MTH 9831 Fall 2015 HW #10 Solutions

Please note that these are *possible* solutions, and that there are often many ways to approach and solve a problem or prove a theorem.

Problem 1 (Shreve #8.7) Since f and g are convex functions, we have for all $0 \le x_1 \le x_2$ and $\lambda \in [0,1]$

$$f((1-\lambda)x_1 + \lambda x_2) \le (1-\lambda)f(x_1) + \lambda f(x_2) \le (1-\lambda)h(x_1) + \lambda h(x_2),$$

and similarly we have

$$g((1-\lambda)x_1 + \lambda x_2) \le (1-\lambda)g(x_1) + \lambda g(x_2) \le (1-\lambda)h(x_1) + \lambda h(x_2).$$

Hence for all $0 \le x_1 \le x_2$ and $\lambda \in [0,1]$ we get

$$\begin{aligned} h((1-\lambda)x_1 + \lambda x_2) &= \max\big(f((1-\lambda)x_1 + \lambda x_2), g((1-\lambda)x_1 + \lambda x_2)\big) \\ &\le (1-\lambda)h(x_1) + \lambda h(x_2). \end{aligned}$$

Problem 2 (Shreve #11.1) Let $M(t)$ be the compensated Poisson process of Thm. 11.2.4: $M(t) = N(t) - \lambda t$, where $N(t) \sim Poisson(\lambda t)$.

(a) $M(t)$ is a martingale by Thm. 11.2.4. Thus, by the conditional Jensen's inequality, since $f(x) = x^2$ is convex, for $s < t$,

$$E(M^2(t)|\mathcal{F}_s) \ge E(M(t)|\mathcal{F}_s)^2 = M(s)^2,$$

which gives us that $M(t)^2$ is a submartingale.

(b) By the fact that $N(t)$ has stationary, independent increments, and by the results of p. 467 and (11.2.9),

$$\begin{aligned} E((M(t) - M(s))^2) &= E([(N(t) - N(s)) - \lambda(t-s)]^2 \\ &= E((N(t) - N(s))^2) - 2\lambda(t-s)E(N(t) - N(s)) + \lambda^2(t-s)^2 \\ &= \lambda^2(t-s)^2 + \lambda(t-s) - 2\lambda^2(t-s)^2 + \lambda^2(t-s)^2 = \lambda(t-s) \\ \implies E(M(t)^2 - \lambda t|\mathcal{F}_s) &= E((M(t) - M(s))^2) + M(s)^2 - \lambda s - \lambda(t-s)|\mathcal{F}_s) \\ &= M(s)^2 - \lambda s + E((M(t) - M(s))^2) - \lambda(t-s) = M(s)^2 - \lambda s. \end{aligned}$$

Problem 3 (Shreve #11.5)
To show that two Poisson processes $N_1(t)$, $N_2(t)$ relative to the same filtration are independent, we'll follow the hint and adapt the proof of Corollary 11.5.3 to show that the joint moment generating function factors. Define

$$X(t) = u_1N_1(t) + u_2N_2(t) - \lambda_1 t(e^{u_1} - 1) - \lambda_2 t(e^{u_2} - 1) = X^c(t) + u_1N_1(t) + u_2N_2(t)$$

so that $Y(t) = e^{X(t)}$. Itô gives us, on $s < u \le t$,

$$e^{X(t)} = e^{X(s)} + \int_s^t e^{X(u)}dX^c(u) + \frac{1}{2}\int_s^t e^{X(u)}d[X^c, X^c](u) + \sum_{s<u\le t}(\Delta e^{X(u)}), \text{ where}$$

$$\Delta e^{X(u)} = e^{X(u)} - e^{X(u-)} = e^{X(u-)}(e^{\Delta X(u)} - 1) = e^{X(u-)}(e^{u_1\Delta N_1(u) + u_2\Delta N_2(u)} - 1).$$

For the continuous terms, we have

$$dX^c(t) = [-\lambda_1(e^{u_1} - 1) - \lambda_2(e^{u_2} - 1)]dt; \qquad d[X^c, X^c](t) = 0.$$

Note that, by #4, only one of the N_js jumps at a time, and $\Delta N_j(t) = 1$ or 0. Hence, this reduces to

$$\Delta e^{X(u)} = e^{X(u-)}\left((e^{u_1} - 1)\Delta N_1(u) + (e^{u_2} - 1)\Delta N_2(u)\right).$$

Combining all these, we have, for any $s \geq 0$,

$$\begin{aligned}
Y(t) = e^{X(t)} &= e^{X(s)} + \int_s^t e^{X(u-)}[-\lambda_1(e^{u_1} - 1) - \lambda_2(e^{u_2} - 1)]du \\
&\quad + \sum_{s<u\leq t} e^{X(u-)}\left((e^{u_1} - 1)\Delta N_1(u) + (e^{u_2} - 1)\Delta N_2(u)\right) \\
&= e^{X(s)} + \int_s^t e^{X(u-)}(e^{u_1} - 1)d(N_1(t) - \lambda_1 t) + \int_s^t e^{X(u-)}(e^{u_2} - 1)d(N_2(t) - \lambda_2 t).
\end{aligned}$$

Therefore, $Y(t)$ is a martingale because $N_j(t) - \lambda_j t$ are martingales, so by the optional stopping theorem,

$$\begin{aligned}
&E[e^{u_1 N_1(t) + u_2 N_2(t) - \lambda_1 t(e^{u_1} - 1) - \lambda_2 t(e^{u_2} - 1)}] = E[Y(0)] = 1 \\
&\qquad \Longrightarrow E[e^{u_1 N_1(t) + u_2 N_2(t)}] = e^{\lambda_1 t(e^{u_1} - 1)} e^{\lambda_2 t(e^{u_2} - 1)} = E[e^{u_1 N_1(t)}]E[e^{u_2 N_2(t)}],
\end{aligned}$$

which implies that $N_1(t)$ and $N_2(t)$ are independent.
Alternatively, we can do this via Theorem 11.5.4 instead of Corollary 11.5.3, using the functions

$$\begin{aligned}
X_j(t) &= u_j N_j(t) - \lambda_j t(e^{u_j} - 1);\ j = 1, 2 \\
f(t, X_1(t), X_2(t)) &= X_1(t) + X_2(t)
\end{aligned}$$

and follow Itô's formula for jump processes. Really, it's the same thing, just reorganized.

Problem 4

(a)

$$\int_0^t N(s)dN(s) = \sum_{s\in(0,t]} N(s)\Delta N(s) = 1 + 2 + \ldots + N(t) = \frac{N(t)(N(t) + 1)}{2}$$

(b)

$$\int_0^t N(s-)dN(s) = \sum_{s\in(0,t]} N(s-)\Delta N(s) = 0 + 1 + \ldots + (N(t) - 1) = \frac{N(t)(N(t) - 1)}{2}$$

(c) Ito's formula for $M^2(t)$ gives

$$\begin{aligned}
M^2(t) &= M^2(0) + 2\int_0^t M(s)dM^c(s) + \sum_{s\in(0,t]} M^2(s) - M^2(s-) \\
&= 2\int_0^t M(s-)dM^c(s) + \sum_{s\in(0,t]} (M(s-)+\Delta N(s))^2 - M^2(s-) \\
&= 2\int_0^t M(s-)dM^c(s) + \sum_{s\in(0,t]} 2M(s-)\Delta N(s) + \sum_{s\in(0,t]} (\Delta N(s))^2 \\
&= 2\int_0^t M(s-)dM(s) + \sum_{s\in(0,t]} (\Delta N(s))^2 \\
&= 2\int_0^t M(s-)dM(s) + N(t).
\end{aligned}$$

Hence we conclude

$$\int_0^t M(s-)dM(s) = \frac{1}{2}(M^2(t) - N(t)).$$

(d) Again using Ito's formula for $M^2(t)$, we have

$$\begin{aligned}
M^2(t) &= M^2(0) + 2\int_0^t M(s)dM^c(s) + \sum_{s\in(0,t]} M^2(s) - M^2(s-) \\
&= 2\int_0^t M(s)dM^c(s) + \sum_{s\in(0,t]} M^2(s) - (M^2(s) - \Delta N(s))^2 \\
&= 2\int_0^t M(s)dM^c(s) + \sum_{s\in(0,t]} 2M(s)\Delta N(s) - \sum_{s\in(0,t]} (\Delta N(s))^2 \\
&= 2\int_0^t M(s)dM(s) - \sum_{s\in(0,t]} (\Delta N(s))^2 \\
&= 2\int_0^t M(s)dM(s) - N(t).
\end{aligned}$$

Hence we conclude

$$\int_0^t M(s)dM(s) = \frac{1}{2}(M^2(t) + N(t)).$$

Problem 5 Given $\sigma > -1$ and a process S satisfying $dS(t) = \sigma S(t-)dM(t)$, we show that $S(t) = S(0)e^{-\lambda\sigma t}(1+\sigma)^{N(t)}$.
We define $X(t) = \sigma M(t)$, which gives $X^c(t) = -\lambda\sigma t$ (the continuous part of X), $d[X^c, X^c]_t = 0$ (the quadratic variation of the continuous part of X) and $\Delta X(t) = \sigma\Delta N(t)$ (the jump part of X). Next we rewrite the above SDE in terms of X, i.e $dS(t) = S(t-)dX(t)$ which is equivalent to

$$S(t) = S(0) + \int_0^t S(s-)dX(s) = \underbrace{S(0) + \int_0^t S(s-)dX^c(s)}_{=S^c(t)} + \underbrace{\sum_{s\in(0,t]} S(s-)\Delta X(s)}_{J^S(t)}.$$

Moreover we have $\Delta S(t) = S(t-)\Delta X(t)$. Now we apply Ito's formula to $\ln S(t)$ and obtain

$$\begin{aligned}
\ln S(t) &= \ln S(0) + \int_0^t \frac{1}{S(s-)} dS^c(s) + \sum_{s\in(0,t]} \ln S(s) - \ln S(s-) \\
&= \ln S(0) + \int_0^t dX^c(s) + \sum_{s\in(0,t]} \ln\left(\frac{S(s)}{S(s-)}\right) \\
&= \ln S(0) + X^c(t) + \sum_{s\in(0,t]} \ln\left(1 + \frac{\Delta S(s)}{S(s-)}\right) \\
&= \ln S(0) + X^c(t) + \sum_{s\in(0,t]} \ln\left(1 + \Delta X(s)\right) \\
&= \ln S(0) + X^c(t) + \sum_{s\in(0,t]} \ln\left(1 + \sigma\Delta N(s)\right).
\end{aligned}$$

We take exp on both sides to finally obtain

$$\begin{aligned}
S(t) &= S(0) e^{X^c(t) + \sum_{s\in(0,t]} \ln(1+\sigma\Delta N(s))} \\
&= S(0) e^{-\lambda\sigma t} \prod_{s\in(0,t]} (1 + \sigma\Delta N(s)) \\
&= S(0) e^{-\lambda\sigma t} (1+\sigma)^{N(t)}.
\end{aligned}$$

Problem 6 First, we rewrite $Z(t)$ as

$$Z(t) = Z(0) \exp\left((\lambda - \tilde{\lambda})t + \ln\left(\frac{\tilde{\lambda}}{\lambda}\right) N(t) \right),$$

which inspires us to define $X(t) = (\lambda - \tilde{\lambda})t + \ln\left(\frac{\tilde{\lambda}}{\lambda}\right) N(t)$. Hence we have $X(t) = X^c(t) + \Delta X(t)$ with $X^c(t) = (\lambda - \tilde{\lambda})t$, $d[X^c, X^c]_t = 0$ and $\Delta X(t) = \ln\left(\frac{\tilde{\lambda}}{\lambda}\right) \Delta N(t)$. We apply Ito's formula to $f(x) = Z(0)e^x$ and get

$$\begin{aligned}
Z(t) = Z(0)e^{X(t)} &= Z(0) + \int_0^t Z(s) dX^c(s) + \sum_{s\in(0,t]} Z(0)(e^{X(s)} - e^{X(s-)}) \\
&= Z(0) + \int_0^t Z(s-) dX^c(s) + \sum_{s\in(0,t]} Z(s-)\left(\exp\left(\ln\left(\frac{\tilde{\lambda}}{\lambda}\right) \Delta N(s) \right) - 1 \right) \\
&= Z(0) + \int_0^t Z(s-)(\lambda - \tilde{\lambda}) ds + \sum_{s\in(0,t]} Z(s-) \frac{\tilde{\lambda} - \lambda}{\lambda} \Delta N(s) \\
&= Z(0) + \int_0^t Z(s-) \frac{\tilde{\lambda} - \lambda}{\lambda} dM(s).
\end{aligned}$$