PDE representations of derivatives with bilateral counterparty risk and funding costs

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Abstract

We derive a partial differential equation (PDE) representation for the value of financial derivatives with bilateral counterparty risk and funding costs. The model is very general in that the funding rate may be different for lending and borrowing and the mark-to-market value at default can be specified exogenously. The buying back of a party's own bonds is a key part of the delta hedging strategy; we discuss how the cash account of the replication strategy provides sufficient funds for this. First, we assume that the mark-to-market value at default is given by the total value of the derivative, which includes counterparty risk. We find that the resulting pricing PDE becomes non-linear, except in special cases, when the non-linear terms vanish and a Feynman-Kac representation of the total value can be obtained. In these cases, the total value of the derivative can be decomposed into the default-free value plus a bilateral credit valuation and funding adjustment. Second, we assume that the mark-to-market value at default is given by the counterparty-riskless value of the derivative. This time, the resulting PDE is linear and the corresponding Feynman-Kac representation is used to decompose the total value of the derivative into the default-free value plus bilateral credit valuation and funding cost adjustments. A numerical example shows that the effect on the valuation adjustments of a non-zero funding spread can be significant.

1 Introduction

Counterparty credit risk implicitly embedded in derivative contracts has become increasingly relevant in recent market conditions. This risk represents the possibility that a counterparty defaults while owing money under the terms of a derivative contract, or more precisely, if the mark-to-market value of the derivative is positive to the seller at the time of default of the counterparty. While for exchange-traded contracts the counterparty credit risk is mitigated by the exchange's presence as intermediary, this is not the case for OTC products. For these, a number of different techniques are being used to mitigate the counterparty risk, most commonly by means of netting agreements and collateral mechanisms. The details of these agreements are for example specified by the International Swaps and Derivatives Association (ISDA) 2002 Master Agreement. However, the counterparty faces a similar risk of the seller defaulting when the mark-to-market value is positive to the counterparty. Taking into account the credit risk of both parties is commonly referred to as considering bilateral counterparty risk. When doing so, the value of the derivative to the seller is influenced by its own credit quality. Papers, books and book chapters that develop techniques for valuation of derivatives and derivative portfolios under counterparty risk include, but are not limited to, Alavian [1], Brigo and Mercurio [2], Gregory [5], Jarrow and Turnbull [6], Jarrow and Yu [7] and Li and Tang [8], Pykhtin and Zhu [10] and Cesari et. al. [4]. There are other areas where the credit of the seller is relevant, in particular in terms of mark-to-market accounting of its own debt, as well as the effect it has on its funding costs. Piterbarg [9] discusses the effect of funding costs on derivative valuations when collateral has to be posted. Here, we combine the effects of the seller's credit on its funding costs with that on the bilateral counterparty risk into a unified framework. We use hedging arguments to derive the extensions to the Black-Scholes PDE in the presence of bilateral counterparty risk in a bilateral jump-to-default model and include funding considerations into the financing of the hedge positions. In addition, we consider two scenarios for the determination of the derivative mark-to-market value at default, namely that recovery is on the total risky value or that it is on the counterparty-riskless value. The latter corresponds to the most common approach taken in the literature. The total value of the derivative will then depend on which of the two mark-to-market rules is used at default. For contracts following the ISDA 2002 Master Agreement, for example, the value of the derivative upon default of one of the counterparties is determined by a dealer poll. There is no reference to the counterparties and one could reasonably expect the derivative value to be the counterparty-riskless value, i.e., the second case considered. In the first case, where we use the default-risky derivative price as mark-to-market, we derive a pricing PDE that in general is non-linear and show that the unknown risky price can be found by solving a non-linear integral equation. Under certain conditions on the payoff, the non-linear terms vanish and we study the Feynman-Kac representation of the solution of the resulting linear PDE. In the second case, where we use the counterparty-riskless derivative price as mark-to-market, the resulting pricing PDE is linear. As in the first case, we use the Feynman-Kac representation to decompose the risky derivative value into a counterparty risk free part, a funding adjustment part, and a bilateral credit valuation adjustment (CVA) part. By using a hedging strategy to derive our results, we ensure that the hedging costs of all considered risk factors are included in the derivative price and our decomposition of the risky price is a generalisation of the result commonly found in the literature. Moreover, we get explicit expressions for the hedges, which is important for risk management. There have been discussions about how a seller can hedge out its own credit risk (see [4] for a summary). The strategy described in this paper includes (re-)purchase by the seller of its own bonds to hedge out its own credit risk. On the face of it, this may seem like a futile approach, since if this bond purchase were funded by issuing more debt (i.e., more bonds), the seller would in effect have achieved nothing in terms of hedging its own credit risk. However, the replication strategy presented shows how the funding for the purchase of the seller's own bond is achieved through the cash account of the hedging strategy. The hedging strategy (including the premium of the derivative) generates the cash needed to fund the repurchase of the seller's own bonds. Although all results in this paper are derived for one derivative on one underlying asset following a particular dynamic, they extend directly to the situation of a netted derivatives portfolio on several underlyings following general diffusion dynamics.

This paper is organised as follows. In Section 2 we summarise the main results of the paper. A general PDE for the counterparty risky derivative value is derived in section 3 but we defer the specification of the boundary condition at default to Sections 4 and 5. In these sections we assume that the mark-to-market value of the derivative at default is given by the risky value and the counterparty-riskless value discounted at the risk-free rate, respectively. We then consider some examples in Section 6 before concluding in Section 7.

2 Main results

We consider a derivative contract $\hat{V}$ on an asset S between a seller B and a counterparty C that may both default. The asset S is not affected by a default of either B or C, and is assumed to follow a Markov process with generator $\mathcal{A}_t$. Similarly we let V denote the same derivative between two parties that cannot default. At default of either the counterparty or the seller, the value of the derivative to the seller $\hat{V}$ is determined with a mark-to-market rule M, which may equal $\hat{V}$ or V (Throughout this paper we use the convention that positive derivative values correspond to seller assets and a counterparty liabilities).

By using replication arguments and including funding costs, we derive the PDEs in Equations (1) and (2) below. The rates, spreads and recoveries used here and throughout the paper are summarised in Table 1 below.

Main result 1. PDE for $\hat{V}$ when $M = \hat{V}$:
When the mark-to-market value at default is given by $M = \hat{V}$, then $\hat{V}$ satisfies the PDE

$$\partial_t \hat{V} + \mathcal{A}_t \hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+ + s_F \hat{V}^+. \tag{1}$$

Main result 2. PDE for $\hat{V}$ when $M = V$:
When the mark-to-market value at default is given by $M = V$, then $\hat{V}$ satisfies the PDE

$$\partial_t \hat{V} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} \quad = \quad -(R_B\lambda_B + \lambda_C)V^- - (\lambda_B + R_C\lambda_C)V^+ + s_F V^+. \tag{2}$$

Main result 3. Credit valuation adjustment (CVA) when $M = V$:
Let $M = V$ and $r_F = r + s_F$. Then $\hat{V} = V + U$, where the credit valuation adjustment U is

Rate	Definition	Choices discussed
r	risk-free rate	
r_B	yield on recovery-less bond of seller B	
r_C	yield on recovery-less bond of counterparty C	
λ_B	$\lambda_B \equiv r_B - r$	
λ_C	$\lambda_C \equiv r_C - r$	
r_F	seller funding rate for borrowed cash on seller's derivatives replication cash account	$r_F = r$ if derivative can be used as collateral $r_F = r + (1 - R_B)\lambda_B$ if derivative cannot be used as collateral
s_F	$s_F \equiv r_F - r$	
R_B	recovery on derivate mark-to-market value in case seller B defaults	
R_C	recovery on derivate mark-to-market value in case counterparty C defaults	

Table 1: Definitions of the rates used throughout the paper.

given by

$$
\begin{aligned}
U(t,S) &= -(1-R_B)\int_t^T \lambda_B D_{r+\lambda_B+\lambda_C}\mathbb{E}_t\left[V^-(u,S(u))\right]du \\
&\quad -(1-R_C)\int_t^T \lambda_C D_{r+\lambda_B+\lambda_C}\mathbb{E}_t\left[V^+(u,S(u))\right]du \\
&\quad -\int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}\mathbb{E}_t\left[V^+(u,S(u))\right]du, \qquad (3)
\end{aligned}
$$

where $D_k(t,u) \equiv \exp\left(-\int_t^u k(v)dv\right)$ is the discount factor between times t and u using rate k. If $s_F = 0$, then U is identical to the regular bilateral CVA derived in many of the papers cited in Section 1.

Another important result of the paper is the justification on which the seller's own credit risk can be taken into account. In the hedging strategy considered, this risk is hedged out by the seller buying back its own bond. It is shown that the cash needed for doing so is generated through the replication strategy.

3 Model setup and derivation of a bilateral risky PDE

We consider an economy with the following four traded assets:

$$
\begin{aligned}
P_R &: \text{ default risk-free zero-coupon bond.} \\
P_B &: \text{ default risky, zero-recovery, zero-coupon bond of party } B. \\
P_C &: \text{ default risky, zero-recovery, zero-coupon bond of party } C. \\
S &: \text{ spot asset with no default risk.}
\end{aligned}
$$

Both the risky bonds P_B and P_C pay 1 at some future date T if the issuing party has not defaulted and zero otherwise. These simplistic bonds are useful for modelling and can be used as building blocks of more complex corporate bonds, including ones with non-zero recovery. We assume that the processes for the assets P_R, P_B, P_C and S under the historical probability measure are specified by

$$
\begin{cases}
\frac{dP_R}{P_R} &= r(t)dt \\
\frac{dP_B}{P_B} &= r_B(t)dt - dJ_B \\
\frac{dP_C}{P_C} &= r_C(t)dt - dJ_C \\
\frac{dS}{S} &= \mu(t)dt + \sigma(t)dW,
\end{cases} \qquad (4)
$$

where $W(t)$ is a Wiener process, $r(t) > 0$, $r_B(t) > 0$, $r_C(t) > 0$, $\sigma(t) > 0$ are deterministic functions of t and J_B and J_C are two independent point processes that jump from 0 to 1 on default of B and C respectively. This assumption will imply that we can hedge using the bonds P_B and P_C alone, but in Section 7 we will discuss how we can relax this. We further stress that the spot asset price S is assumed to be unaffected by a default of party B or C. For the remainder of this paper we will refer to B as the *seller* and C as the *counterparty* respectively.

Now assume that the parties B and C enter a derivative on the spot asset that pays the seller B the amount $H(S) \in \mathbb{R}$ at maturity T. Thus in our convention the payout scenario $H(S) \geq 0$ means that the seller receives cash or an asset from the counterparty. The value of this derivative to the seller at time t is denoted $\hat{V}(t, S, J_B, J_C)$ and depends on the spot S of the underlying and the default states J_B and J_C of the seller B and counterparty C. Analogously we let $V(t, S)$ denote the value to the seller of the same derivative if it were a transaction between two default free parties.

When party B or C defaults, in general the mark-to-market value of the derivative determines the close-out or claim on the position. However, the precise nature of this depends on the contractual details and the mechanism by which the mark-to-market is determined. The 2002 ISDA Master Agreement specifies that the derivative contract will return to the surviving party the recovery value of its positive mark-to-market value (from the view of the surviving party) just prior to default, whereas the full mark-to-market has to be paid to the defaulting party if the mark-to-market value is negative (from the view of the surviving party). The Master Agreement specifies a dealer poll mechanism to establish the mark-to-market to the seller $M(t, S)$ at default, without referring to the names of the counterparties involved in the derivative transaction. In this case, one would expect $M(t, S)$ to be close to $V(t, S)$, even though it is unclear whether the dealers in the poll may or may not include their funding costs in the derivatives price. In other cases, not following the Master agreement, there may be other mechanisms described. Hence we will derive the PDE for the general case $M(t, S)$ and consider the two special cases where $M(t, S) = \hat{V}(t, S, 0, 0)$ and $M(t, S) = V(t, S)$ in Sections 4 and 5 respectively. Let $R_B \in [0, 1]$ and $R_C \in [0, 1]$ denote the recovery rates on the derivative positions of parties B and C respectively. In this paper we take them to be deterministic. From the above discussions it follows that we have the following boundary conditions

$$\begin{aligned} \hat{V}(t, S, 1, 0) &= M^+(t, S) + R_B M^-(t, S) \quad \text{Seller defaults first,} \\ \hat{V}(t, S, 0, 1) &= R_C M^+(t, S) + M^-(t, S) \quad \text{Counterparty defaults first.} \end{aligned} \tag{5}$$

Gregory [5], Li and Tang [8], and the vast majority of papers on valuation of counterparty risk use $M(t, S) = V(t, S)$.

As in the usual Black-Scholes framework, we hedge the derivative with a self-financing portfolio that covers all the underlying risk factors of the model. In our case the portfolio Π that the seller sets up consists of $\delta(t)$ units of S, $\alpha_B(t)$ units of P_B, $\alpha_C(t)$ units of P_C, and $\beta(t)$ units of cash, such that the portfolio value at time t hedges out the value of the derivative contract to the seller, i.e. $\hat{V}(t) + \Pi(t) = 0$. Thus,

$$-\hat{V}(t) = \Pi(t) = \delta(t)S(t) + \alpha_B(t)P_B(t) + \alpha_C(t)P_C(t) + \beta(t). \tag{6}$$

Before proceeding we note that when $\hat{V} \geq 0$ the seller will incur a loss at counterparty default. To hedge this loss, P_C needs to be shorted, so we expect that $\alpha_C \leq 0$. Assuming that the seller can borrow the bond P_C close to the risk free rate r through a repurchase agreement, the spread λ_C between the rate r_C on the bond and the cost of financing the hedge position in C can be approximated to $\lambda_C = r_C - r$. Since we defined P_C to be a bond with zero recovery, this spread corresponds to the default intensity of C.

On the other hand if $\hat{V} \leq 0$ the seller will gain at own default, which can be hedged by buying back P_B bonds, so we expect that $\alpha_B \geq 0$ in this case. For this to work, we need to ensure that enough cash is generated and that any remaining cash (after purchase of P_B) is invested in a way that does not generate additional credit risk to the seller: i.e. any remaining positive cash generates a yield at the risk free rate r.

Imposing that the portfolio $\Pi(t)$ is self-financing implies that

$$-d\hat{V}(t) = \delta(t)dS(t) + \alpha_B(t)dP_B(t) + \alpha_C(t)dP_C(t) + d\bar{\beta}(t), \tag{7}$$

where the growth in cash [1] $d\bar{\beta}$ may be decomposed as $d\bar{\beta}(t) = d\bar{\beta}_S(t) + d\bar{\beta}_F(t) + d\bar{\beta}_C(t)$ with:

- $d\bar{\beta}_S(t)$: The share position provides a dividend income of $\delta(t)\gamma_S(t)S(t)dt$ and a financing cost of $-\delta(t)q_S(t)S(t)dt$, so $d\bar{\beta}_S(t) = \delta(t)(\gamma_S(t) - q_S(t))S(t)dt$. Here the value of $q_S(t)$ depends on the risk-free rate $r(t)$ and the repo-rate of $S(t)$.

- $d\bar{\beta}_F(t)$: From the above analysis, any surplus cash held by the seller after the own bonds have been purchased must earn the risk-free rate $r(t)$ in order not to introduce any further credit risk to the seller. If borrowing money, the seller needs to pay the rate $r_F(t)$. For this rate, we distinguish two cases in the following: where the derivative itself can be used as collateral for the required funding, we assume no haircut and set $r_F(t) = r(t)$. If, however, the derivative cannot be used as collateral, we set the funding rate to the yield of the unsecured seller bond with recovery R_B: i.e. $r_F(t) = r(t) + (1 - R_B)\lambda_B$. In practise the latter case is often the more realistic one. Keeping r_F general for now, we have

$$\begin{aligned} d\bar{\beta}_F(t) &= \{r(t)(-\hat{V} - \alpha_B P_B)^+ + r_F(t)(-\hat{V} - \alpha_B P_B)^-\}dt &(8)\\ &= r(t)(-\hat{V} - \alpha_B P_B)dt + s_F(-\hat{V} - \alpha_B P_B)^- dt, &(9)\end{aligned}$$

  where the *funding spread* $s_F \equiv r_F - r$: i.e. $s_F = 0$, if the derivative can be used as collateral, and $s_F = (1 - R_B)\lambda_B$, if it cannot.

- $d\bar{\beta}_C(t)$: By the arguments above, the seller will short the counterparty bond through a repurchase agreement and incur financing costs of $d\bar{\beta}_C(t) = -\alpha_C(t)r(t)P_C(t)dt$ if we assume a zero haircut.

For the remainder of this paper we will drop the t from our notation, where applicable, to increase clarity. From the above analysis it follows that the change in the cash account is given by

$$d\bar{\beta} = \delta(\gamma_S - q_S)Sdt + \{r(-\hat{V} - \alpha_B P_B) + s_F(-\hat{V} - \alpha_B P_B)^-\}dt - r\alpha_C P_C dt, \tag{10}$$

[1] Note that this growth is the growth in the cash account before rebalancing of the portfolio. The self-financing condition ensures that after dt the rebalancing can happen at zero overall cost. The original version of this paper used notation $d\beta$, suggesting the total change in the cash position. This notation is corrected here. The authors are grateful for Damiano Brigo et al. [3] for pointing this out.

so (7) becomes

$$\begin{aligned}
-d\hat{V} &= \delta dS + \alpha_B dP_B + \alpha_C dP_C + d\bar{\beta} \\
&= \delta dS + \alpha_B P_B(r_B dt - dJ_B) + \alpha_C P_C(r_C dt - dJ_C) \\
&\quad + \{r(-\hat{V} - \alpha_B P_B) + s_F(-\hat{V} - \alpha_B P_B)^- - \alpha_C r P_C - \delta(q_S - \gamma_S)S\}dt \qquad (11) \\
&= \Big\{ - r\hat{V} + s_F(-\hat{V} - \alpha_B P_B)^- + (\gamma_S - q_S)\delta S + (r_B - r)\alpha_B P_B + (r_C - r)\alpha_C P_C \Big\} dt \\
&\quad -\alpha_B P_B dJ_B - \alpha_C P_C dJ_C + \delta dS. \qquad (12)
\end{aligned}$$

On the other hand, by Itô's Lemma for jump diffusions and our assumption that a simultaneous jump is a zero probability event, the derivative value moves by

$$d\hat{V} = \partial_t \hat{V} dt + \partial_S \hat{V} dS + \frac{1}{2}\sigma^2 S^2 \partial_S^2 \hat{V} dt + \Delta\hat{V}_B dJ_B + \Delta\hat{V}_C dJ_C, \qquad (13)$$

where

$$\begin{aligned}
\Delta\hat{V}_B &= \hat{V}(t,S,1,0) - \hat{V}(t,S,0,0), \qquad (14) \\
\Delta\hat{V}_C &= \hat{V}(t,S,0,1) - \hat{V}(t,S,0,0), \qquad (15)
\end{aligned}$$

which can be computed from the boundary condition (5).

Replacing dV in (12) by (13) shows that we can eliminate all risks in the portfolio by choosing δ, α_B and α_C as

$$\begin{aligned}
\delta &= -\partial_S \hat{V}, \qquad (16) \\
\alpha_B &= \frac{\Delta\hat{V}_B}{P_B} \qquad (17) \\
&= -\frac{\hat{V} - (M^+ + R_B M^-)}{P_B}, \qquad (18) \\
\alpha_C &= \frac{\Delta\hat{V}_C}{P_C} \qquad (19) \\
&= -\frac{\hat{V} - (M^- + R_C M^+)}{P_C}. \qquad (20)
\end{aligned}$$

Hence, the cash account evolution (8) can be written as

$$d\bar{\beta}_F(t) = \{-r(t)R_B M^- - r_F(t)M^+\}dt, \qquad (21)$$

so the amount of cash deposited by the seller at the risk-free rate equals $-R_B M^-$ and the amount borrowed at the funding rate r_F equals $-M^+$.

If we now introduce the parabolic differential operator $\mathcal{A}_t$ as

$$\mathcal{A}_t V \equiv \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + (q_S - \gamma_S) S \partial_S V, \qquad (22)$$

then it follows that $\hat{V}$ is the solution of the PDE

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F(\hat{V} + \Delta\hat{V}_B)^+ - \lambda_B \Delta\hat{V}_B - \lambda_C \Delta\hat{V}_C, \\ \hat{V}(T,S) = H(S), \end{cases} \qquad (23)$$

where $\lambda_B \equiv r_B - r$ and $\lambda_C \equiv r_C - r$. Inserting (14) and (15) with boundary condition (5) into (23) finally gives

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}_t \hat{V} - r\hat{V} &= (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B (R_B M^- + M^+) - \lambda_C (R_C M^+ + M^-) \\ \hat{V}(T, S) &= H(S), \end{cases} \tag{24}$$

where we have used $(\hat{V} + \Delta \hat{V}_B)^+ = (M^+ + R_B M^-)^+ = M^+$.

In contrast, the risk-free value V satisfies the regular Black-Scholes PDE

$$\begin{cases} \partial_t V + \mathcal{A}_t V - rV = 0 \\ V(T, S) = H(S), \end{cases} \tag{25}$$

so if one interprets λ_B and λ_C as effective default rates then the differences between (24) and (25) may be interpreted as follows:

- The first term on the right hand side of (24) is the additional growth rate the seller B requires on the risky asset $\hat{V}$ to compensate for the risk that default of either the seller or the counterparty will terminate the derivative contract.
- The second term is the additional funding cost for negative values of the cash account of the hedging strategy.
- The third term is the adjustment in growth rate that the seller can accept because of the cash flow occurring at own default.
- The fourth term is the adjustment in growth rate that the seller can accept because of the cash flow occurring at counterparty default.

Terms one, three and four are related to counterparty risk whereas the second term represents the funding cost. From this interpretation it follows that the PDE for a so-called *extinguisher trade*, whereby it is agreed that no party gets anything at default, is obtained by removing terms three and four from the PDE 24.

In the subsequent sections we will examine the PDE (24) in the following four cases:

- $M(t, S) = \hat{V}(t, S, 0, 0)$ and $r_F = r$.
- $M(t, S) = \hat{V}(t, S, 0, 0)$ and $r_F = r + s_F$.
- $M(t, S) = V(t, S)$ and $r_F = r$.
- $M(t, S) = V(t, S)$ and $r_F = r + s_F$.

Because either the value M at default or the funding rate r_F differ between these four cases, we expect the total derivative value $\hat{V}$ to differ as well.

4 Using $\hat{V}(t,S)$ as mark-to-market at default

Let us consider the case where the payments in case of default are based on $\hat{V}$ so that $M(t,S) = \hat{V}(t,S)$ in the boundary condition (5). Conceptually this is the simpler case, since if the defaulting party is in the money with respect to the derivative contract, then there is no additional effect on the profit and loss at the point of default. Similarly, if the surviving party is in the money with respect to the derivative contract, then its loss is simply $(1-R)\hat{V}$. In this case, the PDE (24) simplifies to

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}_t \hat{V} - r\hat{V} = (1-R_B)\lambda_B \hat{V}^- + (1-R_C)\lambda_C \hat{V}^+ + s_F \hat{V}^+ \\ \hat{V}(T,S) = H(S), \end{cases} \tag{26}$$

where we recall that $s_F = 0$ if the derivative can be posted as collateral and $s_F = (1-R_B)\lambda_B$ if it cannot. Moreover, the hedge ratios α_B and α_C are given by

$$\alpha_B = -\frac{(1-R_B)\hat{V}^-}{P_B}, \tag{27}$$

$$\alpha_C = -\frac{(1-R_C)\hat{V}^+}{P_C}, \tag{28}$$

so $\alpha_B \geq 0$ and $\alpha_C \leq 0$ and the replication strategy generates enough cash $(-\hat{V}^-)$ for the seller to purchase back its own bonds [2].

In the counterparty risk literature it is customary to write $\hat{V} = V + U$, where U is called the credit valuation adjustment, or *CVA*. Inserting this decomposition into (26) and using that V satisfies (25) yields

$$\begin{cases} \partial_t U + \mathcal{A}_t U - rU = (1-R_B)\lambda_B (V+U)^- + (1-R_C)\lambda_C (V+U)^+ + s_F (V+U)^+ \\ U(T,S) = 0, \end{cases} \tag{29}$$

where V is known and acts as a source term. Furthermore, we may formally apply the Feynman-Kac theorem to (29), which with the assumption of deterministic rates gives us the following non-linear integral equation

$$\begin{aligned} U(t,S) = & -(1-R_B)\int_t^T \lambda_B(u) D_r(t,u)\mathbb{E}_t[(V(u,S(u)) + U(u,S(u)))^-]du \\ & -(1-R_C)\int_t^T \lambda_C(u) D_r(t,u)\mathbb{E}_t[(V(u,S(u)) + U(u,S(u)))^+]du \\ & -\int_t^T s_F(u) D_r(t,u)\mathbb{E}_t[(V(u,S(u)) + U(u,S(u)))^+]du. \end{aligned} \tag{30}$$

It follows that we can compute U by first computing V and then solving either the non-linear PDE (29) or the integral equation (30).

Before proceeding with a study of the two cases $s_F = 0$ and $s_F = (1-R_B)\lambda_B$, it is worthwhile to study a few examples, namely where $\hat{V}$ corresponds to bonds of the seller or the counterparty, where those the bonds are either without and with recovery.

[2] For the first term, the cash available to the seller is $(-\hat{V}^-)$, of which a fraction of $(1-R_B)$ is invested in buying back the recovery-less bond B and the fraction R_B is invested risk-free. This is equivalent to investing the total amount $(-\hat{V}^-)$ into purchasing back a seller bond $\bar{B}$ with recovery R_B.

4.1 The seller sells P_B to the counterparty

The first case we consider is a risky, recovery-less bond sold by the seller B to the counterparty C. In this case, we have $\hat{V} = \hat{V}^- = -P_B$ and $R_B = 0$. Since we consider deterministic rates and credit spreads, we do not have any risk with respect to the underlying market factors and the term $\mathcal{A}_t \hat{V}$ vanishes so (26) becomes

$$\begin{cases} \partial_t \hat{V} = (r + \lambda_B)\hat{V} = r_B \hat{V} \\ \hat{V}(T, S) = -1 \end{cases} \tag{31}$$

with the solution

$$\hat{V}(t) = -\exp\left(-\int_t^T r_B(s)ds\right), \tag{32}$$

as expected for $\hat{V} = -P_B(t)$.

If, on the other hand, we consider the bond $P_{\bar{B}}$ that has recovery R_B, then (26) becomes

$$\begin{cases} \partial_t \hat{V} = \{r + (1 - R_B)\lambda_B\}\hat{V} \\ \hat{V}(T, S) = -1 \end{cases} \tag{33}$$

with the solution

$$\hat{V}(t) = -\exp\left(-\int_t^T \{r(s) + (1 - R_B)\lambda_B(s)\}ds\right). \tag{34}$$

As expected, the rate $r + (1 - R_B)\lambda_B$ payable on the bond with recovery is equal to the unsecured funding rate r_F that the seller has to pay on negative cash balances when the derivative cannot be posted as collateral.

4.2 The "Seller" purchases P_C from C

If, on the other hand, $\hat{V}$ describes the purchase of the bond P_C by the seller from the counterparty (i.e. the seller lends to the counterparty without recovery), then $\hat{V} = \hat{V}^+ = P_C$ and $R_C = 0$ and equation (26) becomes

$$\begin{cases} \partial_t \hat{V} = (r_F + \lambda_C)\hat{V} = (r_F + (r_C - r))\hat{V} \\ \hat{V}(T, S) = 1 \end{cases} \tag{35}$$

In this case, if the seller can use the derivative (i.e. the loan asset) as collateral for the funding of its short cash position within its replication strategy, then (neglecting haircuts) we have $r_F = r$, the risk-free rate. The net result in this case is then

$$\partial_t \hat{V} = r_C \hat{V}. \tag{36}$$

so

$$\hat{V}(t) = \exp\left(-\int_t^T r_C(s)ds\right), \tag{37}$$

as expected for $\hat{V}(t) = P_C(t)$. If, on the other hand, we consider a bond $P_{\bar{C}}$ with recovery R_C, then we find

$$\hat{V}(t) = \exp\left(-\int_t^T \{r(s) + (1 - R_C)\lambda_C(s)\}ds\right). \tag{38}$$

as expected.

4.3 The case $r_F = r$

If the derivative can be posted as collateral the PDE (26) becomes

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}_t \hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+ \\ \hat{V}(T, S) = H(S), \end{cases} \tag{39}$$

which is a non-linear PDE that needs to be solved numerically unless $\hat{V} \geq 0$ or $\hat{V} \leq 0$.

Assuming that $\hat{V} \leq 0$ (i.e. the seller sold an option to the counterparty, so $H(S) \leq 0$) and that all rates are deterministic, the Feynman-Kac representation of $\hat{V}$ is given by

$$\hat{V}(t, S) = \mathbb{E}_t[D_{r+(1-R_B)\lambda_B}(t, T)H(S(T))], \tag{40}$$

where $D_k(t, T) \equiv \exp\left(-\int_t^T k(s)ds\right)$ is the discount factor over $[t, T]$ given the rate k.

Alternatively, if for $\hat{V} \leq 0$ we insert the ansatz[3] $\hat{V} = V + U_0$ into (39), apply the Feynman-Kac theorem and finally use that $V(t, s) = D_r(t, u)\mathbb{E}_t[V(u, S(u))]$, then we get

$$U_0(t, S) = -V(t, S)\Big\{\int_t^T (1 - R_B)\lambda_B(u)D_{(1-R_B)\lambda_B}(t, u)du\Big\}. \tag{41}$$

When $\hat{V} \geq 0$, i.e. the "seller" bought an option, symmetry yields that

$$U_0(t, S) = -V(t, S)\Big\{\int_t^T (1 - R_C)\lambda_C(u)D_{(1-R_C)\lambda_C}(t, u)du\Big\}. \tag{42}$$

We conclude by noting that if $\hat{V} \leq 0$, then U_0 depends only on the credit of the seller, whereas if $\hat{V} \geq 0$, then it depends only on the credit of the counterparty.

4.4 The case $r_F = r + (1 - R_B)\lambda_B$

If the derivative cannot be posted as collateral the PDE (26) becomes

$$\begin{cases} \partial_t \hat{V} + \mathcal{A}_t \hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + \{(1 - R_B)\lambda_B + (1 - R_C)\lambda_C\}\hat{V}^+ \\ \hat{V}(T, S) = H(S), \end{cases} \tag{43}$$

which again is a non-linear PDE.

If $\hat{V} \leq 0$ we write $\hat{V} = V + U$ and it is easy to see that $U = U_0$ given in (41) so $\hat{V}$ is given by (40). If $\hat{V} \geq 0$ we have that

[3] We use the zero subscript to indicate that the CVA U_0 has been computed with a zero funding spread s_F

$$\hat{V}(t,S) = \mathbb{E}_t[D_{r+k}(t,T)H(S(T))], \tag{44}$$

with $k \equiv (1-R_B)\lambda_B + (1-R_C)\lambda_C$. Analogously to the case $r_F = r$ we can make the ansatz $\hat{V} = V + U$ and show that

$$U(t,S) = -V(t,S)\Big\{\int_t^T k(u)D_k(t,u)du\Big\}. \tag{45}$$

Comparing (42) and (45) shows that when the "seller" buys an option from the counterparty, then it encounters an additional funding spread $s_F = (1-R_B)\lambda_B$.

5 Using $V(t,S)$ as mark-to-market at default

We will now consider the case where payments in case of default are based on V and hence use $M(t,S) = V(t,S)$ in the boundary condition (5). Equation (24) then becomes

$$\begin{cases} \partial_t\hat{V} + \mathcal{A}_t\hat{V} - (r+\lambda_B+\lambda_C)\hat{V} &= -(R_B\lambda_B+\lambda_C)V^- - (\lambda_B + R_C\lambda_C)V^+ + s_F V^+ \\ \hat{V}(T,S) &= H(S). \end{cases} \tag{46}$$

The PDE (46) is linear and has a source term on the right-hand side. If we write $\hat{V} = V + U$, then the hedge ratios become

$$\alpha_B = \frac{U + (1-R_B)V^-}{P_B}, \tag{47}$$

$$\alpha_C = \frac{U + (1-R_C)V^+}{P_C}. \tag{48}$$

Comparing (27) and (47) shows that in the latter case a default triggers a windfall cashflow of U that needs to be taken into account in the hedging strategy.

Writing $\hat{V} = U + V$ also gives us the following linear PDE for U

$$\begin{cases} \partial_t U + \mathcal{A}_t U - (r+\lambda_B+\lambda_C)U &= (1-R_B)\lambda_B V^- + (1-R_C)\lambda_C V^+ + s_F V^+ \\ U(T,S) &= 0, \end{cases} \tag{49}$$

so again applying the Feynman-Kac theorem yields

$$\begin{aligned} U(t,S) \;=\; & -(1-R_B)\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^-(u,S(u))\right]du \\ & -(1-R_C)\int_t^T \lambda_C(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+(u,S(u))\right]du \\ & -\int_t^T s_F(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+(u,S(u))\right]du. \end{aligned} \tag{50}$$

The CVA U can be calculated by using $V(t,S)$ as a known source term when solving the PDE (49) or computing the integrals (50).

In the case where we can use the derivative as collateral for the funding of our cash account, i.e. $s_F = 0$, the last term of (50) vanishes and the equation reduces to the regular bilateral CVA derived in many of the papers and books cited in Section 1, for example Gregory [5]. The bilateral benefit in this case does not come from any gains at own default, but from being able to use the cash generated by the hedging strategy and buy back own bonds, thus generating an excess return of $(1 - R_B)\lambda_B$. We denote this CVA when $M = V$ and $s_F = 0$ by U_0.

In practice, however, we can normally not use the derivative as collateral and equation (50) gives us a consistent adjustment of the derivatives prices for bilateral counterparty risk and funding costs. In the specific case where the funding spread corresponds to that of the unsecured B bond (with recovery R_B), i.e. $s_F = (1 - R_B)\lambda_B$, we may merge the first and third terms of (50) and rewrite U as

$$\begin{aligned} U(t,S) &= -(1-R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V(u,S(u))\right] du \\ &\quad -(1-R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+(u,S(u))\right] du. \end{aligned} \tag{51}$$

The first term of the last equation now not only contains the bilateral asset described above, but also the funding liability arising from the fact that the higher rate $r_F = (1 - R_B)\lambda_B$ is paid when borrowing for the hedging strategy's cash account.

6 Examples

In this section we calculate the total derivative value $\hat{V}$ for a call option bought by the "seller" in the following four cases:

$$\begin{aligned} &\text{Case 1}: && M = \hat{V},\ s_F = 0 \\ &\text{Case 2}: && M = \hat{V},\ s_F = (1-R_B)\lambda_B \\ &\text{Case 3}: && M = V,\ s_F = 0 \\ &\text{Case 4}: && M = V,\ s_F = (1-R_B)\lambda_B \end{aligned}$$

A bought call is a one-sided trade that satisfies $V \geq 0$ and $\hat{V} \geq 0$, and if we furthermore assume constant rates, the CVAs U_0, U, U_0 and U from Sections 4 and 5 simplify to

$$\begin{aligned} \text{Case 1}: U_0(t,s) &= -\left(1-\exp\{-(1-R_C)\lambda_C(T-t)\}\right) V(t,S) \\ \text{Case 2}: U(t,s) &= -\left(1-\exp\{-((1-R_B)\lambda_B+(1-R_C)\lambda_C)(T-t)\}\right) V(t,S) \\ \text{Case 3}: U_0(t,s) &= -\frac{(1-R_C)\lambda_C\left(1-\exp\{-(\lambda_B+\lambda_C)(T-t)\}\right)}{\lambda_B+\lambda_C} V(t,S) \\ \text{Case 4}: U(t,s) &= -\frac{\{(1-R_B)\lambda_B+(1-R_C)\lambda_C\}\left(1-\exp\{-(\lambda_B+\lambda_C)(T-t)\}\right)}{\lambda_B+\lambda_C} V(t,S) \end{aligned}$$

The results are shown in Figures 1 to 3. Since the four CVAs above are linear in V in all four cases, we have chosen to display their magnitude as a percentage of V. All CVAs are negative since the seller faces counterparty risk and funding costs when $s_F = 0$, but does not have any bilateral asset because of the one-sidedness of the option payoff. From these results we see that the effect of the funding cost is significantly larger than that of choosing $M = \hat{V}$ or $M = V$ for a bought option. For a sold option the impact of the funding cost does not have any effect.

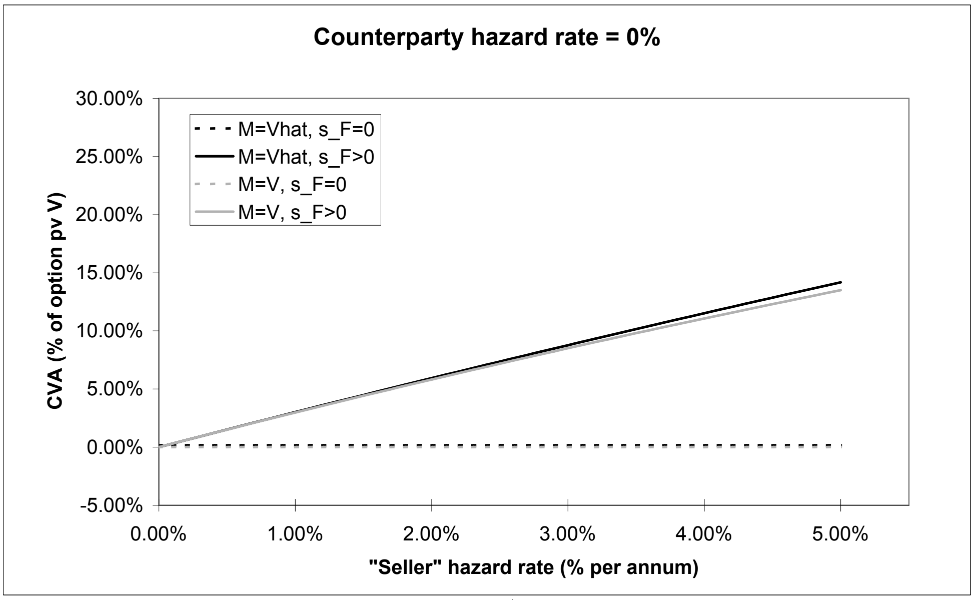

Figure 1: CVA, relative to V, when $M = \hat{V}$ and $M = V$ for different values of λ_B when $\lambda_C = 0$ and $T - t = 5$ years. Other parameters: $R_B = 40\%$, $R_C = 40\%$.

7 Conclusion and possible extensions

The results in this paper extend the standard credit valuation adjustments encountered in the literature by taking all funding costs associated with the hedging strategy into account. Since the seller's funding costs and own credit are intimately related, this results in a consistent treatment of bilateral credit and funding costs in the bilateral CVA calculation. A numerical example of a bought call option shows that taking funding into account is relevant and may result in the CVA being up to 100% higher than if funding is assumed to occur at the risk-free rate. We believe that the results presented here are particularly relevant when pricing interest rate swaps and vanilla options since these markets are very liquid and having an analytical model that does not fully take all costs into account may consume all profits from a deal given today's high funding and credit spreads.

Although we worked within a simple one-derivative, one-asset Black-Scholes framework, the results can be immediately extended as follows:

- Derivatives with more general payments than $H(S(T))$. These could be Asian options or interest rates swaps.

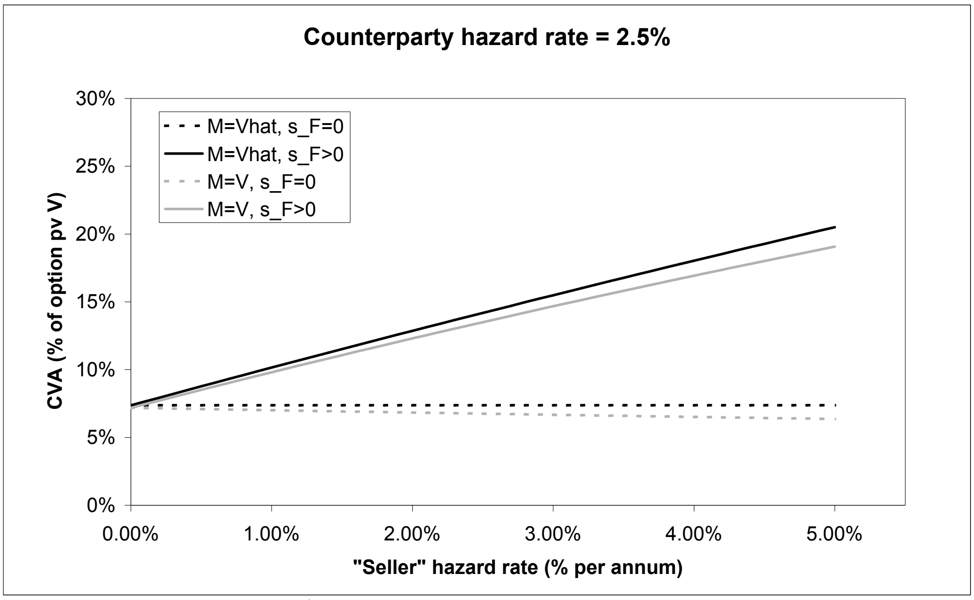

Figure 2: CVA when $M = \hat{V}$ and $M = V$ for different values of λ_B when $\lambda_C = 2.5\%$ and $T - t = 5$ years. Other parameters: $R_B = 40\%$, $R_C = 40\%$.

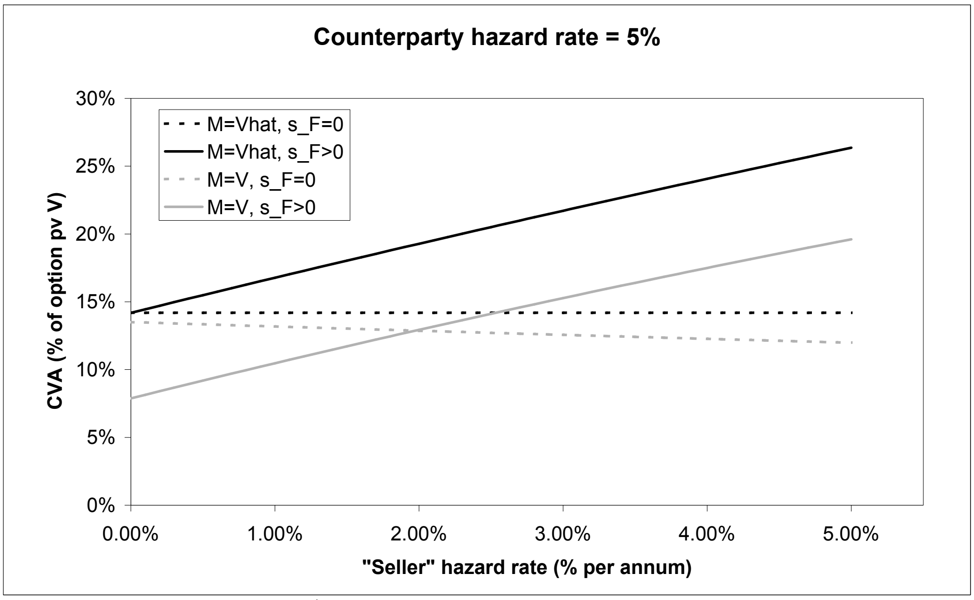

Figure 3: CVA when $M = \hat{V}$ and $M = V$ for different values of λ_B when $\lambda_C = 5\%$ and $T - t = 5$ years. Other parameters: $R_B = 40\%$, $R_C = 40\%$.

- Netted portfolios with many trades. In this case the values $\hat{V}$ and V represent the net derivative portfolio value rather than the value of a single derivative.

- Generalised multi-asset diffusion dynamics for multiple underlyings. The only restriction is that the asset price SDEs satisfy technical conditions such that the option pricing PDE (now multi-dimensional) admits a unique solution given by the Feynman-Kac representation. Note that if the number of assets exceeds two or three it is computationally more efficient to compute the CVA using Monte Carlo simulation combined with numerical integration rather than solving the high dimensional PDE.

- Stochastic interest rates. This is essential for interest rate derivatives, and the effect would be that the discounting in the CVA formula will happen inside the expectation operator.

- Stochastic hazard rates. One way of introducing default time dependence and right/wrong-way risk would be to make λ_B and λ_C stochastic and correlate them with each other and the other market factors. This would simply imply that we would not move the discount factors outside of the expectation operator in (30) and (50). Also, the generator $\mathcal{A}_t$ would incorporate terms corresponding to the new stochastic state variables.

- Direct default time dependence. Another way of introducing default time dependence is by allowing simultaneous defaults. This could be done by letting J_0, J_1 and J_2 be independent point processes and then setting $J_B = J_0 + J_1$ and $J_C = J_0 + J_1$. This approach is known as the Marshall-Olkin copula and would require some kind of basket default instrument for perfect replication. The hazard rates λ_0, λ_1 and λ_2 of J_0, J_1 and J_2 could be made stochastic in which case we can model right and wrong-way risk as well.

Our results can also be readily extended to the case where a collateral agreement is in place by following Piterbarg [9] and introducing a collateral account in the delta hedging strategy.

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