December 9, 2015

MTH 9831. Solutions to Quiz 11.

(1) (2+3 points) Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity λ, $M(t) = N(t) - \lambda t$, $t \geq 0$, be a compensated Poisson process. Compute

$$(a)\ E\left(\int_0^t s\,dM(s)\right)^2; \qquad (b)\ E\left(\int_0^t M(s)\,ds\right)^2.$$

Solution. (a) By problem (2) of HW 11

$$E\left(\int_0^t s\,dM(s)\right)^2 = \lambda\int_0^t s^2\,ds = \frac{1}{3}\,\lambda t^3.$$

(b) The shortest way to solve this problem is probably the following (the same as we used for a BM). By Itô's product formula

$$\int_0^t M(s)\,ds = tM(t) - \int_0^t s\,dM(s) = \int_0^t (t-s)\,dM(s).$$

Then we use the same fact as in part (a):

$$E\left(\int_0^t M(s)\,ds\right)^2 = E\left(\int_0^t (t-s)\,dM(s)\right)^2 = \lambda\int_0^t (t-s)^2\,ds = \frac{1}{3}\,\lambda t^3.$$

(2) (5=1+4 points) Let under $\mathbb{P}$ the process $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity λ, $\{Q(t)\}_{t\geq 0}$ be a compound Poisson process, $Q(t) = \sum_{i=1}^{N(t)} Y_i$, with jump distribution $\mathbb{P}(Y_i = y_m) = p(y_m)$, $y_m > -1$, $m \in \{1, 2, \ldots, M\}$, and the stock price process $S(t)$, $t \geq 0$, be a solution of

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) + S(t-)dQ(t).$$

What is the mean rate of return of the stock? Find a change of measure, such that under the new measure the process $e^{-rt}S(t)$, $t \geq 0$, is a martingale.

Solution. Rewrite the equation as follows:

$$dS(t) = (\mu + \lambda\beta)S(t)dt + \sigma S(t)dB(t) + S(t-)d(Q(t) - \lambda\beta t), \tag{1}$$

where $\beta = \mathbb{E}(Y_1)$. Since $B(t)$ and $(Q(t) - \beta\lambda t)$, $t \geq 0$, are martingales, the mean rate of return of the stock is $\mu + \beta\lambda$.

There are many changes of measure, which satisfy the desired condition. Let us consider a measure $\tilde{\mathbb{P}}$ under which $\tilde{B}(t) = B(t) + \theta t$, $t \geq 0$, is a standard BM and Q has intensity $\tilde{\lambda}$ and the jump distribution $\tilde{p}(y_m)$, $m \in \{1, 2, \ldots, M\}$. Then under $\tilde{\mathbb{P}}$ the process $e^{-rt}S(t)$ satisfies

$$\begin{aligned} d(e^{-rt}S(t)) &= e^{-rt}(-rS(t)dt + \mu S(t)dt + \sigma S(t)d(\tilde{B}(t) - \theta t) + S(t-)dQ(t)) \\ &= e^{-rt}((\mu - r - \theta\sigma + \tilde{\beta}\tilde{\lambda})S(t)dt + \sigma S(t)d\tilde{B}(t) + S(t-)d(Q(t) - \tilde{\lambda}\tilde{\beta}t)), \end{aligned}$$

where $\tilde{\beta} = \tilde{\mathbb{E}}(Y_1) = \sum_{m=1}^{M} y_m\tilde{p}(y_m)$. Setting the dt term to zero we get

$$\begin{aligned} &\mu - r - \theta\sigma + \tilde{\beta}\tilde{\lambda} = 0; \\ &\theta = \frac{1}{\sigma}\,(\mu - r + \tilde{\beta}\tilde{\lambda}); \\ &\theta = \frac{1}{\sigma}\left(\mu - r + \sum_{m=1}^{M} y_m\tilde{\lambda}_m\right), \quad \text{where } \tilde{\lambda}_m = \tilde{\lambda}\tilde{p}(y_m),\ m \in \{1, 2, \ldots, M\}. \end{aligned}$$

The last equation gives the answer: we can pick any strictly positive $\tilde{\lambda}_m$, $m \in \{1, 2, \ldots, M\}$, calculate θ from the last equation, and define $\tilde{\lambda} := \sum_{m=1}^{M} \tilde{\lambda}_m$, $\tilde{p}(y_m) := \tilde{\lambda}_m / \tilde{\lambda}$, $m \in \{1, 2, \ldots, M\}$. Then set

$$Z(t) = e^{(\lambda - \tilde{\lambda})t - \theta B(t) - \theta^2 t/2} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{p}(Y_i)}{\lambda p(Y_i)}, \quad 0 \leq t \leq T,$$

and for a fixed $T > 0$ let

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)\, d\mathbb{P}, \quad A \in \mathcal{F}(T).$$

Under $\tilde{\mathbb{P}}$, the process $e^{-rt} S(t)$, $t \geq 0$, is a martingale.