Linear Algebra Primer

Juan Carlos Niebles and Ranjay Krishna
Stanford Vision and Learning Lab

Another, very in-depth linear algebra review from CS229 is available here:
http://cs229.stanford.edu/section/cs229-linalg.pdf
And a video discussion of linear algebra from EE263 is here (lectures 3 and 4):
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Outline

- Vectors and matrices
  - Basic Matrix Operations
  - Determinants, norms, trace
  - Special Matrices
- Transformation Matrices
  - Homogeneous coordinates
  - Translation
- Matrix inverse
- Matrix rank
- Eigenvalues and Eigenvectors
- Matrix Calculus

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Vectors and matrices are just collections of ordered numbers that represent something: movements in space, scaling factors, pixel brightness, etc. We'll define some common uses and standard operations on them.

Vector

- A column vector $\mathbf{v} \in \mathbb{R}^{n \times 1}$ where

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- A row vector $\mathbf{v}^T \in \mathbb{R}^{1 \times n}$ where

$$\mathbf{v}^T = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

T denotes the transpose operation

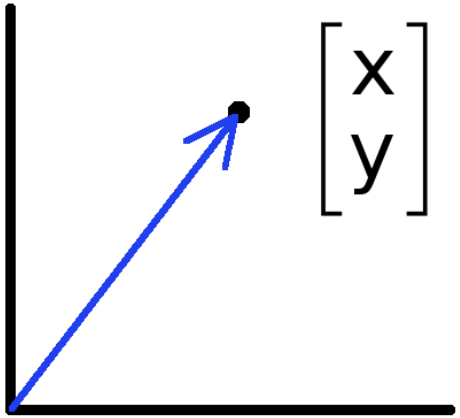
Vector

- We'll default to column vectors in this class

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

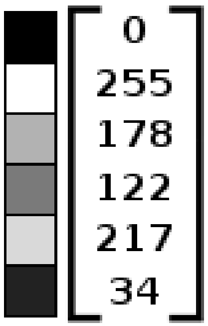
- You'll want to keep track of the orientation of your vectors when programming in python
- You can transpose a vector *V* in python by writing **V.t**. (But in class materials, we will **always** use V^T to indicate transpose, and we will use V' to mean "V prime")

Vectors have two main uses



- Vectors can represent an offset in 2D or 3D space
- Points are just vectors from the origin

- Data (pixels, gradients at an image keypoint, etc) can also be treated as a vector



- Such vectors don’t have a geometric interpretation, but calculations like “distance” can still have value

Matrix

- A matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is an array of numbers with size by , i.e. m rows and n columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

- If $m = n$, we say that $\mathbf{A}$ is square.

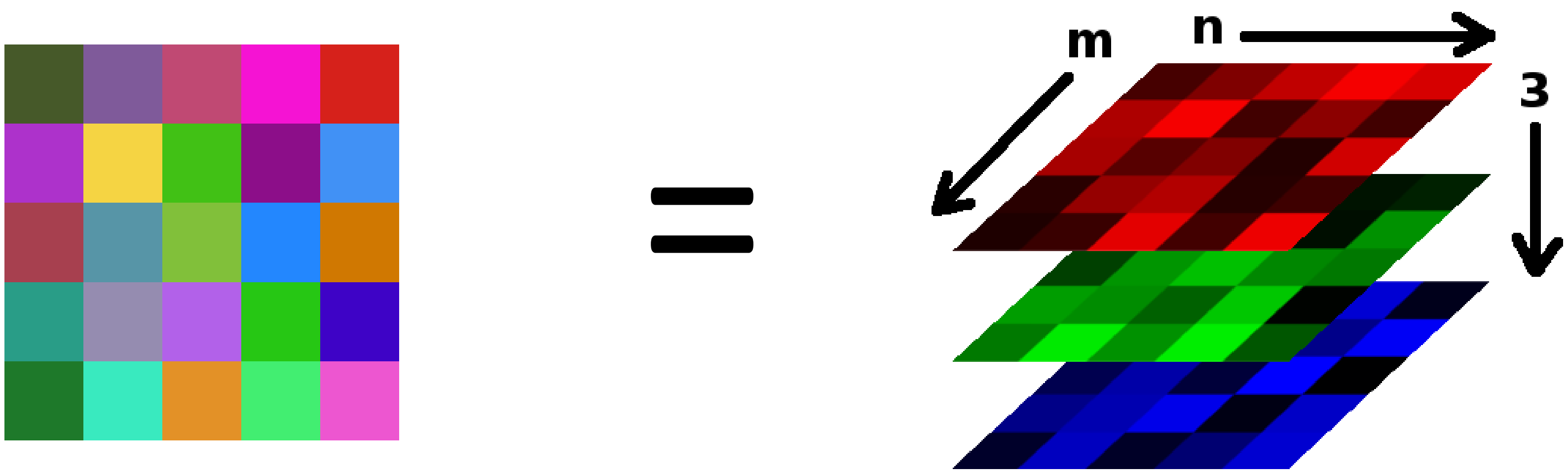
Images

$$= \begin{bmatrix} 193 & 180 & 210 & 112 & 125 \\ 189 & 8 & 177 & 97 & 114 \\ 100 & 71 & 81 & 195 & 165 \\ 167 & 12 & 242 & 203 & 181 \\ 44 & 25 & 9 & 48 & 192 \end{bmatrix}$$

- Python represents an image as a matrix of pixel brightnesses
- Note that the upper left corner is [y,x] = (0,0)

Color Images

- Grayscale images have one number per pixel, and are stored as an m × n matrix.
- Color images have 3 numbers per pixel – red, green, and blue brightnesses (RGB)
- Stored as an m × n × 3 matrix



Basic Matrix Operations

- We will discuss:
  - Addition
  - Scaling
  - Dot product
  - Multiplication
  - Transpose
  - Inverse / pseudoinverse
  - Determinant / trace

Matrix Operations

- Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+1 & b+2 \\ c+3 & d+4 \end{bmatrix}$$

  – Can only add a matrix with matching dimensions, or a scalar.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 7 = \begin{bmatrix} a+7 & b+7 \\ c+7 & d+7 \end{bmatrix}$$

- Scaling

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times 3 = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$$

Vectors

- **Norm** $$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

- More formally, a norm is any function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies 4 properties:

- **Non-negativity:** For all $x \in \mathbb{R}^n,\ f(x) \geq 0$
- **Definiteness**: f(x) = 0 if and only if x = 0.
- **Homogeneity**: For all $x \in \mathbb{R}^n,\ t \in \mathbb{R},\ f(tx) = |t|\, f(x)$
- **Triangle inequality**: For all $x, y \in \mathbb{R}^n,\ f(x + y) \leq f(x) + f(y)$

Matrix Operations

- **Example Norms**

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad\qquad \|x\|_\infty = \max_i |x_i|$$

- General ℓ_p norms:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

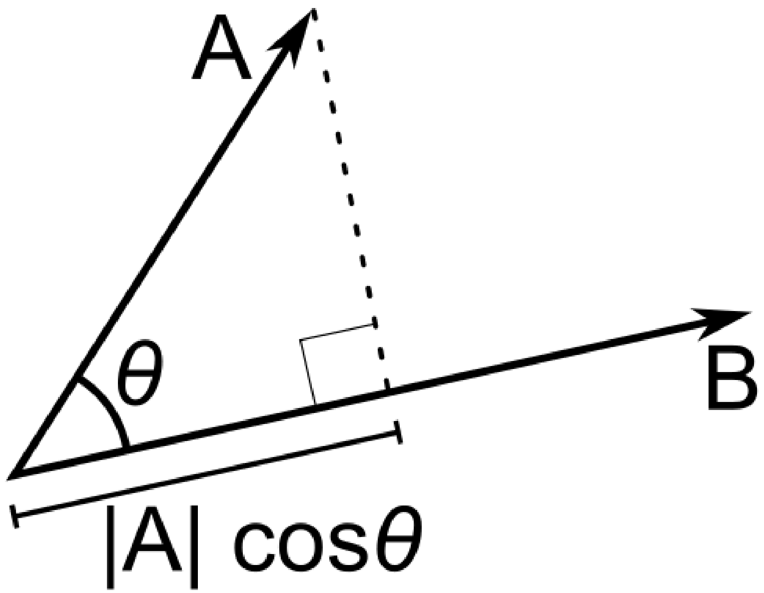
Matrix Operations

- Inner product (dot product) of vectors
  - Multiply corresponding entries of two vectors and add up the result
  - x·y is also |x||y|Cos(*the angle between x and y*)

$$\mathbf{x}^T\mathbf{y} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i \qquad \text{(scalar)}$$

Matrix Operations

- Inner product (dot product) of vectors
  - If B is a unit vector, then A·B gives the length of A which lies in the direction of B



Matrix Operations

- The product of two matrices

$$A \in \mathbb{R}^{m \times n}$$

$$B \in \mathbb{R}^{n \times p}$$

$$C = AB \in \mathbb{R}^{m \times p}$$

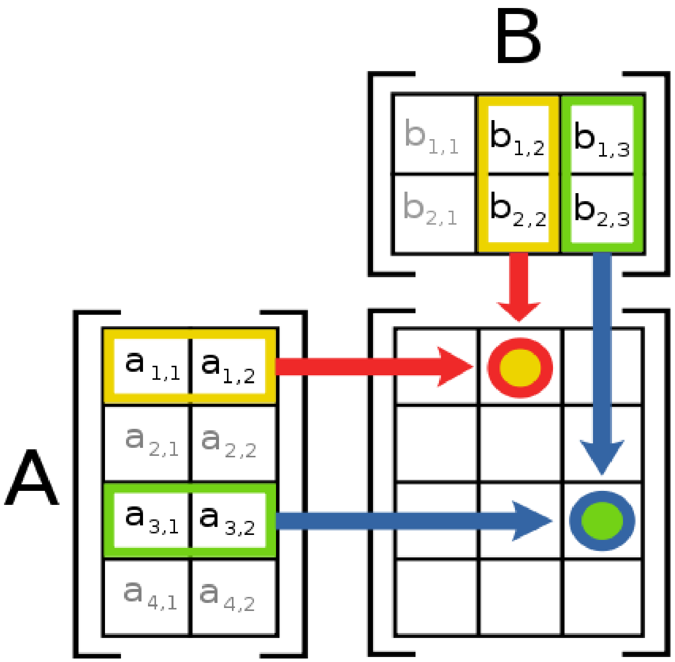
$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

$$C = AB = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_p \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_p \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T b_1 & a_m^T b_2 & \cdots & a_m^T b_p \end{bmatrix}.$$

Matrix Operations

- Multiplication

- The product AB is:



- Each entry in the result is (that row of A) dot product with (that column of B)
- Many uses, which will be covered later

Matrix Operations

- Multiplication example:

A x B

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 4 & 6 \end{bmatrix} \quad \begin{bmatrix} 10 & 14 \\ 34 & 54 \end{bmatrix}$$

$$0 \cdot 3 + 2 \cdot 7 = 14$$

  – Each entry of the matrix product is made by taking the dot product of the corresponding row in the left matrix, with the corresponding column in the right one.

Matrix Operations

- The product of two matrices

Matrix multiplication is associative: $(AB)C = A(BC)$.

Matrix multiplication is distributive: $A(B + C) = AB + AC$.

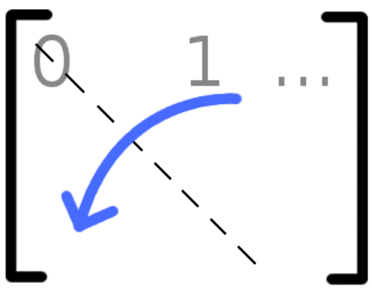
Matrix multiplication is, in general, *not* commutative; that is, it can be the case that $AB \neq BA$. (For example, if $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times q}$, the matrix product BA does not even exist if m and q are not equal!)

Matrix Operations

- Powers
  - By convention, we can refer to the matrix product AA as A^2, and AAA as A^3, etc.
  - Obviously only square matrices can be multiplied that way

Matrix Operations

- Transpose – flip matrix, so row 1 becomes column 1



$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$$

- A useful identity:

$$(ABC)^T = C^T B^T A^T$$

Matrix Operations

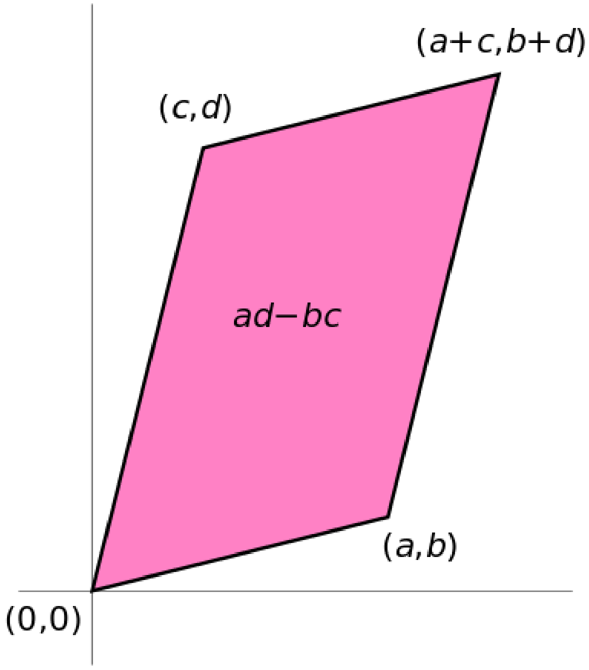
- Determinant
  - $\det(\mathbf{A})$ returns a scalar
  - Represents area (or volume) of the parallelogram described by the vectors in the rows of the matrix
  - For $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(\mathbf{A}) = ad - bc$
  - Properties:

$$\det(\mathbf{AB}) = \det(\mathbf{BA})$$

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

$$\det(\mathbf{A}^T) = \det(\mathbf{A})$$

$$\det(\mathbf{A}) = 0 \Leftrightarrow \mathbf{A} \text{ is singular}$$



Matrix Operations

- Trace

$$\mathrm{tr}(\mathbf{A}) = \text{sum of diagonal elements}$$

$$\mathbf{tr}(\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}) = 1 + 7 = 8$$

  – Invariant to a lot of transformations, so it's used sometimes in proofs. (Rarely in this class though.)

  – Properties:

$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$$

$$\mathrm{tr}(\mathbf{A} + \mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$$

Matrix Operations

- **Vector Norms**

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|x\|_\infty = \max_i |x_i|$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}. \qquad \|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

- Matrix norms: Norms can also be defined for matrices, such as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2} = \sqrt{\operatorname{tr}(A^T A)}.$$

Special Matrices

- Identity matrix **I**
  - Square matrix, 1's along diagonal, 0's elsewhere
  - **I** · [another matrix] = [that matrix]

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Diagonal matrix
  - Square matrix with numbers along diagonal, 0's elsewhere
  - A diagonal · [another matrix] scales the rows of that matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

Special Matrices

- Symmetric matrix

$$\mathbf{A}^T = \mathbf{A}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 7 \\ 5 & 7 & 1 \end{bmatrix}$$

- Skew-symmetric matrix

$$\mathbf{A}^T = -\mathbf{A}$$

$$\begin{bmatrix} 0 & -2 & -5 \\ 2 & 0 & -7 \\ 5 & 7 & 0 \end{bmatrix}$$

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Announcements – part 1

- HW0 submitted last night
- HW1 is due next Monday
- HW2 will be released tonight
- Class notes from last Thursday due before class in exactly 48 hours

Announcements – part 2

- Future homework assignments will be released via github
  - Will allow you to keep track of changes IF they happen.
- Submissions for HW1 onwards will be done all through gradescope.
  - NO MORE CORN SUBMISSIONS
  - You will have separate submissions for the ipython pdf and the python code.

Recap - Vector

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- If $m = n$, we say that $\mathbf{A}$ is square.

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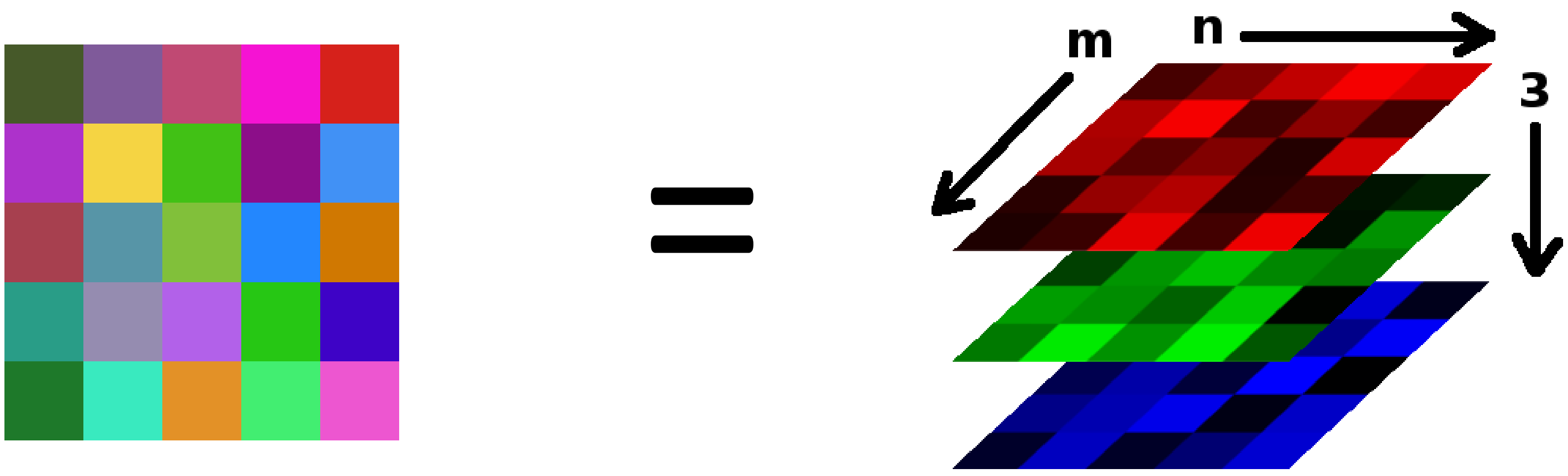

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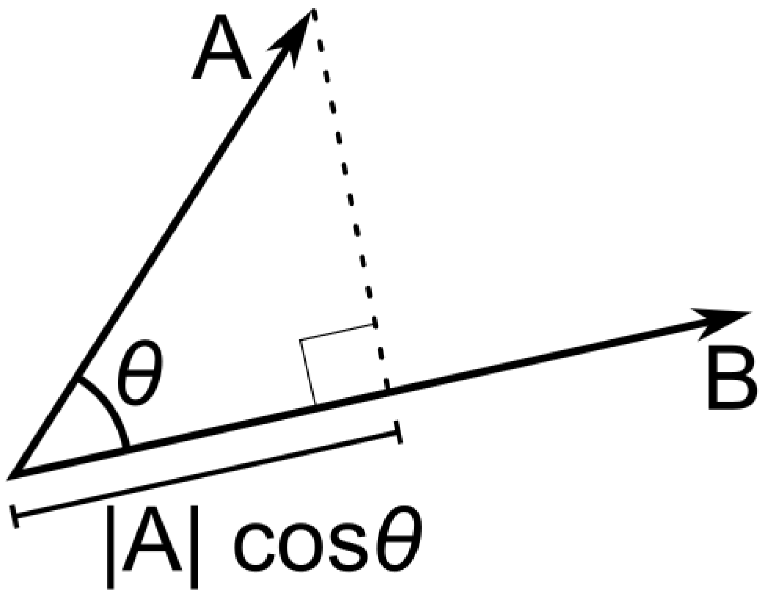

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Matrix multiplication can be used to transform vectors. A matrix used in this way is called a transformation matrix.

Transformation

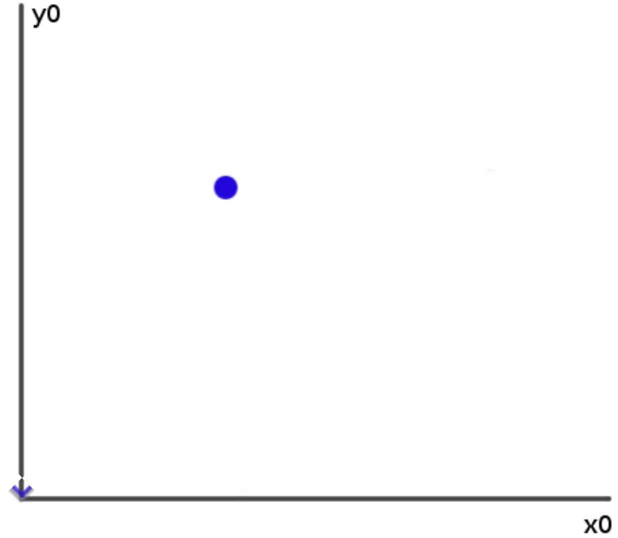
- Matrices can be used to transform vectors in useful ways, through multiplication: x'= Ax
- Simplest is scaling:

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

(Verify to yourself that the matrix multiplication works out this way)

Rotation

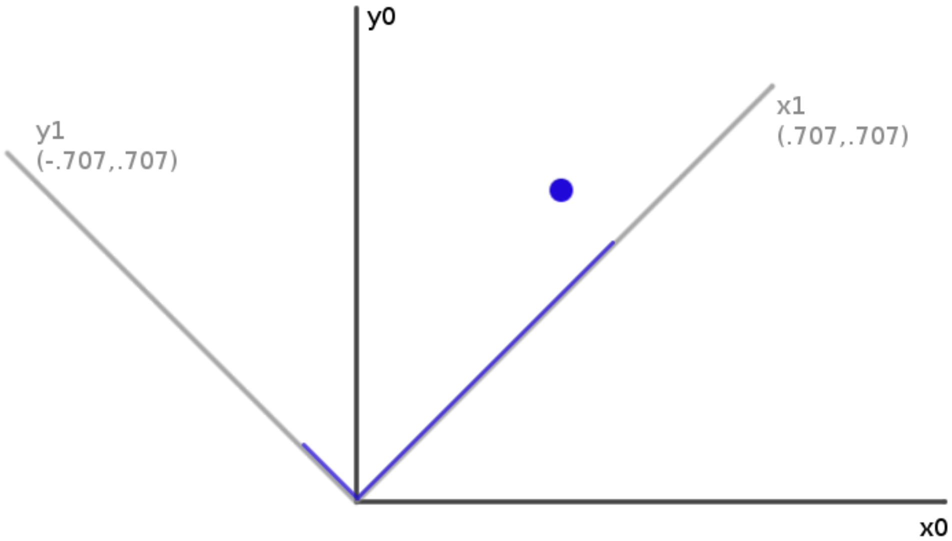
- How can you convert a vector represented in frame "0" to a new, rotated coordinate frame "1"?



Rotation

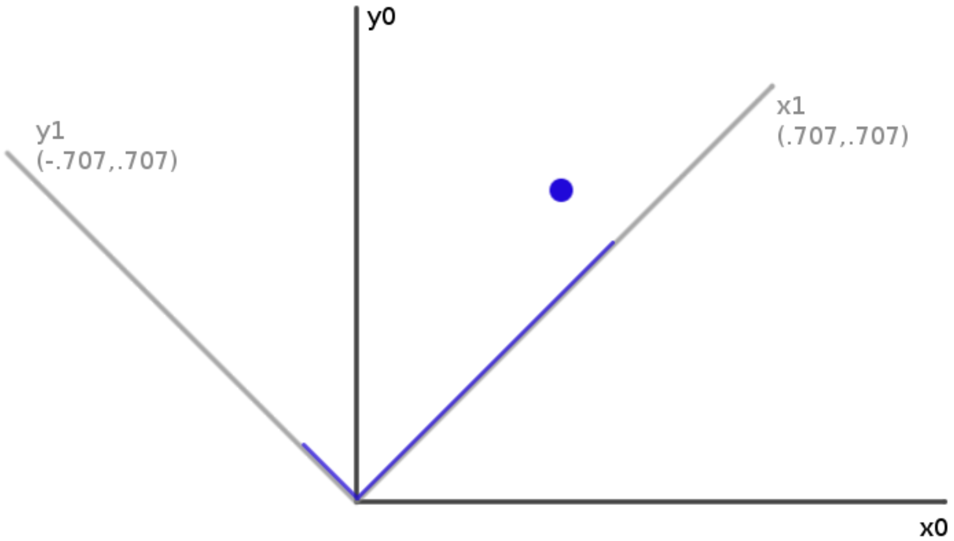
- How can you convert a vector represented in frame "0" to a new, rotated coordinate frame "1"?
- Remember what a vector is:

[component in direction of the frame's x axis, component in direction of y axis]



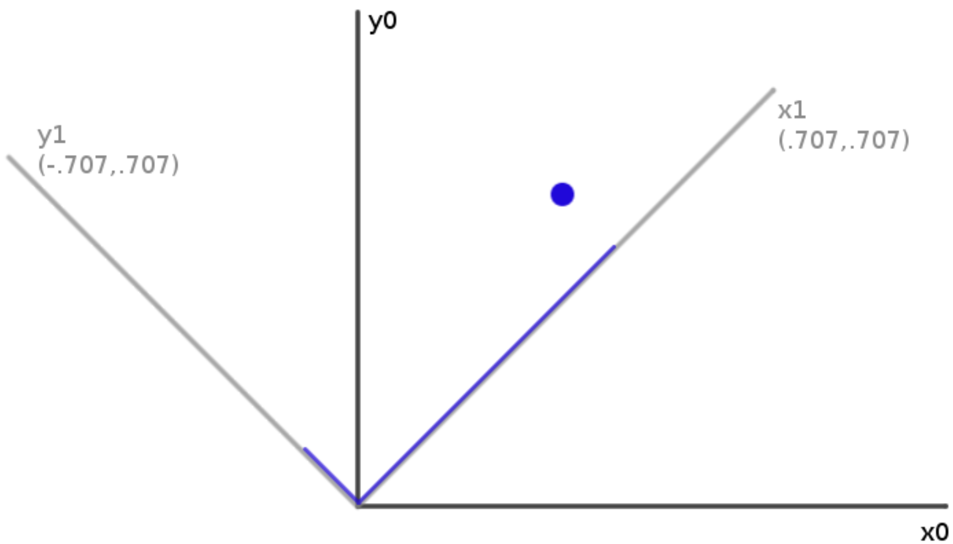
Rotation

- So to rotate it we must produce this vector:
  [component in direction of **new** x axis, component in direction of **new** y axis]
- We can do this easily with dot products!
- New x coordinate is [original vector] **dot** [the new x axis]
- New y coordinate is [original vector] **dot** [the new y axis]



Rotation

- Insight: this is what happens in a matrix*vector multiplication
  - Result x coordinate is: [original vector] **dot** [matrix row 1]
  - So matrix multiplication can rotate a vector p:



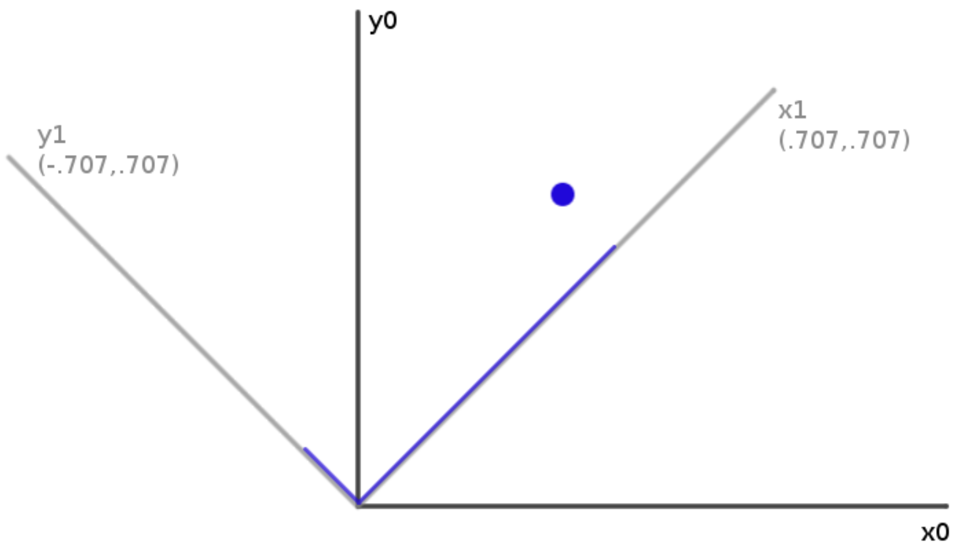
R x p = rotated p'

$$\begin{array}{c} p \\ \begin{bmatrix} px \\ py \end{bmatrix} \end{array}$$

$$\begin{array}{c} R \\ \begin{bmatrix} .707 & .707 \\ -.707 & .707 \end{bmatrix} \end{array} \quad \begin{bmatrix} px' \\ py' \end{bmatrix}$$

Rotation

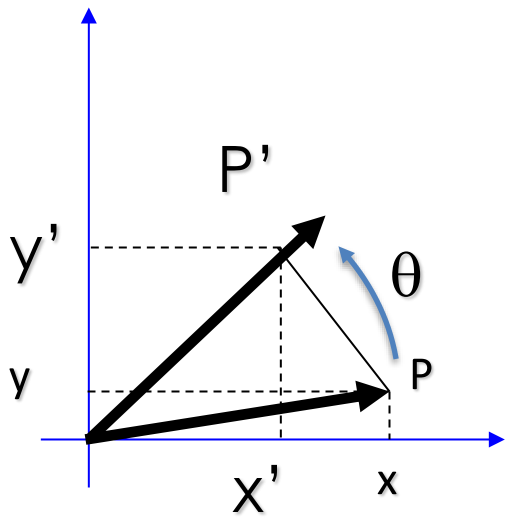
- Suppose we express a point in the new coordinate system which is rotated left
- If we plot the result in the **original** coordinate system, we have rotated the point right



  – Thus, rotation matrices can be used to rotate vectors. We'll usually think of them in that sense-- as operators to rotate vectors

2D Rotation Matrix Formula

Counter-clockwise rotation by an angle θ



$$x' = \cos\theta\ x - \sin\theta\ y$$

$$y' = \cos\theta\ y + \sin\theta\ x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{R}\ \mathbf{P}$$

Transformation Matrices

- Multiple transformation matrices can be used to transform a point:
  $p' = R_2 \, R_1 \, S \, p$

Transformation Matrices

- Multiple transformation matrices can be used to transform a point:
  $p'=R_2 R_1 S p$
- The effect of this is to apply their transformations one after the other, from **right to left**.
- In the example above, the result is
  $(R_2 (R_1 (S p)))$

Transformation Matrices

- Multiple transformation matrices can be used to transform a point:
  $p' = R_2 R_1 S p$
- The effect of this is to apply their transformations one after the other, from **right to left**.
- In the example above, the result is
  $(R_2 (R_1 (S p)))$
- The result is exactly the same if we multiply the matrices first, to form a single transformation matrix:
  $p' = (R_2 R_1 S) p$

Homogeneous system

- In general, a matrix multiplication lets us linearly combine components of a vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

  - This is sufficient for scale, rotate, skew transformations.
  - But notice, we can't add a constant! ☹

Homogeneous system

– The (somewhat hacky) solution? Stick a “1” at the end of every vector:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

– Now we can rotate, scale, and skew like before, AND translate (note how the multiplication works out, above)

– This is called “homogeneous coordinates”

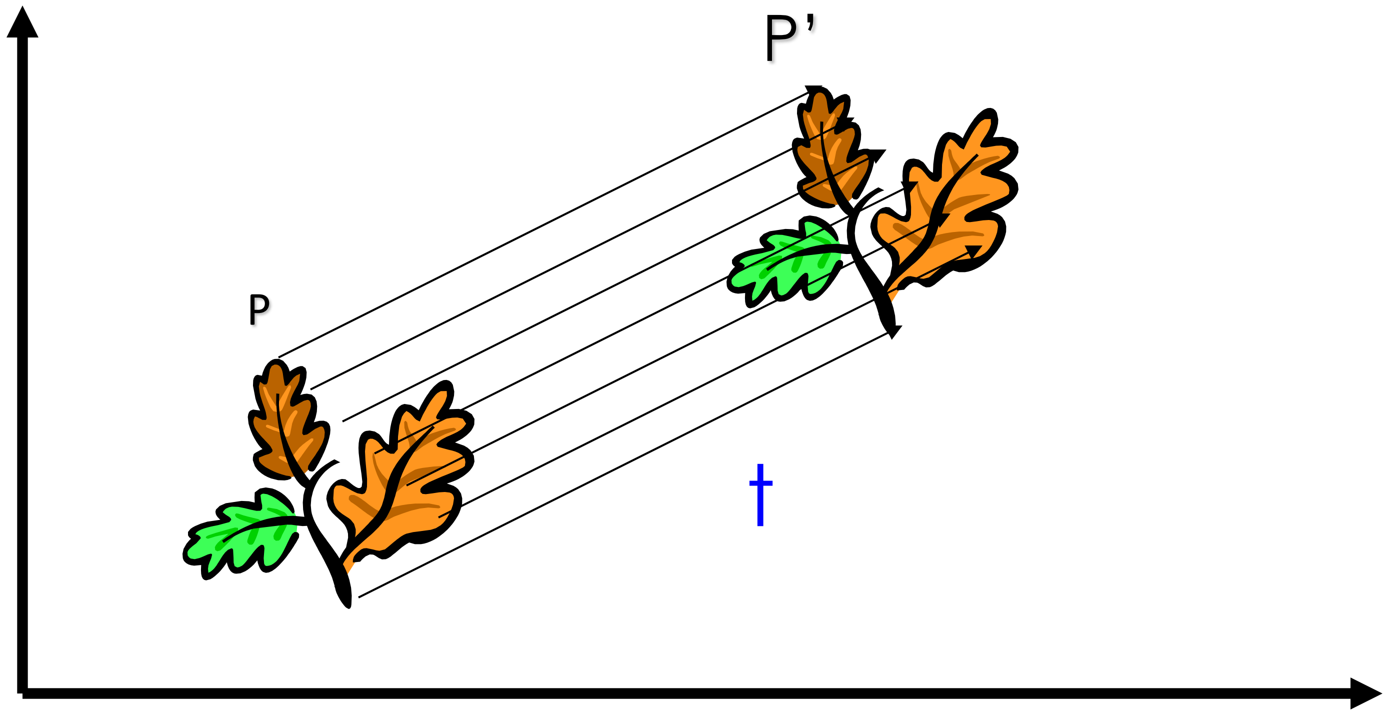
Homogeneous system

– In homogeneous coordinates, the multiplication works out so the rightmost column of the matrix is a vector that gets added.

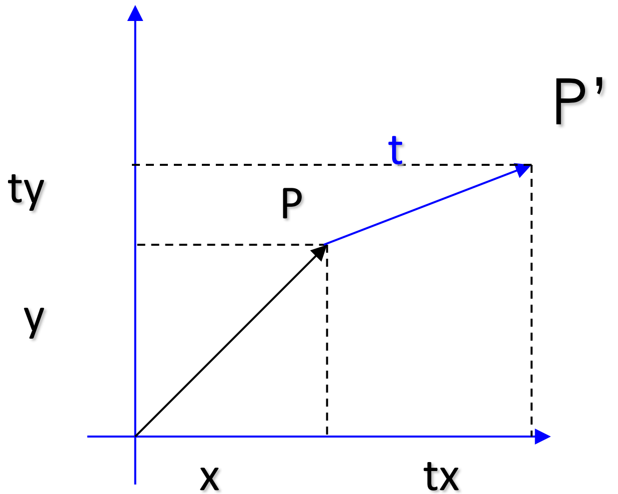
$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

– Generally, a homogeneous transformation matrix will have a bottom row of [0 0 1], so that the result has a “1” at the bottom too.

2D Translation



2D Translation using Homogeneous Coordinates

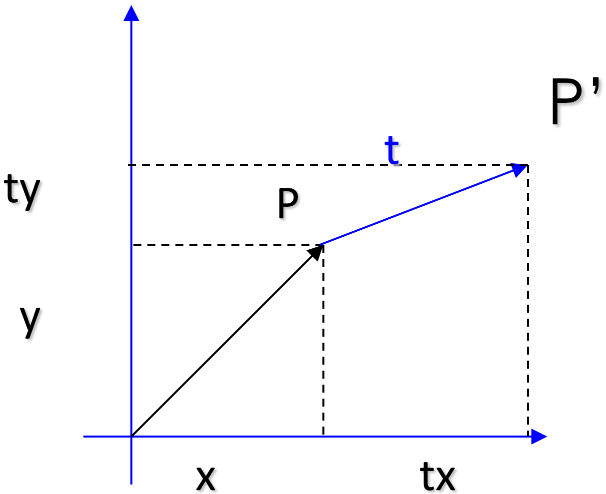


$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \cdot \overset{\text{P}}{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}$$

2D Translation using Homogeneous Coordinates



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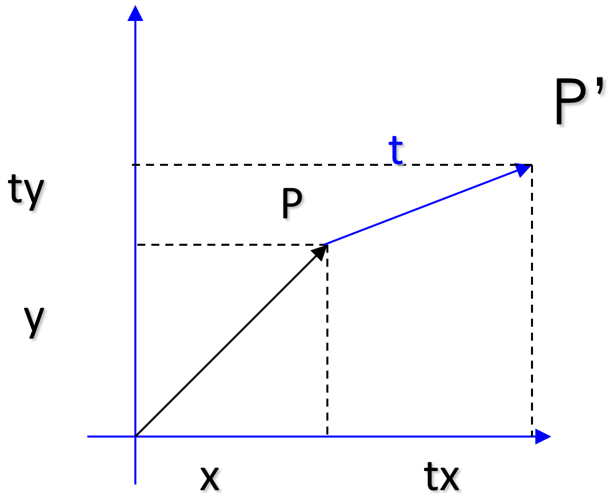
P

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix} \cdot \begin{bmatrix} \boxed{x} \\ \boxed{y} \\ \boxed{1} \end{bmatrix}$$

2D Translation using Homogeneous Coordinates



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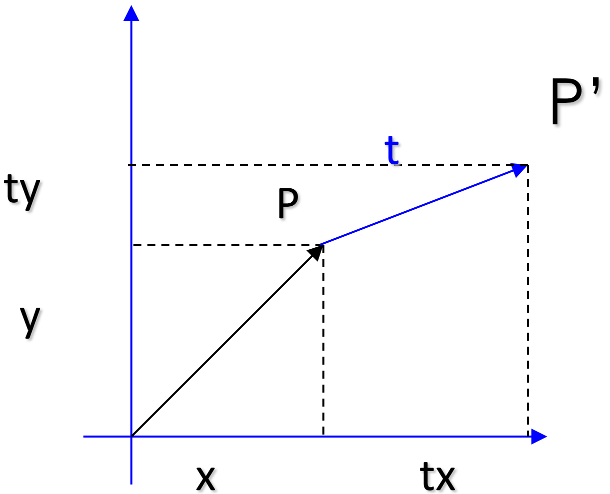
$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \\ & & \\ & & \end{bmatrix} \cdot \overset{P}{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}$$

2D Translation using Homogeneous Coordinates



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p

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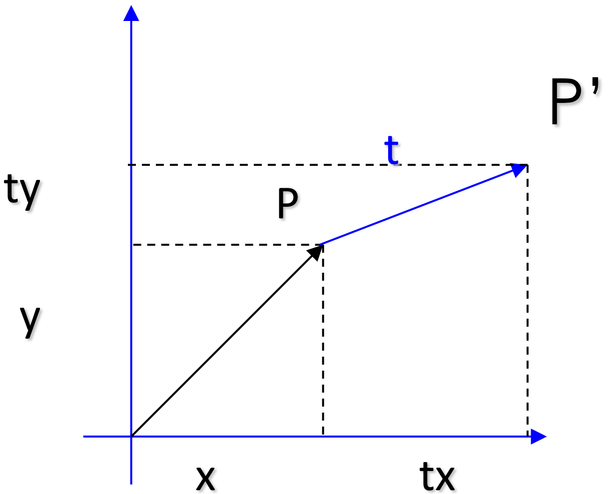
2D Translation using Homogeneous Coordinates



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$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

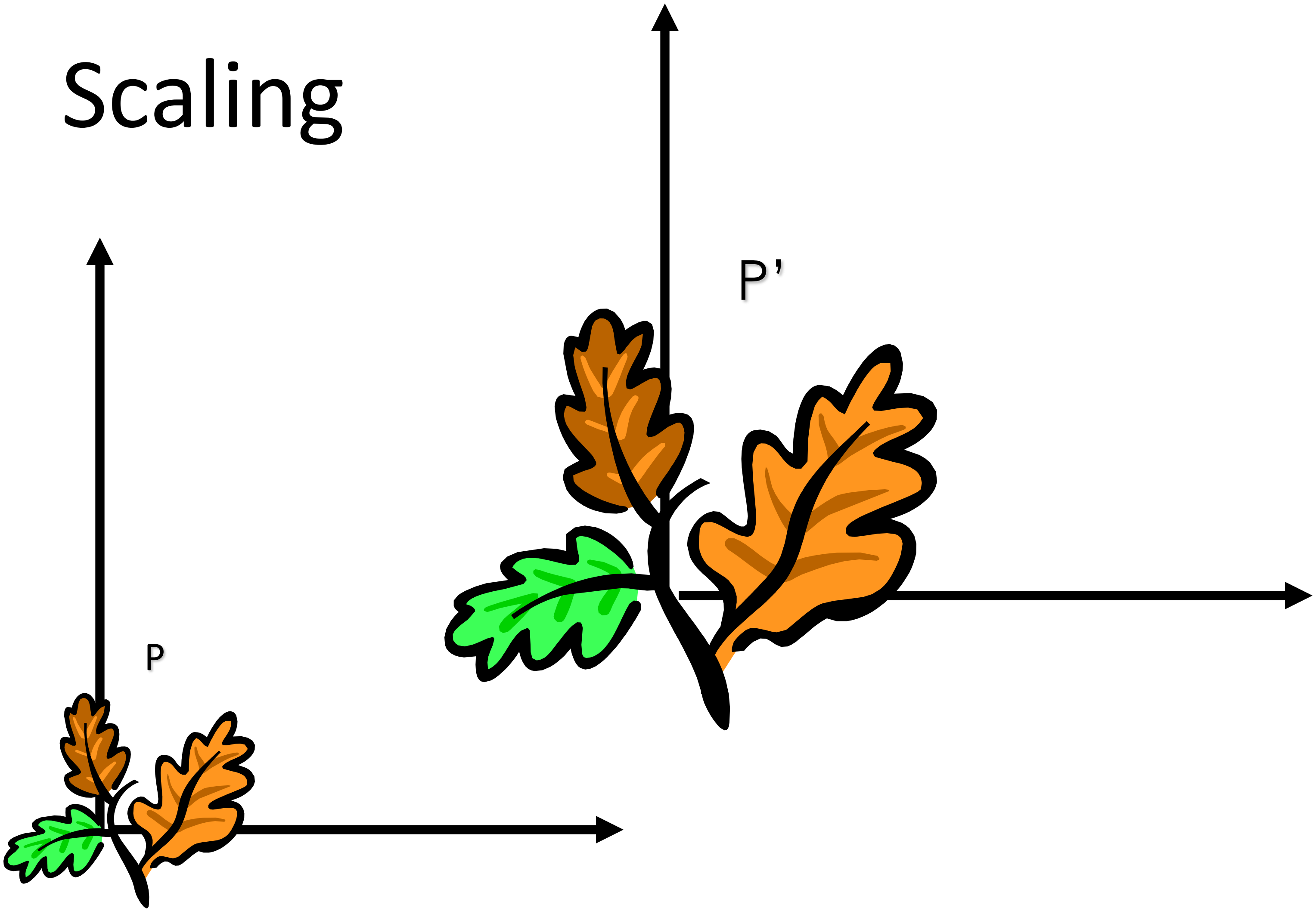
t

P

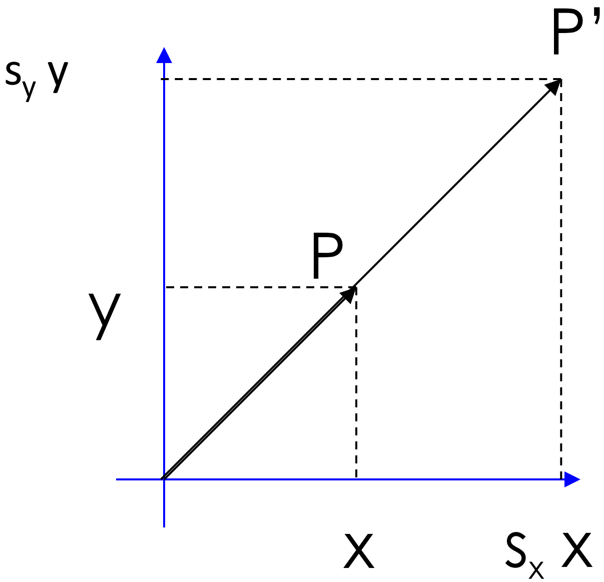
$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{T} \cdot \mathbf{P}$$

Scaling



Scaling Equation

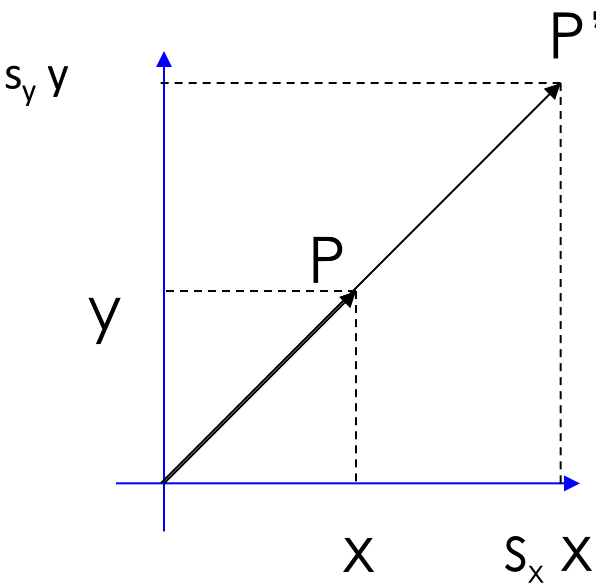


$$\mathbf{P} = (\mathrm{x}, \mathrm{y}) \rightarrow \mathbf{P}' = (\mathrm{s_x x}, \mathrm{s_y y})$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

Scaling Equation



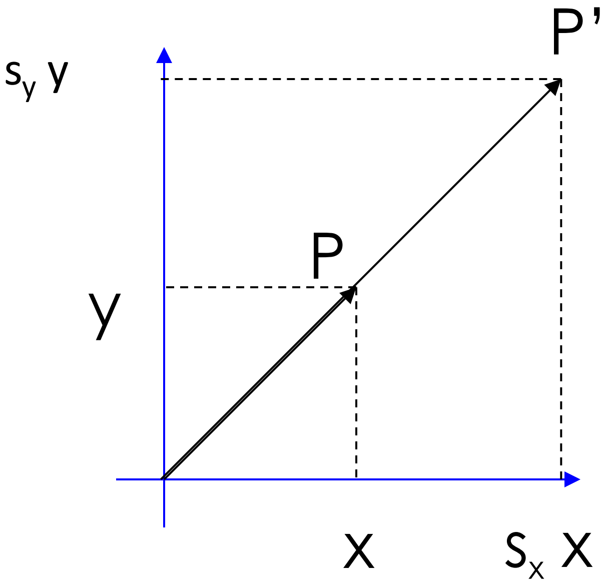
$$\mathbf{P} = (\mathrm{x}, \mathrm{y}) \rightarrow \mathbf{P}' = (\mathrm{s}_\mathrm{x}\mathrm{x}, \mathrm{s}_\mathrm{y}\mathrm{y})$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling Equation



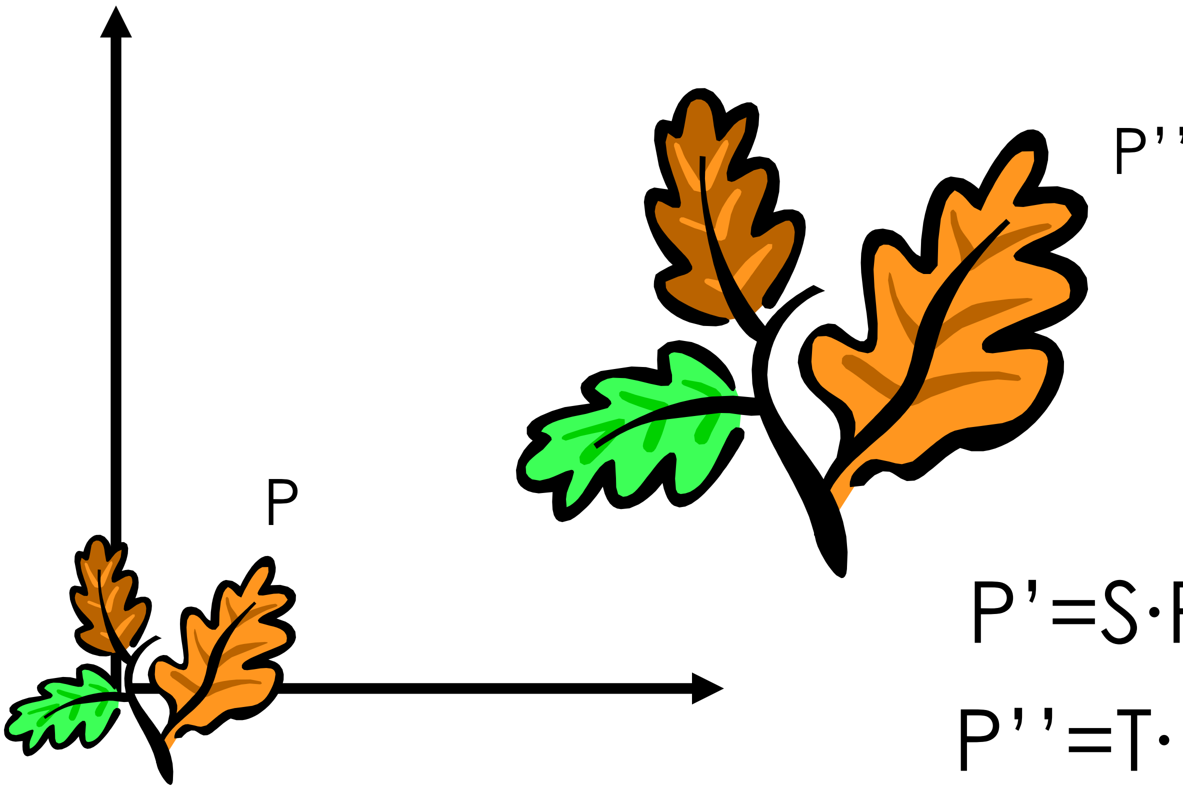
$$\mathbf{P} = (x, y) \rightarrow \mathbf{P}' = (s_x x, s_y y)$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}' & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \cdot \mathbf{P} = \mathbf{S} \cdot \mathbf{P}$$

Scaling & Translating



$P' = S \cdot P$

$P'' = T \cdot P'$

$P'' = T \cdot P' = T \cdot (S \cdot P) = T \cdot S \cdot P$

Scaling & Translating

$$\mathbf{P}'' = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling & Translating

$$\mathbf{P}'' = \mathbf{T} \times \mathbf{S} \times \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} S & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translating & Scaling versus Scaling & Translating

$$\mathbf{P}''' = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

Translating & Scaling != Scaling & Translating

$$\mathbf{P}''' = \mathbf{T}\cdot\mathbf{S}\cdot\mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

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Translating & Scaling != Scaling & Translating

$$\mathbf{P}''' = \mathbf{T}\cdot\mathbf{S}\cdot\mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

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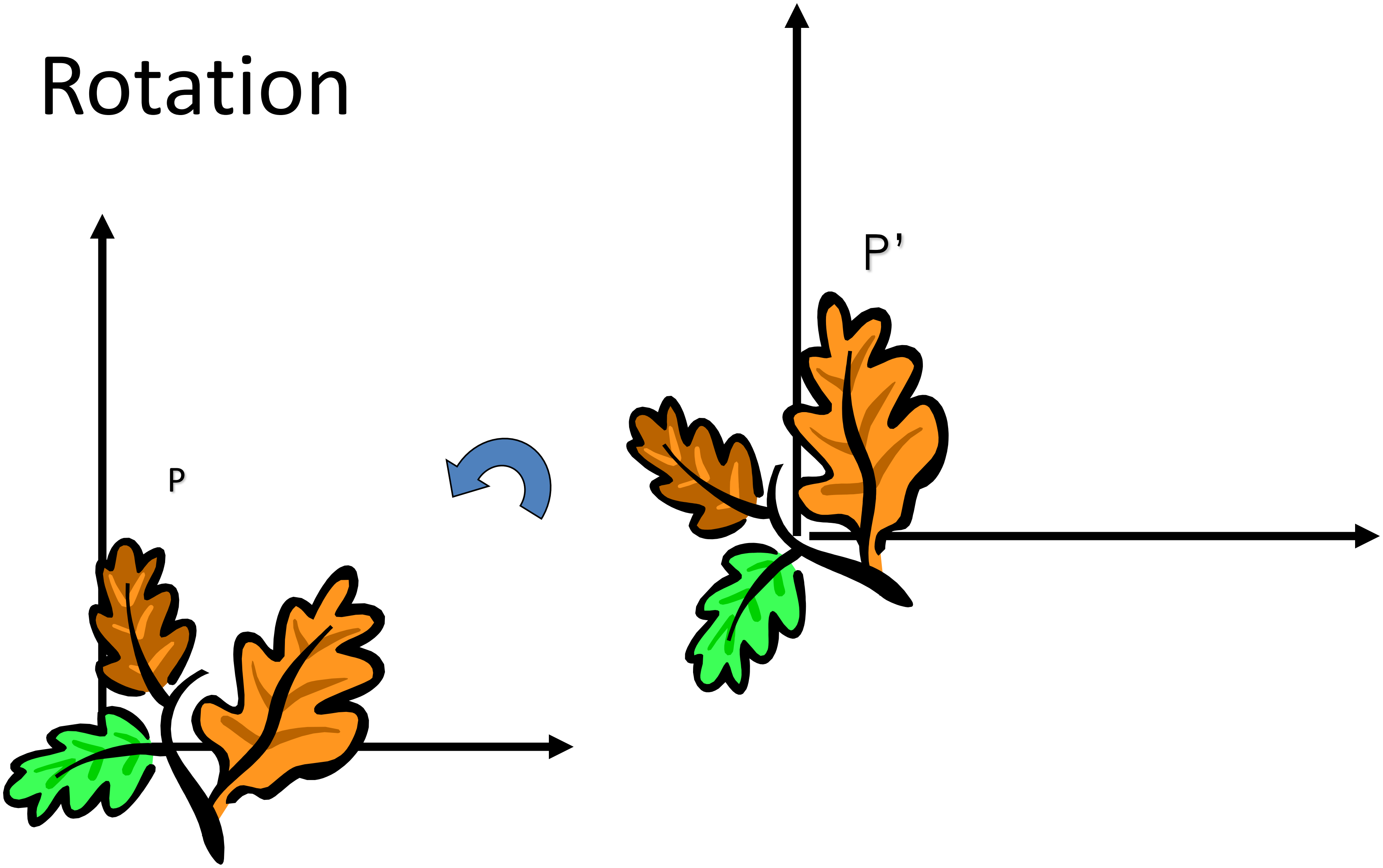
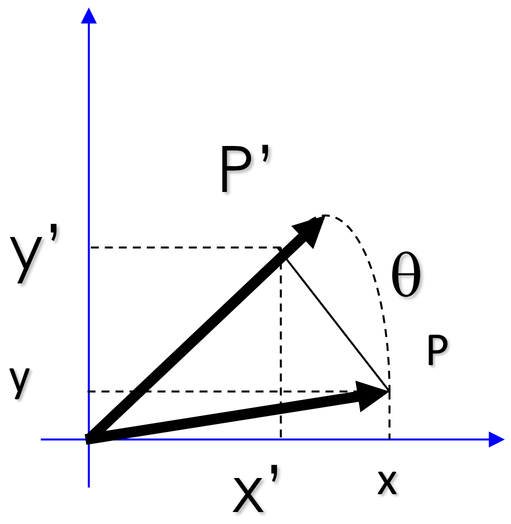
$$= \begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + s_x t_x \\ s_y y + s_y t_y \\ 1 \end{bmatrix}$$

Rotation



Rotation Equations

Counter-clockwise rotation by an angle θ



$$x' = \cos\theta\ x - \sin\theta\ y$$

$$y' = \cos\theta\ y + \sin\theta\ x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{R}\ \mathbf{P}$$

Rotation Matrix Properties

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

A 2D rotation matrix is 2x2

Note: R belongs to the category of *normal* matrices and satisfies many interesting properties:

$$\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

Rotation Matrix Properties

- Transpose of a rotation matrix produces a rotation in the opposite direction

$$\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

- The rows of a rotation matrix are always mutually perpendicular (a.k.a. orthogonal) unit vectors
  - (and so are its columns)

Scaling + Rotation + Translation

P'= (T R S) P

$$\mathbf{P}' = \mathbf{T}\cdot \mathrm{R}\cdot \mathbf{S}\cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} R\,S & t \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This is the form of the general-purpose transformation matrix

Outline

- Vectors and matrices
  - Basic Matrix Operations
  - Determinants, norms, trace
  - Special Matrices
- Transformation Matrices
  - Homogeneous coordinates
  - Translation
- Matrix inverse
- Matrix rank
- Eigenvalues and Eigenvectors
- Matrix Calculate

The inverse of a transformation matrix reverses its effect

Inverse

- Given a matrix **A**, its inverse $\mathbf{A}^{-1}$ is a matrix such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- E.g. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$
- Inverse does not always exist. If $\mathbf{A}^{-1}$ exists, **A** is *invertible* or *non-singular*. Otherwise, it's *singular*.
- Useful identities, for matrices that are invertible:

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$\mathbf{A}^{-T} \triangleq (\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$$

Matrix Operations

- Pseudoinverse
  - Say you have the matrix equation AX=B, where A and B are known, and you want to solve for X

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Matrix Operations

- Pseudoinverse
  - Say you have the matrix equation AX=B, where A and B are known, and you want to solve for X
  - You could calculate the inverse and pre-multiply by it: $A^{-1}AX=A^{-1}B \rightarrow X=A^{-1}B$
  - Python command would be **np.linalg.inv(A)*B**
  - But calculating the inverse for large matrices often brings problems with computer floating-point resolution (because it involves working with very small and very large numbers together).
  - Or, your matrix might not even have an inverse.

Matrix Operations

- Pseudoinverse
  - Fortunately, there are workarounds to solve AX=B in these situations. And python can do them!
  - Instead of taking an inverse, directly ask python to solve for X in AX=B, by typing **np.linalg.solve(A, B)**
  - Python will try several appropriate numerical methods (including the pseudoinverse if the inverse doesn't exist)
  - Python will return the value of X which solves the equation
    - If there is no exact solution, it will return the closest one
    - If there are many solutions, it will return the smallest one

Matrix Operations

- Python example:

$$AX = B$$

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

```
>> import numpy as np
>> x = np.linalg.solve(A,B)
x =
    1.0000
   -0.5000
```

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The rank of a transformation matrix tells you how many dimensions it transforms a vector to.

Linear independence

- Suppose we have a set of vectors $\mathbf{v_1, \ldots, v_n}$
- If we can express $\mathbf{v}_1$ as a linear combination of the other vectors $\mathbf{v}_2 \ldots \mathbf{v}_n$, then $\mathbf{v}_1$ is linearly *dependent* on the other vectors.
  - The direction $\mathbf{v}_1$ can be expressed as a combination of the directions $\mathbf{v}_2 \ldots \mathbf{v}_n$. (E.g. $\mathbf{v}_1 = .7\,\mathbf{v}_2 - .7\,\mathbf{v}_4$)

Linear independence

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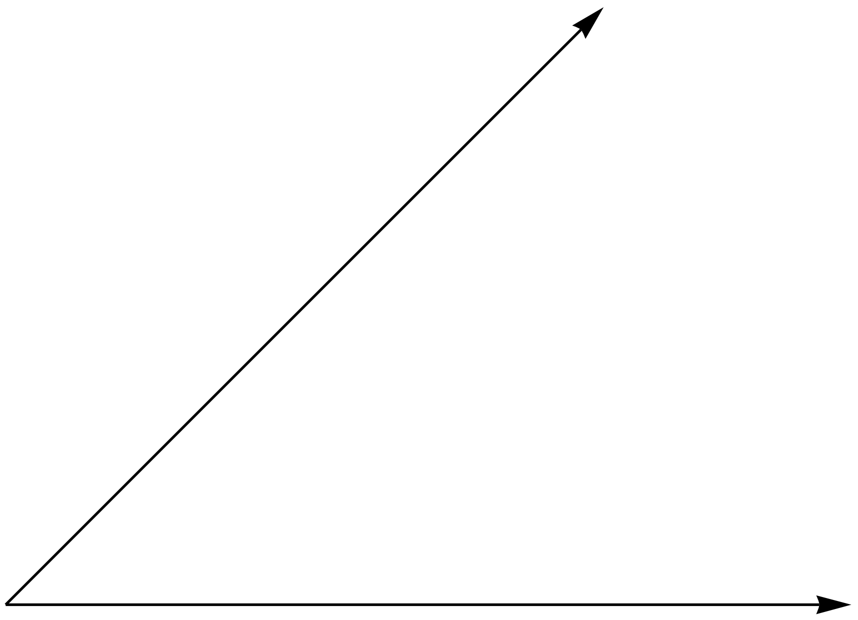
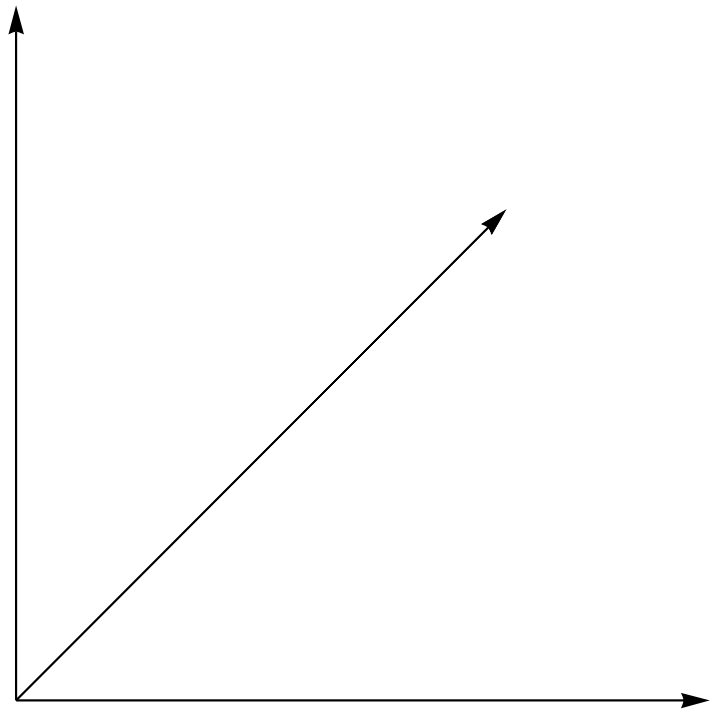
- If no vector is linearly dependent on the rest of the set, the set is linearly *independent*.
  - Common case: a set of vectors $\mathbf{v_1, \ldots, v_n}$ is always linearly independent if each vector is perpendicular to every other vector (and non-zero)

Linear independence

Linearly independent set



Not linearly independent



Matrix rank

- Column/row rank

$$\text{col-rank}(\mathbf{A}) = \text{ the maximum number of linearly independent column vectors of } \mathbf{A}$$
$$\text{row-rank}(\mathbf{A}) = \text{ the maximum number of linearly independent row vectors of } \mathbf{A}$$

  – Column rank always equals row rank

- Matrix rank

$$\text{rank}(\mathbf{A}) \triangleq \text{col-rank}(\mathbf{A}) = \text{row-rank}(\mathbf{A})$$

Matrix rank

- For transformation matrices, the rank tells you the dimensions of the output
- E.g. if rank of **A** is 1, then the transformation

p'=Ap

maps points onto a line.

- Here's a matrix with rank 1:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ 2x + 2y \end{bmatrix}$$

All points get mapped to the line y=2x

Matrix rank

- If an m x m matrix is rank m, we say it's "full rank"
  - Maps an m x 1 vector uniquely to another m x 1 vector
  - An inverse matrix can be found
- If rank < m, we say it's "singular"
  - At least one dimension is getting collapsed. No way to look at the result and tell what the input was
  - Inverse does not exist
- Inverse also doesn't exist for non-square matrices

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Eigenvector and Eigenvalue

- An eigenvector **x** of a linear transformation *A* is a non-zero vector that, when *A* is applied to it, does not change direction.

$$Ax = \lambda x, \quad x \neq 0.$$

Eigenvector and Eigenvalue

- An eigenvector **x** of a linear transformation *A* is a non-zero vector that, when *A* is applied to it, does not change direction.
- Applying *A* to the eigenvector only scales the eigenvector by the scalar value λ, called an eigenvalue.

$$Ax = \lambda x, \quad x \neq 0.$$

Eigenvector and Eigenvalue

- We want to find all the eigenvalues of A:

$$Ax = \lambda x, \quad x \neq 0.$$

- Which can we written as:

$$Ax = (\lambda I)x \quad x \neq 0.$$

- Therefore:

$$(\lambda I - A)x = 0, \quad x \neq 0.$$

Eigenvector and Eigenvalue

- We can solve for eigenvalues by solving:

$$(\lambda I - A)x = 0, \quad x \neq 0.$$

- Since we are looking for non-zero **x**, we can instead solve the above equation as:

$$|(\lambda I - A)| = 0.$$

Properties

- The trace of a A is equal to the sum of its eigenvalues:

$$\mathrm{tr}A = \sum_{i=1}^{n} \lambda_i.$$

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- The determinant of A is equal to the product of its eigenvalues

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- The rank of A is equal to the number of non-zero eigenvalues of A.
- The eigenvalues of a diagonal matrix D = diag(d1, . . . dn) are just the diagonal entries d1, . . . dn

Spectral theory

- We call an eigenvalue λ and an associated eigenvector an **eigenpair**.
- The space of vectors where (A − λI) = 0 is often called the **eigenspace** of A associated with the eigenvalue λ.
- The set of all eigenvalues of A is called its **spectrum**:

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is singular}\}.$$

Spectral theory

- The magnitude of the largest eigenvalue (in magnitude) is called the spectral radius

$$\rho(A) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$$

  – Where C is the space of all eigenvalues of A

Spectral theory

- The spectral radius is bounded by infinity norm of a matrix: $\rho(A) = \lim_{k \to \infty} \|A^k\|^{1/k}$
- Proof: Turn to a partner and prove this!

Spectral theory

- The spectral radius is bounded by infinity norm of a matrix: $\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k}$
- Proof: Let λ and v be an eigenpair of A:

$$|\lambda|^k \|\mathbf{v}\| = \|\lambda^k \mathbf{v}\| = \|A^k \mathbf{v}\| \leq \|A^k\| \cdot \|\mathbf{v}\|$$

and since $\mathbf{v} \neq 0$ we have

$$|\lambda|^k \leq \|A^k\|$$

and therefore

$$\rho(A) \leq \|A^k\|^{\frac{1}{k}}.$$

Diagonalization

- An $n \times n$ matrix A is diagonalizable if it has n linearly independent eigenvectors.
- Most square matrices (in a sense that can be made mathematically rigorous) are diagonalizable:
  - Normal matrices are diagonalizable
  - Matrices with n distinct eigenvalues are diagonalizable

Lemma: Eigenvectors associated with distinct eigenvalues are linearly independent.

Diagonalization

- An $n \times n$ matrix A is diagonalizable if it has n linearly independent eigenvectors.
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Lemma: Eigenvectors associated with distinct eigenvalues are linearly independent.

Diagonalization

- Eigenvalue equation:

$$AV = VD$$

$$A = VDV^{-1}$$

– Where D is a diagonal matrix of the eigenvalues

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Diagonalization

- Eigenvalue equation:

$$AV = VD$$

$$A = VDV^{-1}$$

- Assuming all λ_i's are unique:

$$A = VDV^{T}$$

  - Remember that the inverse of an orthogonal matrix is just its transpose and the eigenvectors are orthogonal

Symmetric matrices

- Properties:
  - For a symmetric matrix A, all the eigenvalues are real.
  - The eigenvectors of A are orthonormal.

$$A = VDV^T$$

Symmetric matrices

- Therefore:

$$x^T A x = x^T V D V^T x = y^T D y \;= \sum_{i=1}^{n} \lambda_i y_i^2$$

  – where $y = V^T x$

- So, if we wanted to find the vector x that:

$$\max_{x \in \mathbb{R}^n} \;\; x^T A x \qquad \text{subject to } \|x\|_2^2 = 1$$

Symmetric matrices

- Therefore:

$$x^T A x = x^T V D V^T x = y^T D y = \sum_{i=1}^{n} \lambda_i y_i^2$$

  – where $y = V^T x$

- So, if we wanted to find the vector x that:

$$\max_{x \in \mathbb{R}^n} \; x^T A x \quad \text{subject to } \|x\|_2^2 = 1$$

  – Is the same as finding the eigenvector that corresponds to the largest eigenvalue.

Some applications of Eigenvalues

- PageRank
- Schrodinger's equation
- PCA

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Matrix Calculus – The Gradient

- Let a function $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ take as input a matrix A of size m × n and returns a real value.
- Then the **gradient** of **f:**

$$\nabla_A f(A) \in \mathbb{R}^{m \times n} = \begin{bmatrix} \frac{\partial f(A)}{\partial A_{11}} & \frac{\partial f(A)}{\partial A_{12}} & \cdots & \frac{\partial f(A)}{\partial A_{1n}} \\ \frac{\partial f(A)}{\partial A_{21}} & \frac{\partial f(A)}{\partial A_{22}} & \cdots & \frac{\partial f(A)}{\partial A_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(A)}{\partial A_{m1}} & \frac{\partial f(A)}{\partial A_{m2}} & \cdots & \frac{\partial f(A)}{\partial A_{mn}} \end{bmatrix}$$

Matrix Calculus – The Gradient

- Every entry in the matrix is: $\nabla_A f(A))_{ij} = \frac{\partial f(A)}{\partial A_{ij}}.$
- the size of $\nabla_A f(A)$ is always the same as the size of A. So if A is just a vector x:

$$\nabla_x f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

Exercise

- Example:

For $x \in \mathbb{R}^n$, let $f(x) = b^T x$ for some known vector $b \in \mathbb{R}^n$

$$f(x) = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Find: $\dfrac{\partial f(x)}{\partial x_k} = ?$

$$\nabla_x f(x) = ?$$

Exercise

- Example:

For $x \in \mathbb{R}^n$, let $f(x) = b^T x$ for some known vector $b \in \mathbb{R}^n$

$$f(x) = \sum_{i=1}^{n} b_i x_i$$

$$\frac{\partial f(x)}{\partial x_k} = \frac{\partial}{\partial x_k} \sum_{i=1}^{n} b_i x_i = b_k.$$

- From this we can conclude that: $\nabla_x b^T x = b$.

Matrix Calculus – The Gradient

- Properties
  - $\nabla_x(f(x) + g(x)) = \nabla_x f(x) + \nabla_x g(x)$.
  - For $t \in \mathbb{R}$, $\nabla_x(t\, f(x)) = t\nabla_x f(x)$.

Matrix Calculus – The Hessian

- The Hessian matrix with respect to x, written $\nabla_x^2 f(x)$ or simply as H is the n × n matrix of partial derivatives

$$\nabla_x^2 f(x) \in \mathbb{R}^{n\times n} = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

Matrix Calculus – The Hessian

- Each entry can be written as: $\nabla_x^2 f(x))_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$.

- Exercise: Why is the Hessian always symmetric?

Matrix Calculus – The Hessian

- Each entry can be written as: $\nabla_x^2 f(x))_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$.

- The Hessian is always symmetric, because

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x)}{\partial x_j \partial x_i}.$$

- This is known as Schwarz's theorem: The order of partial derivatives don’t matter as long as the second derivative exists and is continuous.

Matrix Calculus – The Hessian

- Note that the hessian is not the gradient of whole gradient of a vector (this is not defined). It is actually the gradient of **every entry** of the gradient of the vector.

$$\nabla_x^2 f(x) \in \mathbb{R}^{n\times n} = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

Matrix Calculus – The Hessian

- Eg, the first column is the gradient of $\frac{\partial f(x)}{\partial x_1}$

$$\nabla_x^2 f(x) \in \mathbb{R}^{n\times n} = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

Exercise

- Example:

consider the quadratic function $f(x) = x^T A x$

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

$$\frac{\partial f(x)}{\partial x_k} = \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

Exercise

$$\frac{\partial f(x)}{\partial x_k} = \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

Exercise

$$\begin{aligned}\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k}\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}x_i x_j \\ &= \frac{\partial}{\partial x_k}\left[\sum_{i\neq k}\sum_{j\neq k} A_{ij}x_i x_j + \sum_{i\neq k} A_{ik}x_i x_k + \sum_{j\neq k} A_{kj}x_k x_j + A_{kk}x_k^2\right]\end{aligned}$$

Divide the summation into 3 parts depending on whether:

- i == k or
- j == k

Exercise

$$
\begin{aligned}
\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \\
&= \frac{\partial}{\partial x_k} \left[\boxed{\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j} + \sum_{i \neq k} A_{ik} x_i x_k + \sum_{j \neq k} A_{kj} x_k x_j + A_{kk} x_k^2 \right] \\
&= \sum_{i \neq k} A_{ik} x_i + \sum_{j \neq k} A_{kj} x_j + 2 A_{kk} x_k
\end{aligned}
$$

Exercise

$$
\begin{aligned}
\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \\
&= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \boxed{\sum_{i \neq k} A_{ik} x_i x_k} + \sum_{j \neq k} A_{kj} x_k x_j + A_{kk} x_k^2 \right] \\
&= \boxed{\sum_{i \neq k} A_{ik} x_i} + \sum_{j \neq k} A_{kj} x_j + 2 A_{kk} x_k
\end{aligned}
$$

Exercise

$$
\begin{aligned}
\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \\
&= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \sum_{i \neq k} A_{ik} x_i x_k + \boxed{\sum_{j \neq k} A_{kj} x_k x_j} + A_{kk} x_k^2 \right] \\
&= \sum_{i \neq k} A_{ik} x_i + \boxed{\sum_{j \neq k} A_{kj} x_j} + 2 A_{kk} x_k
\end{aligned}
$$

Exercise

$$
\begin{aligned}
\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \\
&= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \sum_{i \neq k} A_{ik} x_i x_k + \sum_{j \neq k} A_{kj} x_k x_j + \boxed{A_{kk} x_k^2} \right] \\
&= \sum_{i \neq k} A_{ik} x_i + \sum_{j \neq k} A_{kj} x_j + \boxed{2 A_{kk} x_k}
\end{aligned}
$$

Exercise

$$
\begin{aligned}
\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \\
&= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \sum_{i \neq k} A_{ik} x_i x_k + \sum_{j \neq k} A_{kj} x_k x_j + A_{kk} x_k^2 \right] \\
&= \sum_{i \neq k} A_{ik} x_i + \sum_{j \neq k} A_{kj} x_j + 2 A_{kk} x_k
\end{aligned}
$$

Exercise

$$
\begin{aligned}
\frac{\partial f(x)}{\partial x_k} &= \frac{\partial}{\partial x_k} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \\
&= \frac{\partial}{\partial x_k} \left[\sum_{i \neq k} \sum_{j \neq k} A_{ij} x_i x_j + \sum_{i \neq k} A_{ik} x_i x_k + \sum_{j \neq k} A_{kj} x_k x_j + A_{kk} x_k^2 \right] \\
&= \sum_{i \neq k} A_{ik} x_i + \sum_{j \neq k} A_{kj} x_j + 2 A_{kk} x_k \\
&= \sum_{i=1}^{n} A_{ik} x_i + \sum_{j=1}^{n} A_{kj} x_j = 2 \sum_{i=1}^{n} A_{ki} x_i,
\end{aligned}
$$

Exercise

$$f(x) = x^T A x$$

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

$$\frac{\partial^2 f(x)}{\partial x_k \partial x_\ell} = \frac{\partial}{\partial x_k} \left[\frac{\partial f(x)}{\partial x_\ell} \right] = \frac{\partial}{\partial x_k} \left[\sum_{i=1}^{n} A_{\ell i} x_i \right]$$

Exercise

$$f(x) = x^T A x$$

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

$$\frac{\partial^2 f(x)}{\partial x_k \partial x_\ell} = \frac{\partial}{\partial x_k} \left[\frac{\partial f(x)}{\partial x_\ell} \right] = \frac{\partial}{\partial x_k} \left[\sum_{i=1}^{n} A_{\ell i} x_i \right]$$

$$= 2A_{\ell k} = 2A_{k\ell}.$$

Exercise

$$f(x) = x^T A x$$

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j$$

$$\frac{\partial^2 f(x)}{\partial x_k \partial x_\ell} = \frac{\partial}{\partial x_k} \left[\frac{\partial f(x)}{\partial x_\ell} \right] = \frac{\partial}{\partial x_k} \left[\sum_{i=1}^{n} 2 A_{\ell i} x_i \right]$$

$$= 2A_{\ell k} = 2A_{k\ell}.$$

$$\nabla_x^2 f(x) = 2A$$

What we have learned

- Vectors and matrices
  - Basic Matrix Operations
  - Special Matrices
- Transformation Matrices
  - Homogeneous coordinates
  - Translation
- Matrix inverse
- Matrix rank
- Eigenvalues and Eigenvectors
- Matrix Calculate