Homework - 4

Department of Civil, Chemical and Environmental Engineering

Course: Turbulence and CFD Models

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1 Question 1

Derive the incompressible Reynolds-Averaged Navier -Stokes (RANS) equations,

Let start by writing the averaging rules to be followed: In the derivation process both vector notations and Einstein notation are used. Whenever the prior notation is used, **Bold** variables indicate vectors. Similarly, in the later notation, mean values were described with capital letter(i.e U = Time averaged Mean velocity), whereas in the prior notation they are represented by a bold small letter with an overline (i.e $\overline{\mathbf{u}}$ = Time averaged Mean velocity).

- **Time averaging**

  For an instantaneous flow variable f($\mathbf{X}$,t), its time averaging $F_T(\mathbf{X})$, is given by:

$$F_T(\mathbf{X}) = \lim_{t\to\infty} \int_t^{t+T} f(\mathbf{X}, t)dt \tag{1}$$

- **Decomposition of flow variables**

  The instantaneous terms are expressed as the sum of a mean, U($\mathbf{x}$), and a fluctuating part, $u_i^{'}$ ($\mathbf{X}$,t)

$$u_i(\mathbf{X}, t) = U_i(\mathbf{X}) + u_i^{'}(\mathbf{X}, t)(2)$$

  As in equation (1), the quantity $U_i(\mathbf{X})$ is the time averaged,or mean velocity defined by

$$U_i(\mathbf{X}) = \lim_{t\to\infty} \int_t^{t+T} u_i(\mathbf{X}, t)dt \tag{3}$$

  The time averaged of the mean velocity is again the same time-averaged value; i.e.,

$$\overline{U}_i(\mathbf{X}) = \lim_{t\to\infty} \int_t^{t+T} U_i(\mathbf{X}, t)dt = U_i(\mathbf{X}) \tag{4}$$

  where an overbar is shorthand for the time average. The time average of the fluctuating part of the velocity is zero. That is, using equation (4)

$$\overline{u_i^{'}} = \lim_{t\to\infty} \int_t^{t+T} [u_i(\mathbf{X}, t) - U_i(\mathbf{X})]dt = U_i(\mathbf{X}) - \overline{U_i}(\mathbf{X}) = 0 \tag{5}$$

- **Rules of averaging** If ϕ and ψ are instantaneous flow terms their time averaging is given by:

$$
\begin{aligned}
\bar{\phi'} &= 0, \\
\bar{\bar{\phi}} &= \bar{\phi}, \\
\bar{\bar{\phi}} = \overline{\bar{\phi} + \phi'} &= \bar{\phi}, \\
\overline{\phi + \varphi} &= \bar{\phi} + \bar{\varphi}\,, \\
\overline{\bar{\phi}\varphi} = \bar{\bar{\phi}}\bar{\varphi} &= \bar{\phi}\bar{\varphi}, \\
\overline{\bar{\phi}\varphi'} = \bar{\phi}\bar{\varphi'} &= 0, \\
\overline{\frac{\partial \phi}{\partial x}} &= \frac{\partial \bar{\phi}}{\partial x},
\end{aligned}
\qquad
\begin{aligned}
\overline{\phi\varphi} &= \overline{(\bar{\phi} + \phi')(\bar{\varphi} + \varphi')} \\
&= \overline{\bar{\phi}\bar{\varphi} + \bar{\phi}\varphi' + \bar{\varphi}\phi' + \phi'\varphi'} \\
&= \overline{\bar{\phi}\bar{\varphi}} + \overline{\bar{\phi}\varphi'} + \overline{\bar{\varphi}\phi'} + \overline{\phi'\varphi'} \\
&= \bar{\phi}\bar{\varphi} + \overline{\phi'\varphi'}, \\
\overline{\phi'^2} &\neq 0, \\
\overline{\phi'\varphi'} &\neq 0, \\
\overline{\int \phi ds} &= \int \bar{\phi} ds
\end{aligned}
$$

- **Derivation of the Reynolds averaged Navier-Stokes equations** for an incomparable Newtonian fluid
  Continuity equation

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{6}$$

The Navier-Stokes equation

$$\frac{\partial u_i}{\partial t} + u_j \frac{u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} \tag{7}$$

The vectors u_i and x_i are velocity and position, t is time, p is pressure, ρ is density ν in kinematic viscosity.

To simplify the time averaging process, we write the convective term in the "conservative form", i.e.,

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j u_i}{\partial x_j} - u_i \frac{\partial u_j}{\partial x_j} = \frac{\partial u_j u_i}{\partial x_j} \tag{8}$$

Then time averaging equations (6) and (7) yields the **Reynolds averaged equations of motion in conservation form, viz.**

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{9}$$

$$\frac{\partial U_i}{\partial t} + \frac{\partial (U_j U_i + \overline{u_j' u_i'})}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2} \tag{10}$$

Equations (10) can be rearranged and written in their most recognizable form:

$$\frac{\partial U_i}{\partial t} + \frac{\partial U_j U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2} + \frac{1}{\rho}\frac{\partial \tau^R}{\partial x_j} \tag{11}$$

where $\tau^R = -\rho(\overline{u_j^{'} u_i^{'})}$

Using vector notation equations (9) and (10) can be written as:

$$\nabla.\overline{\mathbf{u}} = 0 \tag{12}$$

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \nabla.(\overline{\mathbf{u}}\overline{\mathbf{u}}) = -\frac{1}{\rho}\nabla\overline{p} + \nu\nabla^2\overline{\mathbf{u}} + \frac{1}{\rho}\nabla.\tau^R \tag{13}$$

2 Question 2

Using the Boussinesq hypothesis, derive the solvable incompressible Reynolds-Averaged Navier-Stokes(RANS) equation

Boussinesq hypothesis:

$$\tau^R = 2\mu_T D^R - \frac{2}{3}\rho\kappa\mathbf{I} = \mu_T[\nabla\overline{\mathbf{u}} + (\nabla\overline{\mathbf{u}})^T] - \frac{2}{3}\rho\kappa\mathbf{I} \tag{14}$$

Substituting equation (14) to (13) after expressing the viscous term in equation (13) by the viscous tensor expression indicated by equation (15), we get equation (**??**

$$\nu\nabla^2\overline{\mathbf{u}} = \frac{1}{\rho}\nabla.(2\mu D^R) = \frac{1}{\rho}\nabla.[\nabla\overline{\mathbf{u}} + (\nabla\overline{\mathbf{u}})^T] = \frac{1}{\rho}\nabla.(\nabla\overline{\mathbf{u}}) \tag{15}$$

The transpose of the velocity gradient will vanish in equation (15) due to symmetry of second derivatives and considering continuity equation. This is shown below using Einstein notations. (**N.B: This answers Question 3**)

$$\nabla.(\nabla\overline{\mathbf{u}}^T) = \frac{\partial}{\partial x_j}(\frac{\partial U_j}{\partial x_i}) = \frac{\partial}{\partial x_i}(\frac{\partial U_j}{\partial x_j}) = 0 \tag{16}$$

$$\frac{\partial\overline{\mathbf{u}}}{\partial t} + \nabla.(\overline{\mathbf{u}}\overline{\mathbf{u}}) = -\frac{1}{\rho}\nabla\overline{p} + \frac{1}{\rho}\nabla.(\nabla\overline{\mathbf{u}}) + \frac{1}{\rho}\nabla.(\mu_T[\nabla\overline{\mathbf{u}} + (\nabla\overline{\mathbf{u}})^T] - \frac{2}{3}\rho\kappa\mathbf{I}) \tag{17}$$

Canceling transpose of the velocity gradient in the turbulence stress term and rearranging equation (17) we can write the solvable RANS equation as follows

$$\frac{\partial\overline{\mathbf{u}}}{\partial t} + \nabla.(\overline{\mathbf{u}}\overline{\mathbf{u}}) = -\frac{1}{\rho}\left(\nabla\overline{p} + \frac{2}{3}\rho\kappa\right) + \nabla.\left[\frac{1}{\rho}(\mu + \mu_T)\nabla\overline{\mathbf{u}}\right] \tag{18}$$

The sum of the molecular viscosity and turbulence viscosity shown in equation(18) is collectively called as **effective viscosity** μ_{eff}.

The major reference material used was the book by Wilcox et al. (1998).

References

Wilcox, D. C., et al. (1998). *Turbulence modeling for cfd* (Vol. 2). DCW industries La Canada, CA.