MTH 9831 Assignment 8 (11/04/2015 - 11/11/2015).

Read Lecture 8. Additional references for this material are:

1. S. Shreve, Stochastic Calculus for Finance II, Sections 6.2 - 6.6.
2. A. Etheridge, A Course in Financial Calculus, Section 4.8.

(1) Use the Feynman-Kac formula to solve explicitly (σ and r are given non-negative constants)

$$g_t(t,x) + \frac{1}{2}\sigma^2 g_{xx}(t,x) = rg(t,x), \quad g(T,x) = x^4.$$

Check that your solution is correct by substituting it in the PDE.

(2) (Solving a general linear SDE) Solve Exercise 6.1 following these steps:

(a) Solve $dZ(u) = b(u)Z(u)du + \sigma(u)Z(u)dW(u)$ for $u \geq t$ with the initial condition $Z(t) = 1$.

(b) Solve $dY(u) = k(u)du + \ell(u)dW(u)$ for $u \geq t$, with the initial condition $Y(t) = x$.

(c) Set $X(u) = Y(u)Z(u)$. Find $k(u)$ and $\ell(u)$ such that $X(u)$ solves the original SDE. Express the solution $X(u)$ in terms of the original parameters.

Which of the processes and models that you already know satisfy a linear SDE?

(3) Suppose that for $0 \leq t \leq u \leq T$

$$dX(u) = b(u, X(u))\,du + \sigma(u, X(u))\,dB(u), \quad X(t) = x.$$

Let $f(x)$ and $h(x)$ be given deterministic functions. Find the PDE satisfied by

$$g(t,x) = E^{t,x}[h(X(T)] + \int_t^T E^{t,x}[f(X(u))]\,du.$$

Hint: the game here is to find a relevant martingale, after that the PDE is obtained by applying Ito's formula and setting the drift term to 0. If I give you the martingale, the game is over. Instead, I shall tell you how to see it, since this point of view is applicable to many situations that involve finding a PDE. Start by looking at the following sequence of equalities, think why they are true (all but one are just rewriting of the same thing), and find a martingale:

$$g(t,x) = E^{t,x}[h(X(T)] + \int_t^T E^{t,x}[f(X(u))]\,du;$$

$$g(t,x) = E\Big[h(X(T)] + \int_t^T f(X(u))\,du \,\Big|\, X(t) = x\Big];$$

$$g(t,X(t)) = E\Big[h(X(T)] + \int_t^T f(X(u))\,du \,\Big|\, X(t)\Big];$$

$$g(t,X(t)) = E\Big[h(X(T)] + \int_t^T f(X(u))\,du \,\Big|\, \mathcal{F}(t)\Big];$$

$$g(t,X(t)) = E\Big[h(X(T)] + \int_0^T f(X(u))\,du \,\Big|\, \mathcal{F}(t)\Big] - \int_0^t f(X(u))\,du;$$

$$g(t,X(t)) + \int_0^t f(X(u))\,du = E\Big[g(T,X(T))] + \int_0^T f(X(u))\,du \,\Big|\, \mathcal{F}(t)\Big].$$

Which transition from line to line is non-trivial? Justify this transition.

(4) (From a PDE to an SDE) Find a diffusion process (i.e. an SDE) whose generator $\mathcal{A}$ has the following form:

(a) $(d = 1)$ $\mathcal{A} = (2-x)\dfrac{\partial}{\partial x} + 2x^2\dfrac{\partial^2}{\partial x^2}$. Can you solve the obtained SDE?

(b) $(d = 2)$

$$\mathcal{A} = \frac{1+x^2}{2}\frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial x \partial y} + \frac{1}{2}\frac{\partial^2}{\partial y^2} + 2y\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y}.$$

Give at least two different answers.

(5) (Feynman-Kac+Girsanov) Consider the following terminal value problem:

$$g_t + \frac{1}{2}g_{xx} + \theta(x)g_x = 0, \quad (x,t) \in \mathbb{R} \times [0,T), \quad g(T,x) = h(x).$$

Show that the solution of this problem can be written as

$$g(t,x) = E^{t,x}\left(e^{\int_t^T \theta(B(s))dB(s) - \frac{1}{2}\int_t^T \theta^2(B(s))ds} h(B(T))\right).$$

Hint: use Feynman-Kac and Girsanov theorems. You may assume that h and θ are such that the problem has a unique solution and both theorems are applicable.

(6) (Implying the volatility surface) Solve Exercise 6.10.