Credit Risk Models

4. Swaptions and CMDS

Andrew Lesniewski

Baruch College
New York

Fall 2016

Outline

1. Arbitrage pricing in a nutshell
2. Default swaptions
3. Index options
4. Constant maturity default swaps (CMDS)

Arbitrage free markets

- This is a technical intermezzo in preparation for next few themes: valuation of options on credit default swaps and CMDS based instruments.
- We start by reviewing briefly some basic concepts of arbitrage pricing theory, just enough to cover our upcoming needs.
- We consider a model of a *frictionless* financial market which consists of a number of traded assets. By frictionless we mean that each of the assets is *liquid*, i.e. at each time any bid or ask order can be immediately executed, and there are no *transaction costs*, i.e. each bid and ask order for a security at time t is executed at the same unique price level.
- This is obviously a gross oversimplification of reality. From the conceptual point of view, however, it leads to a profound and workable framework of asset pricing theory.
- We model the price process of each asset as a diffusion process driven by a Brownian motion (or several Brownian motions) on a probability space $(\Omega, \{\mathscr{F}_t\}_{t>0}, \mathsf{P})$.

Arbitrage free market

- A portfolio is *self-financing*, if its price process does not allow for infusion or withdrawal of capital. It is entirely driven by price processes of the constituent instruments and their weights.
- An *arbitrage opportunity* arises if one can construct a self-financing portfolio such that:
  (a) The initial value of the portfolio is zero, $V(0) = 0$.
  (b) With probability one, the portfolio has a non-negative value at maturity, $\mathrm{P}(V(T) \geq 0) = 1$.
  (c) With a positive probability, the value of the portfolio at maturity is positive, $\mathrm{P}(V(T) > 0) > 0$.

  We say the model is *arbitrage free* if it does not allow arbitrage opportunities.
- A random variable X on the probability space $(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathrm{P})$ is square integrable if $\mathrm{E}[X^2] < \infty$. A financial market is called *complete* if each such random variable can be obtained as the terminal value of a self-financing trading strategy, i.e. if there exists a self-financing portfolio such that $X = V(T)$.
- This somewhat technical sounding condition has a natural interpretation. If we think about X as the price of a (possibly very complicated) financial claim, market completeness means that it can be replicated by way of a self-financing trading strategy.

Numeraire

- A *numeraire* is any tradeable asset with price process $\mathcal{N}(t)$ such that $\mathcal{N}(t) > 0$, for all times t. The *relative price* process of an asset with price process $V(t)$ is defined by

$$V^{\mathcal{N}}(t) = \frac{V(t)}{\mathcal{N}(t)}\ .$$

  The relative price of an asset is its price expressed in the units of the numeraire.
- Here are some examples of numeraires that we have already encountered in modeling interest rates.

Examples of numeraires

- A *banking account* defines a numeraire whose value is the value of \$1 deposited in a bank and accruing the riskless instantaneous rate. Its price process $\mathcal{N}(t)$ is

$$\mathcal{N}(t) = \exp\Big(\int_0^t r(s)ds\Big).$$

- A riskless zero coupon bond for maturity T defines a numeraire. Its price at $t < T$ is given by

$$\mathcal{N}(t) = P(t, t, T).$$

- Consider a forward starting annuity (a stream of cash payments) which settles in T_0 and matures in T years from now, respectively. We assume that each cash flow is \$1. Its time t value is given by:

$$A(t, T_{\text{val}}, T_0, T) = \sum_{j=1}^{n} \delta_j P(t, T_{\text{val}}, T_j),$$

where $T_{\text{val}} < T_0$ is the valuation date. The summation runs over the payment dates of the annuity. It defines a numeraire whose value is

$$\mathcal{N}(t) = A(t, T_0, T)$$

(here $A(t, T_0, T) \equiv A(t, t, T_0, T)$.)

Fundamental theorems of arbitrage pricing theory

- A probability measure Q is called an *equivalent martingale measure* (EMM) for the above market, with numeraire $\mathcal{N}(t)$, if it has the following properties:
  (a) Q is equivalent to P, i.e.
  $$dP(\omega) = D_{PQ}(\omega) dQ(\omega),$$
  and
  $$dQ(\omega) = D_{QP}(\omega) dP(\omega),$$
  with some $D_{PQ}(\omega) > 0$ and $D_{QP}(\omega) > 0$.
  (b) The relative price processes $V^{\mathcal{N}}(t)$ are martingales under Q,
  $$V^{\mathcal{N}}(s) = E^{Q}[V^{\mathcal{N}}(t) \,|\, \mathscr{F}_s].$$
- We introduce the notation
  $$D_{PQ} = \frac{dP}{dQ},$$
  and refer to this quantity as the *Radon-Nikodym derivative*.

Fundamental theorems of arbitrage pricing theory

- We now formulate (but not prove) two important theorems.
- *First Fundamental Theorem of arbitrage free pricing*. A market is arbitrage free if and only if, given a numeraire, there exists an equivalent martingale measure Q.
- As an illustration, consider the Black-Scholes model:

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t),$$

  with the bank account numeraire, $B(t) = e^{rt}$.
- With this choice of numeraire,

$$dS^B(t) = (\mu - r)S^B(t)\,dt + \sigma S^B(t)\,dW(t),$$

  where $S^B(t)$ denotes the relative price process,

$$\begin{aligned} S^B(t) &= \frac{S(t)}{B(t)}. \\ &= e^{-rt}S(t). \end{aligned}$$

Fundamental theorems of arbitrage pricing theory

- Next, we use Girsanov's theorem in order to change the probability measure so that the relative price process is driftless,

$$dS^B(t) = \sigma S^B(t)\, dW(t)\,.$$

- Explicitly, this amounts to the following change of measure

$$\frac{d\mathrm{P}}{d\mathrm{Q}}\Big|_t = \exp\Big(-\lambda W(t) - \frac{1}{2}\lambda^2 t\Big),$$

where the quantity

$$\lambda = \frac{\mu - r}{\sigma}$$

is known as the *market price of risk*.

Fundamental theorems of arbitrage pricing theory

- *Second Fundamental Theorem of arbitrage free pricing*. An arbitrage free market is complete if and only if the equivalent martingale measure Q is unique.
- In other words, in a complete, arbitrage free market, to a given numeraire $\mathcal{N}(t)$ corresponds one and only one martingale measure Q.
- Here are some standard EMMs:
    - the EMM associated with the banking account is called the *risk neutral measure*.
    - the EMM associated with the zero coupon bond maturing at T is called the *T-forward measure*.
    - the EMM associated with the forward annuity is called the *annuity measure*.

Change of numeraire technique

- An important consequence of the first fundamental theorem is the arbitrage pricing law. For $s < T$
$$\frac{V(s)}{\mathcal{N}(s)} = \mathsf{E}^{\mathsf{Q}}\Big[\frac{V(T)}{\mathcal{N}(T)}\,\Big|\mathscr{F}_s\Big].$$
- One is, of course, free to use a different numeraire $\mathcal{N}(t) \to \mathcal{N}'(t)$. Girsanov's theorem implies that there exists a martingale measure Q' such that
$$\frac{V(s)}{\mathcal{N}'(s)} = \mathsf{E}^{\mathsf{Q}'}\Big[\frac{V(T)}{\mathcal{N}'(T)}\,\Big|\mathscr{F}_s\Big].$$
- Explicitly, the Radon-Nikodym derivative is given by the ratio of the numeraires:
$$\begin{aligned}\frac{d\mathsf{Q}'}{d\mathsf{Q}}\Big|_t &= \frac{\frac{\mathcal{N}(0)}{\mathcal{N}(t)}}{\frac{\mathcal{N}'(0)}{\mathcal{N}'(t)}}\\ &= \frac{\mathcal{N}(0)}{\mathcal{N}(t)}\frac{\mathcal{N}'(t)}{\mathcal{N}'(0)}.\end{aligned}$$
- The choice of numeraire and the corresponding martingale measure is very much a matter of convenience, and is motivated by the problem at hand.

Default swaption

- A *default swaption* or *CDS option* is an OTC instrument which grants the holder the right, but not obligation, to enter into a (single name) CDS on a contractually specified date T_0 and at a contractually specified spread C.
- An option to buy protection is called a *payer* swaption, while an option to sell protection is called a *receiver* swaption.
- It is common (especially outside of the USD market) that default swaptions have a knock out feature built in.
- Namely, if the reference name underlying the CDS defaults prior to T_0, the swaption is knocked out and the contract is nullified.
- As a result of this knock out provision, single name default swaptions are rather illiquid and, typically, only short dated contracts trade.
- The knock out feature inserts digital risk into a single name swaption, which may have unpleasant consequences.
- For example, a holder of a payer swaption sees its value go through the roof as the spread blows out (and the swaption is deep in the money), and drop to zero if the underlying name defaults prior to T_0.

Valuation of default swaptions

- Consider a payer swaption (with the knock out provision), i.e. an option to buy protection, struck at K. Its payoff (exercise value) is

$$
\begin{aligned}
V_{pay}(T_0, T) &= 1_{\tau > T_0}(V_{prot}(T_0, T) - K\mathcal{A}(T_0, T))^+ \\
&= 1_{\tau > T_0}\mathcal{A}(T_0, T)(C(T_0, T) - K)^+,
\end{aligned}
$$

  where $C(T_0, T) = V_{prot}(T_0, T)/\mathcal{A}(T_0, T)$ is the forward par spread at T_0.
- This expression looks very much like the payoff of an interest rate swaption, except that for the presence of the indicator function $1_{\tau > T_0}$ (reflecting the knock out provision).
- We can include the knock out feature in the risky annuity by defining

$$
\mathcal{A}^*(t, T_0, T) = 1_{\tau > t}\mathcal{A}(t, T_0, T),
$$

  so that the payoff takes the form

$$
V_{pay}(T_0, T) = \mathcal{A}^*(T_0, T)(S_0 - K)^+.
$$

Valuation of default swaptions

- This expression looks exactly like the payoff of an interest rate swaption, and we are inclined to use the methods of arbitrage pricing theory.
- However, while $\mathcal{A}^*$ represents the price of a traded asset (namely a stream of risky cash payments), strictly speaking, *it is not* a a numeraire, as it is not positive.
- From a practical perspective, this is not a problem because the value of the swaption is zero when the numeraire turns zero.
- In order to formulate this mathematically, we start with the valuation formula under the risk neutral measure. Using the results presented in Lecture Notes 2, we find that

$$
\begin{aligned}
V_{pay}(0) &= \mathrm{E}\Big[1_{\tau > T_0} \mathcal{A}(T_0, T)(C(T_0, T) - K)^+ e^{-\int_0^{T_0} r(s)ds}\Big] \\
&= \mathrm{E}\Big[\mathcal{A}(T_0, T)(C(T_0, T) - K)^+ e^{-\int_0^{T_0}(r(s)+\lambda(s))ds}\Big].
\end{aligned}
$$

Valuation of default swaptions

- In the risk neutral measure, we have thus transformed the (non-strictly positive) indicator process $1_{t>\tau}$ into the (strictly positive) process $e^{-\int_0^t (r(s)+\lambda(s))ds}$.
- The factor $e^{-\int_0^{T_0} (r(s)+\lambda(s))ds}$ can be combined with $\mathcal{A}(T_0, T)$, yielding the following numeraire:

$$\begin{aligned}\mathcal{N}(t) &= \mathcal{A}(t, T_0, T)\\ &= \sum_{1\leq i\leq N} \delta_i \mathcal{P}(t, t, T_i).\end{aligned}$$

- Note that this numeraire does not correspond to an actual financial asset (as the risky zero coupon bonds are not modeled correctly), but serves as a useful “pseudo-asset”.
- We let $Q_{T_0,T}$ denote the EMM associated with this numeraire, called the *forward survival measure*.

Valuation of default swaptions

- Using Girsanov's theorem and changing to $Q_{T_0,T}$, we can write the swaption price in the familiar form

$$V_{pay} = \mathcal{A}_0(T_0, T)\mathrm{E}^{Q_{T_0,T}}[(C(T_0, T) - K)^+].$$

- Note that, under the survival measure $Q_{T_0,T}$, the par spread

$$C(t, T_0, T) = \frac{V_{prot}(t, T_0, T)}{\mathcal{A}(t, T_0, T)}$$

is a martingale.

- The precise nature of this martingale is, of course, unknown, and we have to use approximations.

Lognormal model for default swaptions

- The market convention for option pricing is to assume that $C(t)$ follows a lognormal process:
$$\frac{dC(t)}{C(t)} = \sigma dW(t).$$
- As consequence, the swaption price is given by a Black-Scholes type formulas:
$$\begin{aligned} V_{pay} &= \mathcal{A}_0(T)\big(C_0 N(d_1) - KN(d_2)\big), \\ V_{rec} &= \mathcal{A}_0(T)\big(KN(-d_2) - C_0 N(-d_1)\big). \end{aligned}$$
where
$$\begin{aligned} d_1 &= \frac{\log \frac{C_0}{K} + \frac{1}{2}\sigma^2 T_0}{\sigma\sqrt{T_0}}, \\ d_2 &= \frac{\log \frac{C_0}{K} - \frac{1}{2}\sigma^2 T_0}{\sigma\sqrt{T_0}}. \end{aligned}$$

Mechanics of an index option

- There is a liquid market for short dated European options on standardized credit indices. Since the indices roll every 6 months, and the off-the-run indices are less liquid, most of the option activity is in the 1 through 3 months.
- At expiration, the option holder has the right to exercise into a basket of CDSs at a specified spread.
- The content of the basket consists of the components of the underlying index, at the time of option trade.
- Options on indices do not have the knock out feature.
- If there are substitutions or defaults in the basket during the life of the option, this does not affect the deliverable basket in the option contract. In particular, all defaulted CDSs should be delivered at option expiration, with values of one minus recovery.
- This is in contrast with single name CDS swaptions.

Valuation of an index option

- Consider a call option on an index struck at K. For simplicity, we assume that each component CDS has a notional of \$1 (rather than $1/N$ of the face value).
- The payoff of this option is given by

$$\begin{aligned} V_{call}(T_0) &= \Big(\sum_{j=1}^{N} V^j_{prot}(T_0, T) - K\Big)^+ \\ &= \Big(\sum_{j\notin\mathfrak{D}} V^j_{prot}(T_0, T) + \sum_{j\in\mathfrak{D}}(1 - R_j) - K\Big)^+, \end{aligned}$$

where $\mathfrak{D}$ denotes the set of all index components that defaulted prior to T_0.

Valuation of an index option

- Under the risk neutral measure,

$$\mathrm{E}\Big[\sum_{j\notin\mathfrak{D}} V^j_{prot}(T_0,T)\Big] = \mathrm{E}\Big[\sum_{j=1}^{N} 1_{\tau_j>T_0}\mathcal{A}^j(T_0,T)(C^j(T_0,T)-C)\Big]$$
$$= P_0(T_0)^{-1}\sum_{j=1}^{N}\mathcal{A}^j(0,T_0,T)(C^j(T_0,T)-C),$$

  where C^j is the par spread for name j, and C is the spread on the index.
- Also,

$$\mathrm{E}\Big[\sum_{j\in\mathfrak{D}}(1-R_j)\Big] = \mathrm{E}\Big[\sum_{j=1}^{N} 1_{\tau_j<T_0}(1-R_j)\Big]$$
$$= \sum_{j=1}^{N} Q_j(0,T_0)(1-R_j).$$

Valuation of an index option

- Let us now write the payoff as

$$V_{call}(T_0) = (\Phi(T_0) - K)^+.$$

- As a result of the calculations above, we know that

$$\mathrm{E}[\Phi(T_0)] = P_0(T_0)^{-1} \sum_{j=1}^{N} \mathcal{A}^j(0, T_0, T)(C^j(T_0, T) - C) + Q_j(0, T_0)(1 - R_j).$$

- To compute the option value $V_{call}(T_0)$, we need more information about the distribution of $\Phi(T_0)$ beyond its expected value. This can be done by means of the following approximations.

Valuation of an index option

- We assume that $\Phi(T_0)$ can be expressed terms of a suitable "effective" spread $C(T_0)$. A simple approach consists in writing:

$$\begin{aligned}\Phi(T_0) &= \mathfrak{S}(C(T_0)) \\ &= N\mathcal{A}_C(T_0, T)(C(T_0) - C),\end{aligned}$$

where $\mathcal{A}_C(T_0, T)$ is the "effective" risky annuity given in terms of the "effective" survival probabilities $S_C(T_0, T_j)$ introduced below.

- The survival probabilities $S_C(T_0, T_j)$ are defined as follows. We use the credit triangle $C_0 \approx \lambda(1 - R)$ to write

$$S_C(T_0, T_j) = e^{-(C(T_0)/(1-R_{eff}))(T_j - T_0)},$$

where R_{eff} is an effective recovery rate (usually set to 40%).

- The expression for S_C is similar to the way a CDS index is quoted by assuming that all names have identical characteristics.

Valuation of an index option

- We write the payoff of the option as

$$V_{call}(T_0) = \left(\mathfrak{S}(C(T_0)) - K\right)^+,$$

and so

$$V_{call}\,(0) = P_0(T_0)\mathsf{E}\big[\left(\mathfrak{S}(C(T_0)) - K\right)^+\big].$$

- If we know the density φ of the effective coupon $C(T_0)$, we can compute this option value by numerical evaluation of the integral:

$$V_{call} = P_0(T_0)\int_0^\infty (\mathfrak{S}(c) - K)^+\varphi(c)dc.$$

- This is subject to the constraint on the expected value:

$$\begin{aligned}\mathsf{E}\big[\mathfrak{S}(C(T_0))\big] &= \int_0^\infty \mathfrak{S}(c)\varphi(c)dc \\ &= P_0(T)^{-1}\sum_{j=1}^{N}\mathcal{A}_j(C_0^j - C) + Q_j(0, T_j)(1 - R_j).\end{aligned}$$

Valuation of an index option

- The market convention is to assume the lognormal distribution, and so we write

$$C(T_0) = \mu e^{\sigma_C \sqrt{T_0}\, x - \frac{1}{2}\,\sigma_C^2 T_0}, \text{ where } x \sim N(0,1).$$

- The index spread volatility σ_C is a market observable. The mean $\mu = \mathrm{E}[C(T_0)]$ is found by enforcing the condition for $\mathrm{E}\big[\mathfrak{S}(C(T_0))\big]$ above.
- To summarize the above calculations, the call price is given by

$$V_{call} = \frac{P_0(T)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big(\mathfrak{S}\big(\mu e^{\sigma_C \sqrt{T}\, x - \frac{1}{2}\,\sigma_C^2 T}\big) - K\Big)^+ e^{-\frac{1}{2}\,x^2} dx,$$

subject to the condition for μ:

$$\mathrm{E}\big[\mathfrak{S}(C(T_0))\big] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathfrak{S}\big(\mu e^{\sigma_C \sqrt{T}\, x - \frac{1}{2}\,\sigma_C^2 T}\big) e^{-\frac{1}{2}\,x^2} dx,$$

where $\mathrm{E}[\Phi(T_0)]$ is defined above. The integrals can be calculated numerically.
- Note that the strike K is often given in spread terms rather than in price terms. The conversion from spread to strike follows the algorithm presented in Lecture Notes 3 (which is also similar to the definition of $C(T_0)$ above).

CMDS

- A CMDS is a variation on the CDS in which one leg pays a periodic floating (rather than fixed) coupon. This coupon is linked to the fixing of a reference spread on the previous coupon date.
- Denote the coupon dates on the swap by T_i, $i = 1, \ldots, M$. On the date T_i the coupon leg pays the par yield $C(T_{i-1})$ of a reference CDS maturing, say, 5 years later. Since the maturity of the reference CDS remains constant throughout the life of the contract, it i referred to as a *constant maturity default swap*.
- Typically, the other leg on the swap is the standard protection leg. The other leg may also be a standard premium leg, or a CMDS leg corresponding to a different maturity CDS (say, 10 years).
- To develop a pricing model, consider first a single coupon setting at time T_0, on a par rate corresponding to a CDS with payment dates $T_1, \ldots, T_N = T$, so that the maturity of the reference swap is $T - T_0$. The CMDS coupon is paid on T_1.
- For simplicity, we assume that there is no payment at time T_1 if $\tau < T_1$ (no accrued interest).

Valuation of a CMDS

- The value of this payment on the settlement date is

$$V_{cpn}(T_0) = 1_{\tau > T_0} \mathcal{P}(T_0, T_1) C(T_0) \delta,$$

where δ is a day count fraction.

- Today's value of the payment is

$$\begin{aligned} V_{cpn}(0) &= \mathsf{E}[1_{\tau > T_0} \mathcal{P}(T_0, T_1) C(T_0) e^{-\int_0^{T_0} r(s) ds}] \delta \\ &= \mathcal{A}(0) \mathsf{E}^{\mathsf{Q}_{T_0, T}} [\mathcal{P}(T_0, T_1) \mathcal{A}(T_0)^{-1} C(T_0)] \delta, \end{aligned}$$

where $\mathsf{Q}_{T_0,T}$ is the survival measure. We know that $C(t)$ is a martingale under the measure $\mathsf{Q}_{T_0,T}$.

- The quantity $\frac{\mathcal{P}(t, T_1)}{\mathcal{A}(t)}$ is also a martingale.

Andersen's method

- Recall the property of iterated expected value:

$$E[E[X|Y]] = E[X].$$

- Following Leif Andersen, we use this identity to write

$$V_{cpn}(0) = \mathcal{A}(0)\mathrm{E}^{\mathrm{Q}_{T_0,T}}[f(C(T_0))C(T_0)]\delta,$$

where $f(C(T_0))$ is the conditional expectation:

$$f(C(T_0)) \triangleq \mathrm{E}^{\mathrm{Q}_{T_0,T}}[\mathcal{P}(T_0, T_1)\mathcal{A}(T_0)^{-1} \mid C(T_0)].$$

Andersen's method

- Our goal is to find an effective approximation to $f(x)$. Then we set

$$g(x) = f(x)\, x,$$

and compute $V_{cpn}(0)$ by integration:

$$V_{cpn}(0) = \mathcal{A}(0)\delta \int_0^\infty g(x)\,\varphi(x)\,dx,$$

where φ is the probability density of $C(T_0)$.

- The density φ is often known from the assumed model.
- It can also be determined from observed swaption prices by means of the *replication formula*.

Replication formula

- First, we establish an identity which is essentially a restatement of the first order Taylor theorem.
- For a twice continuously differentiable function $F(x)$,

$$F(x) = F(x_0) + F'(x_0)(x - x_0) + \int_{x_0}^{x} F''(u)(x - u)du.$$

- Now, we rewrite the remainder term as follows:

$$\int_{x_0}^{x} F''(u)(x - u)du = \int_{-\infty}^{x_0} F''(u)(u - x)^{+}du + \int_{x_0}^{\infty} F''(u)(x - u)^{+}du,$$

where, as usual, $x^{+} = \max(x, 0)$. Note that the remainder term looks very much like a mixture of payoffs of calls and puts!

Replication formula

- Assume that we are given a diffusion process $X(t) \geq 0$, and use the formula above with $x = X(T)$:

$$F(X(T)) = F(X_0) + F'(X_0)(X(T) - X_0) + \int_0^{X_0} F''(K)(K - X(T))^+ dK + \int_{X_0}^{\infty} F''(K)(X(T) - K)^+ dK.$$

- If Q is a martingale measure such that $\mathrm{E}^{\mathrm{Q}}[X(t)] = X_0$, then

$$\mathrm{E}^{\mathrm{Q}}[F(X(T))] = F(X_0) + \int_0^{X_0} F''(K)\, B^{\mathrm{put}}(T, K, X_0) dK + \int_{X_0}^{\infty} F''(K)\, B^{\mathrm{call}}(T, K, X_0) dK,$$

where

$$B^{\mathrm{call}}(T, K, X_0) = \mathrm{E}^{\mathrm{Q}}[(X(T) - K)^+],$$
$$B^{\mathrm{put}}(T, K, X_0) = \mathrm{E}^{\mathrm{Q}}[(K - X(T))^+].$$

This is the desired replication formula.

Replication formula

- The replication formula states that if $F(x)$ is the payoff of an instrument, its expected value at time T is given by its today's value plus the value of a basket of out of the money calls and puts weighted by the second derivative of the payoff evaluated at the strikes.
- There is an alternative way of writing this. We integrate by parts twice, and note that (i) the boundary terms at 0 and ∞ vanish, and (ii) the following relations hold at X_0:

$$
\begin{aligned}
B^{\text{put}}(T, X_0, X_0) - B^{\text{call}}(T, X_0, X_0) &= 0, \\
\frac{\partial B^{\text{put}}}{\partial K}(T, X_0, X_0) - \frac{\partial B^{\text{call}}}{\partial K}(T, X_0, X_0) &= 1.
\end{aligned}
$$

- As a result,

$$
\mathrm{E}^{\mathrm{Q}}[F(X(T))] = \int_0^\infty F(K) \frac{\partial^2}{\partial K^2} B^{\text{call}}(T, K, X_0)\, dK.
$$

Andersen's method

- Therefore
$$\varphi(K) = \mathcal{A}(0)^{-1} \frac{\partial^2}{\partial K^2} V_{call}(K),$$
so that
$$V_{cpn}(0) = \delta \int_0^\infty g(K) \frac{\partial^2}{\partial K^2} V_{call}(K)\, dK.$$
- This shows that the CMDS payout can be replicated by taking positions in CDS swaptions at many different strikes. The weight on the swaption with strike K is $g''(K)$.
- We still need to find an approximate form of f. To this end, we notice that the graph of $\mathcal{P}(T_0, T_1)\mathcal{A}(T_0)^{-1}$ against $C(T_0)$ is roughly a straight line (in reality, the ratio depends on the entire term structure of spreads...). We thus write
$$f(C) \approx aC + b.$$

Andersen's method

- To determine the constant b, we can use the fact that for $C = 0$,
$$\mathcal{P}(T_0, T_1)\mathcal{A}(T_0)^{-1} = P(T_0, T_1)A(T_0)^{-1},$$
which implies that
$$b = P_0(T_1)A_0^{-1}.$$
- We can then find a from the basic requirement that $\mathcal{P}(t, T_1)/\mathcal{A}(t)$ be a martingale in the annuity measure, such that
$$\begin{aligned} \mathsf{E}^{\mathsf{Q}_{T_0,T}}[aC(T_0) + b] &= aC_0 + b \\ &= \mathcal{P}_0(T_1)/\mathcal{A}(0). \end{aligned}$$
or
$$a = \frac{\mathcal{P}_0(T_1)/\mathcal{A}(0) - b}{C_0}.$$
- So, as a reasonable approximation, we have the formula
$$V_{cpn}(0) = \mathcal{A}(0)\delta \int_0^\infty (ax^2 + bx)\varphi(x)dx.$$
- This integral can be carried out analytically in various models. It can also, in principle, be computed model free, by means of the replication formula.

Pricing CMDS with a cap

- The contract we have considered so far is somewhat simplified. For distressed names, their par spreads can reach very high levels and the CDSs will stop trading.
- As a consequence, it is a standard practice to impose a cap on the forward spread. Also, there is typically a notional scale α on the coupon leg, to allow the total CMDS contract to be issued at par, i.e. at zero value.
- The coupon payment at time T_i is then

$$\alpha\delta 1_{\tau > T_i} \min(C(T_{i-1}), U),$$

  where U is the cap level.
- This change can be easily accommodated: we replace the unconstrained payoff with

$$g(x) = \alpha f(x) \min(x, U),$$

  where $f(x)$ is the same function as before.

Pricing CMDS with a cap

- Using Andersen's method, this yields

$$
\begin{aligned}
V_{cpn}(0) &= \alpha\delta\mathcal{A}(0)\int_0^\infty (ax+b)\min(x,U)\varphi(x)\,dx \\
&= \alpha\delta\mathcal{A}(0)\int_0^U (ax^2+bx)\varphi(x)\,dx + \alpha\delta\mathcal{A}(0)U\int_U^\infty (ax+b)\varphi(x)\,dx.
\end{aligned}
$$

- For CMDS contracts, the value of α that renders the values of both legs of the swap identical is denoted α_{par}; this is the quoted value.
- For the lognormal model, CMDS caps can be priced in closed form.

References

- Andersen, L.: Credit models, lecture notes (2010).
- O'Kane, D.: *Modelling Single-Name and Mult-Name Credit Derivatives*, Wiley (2008).
- Schönbucher, P. J.: Credit Derivatives Pricing Models, Wiley Finance (2003).