# Positive Interest Revisited: **Yield Curve Modeling in Low Rate Environments**


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# Background & Context

- Many needs for internally consistent and tractable term structure models that generate economically plausible behavior over long time periods: valuing long-dated derivatives, mortgage prepayment analysis, hedging annuity and other insurance products, LDI, etc.

- Special problems arise in economic environments with very low interest rates, since the most tractable class of models (Gaussian) will generate future interest rate paths with significantly negative rates with a non-negligible probability

- We will discuss the numerical implementation and some empirical results for a particular specification of the Positive Interest framework introduced by Flesaker & Hughston (1996)

- We will also illustrate the model in the context of the early Brennan-Schwartz long rate/short rate modeling framework

# Positive Interest

- Flesaker & Hughston (FH) characterized positive interest rate models driven by a finite number of Brownian motions as represented in terms of a family of positive martingales M(t,s) (under a pricing measure, which may be $\mathbf{P}$), with M(0,s)=1 for all s, and a deterministic function $\varphi(s)$, such that zero coupon bond and derivatives values are given by:

$$P_{tT} = \frac{\int_T^\infty \varphi(s) M(t,s)\, ds}{\int_t^\infty \varphi(s) M(t,s)\, ds} \qquad V(t) = \frac{E\left[ V(T) \int_T^\infty \varphi(s) M(T,s)\, ds \middle| \mathcal{F}_t \right]}{\int_t^\infty \varphi(s) M(t,s)\, ds}$$

- In this setting, $\varphi$ determines the initial yield curve and the volatility structure of M determines the future rate dynamics

- It is immediately evident that the model keeps rates positive, and it was conjectured by FH and shown by Glasserman and Jin (2001) that any Heath, Jarrow, Morton (HJM) (1992) model restricted to positive rates can be represented in this form

# The Cairns Model

- Cairns (2004) introduced a family of interest rate models based on the original Positive Interest framework of Flesaker & Hughston

- Unlike the HJM approach of specifying the initial yield curve freely, he proposed a parametric fit that is consistent with the future rate dynamics, addressing the critique raised by Björk & Christensen (1999) and Filipovic (2001)

- Cairns established that a 2-factor version of his specification had many appealing qualitative features, but did not estimate model parameters or determine how close the model would fit to actual market data

- Cairns, Leuwattanachotinan & Streftaris (2012) develop a Markov chain Monte Carlo (MCMC) estimation algorithm for the 2-factor model, but only validates the algorithm against simulated data

# Model Specification

- Cairns' model is specified in terms of a set of correlated Ornstein-Uhlenbeck processes with unit annualized volatility, $X_i$, as follows:
- $$\varphi(s)M(t,s) = \exp\left\{-\beta s + \sum_i e^{-\alpha_i(s-t)}\sigma_i X_i(t) - \frac{1}{2}\sum_{i,j}\frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i+\alpha_j}e^{-(\alpha_i+\alpha_j)(s-t)}\right\}$$

- Here, $\alpha_i$ is the mean reversion of $X_i$, and $\rho_{ij}$ is the instantaneous correlation between $dX_i$ and $dX_j$

- The parameter β has a direct financial interpretation as the limiting instantaneous forward rate as the time to maturity goes to infinity; so the model can be readily seen to possess the Dybvig, Ingersoll & Ross (1996) "long rates can never fall" property

- Cairns show that in the 1-factor version of the model, the short rate will be mean-reverting lognormal (Black-Karasinsky) as rates get close to zero and tend toward mean-reverting normal (Vasicek) as rates grow large

# Model Specification - continued

- We define the function H(u,x) by:

$$H(u,x) = \exp\left\{ -\beta u + \sum_i e^{-\alpha_i u} \sigma_i x - \frac{1}{2} \sum_{i,j} \frac{\rho_{ij} \sigma_i \sigma_j}{\alpha_i + \alpha_j} e^{-(\alpha_i + \alpha_j) u} \right\}$$

- Bond prices, the short rate, and the par consol rate are now given by:

$$P(t,T) = \frac{\int_{T-t}^{\infty} H\left(u, X(t)\right) du}{\int_0^{\infty} H\left(u, X(t)\right) du} \quad r_t = \frac{H\left(0, X(t)\right)}{\int_0^{\infty} H\left(u, X(t)\right) du} \quad c_t = \frac{\int_0^{\infty} H\left(u, X(t)\right) du}{\int_0^{\infty} u H\left(u, X(t)\right) du}$$

- We note that for *n=2*, under mild conditions on the model parameters, $\{r_t, c_t\}$ will be a time homogeneous 2-dimensional Ito diffusion, carrying the same information as *X(t)*, which is in the spirit of the Brennan-Schwartz model

# A Digression

- What fills in the term structure between the short rate *r* and the continuously compounded consol rate *c*?
- Consider the par rate on an infinite maturity bond that amortizes continuously at the rate *a*, for all non-negative values of *a*
- Based on the general valuation equation

$$y(a)\int_0^\infty e^{-at}P(0,t)\,dt + a\int_0^\infty e^{-at}P(0,t)\,dt = 1$$

- We get

$$y(a) = \frac{1}{\int_0^\infty e^{-at}P(0,t)\,dt} - a = \frac{\int_0^\infty f(0,t)e^{-at}P(0,t)\,dt}{\int_0^\infty e^{-at}P(0,t)\,dt}$$

$$y(0) = c$$

$$\lim_{a\to\infty} y(a) = r$$

# More about *y(a)*

- The duration of a par bond amortizing at rate $a$ is given by:

$$D(a)=\frac{\int_0^{\infty} te^{-at}P(0,t)\,dt}{\int_0^{\infty} e^{-at}P(0,t)\,dt}$$

- This has units of time and equals $(r+a)^{-1}$ if the yield curve is flat at $r$, creating a natural time scale for $y(a)$
- It is not obvious how to "invert" a yield curve given by $\{y(a)\}$ into more traditional (and, arguably, useful) representations in terms of $\{P(t)\}$ or $\{f(t)\}$

# Numerical Implementation

- The model has a Markovian structure, with all bond prices, swap rates, etc., at each point in time being explicit functions of the contemporaneous value of the state variables
- We have closed form solutions for the joint transition density function of the state variables for arbitrary time steps
- Regardless of the number of factors, the function that maps the state variables into zero coupon bond prices only requires a 1-dimensional integral
- The integrals that need to be evaluated for different maturity zero coupon bond prices have significant overlap, allowing for re-use of interim results
- We use a combination of Simpson's rule and Gauss-Laguerre quadrature to solve for bond prices and swap rates. Option pricing requires integration of the payoff function against the joint density of the state variables, so the number of factor matters

# Swaption Pricing Details

- The payoff of a TxN payer swaption with strike rate K is:

$$V(T) = \frac{1}{2}\sum_{n=1}^{2N} P\left(T, T + {}^{n}\!/_{2}\right)\left[\frac{1 - P\left(T, T+N\right)}{\frac{1}{2}\sum_{n=1}^{2N} P\left(T, T + {}^{n}\!/_{2}\right)} - K\right]^{+}$$

- Using the general valuation operator, and writing the state variables in terms of two iid standard normal variables, the initial swaption value is:

$$V(0) = \frac{e^{-\beta T}}{\int_{0}^{\infty} H\left(X_0, u\right) du} \int_{-\infty}^{\infty} \phi(z_1) \int_{-\infty}^{\infty} \phi(z_2) \left[\int_{0}^{\infty} h(z,u)\, du - \int_{N}^{\infty} h(z,u)\, du - \frac{K}{2}\sum_{n=1}^{2N} \int_{{}^{n}\!/_{2}}^{\infty} h(z,u)\, du\right]^{+} dz_2\, dz_1$$

# Orthogonalizing the State Variables

- Here we have made the substitution:

$$h(z,u) = H\left( \begin{matrix} e^{-\alpha_1 T} X_1(0) + \sqrt{\dfrac{1-e^{-2\alpha_1 T}}{2\alpha_1}} z_1 \\ e^{-\alpha_2 T} X_2(0) + \sqrt{\dfrac{1-e^{-2\alpha_2 T}}{2\alpha_2}} \left( \rho_T z_1 + \sqrt{1-\rho_T^2} z_2 \right) \end{matrix} , u \right)$$

with:

$$\rho_T = \rho \frac{\sqrt{\alpha_1 \alpha_2}}{\dfrac{\alpha_1 + \alpha_2}{2}} \frac{1 - e^{-(\alpha_1+\alpha_2)T}}{\sqrt{\left(1-e^{-2\alpha_1 T}\right)\left(1-e^{-2\alpha_2 T}\right)}}$$

# A conditional Jamshidian trick

- We note that the swap value at the option expiry date is strictly increasing in $z_2$, and that for any value of $z_1$, there is a unique value of $z_2$, $z^*(z_1)$ making it equal to zero
- With all terms positive and bounded, we can exchange the order of integration over u and $z_2$ to get:

$$V(0)=\frac{e^{-\beta T}}{\int_0^\infty H\left(X_0,u\right)du}\int_{-\infty}^{\infty}\phi(z_1)\left[\int_0^\infty\int_{z^*(z_1)}^{\infty}\phi(z_2)h(z,u)\,dz_2\,du-\int_N^\infty\int_{z^*(z_1)}^{\infty}\phi(z_2)h(z,u)\,dz_2\,du-\frac{K}{2}\sum_{n=1}^{2N}\int_{n/2}^{\infty}\int_{z^*(z_1)}^{\infty}\phi(z_2)h(z,u)\,dz_2\,du\right]dz_1$$

- Finally, we can solve the inner integral over $z_2$ analytically:

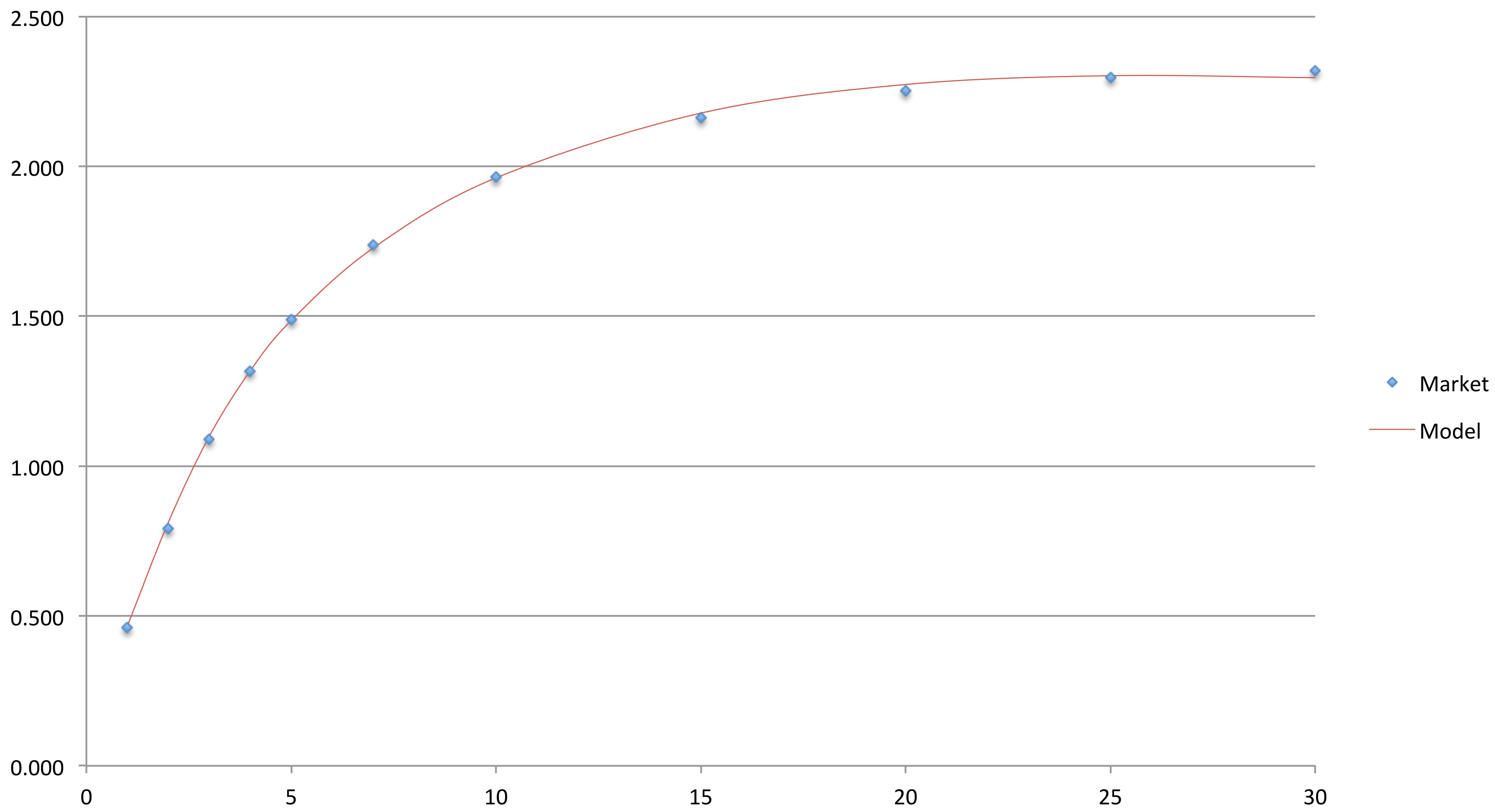
$$\int_{z^*(z_1)}^{\infty}\phi(z_2)h(z,u)dz_2=\int_{z^*(z_1)}^{\infty}\phi(z_2)\exp\left\{t_0(u)+t_1(u)z_1+t_2(u)z_2\right\}dz_2=\exp\left\{t_0(u)+t_1(u)z_1+\frac{1}{2}t_2^2(u)z_2\right\}\Phi\left[t_2(u)-z^*(z_1)\right]$$

- While this process requires the solution of a non-linear equation for each value of $z_1$, it reduces the dimension of the integrals from 3 to 2 and replaces the non-differentiable integrand in the original problem with a smooth one, significantly improving the performance of the Gaussian quadratures we use

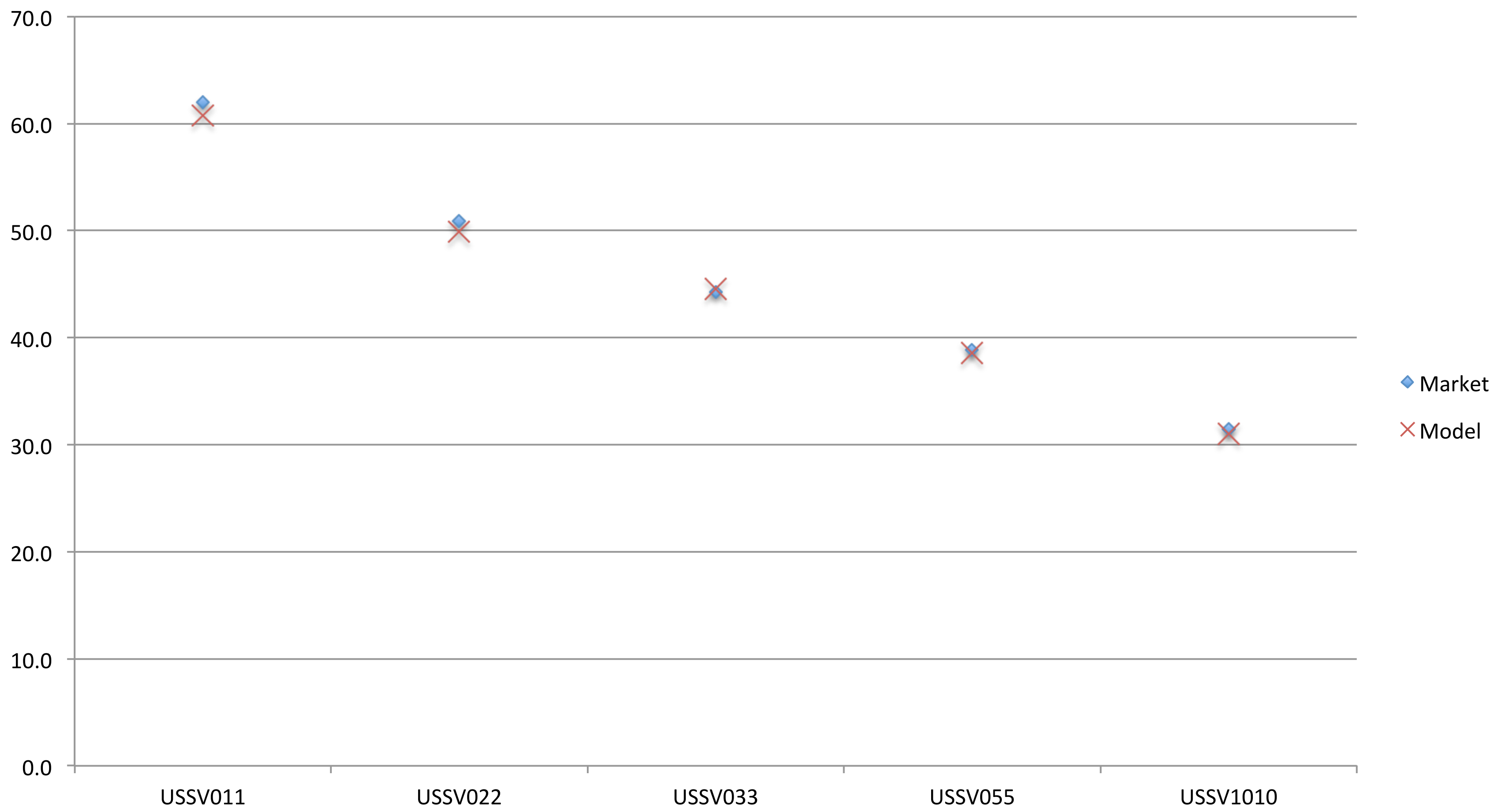
# Empirical Results

- Two perspectives
  - How well does the model fit cross-sectional swap rates and swaption prices?
  - Is the dynamics implied by the cross-sectional parameter calibration consistent with historical rate movements?
- We will look at data for USD swaps and swaptions from July 2014 – April 2016
  - 1, 2, 3, 4, 5, 7,10,15, 20, and 30 year swap rates
  - ATM swaption quotes for 1x1, 2x2, 3x3, 5x5 and 10x10 year maturity x tenor
- The historical rate dynamics is summarized by the first two principal components of weekly swap rate changes for each 4-month period following the dates used for cross-sectional estimation

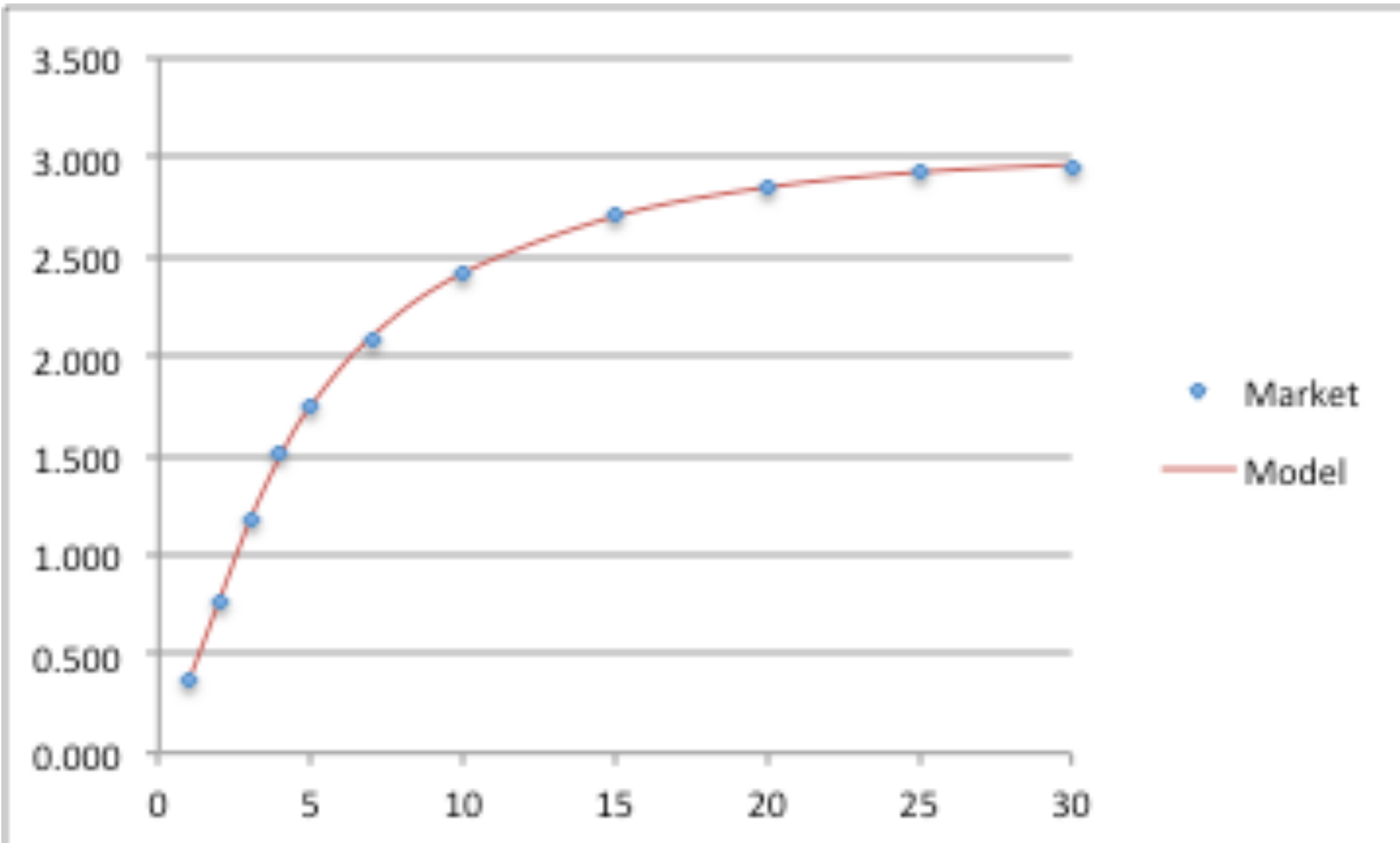
# Swap Rate Fit 4/1/2015



# Implied Vol Fit 4/1/2015



# Swap Rate Fit 12/3/2014

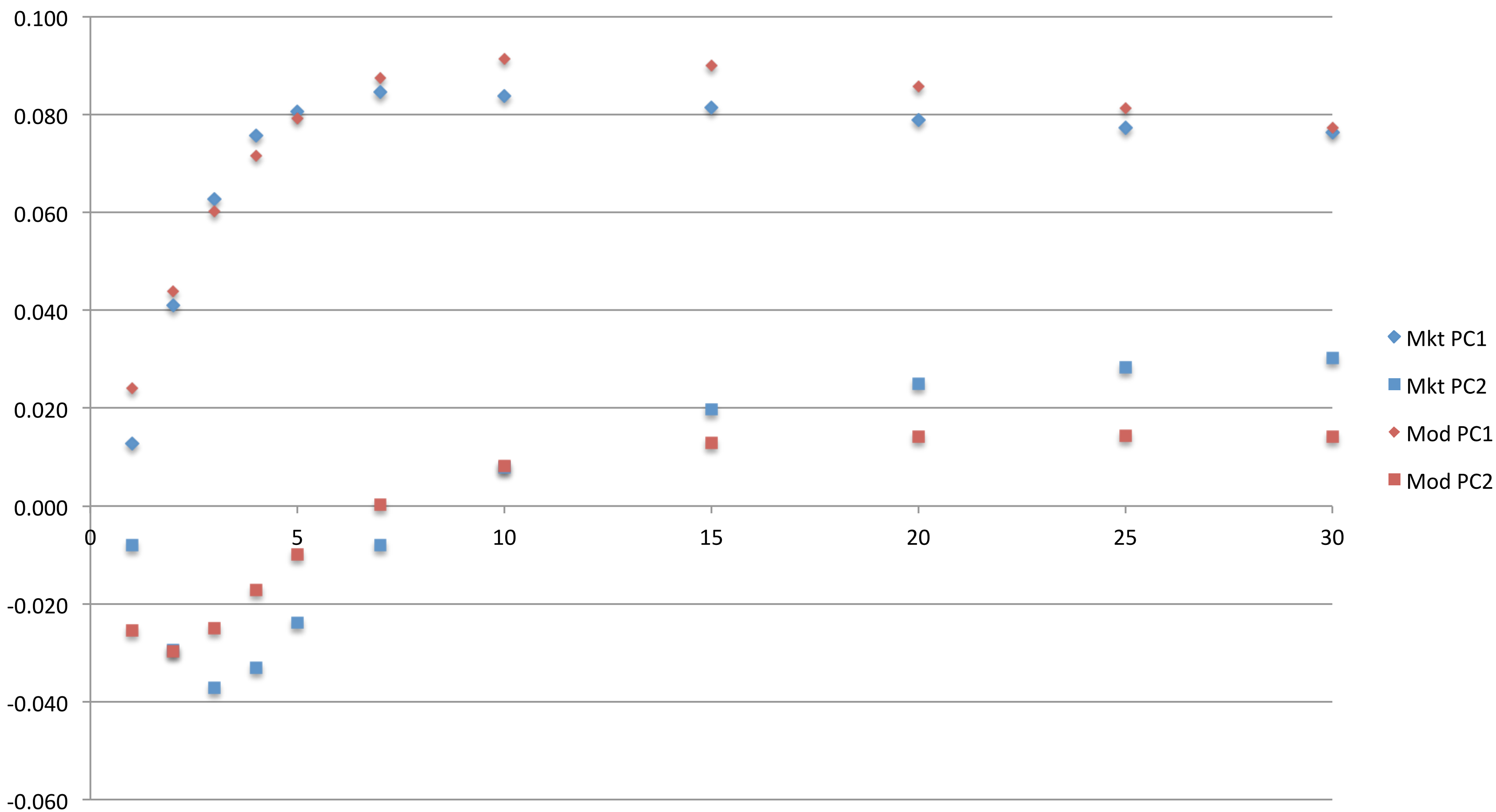


# Implied Vol Fit 12/3/2014

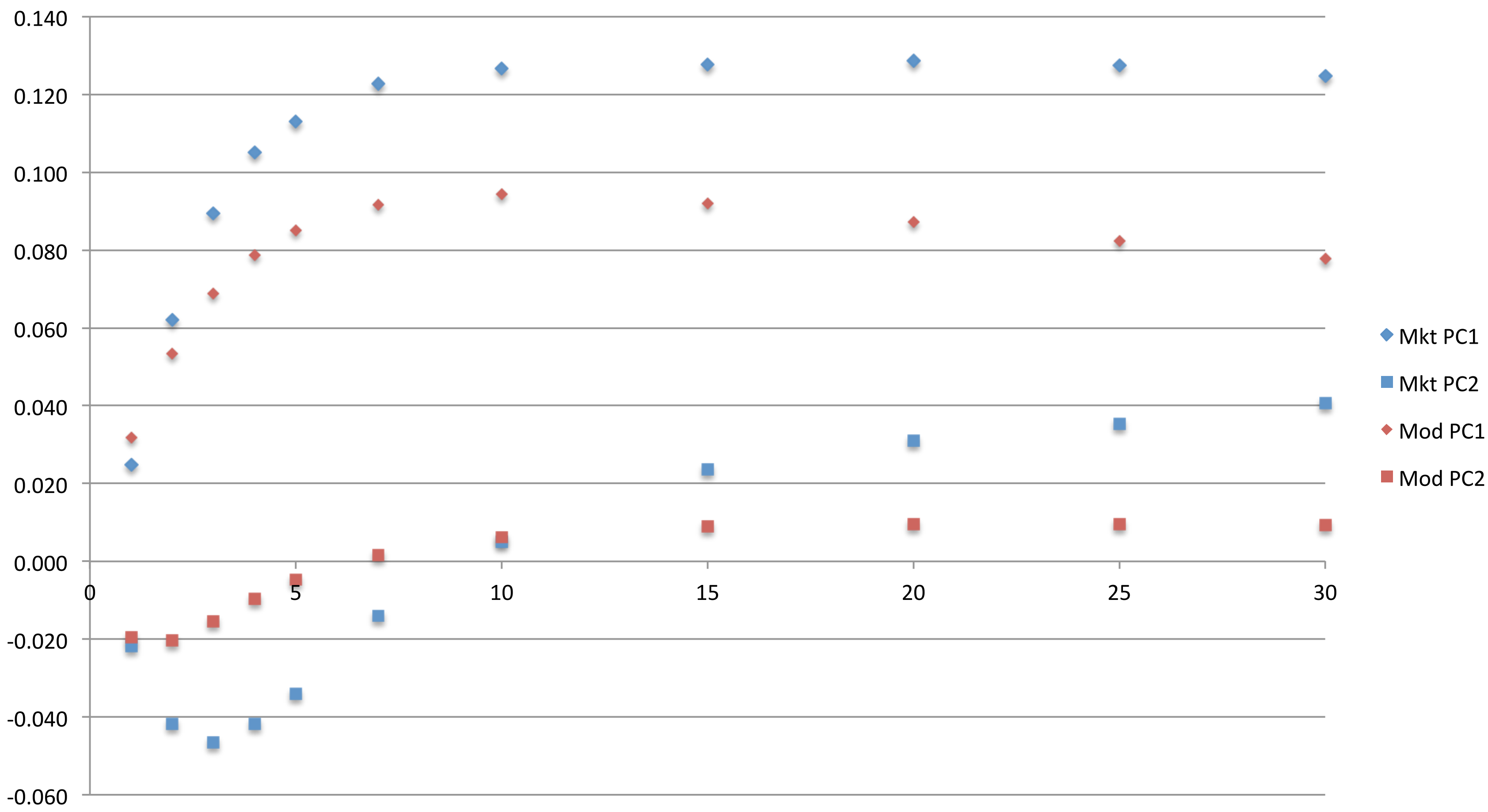
# Parameter Estimates

| | 7/30/14 | 12/3/14 | 4/1/15 | 8/5/15 | 12/2/15 |
|---|---|---|---|---|---|
| **$\beta$** | 2.79% | 2.56% | 1.09% | 1.19% | 2.01% |
| **$\alpha_1$** | 0.806 | 0.914 | 1.079 | 0.782 | 0.851 |
| **$\alpha_2$** | 0.047 | 0.049 | 0.045 | 0.036 | 0.054 |
| **$\sigma_1$** | 1.074 | 0.896 | 0.491 | 0.732 | 0.461 |
| **$\sigma_2$** | 0.564 | 0.591 | 0.775 | 0.650 | 0.709 |
| **$\rho$** | 0.016 | 0.352 | 0.889 | 0.404 | 0.380 |
| **$X_1$** | -2.979 | -3.166 | -3.959 | -2.425 | -2.509 |
| **$X_2$** | 3.784 | 3.421 | 5.573 | 7.013 | 3.405 |
| **Swap RMSE (bps)** | 1.06 | 0.59 | 1.32 | 0.76 | 0.68 |
| **Vol RMSE (points)** | 0.24 | 0.10 | 0.77 | 0.18 | 0.20 |

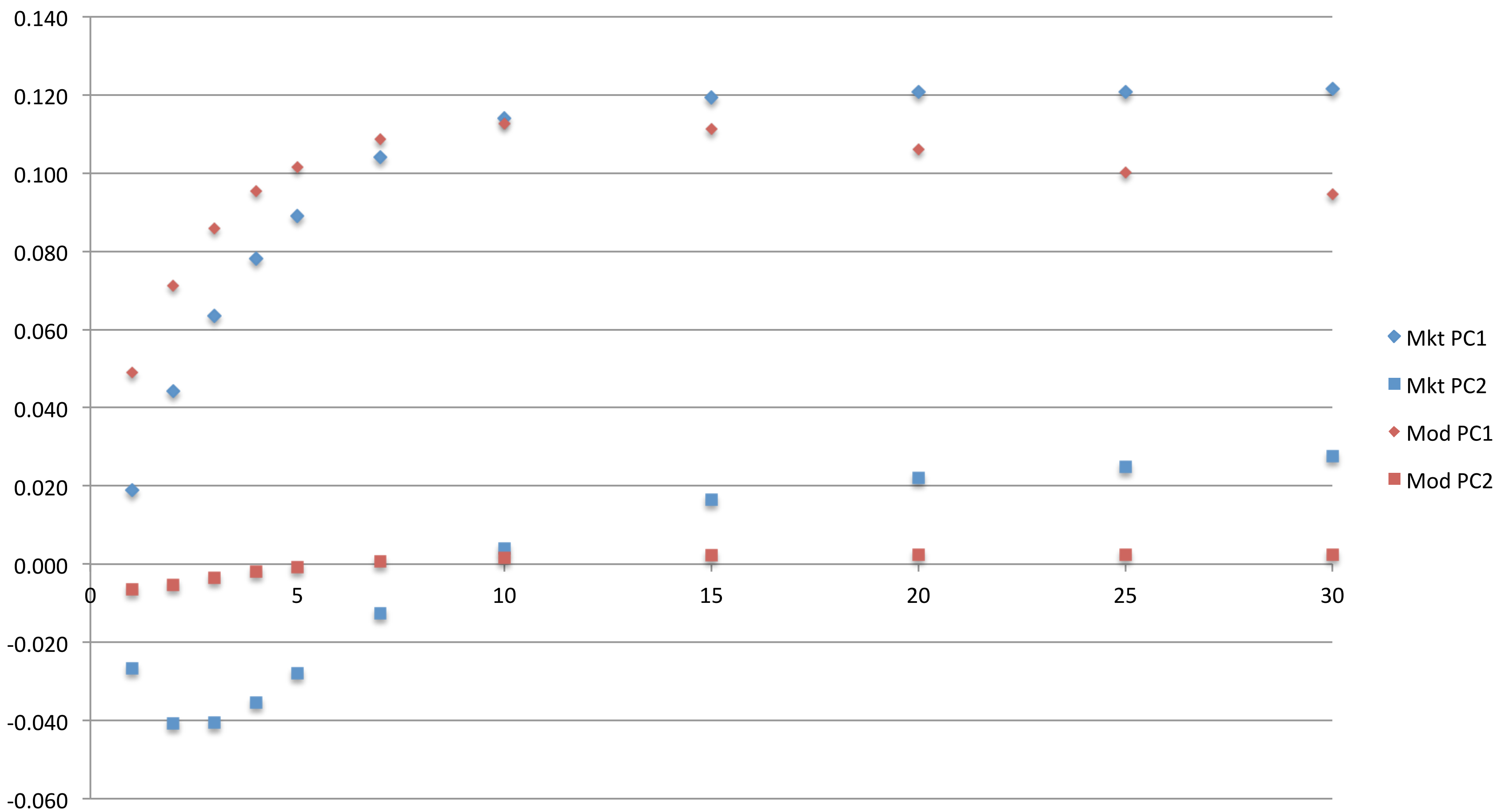
# Model vs Market PCs 7/30/2014



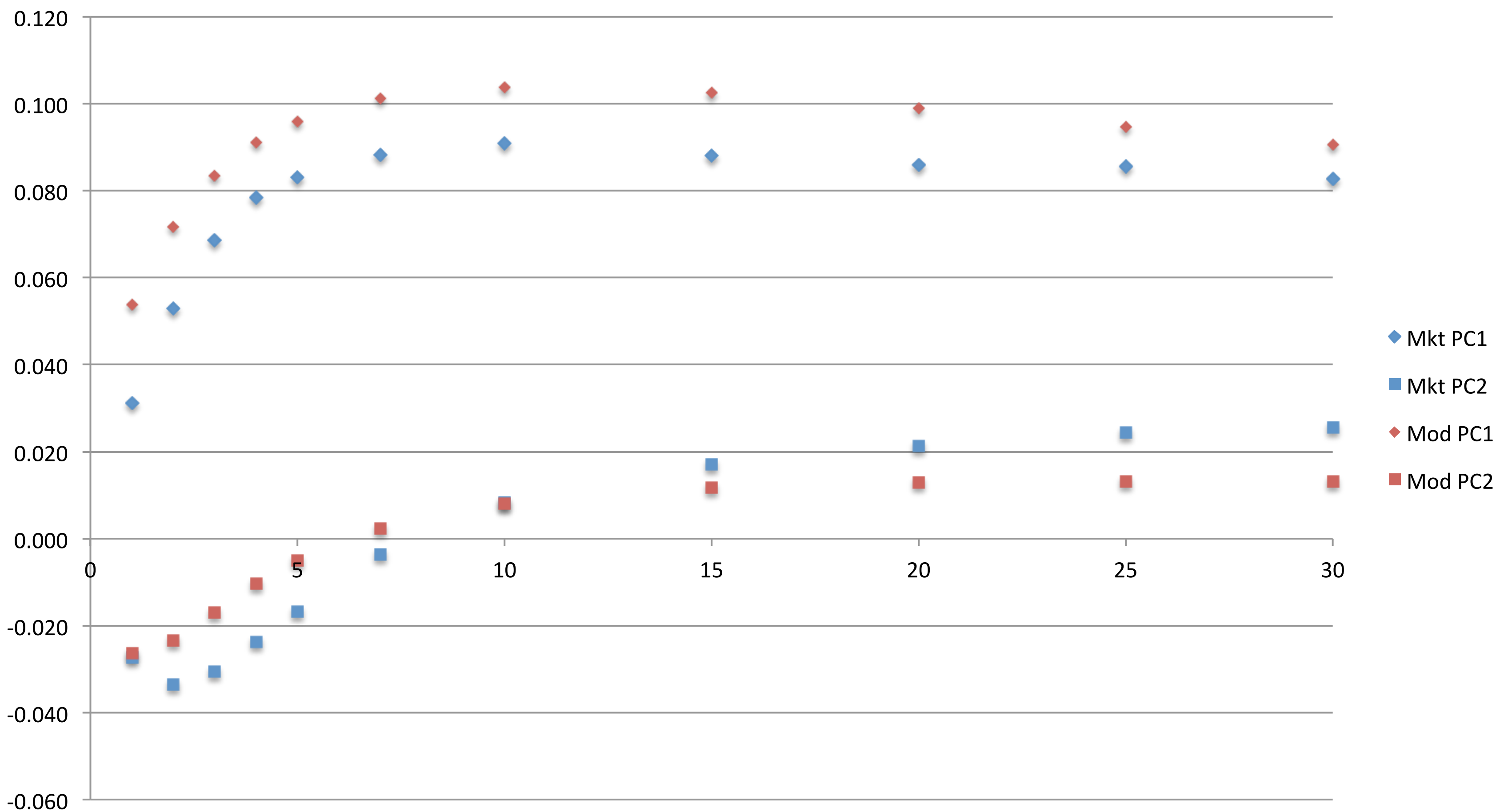
# Model vs Market PCs 12/3/2014



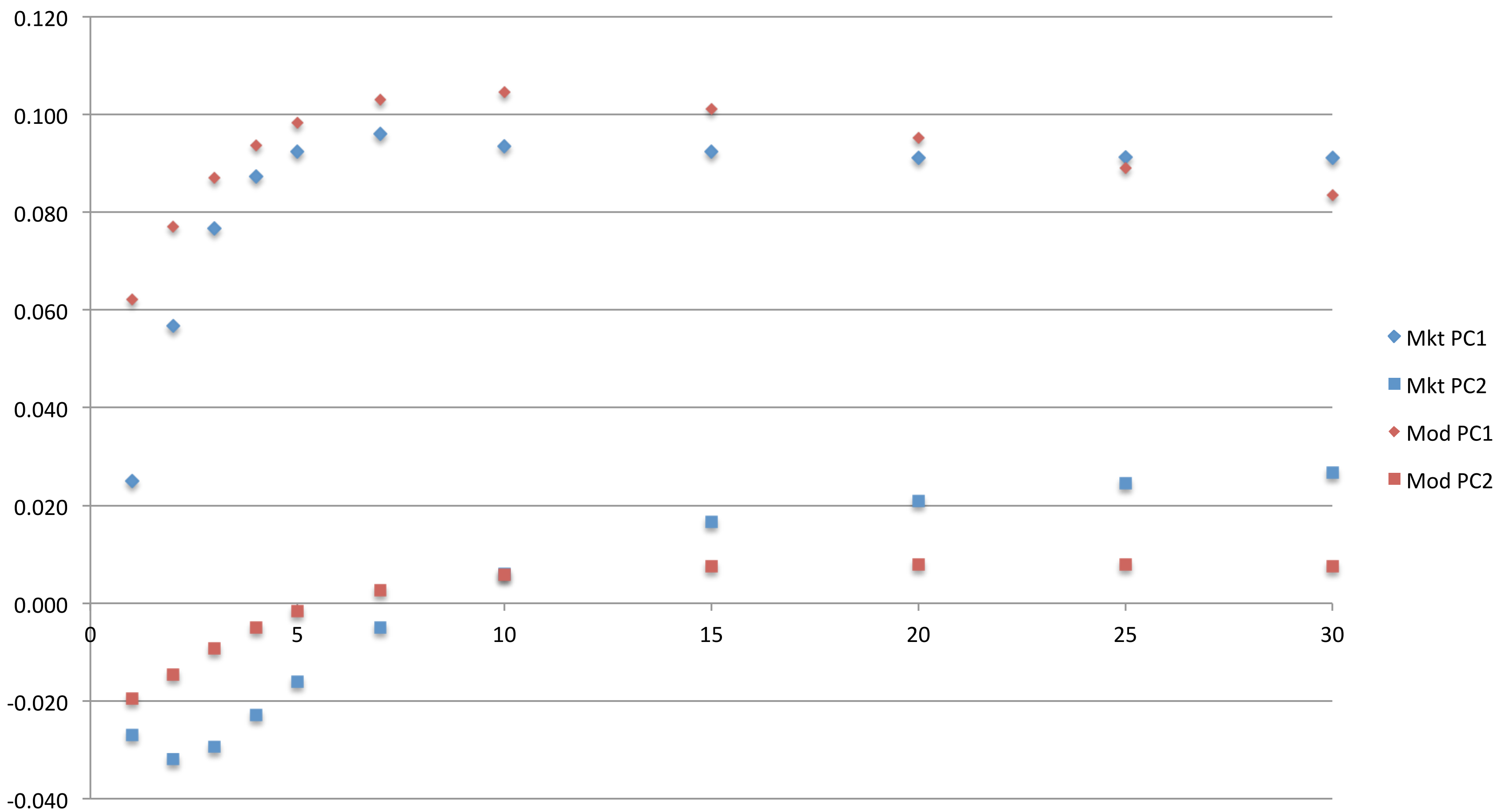
# Model vs Market PCs 4/1/2015



# Model vs Market PCs 8/5/2015



# Model vs Market PCs 12/2/2015



# Conditional Drift & Vol for r(t) and c(t)

- As discussed, we can take the short rate and the consol rate as state variables and express their dynamics as:

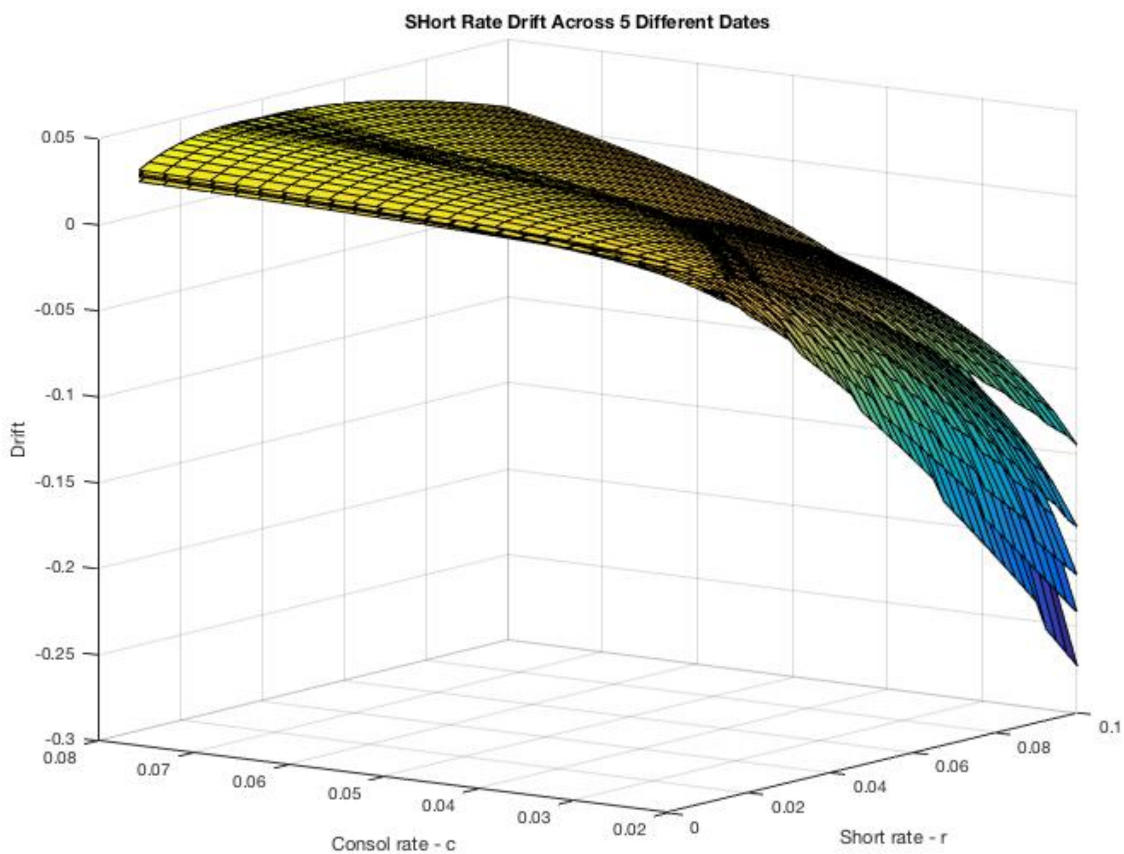
$$dr_t = \mu_r(r_t, c_t)dt + \nu_r(r_t, c_t)'dW_t$$
$$dc_t = \mu_c(r_t, c_t)dt + \nu_c(r_t, c_t)'dW_t$$

- In the following we plot surface graphs for each of the model calibrations of the two drift functions as well as of the relative volatility for each of the two rates, i.e.:

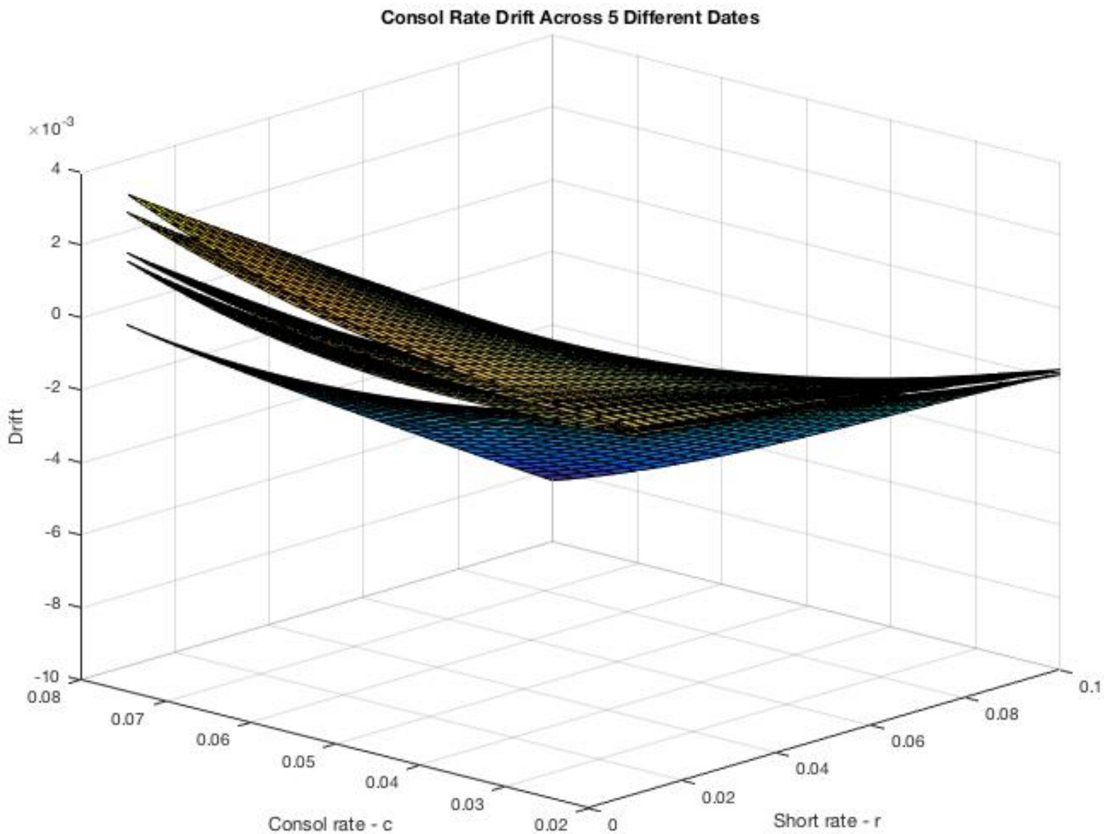
$$vol_r(r_t, c_t) = \frac{\sqrt{\nu_r(r_t, c_t)'\nu_r(r_t, c_t)}}{r_t} \quad vol_c(r_t, c_t) = \frac{\sqrt{\nu_c(r_t, c_t)'\nu_c(r_t, c_t)}}{c_t}$$

- We notice a fairly rich structure to the plotted functions, although with close to constant volatility for the short rate over a wide range of rate levels, corresponding to approximately lognormal behavior

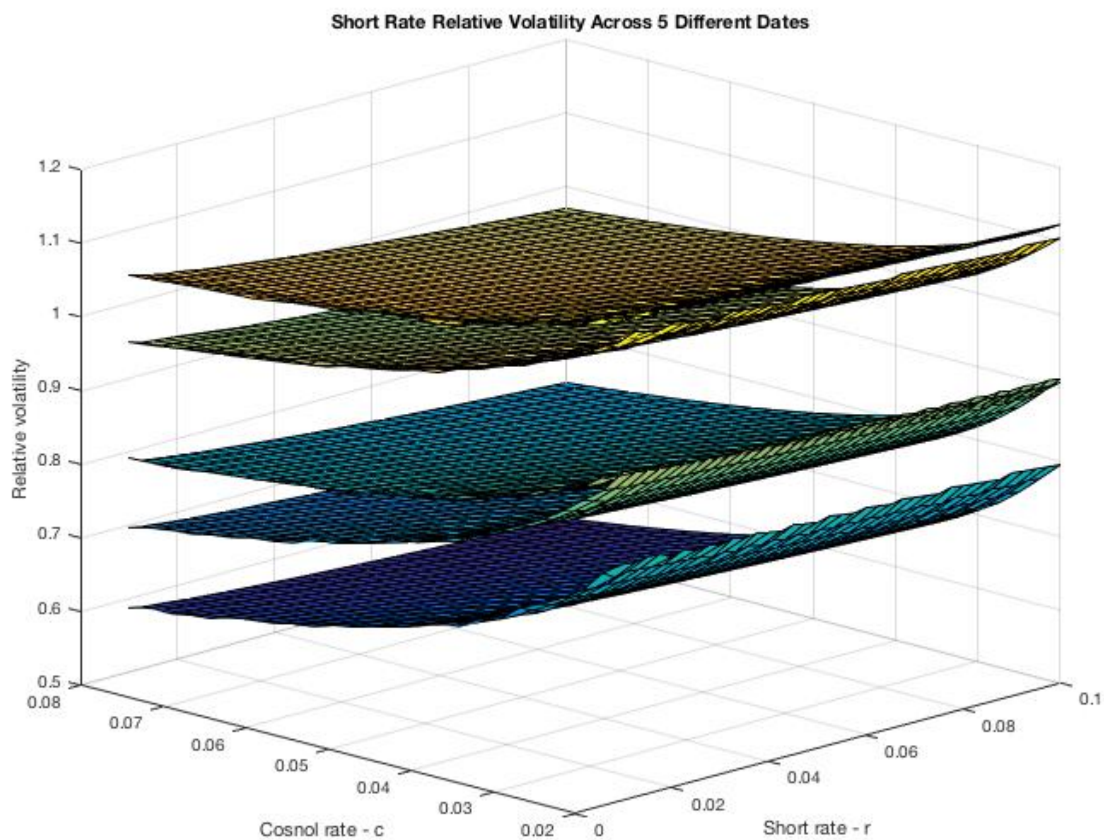
# Short Rate Drift



# Consol Rate Drift



# Short Rate Vol (Relative)



# Consol Rate Vol (Relative)