Comments/Solutions, MTH 9831 Fall 2015 HW #9
Baruch College MFE Program

Please note that these are *possible* solutions, and that there are often many ways to approach and solve a problem or prove a theorem.

Problem 1 (Shreve Vol. 2, Ch.7 #7.8, p 337: Asian call, $r = 0$)

For $r \neq 0$, we are given

$$\text{(7.5.22)} \qquad \gamma(t) = \begin{cases} \frac{1}{rc}(1 - e^{-rc}), & 0 \le t \le T - c \\ \frac{1}{rc}(1 - e^{-r(T-t)}), & T - c \le t \le T. \end{cases}$$

$$\text{(7.5.23)} \qquad X(0) = \frac{1}{rc}(1 - e^{-rc})S(0) - e^{-rT}K.$$

To find these values for $r = 0$, simply use l'Hôpital's rule, since we have $\frac{0}{0}$ situations in both cases:

$$\begin{aligned} \text{(7.5.22)} \qquad \gamma(t) &= \begin{cases} \lim_{r\downarrow 0} \frac{1}{rc}(1 - e^{-rc}), & 0 \le t \le T - c \\ \lim_{r\downarrow 0} \frac{1}{rc}(1 - e^{-r(T-t)}), & T - c \le t \le T \end{cases} \\ &= \begin{cases} \lim_{r\downarrow 0} \frac{ce^{-rc}}{c} = \lim_{r\downarrow 0} e^{-rc} = 1, & 0 \le t \le T - c \\ \frac{T-t}{c}, & T - c \le t \le T \end{cases} \\ \text{(7.5.23)} \qquad X(0) &= \lim_{r\downarrow 0} \left(\frac{1}{rc}(1 - e^{-rc})S(0) - e^{-rT}K \right) \\ &= S(0) - K. \end{aligned}$$

Thus, at time 0 we buy 1 share at $S(0)$ and borrow K. Hold this portfolio for $0 \le t \le T - c$. Then, at $t = T - c$, continuously sell off the 1 share until it has been liquidated at $t = T$. This yields the portfolio

$$X(0) = \begin{cases} S(t) - K, & 0 \le t \le T - c \\ \frac{T-t}{c} S(t) + \frac{1}{c}\int_{T-c}^{t} S(u)du - K, & T - c \le t \le T \end{cases}$$

and so, at $t = T$,

$$X(T) = \frac{1}{c}\int_{T-c}^{T} S(u)du - K.$$

Problem 2 (mimic Shreve pp 324-329: Asian option with different payoff)

We are given the payoff at time T of

$$V(T) = \left(\frac{1}{c}\int_{T-c}^{T} S(t)dt - S(T) \right)^{+}. \tag{1}$$

Our task is to mimic the argument from pp 324-329 of Shreve Vol II to find a portfolio process $\gamma(t)$, $0 \le t \le T$, which gives the number of shares of risky asset held at time t. Since the only thing changed in the payoff function is the value of the stock $S(T)$ instead of a strike price K, we'll change the strategy to borrow nothing and work solely on the value of the stock. Let

$$\gamma(t) = \begin{cases} \frac{1-e^{-rc}}{rc} - 1 & 0 \le t \le T - c, \\ \frac{1-e^{-r(T-t)}}{rc} - 1 & T - c < t < T, \end{cases} \tag{2}$$

with initial capital $X(0) = \gamma(0)S(0)$. Just like in the text, buy these shares at time 0 and hold them until time $T - c$. Then

$$X(t) = \gamma(t)S(t) = \left(\frac{1 - e^{-rc}}{rc} - 1\right) S(t), \;\; 0 \le t \le T - c, \tag{3}$$

and $d\gamma(t) = 0$ for $0 \le t \le T - c$. At time $T - c$, start selling off the stock continuously at rate $d\gamma(t) = -\frac{1}{c}e^{-r(T-t)}$ for $T - c < t \le T$. Then, parroting (7.5.21) and the calculations thereafter, integrating from $T - c$ to t yields

$$\begin{aligned}
d(e^{r(T-t)}X(t)) &= d(e^{r(T-t)}\gamma(t)S(t)) - e^{r(T-t)}S(t)d\gamma(t) \\
\implies e^{r(T-t)}X(t) &= e^{rc}X(T-c) + \int_{T-c}^{t} d(e^{r(T-u)}\gamma(u)S(u)) - \int_{T-c}^{t} e^{r(T-u)}S(u)d\gamma(u) \\
&= e^{rc}\left[\frac{1 - e^{-rc}}{rc} - 1\right] S(T-c) + e^{r(T-t)}\gamma(t)S(t) \\
&\quad - e^{rc}\left[\frac{1 - e^{-rc}}{rc} - 1\right] S(T-c) + \frac{1}{c}\int_{T-c}^{t} S(u)du \\
&= e^{r(T-t)}\gamma(t)S(t) + \frac{1}{c}\int_{T-c}^{t} S(u)du \\
\implies X(t) &= \gamma(t)S(t) + \frac{e^{-r(T-t)}}{c}\int_{T-c}^{t} S(u)du \\
&= \left[\frac{1 - e^{-r(T-t)}}{rc} - 1\right] S(t) + \frac{e^{-r(T-t)}}{c}\int_{T-c}^{t} S(u)du, \;\; T - c < t \le T \\
\implies X(T) &= -S(T) + \frac{1}{c}\int_{T-c}^{T} S(u)du.
\end{aligned}$$

To price an option whose payoff at time T is $V(T) = X(T)^+$, we use a change of numéraire argument to calculate the risk-neutral expectation

$$V(t) = \tilde{\mathbb{E}}[e^{-r(T-t)}V(T) \,|\, \mathcal{F}_t] = \tilde{\mathbb{E}}[e^{-r(T-t)}X(T)^+ \,|\, \mathcal{F}_t].$$

To do so, define $Y(t) = \frac{X(t)}{S(t)} = \frac{e^{-rt}X(t)}{e^{-rt}S(t)}$. Then, all the calculations on pp 326-327 are for a general $\gamma(t)$, so we reach Theorem 7.5.3 of Večeř so that $V(t) = S(t)g(t, \frac{X(t)}{S(t)})$ for the appropriate $g(t, y)$ such that

$$g_t(t,y) + \frac{1}{2}\sigma^2(\gamma(t) - y)^2 g_{yy}(t,y) = 0, \;\; \lim_{y\to-\infty} g(t,y) = 0, \;\; \lim_{y\to\infty}[g(t,y) - y] = 0, \;\; 0 \le t \le T.$$

Problem 3 (Laplace transform of τ_m) Nothing to type here. (Good reading!)

Problem 4 (Shreve Vol. 2, Ch.8 #8.3, p 370: solving the linear complementarity conditions) We are given the conditions

$$v(x) \geq (K-x)^+ \qquad \forall x \geq 0 \tag{4}$$

$$rv(x) - rxv'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \geq 0 \qquad \forall x \geq 0 \tag{5}$$

(i) Substituting $v(x) = Cx^p$ into the PDE (5) with equality at 0, we obviously have

$$\begin{aligned}
&v(x) = Cx^p,\ v'(x) = Cpx^{p-1},\ v''(x) = Cp(p-1)x^{p-2} \\
&\Longrightarrow rv(x) - rxv'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) = Cx^p\left[r - rp - \frac{1}{2}\sigma^2 p(p-1)\right] = 0 \\
&\Longrightarrow -\frac{1}{2}\sigma^2 p^2 + \left(\frac{1}{2}\sigma^2 - r\right)p + r = 0 \Longrightarrow r = -\frac{2r}{\sigma^2} \text{ or } 1
\end{aligned}$$

by solving the quadratic equation.

(ii) Now suppose $v(x) = Ax^{-\frac{2r}{\sigma^2}} + Bx$ satisfies (5) with equality on an interval $[x_1, x_2]$, while $v(x)$ and $v'(x)$ are both continuous $\forall x > 0$. We show that if $v(x)$ also satisfies (4) with equality continuously for $(x_1 - \epsilon, x_1]$ and $[x_2, x_2 + \epsilon)$ for some $\epsilon > 0$, then we must have $v(x) = 0$. There are two cases to consider:

(a) If $x_1 < K < x_2$ or $K \leq x_1 < x_2$, then $v(x) = (K-x)^+ = 0$ for $x \geq K$. But then $A = B = 0$ by continuity of $v(x)$, since for some $x \in [x_1, x_2]$, we have $v(x) = Ax^{-\frac{2r}{\sigma^2}} + Bx = 0$.

(b) If $0 < x_1 < x_2 < K$, then we need equality at x_1, x_2 by continuity of $v(x)$, $v'(x)$:

$$\begin{aligned}
v(x_1) &= Ax_1^{-\frac{2r}{\sigma^2}} + Bx_1 = K - x_1 \\
v(x_2) &= Ax_2^{-\frac{2r}{\sigma^2}} + Bx_2 = K - x_2 \\
v'(x_1) &= -\frac{2Ar}{\sigma^2}x_1^{-\frac{2r}{\sigma^2}-1} + B = -1 \\
v'(x_2) &= -\frac{2Ar}{\sigma^2}x_2^{-\frac{2r}{\sigma^2}-1} + B = -1.
\end{aligned}$$

The last two equations imply $A = 0$, $B = -1$ since $x_1 \neq x_2$. However, plugging these into the first two equations yields $K = 0$, which isn't valid. Hence, equality in both cannot occur as stated.

(iii) We require $v(0) = K$ since this is the value of a put with stock value 0. If $v(x) = Ax^{-\frac{2r}{\sigma^2}} + Bx$, then by continuity of $v(x)$, $v(0) = \lim_{x\to 0}(Ax^{-\frac{2r}{\sigma^2}} + Bx) = \infty$. Thus, since only one of (4) and (5) can have equality at each point, (4) must be the one with equality ($v(0) = (K-0)^+ = K$) and (5) the one with $>$ at $x = 0$. By continuity, then, (5) is $>$ for an interval $[0, x_2]$.

(iv) This is done in (iii)!

(v) $v(x) = (K-x)^+$ does not have $v'(x)$ continuous at $x = K$. Hence, this can't always be an equality.

(vi) To show that the transition from equality of $v(x) = (K-x)^+$ on $[0, x_1]$ (with (5) $>$) and (5) with $=$ on (x_1, ∞) happens at $x_1 = L_* = \frac{2r}{2r+\sigma^2}K$, $A = (K - L_*)(L_*)^{2r/\sigma^2}$, $B = 0$, just check the two linear equations for $v(x_1)$ and $v'(x_1)$ and solve for A and B:

$$\begin{aligned}
v(L_*) &= AL_*^{-\frac{2r}{\sigma^2}} + BL_* = K - L_* \\
v'(L_*) &= -\frac{2Ar}{\sigma^2}L_*^{-\frac{2r}{\sigma^2}-1} + B = -1.
\end{aligned}$$

Problem 5 (Shreve Vol. 2, Ch.8 #8.5(i), (ii), p 372: perpetual American put paying dividends) The SDE for the asset is

$$dS(t) = (r-a)S(t)dt + \sigma S(t)d\tilde{W}(t),$$

a GBM on the risk-neutral BM $\tilde{W}$ $(\tilde{P})$. We define β, set $S(0)$, and give the closed form of $S(t)$ as

$$\beta = r - a - \frac{1}{2}\sigma^2,\ S(0) = x,\ S(t) = x\exp\left[\sigma\tilde{W}(t) + \beta t\right].$$

(i) We exercise at time $\tau = \inf\{t \geq 0 :\ S(t) \leq L\}$. Defining

$$\gamma = \frac{\beta}{\sigma^2} + \frac{1}{\sigma}\sqrt{\frac{\beta^2}{\sigma^2} + 2r}$$

we have

$$S(t) = L \implies \tilde{W}(t) = \frac{1}{\sigma}\left(\ln\left(\frac{L}{x}\right) - \beta t\right).$$

Using Theorem 8.3.2,

$$X(t) = \mu t + \tilde{W}(t),\ m > 0,\ \tau_m = \inf\{t \geq 0 :\ X(t) = m\} \implies \tilde{E}(e^{-\lambda\tau_m}) = e^{-m(-\mu+\sqrt{\mu^2+2\lambda})}\ \forall\lambda > 0,$$

we have

$$\tilde{W}(t) = \frac{1}{\sigma}\left(\ln\left(\frac{L}{x}\right) - \beta t\right) \implies -\tilde{W}(t) - \frac{\beta}{\sigma}t = \frac{1}{\sigma}\ln\left(\frac{x}{L}\right)$$

(where $-\tilde{W}(t)$ is also a BM, $\mu = -\frac{\beta}{\sigma}$, $m = \frac{1}{\sigma}\ln\left(\frac{x}{L}\right) > 0$) and so

$$\tilde{E}(e^{-r\tau}) = e^{-m(-\mu+\sqrt{\mu^2+2r})} = \exp\left(-\frac{1}{\sigma}\ln\left(\frac{x}{L}\right)\sqrt{\frac{\beta^2}{\sigma^2} + 2r}\right) = \exp\left(-\gamma\ln\frac{x}{L}\right) = \left(\frac{x}{L}\right)^{-\gamma}.$$

Hence, the risk-neutral discounted expectation of the payoff is

$$v_L(x) = \tilde{E}^x(e^{-r\tau}(K - S(\tau))) = \begin{cases} K - x & 0 \leq x \leq L \\ (K-L)\left(\frac{x}{L}\right)^{-\gamma} & x > L. \end{cases}$$

(ii) Take the derivative of (i) with respect to L and find the max. Clearly, $\frac{\partial}{\partial L}v_L(x) = 0$ for $0 \leq x \leq L$, so we focus on $x > L$.

$$\frac{\partial}{\partial L}v_L(x) = -\left(\frac{x}{L}\right)^{-\gamma}\left(1 - \frac{\gamma(K-L)}{L}\right) = 0 \implies 1 - \frac{\gamma(K-L_*)}{L_*} = 0 \implies L_* = \frac{\gamma K}{\gamma+1}.$$