Discrete Linear-time Probabilistic Logics: Completeness, Decidability and Complexity

ZORAN OGNJANOVIĆ, *Matematički Institut, Kneza Mihaila 35, 11000 Beograd, Srbija i Crna Gora.*
Email: zorano@mi.sanu.ac.yu

Abstract

We introduce a propositional and a first-order logic for reasoning about discrete linear time and finitely additive probability. The languages of these logics allow formulae that say 'sometime in the future, α holds with probability at least s'. We restrict our study to so-called measurable models. We provide sound and complete infinitary axiomatizations for the logics. Furthermore, in the propositional case decidability is proved by establishing a periodicity argument for ω-sequences extending the decidability proof of standard propositional temporal logic LTL. Complexity issues are examined and a worst-case complexity upper bound is given. Extensions of the presented results and open problems are described in the final part of the paper.

Keywords: Probabilistic logic, temporal logic, axiomatization, decidability.

1 Introduction

Probabilistic reasoning has been recognized as a useful tool in many fields of Computer Science and Artificial Intelligence. The paper [35] stimulated extensive investigations about formal systems for reasoning in the presence of uncertainty [7–10, 12, 13, 20, 21, 24, 33, 34, 36, 42, 45]. In those papers, it is usual to start with classical logic and to add probabilistic operators (in our notation $P_{\geq s}$ with the intended meaning 'the probability is at least s') that behave like modal operators. It is also possible to combine the probabilistic approach with some other (nonclassical) logics. For example, the probabilistic operators are added to modal logic of knowledge in [8], while a probabilistic extension of intuitionistic logic is analysed in [33].

In this paper we describe a way in which probabilistic reasoning can be enriched with some temporal features. The temporal part of the logics is a standard discrete linear-time logic, where the flow of time is isomorphic to natural numbers, i.e. each moment of time has a unique possible future, while the corresponding language contains the 'next' operator ($\bigcirc$) and the reflexive strong 'until' operator (U) (see [2, 14, 26, 28, 30, 32, 44] for further references), while the operators 'sometime' and 'always' are definable from ($\bigcirc$ and U). In our logic the probabilistic operators quantify events along a single time line. It allows us to express sentences such as '(according to the current set of information) the probability that, sometime in the future, α is true is at least s'. And, as the knowledge can evolve during the time, the probability of α might change too. Note that, since the operators 'sometime' and 'always' can be seen as the existential and universal quantifiers over time instants, the probabilistic operators give more refined quantitative characterization of sets of time instants definable by formulas. We may try to motivate the proposed semantics in the following way. A suitable representation of all possible outcomes of an infinite sequence of probabilistic experiments (let us say that experiments A and B are permanently repeated resulting in a or $\neg a$, and b or $\neg b$, respectively) could be an inifinite tree, where every branch corresponds to a possible realization of the sequence of the experiments, and every time instant is described in the form $\pm a, \pm b$

Vol. 16 No. 2,
doi:10.1093/logcom/exi077

depending on obtaining (or not obtaining) a and b in the corresponding experiment. We might be interested in probabilistic properties that hold for all branches. In that case we can reason about an arbitrary branch and need ability to express probabilities of events along it, for example that the probability of the event a is at least s, or some more complicated conditions, like that in every time instant, if the probability of a is less than r, then b must hold forever.

Building on our previous work [39], we analyse completeness, decidability and complexity of a propositional and a first-order discrete linear-time probabilistic logic. In the propositional case, we describe a class of so called measurable models and give a sound and complete infinitary axiomatization. The term infinitary concerns the meta language only, i.e. the object language is countable, and formulae are finite, while only proofs are allowed to be infinite. Thanks to such an approach we are able to obtain the extended completeness theorem ('every consistent set of formulae is satisfiable') which can not be proved using any finitary axiomatization because of the non-compactness of our logic. Following [44] we prove the corresponding decidability theorem and examine complexity of the satisfiability problem. In the first-order case there is no complete and recursive axiomatization at all [1, 15]. Nevertheless, we extend our approach for the propositional logic, and give a complete infinitary first order axiomatization.

Although combining temporal and probabilistic reasoning is of a great interest, there are not many papers considering the corresponding formalizations. In [8] probabilistic operators and modal operators of knowledge are used in a way that is similar to ours. However, some properties of the models analysed there (finite model property, for example) are significantly different from our approach. On the other hand, models that are not measurable are also considered there. Dropping the measurability requirement makes things more complicated. In that case inner and outer measures should be used. The finite additivity does not hold for these measures. Thus, some of our axioms that correspond to this property would not be sound (see [8] for more discussion and some interesting examples). A sound first order axiomatization (which is not complete) is provided in [19] for a logic for representing time, chance and action. This work differs from ours since time intervals and a branching structure of time are considered there. In [17] a propositional logic which can be used for reasoning about probabilistic processes is presented. Despite the differences between our logic and that one (which is motivated by a dynamic generalization of the logic of qualitative probabilities [43]), in [17] an idea to prove completeness using an infinitary rule is used similarly to our approach. In [36] a propositional linear-time probabilistic logic with $\bigcirc$ and F similar to the presented one is analysed, a sound and complete axiomatization is given, but the decidability problem is not solved.

Some papers that either implicitly or explicitly combine probabilistic and temporal reasoning are motivated by the need to analyse probabilistic programs and stochastic systems. In [23, 29] propositional temporal formulae that do not mention probabilities explicitly are interpreted over Markov systems which can simulate the execution of probabilistic programs. The corresponding semantics and axiomatic systems are provided. In [11] the formal language is based on the propositional dynamic logic, the main objects are programs, probabilistic operators can be applied on a limited class of formulae, and the completeness problem is not solved. Another probabilistic propositional dynamic logic (which is a fragment of [11]) is considered in [27]. The branching-time logics PCTL* and PCTL for reasoning about time and probability are described in [3, 22]. The underlying temporal logics are Computational Tree Logics CTL and CTL* with path quantifiers replaced by probabilities. The probability measures are defined on sets of paths, so one can describe properties of a certain fraction of paths which is not possible in our approach. On the other hand, in those logics it is not possible to express features of a particular path. In [5] those logics are extended to allow nondeterminism and model-checking algorithms are presented. To the best of our knowledge, so far no sound and complete proof system for the branching-time probabilistic logics has been proposed. In

[18, 25, 31] probabilistic operators are combined with an interval-based temporal logic to describe behaviors of probabilistic timed automata. In [6] the complexity of testing whether a finite state probabilistic program satisfies its specification described in the standard linear-time logic with $\bigcirc$ and U is determined using automata-theoretic approach.

The rest of the paper is organized as follows. In Section 2 we introduce the Probabilistic Propositional Temporal Logic pLTL and define the class of measurable models. pLTL restricted to the class of all measurable models, denoted pLTL$_{Meas}$, is the very object of study of the paper. Section 3 contains an axiomatization that is proved to be sound and complete with respect to the pLTL$_{Meas}$. In Section 4, using the ideas presented in [44] (see also [30]), we show that any pLTL$_{Meas}$-satisfiable formula is satisfiable in an ultimately periodic model in which various parameters are bounded by functions depending on the size of the formula. In Section 5 we show that the satisfiability problem for the pLTL$_{Meas}$ is **PSPACE**-hard, and that it belongs to **NEXPTIME**. In Section 6 the First Order Probabilistic Temporal Logic FOpLTL is described, an axiomatization and a sketch of its completeness are given. Section 7 contains concluding remarks and directions for further work.

2 Syntax and semantics

Let $[0,1]_Q$ be the set of rational numbers from the real interval $[0,1]$.

pLTL is a propositional probability discrete linear-time temporal logic based on a countably infinite set $\mathrm{PRP} = \{p_i : i \in \omega\}$ of propositional variables, the classical connectives $\neg$ and $\wedge$ (negation and conjunction), the temporal operators $\bigcirc$ (next) and U (until), and the family $(P_{\geq s})_{s \in [0,1]_Q}$ of unary probabilistic operators. The set $\mathrm{FOR}(\mathrm{pLTL})$ of formulae is defined inductively as the smallest set containing propositional variables and closed under formation rules: if α and β are formulae, then $\neg\alpha$, $\bigcirc\alpha$, $P_{\geq s}\alpha$, for every $s \in [0,1]_Q$, $\alpha \wedge \beta$, and $\alpha U \beta$ are formulae.

We use the standard abbreviations for the other classical connectives ($\vee$, $\rightarrow$, and $\leftrightarrow$), and also $\bot$ and $\top$ are the notations for $p_0 \wedge \neg p_0$, and $p_0 \vee \neg p_0$ respectively. The temporal operators F (sometime) and G (always) are defined as $F\alpha \stackrel{\text{def}}{=} \top U \alpha$ and $G\alpha \stackrel{\text{def}}{=} \neg F \neg \alpha$. Furthermore, we use the following notational definitions: $P_{<s}(\alpha) \stackrel{\text{def}}{=} \neg P_{\geq s}(\alpha)$, $P_{\leq s}(\alpha) \stackrel{\text{def}}{=} P_{\geq 1-s}(\neg\alpha)$, $P_{>s}(\alpha) \stackrel{\text{def}}{=} \neg P_{\geq 1-s}(\neg\alpha)$, $P_{=s}(\alpha) \stackrel{\text{def}}{=} P_{\geq s}(\alpha) \wedge P_{\leq s}(\alpha)$, and $\bigcirc^0\alpha \stackrel{\text{def}}{=} \alpha$ and $\bigcirc^{i+1}\alpha \stackrel{\text{def}}{=} \bigcirc\bigcirc^i \alpha$ for $i \geq 0$. If $T = \{\alpha_1, \alpha_2, \ldots\}$ is a set of formulae, then $\bigcirc T$ denotes $\{\bigcirc\alpha_1, \bigcirc\alpha_2, \ldots\}$. We write $Sub(\alpha)$ for the set of subformulae of α, and $|\alpha|$ for the length of (or ***size***) of α, assuming a reasonably succinct encoding. For instance, the rational numbers are represented as pairs of bitstrings. Note that $card(Sub(\alpha)) \leq |\alpha|$.

An example of a formula is $(\bigcirc P_{\geq r} p \wedge F P_{<s}(p \rightarrow q)) \rightarrow G P_{=t} q$ which can be read as 'if the probability of p in the next moment is at least r and sometime in the future q follows from p with the probability less than s, then the probability of q will always be equal to t.'

The semantics for pLTL is a Kripke-style one using ω-sequences as frames. A pLTL-*model* is a triple $\langle W, v, Prob\rangle$ where

1. $W = w_0, w_1, \ldots$ is an ω-sequence of *time instants*,
2. $v : \{w_0, w_1, \ldots\} \rightarrow \mathcal{P}(\mathrm{PRP})$ is a map labeling each state w_i with the set of propositional variables that hold in w_i,
3. $Prob$ associates to every $w_i \in W$, a *probability space*

$$Prob(w_i) = \langle W(w_i), A(w_i), \mu(w_i)\rangle$$

such that:

- $W(w_i) = \{w_j : j \geq i\}$,

- $A(w_i)$ is an algebra of subsets of $W(w_i)$, i.e. it contains $W(w_i)$ and it is closed under complementation and finite union, and
- $\mu(w_i)$ is a finitely additive probability measure on $W(w_i)$:
  - for every $X \in A(w_i)$, $\mu(w_i)(X) \geq 0$,
  - $\mu(w_i)(W(w_i)) = 1$, and
  - $\mu(w_i)(X_1 \cup X_2) = \mu(w_i)(X_1) + \mu(w_i)(X_2)$, for all disjoint sets $X_1, X_2 \in A(w_i)$.

The elements of $A(w_i)$ are called the *measurable sets* of $\mathcal{P}(W(w_i))$. $\mu(w_i)$ does not assign probabilities to all subsets of $W(w_i)$, but only to the measurable ones.

Let M be a pLTL-model, $i \in \omega$ and α be a formula. The satisfiability relation $\models_M$ is inductively defined as follows:

- $w_i \models_M p \overset{\text{def}}{\Leftrightarrow} p \in v(w_i)$,
- $w_i \models_M \neg\alpha \overset{\text{def}}{\Leftrightarrow}$ it is not $w_i \models_M \alpha$,
- $w_i \models_M P_{\geq s}\alpha \overset{\text{def}}{\Leftrightarrow} \mu(w_i)(\{w_{i+j}, j \geq 0 : w_{i+j} \models_M \alpha\}) \geq s$,
- $w_i \models_M \bigcirc\alpha \overset{\text{def}}{\Leftrightarrow} w_{i+1} \models_M \alpha$,
- $w_i \models_M \alpha \wedge \beta \overset{\text{def}}{\Leftrightarrow} w_i \models_M \alpha$ and $w_i \models_M \beta$,
- $w_i \models_M \alpha U \beta \overset{\text{def}}{\Leftrightarrow}$ there is an integer $j \geq 0$ such that $w_{i+j} \models_M \beta$, and for every k such that $0 \leq k < j$, $w_{i+k} \models_M \alpha$.

Note that we concern a reflexive, strong version of the until operator, i.e. if $\alpha U \beta$ holds in a time instant, β must eventually hold. In the above definition the future includes the present, so that:

- $w_i \models_M F\alpha \overset{\text{def}}{\Leftrightarrow}$ there is $j \geq 0$ such that $w_{i+j} \models_M \alpha$, and
- $w_i \models_M G\alpha \overset{\text{def}}{\Leftrightarrow}$ for every $j \geq 0$, $w_{i+j} \models_M \alpha$.

Similarly, the present time instant is included when the probability of formulae are considered. We might also use temporal and probabilistic operators referring to the strict future that does not concern the present. All the presented results then can be proved with essentially no change.

A set of formulae T is pLTL-*satisfiable* if there is a time instant w in a pLTL-model M such that for every formula $\alpha \in T$, $w \models_M \alpha$. A formula α is pLTL-*satisfiable* if the set $\{\alpha\}$ is pLTL-*satisfiable*. A formula α is pLTL-*valid* ($\models \alpha$) if it is satisfied in each time instant in each pLTL-model. We will omit the subscript M from $w \models_M \alpha$ and write $w \models \alpha$ if M is clear from the context.

In a pLTL-model $M = \langle W, v, Prob \rangle$, the set $\{w_{i+j} : j \geq 0,\ w_{i+j} \models_M \alpha\}$, where $w_i \in W$, is denoted by $[\alpha]_{M,w_i}$, or more briefly, $[\alpha]_{w_i}$.

Although the semantics of $P_{\geq s}\alpha$ is quite natural, the only problem is that $[\alpha]_{w_i}$ might not be measurable, i.e. it might not be in $A(w_i)$. That is why in the sequel we consider only so-called measurable models. A pLTL-model M is said to be *measurable* if for every $w_i \in W$, $A(w_i) = \{[\alpha]_{w_i} : \alpha \in \mathrm{FOR}(\mathrm{pLTL})\}$. Observe that $W(w_i) = [\alpha \vee \neg\alpha]_{w_i}$, $W(w_i) \setminus [\alpha]_{w_i} = [\neg\alpha]_{w_i}$, and $[\alpha \vee \beta]_{w_i} = [\alpha]_{w_i} \cup [\beta]_{w_i}$. Consequently, $A(w_i)$ are algebras as is required. Let us denote this class of models by pLTL$_{Meas}$. We write pLTL$_{Meas}$ to denote the probabilistic discrete linear-time temporal logic characterized by the class pLTL$_{Meas}$ of models.

In many places in the sequel, the satisfiability relation $\models$ is coherently extended to structures that are not necessarily models.

In Section 7 certain transformations of pLTL$_{Meas}$-models will be used to prove decidability of pLTL. In some of the transformations parts of the sequence of time instants of a model will be deleted. We will use a concise notation $W' = w_0 \ldots, w_{i-1}, w_j, \ldots$ to denote the sequence of time instants obtained by discarding the part $w_i, \ldots, w_{j-1}$ from $W = w_0, \ldots, w_{i-1}, w_i, \ldots, w_{j-1}, w_j, \ldots$ More strictly, we should write:

- $W' = w'_0, \ldots, w'_{i-1}, w'_i, \ldots, w'_{j-1}, w'_j, \ldots$,
- for $k < i$, $v'(w'_k) \stackrel{\text{def}}{=} v(w_k)$,
- for $k \geq i$, $v'(w'_k) \stackrel{\text{def}}{=} v(w_{k+j-i})$, etc.

However, we will use the former notation, which is in our opinion both shorter and more intuitive.

3 A complete axiomatization

In this section, we give an axiomatization with infinitary rules, and prove the extended completeness theorem with respect to the class of all pLTL$_{Meas}$-models. We define the following deductive system for establishing pLTL$_{Meas}$-validity.

Axiom schemes

A1. all instances of propositional theorems
A2. $\bigcirc(\alpha \to \beta) \to (\bigcirc\alpha \to \bigcirc\beta)$
A3. $\neg \bigcirc \alpha \leftrightarrow \bigcirc\neg\alpha$
A4. $\alpha U \beta \leftrightarrow \beta \vee (\alpha \wedge \bigcirc(\alpha U \beta))$
A5. $\alpha U \beta \to F\beta$
A6. $P_{\geq 0}\alpha$
A7. $P_{\leq s}\alpha \to P_{<t}\alpha$, $t > s$
A8. $P_{<s}\alpha \to P_{\leq s}\alpha$
A9. $(P_{\geq s}\alpha \wedge P_{\geq r}\beta \wedge P_{\geq 1}(\neg\alpha \vee \neg\beta)) \to P_{\geq \min(1,s+r)}(\alpha \vee \beta)$
A10. $(P_{\leq s}\alpha \wedge P_{<r}\beta) \to P_{<s+r}(\alpha \vee \beta)$, $s + r \leq 1$
A11. $G\alpha \to P_{\geq 1}\alpha$

Inference rules

R1. from $\{\alpha, \alpha \to \beta\}$ infer β
R2. from α infer $\bigcirc\alpha$
R3. from $\{\beta \to \bigcirc^i\alpha\}$ for all $i \geq 0$, infer $\beta \to G\alpha$
R4. from $\{\beta \to \bigcirc^m P_{\geq s - \frac{1}{k}}\alpha\}$, for any $m \geq 0$, and for every $k \geq \frac{1}{s}$, infer $\beta \to \bigcirc^m P_{\geq s}\alpha$

Let us briefly discuss some of the above axioms and rules. Note that the axiom system can be divided into four parts. The first three parts deal with propositional, temporal, and probabilistic reasoning, respectively, while the last one concerns mixing of probabilistic and temporal reasoning. The classical propositional logic is a sublogic of the presented logic (by the axiom A1). The axioms A2 and A3 are the usual axioms for the next operator $\bigcirc$, as well as the axioms A4 and A5 for the until operator [30]. The axioms A6 – A10 concern the probabilistic aspect of the logic pLTL$_{Meas}$ [38, 39, 41]. The axiom A6 announces that every formula is satisfied by a set of time instants of the measure at least 0. By substituting $\neg\alpha$ for α in the axiom A6, the formula $P_{\geq 0}\neg\alpha$ is obtained. According to our definition of the operator $P_{\leq 1}$ given in Section 2, we have the following instance of the axiom A6: $P_{\leq 1}\alpha$ ($\stackrel{\text{def}}{=} P_{\geq 1-s}\neg\alpha$, for $s = 1$). It forces every formula to be satisfied by a set of time instants of the measure at most 1, and gives the upper bound for probabilities of formulas in pLTL-models. The axioms A7 and A8 are equivalent to

A7′ $P_{\geq t}\alpha \rightarrow P_{>s}\alpha$, $t > s$
A8′ $P_{>s}\alpha \rightarrow P_{\geq s}\alpha$,

respectively. The axioms A9 and A10 correspond to the finite additivity of probabilities. For example, in the axiom A9, if sets of time instants that satisfy α and β are disjoint, then the measure of the set of time instants that satisfy $\alpha \vee \beta$ is the sum of the measures of the former two sets. The axiom A11 links temporal and probabilistic operators. The rules R1 and R2 are Modus Ponens and Necessitation, respectively. The rules R3 and R4 are infinitary inference rules. The former one characterizes the always operator, while the later one intuitively says that if the probability is arbitrary closed to s, then it is at least s.

A formula α is *derivable from* a set T of formulae ($T \vdash \alpha$) if there is an at most denumerable sequence of formulas $\alpha_0, \alpha_1, \ldots, \alpha$, such that every α_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule, with the exception that Rule R2 can be applied to the theorems only. The sequence $\alpha_0, \alpha_1, \ldots, \alpha$ is the *proof* of $T \vdash \alpha$. If $\emptyset \vdash \alpha$, we say that α is a theorem of the deductive system, also denoted by $\vdash \alpha$. A set T of formulae is *inconsistent* if $T \vdash \alpha$, for every formula α, otherwise it is *consistent*. Equivalently, T is inconsistent iff $T \vdash \perp$. A set T of formulae is *maximal* if for every formula α either $\alpha \in T$ or $\neg\alpha \in T$.

From the above definition of derivability from the set of formulae, the deduction theorem follows.

THEOREM 3.1 (Deduction theorem)
Let α, β be formulae and T be a set of formulae. Then, $T, \alpha \vdash \beta$ implies $T \vdash \alpha \rightarrow \beta$.

PROOF. We use the transfinite induction on the length of the proof of β from $T \cup \{\alpha\}$.

The cases when either $\vdash \beta$ or $\beta = \alpha$ or β is obtained by application of Modus Ponens (R1) are standard. Assume that $T, \alpha \vdash \beta$, where $\beta = \bigcirc\gamma$ is obtained by the inference rule R2. The only premiss of the inference is the formula γ. By definition of the derivability relation $\vdash$, we have $\vdash \gamma$. Thus, $\vdash \beta$. By propositional reasoning we obtained $\vdash \alpha \rightarrow \beta$. *A fortiori*, $T \vdash \alpha \rightarrow \beta$. Now assume that $T, \alpha \vdash \beta$, where $\beta = \gamma \rightarrow G\gamma'$ is obtained by the inference rule R3. Then:

$T, \alpha \vdash \gamma \rightarrow \bigcirc^i\gamma'$, for $i \geq 0$,
$T \vdash \alpha \rightarrow (\gamma \rightarrow \bigcirc^i\gamma')$, for $i \geq 0$, by the induction hypothesis,
$T \vdash (\alpha \wedge \gamma) \rightarrow \bigcirc^i\gamma'$, for $i \geq 0$,
$T \vdash (\alpha \wedge \gamma) \rightarrow G\gamma'$, by the inference rule R3,
$T \vdash \alpha \rightarrow \beta$.

Finally, suppose that $T, \alpha \vdash \gamma \rightarrow \bigcirc^m P_{\geq s}\gamma'$ is obtained from $T, \alpha \vdash \gamma \rightarrow \bigcirc^m P_{\geq s-\frac{1}{k}}\gamma'$, for every $k \geq \frac{1}{s}$, using the inference rule R4. Similarly as above we have $T \vdash \alpha \rightarrow (\gamma \rightarrow \bigcirc^m P_{\geq s-\frac{1}{k}}\gamma')$, and $T \vdash (\alpha \wedge \gamma) \rightarrow \bigcirc^m P_{\geq s-\frac{1}{k}}\gamma'$, for every $k \geq \frac{1}{s}$. By the inference rule R4, we obtain $T \vdash (\alpha \wedge \gamma) \rightarrow \bigcirc^m P_{\geq s}\gamma'$, and $T \vdash \alpha \rightarrow (\gamma \rightarrow \bigcirc^m P_{\geq s}\gamma')$. ∎

The next lemmas contain some auxiliary statements.

LEMMA 3.2
Let α, β be formulae.

1. $\vdash G\alpha \leftrightarrow \alpha \wedge \bigcirc G\alpha$,
2. $\vdash G \bigcirc \alpha \leftrightarrow \bigcirc G\alpha$,

3. $\vdash (\bigcirc\alpha \to \bigcirc\beta) \to \bigcirc(\alpha \to \beta)$,
4. $\vdash \bigcirc(\alpha \wedge \beta) \leftrightarrow (\bigcirc\alpha \wedge \bigcirc\beta)$,
5. $\vdash \bigcirc(\alpha \vee \beta) \leftrightarrow (\bigcirc\alpha \vee \bigcirc\beta)$,
6. $G\alpha \vdash \bigcirc^i\alpha$ for every $i \geq 0$,
7. if $\vdash \alpha$, then $\vdash G\alpha$,
8. if $T \vdash \alpha$, where T is a set of formulae, then $\bigcirc T \vdash \bigcirc\alpha$.
9. for $j \geq 0$, $\bigcirc^j\beta, \bigcirc^0\alpha, \ldots, \bigcirc^{j-1}\alpha \vdash \alpha U\beta$,
10. if $\vdash \alpha$, then $\vdash P_{\geq 1}\alpha$,
11. $\vdash P_{\geq 1}(\alpha \to \beta) \to (P_{\geq s}\alpha \to P_{\geq s}\beta)$,
12. if $\vdash \alpha \leftrightarrow \beta$, then $\vdash P_{\geq s}\alpha \leftrightarrow P_{\geq s}\beta$,
13. $\vdash P_{\geq s}\alpha \to P_{\geq r}\alpha$, $s \geq r$.

PROOF. (1) By the definition of the always operator and the axiom A4, we have $\vdash \neg(\top U\neg\alpha) \leftrightarrow \neg(\neg\alpha\vee(\top\wedge\bigcirc(\top U\neg\alpha)))$. By the propositional reasoning and the axiom A3 we obtain $\vdash \neg(\top U\neg\alpha) \leftrightarrow \alpha \wedge (\bot \vee \bigcirc \neg(\top U\neg\alpha))$. Thus, $\vdash G\alpha \leftrightarrow \alpha \wedge \bigcirc G\alpha$.
(2)–(7) The proofs are easy consequences of the temporal part of the above axiomatization.
(8) We use the induction on the length of the proof of α from T. Let α be obtained by the inference rule R1 from $\beta \to \alpha$ and β. We have:

- $\bigcirc T \vdash \bigcirc(\beta \to \alpha)$ (by the induction hypothesis),
- $\bigcirc T \vdash \bigcirc(\beta \to \alpha) \to (\bigcirc\beta \to \bigcirc\alpha)$ (by the axiom A2),
- $\bigcirc T \vdash \bigcirc\beta \to \bigcirc\alpha$ (by application of Modus Ponens),
- $\bigcirc T \vdash \bigcirc\beta$ (by the induction hypothesis),
- $\bigcirc T \vdash \bigcirc\alpha$ (by application of Modus Ponens).

The case of the inference rule R2 can be solved in a similar way. Suppose that $\alpha = \beta \to G\gamma$ is obtained by the rule R3. Then,

- for $i \geq 0$, $\bigcirc T \vdash \bigcirc(\beta \to \bigcirc^i\gamma)$ (by the induction hypothesis),
- for $i \geq 0$, $\bigcirc T \vdash \bigcirc(\beta \to \bigcirc^i\gamma) \to (\bigcirc\beta \to \bigcirc^i \bigcirc \gamma)$ (by the axiom A2),
- for $i \geq 0$, $\bigcirc T \vdash \bigcirc\beta \to \bigcirc^i \bigcirc \gamma$ (by Modus Ponens),
- $\bigcirc T \vdash \bigcirc\beta \to G \bigcirc \gamma$ (by application of the inference rule R3),
- $\bigcirc T \vdash \bigcirc\beta \to \bigcirc G\gamma$ (by Lemma 3.2(2)),
- $\bigcirc T \vdash (\bigcirc\beta \to \bigcirc G\gamma) \to \bigcirc(\beta \to G\gamma)$ (by Lemma 3.2(3)),
- $\bigcirc T \vdash \bigcirc(\beta \to G\gamma)$ (by Modus Ponens).

Finally, suppose that $\alpha = \beta \to \bigcirc^m P_{\geq s}\gamma$ is obtained by the rule R4. Then,

- $\bigcirc T \vdash \bigcirc(\beta \to \bigcirc^m P_{\geq s-\frac{1}{k}}\gamma)$, for $k > \frac{1}{s}$ (by the induction hypothesis)
- $\bigcirc T \vdash \bigcirc\beta \to \bigcirc^{m+1} P_{\geq s-\frac{1}{k}}\gamma$, for $k > \frac{1}{s}$ (by A2, and Modus Ponens)
- $\bigcirc T \vdash \bigcirc\beta \to \bigcirc^{m+1} P_{\geq s}\gamma$ (by the inference rule R4)
- $\bigcirc T \vdash \bigcirc(\beta \to \bigcirc^m P_{\geq s}\gamma)$ (by Lemma 3.2(3), and Modus Ponens).

(9) By propositional reasoning, we obtain

$$\begin{array}{ll} \bigcirc^j\beta, \bigcirc^0\alpha, \ldots, \bigcirc^{j-1}\alpha \vdash \beta \vee (\alpha\wedge & (\quad \bigcirc\beta \vee (\bigcirc\alpha \wedge \ldots \wedge (\bigcirc^{j-1}\beta \vee (\bigcirc^{j-1}\alpha \wedge \\ & (\quad \bigcirc^j\beta \vee (\bigcirc^j\alpha \wedge \bigcirc^{j+1}(\alpha U\beta)) \ldots). \end{array}$$

Since

$$\vdash \beta \vee (\alpha \wedge (\bigcirc\beta \vee (\bigcirc\alpha \wedge \ldots \wedge (\bigcirc^j\beta \vee (\bigcirc^j\alpha \wedge \bigcirc^{j+1}(\alpha U \beta)) \ldots) \rightarrow (\alpha U \beta)$$

follows from the axiom A4, we have:

$$\bigcirc^j\beta, \bigcirc^0\alpha, \ldots, \bigcirc^{j-1}\alpha \vdash \alpha U \beta.$$

(10) Assume $\vdash \alpha$. By application of the inference rule R2, we get $\vdash \bigcirc^k\alpha$, for every $k \in \omega$. We obtain $\vdash G\alpha$ by the inference rule R3. From the axiom A11 and by application of Modus Ponens, we have $\vdash P_{\geq 1}\alpha$.

(11) First note that using Lemma 3.2(10), we obtain

$$\vdash P_{\geq 1}(\neg\alpha \vee \neg\bot) \tag{3.1}$$

from $\vdash \neg\alpha \vee \neg\bot$. Similarly, from $\vdash (\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha$ we have

$$\vdash P_{\geq 1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha). \tag{3.2}$$

By the axiom A9, we have $\vdash (P_{\geq s}\alpha \wedge P_{\geq 0}\bot \wedge P_{\geq 1}(\neg\alpha \vee \neg\bot)) \rightarrow P_{\geq s}(\alpha \vee \bot)$. Since $\vdash P_{\geq 0}\bot$ by the axiom A6, from (3.1) it follows that

$$\vdash P_{\geq s}\alpha \rightarrow P_{\geq s}(\alpha \vee \bot). \tag{3.3}$$

The expression $P_{\geq s}(\alpha\vee\bot)$ denotes $P_{\geq s}\neg(\neg\alpha\wedge\neg\bot)$, $P_{\geq 1-(1-s)}\neg(\neg\alpha\wedge\neg\bot)$, and $P_{\leq 1-s}(\neg\alpha\wedge\neg\bot)$. Similarly, $\neg P_{\geq s}\neg\neg\alpha$ denotes $P_{<s}\neg\neg\alpha$. By the axiom A10, we have $\vdash (P_{\leq 1-s}(\neg\alpha \wedge \neg\bot) \wedge P_{<s}\neg\neg\alpha) \rightarrow P_{<1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha)$. From (3.2) we obtain that $\vdash (P_{\leq 1-s}(\neg\alpha \wedge \neg\bot) \wedge P_{<s}\neg\neg\alpha) \rightarrow (P_{<1}((\neg\alpha\wedge\neg\bot)\vee\neg\neg\alpha)\wedge\neg P_{<1}((\neg\alpha\wedge\neg\bot)\vee\neg\neg\alpha)$, because $P_{\geq 1}((\neg\alpha\wedge\neg\bot)\vee\neg\neg\alpha)$ denotes $\neg P_{<1}((\neg\alpha \wedge \neg\bot) \vee \neg\neg\alpha)$. It follows that $\vdash P_{\leq 1-s}(\neg\alpha \wedge \neg\bot) \rightarrow \neg P_{<s}\neg\neg\alpha$, i.e.

$$\vdash P_{\geq s}(\alpha \vee \bot) \rightarrow P_{\geq s}\neg\neg\alpha. \tag{3.4}$$

From (3.3) and (3.4) we obtain:

$$\vdash P_{\geq s}\alpha \rightarrow P_{\geq s}\neg\neg\alpha. \tag{3.5}$$

The negation of the formula $P_{\geq 1}(\alpha \rightarrow \beta) \rightarrow (P_{\geq s}\alpha \rightarrow P_{\geq s}\beta)$ is equivalent to $P_{\geq 1}(\neg\alpha \vee \beta) \wedge P_{\geq s}\alpha \wedge P_{<s}\beta$. Since $\vdash P_{\geq s}\alpha \rightarrow P_{\geq s}\neg\neg\alpha$, this formula implies $P_{\geq 1}(\neg\alpha \vee \beta) \wedge P_{\geq s}\neg\neg\alpha \wedge P_{<s}\beta$ which can be rewritten as $P_{\geq 1}(\neg\alpha \vee \beta) \wedge P_{\leq 1-s}\neg\alpha \wedge P_{<s}\beta$. From the axiom A10, $P_{\leq 1-s}\neg\alpha \wedge P_{<s}\beta \rightarrow P_{<1}(\neg\alpha \vee \beta)$, and $P_{<1}\alpha = \neg P_{\geq 1}\alpha$, we have $\vdash \neg(P_{\geq 1}(\alpha \rightarrow \beta) \rightarrow (P_{\geq s}\alpha \rightarrow P_{\geq s}\beta)) \rightarrow P_{\geq 1}(\neg\alpha\vee\beta)\wedge\neg P_{\geq 1}(\neg\alpha\vee\beta)$, a contradiction. It follows that $\vdash P_{\geq 1}(\alpha \rightarrow \beta) \rightarrow (P_{\geq s}\alpha \rightarrow P_{\geq s}\beta)$.

(12) It is an easy consequence of Lemma 3.2(11).
(13) This formula expresses monotonicity of probabilities. It is a consequence of the axioms A7′ and A8′. ∎

LEMMA 3.3
Let T be a maximal consistent set of formulae. Then,

1. for any formula α, exactly one member of $\{\alpha, \neg\alpha\}$ is in T,
2. $\alpha \vee \beta \in T$ iff $\alpha \in T$ or $\beta \in T$,

3. $\alpha \wedge \beta \in T$ iff $\{\alpha, \beta\} \subseteq T$,
4. if $T \vdash \alpha$, then $\alpha \in T$,
5. if $\{\alpha, \alpha \rightarrow \beta\} \subseteq T$, then $\beta \in T$,
6. if $\alpha \in T$ and $\vdash \alpha \rightarrow \beta$, then $\beta \in T$,
7. for any formula α, if $t = \sup_s\{P_{\geq s}\alpha \in T\}$, and $t \in [0,1]_Q$, then $P_{\geq t}\alpha \in T$.

PROOF. Proofs (1)–(6) are standard.
(7) Let $t = \sup_s\{P_{\geq s}\alpha \in T\} \in [0,1]_Q$. Thus, for every $s \in [0,1]_Q$, $s < t$, $T \vdash P_{\geq s}\alpha$. Using the inference rule R4 we have $T \vdash P_{\geq t}\alpha$. Since T is a maximal consistent set, it follows from Lemma 3.3(4) that $P_{\geq t}\alpha \in T$. ■

LEMMA 3.4
Let T be a consistent set of formulae.

1. For any formula α, either $T \cup \{\alpha\}$ is consistent or $T \cup \{\neg\alpha\}$ is consistent.
2. If $\neg(\alpha \rightarrow G\beta) \in T$, then there is $j_0 \geq 0$ such that $T \cup \{\alpha \rightarrow \neg \bigcirc^{j_0} \beta\}$ is consistent.
3. If $\neg(\alpha \rightarrow \bigcirc^m P_{\geq s}\beta) \in T$, then there is $j_0 > \frac{1}{s}$ such that $T \cup \{\alpha \rightarrow \neg \bigcirc^m P_{\geq s - \frac{1}{j_0}}\beta\}$ is consistent.

PROOF. (1) The proof is standard.
(2) Suppose that for every $j \geq 0$, $T, \alpha \rightarrow \neg \bigcirc^j \beta \vdash \perp$. Hence, by the Deduction Theorem, for every $j \geq 0$, $T \vdash \neg(\alpha \rightarrow \neg \bigcirc^j \beta)$. By manipulation at the propositional level, we have $T \vdash \alpha \rightarrow \bigcirc^j\beta$ for every $j \geq 0$. By application of the inference rule R3, $T \vdash \alpha \rightarrow G\beta$. Since $\neg(\alpha \rightarrow G\beta) \in T$, this leads to $T \vdash \perp$, a contradiction.
(3) Similarly to the above, suppose that for every $j > \frac{1}{s}$, $T, \alpha \rightarrow \neg \bigcirc^m P_{\geq s - \frac{1}{j}}\beta \vdash \perp$. By the Deduction Theorem, and manipulation at the propositional level, we have $T \vdash \alpha \rightarrow \bigcirc^m P_{\geq s - \frac{1}{j}}\beta$ for $j > \frac{1}{s}$. By application of the inference rule R4, $T \vdash \alpha \rightarrow \bigcirc^m P_{\geq s}\beta$, a contradiction with the fact that $\neg(\alpha \rightarrow \bigcirc^m P_{\geq s}\beta) \in T$. ■

THEOREM 3.5
Every consistent set T of formulae can be extended to a maximal consistent set.

PROOF. Let $\alpha_0, \alpha_1, \ldots$ be an enumeration of all formulae. Let us define a family $(T_i)_{i \in \omega}$ of sets of formulae, and a set $\mathcal{T}$ as follows:

1. $T_0 \stackrel{\text{def}}{=} T$.
2. For every $i \geq 0$ if $T_i \cup \{\alpha_i\}$ is consistent, then $T_{i+1} \stackrel{\text{def}}{=} T_i \cup \{\alpha_i\}$. Otherwise, if α_i is of the form $\gamma \rightarrow G\beta$, then $T_{i+1} \stackrel{\text{def}}{=} T_i \cup \{\neg\alpha_i, \gamma \rightarrow \neg \bigcirc^{j_0} \beta\}$ for some $j_0 \geq 0$ such that T_{i+1} is consistent. Otherwise, α_i is of the form $\gamma \rightarrow \bigcirc^m P_{\geq s}\beta$, then $T_{i+1} \stackrel{\text{def}}{=} T_i \cup \{\neg\alpha_i, \gamma \rightarrow \neg \bigcirc^m P_{\geq s - \frac{1}{j_0}}\beta\}$ for some $j_0 > 0$ such that T_{i+1} is consistent. Otherwise, $T_{i+1} \stackrel{\text{def}}{=} T_i \cup \{\neg\alpha_i\}$.
3. $\mathcal{T} = \bigcup_{i=0}^{\infty} T_i$.

By Lemma 3.4, the definition of the family $(T_i)_{i \in \omega}$ is correct and each T_i is consistent. It remains to show that $\mathcal{T}$ is maximal and consistent. Let α be an arbitrary formula, and $\alpha = \alpha_i$ for some $i \in \omega$. The step 2 of the construction trivially guaranties that either α or $\neg\alpha$ belongs to T_i and $\mathcal{T}$. Hence $\mathcal{T}$ is maximal.

Let us show that $\mathcal{T}$ is consistent. To do so, we shall establish that $\mathcal{T} \nvdash \perp$ by showing that if $\mathcal{T} \vdash \alpha$, then $\alpha \in \mathcal{T}$ (by induction on the length of the proof of $\mathcal{T} \vdash \alpha$). Indeed, if $\mathcal{T} \vdash \perp$, then $\perp \in \mathcal{T}$ and

$\perp \in T_{u+1}$ with $\perp = \alpha_u$. This would lead to a contradiction with consistency of T_{u+1}.
Base case: α is an axiom or $\alpha \in \mathcal{T}$.
If $\alpha \in \mathcal{T}$, then we are done. Otherwise assume that α is an axiom and $\alpha \notin \mathcal{T}$. Let $\alpha = \alpha_i$. So $\neg\alpha \in T_{i+1}$, $T_{i+1} \vdash \neg\alpha$ and $T_{i+1} \vdash \alpha$ (α is an axiom) which is in contradiction with consistency of T_{i+1}.
Induction step
Case 1: $\mathcal{T} \vdash \alpha$ is obtained from $\mathcal{T} \vdash \beta \to \alpha$ and $\mathcal{T} \vdash \beta$ by Modus Ponens.
By the induction hypothesis, $\beta \to \alpha \in \mathcal{T}$ and $\beta \in \mathcal{T}$. By construction of $\mathcal{T}$, there is $i \geq 0$ such that $\beta \to \alpha \in T_i$, $\beta \in T_i$ and either $\alpha \in T_i$ or $\neg\alpha \in T_i$ (but not both by consistency of T_i). Actually, $\alpha \in T_i$ otherwise this leads to a contradiction by consistency of T_i.
Case 2: $\mathcal{T} \vdash \bigcirc\alpha$ is obtained from $\vdash \alpha$ by the inference rule R2.
By construction of $\mathcal{T}$, there is $i \geq 0$ such that either $\bigcirc\alpha \in T_i$ or $\neg \bigcirc \alpha \in T_i$ (but not both by consistency of T_i). Suppose that $\neg \bigcirc \alpha \in T_i$. So, $T_i \vdash \neg \bigcirc \alpha$, and $T_i \vdash \bigcirc\alpha$ since $\vdash \bigcirc\alpha$. This leads to a contradiction by consistency of T_i. Hence $\alpha \in T_i$ and $\bigcirc\alpha \in \mathcal{T}$.
Case 3: $\mathcal{T} \vdash \gamma \to G\beta$ is obtained from $\mathcal{T} \vdash \gamma \to \bigcirc^j\beta$ for $j \geq 0$ by the inference rule R3.
Suppose that $\gamma \to G\beta \notin \mathcal{T}$, which is equivalent to $\neg(\gamma \to G\beta) \in \mathcal{T}$ by maximality of $\mathcal{T}$. By construction of $\mathcal{T}$, there is $i \geq 0$ such that $\neg(\gamma \to G\beta) \in T_i$. Since T_i is consistent, by Lemma 3.4(2), there is $j_0 \geq 0$ such that $\gamma \to \neg \bigcirc^{j_0} \beta \in T_i$. By the induction hypothesis, for $j \geq 0$, $\gamma \to \bigcirc^j\beta \in \mathcal{T}$. So there is $i' \geq i$ such that $\gamma \to \neg \bigcirc^{j_0} \beta \in T_{i'}$ and $\gamma \to \bigcirc^{j_0}\beta \in T_{i'}$. Since $T_i \subseteq T_{i'}$, we have $\neg(\gamma \to G\beta) \in T_{i'}$. So $T_{i'} \vdash \neg(\gamma \to G\beta)$ or equivalently $T_{i'} \vdash \gamma \wedge \neg G\beta$. Since $T_{i'} \vdash \gamma \to \neg \bigcirc^{j_0} \beta$ and $T_{i'} \vdash \gamma \to \bigcirc^{j_0}\beta$, we get $T_{i'} \vdash \neg \bigcirc^{j_0} \beta \wedge \bigcirc^{j_0}\beta$ which is in contradiction with consistency of $T_{i'}$.
Case 4: $\mathcal{T} \vdash \gamma \to \bigcirc^m P_{\geq s}\beta$ is obtained from $\mathcal{T} \vdash \gamma \to \bigcirc^m P_{\geq s-\frac{1}{k}}\beta$ for $k \geq \frac{1}{s}$ by the inference rule R4.
Suppose that $\gamma \to \bigcirc^m P_{\geq s}\beta \notin \mathcal{T}$, and $\neg(\gamma \to \bigcirc^m P_{\geq s}\beta) \in \mathcal{T}$. By construction of $\mathcal{T}$, there is $i \geq 0$ such that $\neg(\gamma \to \bigcirc^m P_{\geq s}\beta) \in T_i$. Since T_i is consistent, by Lemma 3.4(3), there is $j_0 \geq 1$ such that $\gamma \to \neg \bigcirc^m P_{\geq s-\frac{1}{j_0}}\beta \in T_i$. Thus, there is $i' \geq i$ such that $\gamma \to \bigcirc^m P_{\geq s-\frac{1}{j_0}}\beta \in T_{i'}$ and $\gamma \to \neg \bigcirc^m P_{\geq s-\frac{1}{j_0}}\beta \in T_{i'}$. Since $T_i \subseteq T_{i'}$, we have $\neg(\gamma \to \bigcirc^m P_{\geq s}\beta) \in T_{i'}$. So $T_{i'} \vdash \gamma \wedge \neg \bigcirc^m P_{\geq s}\beta$, $T_{i'} \vdash \bigcirc^m P_{\geq s-\frac{1}{j_0}}\beta$, and $T_{i'} \vdash \neg \bigcirc^m P_{\geq s-\frac{1}{j_0}}\beta$ which is in contradiction with consistency of $T_{i'}$. ∎

Let T be a maximal and consistent set of formulae. Let M_T be a tuple $\langle W, v, Prob\rangle$, where:

- $W \stackrel{\text{def}}{=} w_0, w_1, \ldots$, $w_0 = T$, and for $i > 0$, $w_i = \{\alpha : \bigcirc\alpha \in w_{i-1}\}$,
- for $i \geq 0$, $v(w_i) = \mathrm{PRP} \cap w_i$, and
- for $i \geq 0$, $Prob(w_i) = \langle W(w_i), A(w_i), \mu(w_i)\rangle$ is defined as follows:
  - $W(w_i) \stackrel{\text{def}}{=} \{w_{i+j} : j \geq 0\}$,
  - $A(w_i) \stackrel{\text{def}}{=} \{\{w_{i+j} : j \geq 0, \alpha \in w_{i+j}\} : \alpha \in \mathrm{FOR(pLTL)}\}$, and
  - for $\alpha \in \mathrm{FOR(pLTL)}$, $\mu(w_i)(\{w_{i+j} : j \geq 0, \alpha \in w_{i+j}\}) = \sup\{s : P_{\geq s}\alpha \in w_i\}$.

The first point that we have to guarantee is that the definition of μ is correct.

LEMMA 3.6
For every $i \geq 0$,

1. w_i is a maximal consistent set, and
2. for all formulae α, β, if $\{w_{i+j} : j \geq 0, \alpha \in w_{i+j}\} = \{w_{i+j} : j \geq 0, \beta \in w_{i+j}\}$, then $\sup\{s : P_{\geq s}\alpha \in w_i\} = \sup\{s : P_{\geq s}\beta \in w_i\}$.

PROOF. (1) The proof is by induction on i. By hypothesis, w_0 is maximal and consistent. Let $i \geq 0$ and w_i be maximal and consistent.

Suppose that w_{i+1} is not maximal. There is a formula α such that $\{\alpha, \neg\alpha\} \cap w_{i+1} = \emptyset$. Consequently, $\{\bigcirc\alpha, \bigcirc\neg\alpha\} \cap w_i = \emptyset$. Using Lemma 3.3(4) and Lemma 3.3(5), and the axiom A3, we get that $\{\bigcirc\alpha, \neg\bigcirc\alpha\} \cap w_i = \emptyset$ which is in contradiction with the maximality of w_i. Suppose that w_{i+1} is not consistent, i.e. $w_{i+1} \vdash \alpha \wedge \neg\alpha$, for any formula α. By Lemma 3.2(8), $\bigcirc w_{i+1} \vdash \bigcirc(\alpha \wedge \neg\alpha)$, and $w_i \vdash \bigcirc(\alpha \wedge \neg\alpha)$. By Lemma 3.2(4), and the axiom A3, we get that $w_i \vdash \bigcirc\alpha \wedge \neg\bigcirc\alpha$, which is in contradiction with consistency of w_i.

(2) It is enough to show that $\{w_{i+j} : j \geq 0, \alpha \in w_{i+j}\} \subseteq \{w_{i+j} : j \geq 0, \beta \in w_{i+j}\}$ implies $\sup\{s : P_{\geq s}\alpha \in w_i\} \leq \sup\{s : P_{\geq s}\beta \in w_i\}$. From $\{w_{i+j} : j \geq 0, \alpha \in w_{i+j}\} \subset \{w_{i+j} : j \geq 0, \beta \in w_{i+j}\}$ it follows that for every $j \geq 0$ if $\alpha \in w_{i+j}$, then $\beta \in w_{i+j}$. Thus, by Lemma 3.3, if $\alpha \in w_{i+j}$, then $\alpha \to \beta \in w_{i+j}$. Similarly, if $\alpha \notin w_{i+j}$, i.e., $\neg\alpha \in w_{i+j}$ it follows that $\alpha \to \beta \in w_{i+j}$. Hence, for every $j \geq 0$, $\alpha \to \beta \in w_{i+j}$. By the definition of W, for every $j \geq 0$, $\bigcirc^{i+j}(\alpha \to \beta) \in w_0$, and $\bigcirc^j(\alpha \to \beta) \in w_i$. Since w_i is maximal and consistent, it follows that $G(\alpha \to \beta) \in w_i$. Otherwise, $\neg G(\alpha \to \beta) \in w_i$ by maximality of w_i and there is $j_0 \geq 0$ such that $\neg\bigcirc^{j_0}(\alpha \to \beta) \in w_i$ which leads to a contradiction. By the axiom A11, $P_{\geq 1}(\alpha \to \beta) \in w_i$. Since, $P_{\geq 1}(\alpha \to \beta) \to P_{\geq s}\alpha \to P_{\geq s}\beta$ is a theorem of the axiom system, by Lemma 3.3(5) for every $s \in [0,1]_Q$, $P_{\geq s}\alpha \to P_{\geq s}\beta \in w_i$. It follows that $\sup\{s : P_{\geq s}\alpha \in w_i\} \leq \sup\{s : P_{\geq s}\beta \in w_i\}$. ∎

Next, we can prove that M_T is a pLTL$_{Meas}$-model.

LEMMA 3.7
For every $i \geq 0$, $Prob(w_i)$ is a probability space.

PROOF. Let us show that for every $i \geq 0$ the following hold:

1. $A(w_i)$ is an algebra of subsets of $W(w_i)$,
2. for any set $X \in A(w_i)$, $\mu(w_i)(X) \geq 0$ and $\mu(w_i)(W(w_i)) = 1$,
3. for any set $X \in A(w_i)$, $\mu(w_i)(X) = 1 - \mu(w_i)(W(w_i) \setminus X)$, and
4. for all disjoints sets $X, Y \in A(w_i)$, $\mu(w_i)(X \cup Y) = \mu(w_i)(X) + \mu(w_i)(Y)$.

(1) Since w_{i+j} is maximal and consistent for $j \geq 0$, by Lemma 3.3(4), $\{w_{i+j} : j \geq 0\} = \{w_{i+j} : j \geq 0, \top \in w_{i+j}\}$, and $W(w_i) \in A(w_i)$. Let $X = \{w_{i+j} : j \geq 0, \alpha \in w_{i+j}\}$. By Lemma 3.3(1), $\{w_{i+j} : j \geq 0, \neg\alpha \in w_{i+j}\} = (W(w_i) \setminus X)$. Hence, $W(w_i) \setminus X \in A(w_i)$. Similarly, by using Lemma 3.3(2), one can show that $A(w_i)$ is closed under finite union.

(2) It follows from the axiom A6 that for any set $X \in A(w_i)$, $\mu(w_i)(X) \geq 0$. Moreover, for every $j \geq 0$, $\top \in w_{i+j}$, $G\top \in w_i$ (using the inference rules R2 and R3) and therefore $P_{\geq 1}\top \in w_i$. So $\mu(w_i)(W(w_i)) = 1$.

(3) Suppose that $X = \{w_{i+j} : j \geq 0, \alpha \in w_{i+j}\}$, and $\mu(w_i)(X) = s$.

Let $s = 1$. By the definition of $P_{\leq 0}$, we get $\vdash P_{\geq 1}\alpha \leftrightarrow P_{\leq 0}\neg\alpha$. By the definition of the probabilistic operators, $P_{\leq 0}\neg\alpha = \neg P_{>0}\neg\alpha$, and $\neg P_{>0}\neg\alpha \in w_i$. Using the axiom A7′, we have that for any rational number $r \in (0,1]$, $\neg P_{\geq r}\alpha \in w_i$. Thus, if $\mu(w_i)(X) = 1$, then $\mu(w_i)(W(w_i) \setminus X) = 0$.

Let $s = 0$. It means that for any rational number $r \in (0,1]$, $\neg P_{\geq r}\alpha \in w_i$. By the definition of the probabilistic operators, $\neg P_{\leq 1-r}\neg\alpha \in w_i$, and $P_{>1-r}\neg\alpha \in w_i$. By the axiom A8′, we have that for any rational number r, $P_{\geq 1-r}\neg\alpha \in w_i$. Hence, $\mu(w_i)(W(w_i) \setminus X) = \sup\{s : P_{\geq s}\neg\alpha \in w_i\} = 1$.

Let $s \in [0,1] \setminus \{1, 0\}$. It means that for any rational number $r > s$, $\neg P_{\geq r}\alpha \in w_i$, $P_{>1-r}\neg\alpha \in w_i$, and $P_{\geq 1-r}\neg\alpha \in w_i$. Thus, $\sup\{t : P_{\geq t}\neg\alpha \in w_i\} \geq 1 - s$. On the other hand, if $\sup\{t : P_{\geq t}\neg\alpha \in w_i\} > 1 - s$, there must be some $r' \in [0,1]_Q$ such that $r' < s$ and $P_{\geq 1-r'}\neg\alpha \in w_i$. It follows that for some $r'' \in [0,1]_Q$ such that $r' < r'' < s$, $P_{>1-r''}\neg\alpha \in w_i$, and $\neg P_{\geq r''}\alpha \in w_i$, a contradiction with the assumption that $s = \sup\{t : P_{\geq t}\neg\alpha \in w_i\}$. Thus, if $\mu(w_i)(X) = s$, then $\mu(w_i)(W(w_i) \setminus X) = 1 - s$.

(4) Suppose that $X = \{w_{i+j} : j \geq 0, \alpha \in w_{i+j}\}$, $\mu(w_i)(X) = s$, $Y = \{w_{i+j} : j \geq 0, \beta \in w_{i+j}\}$, $\mu(w_i)(Y) = r$, and $X \cap Y = \emptyset$. Note that $X \subseteq (W(w_i) \setminus Y)$. Thus, $s + r \leq s + (1 - s) = 1$. Also, $(W(w_i) \setminus X) \cup (W(w_i) \setminus Y) = W(w_i)$, and $\mu(w_i)(\{w_{i+j} : j \geq 0, \neg\alpha \vee \neg\beta \in w_{i+j}\}) = 1$. By Lemma 3.3(7) we have that $P_{\geq 1}(\neg\alpha \vee \neg\beta) \in w_i$. For all rational numbers r' and s' such that $r' < r$ and $s' < s$ we have $P_{\geq s'}\alpha \in w_i$ and $P_{\geq r'}\beta \in w_i$. It follows by the axiom A9 that $P_{\geq r'+s'}(\alpha \vee \beta) \in w_i$. Hence, $s + r \leq \sup\{t : P_{\geq t}(\alpha \vee \beta) \in w_i\}$. Obviously, if $s + r = 1$, then $\mu(w_i)(X \cup Y) = 1$. Thus, suppose that $s + r < 1$, and $s + r < t_0 = \sup\{t : P_{\geq t}(\alpha \vee \beta) \in w_i\}$. For every rational number $t'' \in [s + r, t_0]$, we have $P_{\geq t''}(\alpha \vee \beta) \in w_i$. Let us choose two rational numbers $s'' > s$ and $r'' > r$ such that $s'' + r'' = t''$. We have that $\neg P_{\geq s''}\alpha \in w_i$ (by the property of the supremum), $P_{<s''}\alpha \in w_i$ (by the definition of the probabilistic operator $P_{<s''}$), $P_{\leq s''}\alpha \in w_i$ (by the axiom A8), $\neg P_{\geq r''}\beta \in w_i$, and $P_{<r''}\beta \in w_i$. It follows from the axiom A10 that $P_{<t''}(\alpha \vee \beta) \in w_i$, i.e. $\neg P_{\geq t''}(\alpha \vee \beta) \in w_i$, a contradiction with consistency of w_i. Thus, $\mu(w_i)(\{w_{i+j} : j \geq 0, \alpha \vee \beta \in w_{i+j}\}) = r + s$, i.e. $\mu(w_i)(X \cup Y) = \mu(w_i)(X) + \mu(w_i)(Y)$. ∎

THEOREM 3.8
M_T is a pLTL$_{Meas}$-model.

PROOF. From Lemma 3.7 it follows that M_T is a pLTL-model. Now, we have to prove that for every formula α, $\alpha \in w_i$ iff $w_i \models_{M_T} \alpha$. The proof is by induction on the complexity of the formulae. We omit below the obvious base case as well as the cases when the formulae are negations and conjunctions in the induction step.

If $\alpha = \bigcirc\beta$, we have $w_i \models \alpha$ iff $w_{i+1} \models \beta$ iff $\beta \in w_{i+1}$ iff $\alpha \in w_i$ (by construction of w_{i+1}).

Let $\alpha = \beta U\gamma$. Suppose that $w_i \models \beta U\gamma$. There is some $j \geq 0$ such that $w_{i+j} \models \gamma$ and for every k, $0 \leq k < j$, $w_{i+k} \models \beta$. By the induction hypothesis, $\gamma \in w_{i+j}$, for $j \geq 0$, and $\beta \in w_{i+k}$, for $0 \leq k < j$. By construction of M_T, we have $\bigcirc^j\gamma \in w_i$, for $j \geq 0$, and $\bigcirc^k\beta \in w_i$, for $0 \leq k < j$. It follows from Lemma 3.2(9) that $\beta U\gamma \in w_i$. For the other direction, assume that $\beta U\gamma \in w_i$. It follows from the axiom A5 that $F\gamma \in w_i$, i.e. that $\neg F\gamma = G\neg\gamma \notin w_i$. By construction of the model M_T, for some $j \geq 0$, $\bigcirc^j\gamma \in w_i$, i.e. $\gamma \in w_{i+j}$. Let $j_0 = \min\{j : \bigcirc^j\gamma \in w_i\}$. If $j_0 = 0$, $\gamma \in w_i$, and by the induction hypothesis $w_i \models \gamma$. It follows that $w_i \models \beta U\gamma$. Thus, suppose that $j_0 > 0$. For every j such that $0 \leq j < j_0$, $\bigcirc^j\gamma \notin w_i$, i.e. $\gamma \notin w_{i+j}$. From the axiom A4, Lemma 3.2(4), Lemma 3.2(5), and $\beta U\gamma \in w_i$ we have $\gamma \vee (\beta \wedge (\bigcirc\gamma \vee (\bigcirc\beta \wedge \ldots \wedge (\bigcirc^{j_0-1}\gamma \vee (\bigcirc^{j_0-1}\beta \wedge \bigcirc^{j_0}(\beta U\gamma) \ldots) \in w_i$. It follows that for every $j < j_0$, $\bigcirc^j\beta \in w_i$, $\beta \in w_{i+j}$, and by the induction hypothesis $w_{i+j} \models \beta$. From $w_{j_0} \models \gamma$, it follows that $w_i \models \beta U\gamma$.

Let $\alpha = P_{\geq s}\beta$. If $P_{\geq s}\beta \in w_i$, then $\sup\{r : P_{\geq r}\beta \in w_i\} = \mu(w_i)(\{w_{i+j} : j \geq 0, \beta \in w_{i+j}\}) \geq s$. By the induction hypothesis, $\mu(w_i)(\{w_{i+j} : j \geq 0, \beta \in w_{i+j}\}) = \mu(w_i)(\{w_{i+j} : j \geq 0, w_{i+j} \models \beta\})$, and $w_i \models P_{\geq s}\beta$. On the other hand, if $w_i \models P_{\geq s}\beta$, then $t = \sup\{r : P_{\geq r}\beta \in w_i\} \geq s$. If $s = t$, from Lemma 3.3(7) it follows that $P_{\geq s}\beta \in w_i$. If $s < t$, and $P_{\geq s}\beta \notin w_i$, we have that for every rational number $s' \in [s, t]$, $P_{\geq s'}\beta \notin w_i$, a contradiction. It follows that it must be $P_{\geq s}\beta \in w_i$.

Finally, we have that for every $i \geq 0$, and every formula α, $\{w_{i+j} : j \geq 0, \alpha \in w_{i+j}\} = \{w_{i+j} : j \geq 0, w_{i+j} \models \alpha\}$. Thus, for every $i \geq 0$, $A(w_i) = \{[\alpha]_{w_i} : \alpha \in \mathrm{FOR}(\mathrm{pLTL})\}$, and M_T is a pLTL$_{Meas}$-model. ∎

THEOREM 3.9 (Extended completeness theorem)
Every consistent set T of formulae is satisfiable.

PROOF. By Theorem 3.5, T can be extended to a maximal consistent set $\mathcal{T}$ which can be used in the construction of $M_{\mathcal{T}}$. By Theorem 3.8, for every formula α and every time instant w, $\alpha \in w$ iff $w \models \alpha$. Since $w_0 = \mathcal{T}$, T is satisfiable. ∎

Now, the standard completeness theorem ('every valid formula is provable') follows easily: if a formula α is not provable, the set $\{\neg\alpha\}$ is consistent, $\neg\alpha$ is satisfiable (according to Theorem 3.9), and α is not valid.

4 Periodicity

When we investigate the satisfiability of a formula α in a model M we consider only partial information about M. Namely, we are only interested in valuations of primitive propositions from α and probabilities of sets $[\beta]$, for $\beta \in Sub(\alpha)$. Let us formalize this statement and introduce a weaker version of measurable models.

Let α be a formula and $\{\beta_1, \ldots, \beta_t\}$ be the set of subformulae of α. An *atom* of α is defined as a formula $\gamma_1 \wedge \ldots \wedge \gamma_t$ where for every $i \in \{1, \ldots, t\}$, γ_i is either β_i or $\neg\beta_i$. The set of all atoms of α is denoted by $Atoms(\alpha)$. For an atom $a \in Atoms(\alpha)$, and a formula $\beta \in Sub(\alpha)$ we write $\beta \in a$ if β occurs positively in the top conjunction of a, i.e. if $a = \gamma_1 \wedge \ldots \wedge \gamma_t$, and for some i, $\beta = \gamma_i$. Let X_β denote the set $\{a \in Atoms(\alpha) : \beta \in a\}$. Observe that

1. $card(Atoms(\alpha)) \leq 2^{|\alpha|}$, because $card(Sub(\alpha)) \leq |\alpha|$,
2. different atoms are mutually exclusive, i.e. for $a, b \in Atoms(\alpha)$, if a and b are different, then $\vdash \neg(a \wedge b)$, and
3. for $i \in \{1, \ldots, t\}$, $\vdash \beta_i \leftrightarrow \bigvee\{a \in Atoms(\alpha) : \beta_i \in a\}$.

For every formula α and every pLTL$_{Meas}$-model M, in every time instant w from M exactly one atom $a \in Atoms(\alpha)$ holds. Thus, the set $Atoms(\alpha)$ induces a partition of $\{w_0, w_1, \ldots\}$ in any pLTL$_{Meas}$-model. We write $at(\alpha, w)$ to denote the unique atom from $Atoms(\alpha)$ such that $w \models_M at(\alpha, w)$.

A tuple $M = \langle W, v, Prob\rangle$ is α-measurable, or an α-model, if W, v, $W(w)$, and $\mu(w)$ satisfy the same conditions as in the case of a pLTL$_{Meas}$-model, with the additional requirement that $A(w)$ is an algebra of sets generated from the sets $[a]_w$, $a \in Atoms(\alpha)$, only. Obviously, the only change from the definition of pLTL$_{Meas}$-models is that only sets definable by atoms of α and their finite unions are measurable.

THEOREM 4.1
Let α be a pLTL-formula. (1) α is pLTL$_{Meas}$-satisfiable iff (2) there is an α-model M such that $w_0 \models_M \alpha$.

PROOF. (sketch) (1) $\Rightarrow$ (2) can be easily shown by observing that any pLTL$_{Meas}$-model can be appropriately restricted to an α-model while preserving the truth of the subformulae of α. In order to show that (2) implies (1), the extension theorem A.1 for finitely additive measures [4] (see Appendix A) is used. This allows us to establish that for every formula α satisfiable in an α-model M there is an extension M' of M which is a pLTL$_{Meas}$-model, such that α is also satisfiable in M'. ∎

Thus, we can examine the pLTL$_{Meas}$-satisfiability status of a formula α by considering only α-models.

From the computational complexity point of view, it is important that α-models are simpler than pLTL$_{Meas}$-models because in any α-model $M = \langle W, v\ Prob\rangle$ for every $w \in W$, $A(w)$ is a finite set. Note that for every $X \in A(w)$ there is a subset X_{at} of $Atoms(\alpha)$ such that $X = \bigcup\{[a]_w : a \in X_{at}\}$. For example, when $X_{at} = Atoms(\alpha)$, $X = W(w)$, whereas when $X_{at} = \emptyset$, $X = \emptyset$.

Furthermore, if $X, Y \in A(w)$, and $X \neq Y$, then there exists an atom $a \in Atoms(\alpha)$ such that either $[a]_w \subset X \setminus Y$ or $[a]_w \subset Y \setminus X$.

In the sequel, we show that a formula α is satisfiable if and only if it is satisfiable in an α-model such that the ω-sequence of the model has a finite initial sequence of time instants followed by another finite sequence of time instants which permanently repeats and in that way forms the rest of the whole time-line. The lengths of both sequences are bounded by functions of the size of the formula α.

Let α be an arbitrary formula and $M = \langle W, v, Prob\rangle$ be an α-model. We define a relation $\simeq_\alpha \subset W^2$ such that $w \simeq_\alpha u$ iff $at(\alpha, w) = at(\alpha, u)$. Note that $\simeq_\alpha$ is an equivalence relation. There is only a finite number of equivalence classes bounded by $2^{|\alpha|}$ because each time instant w satisfies a unique atom of α, and $card(Atoms(\alpha)) \leq 2^{|\alpha|}$.

We begin by showing that behaviour of subformulae of α does not depend on the time instants between certain instants that are in the relation $\simeq_\alpha$.

Let α be a formula and $M = \langle W, v, Prob\rangle$ be an α-model such that for some $i < j$:

1. $w_i \simeq_\alpha w_j$,
2. for every $k < j$ and for every $a \in Atoms(\alpha)$ if $[a]_{w_k} \cap \{w_i, \ldots, w_{j-1}\} \neq \emptyset$, then $[a]_{w_k} \cap \{w_k, \ldots, w_{i-1}, w_j, \ldots\} \neq \emptyset$ (if $i \leq k$, $\{w_k, \ldots, w_{i-1}, w_j, \ldots\}$ denotes $\{w_j, w_{j+1}, \ldots\}$).

Let $M' = \langle W', v', Prob'\rangle$ be a tuple such that:

- $W' \stackrel{\text{def}}{=} w_0, \ldots, w_{i-1}, w_j, w_{j+1}, \ldots$, and
- for $k \in \omega \setminus \{i, \ldots, j-1\}$
  1. $v'(w_k) \stackrel{\text{def}}{=} v(w_k)$, and
  2. $Prob'(w_k) = \langle W'(w_k), A'(w_k), \mu'(w_k)\rangle$, where
     (a) $W'(w_k) \stackrel{\text{def}}{=} \{w_{k+l} : l \geq 0,\ k+l \notin \{i, \ldots, j-1\}\}$,
     (b) $A'(w_k) = \{X \setminus \{w_i, \ldots, w_{j-1}\} : X \in A(w_k)\}$, and
     (c) for $X \in A(w_k)$, $\mu'(w_k)(X \setminus \{w_i, \ldots, w_{j-1}\}) = \mu(w_k)(X)$.

The first point that we have to guarantee is that the definition of μ' is correct. This is done in the next lemmas.

LEMMA 4.2
For $k \in \omega \setminus \{i, \ldots, j-1\}$, for $X, Y \in A(w_k)$, $X \setminus \{w_i, \ldots, w_{j-1}\} = Y \setminus \{w_i, \ldots, w_{j-1}\}$ implies $\mu(w_k)(X) = \mu(w_k)(Y)$.

PROOF. Assume $X, Y \in A(w_k)$ and $X \setminus \{w_i, \ldots, w_{j-1}\} = Y \setminus \{w_i, \ldots, w_{j-1}\}$. We consider only $k < i$, because for $k \geq j$, $Prob(w_k)$ and $Prob'(w_k)$ coincide. Since M is an α-model, there are atoms $a_1, \ldots, a_s, b_1, \ldots, b_{s'}$ in $Atoms(\alpha)$ such that $X = [a_1]_{w_k} \cup \ldots \cup [a_s]_{w_k}$ and $Y = [b_1]_{w_k} \cup \ldots \cup [b_{s'}]_{w_k}$. Suppose that $\mu(w_k)(X) \neq \mu(w_k)(Y)$. It entails that $X \neq Y$. Then, there exists some atom $a \in Atoms(\alpha)$ such that either $[a]_{w_k} \subset X \setminus Y$ or $[a]_{w_k} \subset Y \setminus X$, and $\mu(w_k)([a]_{w_k}) \neq 0$. Hence, $[a]_{w_k} \neq \emptyset$. If $[a]_{w_k} \not\subset \{w_i, \ldots, w_{j-1}\}$, then it follows that $(X \setminus \{w_i, \ldots, w_{j-1}\}) \neq (Y \setminus \{w_i, \ldots, w_{j-1}\})$, a contradiction. Thus, it must be that $[a]_{w_k} \subset \{w_i, \ldots, w_{j-1}\}$. Since $[a]_{w_k} \neq \emptyset$, $[a]_{w_k} \cap \{w_i, \ldots, w_{j-1}\} \neq \emptyset$ and therefore by assumption on M', $[a]_{w_k} \cap \{w_k, \ldots, w_{i-1}, w_j, \ldots\} \neq \emptyset$, a contradiction. Hence, $\mu(w_k)(X) = \mu(w_k)(Y)$. ∎

LEMMA 4.3
Every $\mu'(w_k)$ is a finitely additive probability.

PROOF. It is easy to prove that for any $k \geq 0$, $A'(w_k)$ is an algebra. By Lemma 4.2 all $\mu'(w_k)$ are well defined. On the other hand, since $\mu(w_k)$ are finitely additive probabilities, the same holds for all $\mu'(w_k)$. For example, one can show that $(X \setminus \{w_i, \ldots, w_{j-1}\}) \cap (Y \setminus \{w_i, \ldots, w_{j-1}\}) = \emptyset$ iff $X \cap Y = \emptyset$, and that for all disjoint sets $X', Y' \in A'(w_k)$, $\mu'(w_k)(X \cup Y) = \mu'(w_k)(X) + \mu'(w_k)(Y)$. ∎

It follows from the Lemmas 4.2 and 4.3 that M' is a pLTL-model. The main result related to the construction of M' is the next theorem.

THEOREM 4.4
For $k \in \omega \setminus \{i, \ldots, j-1\}$, for $\beta \in Sub(\alpha)$, $w_k \models_M \beta$ iff $w_k \models_{M'} \beta$.

PROOF. The proof is by induction on the structure of α.
Base case: If $\beta = p$ is an atomic formula, then by definition of v', $w_k \models_M p$ iff $w_k \models_{M'} p$.
Induction step: the cases when β is a conjunction or a negated formula are by an easy verification. For the other cases, if $k \geq j$, the statement trivially holds. Thus, let $k < j$, i.e. $k \leq i-1$.
Case 1: $\beta = \bigcirc\beta_1$.
If $k \neq i-1$, $w_k \models_M \beta$ iff $w_{k+1} \models_M \beta_1$ iff $w_{k+1} \models_{M'} \beta_1$ (by the induction hypothesis) iff $w_k \models_{M'} \beta$. Furthermore, $w_{i-1} \models_M \beta$ iff $w_i \models_M \beta_1$ iff $w_j \models_M \beta_1$ ($w_i \simeq_\alpha w_j$) iff $w_j \models_{M'} \beta_1$ iff $w_{i-1} \models_{M'} \beta$ ($W' = \ldots w_{i-1} w_j \ldots$).
Case 2: $\beta = \beta_1 U \beta_2$.
Let $w_k \models_M \beta_1 U \beta_2$. It means that there is some $l \geq k$ such that $w_l \models_M \beta_2$, and for every m, $k \leq m < l$, $w_m \models_M \beta_1$. If $l \leq i-1$, by the induction hypothesis, $w_l \models_{M'} \beta_2$, and for every m, $k \leq m < l$, $w_m \models_{M'} \beta_1$. Thus, $w_k \models_{M'} \beta_1 U \beta_2$. If $l \geq j$, then we have $w_j \models_M \beta_1 U \beta_2$. By the induction hypothesis $w_l \models_{M'} \beta_2$, for every m, $k \leq m < l$, $m \notin \{i, \ldots, j-1\}$, $w_m \models_{M'} \beta_1$, and $w_k \models_{M'} \beta_1 U \beta_2$. Finally, if $i \leq l \leq j-1$, we have $w_i \models_M \beta_1 U \beta_2$, and $w_j \models_M \beta_1 U \beta_2$ ($w_i \simeq_\alpha w_j$). Reasoning as above, we conclude that $w_k \models_{M'} \beta_1 U \beta_2$.

For the other direction, suppose that $w_k \models_{M'} \beta_1 U \beta_2$. Then, there is some $l \geq k$ such that $w_l \models_{M'} \beta_2$, and for every m, $k \leq m < l$, $w_m \models_{M'} \beta_1$. If $l \leq i-1$, using the induction hypothesis, we have $w_l \models_M \beta_2$, for every m, $k \leq m < l$, $w_m \models_M \beta_1$, and $w_k \models_M \beta_1 U \beta_2$. If $l \geq j$, it follows that $w_j \models_{M'} \beta_1 U \beta_2$, $w_j \models_M \beta_1 U \beta_2$, and $w_i \models_M \beta_1 U \beta_2$ ($w_i \simeq_\alpha w_j$). It means that there is some $l' \geq i$ such that $w_{l'} \models_M \beta_2$, and for every m, $i \leq m < l'$, $w_m \models_M \beta_1$. From the assumption we also have that for every m, $k \leq m < i$, $w_m \models_{M'} \beta_1$ and, by the induction hypothesis, $w_m \models_M \beta_1$, for every m, $k \leq m < i$. Thus, for every m, $k \leq m < l'$, $w_m \models_M \beta_1$, and $w_{l'} \models_M \beta_2$. It follows that $w_k \models_M \beta_1 U \beta_2$.
Case 3: $\beta = P_{\geq s}\beta_1$.
$w_k \models_M P_{\geq s}\beta_1$ iff $\mu(w_k)(\{w_{k+l} : l \geq 0, w_{k+l} \models_M \beta_1\}) \geq s$ iff $\mu'(w_k)(\{w_{k+l} : l \geq 0, w_{k+l} \models_M \beta_1\} \setminus \{w_i, \ldots, w_{j-1}\}) \geq s$ (by definition of μ') iff $\mu'(w_k)(\{w_{k+l} : l \geq 0,\ k+l \notin \{i, \ldots, j-1\}, w_{k+l} \models_{M'} \beta_1\}) \geq s$ (by the induction hypothesis) iff $w_k \models_{M'} P_{\geq s}\beta_1$. ∎

It is important to remark here that the definition of M' does not straightforwardly entail that M' is an α-model. Indeed one has to check that $Prob'(w_k)$ is a probability space such that the elements of $A'(w_k)$ are of the form $[a_{i_1}]_{w_k} \cup \ldots \cup [a_{i_l}]_{w_k}$ where the $a_{i_1}, \ldots, a_{i_l} \in Atoms(\alpha)$.

LEMMA 4.5
M' is an α-model.

PROOF. We have to prove that $A'(w_k) = \{[a_{i_1}]_{w_k} \cup \ldots \cup [a_{i_l}]_{w_k} : a_{i_1}, \ldots, a_{i_l} \in Atoms(\alpha)\}$. Let $X' \in A'(w_k)$. By definition of $A'(w_k)$, there is $X \in A(w_k)$ such that $X' = X \setminus \{w_i, \ldots, w_{j-1}\}$. Since M is an α-model, there are atoms $a_{i_1}, \ldots, a_{i_l}$ such that $X = [a_{i_1}]_{M,w_k} \cup \ldots \cup [a_{i_l}]_{M,w_k}$. Note that by Theorem 4.4, for every $a \in Atoms(\alpha)$, $w_k \models_M a$ iff $w_k \models_{M'} a$. Consequently, for every $a \in Atoms(\alpha)$, $[a]_{M',w_k} = [a]_{M,w_k} \setminus \{w_i, \ldots, w_{j-1}\}$, and

$$\begin{aligned} X' &= [a_{i_1}]_{M,w_k} \setminus \{w_i, \ldots, w_{j-1}\} \cup \ldots \cup [a_{i_l}]_{M,w_k} \setminus \{w_i, \ldots, w_{j-1}\} \\ &= [a_{i_1}]_{M',w_k} \cup \ldots \cup [a_{i_l}]_{M',w_k}. \end{aligned}$$

∎

An α-model M is *ultimately periodic* (w.r.t. formula α) with starting index i and period m if for every $k \geq i$:

1. $v(w_k) = v(w_{k+m})$,
2. $A(w_{k+m}) = \{\{w_{k'+m} : w_{k'} \in X\} : X \in A(w_k)\}$, and
3. for $X \in A(w_k)$, $\mu(w_{k+m})(\{w_{k'+m} : w_{k'} \in X\}) = \mu(w_k)(X)$.

A *pseudo ultimately periodic* α-model is defined as an ultimately periodic α-model except that the above conditions 1-3 are satisfied but we only require that for $k < i + m$, $\mu(w_k)$ is a map $\mu(w_k) : A(w_k) \to [0, 1]$ not necessarily a finitely additive probability measure. The role of *pseudo ultimately periodic* α-models is similar to the one of *pre models* for linear-time temporal logic (see [30]).

Our next step is to prove that an α-model can be transformed into an ultimately periodic α-model, while the behaviour of the subformulae of α is preserved for time instants before w_{i+m}, where i and m denotes the starting index and the period, respectively.

Let $M = \langle W, v, Prob\rangle$ be an α-model, $i \geq 0$, and $m > 0$ such that:

- $w_i \simeq_\alpha w_{i+m}$, and
- for every atom $a \in Atoms(\alpha)$ and every $k < i+m$ if $[a]_{w_k} \neq \emptyset$, then $[a]_{w_k} \cap (\{w_k, \ldots, w_{i-1}\} \cup \{w_i, \ldots, w_{i+m-1}\}) \neq \emptyset$.

Observe that in the case when $k \geq i$, the set $\{w_k, \ldots, w_{i-1}\} \cup \{w_i, \ldots, w_{i+m-1}\}$ is not equal to $\{w_k, \ldots, w_{i+m-1}\}$.

Let $M'' = \langle W'', v'', Prob''\rangle$ be the pseudo ultimately periodic α-model built from M such that:

- $W'' \stackrel{\text{def}}{=} w''_0, w''_1, \ldots$,
- for every $k < i + m$, $v''(w''_k) \stackrel{\text{def}}{=} v(w_k)$,
- for every $k \geq i + m$, $v''(w''_k) \stackrel{\text{def}}{=} v''(w''_{k-m})$, and
- $W''(w''_k) \stackrel{\text{def}}{=} \{w''_{k+l} : l > 0\}$, while

$A''(w''_k)$ and $\mu''(w''_k)$ are defined as follows. For $k < i + m$, we label each time instant w''_k with $l(w''_k) \stackrel{\text{def}}{=} at(\alpha, w_k)$, while for $k \geq i + m$, $l(w''_k) \stackrel{\text{def}}{=} l(w''_{k-m})$. For $Y \subseteq Atoms(\alpha)$, let $X_{w''_k,Y}$ denote the set $\{w''_{k+l} : l \geq 0,\ l(w''_{k+l}) \in Y\}$. Obviously, for every $k \in \omega$, we have:

- if $a, b \in Atoms(\alpha)$ are different, $X_{w''_k,\{a\}} \cap X_{w''_k,\{b\}} = \emptyset$, and
- $\bigcup_{a \in Atoms(\alpha)} X_{w''_k,\{a\}} = W''(w''_k)$.

The set $A''(w''_k)$ is defined as $A''(w''_k) \stackrel{\text{def}}{=} \{X_{w''_k,Y} : Y \subseteq Atoms(\alpha)\}$. For $k < i+m$, $\mu''(w''_k)(X_{w''_k,Y}) \stackrel{\text{def}}{=} \Sigma_{a \in Y}\mu(w_k)([a]_{w_k})$. Furthermore, for $k \geq i + m$, and $Y \subseteq Atoms(\alpha)$,

$$\mu''(w''_k)(X_{w''_k,Y}) \stackrel{\text{def}}{=} \mu''(w''_{k-m})(X_{w''_{k-m},Y}).$$

Again, we have to prove that the definition of $\mu''(w''_k)$ is correct.

LEMMA 4.6
Let $k \in \omega$, $X_{w''_k,Y_1}, X_{w''_k,Y_2} \in A''(w''_k)$. If $X_{w''_k,Y_1} = X_{w''_k,Y_2}$, then $\mu''(w''_k)(X_{w''_k,Y_1}) = \mu''(w''_k)(X_{w''_k,Y_2})$.

PROOF. It is sufficient to consider the case $k < i + m$. For the time instants w''_k with $k \geq i + m$, we use the properties of w''_{k-m}. Suppose that $\mu''(w''_k)(X_{w''_k,Y_1}) \neq \mu''(w''_k)(X_{w''_k,Y_2})$, i.e. $\Sigma_{a \in Y_1}\mu(w_k)([a]_{w_k}) \neq \Sigma_{a \in Y_2}\mu(w_k)([a]_{w_k})$. Consequently, there is an atom $a \in Atoms(\alpha)$ such that $\mu(w_k)([a]_{w_k}) \neq 0$ (and therefore $[a]_{w_k} \neq \emptyset$) and either $a \in (Y_1 \setminus Y_2)$ or $a \in (Y_2 \setminus Y_1)$. By the assumption on M, there is $l \in \{k, \dots, i-1\} \cup \{i, \dots, i+m-1\}$ such that $w''_l \in [a]_{w_k}$. Hence, either $w''_l \in (X_{w''_k,Y_1} \setminus X_{w''_k,Y_2})$ or $w''_l \in (X_{w''_k,Y_2} \setminus X_{w''_k,Y_1})$, which leads to a contradiction. ∎

LEMMA 4.7
For $k \in \omega$, for $X \in A''(w''_k)$, if $\mu''(w''_k)(X) > 0$, then $X \neq \emptyset$.

PROOF. As in the proof of Lemma 4.6, it is sufficient to consider the case $k < i + m$. Assume $X = X_{w''_k,Y}$ and $\mu''(w''_k)(X) > 0$. Hence, there is $a \in Y$ such that $\mu(w_k)([a]_{w_k}) \neq 0$ and $[a]_{w_k} \neq \emptyset$. By assumption on M, there is $l \in \{k, \dots, i-1\} \cup \{i, \dots, i+m-1\}$ such that $w''_l \in [a]_{w_k}$. Hence $w''_l \in X$. ∎

Now, it is easy to show

LEMMA 4.8
For $k \in \omega$, $Prob''(w_k)$ is a probability space.

By Lemma 4.6, Lemma 4.7, and Lemma 4.8, and using the fact that M is an α-model, we can show the following result.

THEOREM 4.9
For $k \in \omega$, and $\beta \in Sub(\alpha)$:

(1) for every $k < i + m$, $w_k \models_M \beta$ iff $w''_k \models_{M''} \beta$;
(2) for every $k \geq i$, $w''_k \models_{M''} \beta$ iff $w''_{k+m} \models_{M''} \beta$.

PROOF. The proof of (1) and (2) is by simultaneous induction on the complexity of subformulae of the formula α.
Base case: β is a propositional variable occurring in α. (1) and (2) hold by a simple inspection of the definition of M''.
Induction step: the cases when β is a conjunction or a negated formula are by an easy verification. Recall that $X_\gamma = \{a \in Atoms(\alpha) : \gamma \in a\}$, and that for $\gamma \in Sub(\alpha)$, $\vdash \gamma \leftrightarrow \bigvee_{a \in X_\gamma} a$.
Case 1: $\beta = \bigcirc\beta_1$.
(1) If $k \in \{0, \dots, i+m-2\}$, $w_k \models_M \beta$ iff $w_{k+1} \models_M \beta_1$ iff $w''_{k+1} \models_{M''} \beta_1$ (by the induction hypothesis with (1)) iff $w''_k \models_{M''} \beta$. If $k = i + m - 1$, then $w_k \models_M \beta$ iff $w_{i+m} \models_M \beta_1$ iff $w_i \models_M \beta_1$ ($w_i \simeq_\alpha w_{i+m}$) iff $w''_i \models_{M''} \beta_1$ (by the induction hypothesis with (1)) iff $w''_{i+m} \models_{M''} \beta_1$ (by the induction hypothesis with (2)) iff $w''_k \models_{M''} \beta$.
(2) Let $k \geq i$. $w''_k \models_{M''} \beta$ iff $w''_{k+1} \models_{M''} \beta_1$ iff $w''_{k+m+1} \models_{M''} \beta_1$ (by the induction hypothesis with (2)) iff $w''_{k+m} \models_{M''} \beta$.
Case 2: $\beta = \beta_1 U \beta_2$.
(1) Let $k \in \{0, \dots, i+m-1\}$ and assume that $w_k \models_M \beta$. It means that there is some $j \geq k$ such that $w_j \models_M \beta_2$, and for every l, $k \leq l < j$, $w_l \models_M \beta_1$. If $j < i + m$, by the induction hypothesis with (1), $w''_j \models_{M''} \beta_2$, and for every l, $k \leq l < j$, $w''_l \models_{M''} \beta_1$, and $w''_k \models_{M''} \beta_1 U \beta_2$. Next, suppose that $i + m \leq j$. It follows that $w_j \models_M \beta_2$, and for every l, $k \leq l < m$ or $m \leq l < j$, $w_l \models_M \beta_1$,

$w_{i+m} \models_M \beta_1 U \beta_2$, and $w_i \models_M \beta_1 U \beta_2$ ($w_i \simeq_\alpha w_{i+m}$). Obviously, $[\beta_2]_{w_i} \neq \emptyset$, and there is some $a \in X_{\beta_2}$ such that $[a]_{w_i} \neq \emptyset$. By the assumptions about M, $[a]_{w_i} \cap \{w_i, \dots, w_{i+m-1}\} \neq \emptyset$. Let n be the smallest number such that $n \in \{i, \dots, i+m-1\}$, $w_n \models_M \beta_2$, and for every l, $i \leq l < n$, $w_l \models_M \beta_1$. If $k \leq n$, then using the induction hypothesis with (1) as above, we conclude that $w''_k \models_{M''} \beta_1 U \beta_2$. Now, suppose that $i \leq n < k < i+m$. Using the induction hypothesis with (1) we have that for every l, $i \leq l < n$, $w''_l \models_{M''} \beta_1$, and by the induction hypothesis with (2), $w''_{n+m} \models_{M''} \beta_2$, and for every l, $i \leq l < n$, $w''_{l+m} \models_{M''} \beta_1$. By the induction hypothesis with (1), we also have that for every l, $k \leq l < m$, $w''_l \models_{M''} \beta_1$. Thus, $w''_{n+m} \models_{M''} \beta_2$, and for every l, $k \leq l < n+m$, $w''_l \models_{M''} \beta_1$, and $w''_k \models_{M''} \beta_1 U \beta_2$.

For the other direction, suppose that $w''_k \models_{M''} \beta_1 U \beta_2$, namely that there is some $j \geq k$ such that $w''_j \models_{M''} \beta_2$, and for every l, $k \leq l < j$, $w''_l \models_{M''} \beta_1$. If $j < i+m$, then using the induction hypothesis with (1), it follows that $w_j \models_M \beta_2$, and that for every l, $k \leq l < j$, $w_l \models_M \beta_1$. Thus, $w_k \models_M \beta_1 U \beta_2$. If $j \geq i+m$, by the induction hypothesis with (2), $w''_{j-m} \models_{M''} \beta_2$. If $k \leq j-m$, then using the induction hypothesis with (1) as above, it follows that $w_k \models_M \beta_1 U \beta_2$. Finally, let $i \leq j-m < k$. Using the induction hypothesis with (2), we have that for every l, $i \leq l < j-m$, $w''_l \models_{M''} \beta_1$, and by the induction hypothesis with (1), that $w_l \models_M \beta_1$. Thus, $w_i \models_M \beta_1 U \beta_2$, and $w_m \models_M \beta_1 U \beta_2$ ($w_i \simeq_\alpha w_{i+m}$). Using the induction hypothesis with (1), from the above assumption we have that for every l, $k \leq l < m$, $w_l \models_M \beta_1$, and conclude that $w_k \models_M \beta_1 U \beta_2$.

(2) Let $k \geq i$. $w''_k \models_{M''} \beta$ iff there is some $j \geq k$ such that $w''_j \models_{M''} \beta_2$, and for every l, $k \leq l < j$, $w''_l \models_{M''} \beta_1$. By the induction hypothesis with (2), we have that $w''_{j+m} \models_{M''} \beta_2$, and for every l, $k \leq l < j$, $w''_{l+m} \models_{M''} \beta_1$. Thus, $w''_{k+m} \models_{M''} \beta$.

Case 3: $\beta = P_{\geq s}\beta_1$.

(1) Let $k \in \{0, \dots, i+m-1\}$.

We have $w_k \models_M \beta$ iff $\mu(w_k)([\beta_1]_{w_k}) \geq s$ iff $\Sigma_{a \in X_{\beta_1}} \mu(w_k)([a]_{w_k}) \geq s$. By definition of μ'', $\mu''(w''_k)(X_{w''_k, X_{\beta_1}}) = \Sigma_{a \in X_{\beta_1}} \mu(w_k)([a]_{w_k})$. By the induction hypothesis with (1) and (2), $X_{w''_k, X_{\beta_1}} = \{w''_{k+l} : l \geq 0, w''_{k+l} \models_{M''} \beta_1\}$. Consequently, $\Sigma_{a \in X_{\beta_1}} \mu(w_k)([a]_{w_k}) \geq s$ iff $\mu''(w''_k)$ $(\{w''_{k+l} : l \geq 0, w''_{k+l} \models_{M''} \beta_1\}) \geq s$ which is precisely equivalent to $w''_k \models_{M''} P_{\geq s}\beta_1$.

(2) Let $k \geq i$. $w''_k \models_{M''} \beta$ iff (i) $\mu''(w''_k)(\{w''_{k+l} : l \geq 0, w''_{k+l} \models_{M''} \beta_1\}) \geq s$. By the induction hypothesis with (1) and (2), $X_{w''_k, X_{\beta_1}} = \{w''_{k+l} : l \geq 0, w''_{k+l} \models_{M''} \beta_1\}$ and $X_{w''_{k+m}, X_{\beta_1}} = \{w''_{k+m+l} : l \geq 0, w''_{k+m+l} \models_{M''} \beta_1\}$. By definition of μ'', $\mu''(w''_{k+m})(X_{w''_{k+m}, X_{\beta_1}}) = \mu''(w''_k)$ $(X_{w''_k, X_{\beta_1}})$. Hence (i) iff $\mu''(w''_{k+m})(X_{w''_{k+m}, X_{\beta_1}}) \geq s$. ∎

LEMMA 4.10
M'' is an ultimately a periodic α-model with starting index i and period m.

PROOF. It follows from Theorem 4.9 that for every $k \in \omega$,$w''_k \models_{M''} l(w''_k)$. This guarantees that for $Y \subseteq Atoms(\alpha)$, $X_{w''_k, Y} = \{w''_{k+l} : l \geq 0, w''_{k+l} \models \bigvee_{a \in Y} a\}$. It is obvious from the construction of M'' that it is ultimately periodic model with the starting index i and the period m. ∎

Theorem 4.11 below suggests that it is possible to control the satisfaction of formulae by identifying periodic situations in finite time.

THEOREM 4.11
Every pLTL$_{Meas}$-satisfiable formula α is satisfiable in an ultimately periodic α-model with starting index i and period p such that $i \leq 2^{2|\alpha|} + 1$, and $p \leq (2^{|\alpha|} + 1) \times 2^{|\alpha|}$.

PROOF. Assume that α is pLTL$_{Meas}$-satisfiable. Since the considered language contains only future temporal operators, without any loss of generality we can suppose that α holds at the first position of a pLTL$_{Meas}$-model. Thus, using Theorem 4.1, there is an α-model $M = \langle W, v, Prob \rangle$ such that

$w_0 \models_M \alpha$. The proof of the theorem consists in finding an initial sequence of the ω-sequence of M and a period of time line that satisfy the required bounds.

We write $\mathtt{InfAt}$ (resp. $\mathtt{FinAt}$) to denote the set of atoms a of $Atoms(\alpha)$ such that $[a]_{w_0}$ is infinite (resp. finite). For $a \in \mathtt{FinAt}$, we write $k(a)$ to denote the unique natural number such that $w_{k(a)} \models_M a$ and $[a]_{w_{k(a)}} = \emptyset$. In words, $k(a)$ is the index of the last time instant where a holds true.

Let $a, b \in \mathtt{FinAt}$ such that $a \neq b$. Without any loss of generality, we can assume that $k(a) < k(b)$. Assume that $k(b) - k(a) \geq 2^{|\alpha|} + 1$ and there is no a' in $\mathtt{FinAt}$ such that $k(a) < k(a') < k(b)$. So, there exist $i < j \in \{k(a)+1, \ldots, k(b)\}$ such that $w_i \simeq_\alpha w_j$. By construction, for every $k < j$ and for every $c \in Atoms(\alpha)$ if $[c]_{w_k} \cap \{w_i, \ldots, w_{j-1}\} \neq \emptyset$, then $[c]_{w_k} \cap \{w_{k+1}, \ldots, w_{i-1}, w_j, \ldots\} \neq \emptyset$. So let M' be the model obtained from M by deleting $w_i, \ldots, w_{j-1}$. M' is an α-model and $w_0 \models_{M'} \alpha$ (by Theorem 4.4). Similarly, if $a \in \mathtt{FinAt}$, $k(a) \geq 2^{|\alpha|} + 1$ and there is no $a' \in \mathtt{FinAt}$ such that $k(a') < k(a)$, then the deletion of time instants before $w_{k(a)}$ can also be performed. Details are omitted to avoid the boredom of repetitive arguments. By applying such deletions as much as possible, we can easily show that there is an α-model M such that

(i) $w_0 \models_M \alpha$,

(ii) for $a \in \mathtt{FinAt}$, if there is no $a' \in \mathtt{FinAt}$ such that $k(a') < k(a)$, then $k(a) \leq 2^{|\alpha|}$, and

(iii) for $a, b \in \mathtt{FinAt}$ such that $a \neq b$, if $k(a) < k(b)$ and there is no a' in $\mathtt{FinAt}$ such that $k(a) < k(a') < k(b)$, then $k(b) - k(a) \leq 2^{|\alpha|}$.

Let a_{fin} be the unique atom of $\mathtt{FinAt}$ such that for all $k > k(a_{fin})$, $at(w_k, \alpha) \in \mathtt{InfAt}$. If $\mathtt{FinAt} = \emptyset$, then by convention $k(a_{fin}) = 0$. By construction, there are at most $card(Atoms(\alpha))$ successive intervals of at most $card(Atoms(\alpha))$ time instants before $k(a_{fin})$, i.e. $k(a_{fin}) \leq 2^{|\alpha|} \times 2^{|\alpha|}$.

For $a \in \mathtt{InfAt}$, we write $k(a)$ to denote the unique natural number such that $w_{k(a)} \models a$ and for $k \in \{k(a_{fin})+1, \ldots, k(a)-1\}$, $at(w_k, \alpha) \neq a$. In words, $k(a)$ is the index of the first time instant after $w_{k(a_{fin})}$ where a holds true.

Let $a, b \in \mathtt{InfAt}$ such that $a \neq b$. Without any loss of generality, we can assume that $k(a) < k(b)$. Assume that $k(b) - k(a) \geq 2^{|\alpha|} + 1$ and there is no a' in $\mathtt{InfAt}$ such that $k(a) < k(a') < k(b)$. So, there exist $i < j \in \{k(a)+1, \ldots, k(b)\}$ such that $w_i \simeq_\alpha w_j$. By construction, for every $k < j$ and for every $a' \in Atoms(\alpha)$ if $[a]_{w_k} \cap \{w_i, \ldots, w_{j-1}\} \neq \emptyset$, then $[a]_{w_k} \cap \{w_k, \ldots, w_{i-1}, w_j, \ldots\} \neq \emptyset$. So let M' be the model obtained from M by deleting $w_i, \ldots, w_{j-1}$. M' is an α-model and $w_0 \models_{M'} \alpha$ (by Theorem 4.4). By applying such deletions as much as possible, we can easily show that there is an α-model M such that (i), (ii), (iii), and

(iv) for $a, b \in \mathtt{InfAt}$ such that $a \neq b$, if $k(a) < k(b)$ and there is no a' in $\mathtt{InfAt}$ such that $k(a) < k(a') < k(b)$, then $k(b) - k(a) \leq 2^{|\alpha|}$.

Let a_{inf} be the unique atom of $\mathtt{InfAt}$ such that for all $a \in \mathtt{InfAt}$, $k(a) \leq k(a_{inf})$. By construction, there are at most $card(Atoms(\alpha))$ successive intervals of at most $card(Atoms(\alpha))$ time instants between $k(a_{inf})$ and $k(a_{fin})$. Thus, $k(a_{inf}) - k(a_{fin}) \leq 2^{|\alpha|} \times 2^{|\alpha|} + 1$.

Let a_{final} be the unique atom of $\mathtt{InfAt}$ such that $a_{final} = at(w_{k(a_{fin})+1}, \alpha)$. Let k' be the smallest natural number such that $k' > k(a_{inf})$ and $at(w_{k'}, \alpha) = a_{final}$. If $k' - k(a_{inf}) \geq 2^{|\alpha|} + 1$, then there exist $i < j \in \{k(a_{inf})+1, \ldots, k'\}$ such that $w_i \simeq_\alpha w_j$. By construction, for every $k < j$ and for every $a' \in Atoms(\alpha)$ if $[a]_{w_k} \cap \{w_i, \ldots, w_{j-1}\} \neq \emptyset$, then $[a]_{w_k} \cap \{w_k, \ldots, w_{i-1}, w_j, \ldots\} \neq \emptyset$. So let M' be the model obtained from M by deleting $w_i, \ldots, w_{j-1}$. M' is an α-model and $w_0 \models_{M'} \alpha$ (by Theorem 4.4). By applying such deletions as much as possible, we can easily show that there is an α-model M such that (i), (ii), (iii), (iv) and

(v) $k' - k(a_{inf}) \leq 2^{|\alpha|}$.

By construction, $w_{k(a_{fin})+1} \simeq_\alpha w_{k'}$ and $\texttt{InfAt} = \{at(w_{k''}, \alpha) : k(a_{fin}) + 1 \leq k'' < k'\}$. Let $i = k(a_{fin}) + 1$, and $p = k' - i = k' - (k(a_{fin}) + 1)$. Obviously, $i \leq 2^{|\alpha|} \times 2^{|\alpha|} + 1$, and $p \leq (2^{|\alpha|} + 1) \times 2^{|\alpha|}$. By construction, for $a \in Atoms(\alpha)$, and $k < i + p$, if $[a]_{w_k} \neq \emptyset$, then $[a]_{w_k} \cap (\{w_k, \dots, w_{i-1}\} \cup \{w_i, \dots, w_{i+p-1}\}) \neq \emptyset$. Using Lemma 4.10 and Theorem 4.9, α is satisfiable in an ultimately periodic α-model with starting index i and period p. ∎

5 Decidability: towards a computational complexity characterization

In this section, we shall prove decidability of the pLTL$_{Meas}$, and provide its worst-case complexity upper bound, while the lower bound is a simple consequence of [44].

THEOREM 5.1 (Decidability theorem)
The pLTL$_{Meas}$ is decidable.

PROOF. It is proved in Theorem 4.11 that a formula α is satisfiable if and only if it is satisfied in an ultimately periodic α-model whose starting index and period are of bounded sizes. Observe that it does not necessary imply decidability of the pLTL$_{Meas}$ because there are infinitely many such models. Nevertheless, the next procedure decides the satisfiability problem.

Recall that $X_\gamma = \{a \in Atoms(\alpha) : \gamma \in a\}$. Let us call formulae of the form $P_{\geq s}\beta$ and $\neg P_{\geq s}\beta$ that appear in the top conjunction of $a \in Atoms(\alpha)$ positive and negative probabilistic constraints of a, respectively.

Using probabilistic constraints we shall examine whether there is an ultimately periodic α-model $M = \langle W, v, Prob\rangle$ with the corresponding starting index $i \leq 2^{2|\alpha|} + 1$, and period $p \leq (2^{|\alpha|} + 1) \times 2^{|\alpha|}$ such that $w_0 \models \alpha$. The next procedure is applied for every such pair i, p.

We consider a tuple $S_{i,p} = \langle a_0, \dots, a_{i+p-1}\rangle$ of $i + p$ (not necessarily distinct) atoms such that $a_0 \in X_\alpha$. We imagine that every atom $a_k \in S_{i,p}$ corresponds to the time instant w_k of the model we try to find. It means that a_k contains all subformulae of α that hold in w_k. Since $w_k \simeq_\alpha w_{k+lp}$ for $k \geq i$, $l \geq 0$, a_k also corresponds to w_{k+lp}. Thus, if $k \geq i$, $l \geq 0$, every a_{k+lp} denotes the same atom a_k. Our goal is to check whether it is possible that $w_k \models a_k$ for every $k \geq 0$. Note that we consider ultimately periodic models, i.e. that for every $k \geq i + p$, $a_k = a_{k-p}$, and that this examination concerns at most $i + p$ pairs (w_k, a_k).

For every time instant w_k, $k < i+p$ and every propositional variable q, we assume that $q \in v(w_k)$ iff $q \in a_k$. Thus, $w_k \models q$ iff $q \in a_k$. We can easily verify that a_k is propositionally consistent, i.e. that for $\neg\beta \in Sub(\alpha)$, $\neg\beta \in a_k$ iff $\beta \notin a_k$, and that for $\beta_1 \wedge \beta_2 \in Sub(\alpha)$, $\beta_1 \wedge \beta_2 \in a_k$ iff $\beta_1 \in a_k$ and $\beta_2 \in a_k$. Obviously, the above check can be done in a finite number of steps.

We do not try to determine probabilities precisely. Rather, we just check whether there are probabilities such that probabilistic constraints are satisfied in the corresponding time instants. To do that, for every time instant w_k, $k < i + p$, we consider the set of all positive and negative probabilistic constraints that appear in a_k, and a system of linear equalities and inequalities of the form:

$$
\begin{array}{rcll}
\displaystyle\sum_{a \in \mathrm{FutAt}(a_k)} \mu(w_k)(a) & = & 1 & \\
\mu(w_k)(a) & \geq & 0 & \text{, for every } a \in \mathrm{FutAt}(a_k) \\
\displaystyle\sum_{a \in X_\beta \cap \mathrm{FutAt}(a_k)} \mu(w_k)(a) & \geq & s & \text{, for every } P_{\geq s}\beta \in a_k \\
\displaystyle\sum_{a \in X_\beta \cap \mathrm{FutAt}(a_k)} \mu(w_k)(a) & < & s & \text{, for every } \neg P_{\geq s}\beta \in a_k
\end{array}
$$

The first two rows correspond to the general constraints: the probability of all future time instants must be 1, while the probability of every measurable set of future time instants must be nonnegative. The last two rows correspond to the probabilistic constraints, because $\sum_{a \in X_\beta \cap \mathrm{FutAt}(a_k)} \mu(w_k)(a) = \mu(w_k)([\beta]_{w_k})$. Such a system is solvable iff there is a probability $\mu(w_k)$ satisfying all probabilistic constraints that appear in a_k. There are finitely many such systems that can be solved in a finite number of steps.

For every $a_k \in S_{i,p}$, and for every $\bigcirc\beta \in a_k$, we have to verify that $\beta \in a_{k+1}$. It means that, if $w_{k+1} \models \beta$, then $w_k \models \bigcirc\beta$. Similarly, for every $a_k \in S_{i,p}$, and for every $\beta_1 U \beta_2 \in Sub(\alpha)$, we have to check that $\beta_1 U \beta_2 \in a_k$ iff there is some $l \geq k$ such that $\beta_2 \in a_l$, and for every m, $k \leq m < l$, $\beta_1 \in a_m$. Since $a_l = a_{l+p}$, for $l \geq i$, the above conditions can be verified in a finite number of steps.

If the above test is positively solved there is an ultimately periodic α-model $M = \langle W, v, Prob\rangle$ such that $w_k \models a_k$. Since $a_0 \in X_\alpha$, and $w_0 \models \alpha$, we have that α is satisfiable. If the test fails, and there is another possibility of choosing $S_{i,p}$, or there is another possibility of choosing i and p, we continue with the procedure, otherwise we conclude that α is not satisfiable.

It is easy to see that the procedure terminates in a finite number of steps. Thus, the pLTL$_{Meas}$-satisfiability problem is decidable. Since $\models \alpha$ iff $\neg\alpha$ is not satisfiable, the pLTL$_{Meas}$-validity problem is also decidable. ∎

LEMMA 5.2
The pLTL$_{Meas}$-satisfiability problem is **PSPACE**-hard.

PROOF. pLTL$_{Meas}$ contains syntactically the discrete linear-time temporal logic $L(\bigcirc, U)$ which is known to be **PSPACE**-hard [44]. We show that if a formula α in $L(\bigcirc, U)$ is satisfiable in the discrete linear-time temporal logic, then there is an α-model M' such that $w_0 \models_{M'} \alpha$. The point here is mainly to be able to construct properly the probabilistic spaces. Let $M = \langle w_0, w_1, \ldots, v\rangle$ be a plain discrete linear-time structure, and $w_0 \models_M \alpha$. Let us build the following α-model $M' = \langle W', v', Prob'\rangle$:

- $W' \stackrel{\text{def}}{=} w_0, w_1, \ldots$ and $v' \stackrel{\text{def}}{=} v$;
- for $i \in \omega$, $Prob'(w_i) = \langle W'(w_i), A'(w_i), \mu'(w_i)\rangle$ is defined as follows:
  – $W'(w_i) \stackrel{\text{def}}{=} \{w_{i+l} : l \geq 0\}$;
  – $A'(w_i)$ is defined as the smallest subset of $\mathcal{P}(W'(w_i))$ such that $\emptyset \in A'(w_i)$, for $a \in Atoms(\alpha)$, $\{w_{i+l} : l \geq 0, w_{i+l} \models_M a\} \in A'(w_i)$ and $A'(w_i)$ is closed under binary union.
  – for $X \in A'(w_i)$, if $[at(\alpha, w_i)]_{w_i} \subseteq X$, then $\mu'(w_i)(X) \stackrel{\text{def}}{=} 1$ otherwise $\mu'(w_i)(X) \stackrel{\text{def}}{=} 0$.

Since no probabilistic operator occurs in the atoms of α, it is immediate to check that M' is also an α-model and $w_0 \models_{M'} \alpha$. ∎

In order to find a complexity upper bound for the pLTL$_{Meas}$-satisfiability problem, we use the ideas from the proof of [44, Theorem 4.1]. The main difference is that we have to remember some additional information to check conditions related to the probabilistic part of the model. We also use [9, Theorem 2.7] (see Appendix B, Theorem B.1).

THEOREM 5.3
pLTL$_{Meas}$-satisfiability problem is in non-deterministic exponential time.

PROOF. Let α be a formula and $M = \langle W, v, Prob\rangle$ an ultimately periodic α-model with the starting index i and the period p such that $w_0 \models \alpha$. The number of subformulae of α that are of the form $P_{\geq s}\beta$ is less than $|\alpha|$. In every time instant from M exactly one atom from $Atoms(\alpha)$ holds.

It means that for every time instant the corresponding system of linear equalities and inequalities described in Theorem 5.1 is satisfiable. Using Theorem B.1 we conclude that there is another ultimately periodic α-model $M' = \langle W', v', Prob' \rangle$ such that:

- $w'_0 \models \alpha$,
- W and W', and v and v' coincide,
- in every time instant of M' at most $|\alpha|$ sets definable by atoms have non-zero probability, and
- the probabilities assigned to those sets are rational numbers with sizes at most $O(|\alpha| \cdot ||\alpha|| + |\alpha| \log(|\alpha|))$, where $||\alpha||$ denotes the length of the longest coefficient appearing in probabilistic operators in α.

Now, we describe a non-deterministic algorithm that checks whether α is satisfiable. First, we guess two numbers $i \leq 2^{2|\alpha|} + 1$, and $p \leq (2^{|\alpha|} + 1) \times 2^{|\alpha|}$. Then, we guess an ultimately periodic α-model $M = \langle W, v, Prob \rangle$ with the starting index i and the period p such that $w_0 \models \alpha$. It means that for every k, $0 \leq k < i + p$, a set F_k of $card(Sub(\alpha))$ formulae is chosen such that $at(\alpha, w_k) = \wedge_{\beta \in F_k} \beta$. Every F_k, $i \leq k < i + p$, also represents all $at(\alpha, w_{k+lp})$, $l \geq 0$. Thus, for $k \geq i$, $l \geq 0$, F_{k+lp} denotes F_k. We have to verify that F_k represent consistent atoms, namely that:

- for $\neg\beta \in Sub(\alpha)$, $\neg\beta \in F_k$ iff $\beta \notin F_k$,
- for $\beta_1 \wedge \beta_2 \in Sub(\alpha)$, $\beta_1 \wedge \beta_2 \in F_k$ iff $\beta_1 \in F_k$ and $\beta_2 \in F_k$,
- for $\bigcirc\beta \in Sub(\alpha)$, $\bigcirc\beta \in F_k$ iff $\beta \in F_{k+1}$,
- for $\beta_1 U \beta_2 \in Sub(\alpha)$, $\beta_1 U \beta_2 \in F_k$ iff $\beta_2 \in F_k$, or $\beta_1 \in F_k$ and $\beta_1 U \beta_2 \in F_{k+1}$,

and that α belongs to F_0. Everything mentioned above can be done using two sets of formulas representing the atoms in the present and in the next time instant. We also save the set F_i representing the atom which holds at the beginning of the period because it is the same as F_{i+p}. We have to verify that:

- for $\beta_1 U \beta_2 \in Sub(\alpha)$, if $\beta_1 U \beta_2 \in F_i$, then there must be some $l < p$ such that for every k, $i \leq k < i + l$, $\beta_1 \in F_k$, and $\beta_2 \in F_{i+l}$.

Additionally, for every k a set V_k of $|\alpha|$ pairs determining a probability measure is chosen. The first coordinates of the pairs are rational numbers representing probabilities, while the second coordinates are bitstrings representing atoms. The sizes of the rational numbers are at most $O(|\alpha| \cdot ||\alpha|| + |\alpha| \log(|\alpha|))$, while lengths of the bitstrings are at most $card(Atoms(\alpha))$. The only things we have to verify here is that for every V_k the sum of the first coordinates of pairs is equal to 1, and that the bitstrings in the second coordinates are mutually different.

For every k, $0 \leq k < i+p$, we have to verify that all formulae from F_k are true in the time instant w_k. Formulae with a propositional or a temporal main connective can be solved as above. Formulae of the form $P_{\geq s}\beta$ can be checked by computing the sum of probabilities of atoms represented in V_k that contain β in the top conjunction.

All the previous steps can be done in space $O(P(|\alpha|))$ for some suitable polynomial $P(.)$, because there are at most $|\alpha|$ atoms of positive probability in every time instant. However, the next step introduces additional complexity. For every k and every atom a with positive probability given in V_k we have to check that a is represented by an F_l, where:

- $k \leq l$ or
- if $i \leq k$, then $i \leq l < i + p$.

In other words, we have to check that $[a]_{w_k} \neq \emptyset$ if $\mu(w_k)([a]_{w_k}) > 0$. It can be done using a global table with $card(Atoms(\alpha))$ rows and 2 columns. Each row corresponds to an atom. The first field of a row contains the value of the counter k denoting the index of the last time instant such that the corresponding atom appears in V_k. Similarly, the value in the second field denotes the index l of the last set F_l representing the atom. Obviously, the table can be constructed in exponential time.

Having in mind the last step, we conclude that the pLTL$_{Meas}$-satisfiability problem is in **NEXP-TIME**. ∎

We conclude this section by studying the problem of deciding whether a formula α is satisfiable in a given ultimately periodic α-model. We suppose that the considered ultimately periodic α-model M can be finitely described. It means that the starting index i, the period p, $i+p$ valuations defined on propositional variables, and $i+p$ probabilities defined on atoms are given, and that values in ranges of probabilities must be finitely encoded. Let $\|M\|$ denote the size of an encoding of M.

THEOREM 5.4
Let $M = \langle W, v, Prob\rangle$ be a finitely described ultimately periodic α-model of size $\|M\|$ with the starting index $i \geq 0$ and the period $p \geq 1$. There is an algorithm for deciding whether α is satisfiable in M which runs in deterministic time $\mathcal{O}(\|M\|^2 \times |\alpha|)$ and deterministic space $\mathcal{O}(\|M\| \times |\alpha|)$.

PROOF. For $\beta \in Sub(\alpha)$, we compute recursively the label $l(\beta) \subseteq \{0, \ldots, i+p-1\}$ such that for $j \in \{0, \ldots, i+p-1\}$, $j \in l(\beta)$ iff $w_j \models_M \beta$. Each $l(\beta)$ can be encoded as a bitstring of length $i+p$. For a propositional variable β, $j \in l(\beta)$ iff $v(w_j)(\beta) = \top$. Thus, $l(\beta)$ can be computed in time $\mathcal{O}(i+p)$ and space $\mathcal{O}(i+p)$. Similarly, assuming that $l(\beta_1)$ and $l(\beta_2)$ are already defined, for $j \in \{0, \ldots, i+p-1\}$, $j \in l(\beta_1 \wedge \beta_2)$ iff $j \in l(\beta_1)$ and $j \in l(\beta_2)$. $l(\beta_1 \wedge \beta_2)$ can be computed in time $\mathcal{O}(i+p)$ and space $\mathcal{O}(i+p)$. $l(\neg\beta_1)$ is defined similarly. Assuming that $l(\beta)$ and $l(\gamma)$ are already defined, for $j \in \{0, \ldots, i+p-1\}$, $j \in l(\beta U \gamma)$ iff

- there is j', $j \leq j' \leq i+p-1$ such that $j' \in l(\gamma)$ and for all j'', $j \leq j'' < j'$, $j'' \in l(\beta)$ or
- $i \leq j$ and there is j', $i \leq j' < j$ such that $j' \in l(\gamma)$ and for all j'', $j \leq j'' \leq i+p-1$, $i \leq j'' < j'$, $j'' \in l(\beta)$.

$l(\beta U \gamma)$ can be computed in time $\mathcal{O}((i+p)^2)$ and space $\mathcal{O}(i+p)$. Finally, $l(P_{\geq s}\beta)$ can be computed in time $\mathcal{O}(\|M\|)$ and space $\mathcal{O}(\|M\|)$ by computing the sum of probabilities of atoms containing β in the top conjunctions. Since $card(Sub(\alpha)) \leq |\alpha|$, and $\mathcal{O}(i+p) \subset \mathcal{O}(\|M\|)$, the statement easily follows. ∎

6 The first-order case

Now, we turn to the first-order logic, and sketch the main ideas avoiding repetition of technical details mentioned in the case of pLTL. We consider languages that are at most countable extensions of the classical first-order language. Every language contains classical connectives, temporal operators, and a list of unary probabilistic operators as in the propositional case, as well as quantifier $\forall$, and for every integer $k \geq 0$, k-ary relation symbols $P_0^k, P_1^k, \ldots$, and k-ary function symbols $F_0^k, F_1^k, \ldots$ The function symbols of arity 0 are called constant symbols. Terms, the notion of term that is free for a variable, and formulae are defined as usual. Sentences are formulae without free variables. Let $Sentence(L)$ denote the set of all sentences of a first-order language L. The corresponding models are tuples of the form $\langle W, D, I, Prob\rangle$ where:

- W and $Prob$ are as in the propositional case,

- D is a non-empty domain, and
- I associates an interpretation $I(w_i)$ with every $w_i \in W$ such that for all j and k:
  – $I(w_i)(F_j^k)$ is a function from D^k to D,
  – for all m, $I(w_i)(F_j^k) = I(w_m)(F_j^k)$, and
  – $I(w_i)(P_j^k)$ is a relation over D^k.

Let $\langle W, D, I, Prob\rangle$ be a model. A variable valuation v assigns some element of the domain to every variable x, i.e. $v(x) \in D$. If $d \in D$, and v is a valuation, then $v[d/x]$ is a valuation like v except that $v[d/x](x) = d$. The value of a term t in a time instant $w_i \in W$ wrt. v (denoted by $I(w_i)(t)_v$) is:

- if t is a variable x, then $I(w_i)(x)_v = v(x)$, and
- if $t = F_j^k(t_1, \ldots, t_k)$, then $I(w_i)(t)_v = I(w_i)(F_j^k)(I(w_i)(t_1)_v, \ldots, I(w_i)(t_k)_v)$.

Note that we use fixed domain models with rigid terms, which is similar to the objectual interpretation for first order modal logics [16]. As it is pointed there, unexpected consequences may happen if someone rejects these assumptions. For example, in that case the standard first order axiom 13 (see below) is not valid [20].

The truth value of a formula α in a time instant $w_i \in W$ wrt. v (denoted by $I(w_i)(\alpha)_v$) is (we omit the usual conditions for the for the propositional connectives):

- if $\alpha = P_j^k(t_1, \ldots, t_k)$, then $I(w_i)(\alpha)_v = \text{true}$ if $\langle I(w_i)(t_1)_v, \ldots, I(w_i)(t_k)_v\rangle \in I(w_i)(P_j^k)$, otherwise $I(w_i)(\alpha)_v = \text{false}$,
- if $\alpha = P_{\geq s}\beta$, then $I(w_i)(\alpha)_v = \text{true}$ if $\mu(w_i)(\{w_{i+j} \in W(w_i) : I(w_{i+j})(\beta)_v = \text{true}\}) \geq s$, otherwise $I(w_i)(\alpha)_v = \text{false}$,
- if $\alpha = \bigcirc\beta$, then $I(w_i)(\alpha)_v = \text{true}$ if $I(w_{i+1})(\beta)_v = \text{true}$, otherwise $I(w_i)(\alpha)_v = \text{false}$,
- if $\alpha = \beta U \gamma$, then $I(w_i)(\alpha)_v = \text{true}$ if there is an integer $j \geq 0$ such that $I(w_{i+j})(\gamma)_v = \text{true}$, and for every k such that $0 \leq k < j$, $I(w_{i+k})(\beta)_v = \text{true}$, otherwise $I(w_i)(\alpha)_v = \text{false}$,
- if $\alpha = (\forall x)\beta$, then $I(w_i)(\alpha)_v = \text{true}$ if for every $d \in D$, $I(w_i)(\beta)_{v[d/x]} = \text{true}$, otherwise $I(w_i)(\alpha)_v = \text{false}$.

We write $w_i \models_M \alpha$ if for every valuation v, $I(w_i)(\alpha)_v = \text{true}$.

A sentence α is satisfiable if there is a time instant w_i in a model M such that $w_i \models_M \alpha$. A model is measurable if all the sets of time instants definable by sentences are measurable. As in the propositional case we consider the subclass of measurable models denoted by FOpLTL$_{Meas}$. An infinitary axiomatic system for FOpLTL$_{Meas}$ contains all the axiom schemas and inference rules mentioned in Section 3 as well as the following axiom schemas:

A12. $(\forall x)(\alpha \to \beta) \to (\alpha \to (\forall x)\beta)$, where x is not free in α,
A13. $(\forall x)\alpha(x) \to \alpha(t/x)$, where $\alpha(t/x)$ is obtained by substituting all free occurrences of x in $\alpha(x)$ by the term t which is free for x in $\alpha(x)$,
A14. $(\forall x) \bigcirc \alpha(x) \to \bigcirc(\forall x)\alpha(x)$.

and the inference rule

R5. from α infer $(\forall x)\alpha$.

The last axiom is a variant of the well-known Barcan formula, while Rule 5 is Gödel's Generalization. It is easy to check soundness of the axiomatic system. For example, consider the Barcan formula. Suppose that $M = \langle W, D, I, Prob\rangle$ is an FOpLTL$_{Meas}$-model, $w_i \in W$, and

$w_i \models (\forall x) \bigcirc \alpha(x)$, i.e. that for every valuation v, $I(w_i)((\forall x) \bigcirc \alpha(x))_v = \text{true}$. It follows that for every $d \in D$, $I(w_i)(\bigcirc\alpha(x))_{v[d/x]} = \text{true}$, and $I(w_{i+1})(\alpha(x))_{v[d/x]} = \text{true}$. Thus, $w_{i+1} \models (\forall x)\alpha(x)$, and $w_i \models \bigcirc(\forall x)\alpha(x)$.

In the completeness proof we can follow the ideas from the propositional case and prove statements corresponding to the Deduction theorem, and Lemma 3.2. The main difference concerns the proof of Lemma 3.2(8) where an additional case must be considered. Let $(\forall x)\alpha$ be obtained by the inference rule R5. Then we have

- $T \vdash \alpha$,
- $\bigcirc T \vdash \bigcirc\alpha$ (by the induction hypothesis),
- $\bigcirc T \vdash (\forall x) \bigcirc \alpha$ (by the inference rule R5)
- $\bigcirc T \vdash \bigcirc(\forall x)\alpha$ (by Barcan formula).

In the rest of the completeness proof we need a special kind of maximal consistent set of sentences called saturated sets [16] that satisfy (besides maximality and consistency) the following condition:

- if $\neg(\forall x)\alpha(x)$ belongs to the set, then for some term t, $\neg\alpha(t)$ is also in the set.

Using techniques from [39] we can prove a statement which is related to Theorem 3.5:

THEOREM 6.1
Let T be a consistent set of sentences of our language L, and C be a countably infinite set of new constant symbols ($C \cap L = \emptyset$). Then T can be extended to a saturated set $\mathcal{T}$ in the language $L \cup C$.

PROOF. Let $\alpha_0, \alpha_1, \ldots$ be an enumeration of all sentences in the language L, and T_i, $i = 0, 1, 2, \ldots$ and $\mathcal{T}$ denote the sets of sentences obtained by the steps 1–3 of Theorem 3.5, with the additional requirement in the step 2:

- Otherwise, if α_i is of the form $(\forall x)\beta(x)$ (and $T_i \cup \{\alpha_i\}$ is not consistent), then $T_{i+1} \stackrel{\text{def}}{=} T_i \cup \{\neg\alpha_i, \neg\beta(c)\}$, for some $c \in C$ such that T_{i+1} is consistent.

We can prove that every T_i is consistent as before. The new requirement produces consistent sets, too: suppose that for some $i > 0$ the formula α_i is of the form $(\forall x)\beta(x)$ and that $T_i \cup \{(\forall x)\beta(x)\}$ is not consistent. Since T_i is consistent, the same holds for $T_i \cup \{\neg(\forall x)\beta(x)\}$. If there is a constant symbol $c \in C$ such that $\neg\beta(c) \in T_i$, then obviously $T_i \cup \{\neg(\forall x)\beta(x), \neg\beta(c)\}$ is consistent. Suppose that there is no such c. Since the set T does not contain constants from C, and $T_i \cup \{\neg(\forall x)\beta(x)\}$ is obtained by adding only finitely many formulas to the set T, there is at least one constant $c \in C$ such that c does not appear in $T_i \cup \{\neg(\forall x)\beta(x)\}$. If $T_i \cup \{\neg(\forall x)\beta(x), \neg\beta(c)\}$ is not consistent, then $T_i, \neg(\forall x)\beta(x) \vdash \beta(c)$, and since c does not appear in $T_i \cup \{\neg(\forall x)\beta(x)\}$, $T_i, \neg(\forall x)\beta(x) \vdash (\forall x)\beta(x)$. Thus, $T_i \vdash (\forall x)\beta(x)$. It follows that the set T_i is not consistent, a contradiction.

Finally, we have to show that $\alpha \in \mathcal{T}$ follows from $\mathcal{T} \vdash \alpha$, to prove that $\mathcal{T}$ is consistent, while the construction guarantees that $\mathcal{T}$ is maximal and saturated. The only new case which does not appear in the proof of Theorem 3.5 concerns the situation when $\mathcal{T} \vdash (\forall x)\beta(x)$ is obtained from $\mathcal{T} \vdash \beta(x)$ by Rule 5. Since $\beta(x)$ has one free variable, and T_i and $\mathcal{T}$ are sets of sentences, $\beta(x)$ does not belong to $\mathcal{T}$. However, by classical reasoning, we have $\mathcal{T} \vdash \beta(c)$, for every constant $c \in C$, and by the induction hypothesis $\beta(c) \in \mathcal{T}$. If $(\forall x)\beta(x) \notin \mathcal{T}$, by the construction of the set $\mathcal{T}$, there must be some $i > 0$ such that $\beta(c'), \neg\beta(c') \in T_i$ for some $c' \in C$, a contradiction. ∎

Lemma 3.3 holds for saturated sets as well, since they are maximal and consistent. Then, starting from a consistent set T of sentences and its saturated extension $\mathcal{T}$ in the language $L \cup C$, we define a tuple $M_T = \langle W, D, I, Prob \rangle$ as follows:

- $W \stackrel{\text{def}}{=} w_0, w_1, \ldots$, $w_0 = \mathcal{T}$, and for $i > 0$, $w_i = \{\alpha : \bigcirc\alpha \in w_{i-1}\}$,
- D is the set of all variable-free terms in L,
- for $i > 0$, $I(w_i)$ is an interpretation such that:
  - for every function symbol F_j^k, $I(w)(F_j^k)$ is a function from D^m to D such that for all variable-free terms $t_1, \ldots, t_k$ in L, $F_j^k : \langle t_1, \ldots, t_k\rangle \to F_j^k(t_1, \ldots, t_k)$, and
  - for every relation symbol P_j^k, $I(w_i)(P_j^k) = \{\langle t_1, \ldots, t_k\rangle$ for all variable-free terms $t_1, \ldots, t_k$ in L: $P_j^k(t_1, \ldots, t_k) \in w_i\}$.
- for $i \geq 0$, $Prob(w_i) = \langle W(w_i), H(w_i), \mu(w_i)\rangle$ such that:
  - $W(w_i) \stackrel{\text{def}}{=} \{w_{i+j} : j \geq 0\}$,
  - $A(w_i) \stackrel{\text{def}}{=} \{\{w_{i+j} : j \geq 0, \alpha \in w_{i+j}\} : \alpha \in Sentence(L)\}$, and
  - $\mu(w_i)(\{w_{i+j} : j \geq 0, \alpha \in w_{i+j}\}) = \sup\{s : P_{\geq s}\alpha \in w_i\}$.

In a counterpart of Lemma 3.6 we have to prove that every w_i from M is saturated rather than just maximal. To show that suppose that $i \geq 0$ and w_i is saturated, while w_{i+1} is not. It means that there is a formula $\neg(\forall x)\alpha(x) \in w_{i+1}$ such that for every term t, $\neg\alpha(t) \notin w_{i+1}$. Thus, $\bigcirc\neg(\forall x)\alpha(x) \in w_i$, and for every term t, $\bigcirc\neg\alpha(t) \notin w_i$. Using the Barcan formula and the axiom A3, we obtain $\neg(\forall x) \bigcirc \alpha(x) \in w_i$, and for every term t, $\neg \bigcirc \alpha(t) \notin w_i$, a contradiction with saturation of w_i. In the last part of the proof we can show that for every $i \geq 0$, $Prob(w_i)$ is a probability space, and that M_T is a measurable FOpLTL$_{Meas}$-model, as is done in Lemma 3.7 and Theorem 3.8, respectively. The only new cases in Theorem 3.8 concern atomic sentences, and sentences of the form $(\forall x)\beta$. If α is an atomic formula, by the definition of $I(w_i)$, $w_i \models_M \alpha$ iff $\alpha \in w_i$. On the other hand, let $\alpha = (\forall x)\beta$. If $\alpha \in w_i$, then, because of the axiom A13, $\beta(t) \in w_i$ for every $t \in D$. By the induction hypothesis $w_i \models_M \beta(t)$ for every $t \in D$, and $w_i \models_M (\forall x)\beta$. If $\alpha \notin w_i$, there is some $t \in D$ such that $w_i \models_M \neg\beta(t)$, because w_i is saturated. It follows that $w_i \not\models_M (\forall x)\beta$. Thus, we obtain:

THEOREM 6.2 (Extended completeness theorem)
Every consistent set T of sentences is satisfiable.

7 Conclusions

We have introduced two discrete linear-time probability logics: the propositional logic pLTL$_{Meas}$, and the first-order logic FOpLTL$_{Meas}$. Their infinitary axiomatic systems are proved to be complete (see Theorems 3.9 and 6.2). Actually, our results establish extended completeness. In the propositional case that cannot be proved using finitary means. Indeed, there exists a countably infinite set of pLTL-formulae that is unsatisfiable although every finite subset is satisfiable: for instance, consider $\{F^n p : n$ is a positive integer$\} \cup \{FG\neg p\}$ or $\{\neg P_{=0}\alpha\} \cup \{P_{<1/n}\alpha : n$ is a positive integer$\}$. Thus, compactness does not hold for pLTL$_{Meas}$, and it is well known that we cannot hope for the extended completeness having a finitary axiomatic system. In the first-order case we follow the ideas from [39], because the set of all FOpLTL$_{Meas}$-valid sentences is not recursively enumerable [1, 15], and no complete finitary axiomatization is possible. Thus, we believe that it is not only of theoretical interest to give an infinitary and complete first-order proof system. For the pLTL$_{Meas}$ we have proved the decidability theorem. We also give lower and upper complexity bounds. Since there is a gap between them, one of the questions still open is to find a precise characterization of the computational complexity of the pLTL$_{Meas}$-satisfiability problem.

Some of the other directions for further work are as follows. We can specify additional relationships between the flow of time and the probability measures by adding axioms at the proof-theoretical level. For example, the formula $\neg\alpha \to (P_{\geq s}\alpha \to \bigcirc P_{\geq s}\alpha)$, considered as an additional axiom scheme, characterizes models with the property that if α does not hold in a time instant, then in the next time instant its probability will be not decreased. The question is how adding such axiom schemes in the deductive proof system affects completeness and decidability proofs. Our notation differs from the one described in [9, 8]. Namely, the so-called probability terms of the form $w(\alpha)$ and atomic probability formulae of the form $\sum_{i=1}^{n} a_i w(\alpha_i) \geq b$ are allowed there. A formula $w(\alpha) \geq s$ can be rewritten (in our notation) as $P_{\geq s}\alpha$. These restrictions are made here for ease of exposition only. Our syntax can be extended in a straightforward manner, such that the set of well formed formulas of that richer language can be exactly obtained. Reasoning about linear inequalities can be formalized using the axiomatization for reasoning about linear inequalities in the same way as in [9, 8], and all the results can be proved without any essential changes. We can restrict our attention to probabilities that have a fixed finite range, as in [10, 12, 45, 37, 39, 40]. The corresponding logics can be axiomatized similarly to the way done above. It would be worth investigating whether such an assumption can decrease the complexity in the propositional case, for example, along the ideas given in [12]. It is also interesting how to give a finitary axiomatization for the pLTL$_{Meas}$. We believe that combining techniques from [9] and [28, 30] is a promising way to obtain that. Such an axiomatization should additionally have axioms that allow reasoning about linear inequalities. However, note that, as is pointed out in [45, 39], there is an unpleasant logical consequence of finitary axiomatization when lack of compactness is present (which is the case for our logic): there are unsatisfiable sets of formulas that are consistent with respect to the assumed finitary axiomatic system. Finally, the questions of axiomatization, decidability and complexity of branching time probability logics naturally arise.

Acknowledgements

This work was supported by the Ministarstvo nauke i zaštite životne sredine Republike Srbije, through Matematički Institut. I would like to thank Stéphane Demri who was a coauthor of an earlier version of this paper. I would like to thank the anonymous referees for pointing out some errors in the first submitted version of the paper and many useful comments on the text.

References

[1] M. Abadi and J. Y. Halpern. Decidability and expressiveness for first-order logics of probability, *Information and Computation*, **112**, 1–36, 1994.

[2] A. Amir and D. Gabbay. Preservation of expressive completeness in temporal models, *Information and Computation*, **72**, 66–83, 1987.

[3] A. Aziz, V. Singhal, F. Balarin, R. K. Brayton, and A. L. Sangiovanni-Vincentelli. It usually works: The temporal logic of stochastic systems. In *7th International Workshop on Computer-Aided Verification*, volume 939 of *Lecture Notes in Computer Science*, pp. 155–166. Springer-Verlag, 1995.

[4] R. K. Bhaskara and R. M. Bhaskara. *Theory of charges*. Academic Press, 1983.

[5] A. Bianco and L. de Alfaro. Model checking of probabilistic and nondeterministic systems. In *Foundations of Software Technology and Theoretical Computer Science*, volume 1026 of *Lecture Notes in Computer Science*, pp. 499–512. Springer-Verlag, 1995.

[6] C. Courcoubetis and M. Yannakakis. The complexity of probabilistic verification, *Journal of the ACM*, **42**, 857–907, 1995.

[7] R. Djordjević, M. Rašković, and Z. Ognjanović. Completeness theorem for propositional probabilistic models whose measures have only finite ranges, *Archive for Mathematical Logic*, **43**, 557 –563, 2004.

[8] R. Fagin and J. Halpern. Reasoning about knowledge and probability, *Journal of the ACM*, **41**, 340–367, 1994.

[9] R. Fagin, J. Halpern, and N. Megiddo. A logic for reasoning about probabilities, *Information and Computation*, **87**, 78–128, 1990.

[10] M. Fattorosi-Barnaba and G. Amati. Modal operators with probabilistic interpretations I, *Studia Logica*, **46**, 383–393, 1989.

[11] Y. Feldman. A decidable propositional dynamic logic with explicit probabilities, *Information and Control*, **63**, 11–38, 1984.

[12] N. Ferreira, M. Fisher, and W. van der Hoek. Practical reasoning for uncertain agents. In *9th European conference JELIA'04 Logics in Artificial Intelligence*, volume 3229 of *Lecture Notes in Computer Science*, pp. 82–94. Springer-Verlag, 2004.

[13] A. Frish and P. Haddawy. Anytime deduction for probabilistic logic, *Artificial Intelligence*, **69**, 93–122, 1994.

[14] D. Gabbay and I. Hodkinson. An axiomatization of the temporal logic with until and since over the reals numbers, *Journal of Logic and Computation*, **1**, 229–259, 1990.

[15] D. Gabbay, I. Hodkinson, and M. Reynolds. *Temporal Logic. Mathematical Foundations and Computational Aspects (volume 1)*. Clarendon Press, 1994.

[16] J. W. Garson. Quantification in modal logic. In *Handbook of philosophical logic, vol II*, F .Guenthner and D. Gabbay, eds, pp. 249–307. D. Reidel Publishing Comp., 1984.

[17] D. P. Guelev. A propositional dynamic logic with qualitative probabilities, *Journal of Philosophical Logic*, **28**, 575–605, 1999.

[18] D. P. Guelev. Probabilistic neighbourhood logic. volume 1926 of *Lecture Notes in Computer Science*, pp. 264–275. Springer-Verlag, 2000.

[19] P. Haddawy. A logic of time, chance and action for representing plans, *Artificial Intelligence*, **80**, 243–308, 1996.

[20] J. Y. Halpern. An analysis of first-order logics of probability, *Artificial Intelligence*, **46**, 311–350, 1990.

[21] J. Y. Halpern. A logical approach to reasoning about uncertainty: A tutorial. In *Discourse, Interaction, and Communication*, K. Korta X. Arrazola and F. J. Pelletier, eds, pp. 141–155. Kluwer, 1998.

[22] H. Hansson and B. Jonsson. A logic for reasoning about time and reliability, *Formal Aspect of Computing*, **6**, 512–535, 1994.

[23] S. Hart and M. Sharir. Probabilistic temporal logics for finite and bounded models. In *16th ACM Symposium on Theory of Computing*, pp. 1–13. ACM, 1984.

[24] A. Heifetz and Ph. Mongin. Probability logic for type spaces, *Games and Economic Behavior*, **35**, 31–53, 2001.

[25] D.V. Hung and Z. Chaochen. Probabilistic duration calculus for continuous time, *Formal Aspects of Computing*, **11**, 21–44, 1999.

[26] J. Kamp. *Tense Logic and the theory of linear order*. PhD thesis, UCLA, USA, 1968.

[27] D. Kozen. A probabilistic pdl, *Journal of Computer and System Sciences*, **30**, 162–178, 1985.

[28] F. Kröger. *Temporal logic of programs (EATCS Monographs in Computer Science)* Springer-Verlag, 1987.

[29] D. Lehmann and S. Shelah. Reasoning with time and chance, *Information and Control*, **53**, 165–198, 1982.

[30] O. Lichtenstein and A. Pnueli. Propositional temporal logics: Decidability and completeness, *Logic Journal of the IGPL*, **8**, 55–85, 2000.

[31] Z. Liu, A.P. Ravn, E.V. Sorensen, and Z .Chaochen. A probabilistic duration calculus. In *Dependable Computing and Fault-Tolerant Systems*, volume 7 of *Dependable Computing and Fault-Tolerant Systems*, H. Kopetz and Y. Kakuda, eds, pp. 30–52. Springer-Verlag, 1993.

[32] Z. Manna and A. Pnueli. *The Temporal Logic of Reactive and Concurrent Systems: Specification*. Springer-Verlag, 1992.

[33] Z. Marković, Z. Ognjanović, and M. Rašković. A probabilistic extension of intuitionistic logic, *Mathematical Logic Quarterly*, **49**, 415–424, 2003.

[34] M. Meier. An infinitary probability logic for type spaces. In *The Fourth Conferece on Logic and the Foundations of Game and Decision Theory*, 2000.

[35] N. Nilsson. Probabilistic logic, *Artificial Intelligence*, **28**, 71–87, 1986.

[36] Z. Ognjanović. A logic for temporal and probabilistic reasonin. In *Workshop on probabilistic logic and randomized computation, Saabrücken*, 1998.

[37] Z. Ognjanović and M. Rašković. A logic with higher order probabilities, *Publications de l'Institut Mathématique, Nouvelle Série*, **60**, 1–4, 1996.

[38] Z. Ognjanović and M. Rašković. Some probability logics with new types of probability operators, *Journal of Logic and Computation*, **9**, 181–195, 1999.

[39] Z. Ognjanović and M. Rašković. Some first-order probability logics, *Theoretical Computer Science*, **247**, 191–212, 2000.

[40] M. Rašković. Classical logic with some probability operators, *Publications de l'Institut Mathématique, Nouvelle Série*, **53**, 1–3, 1993.

[41] M. Rašković and Z. Ognjanović. A first order probability logic LP_q, *Publications de l'Institut Mathématique, Nouvelle Série*, **65**, 1–7, 1999.

[42] M. Rašković, Z. Ognjanović, and Z. Marković. A logic with conditional probabilities. In *9th European conference JELIA'04 Logics in Artificial Intelligence*, volume 3229 of *Lecture notes in computer science*, J. Leite and J. Alferes, eds, pp. 226–238. Springer-Verlag, 2004.

[43] K. Segerberg. Qualitative probability in a modal setting. In *Second Scandinavian Logic Symposium*, J.E. Fenstad, ed., pp. 341–352. North-Holland, 1971.

[44] A. Sistla and E. Clarke. The complexity of propositional linear temporal logic, *Journal of the ACM*, **32**, 733–749, 1985.

[45] W. van der Hoek. Some considerations on the logic $\mathrm{P}_F\mathrm{D}$: a logic combining modality and probability, *Journal of Applied Non-Classical Logics*, **7**, 287–307, 1997.

A Extension theorem for finitely additive measures

THEOREM A.1 ([4])
Let C be an algebra of subsets of a set Ω and $\mu(w_j)$ a positive bounded charge — a finitely additive measure — on C. Let F be an algebra on Ω containing C. Then there exists a positive bounded charge $\overline{\mu(w_j)}$ on F such that $\overline{\mu(w_j)}$ is an extension of $\mu(w_j)$ from C to F and that the range of $\overline{\mu(w_j)}$ is a subset of the closure of the range of $\mu(w_j)$ on C.

B Theorem about linear systems

THEOREM B.1 ([9])
If a system of r linear equalities and/or inequalities with integer coefficients each of length at most l has a nonnegative solution, then it has a nonnegative solution with at most r entries positive, and where the size of each member of the solution is $O(rl + r\log(r))$.

Received 3 November 2004