BARUCH, MFE

MTH 9876 Assignment Four

Zhou, ShengQuan

November 7, 2016

1 Marshall-Olkin Copula

Consider the Marshall-Olkin copula $C_{\mathrm{MO}}(u_1, u_2)$.

(i) Calculate the Spearman's rho of $C_{\mathrm{MO}}(u_1, u_2)$.
Solution: According to lecture notes,

$$C_{\mathrm{MO}}(u_1, u_2) = u_1 u_2 \min\left(u_1^{-\alpha_1}, u_2^{-\alpha_2}\right) = \min\left(u_1^{1-\alpha_1} u_2, u_1 u_2^{1-\alpha_2}\right),$$

where $\alpha_1 = \frac{\lambda_{12}}{\lambda_1+\lambda_{12}}$ and $\alpha_2 = \frac{\lambda_{12}}{\lambda_2+\lambda_{12}}$. **Note**: there is a typo in lecture notes for α_1 and α_2.

Notice that the two arguments of the min function are equal when

$$u_1^{-\alpha_1} = u_2^{-\alpha_2} \Rightarrow u_2 = u_1^{\frac{\alpha_1}{\alpha_2}}.$$

The Spearman's rho is given by $\rho_S(X, Y) = 12\int\int_{[0,1]^2} C(u_1, u_2)du_1du_2 - 3$, where we calculate the integral in two sub-regions of $(u_1, u_2) \in [0, 1]^2$ separated by the line $u_1^{-\alpha_1} = u_2^{-\alpha_2}$.

$$\begin{aligned}
\int\int_{[0,1]^2} C(u_1, u_2)du_1du_2 &= \int\int_{u_1^{-\alpha_1} > u_2^{-\alpha_2}} u_1 u_2^{1-\alpha_2} du_1 du_2 + \int\int_{u_1^{-\alpha_1} < u_2^{-\alpha_2}} u_1^{1-\alpha_1} u_2 du_1 du_2 \\
&= \int_0^1 du_1 u_1 \int_{u_1^{\alpha_1/\alpha_2}}^1 du_2 u_2^{1-\alpha_2} + \int_0^1 du_1 u_1^{1-\alpha_1} \int_0^{u_1^{\alpha_1/\alpha_2}} u_2 du_2 \\
&= \frac{1}{2-\alpha_2}\int_0^1 du_1 u_1 \left[1 - u_1^{\frac{\alpha_1}{\alpha_2}(2-\alpha_2)}\right] + \frac{1}{2}\int_0^1 du_1 u_1^{1-\alpha_1+2\frac{\alpha_1}{\alpha_2}} \\
&= \frac{1}{2-\alpha_2}\int_0^1 u_1 du_1 + \left(\frac{1}{2} - \frac{1}{2-\alpha_2}\right)\int_0^1 du_1 u_1^{1-\alpha_1+2\frac{\alpha_1}{\alpha_2}} \\
&= \frac{1}{2(2-\alpha_2)} + \left(\frac{1}{2} - \frac{1}{2-\alpha_2}\right)\frac{1}{2-\alpha_1+2\frac{\alpha_1}{\alpha_2}} \\
&= \frac{2\alpha_1 + 2\alpha_2 - \alpha_1\alpha_2 - \alpha_2^2}{2(2-\alpha_2)(2\alpha_1 + 2\alpha_2 - \alpha_1\alpha_2)} \\
&= \frac{\alpha_1 + \alpha_2}{2(2\alpha_1 + 2\alpha_2 - \alpha_1\alpha_2)}.
\end{aligned}$$

Thus,

$$\rho_S(X, Y) = 12 \times \frac{\alpha_1 + \alpha_2}{2(2\alpha_1 + 2\alpha_2 - \alpha_1\alpha_2)} - 3 = \frac{3\alpha_1\alpha_2}{2\alpha_1 + 2\alpha_2 - \alpha_1\alpha_2} = \frac{3\lambda_{12}^2}{2\lambda_1\lambda_{12} + 2\lambda_2\lambda_{12} + 3\lambda_{12}^2}.$$

(ii) Calculate the limits λ_U and λ_L and determine whether the Marshall-Olkin copula has upper and/or lower tail dependence.
Solution: Set $u_1 = u_2 = u$, $C(u,u) = u^{2-\min(\alpha_1,\alpha_2)}$. Thus, the upper tail dependence

$$
\begin{aligned}
\lambda_U &\triangleq \lim_{u\to 1} \frac{1-2u+C(u,u)}{1-u} \\
&= \lim_{u\to 1} \frac{1-2u+u^{2-\min(\alpha_1,\alpha_2)}}{1-u} \\
(v \triangleq 1-u) &= \lim_{v\to 0} \frac{2v-1+(1-v)^{2-\min(\alpha_1,\alpha_2)}}{v} \\
&= \lim_{v\to 0} \frac{2v-1+1-(2-\min(\alpha_1,\alpha_2))\,v}{v} \\
&= \min(\alpha_1,\alpha_2).
\end{aligned}
$$

The lower tail dependence

$$
\begin{aligned}
\lambda_L &\triangleq \lim_{u\to 0} \frac{C(u,u)}{u} \\
&= \lim_{u\to 0} \frac{u^{2-\min(\alpha_1,\alpha_2)}}{u} \\
&= \lim_{u\to 0} u^{1-\min(\alpha_1,\alpha_2)} \\
&= 0.
\end{aligned}
$$

The last equality results from the fact that $\alpha_1 \le 1$ and $\alpha_2 \le 1$.

(iii) Propose a simulation algorithm for $C_{\mathrm{MO}}(u_1,u_2)$. Choose $\lambda_1 = 1.7, \lambda_2 = 0.7, \lambda_{12} = 0.3$, and generate 2000 samples from $C_{\mathrm{MO}}(u_1,u_2)$. Plot the results in a 1×1-square.
Solution: The sampling algorithm is constructed as follows:

- Generate independent exponentially distributed random numbers T_1, T_2, T_{12} with intensities $\lambda_1, \lambda_2, \lambda_{12}$ respectively. This is because T_1, T_2, T_{12} are arriving times of independent Poisson processes. The marginal distributions of the survival probabilities are

$$\mathbb{P}(T_i > t) = e^{-\lambda_i t}, \quad i = 1,2,12.$$

- Take $\tau_1 = \min(T_1, T_{12})$ and $\tau_2 = \min(T_2, T_{12})$. The marginal distributions of the survival probabilities are

$$F_i(t) \triangleq \mathbb{P}(\tau_i > t) = e^{-(\lambda_i+\lambda_{12})t}, \quad i = 1,2.$$

- By definition, $U_i = F_i(\tau_i) = e^{-(\lambda_i+\lambda_{12})\tau_i}$, $i = 1,2$.

Python script:

```
import math
import numpy as np
import matplotlib.pyplot as plt
```

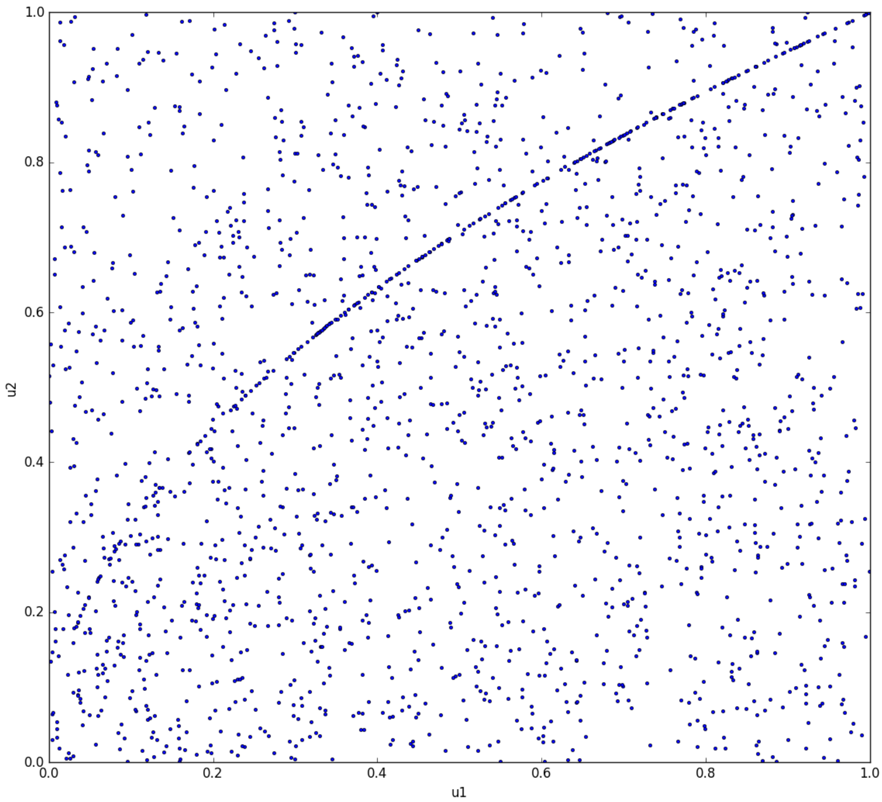
```
lambda1=1.7
lambda2=0.7
lambda12=0.3
ns=2000

T1  = np.random.exponential(1.0/lambda1, ns)
T2  = np.random.exponential(1.0/lambda2, ns)
T12 = np.random.exponential(1.0/lambda12, ns)

tau1 = np.minimum(T1, T12)
tau2 = np.minimum(T2, T12)

u1 = np.exp(-(lambda1+lambda12)*tau1)
u2 = np.exp(-(lambda2+lambda12)*tau2)

plt.xlabel("u1")
plt.ylabel("u2")
plt.show()
```



2 Two-Factor Gaussian Copula Model

Consider the following two-factor generalization of the one-factor Gaussian copula model:

$$Z_i = \beta_{i1} S_1 + \beta_{i2} S_2 + \sqrt{1-\beta_{i1}^2-\beta_{i2}^2}\epsilon_i,$$

where $S_1, S_2 \sim N(0,1)$ are independent, $\beta_{i1}^2 + \beta_{i2}^2 < 1$, for each $i = 1, \cdots, N$, and $\epsilon_i \sim N(0,1)$ are independent of S_1 and S_2.

(i) Determine the correlation structure in this model.
Solution: For $i = 1, \cdots, N$, $\mathbb{E}[Z_i] = 0$. For $i, j = 1, \cdots, N$,

$$\begin{aligned}
\mathbb{E}\left[Z_i Z_j\right] &= \mathbb{E}\left[\left(\beta_{i1} S_1 + \beta_{i2} S_2 + \sqrt{1-\beta_{i1}^2-\beta_{i2}^2}\epsilon_i\right)\left(\beta_{j1} S_1 + \beta_{j2} S_2 + \sqrt{1-\beta_{j1}^2-\beta_{j2}^2}\epsilon_j\right)\right] \\
&= \beta_{i1}\beta_{j1} + \beta_{i2}\beta_{j2} + \sqrt{\left(1-\beta_{i1}^2-\beta_{i2}^2\right)\left(1-\beta_{j1}^2-\beta_{j2}^2\right)}\delta_{ij}.
\end{aligned}$$

In other words,

$$\rho_{ij} = \begin{cases} \beta_{i1}\beta_{j1} + \beta_{i2}\beta_{j2}, & i \neq j, \\ 1, & i = j. \end{cases}$$

(ii) Derive, along the lines of the calculations done in class for the one-factor model, the LHP limit of this model.
Solution: The *homogeneous* part of LHP assumption, $\beta_{i1} = \beta_1$, $\beta_{i2} = \beta_2$, identical LGDs $l_i = l$, and identical default probabilities $Q_i(T) = Q(T)$, $\forall i = 1, \cdots, N$. The conditional loss distribution

$$\mathbb{P}\left(L_N(T) = kl | S_1 = s_1, S_2 = s_2\right) = \binom{N}{k} Q(T|s_1, s_2)^k \left(1 - Q(T|s_1, s_2)\right)^{N-k},$$

where $Q(T|s_1, s_2)$ is the default probability by time T, conditioned on the two driving factors $S_1 = s_1, S_2 = s_2$,

$$\begin{aligned}
Q(T|s_1, s_2) &\triangleq Q(T|S_1 = s_1, S_2 = s_2) \\
&= \mathbb{P}(\tau_i \leq T | S_1 = s_1, S_2 = s_2) \\
&= \mathbb{P}(Z_i \leq H_i(T) | S_1 = s_1, S_2 = s_2), \quad H_i(t) \triangleq N^{-1}(Q_i(t)) \\
&= \mathbb{P}\left(\beta_1 S_1 + \beta_2 S_2 + \sqrt{1-\beta_1^2-\beta_2^2}\epsilon_i \leq H_i(T) \middle| S_1 = s_1, S_2 = s_2\right) \\
&= \mathbb{P}\left(\beta_1 s_1 + \beta_2 s_2 + \sqrt{1-\beta_1^2-\beta_2^2}\epsilon_i \leq H_i(T)\right) \\
&= \mathbb{P}\left(\epsilon_i \leq \frac{H_i(T) - \beta_1 s_1 - \beta_2 s_2}{\sqrt{1-\beta_1^2-\beta_2^2}}\right) \\
&= N\left(\frac{H(T) - \beta_1 s_1 - \beta_2 s_2}{\sqrt{1-\beta_1^2-\beta_2^2}}\right), \quad H(t) \triangleq N^{-1}(Q(t)).
\end{aligned}$$

The total loss distribution

$$\mathbb{P}\left(L_N(T) = kl\right) = \frac{1}{2\pi}\int_{-\infty}^{-\infty}\int_{-\infty}^{-\infty}\mathbb{P}\left(L_N(T) = kl|S_1 = s_1, S_2 = s_2\right)e^{-\frac{s_1^2+s_2^2}{2}}ds_1 ds_2.$$

The *large* part of LHP assumption, $N \to \infty$, let $l_N = \frac{L(T)}{N}$ denote the number of defaulting names per portfolio size. According to lecture notes,

$$\begin{aligned}\mathbb{E}[l_N|s_1, s_2] &= lQ(T|s_1, s_2)\\ \mathrm{Var}[l_N|s_1, s_2] &= \frac{l^2}{N}Q(T|s_1, s_2)(1 - Q(T|s_1, s_2)) \to 0.\end{aligned}$$

The above result indicates that in the large portfolio limit $N \to \infty$, $l_N = \frac{L(T)}{N}$ is non-random when conditioned on $S_1 = s_1, S_2 = s_2$. In other words, all the randomness comes from the two driving factor S_1 and S_2. Thus, replacing the conditional value s_1 and s_2 by the random variables S_1 and S_2,

$$\begin{aligned}\mathbb{P}\left(l_\infty \le xl\right) &= \mathbb{P}\left(Q(T|S_1, S_2) \le x\right)\\ &= \mathbb{P}\left(Q(T|S_1, S_2) \le x\right)\\ &= \mathbb{P}\left(N\left(\frac{H(T) - \beta_1 S_1 - \beta_2 S_2}{\sqrt{1-\beta_1^2-\beta_2^2}}\right) \le x\right)\\ &= \mathbb{P}\left(\frac{H(T) - \beta_1 S_1 - \beta_2 S_2}{\sqrt{1-\beta_1^2-\beta_2^2}} \le N^{-1}(x)\right)\\ &= \mathbb{P}\left(\beta_1 S_1 + \beta_2 S_2 \le H(T) - \sqrt{1-\beta_1^2-\beta_2^2}N^{-1}(x)\right).\end{aligned}$$

Since $\beta_1 S_1 + \beta_2 S_2 \sim N\left(0, \beta_1^2 + \beta_2^2\right)$,

$$\mathbb{P}\left(l_\infty \le xl\right) = N\left(\frac{H(T) - \sqrt{1-\beta_1^2-\beta_2^2}N^{-1}(x)}{\sqrt{\beta_1^2+\beta_2^2}}\right).$$

3 HISTORICAL SIMULATION OF VAR

Assume that the CCP calculates the margin requirements by means of VaR at the $p = 99\%$ confidence level. Regulatory authorities require that the CCP back-test the performance of its risk model by calculating the number of breaches of the VaR model over the test period of the most recent 250 days (one year). Specifically, on each day of the test period, the daily portfolio loss (in reality, it is the n-day portfolio loss, where n is called the market period of risk (MPOR)) is compared to the calculated VaR level. A VaR breach means that the loss on a given day exceeds the VaR level.

(i) Assuming that the daily portfolio returns are independent of each other, find the probability distribution of the number of VaR breaches. What is the expected number of VaR breaches?
Solution: Setting the confidence level at $p = 99\%$ means empirically, one out of 100 sample returns is a VaR breach. Assuming that the daily portfolio returns are independent of each other, find the probability distribution of the number of VaR breaches is

$$\mathbb{P}(\text{\# of VaR breaches} = k) = \binom{250}{k} 0.01^k 0.99^{250-k}.$$

The expected number of VaR breaches is $250 \times 0.01 = 2.5$.

(ii) Determine numerically the probability of $k = 0, 1, \cdots, 10$ VaR breaches.
Solution: Tabulated below. The first row records $\mathbb{P}(\text{\# of VaR breaches} = k)$ and the second row records $\mathbb{P}(\text{\# of VaR breaches} \geq k)$.

k	0	1	2	3	4	5	6	7	8	9	10
$\mathbb{P}(= k)$	0.0811	0.2047	0.2574	0.2149	0.1341	0.0666	0.0273	0.0097	0.0030	0.0008	0.0002
$\mathbb{P}(\geq k)$	0.0811	0.2858	0.5432	0.7581	0.8922	0.9588	0.9861	0.9958	0.9988	0.9996	0.9998

(iii) Portfolio back test showed 4 VaR breaches. Can one reject, at the confidence level of 95%, the hypothesis that the VaR model is working properly?
Solution: From the second row of the above table,

$$\mathbb{P}(\text{\# of VaR breaches} \geq 5) < 5\%.$$

In other words, given a number of 4 VaR breaches, the hypothesis that the VaR model is working properly cannot be rejected at the confidence level of 95%.