1. Problem set 1: C^2-Cubic B-spline Interpolation

   (a) Write the algorithm in Epperson, 2013, Section 2.6, for solving an $n \times n$ tridiagonal matrix. Note that the matrix is stored using three vectors. [10 marks]

   (b) Let $f : [a, b] \to \mathbb{R}$ be sampled at $n+1 \geq 4$ equally spaced points $a = x_0 < x_1 < \ldots < x_n = b$. Write a function that will take as input the function values $\{f(x_i)\}_{i=0}^{n}$ and return the coefficients $\{c_i\}_{i=-1}^{n+1}$ of the *natural* C^2-Cubic B-spline $q_3^n(x)$ that interpolates f. Use your code from the previous question to solve the tridiagonal system. [10 marks]

   (c) Write a function that will take as input the coefficients $\{c_i\}_{i=-1}^{n+1}$, and the nodes $\{x_i\}_{i=0}^{n}$ and output the values of the spline q_3^n evaluated at $20n + 1$ uniformly spaced points $\{\hat{x}_j\}_{j=0}^{20n}$ on the interval $[a, b]$. Your code should also output the evaluation points $\{q_3^n(\hat{x}_j)\}_{j=0}^{20n}$. [10 marks]

   (d) Test your code, from the previous question, for $f(x) = e^{-x}\cos(6\pi x)$ on the interval $[-1, 1]$ for $n = 16, 32, 64, 128$. For each n:

      i Provide a plot of the spline $q_3^n(x)$.

      ii Compute an approximation to the error $\|f - q_3^n\|_{\infty,[-1,1]}$.

   Plot $\log_{10}(\|f - q_3^n\|_{\infty,[-1,1]})$ against $\log_{10}(n)$. Comment on these results. [15 marks]

2. Problem set 2: LSQ approximation + Gaussian quadrature

   (a) Let $f : [-1, 1] \to \mathbb{R}$ be a function to be approximated using LSQ with Legendre basis functions. Write a function to integrate a function using a Gauss–Legendre quadrature with $n = 10$ on the interval $[-1, 1]$. You can either use a built-in function or take the table for $n = 10$ in this link. Integrate $f(x) = e^{-x^2}$ from -1 to 1 using 10 quadrature points. Compare with the analytical solution

$$\int_{-1}^{1} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\,\mathrm{erf}(1). \tag{1}$$

   [5 marks]

   (b) Write a function that will compute the LSQ coefficients for a given f and n. Note that the Legendre polynomials may be obtained using the Bonnet's recurrence relation of Legendre polynomials:

$$n\,P_n(x) = (2n-1)x\,P_{n-1}(x) - (n-1)\,P_{n-2}(x) \tag{2}$$

   with $P_0(x) = 1$ and $P_1(x) = x$. [10 marks]

(c) Test your code for $f(x) = e^{-x}\cos(\pi x)$ on the interval $[-1, 1]$ for $n = 3, 5, 7$. For each n, provide a plot of the LSQ approximation. [15 marks]

3. Problem set 3: Nonlinearity

(a) Using finite differences, write a function to compute the Jacobian of a smooth vector-valued function $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$, given by

$$\boldsymbol{J_F}(\boldsymbol{x}) = \frac{\partial F_i}{\partial x_j}(\boldsymbol{x}) \approx \frac{F_i(\boldsymbol{x} + \epsilon\, \boldsymbol{e}_j) - F_i(\boldsymbol{x})}{\epsilon}, \tag{3}$$

where $\boldsymbol{e}_j$ is the canonical basis in the jth direction. [10 marks]

(b) Write a function to solve the following nonlinear system of equations using Newton's method with the Jacobian approximated with $\epsilon = 0.005, 0.05, 0.5$

$$F_1(x_1, x_2) = x_1^2 - x_2, \qquad F_2(x_1, x_2) = x_1^2 + x_2^2 + 2, \quad \text{with} \quad (x_1^{(0)}, x_2^{(0)}) = (2, 2). \tag{4}$$

print the output (solution) for the last iteration and comment on your results. [15 marks]

End