac{\sqrt{2\pi}}{\sigma} G(\omega \pm \omega_0; \sigma^{-1})$	
unsharp mask		$(1+\gamma)\delta(x)$ $- \gamma G(x; \sigma)$	$\Leftrightarrow$	$(1+\gamma) -$ $\frac{\sqrt{2\pi}\gamma}{\sigma} G(\omega; \sigma^{-1})$	
windowed sinc		$\mathrm{rcos}(x/(aW))$ $\mathrm{sinc}(x/a)$	$\Leftrightarrow$	(see Figure 3.29)	

The small discrete kernels

Name	Kernel	Transform	Plot
box-3	$\frac{1}{3}\boxed{1}\boxed{1}\boxed{1}$	$\frac{1}{3}(1 + 2\cos\omega)$	
box-5	$\frac{1}{5}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}$	$\frac{1}{5}(1 + 2\cos\omega + 2\cos 2\omega)$	
linear	$\frac{1}{4}\boxed{1}\boxed{2}\boxed{1}$	$\frac{1}{2}(1 + \cos\omega)$	
binomial	$\frac{1}{16}\boxed{1}\boxed{4}\boxed{6}\boxed{4}\boxed{1}$	$\frac{1}{4}(1 + \cos\omega)^2$	
Sobel	$\frac{1}{2}\boxed{-1}\boxed{0}\boxed{1}$	$\sin\omega$	
corner	$\frac{1}{2}\boxed{-1}\boxed{2}\boxed{-1}$	$\frac{1}{2}(1 - \cos\omega)$	

2D Fourier Transform

- Oriented sinusoid :

$$s(x, y) = \sin(\omega_x x + \omega_y y)$$

- Two-dimensional Fourier transforms continuous domain

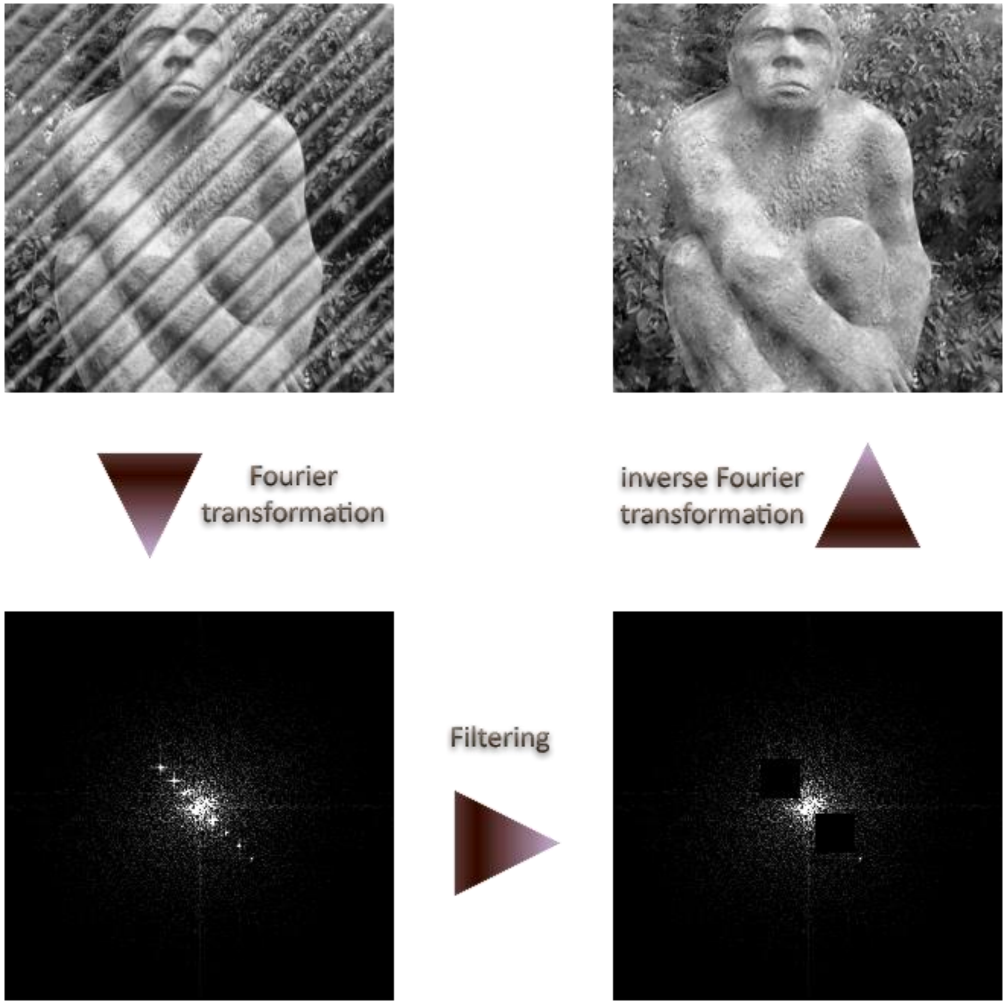
$$H(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) e^{-j(\omega_x x + \omega_y y)} dx\, dy$$

- In the discrete domain

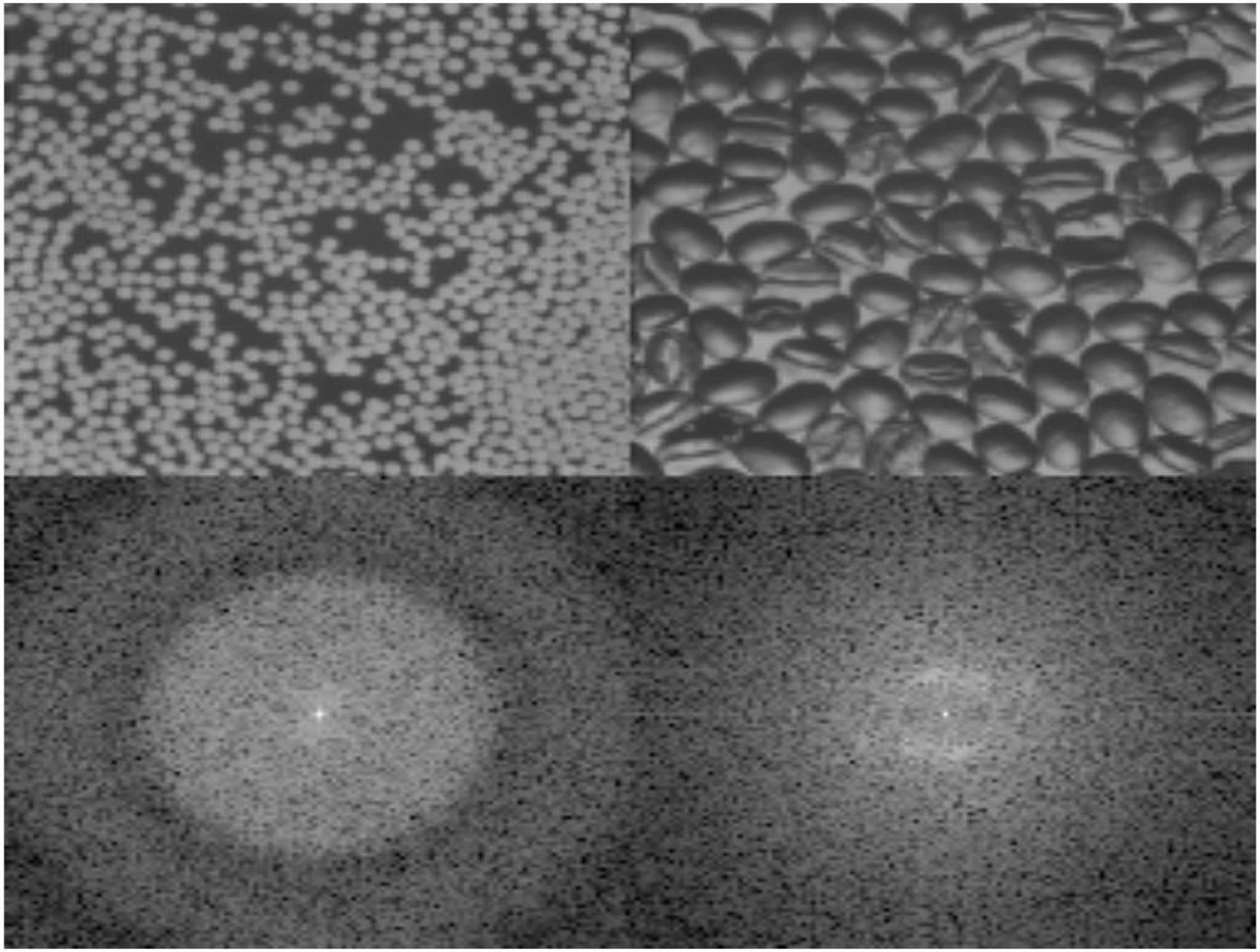
$$H(k_x, k_y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} h(x, y) e^{-j2\pi \frac{k_x x + k_y y}{MN}}$$

  – where M and N are the width and height of the image

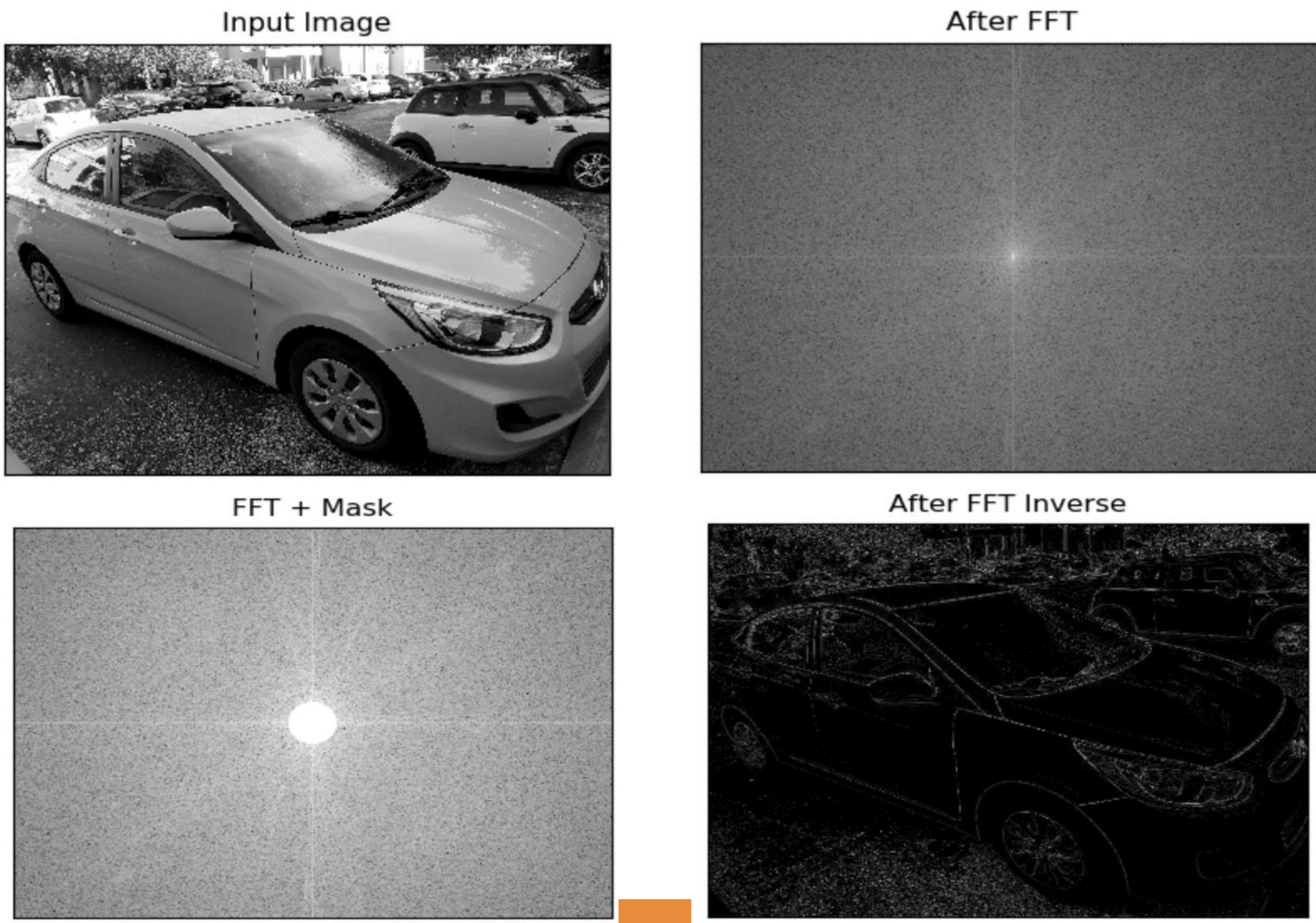
Application: image filtering



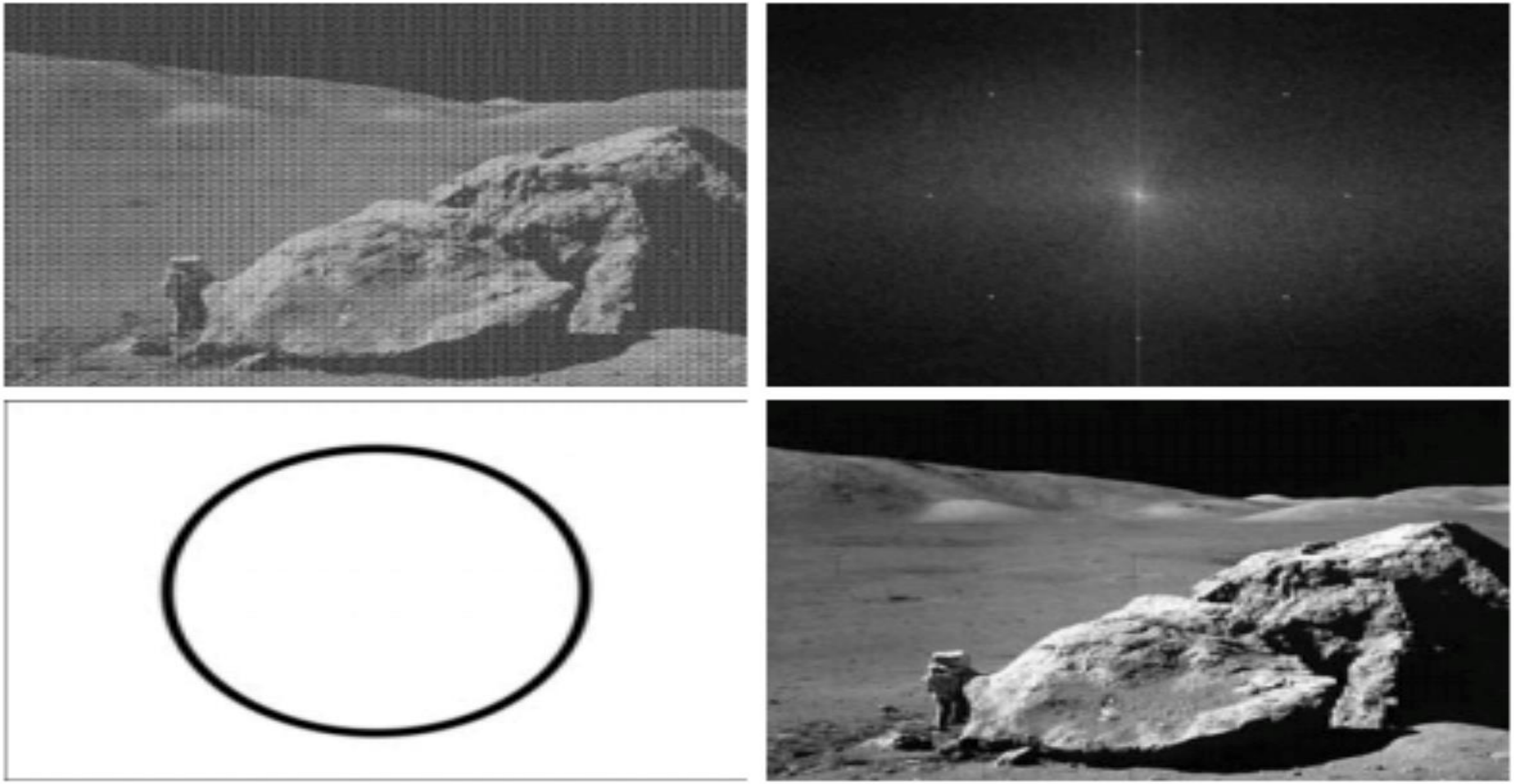
Application: image analysis



Application: Edge detection



Application: image reconstruction



Application: image compression

- Image compression based on 2D Discrete Fourier Transform and matrix minimization algorithm

260 KB
Quantization value=10

138.2 KB
Quantization value=25

88.1 KB
Quantization value=45

(a) Decompressed Lena image, dimension = 1024 x 1024

201 KB
Quantization value=25

108.4 KB
Quantization value=60

71.4 KB
Quantization value=100

(b) Decompressed Lion image, dimension = 1200 x 1200

Summary

- Learn about Fourier transform
- Learn how Fourier transform is applied in image
- Understand and implement applications of Fourier transform such as: image filtering, image analysis, edge detection, image reconstruction, image compression