8. Multiperiod Optimization with Costs

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8.1. **Optimal Execution in the Almgren–Chriss Model.** Suppose we hold a block of X units of a security that we want to completely liquidate before time T. We divide T into N intervals of length $\tau = T/N$, and define the discrete times $t_k = k\tau$, for $k = 0, ..., N$. We define a trading trajectory to be a list $x_0, \ldots, x_N$, where x_k is the number of units that we plan to hold at time t_k. The boundary conditions are:

$$x_0 = X, \quad \text{and} \quad x_N = 0.$$

There are certain simplifications associated to assuming all of the trades are in the same direction, which is always the case when considering executing a fixed order and without short-term alphas. We treat the case of selling shares; however, the order could just as well be a buy – essentially the same reasoning applies with only minor modifications. Let

$$n_k = x_{k-1} - x_k \geq 0$$

be the number of units that we will *sell* between times t_{k-1} and t_k. Clearly, x_k and n_k are related by

$$x_k = X - \sum_{j=1}^{k} n_j = \sum_{j=k+1}^{N} n_j$$

Almgren and Chriss (2001) define a *trading strategy* to be a rule for determining n_k in terms of information available at time t_{k-1}. Broadly speaking we distinguish two types of trading strategies: dynamic and static. Static strategies are determined in advance of trading, that is the rule for determining each n_k depends only on information available at time t_0. Dynamic strategies, conversely, depend on all information up to and including time t_{k-1}.

We distinguish two kinds of market impact. Temporary impact refers to temporary imbalances in supply in demand caused by our trading leading to temporary price movements away from equilibrium. Permanent impact means changes in the equilibrium price due to our trading, which remain at least for the life of our liquidation. In other words, the definitions are such that temporary impact is assumed to revert instantaneously – it is completely undetectable even one period later. In practice, one of the largest determinants of temporary impact is simply spread pay.

The (midpoint) price dynamics are taken to be an arithmetic random walk:

$$S_k = S_{k-1} + \sigma\tau^{1/2}\xi_k - \tau g(n_k/\tau), \tag{8.1}$$

Here σ represents the volatility of the asset, $\xi_k \sim N(0,1)$ are i.i.d. normal, and the permanent impact is a function of the average rate of trading $v_k = n_k/\tau$ during the interval. Note that the innovation in (8.1) is an i.i.d. random term and a deterministic function of the execution, so we can telescope the sum all the way back:

$$S_k = S_0 + \sum_{k=1}^{N} [\sigma \tau^{1/2} \xi_k - \tau g(n_k/\tau)]. \tag{8.2}$$

In this model, we do not explicitly model the bid and the ask as separate processes, but the model does allow for costs associated to the spread as we shall see.

Temporary impact is modeled by assuming the actual price received on the k-th transaction is

$$\tilde{S}_k = S_{k-1} - h(v_k), \quad v_k = n_k/\tau$$

but the effect of $h(v)$ does not appear in S_k. Thus temporary impact literally just means we get a worse price (than the midpoint price) on each of our child fills.

The full trading revenue upon completion of all trades is:

$$\sum_{k=1}^{N} n_k \tilde{S}_k = \sum_{k=1}^{N} n_k [S_{k-1} - h(v_k)] = \sum_{k=1}^{N} n_k S_{k-1} - \sum_{k=1}^{N} n_k h(v_k)$$

The $n_k S_{k-1}$ term needs further simplification. Note that

$$\begin{aligned} \sum_{k=1}^{N} S_{k-1} n_k &= \sum_{k=1}^{N} S_{k-1}(x_{k-1} - x_k) = \sum_{k=0}^{N-1} S_k x_k - \sum_{k=1}^{N} S_{k-1} x_k \\ &= S_0 X + \sum_{k=1}^{N} (S_k - S_{k-1}) x_k \end{aligned}$$

where in the last line we used the boundary conditions $x_0 = X, x_N = 0$.

Hence the full trading revenue is

$$\begin{aligned} \sum_{k=1}^{N} n_k \tilde{S}_k &= S_0 X + \sum_{k=1}^{N} (S_k - S_{k-1}) x_k - \sum_{k=1}^{N} n_k h(v_k) \\ &= S_0 X + \sum_{k=1}^{N} [\sigma \tau^{1/2} \xi_k - \tau g(n_k/\tau)] x_k - \sum_{k=1}^{N} n_k h(v_k) \end{aligned}$$

where we have used eq. (8.1) to obtain an expression for the increment $S_k - S_{k-1}$.

The total slippage is

$$\text{slippage} = X S_0 - \sum_k n_k \tilde{S}_k$$

i.e. the difference between the initial book value and the revenue.

Prior to trading, implementation shortfall is a random variable. We write $E(x)$ for the expected shortfall and $V(x)$ for the variance of the shortfall. These are calculated as

$$\mathbb{E}(\text{slippage}) = \sum_{k=1}^{N} \tau g(n_k/\tau) x_k + \sum_{k=1}^{N} n_k h(v_k) \tag{8.3}$$

$$\mathbb{V}(\text{slippage}) = \sigma^2 \sum_{k=1}^{N} \tau x_k^2 \tag{8.4}$$

For each value of $a > 0$ there corresponds a unique trading trajectory x such that

$$\mathbb{E}(\text{slippage}) + a\,\mathbb{V}(\text{slippage})$$

is minimal. As we know, this trajectory is optimal from the point of view of an investor with Arrow (1971)–Pratt (1964) constant absolute risk aversion parameter $a > 0$. Note that the sign is flipped; usually we would maximize a mean-variance form of profit. In this case case slippage is a cost so we minimize its expectation, but we still don't like variance.

Computing optimal trajectories is significantly easier if we take the permanent and temporary impact functions to be linear in the rate of trading. For linear permanent impact,

$$g(v) = \gamma v$$

Eq. (8.2) then yields

$$S_k = S_0 + \sigma \sum_{j=1}^{k} \tau^{1/2} \xi_j - \gamma(X - x_k)$$

Lemma 8.1. With $g(v) = \gamma v$, the permanent impact term from (8.3) equals

$$\sum_{k=1}^{N} \tau g(n_k/\tau) x_k = \frac{1}{2}\gamma X^2 - \frac{1}{2}\gamma \sum_k n_k^2$$

Proof. The left side is

$$\gamma \sum_{k=1}^{N} n_k x_k = \gamma \sum_{k=1}^{N} n_k \left(\sum_{j=k+1}^{N} n_j \right) = \gamma \sum_{j>k} n_k n_j$$

Noting that $X = \sum_k n_k$, the right side is

$$\frac{1}{2}\gamma \left[\left(\sum_k n_k \right)^2 - \sum_k n_k^2 \right]$$

The desired result now follows easily. □

Similarly, for the temporary impact we take

$$h(v) = \epsilon\, \text{sgn}(v) + \eta v.$$

The units of ϵ are \$/share, and those of η are (\$/share)/(share/time). A reasonable estimate for ϵ is the fixed costs of selling, such as half the bid-ask spread plus fees. It is more difficult to estimate η since it depends on internal and transient aspects of the market microstructure. The linear model above is often called a *quadratic* cost model because the total cost incurred by buying or selling n units in a single unit of time is

$$nh(n/\tau) = \epsilon\,|n| + (\eta/\tau)n^2$$

With both linear cost models,

$$E(x) = \frac{1}{2}\gamma X^2 + \epsilon \sum_{k=1}^{N} |n_k| + \frac{\tilde{\eta}}{\tau}\sum_{k=1}^{N} n_k^2, \quad \tilde{\eta} = \eta - \frac{1}{2}\gamma\tau$$

If all $n_k \geq 0$ as we have assumed, then $\sum_k |n_k| = X$ which means the L^1 term is irrelevant for optimization purposes.

In reality, it's naive to assume the bid-ask spread will be constant over the entire execution path (and one must be wary of simply using formulas from papers without questioning all of the various assumptions that are being made). Sophisticated practitioners would surely use a version of this in which ϵ depends on the time of day.

We need to minimize $U(x) = E(x) + a\,V(x)$, which we do by enforcing the first-order condition:

$$\frac{\partial}{\partial x_j}(E + aV) = 0 \tag{8.5}$$

Keeping only the relevant terms from E, and using $n_k = x_{k-1} - x_k$ we have

$$\begin{aligned}\frac{\partial}{\partial x_j}(E + aV) &= \frac{\partial}{\partial x_j}\left[\frac{\tilde{\eta}}{\tau}\sum_k (x_{k-1} - x_k)^2 + \tau a\sigma^2 \sum_k x_k^2\right] \\ &= \frac{\tilde{\eta}}{\tau}\frac{\partial}{\partial x_j}[(x_{j-1} - x_j)^2 + (x_j - x_{j+1})^2] + 2\tau a\sigma^2 x_j\end{aligned}$$

Setting this equal to zero (and dividing by -2) leads to

$$\frac{\tilde{\eta}}{\tau}[(x_{j-1} - x_j) - (x_j - x_{j+1})] = \tau a\sigma^2 x_j$$

Thus we are led to a linear difference equation

$$\frac{1}{\tau^2}(x_{j-1} - 2x_j + x_{j+1}) = \tilde{\kappa}^2 x_j, \quad \tilde{\kappa}^2 := \frac{a\,\sigma^2}{\tilde{\eta}} \tag{8.6}$$

Solutions of such equations may be written as a combination of exponentials. There exists a constant κ defined as the solution to

$$\frac{2}{\tau^2}(\cosh(\kappa\tau) - 1) = \tilde{\kappa}^2.$$

In terms of this, the precise solution to (8.6) which respects the boundary conditions is

$$x_j = \frac{\sinh(\kappa(T - t_j))}{\sinh(\kappa T)} X \quad \text{where} \quad \frac{2}{\tau^2}(\cosh(\kappa\tau) - 1) = \tilde{\kappa}^2. \tag{8.7}$$

Note that as $\tau \to 0$ we have $\tilde{\eta} \to \eta$ and $\tilde{\kappa} \to \kappa$.

Some comments and discussion are in order. First note we can re-write (8.6) as

$$x_{j+1} = (2 + \tau^2\tilde{\kappa}^2)x_j - x_{j-1},$$

thus representing it in recursive form. You could then, for instance, calculate x_2 from x_1 and x_0, x_3 from x_2 and x_1, etc, but to get the iteration started you need to set the values of the "free parameters" x_0 and x_1 using the boundary conditions, which are $x_0 = X$ and $x_N = 0$. Since the initial boundary condition determines x_0, we are left solving for x_1 such that $x_N = 0$. This is certainly, but a little annoying and potentially prone to floating-point errors. For this reason, (8.7) (and more generally, continuous-time solutions) are to be preferred when available.

Equation (8.6) resembles the simplest non-trivial second-order differential equation,

$$x''(t) = c\,x(t)$$

and it is natural to wonder whether one might not have gotten to this equation directly, without having to discretize time. This is, indeed, possible and leads to an illuminating parallel with the Lagrangian formulation of classical mechanics. One ends up representing the trading path as a twice-differentiable function $x(t)$ from the outset, and applying a method for minimizing "functionals" $\mathcal{F}[x(t)]$ which are functions fo the entire path. This technique is known as the "calculus of variations" and is treated in my paper with Jerome Benveniste: `https://ssrn.com/abstract=3057570`.

The parameter a is subjective and specific to the investor/trader using the model. Its interpretation, however, is very intuitive: higher a means you are more averse to the variance you will incur by holding onto the position rather than liquidating it quickly, so it naturally speeds up trading. Conversely, lower a means you don't care about this and simply want to minimize total impact while satisfying the boundary conditions. This leads to trading paths which resemble a straight line.

Note that all of the permanent impact terms magically dropped out. This is because as long as permanent impact is linear, and you are definitely going to liquidate the entire block of X shares, then you are always going to incur the same total permanent impact no matter the shape of the liquidation curve (the n_j). Since there's nothing you can do to control this, it doesn't enter into the optimization. This would not be the case if the total order size X were also a variable you could control (such as for an alpha strategy).

For a similar reason, the solution does not depend on ϵ (which you can think of as one-half the bid-offer spread). This is again because the total cost due to this term is simply proportional to X no matter the shape of the trading path. This would be different if, say, you had a predictive spread model which had time-of-day dependence, or if you were trading an alpha strategy and could control X.

8.2. **General Multiperiod Problems.** Consider a rational agent planning a sequence of trades beginning presently and extending into the future. Specifically, a *trading plan* for the agent is modeled as a specific portfolio sequence

$$\mathbf{x} = (x_1, x_2, \dots, x_T),$$

where x_t is the portfolio the agent plans to hold at time t in the future. If r_{t+1} is the vector of asset returns over $[t, t+1]$, then the trading profit (ie. difference between initial and final wealth) associated to the trading plan $\mathbf{x}$ is given by

$$\pi(\mathbf{x}) = \sum_t [x_t \cdot r_{t+1} - c_t(x_{t-1}, x_t)] \tag{8.8}$$

where $c_t(x_{t-1}, x_t)$ is the total cost (including but not limited to market impact, spread pay, borrow costs, ticket charges, financing, etc.) associated with holding portfolio x_{t-1} at time $t-1$ and ending up with x_t at time t.

Trading profit $\pi(\mathbf{x})$ is a random variable, since many of its components are future quantities unknowable at time $t = 0$. The distribution of $\pi(\mathbf{x})$ need not be normal, and we do not assume normality in this paper. However, a number of important calculations are only tractable if we assume that the investor's utility function can be approximated by the first two terms in its Taylor series. Thus the problem we treat initially is that of maximizing $u(\mathbf{x})$, where

$$u(\mathbf{x}) := \mathbb{E}[\pi(\mathbf{x})] - (\kappa/2)\mathbb{V}[\pi(\mathbf{x})] \tag{8.9}$$

Just after Intuition 8.1 below, we discuss how our framework can be applied to more general problems which transcend (8.9).

We will often refer to a planned portfolio sequence $\mathbf{x} = (x_1, x_2, \dots, x_T)$ simply as a "path." Similarly we sometimes refer to (8.9) as the "utility of the path $\mathbf{x}$," while remembering the more complex link to utility theory noted above. Our task, in this simpler language, is to find the maximum-utility path $\mathbf{x}^* = \operatorname{argmax}_{\mathbf{x}} u(\mathbf{x})$.

Combining (8.8) with (8.9), and defining

$$\alpha_t := \mathbb{E}[r_{t+1}] \quad \text{and} \quad \Sigma_t := \boldsymbol{V}[r_{t+1}],$$

one has

$$u(\mathbf{x}) = \sum_t \left[x_t' \alpha_t - \frac{\kappa}{2} x_t' \Sigma_t x_t - c_t(x_{t-1}, x_t) \right] \tag{8.10}$$

Note that any symmetric, positive-definite matrix Q defines a norm

$$N_Q(x) = x'Qx$$

and an associated metric $d_Q(x, y) := N_Q(x - y)$, and bilinear form

$$b_Q(x, y) = N_Q(x - y).$$

The bilinear form associated to $\kappa\Sigma_t$ will be useful to us. It is

$$b_{\kappa\Sigma_t}(x_t, y_t) := \frac{1}{2}(y_t - x_t)'\kappa\Sigma_t(y_t - x_t) \tag{8.11}$$

The bilinear form $b_{\kappa\Sigma_t}$ is called the *tracking error variance.*

Now, let us take a certain specific choice of y_t, namely the unconstrained Markowitz portfolio

$$y_t := (\kappa\Sigma_t)^{-1}\alpha_t. \tag{8.12}$$

and evaluate the quadratic form at the pont (8.12)

$$\begin{aligned} b_{\kappa\Sigma_t}(x_t, y_t) &= \frac{1}{2}(y_t - x_t)'\kappa\Sigma_t(y_t - x_t) \\ &= \frac{1}{2}\Big[y_t'(\kappa\Sigma_t)y_t - 2x_t'(\kappa\Sigma_t)y_t + x_t'(\kappa\Sigma_t)x_t\Big] \\ &= \frac{1}{2}y_t'(\kappa\Sigma_t)y_t - x_t'\alpha_t + \frac{1}{2}x_t'(\kappa\Sigma_t)x_t \end{aligned}$$

We have shown

$$-b_{\kappa\Sigma_t}(x_t, y_t) = O(y^2) + x_t'\alpha_t - \frac{1}{2}x_t'(\kappa\Sigma_t)x_t$$

The right-hand side here is the first two terms in the utility calculation (8.10), which are all the terms not dealing with costs. Note that $b_{\kappa\Sigma_t}$ measures variance of the tracking error to the Markowitz portfolio. Then up to x-independent terms,

$$u(\mathbf{x}) = -\sum_t \left[b_{\kappa\Sigma_t}(x_t, y_t) + c_t(x_{t-1}, x_t)\right] \tag{8.13}$$

In any multiperiod optimization problem with transaction costs, one can always ask what the solution would be in an ideal world without transaction costs, or equivalently, in the limit as costs tend to zero. We call this solution the *ideal sequence*, and always denote it by y_t. By mean-variance equivalence, if asset returns follow an elliptical distribution, then the ideal sequence is (8.12).

Intuition 8.1. Multiperiod portfolio optimization is mathematically equivalent to optimally tracking a sequence y_t, called the *ideal sequence*, which is the portfolio sequence that would be optimal in a transaction-cost free world.

The general guiding principle expressed as Intuition 8.1 extends beyond the case in which the ideal sequence is $(\kappa\Sigma_t)^{-1}\alpha_t$, and indeed, beyond the case in which y_t has a clean derivation from a utility function. For computing optimal liquidation paths in the spirit of Almgren and Chriss (1999), the ideal sequence is clearly $y_t = 0$ for all t. For hedging exposure to derivatives, y_t should be our expectation of the offsetting replicating portfolio at all future times until expiration.

Tracking the portfolios of Black and Litterman (1992) is also a special case of our framework in which y_t is the solution to a mean-variance problem with a Bayesian posterior distribution for the expected returns. Since the posterior is Gaussian in the original Black-Litterman model, the two-moment approximation to utility is exact, and one simply replaces α_t and Σ_t with the appropriate quantities.

8.3. **Non-differentiable Optimization.** Given a convex, differentiable map $f : \mathbb{R}^n \to \mathbb{R}$, if we are at a point x such that $f(x)$ is minimized along each coordinate axis, have we found a global minimizer? In other words, does

$$f(x + d \cdot e_i) \geq f(x) \quad \text{for all} \quad d, i$$

imply that $f(x) = \min_z f(z)$? Here $e_i = (0, ..., 1, ...0) \in \mathbb{R}^n$, the i-th standard basis vector.

The answer is: Yes!

Now consider the same question, but without the differentiability assumption.

The answer changes to no, and Fig. 8.1 below gives a counterexample.

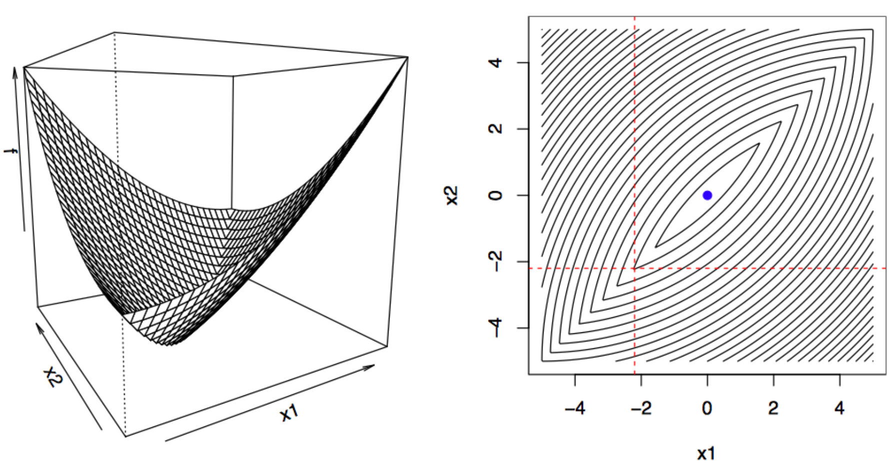

FIGURE 8.1. A convex function for which coordinate descent will get "stuck" before finding the global minimum.

Consider the same question again: "if we are at a point x such that $f(x)$ is minimized along each coordinate axis, have we found a global minimizer?" only now

$$f(x) = g(x) + \sum_{i=1}^{n} h_i(x_i)$$

with g convex, differentiable and each h_i convex? (In this case, we say the non-differentiable part is *separable.*)

If the non-differentiable term is separable, the answer is yes once again. This is a special case of a deep general result proved by Tseng (2001), which we will call "Tseng's theorem." The main take-away is: we can easily optimize

$$f(x) = g(x) + \sum_{i=1}^{n} h_i(x_i)$$

with g convex, differentiable and each h_i convex, by coordinate-wise optimization.

Tseng's results also suggest an algorithm, called *blockwise coordinate descent* (BCD).

Algorithm 8.1. Chose an initial guess for x. Repeatedly iterate cyclically through $i = 1, \dots, N$, and perform the following optimization and update:

$$x_i = \underset{\omega}{\operatorname{argmin}}\, f(x_1, \dots, x_{i-1},\ \omega,\ x_{i+1}, \dots, x_N)$$

Tseng (2001) shows that for functions of the form above, any limit point of the BCD iteration is a minimizer of f. The order of cycling through coordinates is arbitrary; and we can use any scheme that visits each of $\{1, 2, \dots, n\}$ every M steps for fixed constant M. We can also everywhere replace individual coordinates with blocks of coordinates.

8.4. **Single-asset trading paths.** Now let us consider the multiperiod problem for a single asset, in which case the ideal sequence $\mathbf{y} = (y_t)$ and the optimal holdings (or equivalently, hidden states) $\mathbf{x} = (x_t)$ are both univariate time series. Since the multiperiod many-asset problem can be reduced to iteratively solving a sequence of single-asset problems, the methods we develop in this section are important even if our main interest is in multi-asset portfolios.

A very important class of examples arises when there are no constraints, but the cost function is a convex and non-differentiable function of the difference

$$\delta_t := x_t - x_{t-1}.$$

This allows for non-quadratic terms as in Almgren et al. (2005) and non-differentiable terms such as linear proportional costs.

In this case, we can use Tseng's theorem, applied to *trades* rather than *positions.* Writing

$$x_t = x_0 + \sum_{s=1}^{t} \delta_s,$$

the utility function becomes

$$u(\mathbf{x}) = -\sum_t \left[b\Big(x_0 + \sum_{s=1}^{t} \delta_s, y_t\Big) + c_t(\delta_t) \right] \tag{8.14}$$

Eq. (8.14) satisfies the convergence criteria of Tseng (2001) that the non-differentiable term is separable *across time*, while the non-separable term is differentiable. One then performs coordinate descent over the trades $\delta_1, \delta_2, \ldots, \delta_T$. Almost any reasonable starting point will do to initialize the iteration, but if a warm start from a previous optimization is available, that may speed things along. This approach was introduced in Kolm and Ritter (2015).

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