Modal logic

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Modal logic is a type of formal logic primarily developed in the 1960s that extends classical propositional and predicate logic to include operators expressing modality. A modal—a word that expresses a modality—qualifies a statement. For example, the statement "John is happy" might be qualified by saying that John is usually happy, in which case the term "usually" is functioning as a modal. The traditional alethic modalities, or modalities of truth, include possibility ("Possibly, *p*", "It is possible that *p*"), necessity ("Necessarily, *p*", "It is necessary that *p*"), and impossibility ("Impossibly, *p*", "It is impossible that *p*").[1] Other modalities that have been formalized in modal logic include temporal modalities, or modalities of time (notably, "It was the case that *p*", "It has always been that *p*", "It will be that *p*", "It will always be that *p*"),[2][3] deontic modalities (notably, "It is obligatory that *p*", and "It is permissible that *p*"), epistemic modalities, or modalities of knowledge ("It is known that *p*")[4] and doxastic modalities, or modalities of belief ("It is believed that *p*").[5]

A formal modal logic represents modalities using modal operators. For example, "It might rain today" and "It is possible that rain will fall today" both contain the notion of possibility. In a modal logic this is represented as an operator, "Possibly", attached to the sentence "It will rain today".

The basic unary (1-place) modal operators are usually written "□" for "Necessarily" and "◇" for "Possibly". In a classical modal logic, each can be expressed by the other with negation:

$$\Diamond P \leftrightarrow \lnot \Box \lnot P;$$
$$\Box P \leftrightarrow \lnot \Diamond \lnot P.$$

Thus it is *possible* that it will rain today if and only if it is *not necessary* that it will *not* rain today; and it is *necessary* that it will rain today if and only if it is *not possible* that it will *not* rain today. Alternative symbols used for the modal operators are "L" for "Necessarily" and "M" for "Possibly".[6]

Contents

Development of modal logic

In addition to his non-modal syllogistic, Aristotle also developed a modal syllogistic in Book I of his *Prior Analytics* (chs 8 – 22), which Theophrastus attempted to improve.[7] There are also passages in Aristotle's work, such as the famous sea-battle argument in *De Interpretatione* § 9, that are now seen as anticipations of the connection of modal logic with potentiality and time. In the Hellenistic period, the logicians Diodorus Cronus, Philo the Dialectician and the Stoic Chrysippus each developed a modal system that accounted for the interdefinability of possibility and necessity, accepted axiom **T** (see below), and combined elements of modal logic and temporal logic in attempts to solve the notorious Master Argument.[8] The earliest formal system of modal logic was developed by Avicenna, who ultimately developed a theory of "temporally modal" syllogistic.[9] Modal logic as a self-aware subject owes much to the writings of the Scholastics, in particular William of Ockham and John Duns Scotus, who reasoned informally in a modal manner, mainly to analyze statements about essence and accident.

C. I. Lewis founded modern modal logic in his 1910 Harvard thesis[10] and in a series of scholarly articles beginning in 1912. This work culminated in his 1932 book *Symbolic Logic* (with C. H. Langford),[11] which introduced the five systems *S1* through *S5*.

Ruth C. Barcan (later Ruth Barcan Marcus) developed the first axiomatic systems of quantified modal logic — first and second order extensions of Lewis' *S2*, *S4*, and *S5*.[12][13][14]

The contemporary era in modal semantics began in 1959, when Saul Kripke (then only a 19-year-old Harvard University undergraduate) introduced the now-standard Kripke semantics for modal logics. These are commonly referred to as "possible worlds" semantics. Kripke and A. N. Prior had previously corresponded at some length. Kripke semantics is basically simple, but proofs are eased using semantic-tableaux or analytic tableaux, as explained by E. W. Beth.

A. N. Prior created modern temporal logic, closely related to modal logic, in 1957 by adding modal operators [F] and [P] meaning "eventually" and "previously". Vaughan Pratt introduced dynamic logic in 1976. In 1977, Amir Pnueli proposed using temporal logic to formalise the behaviour of continually operating concurrent programs. Flavors of temporal logic include propositional dynamic logic (PDL), propositional linear temporal logic (PLTL), linear temporal logic (LTL), computational tree logic (CTL), Hennessy – Milner logic, and *T*.

The mathematical structure of modal logic, namely Boolean algebras augmented with unary operations (often called modal algebras), began to emerge with J. C. C. McKinsey's 1941 proof that *S2* and *S4* are decidable,[15] and reached full flower in the work of Alfred Tarski and his student Bjarni Jónsson (Jónsson and Tarski 1951 – 52). This work revealed that *S4* and *S5* are models of interior algebra, a proper extension of Boolean algebra originally designed to capture the properties of the interior and closure operators of topology. Texts on modal logic typically do little more than mention its connections with the study of Boolean algebras and topology. For a thorough survey of the history of formal modal logic and of the associated mathematics, see Robert Goldblatt (2006).[16]

Semantics

Model theory

The semantics for modal logic are usually given as follows:[17] First we define a *frame*, which consists of a non-empty set, *G*, whose members are generally called possible worlds, and a binary relation, *R*, that holds (or not) between the possible worlds of *G*. This binary relation is called the *accessibility relation*. For example, *w R u* means that the world *u* is accessible from world *w*. That is to say, the state of affairs known as *u* is a live possibility for *w*. This gives a pair, $\langle G, R\rangle$. Some formulations of modal logic also include a constant term in *G*, conventionally called "the actual world", which is often symbolized as $w*$.

Next, the *frame* is extended to a *model* by specifying the truth-values of all propositions at each of the worlds in *G*. We do so by defining a relation *v* between possible worlds and positive literals. If there is a world *w* such that $v(w, P)$, then *P* is true at *w*. A model is thus an ordered triple, $\langle G, R, v\rangle$.

Then we recursively define the truth of a formula at a world in a model:

- if $v(w, P)$ then $w \models P$
- $w \models \neg P$ if and only if $w \not\models P$
- $w \models (P \wedge Q)$ if and only if $w \models P$ and $w \models Q$
- $w \models \Box P$ if and only if for every element *u* of *G*, if *w R u* then $u \models P$
- $w \models \Diamond P$ if and only if for some element *u* of *G*, it holds that *w R u* and $u \models P$
- $\models P$ if and only if $w* \models P$

According to these semantics, a truth is *necessary* with respect to a possible world *w* if it is true at every world that is accessible to *w*, and *possible* if it is true at some world that is accessible to *w*. Possibility thereby depends upon the accessibility relation *R*, which allows us to express the relative nature of possibility. For example, we might say that given our laws of physics it is not possible for humans to travel faster than the speed of light, but that given other circumstances it could have been possible to do so. Using the accessibility relation we can translate this scenario as follows: At all of the worlds accessible to our own world, it is not the case that humans can travel faster than the speed of light, but at one of these accessible worlds there is *another* world accessible from *those* worlds but not accessible from our own at which humans can travel faster than the speed of light.

It should also be noted that the definition of □ makes vacuously true certain sentences, since when it speaks of "every world that is accessible to *w*" it takes for granted the usual mathematical interpretation of the word "every" (see vacuous truth). Hence, if a world *w* doesn't have any accessible worlds, any sentence beginning with □ is true.

The different systems of modal logic are distinguished by the properties of their corresponding accessibility relations. There are several systems that have been espoused (often called *frame conditions*). An accessibility relation is:

- **reflexive** iff *w R w*, for every *w* in *G*
- **symmetric** iff *w R u* implies *u R w*, for all *w* and *u* in *G*
- **transitive** iff *w R u* and *u R q* together imply *w R q*, for all *w*, *u*, *q* in *G*.
- **serial** iff, for each *w* in *G* there is some *u* in *G* such that *w R u*.
- **Euclidean** iff, for every *u*, *t*, and *w*, *w R u* and *w R t* implies *u R t* (note that it also implies: *t R u*)

The logics that stem from these frame conditions are:

- K := no conditions
- D := serial
- T := reflexive
- S4 := reflexive and transitive
- S5 := reflexive and Euclidean

The Euclidean property along with reflexivity yields symmetry and transitivity. (The Euclidean property can be obtained, as well, from symmetry and transitivity.) Hence if the accessibility relation *R* is reflexive and Euclidean, *R* is provably symmetric and transitive as well. Hence for models of S5, *R* is an equivalence relation, because *R* is reflexive, symmetric and transitive.

We can prove that these frames produce the same set of valid sentences as do the frames where all worlds can see all other worlds of *W* (*i.e.*, where *R* is a "total" relation). This gives the corresponding *modal graph* which is total complete (*i.e.*, no more edges (relations) can be added). For example, in any modal logic based on frame conditions:

$w \models \Diamond P$ if and only if for some element *u* of *G*, it holds that $u \models P$ and *w R u*.

If we consider frames based on the total relation we can just say that

$w \models \Diamond P$ if and only if for some element *u* of *G*, it holds that $u \models P$.

We can drop the accessibility clause from the latter stipulation because in such total frames it is trivially true of all *w* and *u* that *w R u*. But note that this does not have to be the case in all S5 frames, which can still consist of multiple parts that are fully connected among themselves but still disconnected from each other.

All of these logical systems can also be defined axiomatically, as is shown in the next section. For example, in S5, the axioms $P \implies \Box\Diamond P$, $\Box P \implies \Box\Box P$ and $\Box P \implies P$ (corresponding to *symmetry*, *transitivity* and *reflexivity*, respectively) hold, whereas at least one of these axioms does not hold in each of the other, weaker logics.

Axiomatic systems

The first formalizations of modal logic were axiomatic. Numerous variations with very different properties have been proposed since C. I. Lewis began working in the area in 1910. Hughes and Cresswell (1996), for example, describe 42 normal and 25 non-normal modal logics. Zeman (1973) describes some systems Hughes and Cresswell omit.

Modern treatments of modal logic begin by augmenting the propositional calculus with two unary operations, one denoting "necessity" and the other "possibility". The notation of C. I. Lewis, much employed since, denotes "necessarily *p*" by a prefixed "box" ($\Box p$) whose scope is established by parentheses. Likewise, a prefixed "diamond" ($\Diamond p$) denotes "possibly *p*". Regardless of notation, each of these operators is definable in terms of the other in classical modal logic:

- $\Box p$ (necessarily *p*) is equivalent to $\neg\Diamond\neg p$ ("not possible that not-*p*")
- $\Diamond p$ (possibly *p*) is equivalent to $\neg\Box\neg p$ ("not necessarily not-*p*")

Hence $\Box$ and $\Diamond$ form a dual pair of operators.

In many modal logics, the necessity and possibility operators satisfy the following analogues of de Morgan's laws from Boolean algebra:

> "It is **not necessary that** *X*" is logically equivalent to "It is **possible that not** *X*".
> "It is **not possible that** *X*" is logically equivalent to "It is **necessary that not** *X*".

Precisely what axioms and rules must be added to the propositional calculus to create a usable system of modal logic is a matter of philosophical opinion, often driven by the theorems one wishes to prove; or, in computer science, it is a matter of what sort of computational or deductive system one wishes to model. Many modal logics, known collectively as normal modal logics, include the following rule and axiom:

- N, **Necessitation Rule:** If *p* is a theorem (of any system invoking N), then $\Box p$ is likewise a theorem.
- K, **Distribution Axiom:** $\Box(p \to q) \to (\Box p \to \Box q)$.

The weakest normal modal logic, named *K* in honor of Saul Kripke, is simply the propositional calculus augmented by $\Box$, the rule N, and the axiom K. *K* is weak in that it fails to determine whether a proposition can be necessary but only contingently necessary. That is, it is not a theorem of *K* that if $\Box p$ is true then $\Box\Box p$ is true, i.e., that necessary truths are "necessarily necessary". If such perplexities are deemed forced and artificial, this defect of *K* is not a great one. In any case, different answers to such questions yield different systems of modal logic.

Adding axioms to *K* gives rise to other well-known modal systems. One cannot prove in *K* that if "*p* is necessary" then *p* is true. The axiom **T** remedies this defect:

- T, **Reflexivity Axiom:** $\Box p \to p$ (If *p* is necessary, then *p* is the case.)

T holds in most but not all modal logics. Zeman (1973) describes a few exceptions, such as $S1^0$.

Other well-known elementary axioms are:

- 4: $\Box p \to \Box\Box p$
- B: $p \to \Box\Diamond p$
- D: $\Box p \to \Diamond p$
- 5: $\Diamond p \to \Box\Diamond p$

These yield the systems (axioms in bold, systems in italics):

- *K* := **K** + **N**
- *T* := *K* + **T**

- *S4* := *T* + 4
- *S5* := *S4* + 5
- *D* := *K* + D.

K through *S5* form a nested hierarchy of systems, making up the core of normal modal logic. But specific rules or sets of rules may be appropriate for specific systems. For example, in deontic logic, $\Box p \to \Diamond p$ (If it ought to be that *p*, then it is permitted that *p*) seems appropriate, but we should probably not include that $p \to \Box\Diamond p$. In fact, to do so is to commit the naturalistic fallacy (i.e. to state that what is natural is also good, by saying that if *p* is the case, *p* ought to be permitted).

The commonly employed system *S5* simply makes all modal truths necessary. For example, if *p* is possible, then it is "necessary" that *p* is possible. Also, if *p* is necessary, then it is necessary that *p* is necessary. Other systems of modal logic have been formulated, in part because *S5* does not describe every kind of modality of interest.

Structural proof theory

Sequent calculi and systems of natural deduction have been developed for several modal logics, but it has proven hard to combine generality with other features expected of good structural proof theories, such as purity (the proof theory does not introduce extra-logical notions such as labels) and analyticity (the logical rules support a clean notion of analytic proof). More complex calculi have been applied to modal logic to achieve generality.

Decision methods

Analytic tableaux provide the most popular decision method for modal logics.

Alethic logic

Modalities of necessity and possibility are called *alethic* modalities. They are also sometimes called *special* modalities, from the Latin *species*. Modal logic was first developed to deal with these concepts, and only afterward was extended to others. For this reason, or perhaps for their familiarity and simplicity, necessity and possibility are often casually treated as *the* subject matter of modal logic. Moreover, it is easier to make sense of relativizing necessity, e.g. to legal, physical, nomological, epistemic, and so on, than it is to make sense of relativizing other notions.

In classical modal logic, a proposition is said to be

- **possible** if and only if it is *not necessarily false* (regardless of whether it is actually true or actually false);
- **necessary** if and only if it is *not possibly false*; and
- **contingent** if and only if it is *not necessarily false* and *not necessarily true* (i.e. possible but not necessarily true);
- **impossible** if and only if it is *not possibly true* (i.e. false and necessarily false).

In classical modal logic, therefore, either the notion of possibility or necessity may be taken to be basic, where these other notions are defined in terms of it in the manner of De Morgan duality. Intuitionistic modal logic treats possibility and necessity as not perfectly symmetric.

For those with difficulty with the concept of something being possible but not true, the meaning of these terms may be made more comprehensible by thinking of multiple "possible worlds" (in the sense of Leibniz) or "alternate universes"; something "necessary" is true in all possible worlds, something "possible" is true in at least one possible world. These "possible world semantics" are formalized with Kripke semantics.

Alternatively think of configurations of objects and materials that could have been differently arranged. E.g. the tea is on the top shelf and the coffee is on a lower shelf, but it might have been the case that both were on the lower shelf, or both on the top shelf, or the tea on the lower shelf and the coffee on the top shelf. Moreover, there might have been nothing on the top shelf (at that time), or there might have been nothing but a mouse there. Anyone who considers rearranging contents of a cupboard, or furniture of a room, or possible items to serve for a meal is using this concept of alethic possibility. If you try to draw a planar convex closed figure with four sides and exactly three corners you will find that is impossible: an example of alethic impossibility. Such a figure with four sides necessarily has four corners: that's an example of alethic necessity. Ordinary uses of these concepts do not seem to require the ability to think about collections of complete possible universes, merely rearrangements of portions of this universe.

Physical possibility

Something is physically, or nomically, possible if it is permitted by the laws of physics. For example, current theory is thought to allow for there to be an atom with an atomic number of 126,[18] even if there are no such atoms in existence. In contrast, while it is logically possible (i.e. probably via Alcubierre drive or worm holes) to accelerate beyond the speed of light,[19] modern science stipulates that it is not physically possible for material particles or information.[20]

Metaphysical possibility

Philosophers ponder the properties that objects have independently of those dictated by scientific laws. For example, it might be metaphysically necessary, as some who advocate physicalism have thought, that all thinking beings have bodies[21] and can experience the passage of time. Saul Kripke has argued that every person necessarily has the parents they do have: anyone with different parents would not be the same person.[22]

Metaphysical possibility has been thought to be more restricting than bare logical possibility[23] (i.e., fewer things are metaphysically possible than are logically possible). However, its exact relation (if any) to logical possibility or to physical possibility is a matter of dispute. Philosophers also disagree over whether metaphysical truths are necessary merely "by definition", or whether they reflect some underlying deep facts about the world, or something else entirely.

Confusion with epistemic modalities

Alethic modalities and epistemic modalities (see below) are often expressed in English using the same words. "It is possible that bigfoot exists" can mean either "Bigfoot *could* exist, whether or not bigfoot does in fact exist" (alethic), or more likely, "For all I know, bigfoot exists" (epistemic).

It has been questioned whether these modalities should be considered distinct from each other. The criticism states that there is no real difference between "the truth in the world" (alethic) and "the truth in an individual's mind" (epistemic).[24] An investigation has not found a single language in which alethic and epistemic modalities are formally distinguished, as by the means of a grammatical mood.[25]

Epistemic logic

Epistemic modalities (from the Greek *episteme*, knowledge), deal with the *certainty* of sentences. The $\Box$ operator is translated as "x knows that⋯", and the $\Diamond$ operator is translated as "For all x knows, it may be true that⋯" In ordinary speech both metaphysical and epistemic modalities are often expressed in similar words; the following contrasts may help:

A person, Jones, might reasonably say *both*: (1) "No, it is *not* possible that Bigfoot exists; I am quite certain of that"; *and*, (2) "Sure, Bigfoot possibly *could* exist". What Jones means by (1) is that given all the available information, there is no question remaining as to whether Bigfoot exists. This is an epistemic claim. By (2) he makes the *metaphysical* claim that it is *possible for* Bigfoot to exist, *even though he does not* (which is not equivalent to "it is *possible that* Bigfoot exists − for all I know", which contradicts (1)).

From the other direction, Jones might say, (3) "It is *possible* that Goldbach's conjecture is true; but also *possible* that it is false", and *also* (4) "if it *is* true, then it is necessarily true, and not possibly false". Here Jones means that it is *epistemically possible* that it is true or false, for all he knows (Goldbach's conjecture has not been proven either true or false), but if there *is* a proof (heretofore undiscovered), then it would show that it is not *logically* possible for Goldbach's conjecture to be false—there could be no set of numbers that violated it. Logical possibility is a form of *alethic* possibility; (4) makes a claim about whether it is possible (i.e., logically speaking) that a mathematical truth to have been false, but (3) only makes a claim about whether it is possible, for all Jones knows, (i.e., speaking of certitude) that the mathematical claim is specifically either true or false, and so again Jones does not contradict himself. It is worthwhile to observe that Jones is not necessarily correct: It is possible (epistemically) that Goldbach's conjecture is both true and unprovable.[26]

Epistemic possibilities also bear on the actual world in a way that metaphysical possibilities do not. Metaphysical possibilities bear on ways the world *might have been,* but epistemic possibilities bear on the way the world *may be* (for all we know). Suppose, for example, that I want to know whether or not to take an umbrella before I leave. If you tell me "it is *possible that* it is raining outside" − in the sense of epistemic possibility − then that would weigh on whether or not I take the umbrella. But if you just tell me that "it is *possible for* it to rain outside" − in the sense of *metaphysical possibility* − then I am no better off for this bit of modal enlightenment.

Some features of epistemic modal logic are in debate. For example, if *x* knows that *p*, does *x* know that it knows that *p*? That is to say, should $\Box P \rightarrow \Box\Box P$ be an axiom in these systems? While the answer to this question is unclear,[27] there is at least one axiom that is generally included in epistemic modal logic, because it is minimally true of all normal modal logics (see the section on axiomatic systems):

- K, *Distribution Axiom*: $\Box(p \to q) \to (\Box p \to \Box q)$.

Temporal logic

Temporal logic is an approach to the semantics of expressions with tense, that is, expressions with qualifications of when. Some expressions, such as '2 + 2 = 4', are true at all times, while tensed expressions such as 'John is happy' are only true sometimes.

In temporal logic, tense constructions are treated in terms of modalities, where a standard method for formalizing talk of time is to use *two* pairs of operators, one for the past and one for the future (P will just mean 'it is presently the case that P'). For example:

F*P* : It will sometimes be the case that *P*
G*P* : It will always be the case that *P*
P*P* : It was sometime the case that *P*
H*P* : It has always been the case that *P*

There are then at least three modal logics that we can develop. For example, we can stipulate that,

$\Diamond P$ = *P* is the case at some time *t*
$\Box P$ = *P* is the case at every time *t*

Or we can trade these operators to deal only with the future (or past). For example,

$\Diamond_1 P$ = F*P*
$\Box_1 P$ = G*P*

or,

$\Diamond_2 P$ = *P* and/or F*P*
$\Box_2 P$ = *P* and G*P*

The operators F and G may seem initially foreign, but they create normal modal systems. Note that F*P* is the same as ¬G¬*P*. We can combine the above operators to form complex statements. For example, P*P* → □P*P* says (effectively), *Everything that is past and true is necessary*.

It seems reasonable to say that possibly it will rain tomorrow, and possibly it won't; on the other hand, since we can't change the past, if it is true that it rained yesterday, it probably isn't true that it may not have rained yesterday. It seems the past is "fixed", or necessary, in a way the future is not. This is sometimes referred to as accidental necessity. But if the past is "fixed", and everything that is in the future will eventually be in the past, then it seems plausible to say that future events are necessary too.

Similarly, the problem of future contingents considers the semantics of assertions about the future: is either of the propositions 'There will be a sea battle tomorrow', or 'There will not be a sea battle tomorrow' now true? Considering this thesis led Aristotle to reject the principle of bivalence for assertions concerning the future.

Additional binary operators are also relevant to temporal logics, *q.v.* Linear Temporal Logic.

Versions of temporal logic can be used in computer science to model computer operations and prove theorems about them. In one version, ◇*P* means "at a future time in the computation it is possible that the computer state will be such that P is true"; □*P* means "at all future times in the computation P will be true". In another version, ◇*P* means "at the immediate next state of the computation, *P* might be true"; □*P* means "at the immediate next state of the computation, P will be true". These differ in the choice of Accessibility relation. (P always means "P is true at the current computer state".) These two examples involve nondeterministic or not-fully-understood computations; there are many other modal logics specialized to different types of program analysis. Each one naturally leads to slightly different axioms.

Deontic logic

Likewise talk of morality, or of obligation and norms generally, seems to have a modal structure. The difference between "You must do this" and "You may do this" looks a lot like the difference between "This is necessary" and "This is possible". Such logics are called *deontic*, from the Greek for "duty".

Deontic logics commonly lack the axiom **T** semantically corresponding to the reflexivity of the accessibility relation in Kripke semantics: in symbols, $\Box\phi \to \phi$. Interpreting $\Box$ as "it is obligatory that", **T** informally says that every obligation is true. For example, if it is obligatory not to kill others (i.e. killing is morally forbidden), then **T** implies that people actually do not kill others. The consequent is obviously false.

Instead, using Kripke semantics, we say that though our own world does not realize all obligations, the worlds accessible to it do (i.e., **T** holds at these worlds). These worlds are called idealized worlds. *P* is obligatory with respect to our own world if at all idealized worlds accessible to our world, *P* holds. Though this was one of the first interpretations of the formal semantics, it has recently come under criticism.[28]

One other principle that is often (at least traditionally) accepted as a deontic principle is *D*, $\Box\phi \to \Diamond\phi$, which corresponds to the seriality (or extendability or unboundedness) of the accessibility relation. It is an embodiment of the Kantian idea that "ought implies can". (Clearly the "can" can be interpreted in various senses, e.g. in a moral or alethic sense.)

Intuitive problems with deontic logic

When we try and formalize ethics with standard modal logic, we run into some problems. Suppose that we have a proposition *K*: you have stolen some money, and another, *Q*: you have stolen a small amount of money. Now suppose we want to express the thought that "if you have stolen some money, it ought to be a small amount of money". There are two likely candidates,

(1) $(K \to \Box Q)$
(2) $\Box(K \to Q)$

But (1) and *K* together entail $\Box Q$, which says that it ought to be the case that you have stolen a small amount of money. This surely isn't right, because you ought not to have stolen anything at all. And (2) doesn't work either: If the right representation of "if you have stolen some money it ought to be a small amount" is (2), then the right representation of (3) "if you have stolen some money then it ought to be a large amount" is $\Box(K \to (K \land \neg Q))$. Now suppose (as seems reasonable) that you ought not to steal anything, or $\Box\neg K$. But then we can deduce $\Box(K \to (K \land \neg Q))$ via $\Box(\neg K) \to \Box(K \to K \land \neg K)$ and $\Box(K \land \neg K \to (K \land \neg Q))$ (the contrapositive of $Q \to K$); so sentence (3) follows from our hypothesis (of course the same logic shows sentence (2)). But that can't be right, and is not right when we use natural language. Telling someone they should not steal certainly does not imply that they should steal large amounts of money if they do engage in theft.[29]

Doxastic logic

Doxastic logic concerns the logic of belief (of some set of agents). The term doxastic is derived from the ancient Greek *doxa* which means "belief". Typically, a doxastic logic uses $\Box$, often written "B", to mean "It is believed that", or when relativized to a particular agent s, "It is believed by s that".

Other modal logics

Significantly, modal logics can be developed to accommodate most of these idioms; it is the fact of their common logical structure (the use of "intensional" sentential operators) that make them all varieties of the same thing.

The ontology of possibility

In the most common interpretation of modal logic, one considers "logically possible worlds". If a statement is true in all possible worlds, then it is a necessary truth. If a statement happens to be true in our world, but is not true in all possible worlds, then it is a contingent truth. A statement that is true in some possible world (not necessarily our own) is called a possible truth.

Under this "possible worlds idiom," to maintain that Bigfoot's existence is possible but not actual, one says, "There is some possible world in which Bigfoot exists; but in the actual world, Bigfoot does not exist". However, it is unclear what this claim commits us to. Are we really alleging the existence of possible worlds, every bit as real as our actual world, just not actual? Saul Kripke believes that 'possible world' is something of a misnomer – that the term 'possible world' is just a useful way of visualizing the concept of possibility.[30] For him, the sentences "you could have rolled a 4 instead of a 6" and "there is a possible world where you rolled a 4, but you rolled a 6 in the actual world" are not significantly different statements, and neither commit us to the existence of a possible world.[31] David Lewis, on the other hand, made himself notorious by biting the bullet, asserting that all merely possible worlds are as real as our own, and that what distinguishes our world as *actual* is simply that it is indeed our world – *this* world.[32] That position is a major tenet of "modal realism". Some philosophers decline to endorse any version of modal realism, considering it ontologically extravagant, and prefer to seek various ways to paraphrase away these ontological commitments. Robert Adams holds that 'possible worlds' are better thought of as 'world-stories', or consistent sets of propositions. Thus, it is possible that you rolled a 4 if such a state of affairs can be described coherently.[33]

Computer scientists will generally pick a highly specific interpretation of the modal operators specialized to the particular sort of computation being analysed. In place of "all worlds", you may have "all possible next states of the computer", or "all possible future states of the computer".

Further applications

Modal logics have begun to be used in areas of the humanities such as literature, poetry, art and history.[34][35][36]

Controversies

Nicholas Rescher has argued that Bertrand Russell rejected modal logic, and that this rejection led to the theory of modal logic languishing for decades.[37] However, Jan Dejnozka has argued against this view, stating that a modal system which Dejnozka calls *MDL* is described in Russell's works, although Russell did believe the concept of modality to "come from confusing propositions with propositional functions," as he wrote in *The Analysis of Matter*.[38]

Arthur Norman Prior warned his protégé Ruth Barcan to prepare well in the debates concerning quantified modal logic with Willard Van Orman Quine, due to the biases against modal logic.[39]

See also

- Accessibility relation
- Counterpart theory
- David Kellogg Lewis
- De dicto and de re
- Description logic
- Doxastic logic
- Dynamic logic
- Enthymeme
- Hybrid logic
- Interior algebra
- Interpretability logic
- Kripke semantics
- Modal verb
- Multi-valued logic
- Possible worlds
- Provability logic
- Regular modal logic
- Relevance logic
- Research Materials: Max Planck Society Archive
- Rhetoric
- Strict conditional
- Two dimensionalism

Notes

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External links

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  - "Rudolf Carnap's Modal Logic (http://www.iep.utm.edu/cmlogic)" – by MJ Cresswell.
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- List of Logic Systems (http://www.cc.utah.edu/~nahaj/logic/structures/systems/index.html) List of many modal logics with sources, by John Halleck.
- Advances in Modal Logic. (http://aiml.net/) Biannual international conference and book series in modal logic.
- S4prover (http://teachinglogic.imag.fr/TableauxS4) A tableaux prover for S4 logic
- "Some Remarks on Logic and Topology (http://www.labri.fr/perso/moot/talks/TopologyNotes.pdf)" – by Richard Moot; exposits a topological semantics for the modal logic S4.
- LoTREC (http://www.irit.fr/Lotrec/) The most generic prover for modal logics from IRIT/Toulouse University

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