12. FX Derivatives

FX forwards

An FX forward is the simplest foreign exchange derivative contract. It is an agreement at some future time T to receive the amount N_A in currency A and pay the amount N_B in currency B. As such, its value depends upon what the exchange rate between the two currencies is when time T arrives.

The simplest approach to pricing this derivative is to discount the cash flows. We suppose that the relevant discount factors $P_{A,T}$ and $P_{B,T}$ are known and can discount each. For convenience, it is often useful to express these values in terms of the risk-free zero rates r_A and r_B in each currency.

Denominated in currency A, the present value of receiving N_A at T is...

$$V_{receive} = N_A P_{A,T} = N_A e^{-r_A T}$$

...while the present value denominated in currency B of paying N_B at T is...

$$V_{pay} = -N_B P_{B,T} = -N_B e^{-r_B T}$$

Since the two values are denominated in different currencies, to determine the value of the deal we must choose in which currency to express the value. Take $X_{A,0}$ to be the value of one unit of currency B denominated in currency A. Then the value of the deal is:

$$V_A = N_A e^{-r_A T} - X_{A,0} N_B e^{-r_B T}$$

This can be more conveniently seen if we express the terms of the deal in terms of a strike FX rate $X_{A,K}$ by...

$$X_{A,K} = \frac{N_A}{N_B}$$

...which is the the value of one unit of currency B, expressed in currency A, at which the exchange will take place. Then we may express the present value of the contract in currency A as:

$$V_A = N_A e^{-r_A T} - X_{A,0} N_B e^{-r_B T} = N_B \left(X_{A,K} e^{-r_A T} - X_{A,0} e^{-r_B T} \right)$$

That is, the FX forward can be seen in currency A as a contract on a quantity N_B of the foreign currency, and its value can be expressed in terms of the spot exchange rate and the strike exchange rate defined by the contract.

It is worthwhile wondering what the fair strike exchange rate $X^*_{A,K}$ causes the initial value of this contract to be zero. Since the contract is linear in the constant N_B, this condition holds when...

$$X^*_{A,K} e^{-r_A T} - X_{A,0} e^{-r_B T} = 0$$

$$X^*_{A,K} = X_{A,0} e^{(r_A - r_B)T} = X_{A,F}$$

This quantity is known as the forward exchange rate for the given time T.

Symmetry of the value of an FX forward

In the previous discussion, we denominated all quantities to currency A. Under this choice, if we denote the quantity $X_{A,0} = X_0$ as the spot exchange rate, $r_A = r$ as the domestic-currency interest rate, and $r_B = r_f$ as the foreign-currency interest rate, then the forward FX rate $X_{A,F} = X_F$ for time T is given by:

$$X_F = X_0 e^{(r-r_f)T}$$

In currency A, our valuation formula arrived at above describes the value of a contract to pay foreign in the quantity $N_B = N_f$, the foreign-currency quantity, at the strike rate $X_{A,K} = X_K$. Thus, restating the formula, the value of the contract is:

$$V_{pay} = N_f \left(X_K e^{-rT} - X_0 e^{-r_f T} \right)$$

It seems sensible to wonder whether analogous expressions hold if we instead choose to value the contract in currency B. To do this, we note that if a given exchange rate X_A at some time describes the value of one unit of currency B in currency A, then the corresponding rate in currency B $X_B = 1 / X_A$ gives the value of one unit of currency A denominated in currency B.

Thus in converting our original valuation formula, which is denominated in currency A...

$$V_A = N_A e^{-r_A T} - X_{A,0} N_B e^{-r_B T}$$

...into currency B, we need to divide by the quantity we have denoted above as $X_{A,0}$, the value of one unit of currency B in currency A at pricing time. Under this choice, we see:

$$V_B = \frac{V_A}{X_{A,0}} = \frac{1}{X_{A,0}} N_A e^{-r_A T} - N_B e^{-r_B T}$$

As we have seen already:

$$X_{B,0} = \frac{1}{X_{A,0}}$$

Also...

$$X_{B,K} = \frac{1}{X_{A,K}}$$

while our original definition of $X_{A,K}$ was given by...

$$X_{A,K} = \frac{N_A}{N_B}$$

...and so:

$$X_{B,K} = \frac{N_B}{N_A}$$

Thus, we may rewrite our valuation formula:

$$V_B = N_A\left(X_{B,0}e^{-r_A T} - X_{B,K}e^{-r_B T}\right)$$

When we price in currency B, the foreign-currency quantity $N_f = N_A$, while the domestic and foreign rates are respectively r_B and r_A. So we may rewrite this expression as...

$$V_B = N_f\left(X_0 e^{-r_f T} - X_K e^{-rT}\right)$$

Finally, we note that since the agreement is to receive the currency A quantity and pay the currency B quantity, then from the standpoint of a currency B investor this is a contract to receive foreign. Thus we see that:

$$V_{receive} = N_f\left(X_0 e^{-r_f T} - X_K e^{-rT}\right) = -N_f\left(X_K e^{-rT} - X_0 e^{-r_f T}\right) = -V_{pay}$$

...which establishes the desired result. In either currency, the expressions for valuing an FX forward contract to pay/receive are the same, and converting the result of a formula in one currency into the other gives the same result. Thus, the valuation method described here is fully consistent.

Forward contracts revisited

Earlier, we used a hedging argument to establish that the forward price at future time T of an asset with spot price S_0 paying a continuous dividend at the rate q when the interest rate is r is given by...

$$F = S_0 e^{(r-q)T}$$

We went on to note that a foreign currency behaves very much like a dividend-paying asset, with the foreign interest rate taking on the role of the dividend rate q. Thus, it should come as no surprise that in the above we found by other means that the forward exchange rate is given by...

$$X_F = X_0 e^{(r-r_f)T}$$

While many concepts relating for foreign exchange can become confusing at times, it must be reiterated that there nothing essentially more difficult with pricing these derivatives than on the more intuitive derivatives of equity prices. After all, if we enter into a forward contract to pay strike K on a quantity Q of the asset described above, we saw relatively easily that the present value of this contract to receive the asset is:

$$V_{receive} = Q\left(S_0 e^{-qT} - Ke^{-rT}\right)$$

In precisely the same way, the contract to receive the amount N_f of foreign currency has value...

$$V_{receive} = N_f\left(X_0 e^{-r_f T} - X_K e^{-rT}\right)$$

Any additional confusion caused by the FX case arises from the fact that either of the two currencies involved may be taken to be the pricing currency, while there seems to be no choice of currency available in the case of an equity forward. We will see in a later chapter that, on the contrary, it is not only possible but sometimes quite convenient to price derivatives on an equity in the "currency" of the underlying or some closely related asset rather than in the currency of the payoff. Thus, the concepts we will apply here in the context of currency exchanges apply generally across a whole range of assets, not only currencies.

Vanilla FX options and put-call duality

Given the correspondence noted above between derivatives on an equity and derivatives on a currency, it seems sensible to imagine that the results we have previously developed for vanilla equity options apply likewise to options on currencies. While the terms in which the deal are expressed may differ somewhat, indeed it is true that all of the dynamic hedging arguments we have seen so far, and the results we were able to obtain from them, hold in the case of FX options as well.

As in the equity case, a forward contract on an FX rate is linear and has both upside and downside realizations. We construct European options by considering only the positive part of the forward contract's payoff. Thus, in an FX option one has the right, but not the obligation, to receive the quantity N_A in currency A while paying N_B in currency B at the future time T.

What option is this from the standpoint of an investor whose native currency is currency A? To this investor, the option is to receive a fixed payment in his or her currency in exchange for giving up a certain amount of the foreign asset. Seen this way, the investor will only exercise if the foreign asset's price has *declined* enough to make the exchange favorable. Thus, this deal is a European *put option* on currency B, as seen from the standpoint of a currency A investor.

Taking this as our guide, we define the foreign-currency quantity $N_f = N_B$ and the strike FX rate of the put $X_K = N_A / N_B$, which like all exchange rates we express as the price of one unit of currency B denominated in our chosen pricing currency. As before, r is the pricing currency (currency A) risk-free rate to T and r_f is the foreign currency (currency B) risk-free rate to the same maturity. Then the Black-Scholes value of this option, denominated in currency A, is:

$$V_{put} = N_f\left[X_K e^{-rT} N(-d_2) - X_0 e^{-r_f T} N(-d_1)\right]$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{X_0}{X_K}\right) + \left(r - r_f + \frac{1}{2}\sigma^2\right)T\right]$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

As suggested by our discussion of forwards above, if from the standpoint of a currency A investor this is a contract to pay some quantity of foreign asset, then from the standpoint of a currency B investor this same deal will be a contract to *receive* some quantity of foreign asset. Seen from the perspective of currency B, the deal offers the option to pay a fixed amount at T and receive a quantity of foreign asset. Obviously, the currency-B investor will only exercise this option if the foreign asset's price has *increased* enough to make the exchange favorable. Thus, this deal is a European *call option* on currency A, as seen from the standpoint of a currency B investor. Then the Black-Scholes pricing formula for the deal is:

$$V_{call} = N_f\left[X_0 e^{-r_f T} N(d_1) - X_K e^{-rT} N(d_2)\right]$$
$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{X_0}{X_K}\right) + \left(r - r_f + \frac{1}{2}\sigma^2\right)T\right]$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

That is, depending upon the currency taken as the pricing currency, this deal is either a put or a call. Furthermore, it can be shown that, as expected, the value obtained using one of these formulas, if converted to the other currency at the spot FX rate, is equal to the value obtained using the other. This should in the end come as no surprise; they are, after all, the very same deal.

A problem with volatility: Siegel's FX paradox

We have previously seen that the forward FX rate obtained in currency A for a cross with currency B...

$$X_{F,A} = X_{0,A} e^{(r_A - r_B)T}$$

...agrees with the one obtained in currency B for a cross with currency A...

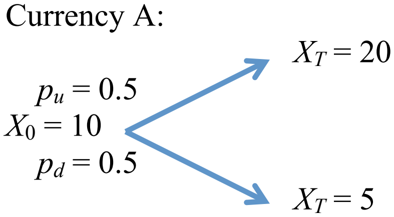
$$X_{F,B} = X_{0,B} e^{(r_B - r_A)T} = X_{0,A}^{-1}\left(e^{(r_A - r_B)T}\right)^{-1} = X_{F,A}^{-1}$$

...as expected in any arbitrage-free pricing framework. Further, we have asserted that option prices in the two currencies on the same cross are consistent with one another.

The introduction of volatility in the framework of option pricing raises a sometimes confusing set of questions about how one accommodates the change from one currency to another. So, for example, consider a simple single-period binomial model for a cross between currencies A and B.

Suppose the spot price of a single unit of currency B is 10 units of currency A. Under our model, at some future time the price of unit of currency B may either be 20 units of currency A or 5 units of currency A. From interest rates we know that the forward FX rate is 12.5; therefore, the risk-neutral probabilities of an up or a down transition are the same: one half.

Currency A:



For each of these realizations, we can of course take the reciprocal and determine the value of one unit of currency A denominated in currency B. Using this fact alone gives the following speculative model of the cross seen in currency B:

Currency B(?):

$$
\begin{array}{ccc}
 & & X_T = 0.2 \\
p_u = 0.5 & \nearrow & \\
X_0 = 0.1 & & \\
p_d = 0.5 & \searrow & \\
 & & X_T = 0.05
\end{array}
$$

We can easily enough determine what the forward is under this model, since 0.5*(0.2+0.05) = 0.125. Taking the reciprocal, we see that this model produces the forward FX rate of 8, expressed as the price in currency A of one unit of currency B. Unfortunately, this differs from the forward obtained from the original model.

How can this be? Even though we have shown by other means that the forwards calculated in either currency must agree, our introduction of a random character to the FX rate appears to have caused some problems with the model. The fact that these two views of the situation do not agree is known as Siegel's FX paradox, and in this simple situation it is not difficult to see what the solution must be.

Recall that in our earlier discussions of binomial models, we were forced to choose a particular set of probabilities for the up and down moves. These *risk-neutral* probabilities were chosen not because they were the actual likelihoods of these moves, but because they were the unique probabilities that caused the forward price returned by the model to match the arbitrage-free forward price. This was the fundamental requirement of constructing the model so that it could be used to return arbitrage-free prices of other derivatives on the underlying.

In our original currency A model, we chose up and down probabilities in just such a way. The fact that transferring this same model with its risk-neutral probabilities to currency B did not have the same result is, of course, not due to some error in how the FX rate is calculated in each state. It is due to the fact that the risk-neutral probabilities of various outcomes in currency B *are not the same* as those in currency A.

Indeed, it is our knowledge of the forward that allows us to construct the correct model:

Currency B:

$$
\begin{array}{ccc}
 & & X_T = 0.2 \\
p_u = 0.2 & \nearrow & \\
X_0 = 0.1 & & \\
p_d = 0.8 & \searrow & \\
 & & X_T = 0.05
\end{array}
$$

This fact—that one must choose different risk-neutral probabilities of *the same outcomes* in order to obtain arbitrage-free prices in different currencies, may be difficult to accept. Still, it seems no less plausible than the previously encountered assertion that, in order for prices to be arbitrage-free, all non-dividend-paying assets must grow in expectation at the risk-free rate. It is a consequence of the formal requirements of risk-neutral pricing, not a statement of the real-world probabilities involved.

FX rates in continuous time

The above paradox has an analogous version in continuous time. Reasoning once again by analogy with equities, if we assume a geometric Brownian motion for the price of a unit of currency B denominated in currency A, then we would expect the stochastic differential equation governing its evolution to be:

$$dX_{A,t} = (r_A - r_B) X_{A,t} dt + \sigma X_{A,t} dW_{A,t}$$

Likewise, we would expect that the price of a unit of currency A denominated in currency B shows the analogous dynamics:

$$dX_{B,t} = (r_B - r_A) X_{B,t} dt + \sigma X_{B,t} dW_{B,t}$$

It is worth wondering merely on the basis of these two expressions whether the parameter σ is indeed, as we have represented it here, the same in the two cases. At least as important, it is also worth wondering what relationship must exist between the random increments we have her labeled $dW_{A,t}$ and $dW_{B,t}$, since common sense suggests that the two cannot be utterly different. Since in the end each of these describes the dynamics of the same cross, simply seen from different sides, we would expect that one is a function of the other.

To investigate the question, we will take advantage once again of the previously seen fact that $X_B = 1 / X_A$ at each moment in time. Since $1 / X_A$ is a function of a random variable whose dynamics are known, we may take this to be the function f in the previously encountered expression for Ito's Lemma:

$$df = f_T dt + f_X dX + \frac{1}{2} f_{XX} dX dX$$

In this case (omitting most subscripts for the sake of compactness):

$$f = \frac{1}{X}$$

$$f_T = 0$$

$$f_X = -\frac{1}{X^2}$$

$$f_{XX} = \frac{2}{X^3}$$

$$dX = (r_A - r_B) X dt + \sigma X dW_A$$

$$dX dX = \sigma^2 X^2 dt$$

With these results in mind, we conclude that...

$$d\left[\frac{1}{X}\right] = -\frac{1}{X^2}\left[\left(r_A - r_B\right)Xdt + \sigma X dW_A\right] + \frac{1}{2}\frac{2}{X^3}\sigma^2 X^2 dt$$

$$d\left[\frac{1}{X}\right] = \left(r_B - r_A\right)\frac{1}{X}dt - \sigma\frac{1}{X}dW + \sigma^2\frac{1}{X}dt$$

$$d\left[\frac{1}{X}\right] = \left(r_B - r_A + \sigma^2\right)\frac{1}{X}dt - \sigma\frac{1}{X}dW_A$$

This expression is actually quite close to the one we sought for the alternative exchange rate $dX_{B,t}$. Note that the volatility σ is preserved by this transformation using Ito's Lemma. The negative coefficient is of little concern, thanks to the symmetry of Wiener processes; the properties of the process dW coincide precisely with those of its reflection $-dW$.

The only remaining difference is the additional nonrandom return term $\sigma^2 dt$. Since the function we have applied, $1 / x$, is convex, it comes as no surprise that the "extra" return that appears in this expression is positive, thanks to Jensen's inequality. And as in our simple binomial case, this expression suggests a possible method of eliminating this excess drift: Choose a different probability measure—the risk-neutral measure for currency B—under which the unwanted drift term disappears.

Change of measure: Discrete Case

The change-of-measure technique in stochastic calculus is a fundamental and rather technical aspect of arbitrage-free pricing. A full treatment of its formal nature and justification is not appropriate here. Instead, it is important for us to understand the practical value of the technique.

Generally, as we have already seen the technique is one that assigns alternate probabilities to a set of outcomes. This can be thought of as a random variable Y that serves as a multiplier for the original probabilities. In order for the resulting set of probabilities to be consistent[1] with the original, we must have a few features that a look at our binomial model from the previous section may make clearer.

In the original currency A case, in the state we will call "up" the value of a unit of currency B became 20; in the state we will call "down" it became 5. Now consider the random variable:

$$Y = \begin{cases} 1.6 & up \\ 0.4 & down \end{cases}$$

---

[1] Again, in the interest of not getting bogged down in formalism we will not define precisely what is meant by "consistency." This and other technical features alluded to here are discussed at sufficient length in any stochastic calculus course.

Two properties of this random variable are worth noting before we proceed. First, both are strictly positive. Second, the expected value of $Y = 1$. Any Y satisfying these properties can be used to create a valid change of measure for this model.

We can then take the expectation of any function g of the final price, which we call X here, under the new measure by:

$$E_Z\left[g(X)\right] = E\left[g(X)Y\right]$$

The properties above of Y ensure that the expectation is valid and that its value is in some sense consistent with the original (no possible outcome from the original measure is assigned a probability of zero). That is:

$$E\left[g(X)Y\right] = g\left(X_{up}\right)Y_{up}p_{up} + g\left(X_{down}\right)Y_{down}p_{down}$$

$$= g\left(X_{up}\right)p_{Y,up} + g\left(X_{down}\right)p_{Y,down} = E_Y\left[g\left(X\right)\right]$$

Of course this value is not the same as the expectation of $g(X)$ under the original measure. Y has been chosen specifically so that it satisfies the needed properties *and* results in a risk-neutral measure when we consider the question from the standpoint of currency B. The advantage, though, is that we have been able to make the change by redefining the *proababilities* and leaving the rest of the model intact.

To verify that our Y accomplishes the change of measure, we take $g(X_T) = 1 / X_T$. Then:

$$E\left[g\left(X_T\right)Y\right] = \frac{1}{X_{T,up}}p_{up}Y_{up} + \frac{1}{X_{T,down}}p_{down}Y_{down} = \frac{1}{20}*0.5*1.6 + \frac{1}{5}*0.5*0.4$$

$$E_Y\left[g\left(X_T\right)\right] = \frac{1}{20}*0.8 + \frac{1}{5}*0.2 = 0.08 = \frac{1}{12.5}$$

...confirming that this change of measure leads to the desired result.

Girsanov's Theorem: Continuous Case

The famous theorem that shows how this is done for Brownian motion is known as Girsanov's theorem. As one might expect in the case of a continuous-time process that can take on any real value, the form this technique takes is a bit different than in the discrete case. Nevertheless, in essence the method is the same: It preserves the fundamental structure of the model while reweighting the probabilities of each outcome.

Returning to our original dynamics for $1 / X_A$ and rewriting slightly, we have the expression:

$$dX_B = \left(r_B - r_A + \sigma^2\right)X_B dt - \sigma X_B dW_A$$

Our goal is to define the measure-changed increment dW_B in such a way that the additional drift term involving the volatility is canceled. This can be done by defining:

$$dW_B = dW_A - \sigma dt$$

...making...

$$dX_B = (r_B - r_A)X_B dt + \sigma^2 X_B dt - \sigma X_B dW_A = (r_B - r_A)X_B dt - \sigma X_B (dW_A - \sigma dt)$$
$$dX_B = (r_B - r_A)X_B dt + \sigma X_B dW_B$$

It may be pointed out with some justice that this looks more like a formal trick than any real change. After all, as in our discrete case we have merely chosen the new measure in order to arrive at the result we wanted. The value of the theorem is, first, in assuring us that this change is valid. More usefully, it makes explicit what the process Y looks like in this case, and can in some cases even allow us to transform expectations from one measure relatively easily into another. Of particular importance is the fact that this measure change doesn't alter the volatility of the Brownian motion.

Returning to the measure defined by currency A for a moment, consider our original view of $X_B = 1 / X_A$:
$$dX_B = (r_B - r_A + \sigma^2)X_B dt - \sigma X_B dW_A$$
$$X_{B,T} = X_{B,0} \exp\left\{\left(r_B - r_A + \frac{1}{2}\sigma^2\right)T - \sigma W_A\right\}$$

W_A is as usual a normally distributed random variable with mean zero and variance T. Thus, taking the expectation in this measure yields:
$$E_A[X_{B,T}] = \int X_{B,0} \exp\left\{\left(r_B - r_A + \frac{1}{2}\sigma^2\right)T - \sigma W_A\right\} dP_A =$$
$$\int X_{B,0} \exp\left\{\left(r_B - r_A + \frac{1}{2}\sigma^2\right)T - \sigma\sqrt{T}z\right\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}z^2\right\} dz =$$
$$X_{B,0} \exp\left\{\left(r_B - r_A + \frac{1}{2}\sigma^2\right)T\right\} \frac{1}{\sqrt{2\pi}} \int \exp\left\{-\frac{1}{2}\left(z^2 + 2\sigma\sqrt{T}z\right)\right\} dz =$$
$$X_{B,0} \exp\left\{\left(r_B - r_A + \sigma^2\right)T\right\} \frac{1}{\sqrt{2\pi}} \int \exp\left\{-\frac{1}{2}\left(z^2 + 2\sigma\sqrt{T}z + \sigma^2 T\right)\right\} dz =$$
$$X_{B,0} \exp\left\{\left(r_B - r_A + \sigma^2\right)T\right\} \frac{1}{\sqrt{2\pi}} \int \exp\left\{-\frac{1}{2}\left(z + \sigma\sqrt{T}\right)^2\right\} dz =$$
$$X_{B,0} \exp\left\{\left(r_B - r_A + \sigma^2\right)T\right\}$$

As we did in the discrete case, we define a new measure using the random variable Y. Without expressing Girsanov's theorem in its more general form, here we simply state the result for the particular change of measure defined above:
$$Y_T = \exp\left\{\sigma W_A - \frac{1}{2}\sigma^2 T\right\}$$

It's quite clear that the change of measure results in what we know to be the proper forward price:

$$E_A\left[X_{B,T}Y_T\right] = \int X_{B,0}\exp\left\{\left(r_B - r_A + \frac{1}{2}\sigma^2\right)T - \sigma W_A\right\}\exp\left\{\sigma W_A - \frac{1}{2}\sigma^2 T\right\}dP_A =$$

$$X_{B,0}\exp\left\{\left(r_B - r_A\right)T\right\}\int\exp\left\{\frac{1}{2}\sigma^2 T - \sigma W_A + \sigma W_A - \frac{1}{2}\sigma^2 T\right\}dP_A =$$

$$X_{B,0}\exp\left\{\left(r_B - r_A\right)T\right\}\int dP_A = X_{B,0}\exp\left\{\left(r_B - r_A\right)T\right\}$$

Furthermore, Y is clearly positive everywhere. The only property remaining to verify is its expected value under A:

$$E_A\left[Y_T\right] = \int\exp\left\{\sigma W_A - \frac{1}{2}\sigma^2 T\right\}dP_A = \int\exp\left\{\sigma\sqrt{T}z - \frac{1}{2}\sigma^2 T\right\}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}z^2\right\}dz =$$

$$\exp\left\{-\frac{1}{2}\sigma^2 T\right\}\frac{1}{\sqrt{2\pi}}\int\exp\left\{-\frac{1}{2}\left(z^2 - 2\sigma\sqrt{T}z\right)\right\}dz = 1$$

...using our by now very familiar method of completing the square.

FX Options: Delta and Risk-Neutral Probability of Exercise

We've seen that the particular change of measure above is required in order to move from the risk-neutral measure induced by currency A to the risk-neutral measure induced by currency B. In the process we assign new probabilities to the various outcomes at each future time T upon which a vanilla option's value depends.

Further, we have previously seen that a put on currency B denominated in currency A is equivalent to a call on currency A denominated in currency B.

The delta of our put on currency B is the usual Black-Scholes delta, which we can express as:

$$\Delta_{A,put} = e^{-r_B T}N\left(-\frac{1}{\sigma\sqrt{T}}\left(\ln\frac{X_{A,0}}{X_{A,K}} + \left(r_A - r_B + \frac{1}{2}\sigma^2\right)T\right)\right)$$

We note that the portion of the d_1 term that is related to the drift, $(r_A - r_B + \frac{1}{2}\sigma^2)T$, differs from the actual drift of the log of the exchange rate in this measure, $(r_A - r_B - \frac{1}{2}\sigma^2)T$, by $\sigma^2 T$, which curiously is also the drift adjustment we had to make in the measure change to currency B. Further, as we have already noted the strike and spot price of the deal when represented in currency B are the reciprocals of their corresponding values in currency A. So it follows that:

$$\Delta_{A,put} = -e^{-r_B T}N\left(-\frac{1}{\sigma\sqrt{T}}\left(-\ln\frac{X_{B,0}}{X_{B,K}} - \left(r_B - r_A - \frac{1}{2}\sigma^2\right)T\right)\right) =$$

$$-e^{-r_B T}N\left(\frac{1}{\sigma\sqrt{T}}\left(\ln\frac{X_{B,0}}{X_{B,K}} + \left(r_B - r_A - \frac{1}{2}\sigma^2\right)T\right)\right) = -e^{-r_B T}P_B\left(X_{B,T} > X_{B,K}\right)$$

That is, the argument $-d_1$ as seen in currency A is exactly the argument d_2 as seen in currency B, and as we have already seen $N(d_2)$ is the risk-neutral probability (in currency B) that the call option ends in the money.

With additional thought it will hopefully come to seem not only unsurprising, but absolutely essential, that these values agree. As we have already seen, the Black-Scholes pricing formula presents an explicit recipe for locally replicating the option: It consists of delta shares of the underlying plus this other quantity, related to the probability of exercise, of the money market account in the valuation currency.

Since as we have seen the two trades (put on B in currency A, call on A in currency B) are in fact the same trade, it would be surprising indeed if the replication recipe differed across the two currencies. To hedge either, we are short some quantity of currency B and long some quantity of currency A. Depending upon the viewpoint from which the pricing is done, one currency looks random while the other looks to have constant value, but there is no difference in the positions that ought to be taken to hedge the position across the two currencies.

As a consequence, although it may seem strange to think of delta as being a (discounted) probability, it is—a probability expressed in the measure induced by the asset which is taken to be random. The important point is that the change of measure we performed in the previous section—which may have seemed arbitrary or simply a trick—is anything but: It is necessary to guarantee that the instrument's essential properties do not depend upon the choice of currency in which it is priced.

One-Touch Options

A common instrument in FX derivatives markets is a form of barrier option known as a one-touch option. Far simpler than the standard barrier option, this option pays a fixed cash amount in the event that the FX rate touches a particular barrier level before expiry, and pays nothing otherwise.

One-touch options may specify payment at hit (i.e., the cash amount is paid when the touch occurs) or payment at expiry. The counterpart of a one-touch option is a no-touch option, which pays the cash amount at expiry as long as the specified level is not touched. In the case of a one-touch with payment at expiry, there is a form of parity that holds for these options:

$$V_{one-touch} + V_{no-touch} = Ce^{-rT}$$

...if each option pays the fixed amount C in the domestic (pricing) currency, since the portfolio of the two together results in a certain cash flow in this currency at expiry.

As with standard barrier options, trading-quality pricing of these instruments must take account of the differences in implied volatility at different strikes and expiries, since this feature has a significant effect on the option's value.

If a simpler method is suitable, however, it is possible to derive closed-form pricing formulas for these options under Black-Scholes assumptions. The foundation of these expressions is a well known result concerning Brownian motion.

The Reflection Principle and First Passage Time of a Brownian Motion
In order to price this option, we must know the distribution of the first passage time—the first time at which it crosses a particular level—for an asset following geometric Brownian motion with drift. The full derivation of the pricing formula relies on several results, some of which are too involved to present here. We will show the first of these, however, in a form that illustrates an interesting and useful property of Brownian motion.

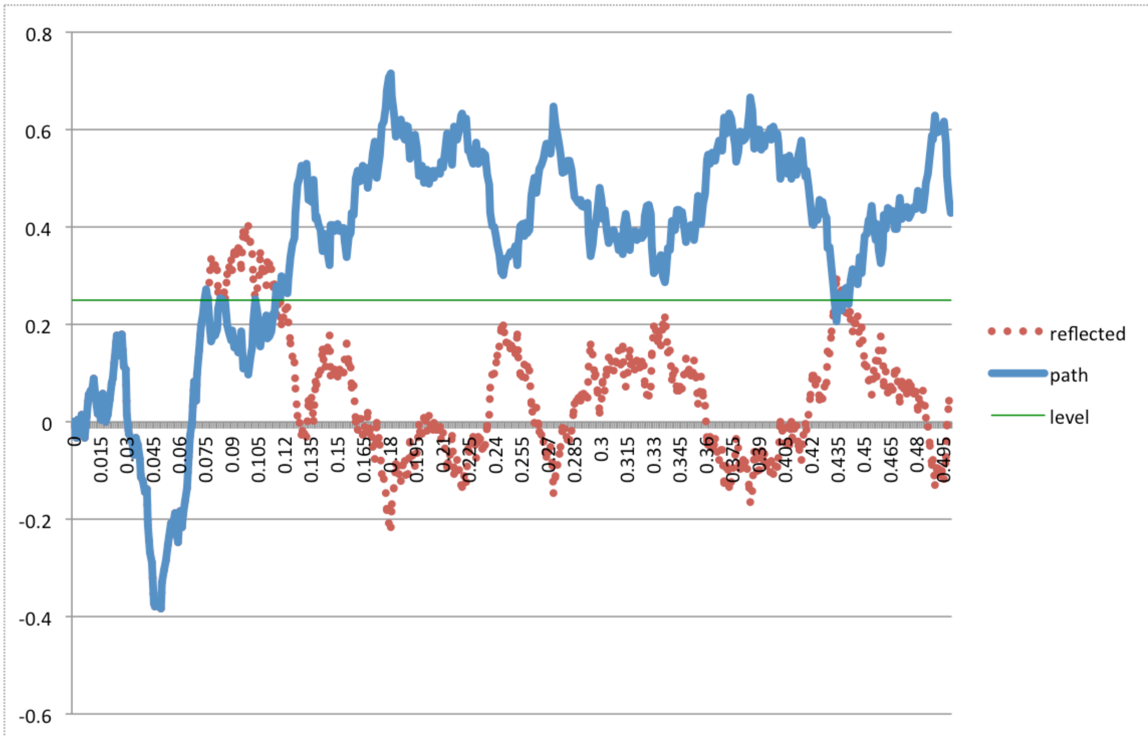
For this section, we consider the simple form of an arithmetic Brownian motion:
$dB = \sigma dW$
...with volatility but no drift. We would like to know the distribution of the random variable τ_a, representing the first time at which the Brownian motion touches this level. We will assume that $a > 0$; the resulting expression can easily be extended to negative values.

To find the distribution, we wish to know $P(\tau_a < T)$ for arbitrary time T. A simple starting point is to consider the related probability $P(B_T < a)$. We know that the terminal point of the path is normally distributed with mean zero and standard deviation $\sigma\sqrt{T}$, so clearly:
$$P\left(B_T < a\right) = 1 - N\left(\frac{a}{\sigma\sqrt{T}}\right) = N\left(\frac{-a}{\sigma\sqrt{T}}\right)$$

Each of the paths represented by this probability certainly crosses the level a, since the paths are continuous. There must, however, be more paths than just these; some must cross the level of interest and then move back below it.

It is in finding these paths that we can take advantage of the symmetry of Brownian motion. For each path that crosses the barrier and ends above it, we take the first point where the path touches the barrier. From this point onward, we reflect the path, so that each increment of the new path is exactly the opposite of the increments on the original path.

This new reflected path clearly ends below the level. Also, since our motion is symmetric, it is equally probable as the first path. The below, as an example, shows one such reflected path:



Certainly every path ending above the barrier has such a reflection; and, once again, the symmetry of Brownian motion guarantees that no path that touches the barrier and ends below it is missed by this counting method, since we can explicitly construct a path ending above the barrier from any path that touches but ends below using the same reflection method.

As a consequence, we conclude there must be a one-to-one correspondence between paths ending above the barrier and those that cross the barrier but end below. We can therefore conclude that:

$$P\left(\tau_a < T\right) = 2P\left(B_T < a\right) = 2N\left(\frac{-a}{\sigma\sqrt{T}}\right)$$

Extending this result to the case when our Brownian motion includes a drift term...

$dR = \mu dt + \sigma dW$

...proves to be more of a challenge. One way of doing so employs the Girsanov theorem to change measure, transforming the problem into one involving a driftless Brownian motion. Even so, taking the required expectation proves a good deal more difficult than the example we have previously seen of measure change.

Omitting this elaboration, we simply move ahead to the result, derived in many stochastic calculus textbooks, that in the presence of drift with $a > 0$ the resulting expression is:

$$P\left(\tau_a < T\right) = N\left(\frac{-a + \mu T}{\sigma\sqrt{T}}\right) + e^{2\mu a}N\left(\frac{-a - \mu T}{\sigma\sqrt{T}}\right)$$

...which, satisfyingly, reduces back to the original expression in the driftless case.

When $a < 0$, note that by symmetry this is equivalent to the $a > 0$ case with the sign of the drift switched. Thus in this case we conclude:

$$P\left(\tau_a < T; a < 0\right) = N\left(\frac{a - \mu T}{\sigma\sqrt{T}}\right) + e^{2\mu a} N\left(\frac{a + \mu T}{\sigma\sqrt{T}}\right)$$

One-Touch Options with Payment at Expiry

The above can easily be used to determine a pricing formula for a one-touch with payment at expiry in domestic currency. We take:

C = the payment amount in the event of a touch
r = the domestic currency risk-free rate
r_f = the foreign currency risk-free rate
T = time to expiry
X_0 = the price of one unit of foreign in the pricing currency
H = the barrier level, expressed as the price of one unit of foreign in the pricing currency
σ = the volatility of the exchange rate

Since the currency follows a geometric Brownian motion, the level a is given by...

$$a = \ln\frac{H}{X_0}$$

...and the drift by...

$$\mu = r - r_f - \frac{\sigma^2}{2}$$

In the case where $H > X_0$ and $\mu > 0$, the value of the option is then:

$$V_{one-touch} = E\left[Ce^{-rT} I_{\tau<T}\right] = Ce^{-rT} P\left(\tau < T\right) =$$

$$Ce^{-rT}\left[N\left(\frac{\ln\frac{X_0}{H} + \left(r - r_f - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) + e^{2\ln\frac{H}{X_0}\left(r - r_f - \frac{\sigma^2}{2}\right)} N\left(\frac{-\ln\frac{X_0}{H} + \left(r - r_f - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right] =$$

$$Ce^{-rT}\left[N\left(\frac{\ln\frac{X_0}{H} + \left(r - r_f - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) + \left(\frac{H}{X_0}\right)e^{2\left(r - r_f - \frac{\sigma^2}{2}\right)} N\left(\frac{-\ln\frac{X_0}{H} + \left(r - r_f - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right]$$

In cases where H is a lower barrier, or the drift is negative, slight adjustments are required. Further, when the payment occurs at hit rather than at expiry, we must discount from the hit time and thus take a slightly different expectation.[2]

[2] For further details on this and many other excellent FX results, see "Pricing Formulae for Foreign Exchange Options" by Weber and Wystup, available at: *https://mathfinance2.com/MF_website/download.aspx?AttachmentRef=30*

The above holds in cases where the payment occurs in the native (pricing) currency. That is, if a dollar-based investor is paid a fixed amount in dollars if the USDEUR barrier is touched, then these are the appropriate formulas. Likewise, if a Euro-based investor is paid Euros with a touch of the same barrier, these would be appropriate (with the spot price, rates, and barrier level of course adjusted accordingly).

If a dollar-based investor is paid in Euros based on the touch of a USDEUR barrier, then the matter may be more complicated. If the payment occurs at the time of hit, then the conversion rate at the time of the payment is of course known, and this amount may be converted to USD at the barrier rate, and the one-touch formula for payment at hit used accordingly.

If, however, payment occurs in EUR at expiry, then the value in dollars of the asset that will be delivered at expiry has a random component. From the dollar investor's perspective, the payment received is not a fixed quantity of cash, but a fixed quantity of the foreign asset.

Since the price returned by the method above is arbitrage-free, however, this in the end turns out not to be a serious obstacle. By taking the reciprocal of the spot and barrier levels, exchanging the roles of the domestic and foreign interest rates, and employing a lower-barrier valuation if the original deal includes an upper barrier (or vice-versa), the deal can be priced in EUR and converted to USD at the spot exchange rate to determine the correct USD price.