MTH 9831, FALL 2015. LECTURE 5

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Abstract. In this lecture we continue to discuss stochastic calculus and its applications. We shall see, in particular, generalizations to higher dimensions. The exposition follows to a large extent parts of Sections 4.5, 4.6, and 7.2 of the textbook.

1. Application: derivation of Black-Scholes-Merton PDE.
2. Multidimensional Itô-Doeblin formula.
3. Application: finding the covariance of stock prices driven by correlated Brownian motions.
4. Application: Lévy's characterization of Brownian motion (2 dimensions).
5. Change of measure: Girsanov's theorem in one dimension.

1. Application: Derivation of Black-Scholes-Merton PDE.

Let us assume that the market consists of one risky asset, stock, whose price $S(t)$ at time $t \geq 0$ satisfies

$$dS(t) = \alpha S(t)\, dt + \sigma S(t)\, dB(t) \tag{1.1}$$

and a riskless asset, money market account (MMA) which has a constant interest rate r (continuous compounding). Denote the portfolio value at time t by $X(t)$. Suppose that at time t the investor holds $\Delta(t)$ shares of the stock (random but adapted to the filtration of the Brownian motion). The remainder $X(t) - \Delta(t)S(t)$ is invested in the MMA. Then the evolution of the portfolio value is described by the equation

$$dX(t) = \Delta(t)\, dS(t) + r(X(t) - \Delta(t)S(t))dt. \tag{1.2}$$

Substituting $dS(t)$ into (1.2) we get

$$\begin{aligned} dX(t) &= \Delta(t)(\alpha S(t)\, dt + \sigma S(t)\, dB(t)) + r(X(t) - \Delta(t)S(t))dt \\ &= \sigma\Delta(t)dB(t) + rX(t)\, dt + (\alpha - r)\Delta(t)S(t)\, dt =: (I) + (II) + (III). \end{aligned}$$

The three terms above are:

(I) the volatility term proportional to the size of the stock investment;
(II) an average underlying rate of return r of the portfolio;
(III) a risk premium $(\alpha - r)$ for investing in stock.

Consider now a European call option on the stock with payoff $(S(T) - K)_+$ at time T. Black, Scholes, and Merton argued that the value of this call at time t should depend only on $T - t$ and the price of the stock at time t (other parameters $r,\ \sigma,\ K$ being fixed). If we let $c(t, x)$ denote the price of the option at time t when $S(t) = x$ then $c(t, x)$ is a non-random function. The stochastic process $c(t, S(t))$ is the price of the option at time t. Assuming that $c(t, x) \in C^{1,2}([0, T) \times [0, \infty))$ we get for all $0 \leq t < T$ that

$$\begin{aligned} dc(t, S(t)) &= c_t(t, S(t))\, dt + c_x(t, S(t))\, dS(t) + \frac{1}{2}\, c_{xx}(t, S(t))\, d[S]_t \\ &= \left(c_t(t, S(t)) + \alpha S(t) c_x(t, S(t) + \frac{1}{2}\, c_{xx}(t, S(t))\right) dt + \sigma c_x(t, S(t))S(t)\, dB(t). \end{aligned} \tag{1.3}$$

The above equation represents the evolution of the option value.

A hedging portfolio for a short position on an option has to satisfy the equation $X(t) = c(t, S(t))$ for all $t \in [0, T]$. If we equate (1.2) and (1.3) then the equation will contain the term $rX(t)dt$ in addition to terms containing c and its partial derivatives. To get rid of this term we shall equate the present values instead:

$$e^{-rt}X(t) = c^{-rt}c(t,S(t)), \quad t \in [0,T]. \tag{1.4}$$

The corresponding evolutions are given by

$$\begin{aligned} d(e^{-rt}X(t)) &= -re^{-rt}X(t)\,dt + e^{-rt}dX(t) \\ &= e^{-rt}\Delta(t)(\alpha - r)S(t)\,dt + \sigma e^{-rt}\Delta(t)S(t)\,dB(t) \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} d(e^{-rt}c(t,S(t))) &= -re^{-rt}c(t,S(t))\,dt + e^{-rt}dc(t,S(t)) \\ &= e^{-rt}\left(-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\,\sigma^2 S^2(t)c_{xx}(t,S(t))\right)dt \\ &\qquad + e^{-rt}\sigma c_x(t,S(t))S(t)\,dB(t). \end{aligned} \tag{1.6}$$

Equating the stochastic integral parts and regular parts in (1.5) and (1.6) we obtain

- from the stochastic integral parts: $\Delta(t) = c_x(t,S(t))$, $t \in [0,T)$, a.s.;
- from the regular parts and the previous line:

$$c_x(t,S(t))(\alpha - r)S(t) = -rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t)),$$

  $t \in [0,T)$, a.s. and, after cancellations,

$$rc(t,S(t)) = c_t(t,S(t)) + rS(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t)), \quad t \in [0,T), \;\text{ a.s..}$$

The last equation tells us that we need to find a smooth function $c(t,x)$ which satisfies the following Black-Scholes-Merton partial differential equation (BSM PDE)

$$rc(t,x) = c_t(t,x) + \alpha x c_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x), \quad x \geq 0,\; t \in [0,T),$$

and the terminal condition $c(T,x) = (x-K)_+$. For the solution to be unique we need additional boundary conditions at $x = 0$ and as $x \to \infty$. The first one is obtained by plugging $x = 0$ in the BSM PDE to get $c_t(t,0) = rc(t,0)$. Solving this ODE with $c(0,0) = 0$ we get the condition at $x = 0$: $c(t,0) = 0$ for all $t \in [0,T]$. The condition at infinity can be stated in the following form:

$$\lim_{x\to\infty}(c(t,x) - (x - e^{-(T-t)}K)) = 0, \quad t \in [0,T].$$

As x grows large, the call option will be deep in the money, and it will very likely finish in the money. In this case, the price of the call at time t is almost as much as the time t price of the forward contract with delivery price K and expiration T, i.e. $S(t) - e^{-(T-t)}K$. This explains the condition at infinity.

Suppose that we have found such a function $c(t,x)$. Then at each time t we have both the option price $c(t,S(t))$ ($S(t)$ is known at time t) and the replicating portfolio. Indeed, if the investor starts with initial capital $X(0) = c(0,S(0))$ and at time t has $\Delta(t) = c_x(t,S(t))$ shares of the underlying asset in his portfolio, then the $dB(t)$ terms in (1.5) and (1.6) agree. By the BSM PDE the dt terms in (1.5) and (1.6) also agree. This and the equality of the initial values imply that the equation in (1.4) holds for all $t \in [0,T)$. As $t \uparrow T$ we get by continuity of $X(t)$ and $c(t,S(t))$ that $X(T) = c(T,S(T)) = (S(T)-K)_+$, i.e. the short position is successfully hedged.

2. Multidimensional Itô-Doeblin formula.

We shall consider dimension $d = 2$. Higher dimensions are handled in exactly the same way. Let $f(t, x, y)$ be a smooth function and $x(t)$, $y(t)$ be continuous paths. Then as $h \to 0$

$$\begin{aligned} f(t+h, x(t+h), y(t+h)) &- f(t, x(t), y(t)) \\ &= f_t(t, x(t), y(t))h + f_x(t, x(t), y(t))(x(t+h) - x(t)) + f_y(t, x, y)(y(t+h) - y(t)) \\ &+ \frac{1}{2} f_{xx}(t, x, y)(x(t+h) - x(t))^2 + \frac{1}{2} f_{yy}(t, x, y)(y(t+h) - y(t))^2 \\ &\qquad + f_{xy}(t, x, y)(x(t+h) - x(t))(y(t+h) - y(t)) \\ &\qquad\quad + o(|h| + (x(t+h) - x(t))^2 + (y(t+h) - y(t))^2). \end{aligned}$$

Fix a partition of $[0, T]$ and write

$$f(t, x(t), y(t)) - f(0, x(0), y(0)) = \sum_{i=1}^{n} (f(t_i, x(t_i), y(t_i)) - f(t_{i-1}, x(t_{i-1}), y(t_{i-1}))).$$

Let us apply the above expansion to each term of the sum. As before, we would like to replace $x(t)$ and $y(t)$ with continuous stochastic processes $X(t)$ and $Y(t)$ and pass to the limit (in L^2) as $h \to 0$. What is new in comparison to the one dimensional case? The cross terms are new. Recall the definition of the cross variation of stochastic processes X and Y. The cross variation of X and Y is a stochastic process denoted by $[X, Y]_t$, $t \geq 0$, such that

$$\lim_{\|\Pi\| \to 0} E\left[\left(\sum_{i=1}^{n} (X(t_i) - X(t_{i-1})(Y(t_i) - Y(t_{i-1})) - [X, Y]_t \right)^2 \right] = 0.$$

For which processes can we get a meaningful cross variation? Recall that in dimension one we worked with Itô processes. Thus, it is natural to let X and Y be Itô processes driven by a pair of Brownian motions.

Definition 2.1. *A two-dimensional Brownian motion* $B(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}$, $t \geq 0$, *is a stochastic process which has the following properties:*

(i) *each of* $B_1 := (B_1(t))_{t \geq 0}$ *and* $B_2 := (B_2(t))_{t \geq 0}$ *is a one-dimensional Brownian motion;*
(ii) *The processes* B_1 *and* B_2 *are independent.*

Definition 2.2. *Let* $(B(t))_{t \geq 0}$ *be a two-dimensional Brownian motion. A filtration* $(\mathcal{F}(t))_{t \geq 0}$ *is said to be an associated filtration if*

(i) $\mathcal{F}(s) \subset \mathcal{F}(t)$ *for all* $0 \leq s \leq t$;
(ii) *for every* $t \geq 0$ *the random vector* $B(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}$ *is* $\mathcal{F}(t)$*-measurable*[1]*;*

- *for all* $0 \leq s \leq t$ *the random vector* $\begin{bmatrix} B_1(t) - B_1(s) \\ B_2(t) - B_2(s) \end{bmatrix}$ *is independent from* $\mathcal{F}(s)$.

Definition 2.3. *Let* $(B(t))_{t \geq 0}$ *be a two-dimensional Brownian motion,* $(\mathcal{F}(t))_{t \geq 0}$ *be the associated filtration. Let* $\Theta(t) = \begin{bmatrix} \Theta_1(t) \\ \Theta_2(t) \end{bmatrix}$, $t \geq 0$ *and* $\sigma(t) = \begin{bmatrix} \sigma_{11}(t) & \sigma_{12}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) \end{bmatrix}$, $t \geq 0$ *be processes adapted to the filtration* $(\mathcal{F}(t))_{t \geq 0}$ *and such that all integrals below are well-defined. The process* $\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}$ *is a two-dimensional Itô process if*

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \begin{bmatrix} X(0) \\ Y(0) \end{bmatrix} + \int_0^t \Theta(u)\, du + \int_0^t \sigma(u)\, dB(u), \tag{2.1}$$

[1] the pre-image of every two-dimensional Borel set is in $\mathcal{F}(t)$.

where

$$\Theta(u)du = \begin{bmatrix}\Theta_1(u)\\ \Theta_2(u)\end{bmatrix} du, \quad \sigma(u)\, dB(u) = \begin{bmatrix}\sigma_{11}(t) & \sigma_{12}(t)\\ \sigma_{21}(t) & \sigma_{22}(t)\end{bmatrix}\begin{bmatrix}dB_1(u)\\ dB_2(u)\end{bmatrix}.$$

We need to calculate $[X]_t$, $[Y]_t$, and $[X,Y]_t$. Let us try to do a formal computation and see what should be the answer. Rewrite (2.1) in differential form,

$$\begin{aligned} dX(t) &= \Theta_1(t)\, dt + \sigma_{11}(t)\, dB_1(t) + \sigma_{12}(t)\, dB_2(t);\\ dY(t) &= \Theta_2(t)\, dt + \sigma_{21}(t)\, dB_1(t) + \sigma_{22}(t)\, dB_2(t), \end{aligned}$$

and get

$$\begin{aligned} dX(t)dX(t) &= \Theta_1^2(t)\, dtdt + \sigma_{11}^2(t)\, dB_1(t)dB_1(t) + \sigma_{12}^2(t)\, dB_2(t)dB_2(t)\\ &+ 2\Theta_1(t)\sigma_{12}(t)\, dtdB_2(t) + 2\Theta_1(t)\sigma_{11}(t)\, dtdB_1(t) + 2\sigma_{11}(t)\sigma_{12}(t)\, dB_1(t)dB_2(t), \end{aligned} \tag{2.2}$$

a similar expression for $dY(t)dY(t)$, and

$$\begin{aligned} dX(t)dY(t) &= \Theta_1(t)\Theta_2(t)\, dtdt + \sigma_{11}(t)\sigma_{21}(t)\, dB_1(t)dB_1(t) + \sigma_{12}(t)\sigma_{22}(t)\, dB_2(t)dB_2(t)\\ &+ \Theta_1(t)\sigma_{12}(t)\, dtdB_1(t) + \Theta_1(t)\sigma_{22}(t)\, dtdB_2(t) + \Theta_1(t)\sigma_{22}(t)\, dtdB_2(t) + \Theta_2(t)\sigma_{12}(t)\, dtdB_1(t)\\ &+ \sigma_{11}(t)\sigma_{22}(t)\, dB_1(t)dB_2(t) + \sigma_{12}(t)\sigma_{21}(t)\, dB_1(t)dB_2(t). \end{aligned} \tag{2.3}$$

Notice that we have already dealt with terms of the form "$dtdt$", "$dB(t)dB(t)$", "$dtdB(t)$". The only new expression is "$dB_1(t)dB_2(t)$". The next two lemmas take care of the terms involving $dB_1(t)dB_2(t)$.

Lemma 2.4. *Let $(B(t))_{t\geq 0}$ be a two-dimensional Brownian motion. Then $[B_1,B_2]_t \equiv 0$ a.s..*

We postpone the proof of Lemma 2.4 until the end of this subsection.

Lemma 2.5. *Let $(B(t))_{t\geq 0}$ be a two-dimensional Brownian motion,*

$$I_1(t) = \int_0^t \sigma_1(t)\, dB_1(t), \quad I_2(t) = \int_0^t \sigma_2(t)\, dB_2(t).$$

Then $[I_1,I_2]_t \equiv 0$ a.s..

The proof of Lemma 2.5 is omitted[2]. Returning to the computation we formally conclude from (2.2), (2.3), Lemmas 2.4 and 2.5 that

$$\begin{aligned} d[X]_t &= (\sigma_{11}^2(t) + \sigma_{12}^2(t))dt;\\ d[Y]_y &= (\sigma_{12}^2(t) + \sigma_{22}^2(t))dt;\\ d[X,Y]_t &= (\sigma_{11}(t)\sigma_{22}(t) + \sigma_{21}(t)\sigma_{21}(t))dt. \end{aligned}$$

Putting everything together we arrive at

Theorem 2.6 (Itô-Doeblin formula for two-dimensional Itô processes)**.** *Let $f \in C^2([0,\infty)\times\mathbb{R}^2)$ and $(X(t),Y(t))^T$, $t \geq 0$, be a two-dimensional Itô process. Then*

$$\begin{aligned} &f(t,X(t),Y(t)) - f(0,X(0),Y(0)) =\\ &\quad \int_0^t f_t(s,X(s),Y(s))\, ds + \int_0^t f_x(s,X(s),Y(s))\, dX(s) + \int_0^t f_y(s,X(s),Y(s))\, dY(s)\\ &+ \frac{1}{2}\int_0^t f_{xx}(s,X(s),Y(s))\, d[X]_s + \frac{1}{2}\int_0^t f_{yy}(s,X(s),Y(s))\, d[Y]_s + \int_0^t f_{xy}(s,X(s),Y(s))\, d[X,Y]_s. \end{aligned}$$

The proof is omitted.

[2] The proof is done first for elementary functions $\sigma_i(t)$, $i = 1,2$. Then the result is extended to general integrands using approximations of σ_i, $i = 1,2$, by simple functions.

Corollary 2.7 (Integration by parts or Itô product rule)**.** *Let $(X(t), Y(t))^T$, $t \geq 0$, be a two-dimensional Itô process. Then*

$$\int_0^t X(s)\,dY(s) = X(t)Y(t) - X(0)Y(0) - \int_0^t Y(s)\,dX(s) - [X,Y]_t.$$

Proof. Apply Itô-Doeblin formula to $f(t,x,y) = xy$. We have $f_t = 0,\ f_x = y,\ f_y = x,\ f_{xx} = f_{yy} = 0,\ f_{xy} = 1$, and

$$X(t)Y(t) - X(0)Y(0) = \int_0^t X(s)\,dY(s) + \int_0^t Y(s)\,dX(s) + \int_0^t 1\,d[X,Y]_t.$$

Rearranging the terms we get the desired statement. □

Proof of Lemma 2.4. We need to show that (note: Π is a partition of $[0,t]$: $0 = t_0 < t_1 < \cdots < t_n = t$)

$$\lim_{\|\Pi\|\to 0} E\left[\left(\sum_{i=1}^n (B_1(t_i) - B_1(t_{i-1}))(B_2(t_i) - B_2(t_{i-1}))\right)^2\right] = 0.$$

Computing the square of the sum we get two types of terms

$$(B_1(t_i) - B_1(t_{i-1}))^2(B_2(t_i) - B_2(t_{i-1}))^2$$

and

$$(B_1(t_i) - B_1(t_{i-1}))(B_2(t_i) - B_2(t_{i-1}))(B_1(t_j) - B_1(t_{j-1}))(B_2(t_j) - B_2(t_{j-1})),\quad i \neq j.$$

Since B_1 and B_2 are independent and each of them has independent mean zero increments, the expectation of every cross term is zero and the expectation of every term with squares splits into the product of expectations. Therefore, we get that the expectation under the limit is equal to

$$\sum_{i=1}^n (t_i - t_{i-1})^2 \leq \|\Pi\| \sum_{i=1}^n (t_i - t_{i-1}) = \|\Pi\| t \to 0 \text{ as } \|\Pi\| \to 0.$$

□

3. Application: finding the covariance of stock prices driven by correlated Brownian motions.

Let B be a two-dimensional Brownian motion and S_1 and S_2 satisfy the following system of SDEs:

$$dS_1(t) = \alpha_1 S_1(t)\,dt + \sigma_1 S_1(t)\,dB_1(t)$$
$$dS_2(t) = \alpha_2 S_2(t)\,dt + \sigma_2\rho S_2(t)\,dB_1(t) + \sigma_2\sqrt{1-\rho^2}S_2(t)\,dB_2(t),$$

where $\alpha_i \in \mathbb{R}$, $\sigma_i > 0$, $i = 1,2$, and $\rho \in [-1,1]$. If $\rho \neq 0$ then S_1 and S_2 are correlated, since dB_1 appears in both equations. Let us find the correlation between $S_1(t)$ and $S_2(t)$.

We already know that S_1 is a geometric Brownian motion (GBM) and $S_1(t) = S_1(0) \times e^{(\alpha_1 - \sigma_1^2/2)t + \sigma_1 B_1(t)}$. What kind of process is S_2? To see the answer rewrite the equation for S_2 as

$$\frac{dS_2(t)}{S_2(t)} = \alpha_2\,dt + \sigma_2\left(\underbrace{\rho\,dB_1(t) + \sqrt{1-\rho^2}\,dB_2(t)}_{=:dB_3(t)}\right) \tag{3.1}$$

and define $B_3(t) = \rho B_1(t) + \sqrt{1-\rho^2}B_2(t)$. If we show that B_3 is a standard Brownian motion then we shall get that S_2 is a GBM and $S_2(t) = S_2(0)e^{(\alpha_2 - \sigma_2^2/2)t + \sigma_2 B_3(t)}$. We shall use Lévy's characterization of Brownian motion. We have that $B_3(0) = 0$, B_3 has continuous paths and is a an $\mathcal{F}(t)$-martingale (as a linear combination of $\mathcal{F}(t)$-martingales). All we need to check is the quadratic variation.

$$d[B_3]_t = \rho^2 d[B_1]_t + (1-\rho^2)d[B_2]_t + 2\rho\sqrt{1-\rho^2}\underbrace{d[B_1,B_2]_t}_{=0} = \rho^2\,dt + (1-\rho^2)\,dt = dt.$$

We conclude that B_3 is a Brownian motion. Observe that the correlation between $B_1(t)$ and $B_3(t)$ is ρ, so we can say that stock prices S_1 and S_2 are driven by two correlated Brownian motions, B_1 and B_3.

To compute the correlation between the stock prices we note that the SDEs (alternatively, the explicit formulas) imply that $E(S_i(t)) = S(0)e^{\alpha_i t}$, $i = 1, 2$. We only need to compute $E(S_i(t)S_j(t))$. We shall deal with the case $i = 1,\ j = 2$ (when $i = j$ we just need to set $\rho = 1$). By the Itô product rule,

$$\begin{aligned}
d(S_1(t)S_2(t)) &= S_1(t)\,dS_2(t) + S_2(t)\,dS_1(t) + d[S_1, S_2]_t \\
&\overset{(3.1)}{=} S_1(t)S_2(t)(\alpha_2\,dt + \sigma_2\,dB_3(t)) + S_1(t)S_2(t)(\alpha_1\,dt + \sigma_1\,dB_1(t)) + \rho\sigma_1\sigma_2 S_1(t)S_2(t)\,dt \\
&= (\alpha_1 + \alpha_2 + \rho\sigma_1\sigma_2)S_1(t)S_2(t)\,dt + S_1(t)S_2(t)(\sigma_2\,dB_3(t) + \sigma_1\,dB_1(t)).
\end{aligned}$$

Integrating from 0 to t and taking the expectation we find that

$$E(S_1(t)S_2(t)) - S_1(0)S_2(0) = (\alpha_1 + \alpha_2 + \rho\sigma_1\sigma_2)\int_0^t E(S_1(u)S_2(u))\,du.$$

Solving the above equation we get that

$$\mathrm{Cov}(S_1(t), S_2(t)) = E(S_1(t)S_2(t)) - S_1(0)S_2(0) = S_1(0)S_2(0)(e^{(\alpha_1+\alpha_2+\rho\sigma_1\sigma_2)t} - 1).$$

From here there is only a short step to computing the correlation: and

$$\rho(S_1(t), S_2(t)) = \frac{S_1(0)S_2(0)e^{(\alpha_1+\alpha_2)t}(e^{\rho\sigma_1\sigma_2 t} - 1)}{S_1(0)e^{\alpha_1 t}\sqrt{e^{\sigma_1^2 t} - 1}S_2(0)e^{\alpha_2 t}\sqrt{e^{\sigma_2^2 t} - 1}} = \frac{e^{\rho\sigma_1\sigma_2 t} - 1}{\sqrt{e^{\sigma_1^2 t} - 1}\sqrt{e^{\sigma_2^2 t} - 1}}.$$

4. Application: Lévy's characterization of Brownian motion (2 dimensions).

Theorem 4.1. *Let $M_i(t)$, $t \ge 0$, $i = 1, 2$, be continuous $\mathcal{F}(t)$-martingales. Assume that $M_i(0) = 0$, $[M_i]_t = t$, $t \ge 0$ a.s. ($i = 1, 2$), $[M_1, M_2]_t = 0$, $t \ge 0$, a.s.. Then $M(t) := \begin{bmatrix} M_1(t) \\ M_2(t) \end{bmatrix}$ is a standard two-dimensional Brownian motion.*

Proof. By the one dimensional theorem, each M_1 and M_2 is a standard Brownian motion. We need to show that they are independent.

Method: We shall show that for $0 \le s \le t$ the vector $\begin{pmatrix} M_1(t) - M_1(s) \\ M_2(t) - M_2(s) \end{pmatrix}$ is independent of $\mathcal{F}(s)$ and is normally distributed with mean 0 and covariance matrix $\begin{pmatrix} t-s & 0 \\ 0 & t-s \end{pmatrix}$. This will imply that the joint distribution of increments of processes M_1 and M_2 is normal. Then the fact that the above covariance matrix is diagonal will imply the independence of the processes M_1 and M_2.

Since $M_i(t)$ is a Brownian motion, $E(e^{u_i(M_i(t)-M_i(s))}) = e^{u^2(t-s)/2}$, $i = 1, 2$. Therefore, it is sufficient to show that

$$E\left(e^{u_1(M_1(t)-M_1(s))+u_2(M_2(t)-M_2(s))} \,|\, \mathcal{F}(s)\right) = e^{u_1^2(t-s)/2+u_2^2(t-s)/2} \quad \text{for all } 0 \le s \le t. \tag{4.1}$$

This gives us an idea to try to show that the process

$$f(t, M_1(t), M_2(t)) = e^{u_1 M_1(t)+u_2 M_2(t)-u_1^2 t/2-u_2^2 t/2}, \quad t \ge 0,$$

is a martingale. Then we would have $E(f(t, M_1(t), M_2(t)) \,|\, \mathcal{F}(s)) = Ef(s, M_1(s), M_2(s))$, and this would immediately imply (4.1).

To compute $df(t, M_1(t), M_2(t))$ we calculate for $f(t, x, y) = e^{u_1 x+u_2 y-(u_1^2+u_2^2)t/2}$:

$$\begin{aligned}
f_t &= -\frac{u_1^2 + u_2^2}{2} f; \quad f_x = u_1 f; \quad f_y = u_2 f; \\
f_{xy} &= u_1 u_2 f; \quad f_{xx} = u_1^2 f; \quad f_{yy} = u_2^2 f,
\end{aligned}$$

and apply Itô-Doeblin formula for martingales

$$df(t, M_1(t), M_2(t)) = -\frac{u_1^2 + u_2^2}{2} f\,dt + u_1 f dM_1(t) + u_2 f\, dM_2(t)$$
$$+ \frac{1}{2}\, u_1^2 f \underbrace{d[M_1]_t}_{=dt} + \frac{1}{2}\, u_2^2 f \underbrace{d[M_2]_t}_{=dt} + u_1 u_2 f \underbrace{d[M_1, M_2]_t}_{=0}.$$

All "dt" terms cancel out, and we are left only with stochastic integrals. Therefore $f(t, M_1(t), M_2(t))$ is a martingale. □

5. Change of measure: Girsanov's theorem in one dimension.

Recall the following fact from the refresher (review Theorem 1.6.1 and Example 1.6.6 of the textbook).

Theorem 5.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let Z be an a.s. non-negative random variable with $E(Z) = 1$. For $A \in \mathcal{F}$ define*

$$\widetilde{P}(A) = \int_A Z(\omega) dP(\omega).$$

Then $\widetilde{P}$ is a probability measure. Moreover,

1. *If X is an a.s. non-negative random variable then $\widetilde{E}(X) = E(XZ)$.*
2. *If $P(Z > 0) = 1$ then we also have $E(X) = \widetilde{E}(XZ^{-1})$.*
3. *If $P(Z > 0) = 1$ then P and $\widetilde{P}$ are equivalent and Z is called the Radon-Nikodym derivative of $\widetilde{P}$ with respect to P and is denoted by $d\widetilde{P}/dP$.*

In examples considered in the refresher we changed the measure for a single random variable. Now the goal is to perform the change of measure for a whole stochastic process on a finite time interval.

Theorem 5.2 (Girsanov, $d = 1$)**.** *Let B be a standard Brownian motion and $\mathcal{F}(t)$, $t \geq 0$, be a filtration for this Brownian motion. Let $\Theta(t)$, $t \in [0, T]$, be an $\mathcal{F}(t)$-adapted process and define*

$$Z(t) = \exp\left(-\int_0^t \Theta(s)\, dB(s) - \frac{1}{2}\int_0^t \Theta^2(s)\, ds\right), \quad \widetilde{B}(t) = B(t) + \int_0^t \Theta(s)\, ds, \quad t \in [0, T].$$

Assume that

$$E\int_0^T \Theta^2(t) Z^2(t)\, dt < \infty. \tag{5.1}$$

Set $Z = Z(T)$. Then $E(Z) = 1$, and under the probability measure

$$\widetilde{P}(A) = \int_A Z(\omega)\, dP(\omega) \tag{5.2}$$

the process $\widetilde{B}(t)$, $0 \leq t \leq T$, is a standard Brownian motion.

Remark 5.3. The condition (5.1) is not easy to check. The following conditions are known to be sufficient to infer the conclusions of the above theorem:

$$E\left[\exp\left(\frac{1}{2}\int_0^T \Theta^2(t)\, dt\right)\right] < \infty \qquad \text{(Novikov's condition);}$$
$$E\left[\exp\left(\frac{1}{2}\int_0^T \Theta(t)\, dB(t)\right)\right] < \infty \qquad \text{(Kazamaki's condition).}$$

Kazamaki's condition implies Novikov's condition but Novikov's is easier to verify.

To discuss the idea of the proof we need the following two technical lemmas. They have a common set up as follows. Let $(\Omega, \mathcal{F}, P)$ be a probability space Z be a random variable such that $P(Z>0)=1$ and $E(Z)=1$. Let $\mathcal{F}(t)$, $t\in[0,T]$, be a filtration, $\mathcal{F}(T)\subset\mathcal{F}$. Define $\widetilde{P}(A)$ by (5.2) and set

$$Z(t)=E(Z\,|\,\mathcal{F}(t)),\quad 0\le t\le T. \tag{5.3}$$

Lemma 5.4. *Let X be $\mathcal{F}(t)$-measurable for some $t\le T$ and either $X\ge 0$ a.s. or $E(Z(t)|X|)<\infty$. Then*

$$\widetilde{E}(X)=E(XZ(t)) \tag{5.4}$$

Proof. By Theorem 5.1

$$\widetilde{E}(X)=E(XZ)=E(E(XZ\,|\,\mathcal{F}(t)))=E(XE(Z\,|\,\mathcal{F}(t)))\overset{(5.3)}{=}E(XZ(t)).$$

□

Lemma 5.5. *Let Y be $\mathcal{F}(t)$-measurable for some $t\le T$ and either $Y\ge 0$ a.s. or $E(Z(t)|Y|)<\infty$. Then for all $0\le s\le t$*

$$\widetilde{E}(Y\,|\,\mathcal{F}(s))=\frac{1}{Z(s)}E(YZ(t)\,|\mathcal{F}(s)).$$

Proof. Fix $s\in[0,t]$ and define

$$X=\frac{1}{Z(s)}E(YZ(t)\,|\mathcal{F}(s)). \tag{5.5}$$

Then X is $\mathcal{F}(s)$-measurable. We want to show that $\widetilde{E}(Y\,|\,\mathcal{F}(s))=X$ a.s.. We use the definition of conditional expectation. We need to show that for every $A\in\mathcal{F}(s)$

$$\int_A Y(\omega)\,d\widetilde{P}(\omega)=\int_A X(\omega)\,d\widetilde{P}(\omega).$$

We start with the right-hand side and transform it into the left-hand side.

$$\int_A X(\omega)\,d\widetilde{P}(\omega)=\widetilde{E}(\underbrace{\mathbb{1}_A X}_{\mathcal{F}(s)-\text{meas.}})\overset{\text{L. }5.4}{=}E(\mathbb{1}_A XZ(s))\overset{(5.5)}{=}E(\mathbb{1}_A(E(YZ(t)\,|\,\mathcal{F}(s)))$$
$$E(E(\mathbb{1}_A Y(Z(t)\,|\,\mathcal{F}(s)))=E(\underbrace{\mathbb{1}_A Y}_{\mathcal{F}(t)-\text{meas.}}Z(t))\overset{\text{L. }5.4}{=}\widetilde{E}(\mathbb{1}_A Y)=\int_A Y(\omega)\,d\widetilde{P}(\omega).$$

□

Sketch of proof of Theorem 5.2. We shall use Lévy's characterization of Brownian motion ($d=1$). We see immediately from the definition of $\widetilde{B}(t)$ that $\widetilde{B}(0)=0$ and $\widetilde{B}(t)$, $0\le t\le T$, has continuous paths. We need to check the martingale property (Steps 1-3 below) and and that $[\widetilde{B}]_t=t$, $0\le t\le T$, $\widetilde{P}$-a.s. (Step 4).

Step 1. $Z(t)$, $0\le t\le T$, is an $\mathcal{F}(t)$ martingale with respect to P.[3] Write $Z(t)=e^{X(t)}$, where

$$X(t)=-\int_0^t\Theta(s)\,dB(s)-\frac{1}{2}\int_0^t\Theta^2(s)\,ds.$$

Then by the Itô-Doeblin formula for Itô processes we have

$$dZ(t)=Z(t)\,dX(t)+\frac{1}{2}Z(t)d[X]_t=-\Theta(t)Z(t)\,dB(t)-\frac{1}{2}Z(t)\Theta^2(t)\,dt+\frac{1}{2}Z(t)\Theta^2(t)\,dt$$
$$=-\Theta(t)Z(t)\,dB(t).$$

[3]This was in Exercise (8) (see (1)(f) with $\Delta(t)=-\Theta(t)$) in the handout on Itô formula, part II.

Integrating from 0 to t we get

$$Z(t) = 1 - \int_0^t \Theta(s) Z(s)\, dB(s).$$

We conclude that $Z(t)$, $0 \le t \le T$, is a martingale[4] and $E(Z(t)) = 1$ for all $t \in [0,T]$.

Step 2. $\widetilde{B}(t)Z(t)$, $0 \le t \le T$, is an $\mathcal{F}(t)$ martingale with respect to P. indeed, by the Itô product rule

$$\begin{aligned} d(\widetilde{B}(t)Z(t)) &= \widetilde{B}(t)\, dZ(t) + Z(t)\, d\widetilde{B}(t) + d[Z, \widetilde{B}]_t \\ &= \widetilde{B}(t)(-\Theta(t)Z(t))\, dB(t) + Z(t)(dB(t) + \Theta(t)\, dt) - \Theta(t)Z(t)\, dt = Z(t)(1 - \widetilde{B}(t)\Theta(t))\, dB(t), \end{aligned}$$

and the martingale property follows.

Step 3. $\widetilde{B}(t)$, $0 \le t \le T$, is an $\mathcal{F}(t)$ martingale with respect to $\widetilde{P}$. Integrability conditions are satisfied by the conditions imposed on Θ. We shall only check that $\widetilde{E}(\widetilde{B}(t) \mid \mathcal{F}(s)) = \widetilde{B}(s)$, $0 \le s \le t \le T$.

$$\widetilde{E}(\widetilde{B}(t) \mid \mathcal{F}(s)) \overset{\text{L. 5.5}}{=} \frac{1}{Z(s)} E(\widetilde{B}(t)Z(t) \mid \mathcal{F}(s)) \overset{\text{Step 2}}{=} \frac{1}{Z(s)} \widetilde{B}(s)Z(s) = \widetilde{B}(s).$$

Step 4. The computation of the quadratic variation of $\widetilde{B}$. From the definition of $\widetilde{B}$ we have $d\widetilde{B}(t) = dB(t) + \Theta(t)dt$. Thus, $d[\widetilde{B}]_t = d[B]_t = t$ and $[\widetilde{B}]_t = t$, $t \in [0,T]$, (P-a.s.). But P and $\widetilde{P}$ are equivalent, therefore $[\widetilde{B}]_t = t$, $t \in [0,T]$, ($\widetilde{P}$-a.s.). □

[4] In general, the absence of the "dt" term only guarantees that the process is a so called local martingale. Some integrability condition is needed to conclude that Z is a "true" martingale and $E(Z(t)) = 1$. The condition (5.1) serves this purpose.