Note Title

MTH 9831 Lecture 8

Connections with PDEs.

1. SDE: definition, Markov property of solutions, associated differential operators.
2. Feynman-Kac formula.
3. Kolmogorov's backward and forward equations.

1. **Def.** Let $b(t,x)$ be a deterministic d-dimensional vector valued function on $[0,\infty)\times\mathbb{R}^d$ and $\sigma(t,x)$ be a deterministic $d\times r$ matrix for all $(t,x)\in[0,\infty)\times\mathbb{R}^d$. Denote by $B(t)$ the r-dimensional standard BM.

A stochastic differential equation with drift $b(t,x)$ and volatility $\sigma(t,x)$ is an equation of the form

$$dX(t) = b(t,X(t))\,dt + \sigma(t,X(t))\,dB(t). \qquad (\mathcal{E})$$

A (strong) solution to $(\mathcal{E})$ with initial condition $x\in\mathbb{R}^d$ is a stochastic process $(X(t))_{t\geq 0}$ with continuous sample paths and with the following properties.

(i) $X(0)=x$ a.s. (ii) $(X(t))_{t\geq0}$ is adapted to the filtration $(\mathcal{F}(t))_{t\geq0}$ generated by $(B(t))_{t\geq0}$

(iii) $\int_0^t |b_i(s,X(s))| + \sigma_{ij}^2(s,X(s))\,ds < \infty$ a.s. for $1\leq i\leq d$, $1\leq j\leq r$, $0\leq t<\infty$

(iv) for $t\in[0,T]$

$$X(t) = X(0) + \int_0^t b(u,X(u))\,du + \int_0^t \sigma(u,X(u))\,dB(u) \quad \text{a.s.}$$

Remark: often in applications we need to start at time t from a given point $x\in\mathbb{R}^d$ and find X: $X(t)=x$ and

$$X(u) = X(t) + \int_t^u b(a,X(a))\,da + \int_t^u \sigma(a,X(a))\,dB(a), \text{ a.s.,}$$

for $u\geq t$.

Some conditions on $b(t,x)$ and $\sigma(t,x)$ are needed to ensure the existence and uniqueness of solutions of $(\mathcal{E})$.

<u>Example</u>. $dX(t) = X^3(t)\,dt - X^2(t)\,dB(t)$, $X(0)=1$.
Let us try to find $X(t)$ in the form $f(t,B(t))$.
Then

$$dX(t) = df(t,B(t)) = f_t\,dt + f_x\,dB(t) + \frac{1}{2} f_{xx}\,dt$$
$$= \left(f_t + \frac{1}{2} f_{xx}\right) dt + f_x\,dB(t).$$

f has to satisfy : $f_x = -f^2$ and $f_t + \frac{1}{2} f_{xx} = f^3$

$$f_x = -f^2 \Rightarrow f(t,x) = \frac{1}{x + C(t)}$$

Substituting in the second equation gives

$$\frac{-c'(t)}{(x+c(t))^2} + \frac{1}{\cancel{2}} \frac{\cancel{2}}{(x+c(t))^3} = \frac{1}{(x+c(t))^3}$$

$$\Rightarrow c'(t) = 0 \Leftrightarrow c(t) \equiv \text{const}.$$

$$f(t,x) = (x+C)^{-1}$$
$$X(0) = f(0,0) = \frac{1}{C} = 1 \Rightarrow C = 1$$
$$X(t) = f(t,B(t)) = (B(t)+1)^{-1}.$$

The problem is that with probability 1 $B(t)$ hits -1, i.e. $P(T_{-1} < \infty) = 1$.
Thus, the solution exists only up to the "explosion time" T_1.

Typical conditions which prevent explosions and guarantee existence and uniqueness of $(\mathcal{E})$ are

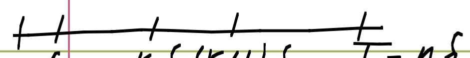
$$|b(t,x) - b(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \le K|x-y|$$
$$|b(t,x)|,\ \|\sigma(t,x)\| \le K(1+|x|)$$

for all $t \geq 0$, $x, y \in \mathbb{R}$. In fact, for $d=1$ the conditions can be weakened (for example CIR is included). Here $\|A\| = \sqrt{\sum_{i,j=1}^{d} a_{ij}^2}$.

<u>Markov property</u>. Intuition. $(d=1)$

How can one try to simulate $X(t)$, $0 \leq t \leq T$? Suppose $X(0) = x$ is given. Consider a small increment of time δ. Then according to $(\mathcal{E})$

$$X(\delta) \simeq x + b(0,x)\delta + \sigma(0,x)\underbrace{B(\delta)}_{\sim N(0,\delta)}$$



$$= x + b(0,x)\delta + \sigma(0,x)\sqrt{\delta}\, Z_1, \quad Z_1 \sim N(0,1)$$

Then
$$X((k+1)\delta) = X(k\delta) + b(k\delta, X(k\delta))\delta + \sigma(k\delta, X(k\delta))\left(B((k+1)\delta) - B(k\delta)\right)$$
$$= X(k\delta) + b(k\delta, X(k\delta))\delta + \sigma(k\delta, X(k\delta))\sqrt{\delta}\, Z_{k+1};$$
and $Z_1, \dots, Z_k, \dots$ are i.i.d. $N(0,1)$

Then $X(n\delta) = X(T)$ gives us a simulated value at time T. If we could justify passing to the limit (in some sense) as $\delta \to 0$ then the limiting process would give us a solution to $(\mathcal{E})$. Moreover, our procedure suggests that the limiting process would be memoryless ($\equiv$ Markov process), since given $X(k\delta)$ we can simulate $X((k+1)\delta), \dots, X(n\delta)$ without knowing $X(0), X(\delta), \dots, X((k-1)\delta)$.

More rigorously, the following theorem holds:

Theorem 1. Let $0 \le t \le T$ and h be a Borel measurable function. Set

$$g(t,x) = E^{t,x} h(X(T)).$$

(take $X(t) = x$, solve $(\mathcal{E})$ for $u \in [t,T]$, get $X(T)$, and compute the expectation of $h(X(T))$, repeat for all $t \in [0,T]$, $x \in \mathbb{R}$. Get a deterministic function $g(t,x)$). Then for $t \in [0,T]$

$$\mathbb{E}\big(h(X(T)) / \mathcal{F}(t)\big) = g(t, X(t)),$$

i.e. solutions to SDEs are Markov processes.

Proof is omitted.

Heuristics:

Meanings of $b(t,x)$ and $\sigma^2(t,x)$ $(d=1)$

$$E\big(X(t+\delta) - X(t) / X(t) = x\big) \approx$$

$$E\big(b(t,X(t))\delta + \sigma(t,X(t))\sqrt{\delta}\, Z / X(t) = x\big)$$

$$= b(t,x)\delta \ ; \quad \lim_{\delta \to 0} \delta^{-1} E\big(X(t+\delta) - X(t) / X(t) = x\big) = b(t,x)$$

$$E\big((X(t+\delta) - X(t))^2 / X(t) = x\big)$$

$$\approx E\big((b(t,X(t))\delta + \sigma(t,X(t))\sqrt{\delta}\, Z)^2 / X(t) = x\big)$$

$$= b^2(t,x)\delta^2 + \delta\sigma^2(t,x) \ ;$$

$$\lim_{\delta \to 0} \frac{1}{\delta} E\big((X(t+\delta) - X(t))^2 / X(t) = x\big) = \sigma^2(t,x) =: a(t,x)$$

$b(t,x)$ is called the drift coefficient
$a(t,x)$ is called the diffusion coefficient.

If $u(t,x)$ is a smooth function then

$$E(u(t+\delta, X(t+\delta)) - u(t, X(t)) / X(t) = x)$$
$$= E\Big(u_t(t, X(t))\delta + u_x(t, X(t))(X(t+\delta) - X(t)) + \tfrac{1}{2} u_{xx}(t, X(t))(X(t+\delta) - X(t))^2 + \dots / X(t) = x \Big)$$
$$= \delta\,(u_t(t,x) + b(t,x)\, u_x(t,x) + \tfrac{1}{2} u_{xx}(t,x)\, a(t,x)) + \bar{o}(\delta)$$

$$\lim_{\delta \to 0} \delta^{-1} E(u(t+\delta, X(t+\delta)) - u(t, X(t)) / X(t) = x)$$
$$= u_t(t,x) + b(t,x)\, u_x(t,x) + \tfrac{1}{2} a(t,x)\, u_{xx}(t,x)$$
$$=: u_t(t,x) + (\mathcal{A}_t u)(t,x), \text{ where}$$

$$\mathcal{A}_t := b(t,x)\frac{\partial}{\partial x} + \frac{1}{2} a(t,x) \frac{\partial^2}{\partial x^2} \qquad (t \geq 0) \quad (1)$$

Similarly, for $d \geq 1$ we set

$$\mathcal{A}_t := \sum_{i=1}^{d} b_i(t,x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} \quad (t \geq 0) \quad (2)$$

where $b^T(t,x) = \lim_{\delta \to 0} \delta^{-1} E((X(t+\delta) - X(t))^T / X(t) = x)$

$$a(t,x) = \lim_{\delta \to 0} \delta^{-1} E\Big((X(t+\delta) - X(t))(X(t+\delta) - X(t))^T / X(t) = x \Big) = (\sigma\sigma^T)(t,x),$$

Remark.
When b and σ do not depend on t, i.e. $b = b(x)$, $\sigma = \sigma(x)$
X is called a (time homogeneous, or Itô) diffusion process and
$\mathcal{A}$ (now it does not depend on t) is called the generator of X.

② Feynman–Kac formula.

This formula gives a stochastic representation of a solution to a PDE. More precisely,

__Theorem 2__ (Feynman–Kac formula)
Let $g(t,x)$ be the solution of

$$g_t + b(t,x) g_x + \frac{1}{2}\sigma^2(t,x)\, g_{xx} = r(t,x) g$$
$$g(T,x) = h(x).$$

Let $(X(u))_{t \le u \le T}$ solve $(\mathcal{E})$ with $X(t) = x$. Then

$$g(t,x) = E^{t,x}\left(e^{-\int_t^T r(s,X(s))ds} h(X(T)) \right)$$

(We assume that some mild integrability conditions are satisfied so that all integrals make sense.)

The idea of the proof. For $0 \le t \le u \le T$ we compute (t is fixed and u is changing)

$$d\left(e^{-\int_t^u r(s,X(s))ds} g(u, X(u)) \right) =$$

$$= -r(u,X(u)) e^{-\int_t^u r(s,X(s))ds} g(u,X(u))\,du$$

$$+ e^{-\int_t^u r(s,X(s))ds}\, dg(u,X(u))$$

$$= e^{-\int_t^u r(s,X(s))ds}\Big[\big(-r(u,X(u))\, g(u,X(u)) +$$

$$+ g_t(u,X(u)) + b(u,X(u))\, g_x(u,X(u)) + \frac{1}{2}\sigma^2(u,X(u)) \times$$

$$g_{xx}(u,X(u))\big)\, du + g_x(u,X(u))\, \sigma(u,X(u))\, dB(u) \Big]$$

Since g solves the PDE, the drift term is 0.
Then

$$e^{-\int_t^T r(s,X(s))\,ds} g(T,X(T)) - g(t,X(t))$$
$$= \int_t^T e^{-\int_t^u r(s,X(s))\,ds} g_x(u,X(u))\,\sigma(u,X(u))\,dB(u).$$

Taking conditional expectation (given $X(t)=x$) we get

$$\mathbb{E}^{t,x}\left(e^{-\int_t^T r(s,X(s))\,ds} h(X(T))\right) = g(t,x) \quad \blacksquare.$$

<u>Theorem 3</u> (Multidimensional Feynman-Kac Formula). Let $a(t,x) = \sigma\sigma^T(t,x)$ and assume that $g(t,x)$ satisfies on $[0,T)\times\mathbb{R}^d$

$$\partial_t g + \sum_{j=1}^d b_j(t,x)\frac{\partial}{\partial x_j} g + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d a_{ij}(t,x)\frac{\partial^2 g}{\partial x_i \partial x_j} = r(t,x) g$$

with the terminal condition $g(T,x) = h(x)$, $x\in\mathbb{R}^d$
Then (under integrability conditions)

$$g(t,x) = \mathbb{E}^{t,x}\left(e^{-\int_t^T r(s,X(s))\,ds} h(X(T))\right), \text{ where}$$

$(X(u))_{t\le u\le T}$ solves $(\mathcal{E})$ with $X(t)=x$; i.e. $\forall j=1,2,..,d$.

$$dX_j(u) = b_j(u,X(u))\,du + \sum_{k=1}^d \sigma_{jk}(u,X(u))\,dB_k(u)$$

$$X_j(t) = x_j, \quad j=1,2,..,d; \quad t\le u\le T.$$

Example: (Heston stochastic volatility model; $d=2$)
Suppose that under $\widetilde{\mathbb{P}}$ the stock price follows

$$dS(t) = rS(t)\,dt + \sqrt{V(t)}\,S(t)\,d\widetilde{B}_1(t), \text{ where}$$

$V(t)$ is a stochastic process and

$$dV(t) = (a - b\,V(t))\,dt + \sigma\sqrt{V(t)}\,d\widetilde{B}_2(t),$$

$a, b, \sigma > 0$ and $d[B_1, B_2](t) = \rho\,dt$, $-1 < \rho < 1$.

- Find the corresponding generator $\mathcal{A}$
- What does the FK formula say about the solution of $\frac{\partial g}{\partial t} + \mathcal{A}g = rg$; $g(T, x, v) = h(x, v)$?

It is easy to find the drift $b(t, x, v) = \begin{pmatrix} rx \\ a - bv \end{pmatrix}$.

There are 2 ways to find $\|a_{ij}(t, x, v)\|$.

I. Find $\sigma(t, x, v)$ and compute $\sigma\sigma^T(t, x, v)$

For this we have to "decorrelate" our BM

$$d\widetilde{B}_1(t) = d\widetilde{W}_1(t)$$
$$d\widetilde{B}_2(t) = \rho\,d\widetilde{W}_1(t) + \sqrt{1-\rho^2}\,d\widetilde{W}_2(t), \text{ where}$$

$(\widetilde{W}_1(t), \widetilde{W}_2(t))$ is a standard 2-dimensional BM

$$dS(t) = rS(t)\,dt + \sqrt{V(t)}\,S(t)\,d\widetilde{W}_1(t)$$
$$dV(t) = (a - b\,V(t))\,dt + \sigma\sqrt{V(t)}\left(\rho\,d\widetilde{W}_1(t) + \sqrt{1-\rho^2}\,d\widetilde{W}_2(t)\right)$$

$$\sigma(t, x, v) = \begin{Vmatrix} \sqrt{v}\,x & 0 \\ \rho\sigma\sqrt{v} & \sqrt{1-\rho^2}\,\sigma\sqrt{v} \end{Vmatrix}$$

$$\sigma\sigma^T(t,x,v) = \left\| \begin{matrix} v x^2 & \sigma x v \rho \\ \sigma x v \rho & \sigma^2 v \end{matrix} \right\|$$

$\underline{\text{II}}$. Compute directly $\| a_{ij}(t,x,v) \|$.

$$d[S,S](t) = V(t)\, S^2(t)\, dt \Rightarrow a_{11}(t,x,v) = v x^2$$

$$d[S,V](t) = \sigma V(t) S(t) \rho\, dt \Rightarrow a_{12}(t,x,v) = a_{21}(t,x,v) = \sigma \rho v x$$

$$d[V,V](t) = \sigma^2 V(t)\, dt \Rightarrow a_{22}(t,x,v) = \sigma^2 v$$

Why does this work? Recall our heuristics:

$$E\left(\left(S(t+\delta) - S(t)\right)^2 \middle| S(t)=x,\ V(t)=v\right)$$

$$\approx E\left(\left(r S(t)\delta + \sqrt{V(t)}\, S(t) \sqrt{\delta}\, Z\right)^2 \middle| S(t)=x,\ V(t)=\delta\right)$$

$$\approx v x^2 \delta + \bar{o}(\delta) \Rightarrow a_{11}(t,x,v) = v x^2$$

$$E\left(\left(S(t+\delta) - S(t)\right)\left(V(t+\delta) - V(t)\right) \middle| S(t)=x,\ V(t)=v\right)$$

$$\approx E\left(\left(r S(t)\delta + \sqrt{V(t)}\, S(t)\sqrt{\delta}\, Z_1\right)\left((a - b V(t))\delta + \sigma\sqrt{V(t)}\sqrt{\delta}\, Z_2\right) \middle| S(t)=x,\ V(t)=v\right)$$

$$= \sigma v x \delta \underbrace{E(Z_1 Z_2)}_{\rho} + \bar{o}(\delta) \Rightarrow a_{12}(t,x,v) = \sigma v x \rho.$$

$a_{22}(t,x,v)$ is determined similarly. Thus,

$$\mathcal{A} := r x \frac{\partial}{\partial x} + (a - b v)\frac{\partial}{\partial v} + \frac{1}{2} v x^2 \frac{\partial^2}{\partial x^2} + \sigma \rho v x \frac{\partial^2}{\partial x \partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2}{\partial v^2}.$$

- By the FK formula $g(t,x,v) = \mathbb{E}^{t,x,v}\, h(S(T), V(T))$ represents the solution to

$$\frac{\partial g}{\partial t} + \mathcal{A} g = 0 \quad \text{with the terminal condition}$$

$$g(T,x,v) = h(x,v).$$

(3) Kolmogorov backward and forward equations.

Main point: we have an SDE $(\mathcal{E})$ and its solution X such that $X(t) = x$.
For a given T, $X(T)$ is a random variable, whose distribution depends on parameters of $(\mathcal{E})$ and also on t, x.

- $X(T)$ has a density denoted by $p(\underline{t}, \underline{x}; \underline{T}, y)$. Here t, x, T are $\underline{\text{fixed}}$ parameters, and to get the density of $X(T)$ you have to consider it as a function of y.

Question: how to find this density?

- Variables (x,t) are called "backward variables" as they correspond to the starting point of the process X: $X(t) = x$.
- Variables (T, y) are called "forward variables" as y is the "location" of the process at a future time T.
- $(X(u))_{t \le u \le T}$ is a Markov process, and $p(t,x;T;y)$ is its transition probability density, that is

$$P(X(T) \in B \mid X(t) = x) = \int_B p(t,x;T,y)\,dy, \quad \forall B \in \mathcal{B}.$$

Remark. When the coefficients b and σ do not depend on t, that is when $b = b(x)$ and $\sigma = \sigma(x)$ then p is a function of $(T-t), x, y$ (as for BM)

- all in all $p(t,x;T,y)$ has 2 pairs of variables (t,x) – backward and (T,y) – forward.
- Kolmogorov <u>backward</u> equation is the PDE to which $p(t,x;\underline{T},\underline{y})$ is a solution (with some special <u>terminal</u> data) when T,y are treated as fixed parameters and t,x as variables.
- Kolmogorov <u>forward</u> equation is the PDE to which $p(\underline{t},\underline{x};T,y)$ is a solution (with some special initial data) when x,t are treated as fixed parameters ant T,y as variables.

  In particular, the density of $X(T)$ satisfies the forward equation in <u>variables</u> T,y.

<u>Example</u>. (heat equation)

$$\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2} = 0 \; ; \; g(x,T) = h(x) \qquad (3)$$

$(d=1)$ $\quad b \equiv 0$; $a \equiv 1$ $\Rightarrow$ the corresponding SDE is

$$dX(u) = dB(u) \; ; \; X(t) = x$$

Thus $X(u) = x + B(u) - B(t)$, $\quad t \le u \le T$

Generator

$$\mathcal{A} = \frac{\partial^2}{\partial x^2}.$$

$$g(t,x) = \mathbb{E}^{t,x} h(X(T)) \quad \text{– solution to (3).}$$

$$g(t,x) = \mathbb{E}^{t,x} h\left(x + B(T) - B(t)\right)$$

Since $B(T) - B(t) \sim N(0, T-t)$, we can compute $B(T)-B(t) \overset{d}{=} \sqrt{T-t}\, Z$

(4) $$g(t,x) = \int_{-\infty}^{\infty} h(x + \sqrt{T-t}\, z) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

Look at it now from the perspective of $p(t,x;T,y)$

$$g(t,x) = \mathbb{E}^{t,x} h(X(T)) =$$

(5) $$\int_{-\infty}^{\infty} h(y)\, p(t,x;T,y)\, dy$$

- If $h(x) = 1_B(x)$ for some set B then

$$g(t,x) = \int_B p(t,x;T,y)\, dy = P(X(T) \in B \,/\, X(t) = x).$$
$$= \mathbb{E}^{t,x}\left(1_B(X(T))\right) \quad \text{(FK formula)}$$

- Comparing (4) & (5) (setting $y = x + \sqrt{T-t}\, z$ in (4)) we get that

$$p(t,x;T,y) = \frac{1}{\sqrt{(2\pi)(T-t)}} e^{-\frac{(x-y)^2}{2(T-t)}}.$$

– the transition probability density of BM;

$p(t,x;T,y)\,dy$ is "the probability that the BM that started at x at time t will be in $(y, y+dy)$ at time T."

$(d=1)$ <u>Theorem.</u> Let X solve $(\mathcal{E})$ on $[t,T]$ and $X(t)=x$. Let

$$p(t,x;T;y)=\lim_{|B|\to 0}\frac{1}{|B|}P(X(T)\in B \,/\, X(t)=x)$$

($|B|$ is the Lebesgue measure of set $B\in\mathcal{B}$)

Then for $x\in\mathbb{R}^d$, $0\le t<T$

(6) $\quad \frac{\partial p}{\partial t}+\mathcal{A}_t p=0$ with the terminal condition (in variables t,x) $\quad p(T,x;T,y)=\delta_y(x)$.

The latter means that $\forall f\in C_B(\mathbb{R})$

$$\lim_{t\uparrow T}\int f(x)\,p(t,x;T,y)\,dx=f(y),$$ or

equivalently, that for every $B\in\mathcal{B}$

$$\lim_{t\uparrow T}\int_B p(t,x;T,y)\,dx=1_B(y)$$

((6) is Kolmogorov backward equation, $\mathcal{A}$ is given by (2), p. 5)

<u>Theorem.</u> (Kolmogorov forward equation)

Define $\mathcal{A}_T^* f(T,y)=-\frac{\partial}{\partial y}\big(b(T,y)f(T,y)\big)$

$$+\frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\sigma^2(T,y)f(T,y)\big) \quad \text{(adjoint operator)}$$

Then $p(t,x;T,y)$ satisfies for $T>t$, $x\in\mathbb{R}^d$

$\frac{\partial p}{\partial T} = \mathcal{A}^* p$ with the initial condition
(in variables T, y) $\quad p(t,x;t,y) = \delta_x(y)$,
meaning that $\forall f \in C_B(\mathbb{R})$

$$\lim_{T \downarrow t} \int_{\mathbb{R}} f(y)\, p(t,x;T,y)\, dy = f(x), \text{ or}$$

equivalently, that $\forall B \in \mathcal{B}$

$$\lim_{T \downarrow t} \int_B p(t,x;T,y)\, dy = 1_B(x).$$

An example to have in mind: $X(t)$ is a standard Brownian motion (d-dimensional), $\mathcal{A}_t = \frac{1}{2}\Delta_x = \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$.

$$p(t,x,T,y) = \frac{1}{(2\pi)^{d/2}(T-t)^{1/2}} \exp\left(-\frac{1}{2(T-t)} \sum_{i=1}^{d} (y_i - x_i)^2\right)$$

Then the backward equation: $\frac{\partial p}{\partial t} + \frac{1}{2}\Delta_x p = 0$, $0 \le t < T$, $x \in \mathbb{R}^d$

$$\lim_{t \uparrow T} \int_{\mathbb{R}^d} f(x)\, p(t,x,T,y)\, dx \underbrace{=}_{\text{set } z_i = \frac{x_i - y_i}{\sqrt{T-t}}} \lim_{t \uparrow T} \int_{\mathbb{R}^d} f\left(y + \sqrt{2(T-t)}\, z\right) \frac{1}{(2\pi)^{d/2}} e^{-\frac{\|z\|^2}{2}}\, dz = f(y).$$

for all continuous bounded functions f.
Computing the adjoint operator we get that
in this case $\mathcal{A}_T^* = \frac{1}{2}\Delta_y = \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2}{\partial y_i^2}$ and
the forward equation is

$$\frac{\partial p}{\partial T} = \frac{1}{2}\Delta_y p; \quad t < T,\ x \in \mathbb{R}^d, \text{ and}$$

$$\lim_{T \downarrow t} \int_{\mathbb{R}^d} f(y)\, p(t,x,T,y) \underbrace{=}_{\text{set } z_i = \frac{y_i - x_i}{\sqrt{T-t}}} \lim_{T \downarrow t} \int_{\mathbb{R}^d} f\left(x + \sqrt{T-t}\, z\right) \frac{1}{(2\pi)^{d/2}} e^{-\frac{\|z\|^2}{2}}\, dz = f(x)$$

for all $f \in C_B(\mathbb{R}^d)$.