2. Yield and Duration

Notes
Interest rates are expressed with continuous compounding unless otherwise specified.

We will not concern ourselves with industry daycount or business-day conventions. To translate maturities into times in years…

For maturities expressed in days, assume a 365-day year (i.e., a 1D maturity corresponds to $T = 1 / 365$).

For maturities expressed in weeks, assume 52 weeks per year.

For maturities expressed in months, assume 12 months per year.

Exercises
1. The term structure of risk-free zero rates is as follows:

t	r (continuous)
0.5	0.011
1	0.015
2	0.022
3	0.024
5	0.029

(a) What is the present value of a risk-free bond with maturity 3 years paying annual coupons of 3.5%? Express your answer per 100 face.
(b) What is the 2-year semiannual par yield? Use linear interpolation in the continuously compounded risk-free zero rate.
(c) What is the yield (expressed as a continuously compounded rate) of a 1-year risk-free bond with a 5% coupon rate per year that pays semiannually?

2. The following problems describe bond prices or yields observed at the same time. All bond prices are expressed per 100 face.

(a) A zero-coupon bond with a 6-month maturity trades at 99.875. What is the risk-free zero rate, expressed with continuous compounding, to this bond's maturity?
(b) A bond paying a coupon of 4% per year semiannually with a maturity of 1 year has a yield of 50 basis points, expressed in the bond's natural compounding frequency. What is the risk-free zero rate, expressed with continuous compounding, to this bond's maturity? (A basis point is 1% of 1%, or 0.0001.)
(c) A bond paying a 2% coupon per year semiannually with a maturity of 2 years has a yield of 1.05%, expressed in the bond's natural compounding frequency. Using linear interpolation in the continuously compounded risk-free zero rates, find the risk-free zero rate, expressed with continuous compounding, to this bond's maturity.

3. A bond with a 5-year maturity pays a 4% coupon semiannually. At the moment, it trades with a yield of 2.5%, expressed with continuous compounding.

(a) What is the price of the bond? Express your answer assuming a 100 face amount.
(b) What is the (Macaulay) duration of the bond? What is its convexity?
(c) What is the modified duration of the bond?
(d) Use the discussion in the chapter to derive an expression for a bond's modified convexity, and calculate it for this bond.

Applications

1. *A Lower Bound on the Convexity of a Collection of Deterministic Positive Cash Flows*

In the chapter, it was asserted that for any collection of deterministic and strictly positive cash flows with a particular duration, a zero-coupon bond exhibits the least convexity. The goal of this problem is to formalize that assertion.

For the purposes of this problem, we will take advantage of the fact, discussed more thoroughly in a later chapter, that for a portfolio Π consisting of several instruments with values $V_1, V_2, \ldots, V_n$ having durations D_1, D_2, D_n and convexities $C_1, C_2, \ldots, C_n$, we may say that…

$$V_\Pi = \sum_{i=1}^{n} V_i$$

$$D_\Pi = \frac{1}{V_\Pi} \sum_{i=1}^{n} V_i D_i$$

$$C_\Pi = \frac{1}{V_\Pi} \sum_{i=1}^{n} V_i C_i$$

…for certain suitable definitions of duration and convexity. For the moment, we will not concern ourselves with what those definitions are, but will simply use the result.

(a) Show that for an arbitrary portfolio Π consisting of n strictly positive cash flows occurring at different times, the following inequality holds:

$$C_\Pi > D_\Pi^2$$

Suggestion: This can easily be done by induction. Begin with the case of two deterministic positive cash flows at times T_1 and T_2, and show that…

$$C_\Pi \geq D_\Pi^2$$

…with equality holding if and only if $T_1 = T_2$.

Next, assume that we have a portfolio of n cash flows Π_n satisfying…

$$C_{\Pi_n} > D_{\Pi_n}^2$$

…and show that the addition of any deterministic positive cash flow to this portfolio results in a new portfolio Π_{n+1} that satisfies:

$$C_{\Pi_{n+1}} > D_{\Pi_{n+1}}^2$$

(b) Explain why your argument from above in (a) suffices to prove the desired result concerning zero-coupon bonds.

(c) Can the result be extended to portfolios in which some of the cash flows are negative—that is, levered portfolios or portfolios that also include deterministic liabilities? Explain the reasoning behind your answer.

2. *Approximation of Portfolio Yield from Yields of Its Constituent Instruments*
A portfolio consists of two fixed-coupon bonds: The first has maturity 5 years, pays interest semiannually at the rate of 4% per year, and has a yield (expressed with continuous compounding) of 2.5%. The second has maturity 3 years, pays interest annually at the rate of 1.5% per year, and has a yield (expressed with continuous compounding) of 5%.

(a) Compute the prices and durations of these bonds, expressing each price per 100 face amount.
(b) Suppose we have 100 total to invest. We construct a portfolio by choosing weight w to invest in the first bond, and therefore weight $1 - w$ to invest in the second. For w running from 0 to 1 inclusive, spaced 0.025 apart, calculate the quantity (face amount) held of each bond given w.
(c) Using your nonlinear solver of choice, for each of the portfolios above in part b, calculate the yield of the portfolio's cash flows, expressed with continuous compounding.
(d) For each of the portfolios whose yield you calculated above in c, compare your result to the following approximation:

$$\tilde{y} = \frac{V_1 D_1 y_1 + V_2 D_2 y_2}{V_1 D_1 + V_2 D_2}$$

...where V is the present value of each bond, D is its duration, and y is its yield. How does the approximation perform in this example?
(e) Consider a portfolio Π of N fixed-coupon bonds, with known V_i, D_i, and y_i for each bond, $i = 1, 2, \ldots, N$. Using the first-order approximation of the bond's price at a given yield...

$$V_i(y) \approx V_i - D_i V_i (y - y_i)$$

...derive the approximation of the portfolio yield:

$$y_\Pi \approx \frac{\sum_{i=1}^{N} V_i D_i y_i}{\sum_{i=1}^{N} V_i D_i}$$

Sandbox Application
Interest Rate Sensitivities and Hedging
Sensitivities to yield curve moves are the most fundamental risk measures for fixed income portfolios. In addition to DV01 and duration, which describe instruments' sensitivities to parallel shifts in the yield curve, fixed income portfolio managers are typically interested in knowing the portfolio's sensitivities to moves at individual maturities on the yield curve—so-called "key rate" DV01 and duration. Knowing the particular profile of interest rate sensitivity allows for more targeted hedging. In this exercise, we will use a sample bond portfolio and the sandBox toolkit to calculate analytics that can be used to construct hedges of fixed income portfolios.

(i) The base interest rate curve is stripped from traded securities. Suppose that today, we observe a collection of bonds with the following attributes:

maturity	coupon	coupon frequency	yield	yield quote basis
3M	0.000%	0	0.3440%	0
6M	0.000%	0	0.4083%	0
1Y	0.000%	0	0.5389%	0
2Y	0.625%	2	0.8409%	2
5Y	1.375%	2	1.5831%	2
10Y	2.000%	2	2.2247%	2
30Y	2.875%	2	2.6903%	2

The bonds up to and including the 1Y maturity are zero-coupon bonds. The bonds thereafter are vanilla fixed-coupon bonds. Assume for the purposes of this problem that all of these bonds are dated today (that is, all of the coupon-bearing bonds have zero accrued interest now).

Create a function that bootstraps a term structure of zero rates from a vector of provided bond data, and show its output for this collection of bonds. (Assume linear interpolation in the continuously compounded zero rates in the implementation of your function. Also assume that any rate before the 3M point, including the point at time zero, has the same zero rate as the 3M point.)

(ii) You have a portfolio consisting of the following bonds:

bond maturity	bond face	bond coupon	coupon frequency
2M	$10,000.00	0.000%	0
8M	$22,000.00	2.500%	2
10M	$10,000.00	0.000%	0
1Y3M	$25,000.00	0.850%	2
1Y11M	$15,000.00	1.015%	2
2Y6M	$28,000.00	2.200%	2
3Y3M	$30,000.00	1.200%	2
4Y6M	$35,000.00	3.500%	2
6Y2M	$50,000.00	4.500%	2
9Y	$68,000.00	3.000%	2
14Y	$80,000.00	2.750%	2
22Y5M	$65,000.00	2.500%	2

Assume that all coupons are dated at least one coupon period in the past, and that any cash flows due today are included in your present value calculation.

Using the interest rate curve from part (i), determine the values of the following analytics for each bond individually, and for the bond portfolio overall:

NPV—This is the so-called "dirty price," which is simply the present value of each bond's future cash flows; this value includes any accrued interest since the most recent coupon payment.
DV01—Calculate this using a 1bp parallel shift in the zero rates from which you determined the bond values.

(iii) Calculate, for each bond individually and for the portfolio overall:
Yield (continuous)
Macaulay Duration
(Macaulay) Convexity

Also calculate what we will here call the effective duration and convexity of each instrument and of the portfolio overall: Whereas above we used the (flat) yield to calculate these sensitivities, here we use the full term structure of the interest rate curve assuming a parallel shift in zero rates.

(iv) A simple hedge for a fixed income portfolio is to short 10Y bonds in a quantity sufficient to make the overall portfolio DV01-neutral. This is the so-called "Ten-Year Equivalent" of the portfolio. Using the information you were originally given about the 10Y bond, calculate the 10Y equivalent face amount and market value for each bond in the portfolio, and for the portfolio overall. (Be sure to calculate the DV01 using the interest rate curve you bootstrapped, not by merely shocking the instrument's yield.)

(v) The "key-rate" sensitivities of a portfolio are calculated with respect to moves in single points on the interest rate curve. The key-rate DV01's, for example, are the value changes under 1bp shifts up in the interest rate curve at each point individually. Key rate duration is typically calculated using a central finite difference approximation of the derivative, by repricing twice—once under a 1bp shock up, and a second time under a 1bp shock down—at each point individually.

Calculate the key rate DV01's and Durations of the individual bonds in the portfolio and in the portfolio overall using shifts in the zero rates you bootstrapped.

(vi) An alternate method that many traders find more intuitive is to calculate key rate sensitivities not from the bootstrapped zero rates, but from shifts in yield on the instruments from which the curve was initially constructed. Do the same as above in part (v), only this time calculate the value change under 1bp shocks in yield to the instruments from which you strip the interest rate curve.

(vii) Comment on the differences you see in results. In particular, there is one surprising feature of some of the yield-based key rate sensitivities in part (vi). Explain how these odd-seeming results arise.