MAT244-final-report

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1 Background Assumptions

Global assumptions:

G-1. The government is able to impose its policy on rents, and rent reacts in response to government policy immediately.

Justification: The government has the power to control market rent. For example, it can impose price ceiling, increase tax, set up entry quota, or provide subsidies to tenants or landlords.

G-2. The government cannot directly control population, but it has access to the population data of the city.

Justification: The government doesn't want to impose policies on rents blindly; it wants to make changes according to population changes. The population data would be available from Registry Departments, and the government can calculate rate of change of population.

G-3. The government aims to control outflow of population taking rent control as a tool.

Justification: The government want to maintain a reasonable population inside the city, and its only available tool is controlling rent in this model.

G-4. The new policy is only responsible for at most future 50 years.

Justification: Super long-term plans are unrealistic, and macro changes are also unpredictable. A 50-year policy with positive changes to the city is already beneficial.

Original assumptions of HIGO model from which we inherit:

A-1. The city is capable of a maximum of 10 million people.

A-2. People are attracted by people and move to the city if many other people already live in the city.

A-3. People will move away from the city if the rent is too high.

A-4. Rent increases with inflation.

2 Executive Summary

To the city of Oronto,

With the soaring rent and decreasing population, business owners are concerned about the future of the city, especially its economic prospect. In this report, we are commissioned by the Housing Institute of Greater Oronto (HIGO) to analyze the provided population-rent model and offer a potential improvement in order to control the rent price and population in Oronto.

We first considered the original model provided by HIGO. While the original model on rent is mathematically correct, it only consider the effect of inflation on rent price and ignore the other factors such as population and governmental policies. This causes the predicted rent price to increase infinitely. On the other hand, the prediction of population is very accurate because it successfully limit the population to 10 Millions and reflect the interaction between population and rent price. Based on the given model, we interpret the model prediction as the following: Between 2024 and 2029, we expect city of Oronto to attract new residents to the city thanks to the low average rent price. Right after the population reached its maximum in 2029, residents of Oronto start moving out of the city at a rather slow rate due to the increasing rent price driven by inflation and lack of external intervention by government.

Then, we proposed a new model which modifies the existing model by adding additional factors into the model for the rent to reflect the impact of population and governmental policies on the average monthly rent price of Oronto. We decided not to change the model for population because of the assumption G-2 which forbids direct attempt to modify the population growth. This additional factor is proportional to how fast population is growing at a given point: if the population is growing faster, the new factor is going to slow down the growth of rent and vice versa. This modification of the model removes the massive outflow of population from Oronto observed in the HIGO model. The modification also change the behavior of rent price development. Between 2024 and 2029, the new model predicts a drop in average monthly rent price (this is going to be discussed in detail later) followed by a constant perpetual growth of average rent. The new model can be interpret as such: At the beginning, in order to attract more resident to the city, the average rent price is artificially suppressed against inflation; once the number of resident reaches the upper limit of the capacity of Oronto, the intervention is loosen and the rent price is driven by inflation again. However, the reduction of intervention does not cause the massive outflow of population from Oronto because the effect of rent reduction remains effective even after the reduction is terminated, which compensates the inflation for the residents. However, similar to the original model proposed by HIGO, we still cannot expect a perfectly stable average rent price and population, so HIGO should expect volatility within the interaction of rent price and population and prepare for necessary intervention.

The biggest advantage of our modified model is that massive population outflow is deterred while rent is controlled. However, on the executive side, significantly more effort (not necessarily money) should be put on calculating the exact subsidizing amount in order to precisely control the rent, and accordingly, the population. At the initial stage of the program, the government should be prepared to investment on subsidy (as calculated later, the total subsidy needed throughout 5 decades is $18.46 billion). Another important arrangement is to prepare enough housing for the massive population inflow in the first decade. After the first decade, the population will stay stable, but during the first decade, about 5 million people attracted by the subsidy will flow in the city.

Our core proposal is that the city should intervene the rent by monitoring population change and act accordingly. The specific formulae of subsidy amounts are given in the following chapter. We believe the model will assist the city in ensuring controlled population and rent, thus promoting healthy development of the city's economy.

3 Technical Report

3.1 Analysis on HIGO Model

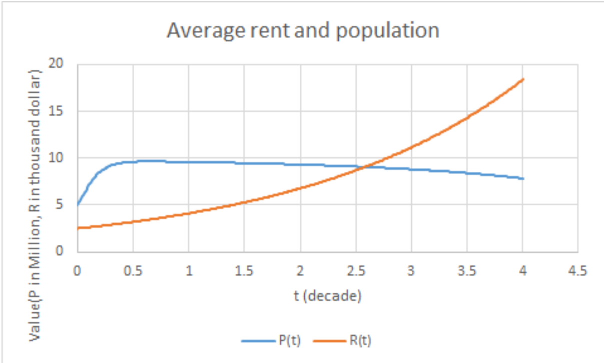

Figure 1: Development of average rent and population based on the HIGO model

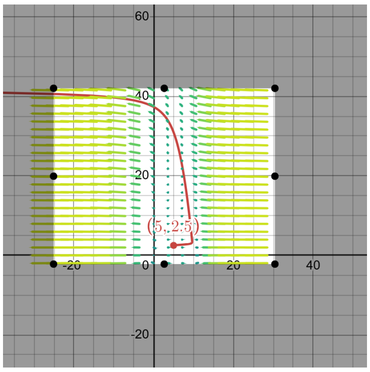

Figure 2: Phase Portrait of HIGO (x-axis: P(t), y-axis: R(t))

In the original HIGO model, by using Desmos simulation, we observe a rise in population in the first 5 years, which drops constantly after that, which aligns with assumption A-2 and A-3. Therefore, an equilibrium is not expected to be found for population within a reasonable period of time.

On the other hand, the rent rises constantly as assumed by A-4. This can be shown by solving the differential equation $R'(t)$

$$\begin{aligned}\frac{dR}{dt} &= 0.5R(t)\\ \frac{1}{R(t)}dR &= 0.5dt\\ \int \frac{1}{R(t)}dR &= \int 0.5dt\\ \ln(R(t)) &= 0.5t + c\\ R(t) &= e^{0.5t+c}\end{aligned}$$

and the monotonically increasing behavior of rent can be explained by the property of exponential function.

Therefore, we believe that the original HIGO model fails to reflect the interaction between population and rent price as well as potential governmental intervention since it does not contain any information about either $P(t)$ or $P'(t)$. Aside from this, the original model successfully maintains the assumptions A-1 by applying the term $(10 - P(t))$ to $P'(t)$ and the assumption of non-zero average rent by setting the initial average rent to positive \$2500.

3.2 Proposed changes

With problems in the original model, by government's power assumed in G-1 and G-2, we propose changes as follows:

> The government should monitor the population change each month, and subsidize/charge the rent accordingly to let the rent change rate drop/rise, in addition to inflation, by amount equal to half of population change rate (subsidize when population increases, and charge when it decreases).

3.2.1 Modified model with the change

$$P'(t) = P(t) \cdot (10 - P(t)) - R(t)$$

$$R'(t) = 0.5 \cdot R(t) \underline{\underline{-\mathbf{0.5} \cdot \mathbf{P}'(\mathbf{t})}}$$

Or equivalently, to write it in a standard ODE system:

$$P'(t) = P(t) \cdot (10 - P(t)) - R(t)$$

$$R'(t) = R(t) - 0.5 \cdot P(t) \cdot (10 - P(t))$$

For more intuitive analysis we'll use the first system.

3.2.2 Consequences of this change

In this model, the rent responses in inverse proportion to population change rate. Therefore, rents will not behave as drastically as before.

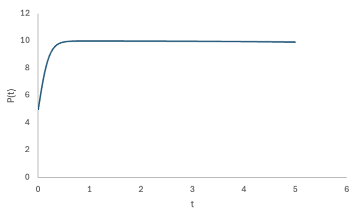

Figure 3: Component graph of P(t)

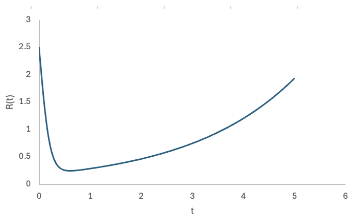

Figure 4: Component graph of R(t)

From the component graphs we observe that after the first decade, the population remain stable, and the mass population outflow in the previous model does not exist anymore. The rent decreases until about the fifth year, and then rises back to its original level at the end of 5th decade, which also shows success comparing to the drastic increase of rent in the previous model. Both population and rent are successfully controlled, reaching government's goal assumed in G-3.

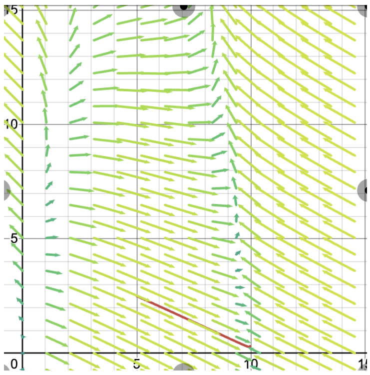

Figure 5: Phase portrait for the 1st decade

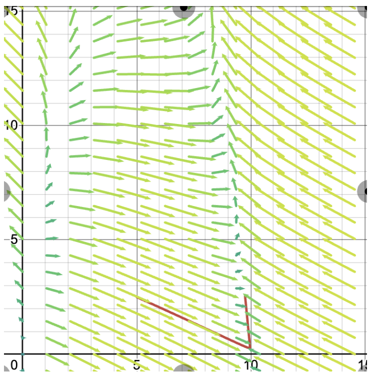

Figure 6: Phase portrait until 5th decade

It's impossible to find a reasonable equilibrium point, because if we let $P'(t) = 0$ and $R'(t) = 0$, the second condition would give $R(t) = P'(t) = 0$, but zero rent is unreasonable. Indeed, as stated in G-4, we're not responsible for long-term behaviors of the plan.

3.2.3 Specific steps the government should take

The specific actions proposed for the government would be:
(1) Monitor population change each month and calculate $P'(t)$.
(2) Calculate the original rent by Euler's method without subsidy

$$R_{\text{original}}(t) = R(t - \frac{1}{10 \times 12}) + \frac{1}{10 \times 12} R'_{\text{original}}(t - \frac{1}{10 \times 12})$$

and target subsidy with the new model

$$R(t) = R(t - \frac{1}{10 \times 12}) + \frac{1}{10 \times 12} R'(t - \frac{1}{10 \times 12})$$

where $R'_{\text{original}}(t) = 0.5R(t)$ and $R'(t) = 0.5R(t) - 0.5P'(t)$.
(3) Calculate monthly total subsidy

$$P(t)\Delta R(t)$$

where $\Delta R(t) = R_{\text{original}}(t) - R(t)$. This is the amount the government need to spend (or charge, if negative) each month.

3.2.4 Would there be severe budget burden for the government with this change?

No. We can evaluate this by summing up all subsidies.
For each rental unit, the subsidy is $R_{\text{original}} - R$, in thousands of dollars.

t	P(t)	R(t)	R(t)_original	difference_R	Subsidy
0	5	2.5	2.5	0	0
0.008333333	5.19791667	2.41145833	2.51041667	0.09895833	0.51437717
0.016666667	5.39587583	2.32252649	2.4219184	0.09939191	0.5363064
0.025	5.59322599	2.23352861	2.33261782	0.09908921	0.55422836
0.033333333	5.78932031	2.14478782	2.24324785	0.09846003	0.57001667
0.041666667	5.98352514	2.05662202	2.15413468	0.09751266	0.58346948
0.05	6.1752282	1.96933975	2.06559758	0.09625783	0.59441409
...					
1	9.98567987	0.2920389	0.2920279	-1.1E-05	-0.0001098
...					
5	9.89167465	2.25516921	2.25493565	-0.0002336	-0.0023103
Total					18.4586054

Figure 7: Subsidies needed from the government

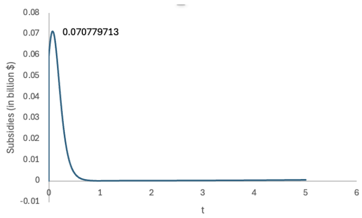

Figure 8: Subsidies needed from the government

For a month, the subsidy is the subsidy of each rental unit multiplied by number of rental units (population), thus $P(R_{\text{original}} - R)$, in billions of dollars (because population is in millions).
The total subsidy over the 5 decades is effectively the sum of subsidies over all months.

By calculation, the total subsidy required over the 5 decades is \$18.46 billion, which should not be a large amount for a large city like Oronto with 10 million residents. The average subsidy per year is \$369.2 million, and the average subsidy per year per person is \$36.92.