MTH 9831 Fall 2015 HW #3 Solutions

Please note that these are *possible* solutions, and that there are often many ways to approach and solve a problem or prove a theorem.

Problem 1 Show that the cross-variation of a Brownian motion $(B(t))_{t\geq 0}$ and t on $[0,T]$ is equal to zero (L^2-limit).

Let $||\Pi|| = \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T\}$. We have

$$\begin{aligned}
&\left(\sum_{i=1}^{n}(B(t_i) - B(t_{i-1}))(t_i - t_{i-1})\right)^2 \\
&= \sum_{i=1}^{n}(B(t_i) - B(t_{i-1}))^2(t_i - t_{i-1})^2 \\
&\quad + 2\sum_{i<j=1}(B(t_i) - B(t_{i-1}))(t_i - t_{i-1})(B(t_j) - B(t_{j-1}))(t_j - t_{j-1}).
\end{aligned}$$

Since $E[(B(t_i) - B(t_{i-1}))(B(t_j) - B(t_{j-1}))] = 0$ for $i \neq j$ and for $t_i - t_{i-1} < 1$, $(t_i - t_{i-1})^2 < (t_i - t_{i-1})$, we have

$$\begin{aligned}
&E\left[\left(\sum_{i=1}^{n}(B(t_i) - B(t_{i-1}))(t_i - t_{i-1})\right)^2\right] \\
&= \sum_{i=1}^{n}E[(B(t_i) - B(t_{i-1}))^2](t_i - t_{i-1})^2 \\
&\quad + 2\sum_{i<j}E[(B(t_i) - B(t_{i-1}))(B(t_j) - B(t_{j-1}))](t_i - t_{i-1})(t_j - t_{j-1}) \\
&= \sum_{i=1}^{n}(t_i - t_{i-1})^3 \leq ||\Pi||\sum_{j=1}^{n}(t_i - t_{i-1})^2 \leq ||\Pi||T \to 0 \text{ as } ||\Pi|| \to 0.
\end{aligned}$$

Problem 2 Show that if we let $t_i^* = (t_i + t_{i-1})/2$, $i = 1, \ldots, n$, in (2.1) in the notes of Lecture 3, then

$$\int_0^T B(t) \circ dB(t) = \frac{1}{2}B^2(T).$$

Let $0 = t_0 < t_1 < \ldots < t_n = T$, then

$$\begin{aligned}
&\int_0^T B(t) \circ dB(t) \\
&\overset{L^2}{=} \lim_{||\Pi||\to 0}\sum_{i=1}^{n}B(t_i^*)(B(t_i) - B(t_{i-1}) \\
&\overset{L^2}{=} \lim_{||\Pi||\to 0}\sum_{i=1}^{n}\left[B(t_i^*)\left(B(t_i) - B(t_i^*)\right) + B(t_{i-1})\left(B(t_i^*) - B(t_{i-1})\right)\right] + \lim_{||\Pi||\to 0}\sum_{i=1}^{n}\left(B(t_i^*) - B(t_{i-1})\right)^2.
\end{aligned}$$

The first sum evaluated on the partition $0 = t_0 < t_1^* < t_1 < \ldots < t_n^* < t_n = T$ converges in L^2 to the Ito integral and hence we have

$$\int_0^T B(t) \circ dB(t) \overset{L^2}{=} \frac{1}{2}(B^2(T) - T) + \lim_{||\Pi||\to 0}\sum_{i=1}^{n}\left(B(t_i^*) - B(t_{i-1})\right)^2.$$

If we show that

$$\lim_{||\Pi||\to 0} E\left[\sum_{i=1}^{n}\left((B(t_i^*)-B(t_{i-1}))^2-\frac{T}{2}\right)^2\right]=0$$

we are done. We define $\delta_i := (B(t_i^*)-B(t_{i-1}))^2 - \frac{t_i-t_{i-1}}{2}$ and note that

$$\begin{aligned}
E\left[\delta_i^2\right] &= E\left[(B(t_i^*)-B(t_{i-1}))^4\right] + (t_i-t_{i-1})E\left[(B(t_i^*)-B(t_{i-1}))^2\right] + \frac{(t_i-t_{i-1})^2}{4} \\
&= 3\frac{(t_i-t_{i-1})^2}{4} - \frac{(t_i-t_{i-1})^2}{2} + \frac{(t_i-t_{i-1})^2}{4} = \frac{(t_i-t_{i-1})^2}{2} \\
E\left[\delta_i\right] &= 0.
\end{aligned}$$

With that we finally obtain

$$\begin{aligned}
\lim_{||\Pi||\to 0} E\left[\sum_{i=1}^{n}\left((B(t_i^*)-B(t_{i-1}))^2-\frac{T}{2}\right)^2\right] &= \lim_{||\Pi||\to 0} E\left[\left(\sum_{i=1}^{n}\delta_i\right)^2\right] \\
&= \lim_{||\Pi||\to 0}\sum_{i=1}^{n} E\left[\delta_i^2\right] + 2\lim_{||\Pi||\to 0}\sum_{i<j} E\left[\delta_i\delta_j\right] \\
&= \lim_{||\Pi||\to 0}\sum_{i=1}^{n}\frac{(t_i-t_{i-1})^2}{2} + 2\lim_{||\Pi||\to 0}\sum_{i<j} E\left[\delta_i\right]E\left[\delta_j\right] \\
&= \lim_{||\Pi||\to 0}\sum_{i=1}^{n}\frac{(t_i-t_{i-1})^2}{2} + 0 \\
&= 0.
\end{aligned}$$

Problem 3 Prove that, for $\Delta(\omega, s)$ a simple function on $(\Omega, [0,T])$, the stochastic integral $I(t) = \int_0^t \Delta(s)dB(s)$ is a martingale. In particular $E(I(t)) = E(I(0)) = 0$.

We have checked for finiteness and measurability in the TA session, hence only check the martingale property. For $t_j \leq s < t_{j+1}$, $t_k < t \leq t_{k+1} \leq T$, and by various instances of the tower property measurable so that each Δ_i pops out, and lets each independent Brownian increment be mean 0,

$$\begin{aligned}
&E(I(t)|\mathcal{F}(s)) \\
&= I(s) + E(I(t)-I(s)) \\
&= I(s) + E(\Delta_j(B(t_{j+1})-B(s))) + \sum_{i=j+1}^{k} E(\Delta_{i-1}(B(t_i)-B(t_{i-1}))) + E(\Delta_k(B(t)-B(t_k))) \\
&= I(s) + E(E(\Delta_j(B(t_{j+1})-B(s))|\mathcal{F}_{t_j})) + \sum_{i=j+1}^{k} E(E(\Delta_{i-1}(B(t_i)-B(t_{i-1}))|\mathcal{F}_{t_j})) \\
&\quad + E(E(\Delta_k(B(t)-B(t_k))|\mathcal{F}_{t_j})) \\
&= I(s) + E(\Delta_j E((B(t_{j+1})-B(s)))) + \sum_{i=j+1}^{k} E(\Delta_{i-1}E((B(t_i)-B(t_{i-1})))) \\
&\quad + E(\Delta_k E((B(t)-B(t_k)))) = I(s).
\end{aligned}$$

In particular we have $E[I(t)] = 0$ as $I(t)$ is a martingale and $I(0) = 0$.

Problem 4 The base case $n = 1$ is easy: $E(B(t)^2) = t^1(2(1) - 1)!! = t$ is well known. Next, we assume $E(B(t)^{2n}) = t^n(2n-1)!!$ and prove the $n+1$ case with a simple integration by parts:

$$u = -x^{2n+1} \implies du = -(2n+1)x^{2n}dx; \;\; v = e^{-\frac{x^2}{2t}} \implies dv = -\frac{x}{t}e^{-\frac{x^2}{2t}}dx$$

$$\begin{aligned} E(B(t)^{2n+2}) &= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} x^{2n+2}e^{-\frac{x^2}{2t}}dx = \frac{t}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} u\,dv \\ &= \frac{t}{\sqrt{2\pi t}}\left(uv|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\,du\right) = \frac{t(2n+1)}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} x^{2n}e^{-\frac{x^2}{2t}}dx = t(2n+1)E(B(t)^{2n}). \end{aligned}$$

Problem 5

(i) We show that $I(t) - I(s)$ is independent of $\mathcal{F}(s)$ for $0 \le s < t \le T$. We follow the suggested simplification in the hint and consider $I(t_k) - I(t_l)$ for two partition points with $t_l < t_k$. Then the increment can be expressed as

$$I(t_k) - I(t_l) = \sum_{j=l}^{k-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)).$$

As $(\Delta(t))_{t\ge 0}$ is a deterministic function, increments of Brownian motion $W(t_{j+1}) - W(t_j)$ are independent of $\mathcal{F}(t_j)$ and $\mathcal{F}(t_l) \subset \mathcal{F}(t_j)$ for all $j = l, \ldots, k-1$, we have that $I(t_k) - I(t_l)$ is independent of $\mathcal{F}(t_l)$.

(ii) We use the same simplification as in (i). The increment $I(t_k) - I(t_l)$ is a linear combination of independent normal random variables and hence is normal itself. We have

$$E[I(t_k)-I(t_l)] = E[\sum_{j=l}^{k-1} \Delta(t_j)(W(t_{j+1})-W(t_j))] = \sum_{j=l}^{k-1} \Delta(t_j)E[(W(t_{j+1})-W(t_j))] = 0$$

and

$$\begin{aligned} Var[I(t_k) - I(t_l)] &= Var[\sum_{j=l}^{k-1} \Delta(t_j)(W(t_{j+1}) - W(t_j))] \\ &= \sum_{j=l}^{k-1} \Delta^2(t_j)Var[(W(t_{j+1}) - W(t_j))] = \sum_{j=l}^{k-1} \Delta^2(t_j)(t_{j+1} - t_j)] \\ &= \int_{t_k}^{t_l} \Delta(t)^2 dt. \end{aligned}$$

(iii) For $0 \le s \le t \le T$, we have $E[I(t) - I(s)|\mathcal{F}(s)] = E[I(t) - I(s)] = 0$.

(iv) For $0 \le s \le t \le T$, we get

$$\begin{aligned}
&E\left[I^2(t) - \int_0^t \Delta^2(u)du - I^2(s) + \int_0^s \Delta^2(u)du \mid \mathcal{F}(s)\right] \\
&= E\left[I^2(t) - I^2(s) - \int_s^t \Delta^2(u)du \mid \mathcal{F}(s)\right] \\
&= E\left[(I(t)-I(s))^2 + 2I(s)(I(t)-I(s)) \mid \mathcal{F}(s)\right] - \int_s^t \Delta^2(u)du \\
&= E\left[(I(t)-I(s))^2\right] + 2I(s)E\left[I(t)-I(s) \mid \mathcal{F}(s)\right] - \int_s^t \Delta^2(u)du \\
&= \int_s^t \Delta^2(u)du + 0 - \int_s^t \Delta^2(u)du = 0,
\end{aligned}$$

where we have used the independence of $I(t) - I(s)$ of $\mathcal{F}(s)$ and that $I(s)$ is $\mathcal{F}(s)$-measurable in the third equality.

Problem 6 From the definition of the integral we have that

$$I(t) - I(s) = \Delta(0)(W(s)-W(0)) + \Delta(s)(W(t)-W(s)) - \Delta(0)(W(s)-W(0)) = W(s)(W(t)-W(s)).$$

(i) False, as $W(s) \in \mathcal{F}(s)$.

(ii) False, we have

$$\begin{aligned}
&E[(I(t)-I(s))^4] = E[W^4(s)(W(t)-W(s))^4] = E[W^4(s)]E[(W(t)-W(s))^4] = 3s^2 \cdot 3(t-s)^2 \\
&3Var[I(t)-I(s)] = 3E[(I(t)-I(s))^2] = 3E[W^2(s)]E[(W(t)-W(s))^2] = 3s(t-s).
\end{aligned}$$

(iii) True, as $E[I(t) - I(s) \mid \mathcal{F}(s)] = W(s)E[W(t) - W(s)] = 0$.

(iv) True, as we have

$$\begin{aligned}
&E\left[I^2(t) - \int_0^t \Delta^2(u)du - I^2(s) + \int_0^s \Delta^2(u)du \mid \mathcal{F}(s)\right] \\
&= E\left[I^2(t) - I^2(s) - \int_s^t \Delta^2(u)du \mid \mathcal{F}(s)\right] \\
&= E\left[(I(t)-I(s))^2 + 2I(s)(I(t)-I(s)) - \int_s^t \Delta^2(u)du \mid \mathcal{F}(s)\right] \\
&= E\left[W(s)^2(W(t)-W(s))^2 + 2\Delta(0)W^2(s)(W(t)-W(s)) - W^2(s)(t-s) \mid \mathcal{F}(s)\right] \\
&= W(s)^2E\left[(W(t)-W(s))^2\right] + 2\Delta(0)W^2(s)E[W(t)-W(s)] - W^2(s)(t-s) \\
&= W(s)^2(t-s) + 0 - W^2(s)(t-s) = 0.
\end{aligned}$$

Problem 7 Find the expectation and variance of $E(\int_0^T B(t)^2 dt)$.

First, we'll use the technique from the lecture notes: since we know that $d(B(t)^2) = 2B(t)dB(t) + dt$, we have for differentiable $f(t)$ the product rule result

$$\begin{aligned}
d(f(t)B(t)^2) = f'(t)B(t)^2dt + f(t)d(B(t)^2) = f'(t)B(t)^2dt + 2f(t)B(t)dB(t) + f(t)dt \\
f(t) = t \implies d(tB(t)^2) = (B(t)^2 + t)dt + 2tB(t)dB(t),
\end{aligned}$$

which yields $B(t)^2dt = d(tB(t)^2) - tdt - 2tB(t)dB(t)$. Integrating and taking expectations, we have

$$E(\int_0^T B(t)^2dt) = E(\int_0^T d(tB(t)^2)) - \int_0^T tdt - E(\int_0^T 2tB(t)dB(t))$$
$$= E(TB(T)^2) - \frac{T^2}{2} \text{ (since the mean of the stochastic integral is zero)} = \frac{T^2}{2}.$$

We could have found this more easily via stochastic Fubini's theorem, switching $E(\cdot)$ and $\int_0^T \cdot dt$:

$$E(\int_0^T B(t)^2dt) = \int_0^T E(B(t)^2)dt = \int_0^T t\,dt = \frac{T^2}{2}.$$

The variance requires the second moment of this integral. By stochastic Fubini,

$$E\left(\left(\int_0^T B(t)^2dt\right)^2\right) = E\left(\left(\int_0^T B(t)^2dt\right)\left(\int_0^T B(s)^2ds\right)\right) = \int_0^T \int_0^T E(B(t)^2B(s)^2)dt\,ds.$$

We are then integrating over the square $[0,T]^2$, and so can ignore the probability-zero event $B(t) = B(s)$ on $s = t$ and focus on the two triangles

$$\{(s,t) : 0 \le s \le T, 0 \le t \le s\} \cup \{(s,t) : 0 \le t \le T, 0 \le s \le t\}.$$

We'll consider this simply as twice the integral over one of these triangles, since they are symmetric. Thus, rewriting for $s < t$

$$B(t)^2B(s)^2 = (B(t)-B(s)+B(s))^2B(s)^2 = (B(t)-B(s))^2B(s)^2+2(B(t)-B(s))^2B(s)^3+B(s)^4$$

and noticing that, by independence of increments, the expectations of these terms are

$$E[(B(t) - B(s))^2B(s)^2] = (t-s)s;\;\; E[(B(t) - B(s))^2B(s)^3] = 0;\;\; E[B(s)^4] = 3s^2,$$

we have the second moment

$$\int_0^T \int_0^T E(B(t)^2B(s)^2)dt\,ds = 2\int_0^T \int_0^t ((t-s)s + 3s^2)ds\,dt = 2\int_0^T \int_0^t (ts + 2s^2)ds\,dt$$
$$= 2\int_0^T \left[\frac{t^3}{2} + \frac{2t^3}{3}\right] dt = \frac{7}{3}\int_0^T t^3dt = \frac{7}{12}T^4,$$

which yields the variance

$$Var\left(\int_0^T B(t)^2dt\right) = E\left(\left(\int_0^T B(t)^2dt\right)^2\right) - E\left(\int_0^T B(t)^2dt\right)^2 = \frac{7}{12}T^4 - \left(\frac{1}{2}T^2\right)^2 = \frac{1}{3}T^4.$$