BARUCH, MFE

MTH 9878 Assignment Two *

He, Jing Qiu
Huang, Qing
Zhou, Minzhe
Zhou, ShengQuan

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* We agree for our work to be posted on the forum.

1 Numéraire Invariance Theorem

Consider a frictionless market $(S_1(t),...,S_N(t))$ and a self-financing portfolio with weights $(w_1(t),...,w_N(t))$. Given a numeraire $\mathcal{N}(t)$, prove that the portfolio expressed in terms of the relative prices $\left(S_1^{\mathcal{N}}(t),...,S_N^{\mathcal{N}}(t)\right)$ is self-financing.
Proof: The portfolio is written as

$$V(t) = \sum_{i=1}^{N} w_i(t) S_i(t).$$

The self-financing condition is

$$dV(t) = \sum_{i=1}^{N} w_i(t) dS_i(t).$$

For the portfolio expressed in terms of relative prices, a direct computation follows

$$\begin{aligned}
d\left(\frac{V(t)}{\mathcal{N}(t)}\right) &= V(t) d\left(\frac{1}{\mathcal{N}(t)}\right) + \frac{1}{\mathcal{N}(t)} dV(t) + d\left(\frac{1}{\mathcal{N}(t)}\right) dV(t) \\
&= \sum_{i=1}^{N} w_i(t) S_i(t) d\left(\frac{1}{\mathcal{N}(t)}\right) + \frac{1}{\mathcal{N}(t)} \sum_{i=1}^{N} w_i(t) dS_i(t) + d\left(\frac{1}{\mathcal{N}(t)}\right) \sum_{i=1}^{N} w_i(t) dS_i(t) \\
&= \sum_{i=1}^{N} w_i(t) \left[S_i(t) d\left(\frac{1}{\mathcal{N}(t)}\right) + \frac{1}{\mathcal{N}(t)} dS_i(t) + d\left(\frac{1}{\mathcal{N}(t)}\right) dS_i(t) \right] \\
&= \sum_{i=1}^{N} w_i(t) d\left(\frac{S_i(t)}{\mathcal{N}(t)}\right).
\end{aligned}$$

In other words, the portfolio expressed in the new Numéraire remains self-financing. $\square$

2 Differential of Radon-Nikodým Derivative

Prove that

$$d\left(\frac{\mathcal{M}(t)}{\mathcal{N}(t)}\right) = \frac{\mathcal{M}(t)}{\mathcal{N}(t)}\left(\frac{B_{\mathcal{M}}(t)}{\mathcal{M}(t)} - \frac{B_{\mathcal{N}(t)}}{\mathcal{N}(t)}\right) dW^{\mathbb{P}}(t).$$

Proof: Because

$$D(t) = \frac{\mathcal{N}(0)}{\mathcal{M}(0)} \cdot \frac{\mathcal{M}(t)}{\mathcal{N}(t)}$$

is a martingale, we can ignore all the differential terms except those proportional to $dW^{\mathbb{P}}(t)$.

Let $X = \log \frac{\mathcal{M}(t)}{\mathcal{N}(t)}$,

$$\begin{aligned}
d\left(\frac{\mathcal{M}(t)}{\mathcal{N}(t)}\right) &= de^X \\
&= e^X dX + \frac{1}{2} e^X dX dX \\
&= \frac{\mathcal{M}(t)}{\mathcal{N}(t)}\left(dX + \frac{1}{2}\underbrace{dXdX}_{O(dt)}\right) \\
&= \frac{\mathcal{M}(t)}{\mathcal{N}(t)} d\log\frac{\mathcal{M}(t)}{\mathcal{N}(t)} + O(dt) \\
&= \frac{\mathcal{M}(t)}{\mathcal{N}(t)}\left[d\log\mathcal{M}(t) - d\log\mathcal{N}(t)\right] + O(dt) \\
&= \frac{\mathcal{M}(t)}{\mathcal{N}(t)}\left[\frac{d\mathcal{M}(t)}{\mathcal{M}(t)} - \frac{d\mathcal{N}(t)}{\mathcal{N}(t)} + O(dt)\right] + O(dt) \\
\text{(martingale property)} &= \frac{\mathcal{M}(t)}{\mathcal{N}(t)}\left[\frac{B_{\mathcal{M}}(t)}{\mathcal{M}(t)} - \frac{B_{\mathcal{N}(t)}}{\mathcal{N}(t)}\right] dW^{\mathbb{P}}(t) + \cancelto{0}{O(dt)}.
\end{aligned}$$