

MTH 9831, FALL 2015. LECTURE 6

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ABSTRACT. We start the lecture with applications of Girsanov's theorem.

1. Application of Girsanov's theorem: finding the joint distribution of the Brownian motion with a constant drift and its running maximum.
2. Application of Girsanov's theorem: risk-neutral measure for a simple market (stock + MMA).
3. Martingale representation theorem. Application to a single stock model: pricing and hedging.
4. Girsanov's theorem and martingale representation theorem for $d > 1$.
5. Multidimensional market model.

1. APPLICATION: FINDING THE JOINT DISTRIBUTION OF THE BROWNIAN MOTION WITH A CONSTANT DRIFT AND ITS RUNNING MAXIMUM.

Let $\tilde{B}(t) = \alpha t + B(t)$, where $(B(t))_{t \geq 0}$ is a standard Brownian motion and $\alpha > 0$ is a constant. Denote by $\tilde{B}^*(T)$ the maximum of \tilde{B} up to time T , that is $\tilde{B}^*(T) = \max_{0 \leq t \leq T} \tilde{B}(t)$. We are going to compute the joint density $f(x, a)$ of $(\tilde{B}(T), \tilde{B}^*(T))$.

Note that since $\tilde{B}^*(T) \geq 0$ and $\tilde{B}^*(T) \geq \tilde{B}(T)$, the density $f(x, a)$ is equal to zero on the set $\{a < 0\} \cup \{a < x\}$. Thus, we need to compute $f(x, a)$ on the set $\{a \geq 0\} \cap \{a \geq x\}$. The idea is to perform a change of measure so that the process \tilde{B} becomes a standard Brownian motion for which we already know the joint distribution of its end point at time T and its maximum of time T .¹ We write

$$(1.1) \quad \begin{aligned} P(\tilde{B}(T) \leq x, \tilde{B}^*(T) \leq a) &= E(\mathbb{1}_{\{\tilde{B}(T) \leq x, \tilde{B}^*(T) \leq a\}}) \\ &= E(\mathbb{1}_{\{\tilde{B}(T) \leq x, \tilde{B}^*(T) \leq a\}} (Z(T))^{-1} Z(T)) = \tilde{E}(\mathbb{1}_{\{\tilde{B}(T) \leq x, \tilde{B}^*(T) \leq a\}} (Z(T))^{-1}), \end{aligned}$$

where $Z(T) = e^{-\alpha B(T) - \alpha^2 T/2}$ and \tilde{E} is the expectation with respect to the new probability measure

$$\tilde{P}(A) = \int_A Z(T) dP.$$

This can be done according to Theorem 5.1(2) of lecture 5 since $E(Z(T)) = 1$ and $P(Z(T) > 0) = 1$. Note that this choice of measure corresponds to the case $\Theta(t) \equiv \alpha$ in Girsanov's theorem. Then the Radon-Nikodym derivative which appears there is exactly

$$Z(T) = \exp \left(- \int_0^T \Theta(t) dB(t) - \frac{1}{2} \int_0^T \Theta^2(t) dt \right) = e^{-\alpha B(T) - \alpha^2 T/2}.$$

Therefore, under \tilde{P} the process \tilde{B} is a standard Brownian motion and

$$\begin{aligned} \tilde{P}(\tilde{B}(T) \leq x, \tilde{B}^*(T) \leq a) &= N\left(\frac{2a-x}{\sqrt{T}}\right) - N\left(-\frac{x}{\sqrt{T}}\right); \\ \tilde{f}(x, a) &= \frac{\partial^2}{\partial x \partial a} \left(N\left(\frac{2a-x}{\sqrt{T}}\right) - N\left(-\frac{x}{\sqrt{T}}\right) \right) = \frac{\partial}{\partial x} \left(N'\left(\frac{2a-x}{\sqrt{T}}\right) \frac{2}{\sqrt{T}} \right) \\ &= \frac{2}{\sqrt{T}} \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{2\pi}} e^{-(2a-x)^2/(2T)} \right) = \frac{2(2a-x)}{T\sqrt{2\pi T}} e^{-(2a-x)^2/(2T)}, \quad x \leq a, a \geq 0. \end{aligned}$$

¹See Theorem 1.8 and Exercise 2 from lecture 2

Since we know the joint density and

$$(Z(T))^{-1} = e^{\alpha B(T) + \alpha^2 T/2} = e^{\alpha(\tilde{B}(t) - \alpha T) + \alpha^2 T/2} = e^{\alpha \tilde{B}(T) - \alpha^2 T/2},$$

(1.1) can be written as

$$\int_{-\infty}^x \int_{-\infty}^a e^{\alpha y - \alpha^2 T/2} \tilde{f}(y, m) dy dm.$$

Differentiating in x and then a we find that

$$f(x, a) = e^{\alpha x - \alpha^2 T/2} \frac{2(2a - x)}{T\sqrt{2\pi T}} e^{-(2a - x)^2/(2T)}, \quad x \leq a, \quad a \geq 0.$$

It is now a just calculus exercise to compute the distribution function and the density of $\tilde{B}^*(T)$.

Exercise 1. Compute the distribution function and the density of $\tilde{B}^*(T)$.

These distributions and, more generally, the change of measure play key roles in pricing of exotic options (see Chapter 7 of the textbook).

2. APPLICATION OF GIRSANOV'S THEOREM: RISK-NEUTRAL MEASURE FOR A SIMPLE MARKET.

Let $(B(t))_{t \geq 0}$ be a standard Brownian motion on Ω, \mathcal{F}, P , $(\mathcal{F}(t))_{t \geq 0}$ be the filtration for this Brownian motion, $\mathcal{F}(t) \subset \mathcal{F}$ for all $t \geq 0$. We shall consider a market with

- an MMA account with an adapted interest rate process $(R(t))_{t \geq 0}$ so that the discounting process $(D(t))_{t \geq 0}$ satisfies

$$(2.1) \quad dD(t) = -R(t)D(t) dt, \quad t \geq 0,$$

$$\text{i.e. } D(t) = e^{-\int_0^t R(u) du},$$

- a single stock whose price under P satisfies the equation

$$(2.2) \quad dS(t) = \alpha(t)S(t) dt + \sigma(t)S(t) dB(t),$$

where $\alpha(t)$, $\sigma(t)$, $t \geq 0$, are adapted processes, $\sigma(t) \neq 0$ a.s.. We know that the solution of (2.2) is given by

$$S(t) = S(0) \exp \left(\int_0^t \sigma(u) dB(u) + \int_0^t \alpha(u) - \sigma^2(u)/2 du \right)$$

which a *generalized geometric Brownian motion (GBM)*

We call the above model a *simple market model*.

Definition 2.1. A risk-neutral measure \tilde{P} (also called an equivalent martingale measure) is a measure on (Ω, \mathcal{F}, P) that is

- equivalent to P and
- under which the discounted stock price process $(D(t)S(t))_{t \geq 0}$ is an $\mathcal{F}(t)$ -martingale.

To find \tilde{P} we calculate $d(D(t)S(t))$ and see if we can get rid of the “ dt ” term. Since $D(t)$ is a regular process, $d[S, D]_t = 0$ and

$$d(D(t)S(t)) = D(t)dS(t) + S(t)dD(t) = \sigma(t)D(t)S(t) \left(dB(t) + \frac{\alpha(t) - R(t)}{\sigma(t)} dt \right).$$

Let $\Theta(t) = (\sigma(t))^{-1}(\alpha(t) - R(t))$ be the market price of risk. It is the excess instantaneous rate of return of the stock (over MMA) per unit of volatility. Set

$$\tilde{P}(A) = \int_A Z dP, \quad \text{where } Z = Z(T) = \exp \left(- \int_0^T \Theta(t) dB(t) - \frac{1}{2} \int_0^T \Theta^2(t) dt \right).$$

Then under \tilde{P} the process $\tilde{B}(t) = B(t) + \int_0^t \Theta(s) ds$, $0 \leq t \leq T$, is a standard Brownian motion, and the process

$$(2.3) \quad d(D(t)S(t)) = \sigma(t)D(t)S(t)d\tilde{B}(t), \quad 0 \leq t \leq T,$$

is an $\mathcal{F}(t)$ -martingale under \tilde{P} . Since $P(Z > 0) = 1$, \tilde{P} is equivalent to P . Moreover, from (2.3) we also get that

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\tilde{B}(t).$$

This shows that the instantaneous rate of return of the stock under \tilde{P} is the same as for the MMA,

$$M(t) := (D(t))^{-1} = \exp\left(\int_0^t R(u)du\right); \quad dM(t) = R(t)M(t)dt.$$

This should explain the name “risk-neutral” measure for \tilde{P} .

Let $X(t)$ be the value of a portfolio which consists of $\Delta(t)$ shares of stock and $\Gamma(t)$ shares of MMA. We shall show that under the self-financing condition (see problem 4.10 of the textbook included in HW 5) not only the discounted stock price but the discounted portfolio value process $(D(t)X(t))_{0 \leq t \leq T}$ is an $\mathcal{F}(t)$ -martingale under \tilde{P} constructed above. We have $X(t) = \Delta(t)S(t) + \Gamma(t)M(t)$ and

$$dX(t) = \Delta(t)dS(t) + S(t)d\Delta(t) + d[S, \Delta]_t + \Gamma(t)dM(t) + M(t)d\Gamma(t) + d[\Gamma, M]_t.$$

The self-financing condition states that

$$S(t)d\Delta(t) + d[S, \Delta]_t + M(t)d\Gamma(t) + d[\Gamma, M]_t = 0, \quad 0 \leq t \leq T.$$

Therefore,

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + \Gamma(t)dM(t) = \Delta(t)dS(t) + (X(t) - \Delta(t)S(t))\frac{dM(t)}{M(t)} \\ &= \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt \\ &= R(t)X(t)dt + \Delta(t)S(t)\sigma(t)\underbrace{(\Theta(t)dt + dB(t))}_{=d\tilde{B}(t)} = R(t)X(t)dt + \Delta(t)S(t)\sigma(t)d\tilde{B}(t). \end{aligned}$$

Finally, since D is a regular process, using the above equation we find that

$$d(D(t)X(t)) = D(t)dX(t) + X(t)\underbrace{\frac{dD(t)}{D(t)}}_{=-R(t)dt} = \Delta(t)D(t)\sigma(t)S(t)d\tilde{B}(t) \stackrel{(2.3)}{=} \Delta(t)d(D(t)S(t))$$

From the last two equalities in the above formula we conclude that

- $(D(t)X(t))_{0 \leq t \leq T}$ is an $\mathcal{F}(t)$ -martingale under \tilde{P} ;
- changes to $\tilde{D}(t)X(t)$ are entirely due to the changes in $D(t)S(t)$.

3. MARTINGALE REPRESENTATION THEOREM (MRT)

Bottom line: we know that stochastic integrals with respect to Brownian motion are martingales. The MRT states that (under conditions) the converse is also true.

Theorem 3.1 (Martingale representation theorem). *Let $(B(t))_{t \geq 0}$ be a Brownian motion on the probability space (Ω, \mathcal{F}, P) and $(\mathcal{F}(t))_{t \geq 0}$ be the filtration generated by the Brownian motion. Let $(M(t))_{0 \leq t \leq T}$ be a square-integrable $\mathcal{F}(t)$ -martingale. Then there is an adapted process $(\Gamma(t))_{0 \leq t \leq T}$ such that*

$$(3.1) \quad M(t) = M(0) + \int_0^t \Gamma(u)dB(u), \quad 0 \leq t \leq T.$$

The proof is omitted.

Remark 3.2. *There is no a priori imposed condition that M has continuous paths. This is one of the conclusions of this theorem, since (3.1) implies that M has continuous paths. Unlike the discrete case (see HW 6, exercise 1), there is no explicit formula for the process Γ .*

Return to a simple market model. While Girsanov's theorem guarantees the existence of the risk-neutral measure for this model, MRT ensures that every contingent claim can be hedged (even though MRT does not say how). We have to assume though that $(\mathcal{F}(t))_{t \geq 0}$ is the natural filtration of the driving Brownian motion. Let $V(T)$ be $\mathcal{F}(T)$ -measurable. $V(T)$ represents the payoff at time T of some (possibly path dependent) derivative security.

Question: what initial capital $X(0)$ and portfolio process $(\Delta(t))_{0 \leq t \leq T}$ ensure a perfect hedge for a short position in the derivative security, i.e. ensure that $X(T) = V(T)$ a.s.?

We need a portfolio process $(\Delta(t))_{0 \leq t \leq T}$ such that $X(T) = V(T)$ a.s., or equivalently, $D(T)X(T) = D(T)V(T)$ a.s.. Define

$$(3.2) \quad V(t) = \frac{1}{D(t)} \tilde{E}(D(T)V(T) | \mathcal{F}(t)), \quad 0 \leq t \leq T.$$

Then $(D(t)V(t))_{0 \leq t \leq T}$ is an $\mathcal{F}(t)$ -martingale under \tilde{P} . By the MRT, there is an adapted process $(\tilde{\Gamma}(t))_{0 \leq t \leq T}$ such that

$$D(t)V(t) = \underbrace{D(0)}_{=1} V(0) + \int_0^t \tilde{\Gamma}(u) \tilde{B}(u), \quad 0 \leq t \leq T.$$

In order to have $V(t) = X(t)$, $0 \leq t \leq T$, we should choose $V(0) = X(0)$ and, since

$$D(t)X(t) = X(0) + \int_0^t D(u)\Delta(u)\sigma(u)S(u) d\tilde{B}(u),$$

we should also choose $\Delta(t)$ so that

$$D(t)\Delta(t)\sigma(t)S(t) = \tilde{\Gamma}(t), \quad 0 \leq t \leq T.$$

Since $D(t)$ and $S(t)$ are positive and $\sigma(t) \neq 0$ a.s. by assumption

$$\Delta(t) = \frac{\tilde{\Gamma}(t)}{D(t)\sigma(t)S(t)}, \quad 0 \leq t \leq T.$$

Thus, this choice of $X(0)$ and $(\Delta(t))_{0 \leq t \leq T}$ provides a perfect hedge for this derivative security. Since the claim $V(T)$ was arbitrary, we conclude that every contingent claim can be hedged, i.e. the model is complete.

Let us turn now to pricing. Once the portfolio process $(\Delta(t))_{0 \leq t \leq T}$ is shown to exist, we have that

$$D(t)X(t) = X(0) + \int_0^t D(u)\Delta(u)S(u)\sigma(u) d\tilde{B}(u)$$

is a martingale, and, thus,

$$D(t)X(t) = \tilde{E}(D(T)X(T) | \mathcal{F}(t)) = \tilde{E}(D(T)V(T) | \mathcal{F}(t)) = D(t)V(t).$$

$X(t)$ is the capital needed at time t to hedge a short position in the derivative security with payoff $V(T)$. Thus, we can refer to $X(t)$ (and $V(t)$) as the price of the derivative security at time t . In short, (3.2) becomes our risk-neutral pricing formula for a single stock model:

$$\begin{aligned} V(t) &= \frac{1}{D(t)} \tilde{E}(D(T)V(T) | \mathcal{F}(t)) = \tilde{E}\left(\frac{D(T)}{D(t)} V(T) | \mathcal{F}\right) \\ &= \tilde{E}\left(e^{-\int_t^T R(u) du} V(T) | \mathcal{F}(t)\right), \quad 0 \leq t \leq T. \end{aligned}$$

Example 3.3 (Black-Scholes-Merton (BSM) model). We assume that $R(t) \equiv r \geq 0$, $\sigma(t) \equiv \sigma > 0$, and $\alpha(t)$ a stochastic process adapted to the filtration generated by the driving Brownian motion. We shall compute the price of a European derivative security with payoff $V(T) = f(S(T))$, where f is a non-negative Borel function. We already know that under the risk-neutral measure \tilde{P}

$$dS(t) = rS(t) dt + \sigma S(t) d\tilde{B}(t),$$

and $S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma\tilde{B}(t)}$. We need to compute

$$V(t) = \tilde{E} \left(e^{-r(T-t)} f(S(T)) \mid \mathcal{F}(t) \right) = e^{-r(T-t)} \tilde{E} (f(S(T)) \mid \mathcal{F}(t)), \quad 0 \leq t \leq T.$$

Writing $S(T) = S(t)e^{(r-\sigma^2/2)(T-t)+\sigma(\tilde{B}(T)-\tilde{B}(t))}$ and using the independence lemma ($\tilde{B}(T) - \tilde{B}(t) =: \sqrt{T-t}Z$ is independent of $\mathcal{F}(t)$ and $S(t)$ is $\mathcal{F}(t)$ -measurable) we get that

$$\tilde{E} \left(f \left(S(t)e^{(r-\sigma^2/2)(T-t)+\sqrt{T-t}Z} \right) \mid \mathcal{F}(t) \right) = g(t, S(t)),$$

where

$$g(t, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f \left(x e^{(r-\sigma^2/2)(T-t)+\sqrt{T-t}z} \right) e^{-z^2/2} dz.$$

We have accomplished two things: (1) we have found a closed form formula for the price

$$V(t) = e^{-r(T-t)} g(t, S(t)), \quad 0 \leq t \leq T,$$

and (2) we have shown that a GBM with a constant drift and volatility parameters is a Markov process.

Taking $f(x) = (x - K)_+$ and computing the integral will give a standard BSM formula. The details are left as an exercise.²

4. GIRSANOV'S THEOREM AND MARTINGALE REPRESENTATION THEOREM FOR $d > 1$

Let (Ω, \mathcal{F}, P) be a probability space and $B = (B(t))_{t \geq 0}$, $B(t) = (B_1(t), B_2(t), \dots, B_d(t))^T$ be a standard d -dimensional Brownian motion on it. Measure P will be called “market”, “real world”, or “actual” probability measure, i.e. the one that can be observed from the market data. Denote by $(\mathcal{F}(t))_{t \geq 0}$ a filtration associated with Brownian motion B (not necessarily the one generated by B). Time horizon, or expiration date, will be denoted by T . We shall restrict everything to times $0 \leq t \leq T$ and also assume that $\mathcal{F}(T) = \mathcal{F}$.

Theorem 4.1 (Girsanov, $d \geq 1$). *Let $\Theta(t) = (\Theta_1(t), \Theta_2(t), \dots, \Theta_d(t))$, $0 \leq t \leq T$, be an $\mathcal{F}(t)$ -adapted process. Define*

$$Z(t) = \exp \left(- \int_0^t \Theta(u) dB(u) - \frac{1}{2} \int_0^t \|\Theta(u)\|^2 du \right), \quad \tilde{B}(t) = B(t) + \int_0^t \Theta(u) du,$$

where $\|\Theta(t)\|^2 = \sum_{i=1}^d \Theta_i^2(t)$. Assume that

$$(4.1) \quad E \int_0^T \|\Theta(t)\|^2 Z^2(t) dt < \infty.$$

Set $Z = Z(T)$. Then $E(Z) = 1$, and under the probability measure

$$(4.2) \quad \tilde{P}(A) = \int_A Z(\omega) dP(\omega)$$

the process $\tilde{B}(t)$, $0 \leq t \leq T$, is a standard d -dimensional Brownian motion.

The proof is similar to the one-dimensional version and uses a d -dimensional version of Lévy's characterization.

²Or see p. 219-220 of the textbook, where you have to use $y = -z$ as a variable of integration.

Theorem 4.2 (MRT, $d \geq 1$). *Assume that $(\mathcal{F})_{0 \leq t \leq T}$ is the filtration generated by the standard d -dimensional Brownian motion $(B(t))_{0 \leq t \leq T}$, $B(t) = (B_1(t), B_2(t), \dots, B_d(t))^T$. Let $(M(t))_{0 \leq t \leq T}$ be a (one-dimensional) $\mathcal{F}(t)$ -martingale with respect to P . Then there is an adapted d -dimensional process $(\Gamma(t))_{0 \leq t \leq T}$, $\Gamma(t) = (\Gamma_1(t), \Gamma_2(t), \dots, \Gamma_d(t))$, such that*

$$M(t) = M(0) + \int_0^t \Gamma(u) dB(u), \quad 0 \leq t \leq T.$$

If, in addition, we assume the notation and conditions of Theorem 4.1 and if $(\widetilde{M}(t))_{0 \leq t \leq T}$ is an $\mathcal{F}(t)$ -martingale with respect to \widetilde{P} , then there is an adapted d -dimensional process $(\widetilde{\Gamma}(t))_{0 \leq t \leq T}$, such that

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u) d\widetilde{B}(u), \quad 0 \leq t \leq T.$$

5. MULTIDIMENSIONAL MARKET MODEL

We shall consider a model which consists of m stocks (driven by d independent Brownian motions with the associated filtration $(\mathcal{F}(t))_{0 \leq t \leq T}$) and a money market account (MMA).

- MMA: interest rate $(R(t))_{0 \leq t \leq T}$, a non-negative adapted process. As before we introduce the discounting factor $D(t) = e^{-\int_0^t R(u) du}$, $0 \leq t \leq T$, which satisfies $D'(t) = -R(t)D(t)$, or $dD(t) = -R(t)D(t)dt$, and is a regular process.
- m stocks with the price vector $S(t) = (S_1(t), S_2(t), \dots, S_m(t))^T$, $0 \leq t \leq T$. We assume that for some adapted vector process $\alpha(t) = (\alpha_1(t), \alpha_2(t), \dots, \alpha_m(t))^T$, $0 \leq t \leq T$, and an $m \times d$ matrix process $\sigma(t) = \|\sigma_{ij}(t)\|_{i=1,2,\dots,m;j=1,2,\dots,d}$ such that

$$\sigma_i^2(t) := \sum_{j=1}^d \sigma_{ij}^2(t) > 0 \quad \text{for all } 0 \leq t \leq T \text{ (a.s.) and } i = 1, 2, \dots, m,$$

the price vector satisfies

$$(5.1) \quad dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t) \sum_{j=1}^d \sigma_{ij}(t) dB_j(t), \quad m = 1, 2, \dots, m.$$

Just as in an example with two stocks we shall first show that each of $(S_i(t))_{0 \leq t \leq T}$, $i = 1, 2, \dots, m$, is a generalized geometric Brownian motion. To see this, define $W_i(0) = 0$,

$$(5.2) \quad dW_i(t) = \frac{1}{\sigma_i(t)} \sum_{j=1}^d \sigma_{ij}(t) dB_j(t), \quad i = 1, 2, \dots, m.$$

Then $(W_i(t))_{0 \leq t \leq T}$ is a continuous martingale, and (using $d[B_i, B_j]_t = 0$, $k \neq j$, and $d[B_j]_t = dt$)

$$d[W_i]_t = \frac{1}{\sigma_i^2(t)} \sum_{j=1}^d \sigma_{ij}^2(t) dt = dt.$$

By Lévy's theorem we conclude that $(W_i(t))_{0 \leq t \leq T}$ is a standard Brownian motion. Rewriting the equation for $dS_i(t)$ in terms of the newly defined Brownian motion we get

$$(5.3) \quad dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t)\sigma_i(t)dW_i(t), \quad i = 1, 2, \dots, m,$$

and conclude that each S_i is a generalized geometric Brownian motion.

Next we discuss how different stock prices, S_k and S_ℓ , are correlated. They are correlated through their driving Brownian motions W_k and W_ℓ ,

$$d[W_k, W_\ell]_t \stackrel{(5.2)}{=} \frac{1}{\sigma_k(t)\sigma_\ell(t)} \sum_{j=1}^d \sum_{n=1}^d \sigma_{kj}(t)\sigma_{\ell n}(t)d[B_j, B_n]_t = \frac{1}{\sigma_k(t)\sigma_\ell(t)} \sum_{j=1}^d \sigma_{kj}(t)\sigma_{\ell j}(t)dt =: \rho_{k\ell}(t)dt.$$

The process $(\rho_{k\ell}(t))_{0 \leq t \leq T}$ is called *instantaneous correlation* of W_k and W_ℓ .

Exercise 2. Use Itô's product rule to show that $\text{Cov}(W_k(t), W_\ell(t)) = E \int_0^t \rho_{k\ell}(u) du$.

Computing the cross variation between S_k and S_ℓ we get

$$d[S_k, S_\ell]_t \stackrel{(5.3)}{=} \rho_{k\ell}(t) S_k(t) S_\ell(t) \sigma_k(t) \sigma_\ell(t) dt.$$

In the case when $\rho_{k\ell}(t)$, $\sigma_k(t)$ and $\sigma_\ell(t)$, $0 \leq t \leq T$, are non-random we can explicitly compute the covariance and correlation of $S_k(t)$ and $S_\ell(t)$ just as we did in Section 3 of lecture 5.

Finally, using Itô-Doeblin formula and (5.3) we can show that

$$d(D(t)S_i(t)) = D(t)S_i(t)((\alpha_i(t) - R(t))dt + \sigma_i(t)dW_i(t)), \quad i = 1, 2, \dots, m.$$

Exercise 3. Derive the above formula.

Next time we shall address the question of existence and uniqueness of a risk-neutral measure for this model.