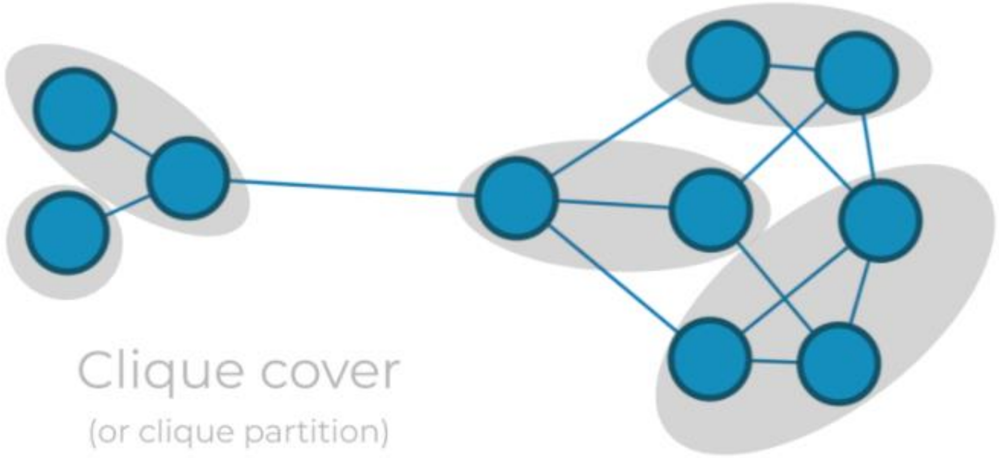
Clique Cover

What is Clique?

A clique of a graph *G* is a set *X* of vertices of *G* with the property that every pair of distinct vertices in *X* are adjacent in *G*.

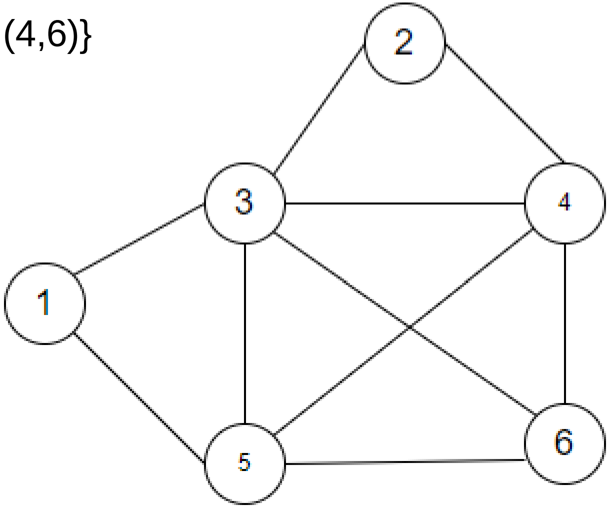


Clique Cover

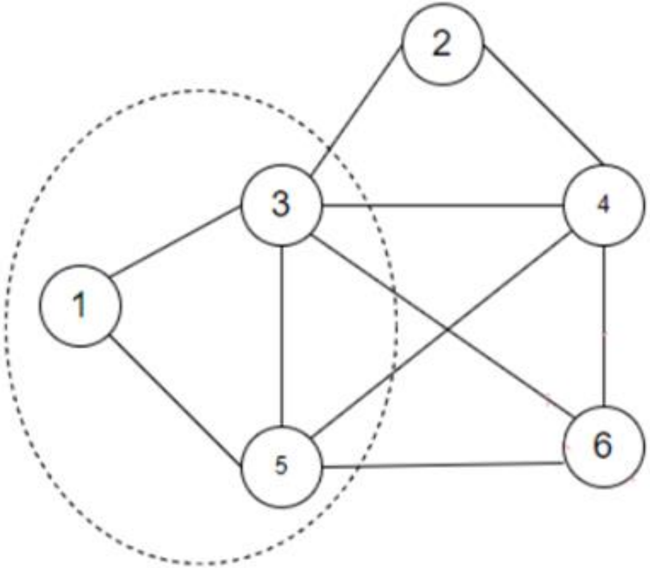
In graph theory, a **clique cover** or **partition into cliques** of a given undirected graph is a partition of the vertices of the graph into cliques, subsets of vertices within which every two vertices are adjacent. A **minimum clique cover** is a clique cover that uses as few cliques as possible. The minimum *k* for which a clique cover exists is called the **clique cover number** of the given graph.

Example

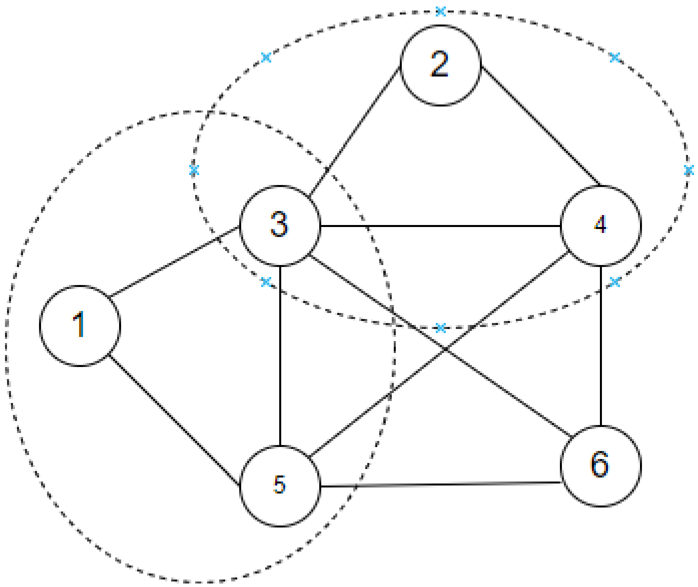
Let's Assume an undirected connected graph Graph G = (V,E)
V = {1,2,3,4,5,6}
E = {(1,3),(1,5),(3,5),(3,2),(2,4),(3,4),(4,5),(3,6),(5,6),(4,6)}



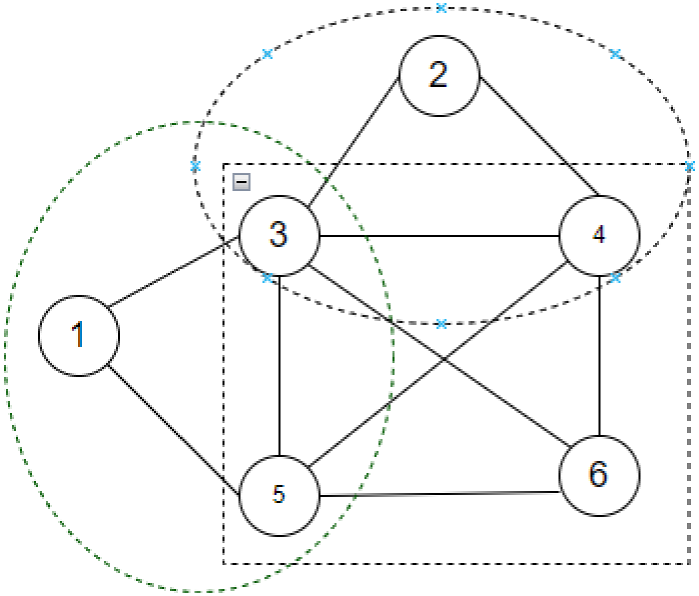
Example



Example

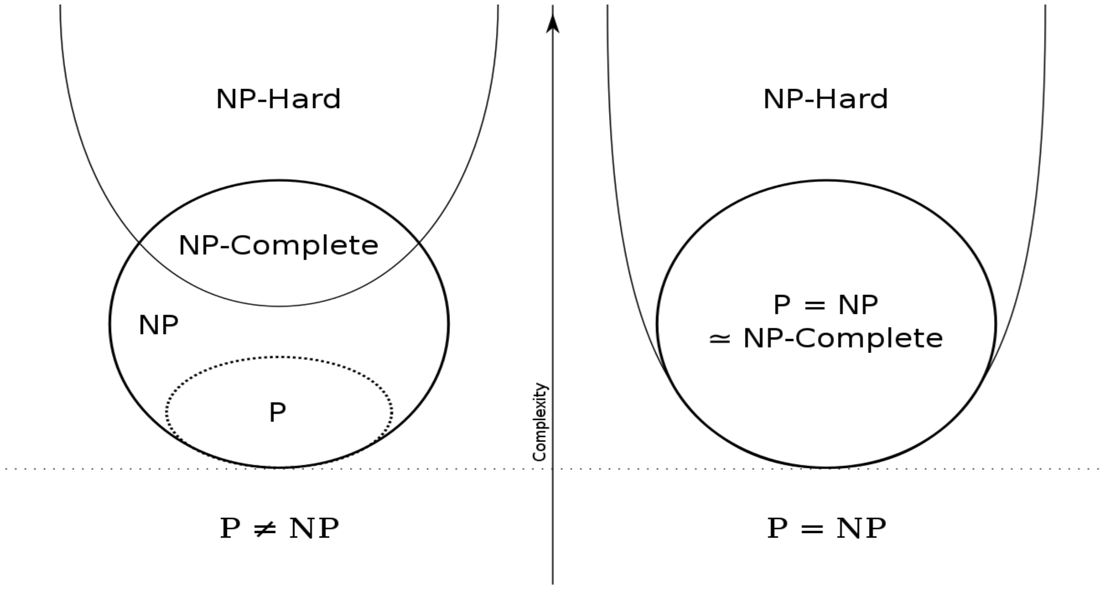


Example



Clique Cover problem is NP- Hard

- Firstly, We are going to prove that this problem is NP-Complete.
  As NP-complete is a subset of NP & NP-Hard. So proving the problem to be NP-Complete sufficient.



Clique Cover problem is NP- Complete

Given a new problem X, here is the basic strategy for proving it is NP-Complete.

1. Prove that X ∈ NP.

2. Choose a problem Y that is known to be NP-complete.

3. Prove that $Y \leq_P X$.

Clique Cover problem is in NP

- It is easy to see why the problem is in NP. Given graph G and integer k, one certificate that the answer is yes is one can verify in polynomial time that at most k cliques can cover the whole graph.

Reduction from k-coloring problem

- **If a graph is k-colorable then the graph has k-independent set.**
- Normal reduction between Independent Set and Clique by taking the complement graph so any independent set becomes a clique.

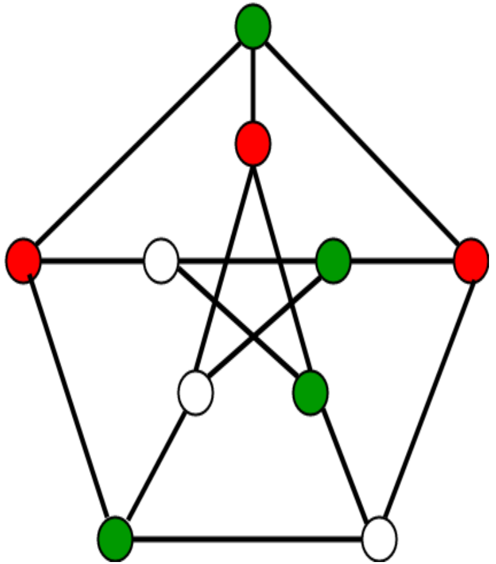
Relation to coloring

A clique cover of a graph *G* may be seen as a graph coloring of the complement graph of *G*, the graph on the same vertex set that has edges between non-adjacent vertices of *G*. Like clique covers, graph colorings are partitions of the set of vertices, but into subsets with no adjacencies (independent sets) rather than cliques. A subset of vertices is a clique in *G* if and only if it is an independent set in the complement of *G*, so a partition of the vertices of *G* is a clique cover of *G* if and only if it is a coloring of the complement of *G*.

Lemma

If a graph is k-colourable, then it can be partitioned into k independent sets.

- A coloring using at most *k* colors is called a ***k*-coloring**.
- A subset of vertices assigned to the same color is called a ***color class***, every such class forms an independent set. Thus, a *k*-coloring is the same as a partition of the vertex set into *k* independent sets, and the terms *k-partite* and *k-colorable* have the same meaning.



Reduction from k-coloring problem

- If a graph is k-colorable then the graph has k-independent set
- **Normal reduction between Independent Set and Clique by taking the complement graph so any independent set becomes a clique.**

Reduction Between Independent set & Clique

Clique Problem:

- For a given Graph G = (V,E) and integer k, the clique problem is to find whether G contains a clique of size >= k

Independent Set Problem:

- For a given graph G’ = (V’,E’) and integer k’ the independent set problem is to find whether G’ contains an independent set of size >= k’

Reduction Between Independent set & Clique

To Reduce an Independent set problem to a Clique problem for a given graph G = (V,E) , construct a complementary graph G'=(V',E') such that

1. V = V' that is the complement graph will have the same vertices as the original graph
2. E' is the complement of E that is G' has all the edges that is not present in G

Reduction Between Independent set & Clique

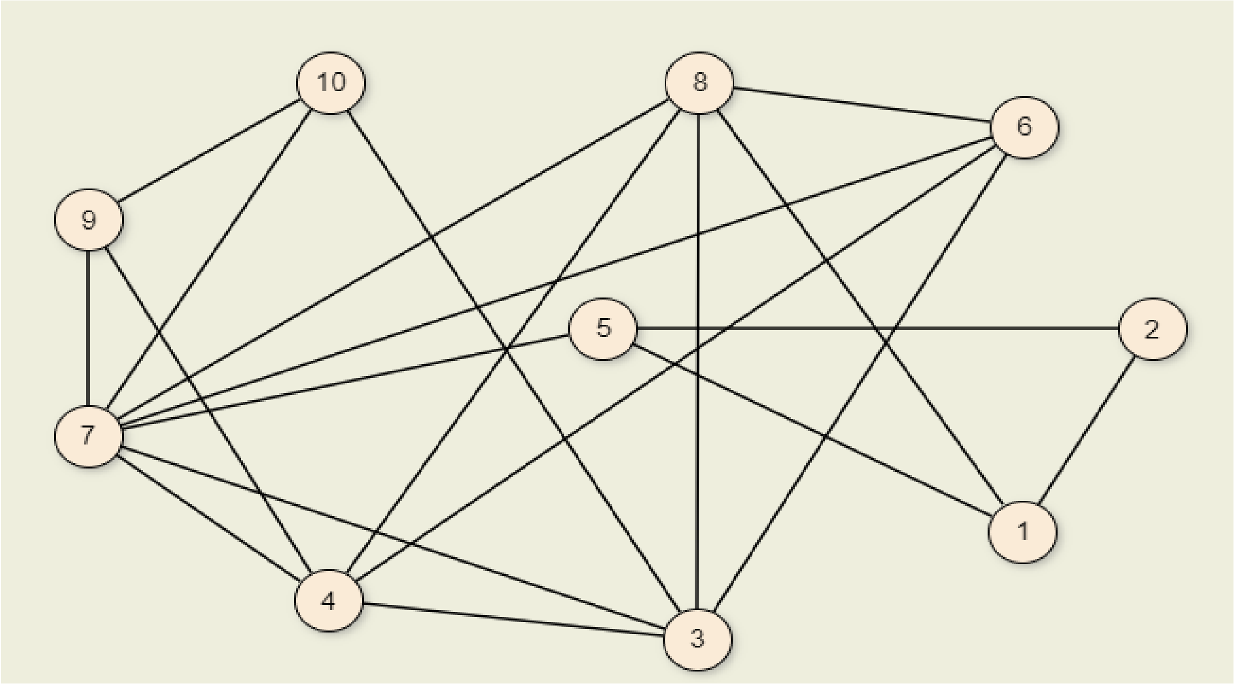


Fig: Graph **G**

Reduction Between Independent set & Clique

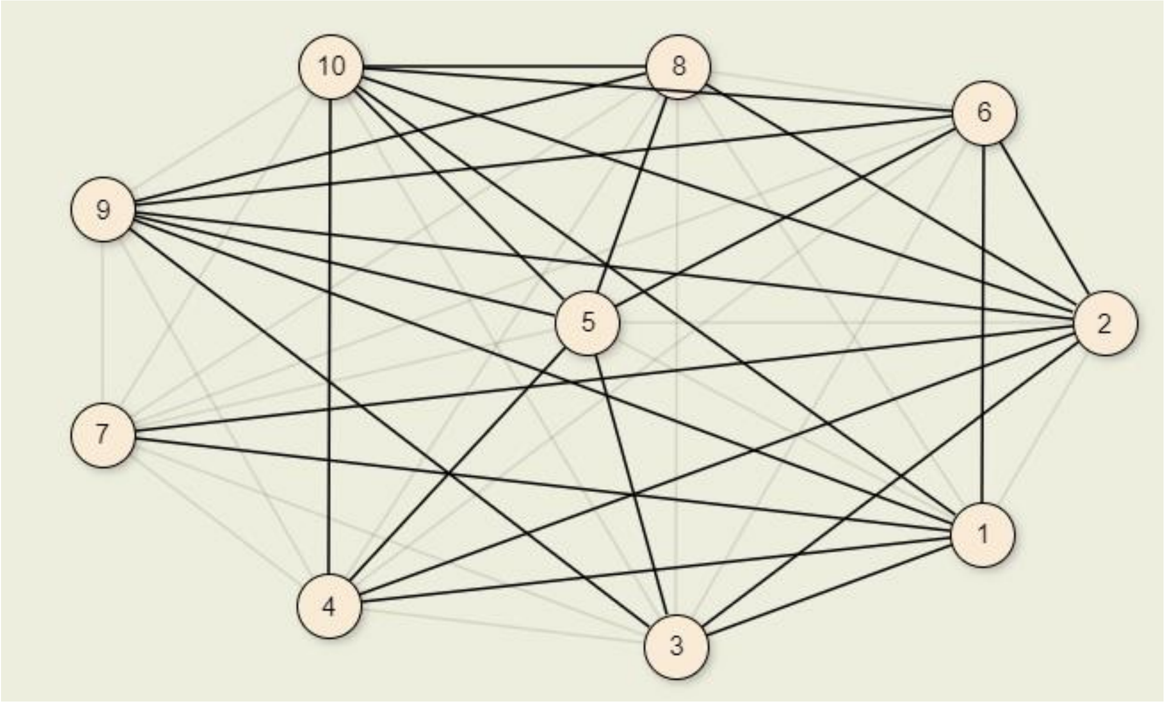


Fig: Graph G’

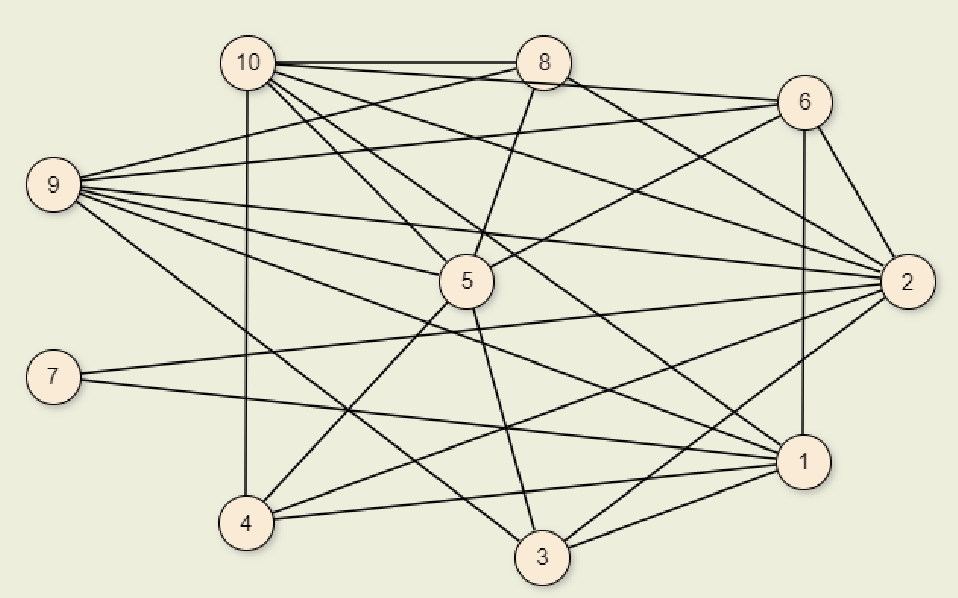
Reduction Between Independent set & Clique

1. **If there is an independent set of size k in the complement graph G’**, it implies no two vertices share an edge in G’ which further implies all of these vertices share an edge with all others in G forming a clique, that is **there exists a clique of size k in G.**
2. **If there is a clique of size k in the graph G** , it implies all vertices share an edge with all others in G which further implies no two of those vertices share an edge in G’ forming an independent set, that is **there exists an independent set of size k in G’.**

Reduction Between Independent set & Clique

Does G’ below have an independent set of size 8?

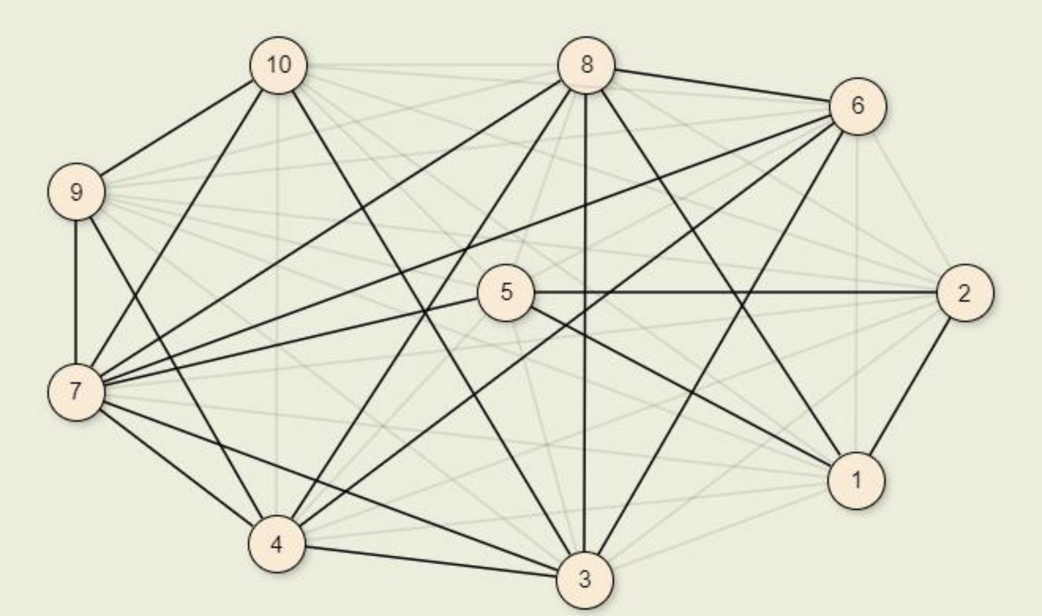
No



Reduction Between Independent set & Clique

Does G below have a clique of size 8?

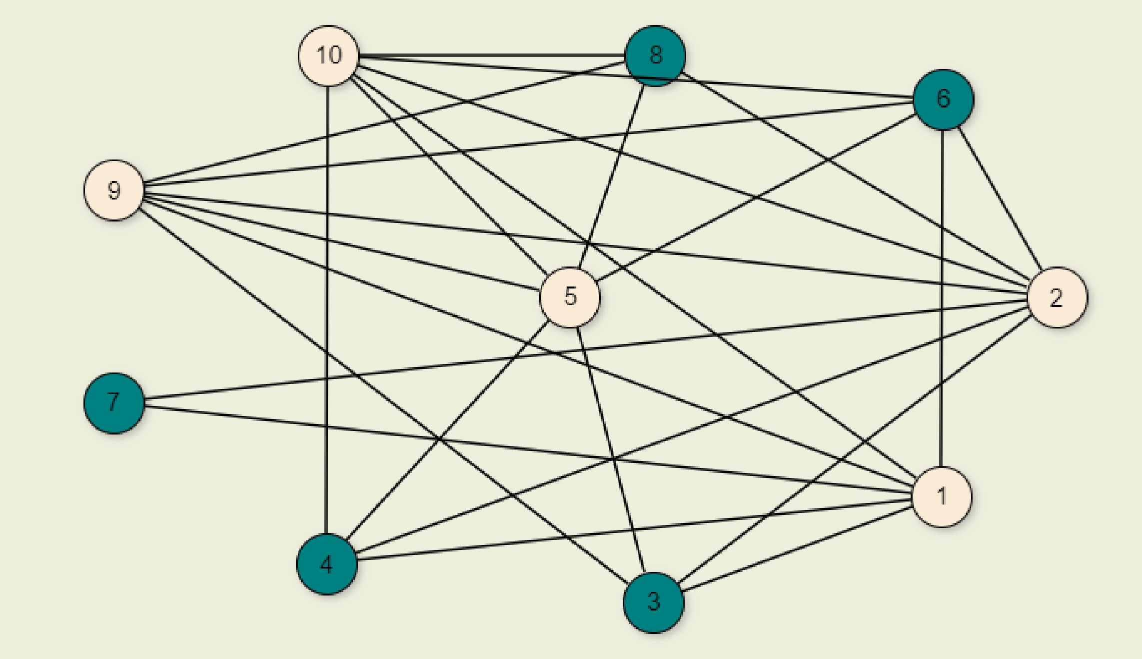
No



Reduction Between Independent set & Clique

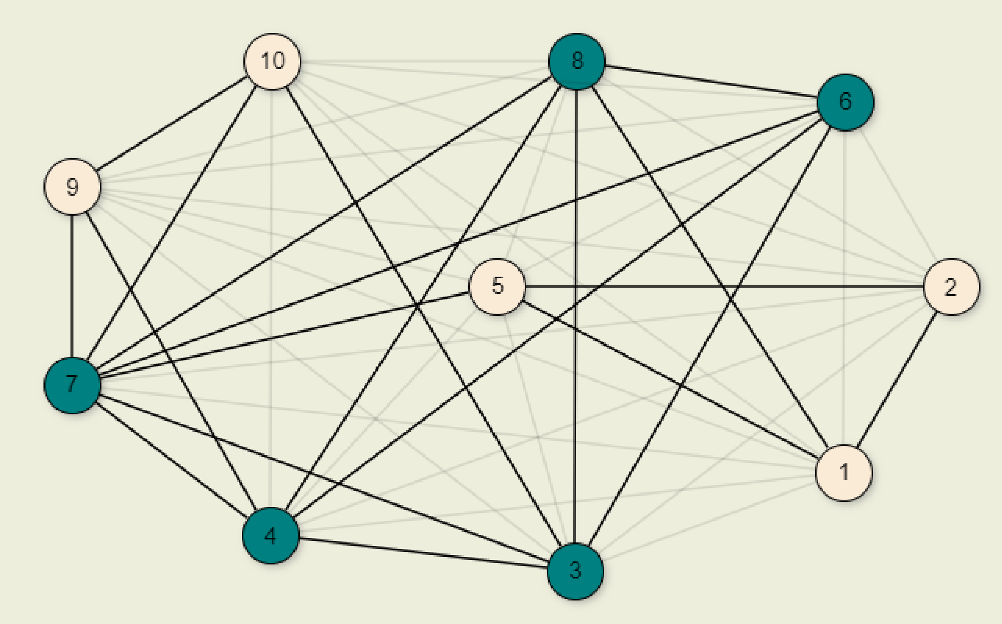
Does G’ below have an independent set of size 5?

YES



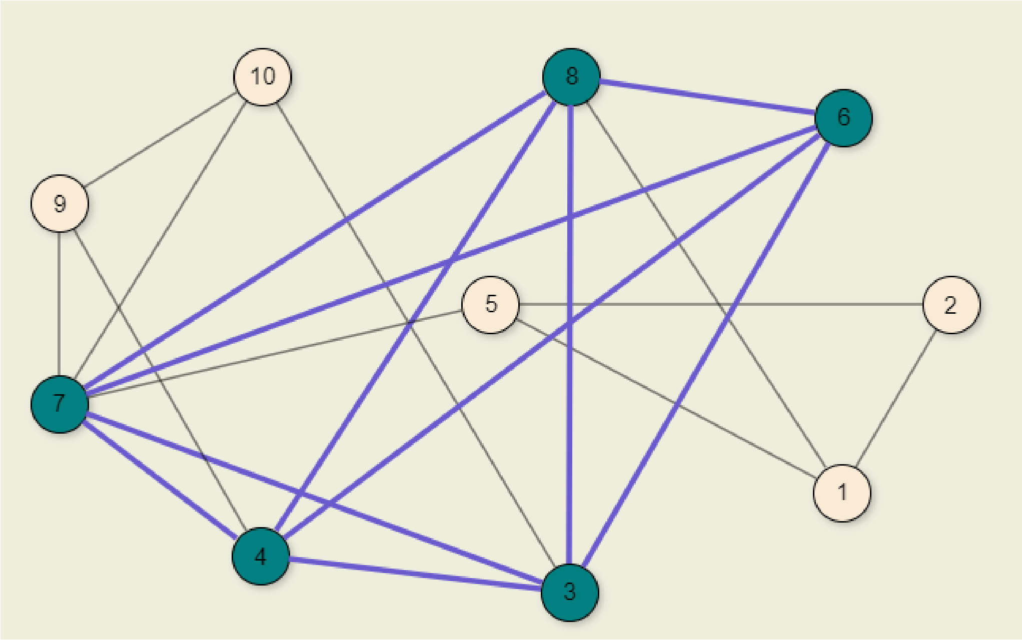
Reduction Between Independent set & Clique

Does G below have a clique of size 5?



Reduction Between Independent set & Clique

YES, It forms a clique of size 5 in G



Reduction Between Independent set & Clique

- So, we have proved that if there is an independent set of size k in the complement graph G’, there exists a clique of size k in G
- In simple terms, any independent set becomes a clique in the complement graph.

Reduction from k-coloring problem

- If G is k-colourable, it can be partitioned into k independent sets.Hence G’ can be partitioned into or covered by k cliques.
- Conversely if G’ can be covered by k cliques G has a partition into k independent sets and hence is k-colourable.
- Hence k-colouring problem which is a NP-Complete problem has been reduced to clique cover problem. So we can say that clique cover is also a NP-Complete problem.
- As NP-Complete is a subset of NP-Hard, we can say that clique cover problem is a NP-Hard problem.