# MTH 9878, Spring 2016
# Review

1. If $\tilde{W}(t) = W(t) - \int_0^t \theta(s)\,ds$, and if $\tilde{W}(t)$ is a Brownian motion under the measure $Q$, what is $\frac{d\mathbb{Q}}{d\mathbb{P}}$?

2. Prove that $D(t) = e^{\int_0^t \theta(s)\,dW(s) - \frac{1}{2}\int_0^t \theta^2(s)\,ds}$ is a martingale under $\mathbb{P}$.

3. Prove that $D(t) = e^{\int_0^t \theta(s)\,dW(s) - \frac{1}{2}\int_0^t \theta^2(s)\,ds}$ satisfies

$$dD(t) = \theta(t)D(t)\,dW(t).$$

4. What is a numeraire?

5. What is $T$-forward measure?

6. What is the numeraire that corresponds to swap measure?

7. What is a 7 year spot starting cap struck at 3%?

8. What is normal model?

9. What is a shifted log-normal model?

10. What is CEV model?

11. Prove that the price of call in the normal model is

$$\begin{aligned} P^{\text{call}}(T,K,F_0,\sigma) &= \mathcal{N}(0)B_{\text{n}}^{\text{call}}(T,K,F_0,\sigma), \quad \text{where} \\ B_{\text{n}}^{\text{call}}(T,K,F_0,\sigma) &= \sigma\sqrt{T}\left(d_+ N(d_+) + N'(d_-)\right), \\ d_\pm &= \pm\frac{F_0 - K}{\sigma\sqrt{T}}. \end{aligned}$$

12. Prove that

$$P^{\text{floor}} - P^{\text{cap}} = K\sum_{j=1}^{n} \delta_j P_0(0,T_j) - \sum_{j=1}^{n} \delta_j P_0(0,T_j) L_0(T_{j-1},T_j).$$

13. Explain how to obtain the implied log-normal volatilities if we know the prices of call options for different strikes.

14. Define the SABR model.

15. (a) What is LIBOR in arrears?

    (b) Prove that if $(\mathbb{P},\mathcal{M})$ and $(\mathbb{Q},\mathcal{N})$ are two EMM-numeraire pairs then

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\mathcal{N}(T)\mathcal{M}(0)}{\mathcal{N}(0)\mathcal{M}(T)}.$$

(c) The present value of the LIBOR in arrears is

$$P = P_0(0,S) \cdot \mathbb{E}^{\mathbb{Q}_S}\left[L(S,T)\right].$$

Prove that this present value also satisfies the equation

$$P = P_0(0,T)\mathbb{E}^{\mathbb{Q}_T}\left[\frac{L(S,T)}{P(S,T)}\right].$$

16. Assume that $S_5$ and $S_{10}$ are 5 and 10 years CMS rates. A call option on $S_5$ with strike $K_5$ has price $P_5$. A call option on $S_{10}$ with strike $K_{10}$ has price $P_{10}$. Prove that the price $P'$ of the call option on the spread $S_{10} - S_5$ with strike $K_{10} - K_5$ must satisfy $P' \geq P_{10} - P_5$.

17. (a) If the interest rate follows the Hull-White model,

$$dr(t) = \left(\frac{d\mu(t)}{dt} + \lambda\left(\mu(t) - r(t)\right)\right) dt + \sigma(t)\, dW(t).$$

prove that it can be expressed by

$$r(t) = r_0 e^{-\lambda t} + \mu(t) - \mu(0)e^{-\lambda t} + e^{-\lambda t}\int_0^t \sigma(s)e^{\lambda s}\, dW(s).$$

(b) Determine the formulas for the expectation and variance for the interest rate.

(c) Why is Hull-White model considered mean-reverting?

18. (a) Explain why the LIBOR from FRAs and Eurodollar futures are given by the following two formulas

$$\begin{aligned} L_0^{\text{FRA}}(T_1,T_2) &= \frac{1}{\delta}\left(\frac{P_0(0,T_1)}{P_0(0,T_2)} - 1\right) + B_0(T_1,T_2), \\ L_0^{\text{fut}}(T_1,T_2) &= \frac{1}{\delta}\left(\mathbb{E}^{\mathbb{Q}_0}\left[e^{\int_{T_1}^{T_2} r(u)\, du}\right] - 1\right) + B_0(T_1,T_2). \end{aligned}$$

(b) Use the formulas

$$\begin{aligned} L_0^{\text{FRA}}(T_1,T_2) &= \frac{1}{\delta}\left(e^{\int_{T_1}^{T_2} r_0(u)\, du} \cdot e^{-\frac{1}{2}\int_0^{T_2} h_\lambda^2(T_2-u)\sigma^2(u)\, du} \cdot e^{\frac{1}{2}\int_0^{T_1} h_\lambda^2(T_1-u)\sigma^2(u)\, du} - 1\right) + B_0(T_1,T_2), \\ r(u) &= r_0(u) + \int_0^u e^{-\lambda(u-v)}\sigma(v)\, dW(v), \quad \text{and} \quad h_\lambda(z) = \frac{1-e^{-\lambda z}}{\lambda} \end{aligned}$$

to calculate the Eurodollar/FRA convexity correction.

19. What is LMM?

20. Why is the rate $L_j(t)$ killed at $t = T_j$?

21. Prove that the numeraire $B(t)$ for the spot measure can be represented as

$$B(t) = \frac{P(t, T_{\gamma(t)})}{\prod_{i=1}^{\gamma(t)} P(T_{i-1}, T_i)},$$

for certain number $\gamma(t)$. What is $\gamma(t)$?

22. (a) If $(\mathbb{P}, \mathscr{N})$ and $(\mathbb{Q}, \mathscr{M})$ are two associated measure-numeraire pairs and if the asset $X$ satisfies

$$\begin{aligned} dX(t) &= \Delta^{\mathbb{P}}(t)\,dt + C(t)dW^{\mathbb{P}}(t) \\ dX(t) &= \Delta^{\mathbb{Q}}(t)\,dt + C(t)dW^{\mathbb{Q}}(t), \end{aligned}$$

prove that

$$\Delta^{\mathbb{Q}}(t) - \Delta^{\mathbb{P}}(t) = \frac{d}{dt}\left[X, \log\frac{\mathscr{M}}{\mathscr{N}}\right](t).$$

(b) Prove that if $j > k$ then the Libor rate $L_j$ satisfies the following equation under the EMM corresponding to the numeraire $P(\cdot, T_{k+1})$:

$$dL_j = C_j(t) \cdot \sum_{i=k+1}^{j} \frac{\rho_{ij}\delta_i C_i(t)}{1+\delta_i F_i(t)}\,dt + C_j(t)\,dW_j(t).$$

23. Assume that a random number generator produces a random variable $X$ with uniform $[0,1]$ distribution. Using only this random variable $X$, create a random variable with normal $N(\mu, \sigma^2)$ distribution for given $\mu$ and $\sigma$.

24. Assume that $\overrightarrow{Z} = (Z_1, Z_2, \dots, Z_d)$ is a $d$-dimensional vector of normal random variables whose covariance matrix has the Cholesky decomposition $\rho = LL^T$. What formula can be used to express $\overrightarrow{Z}$ in terms of a vector $\overrightarrow{\xi} = (\xi_1, \xi_2, \dots, \xi_d)$ of independent standard normal random variables?

25. Explain what are Euler and Milstein schemes for solving stochastic differential equations.

26. What is a callable swap?

27. What is a Bermudan swaption?

28. Give an example of a callable swap and explain how it can be expressed as a portfolio of a vanilla swap and a Bermudan swaption.

29. Which swap rate is higher: The swap rate of a vanilla swap or that of a callable swap?

30. How is a function approximated using Hermite polynomials?

31. If $h_k(x)$ are normalized Hermite polynomials, calculate

$$\int_{-\infty}^{+\infty} (h_1(x) + h_3(x)) \cdot (h_2(x) + h_3(x) + h_4(x)) \cdot e^{-\frac{x^2}{2}}\,dx.$$

32. Assume that $c_k$ are coefficients of the function $f(X_1, X_2, \dots, X_j)$ in the Fourier expansion with respect to the basis formed by Hermite polynomials. Explain how the functions $(c_k)_{k=1}^{\kappa}$ can be approximately calculated by minimizing the functional

$$\zeta(c_0, c_1, \dots, c_\kappa) = \sum_{i=1}^{N} \left( f(X^i_{1:j}) - \sum_{k=0}^{\kappa} c_k h_k(X^i_j) \right)^2.$$