Name: *October 14, 2015*

MTH 9831. Solutions to Quiz 5.

(1) (3 points) Let $B(t)$ be a vector of two correlated Brownian motions such that

$$dB(t)(dB(t))^T = \begin{pmatrix} 1 & 1/\sqrt{5} \\ 1/\sqrt{5} & 1 \end{pmatrix} dt = \begin{pmatrix} 1 & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & 1/\sqrt{5} \\ 0 & 2/\sqrt{5} \end{pmatrix} dt.$$

(a) Find the matrix A such that $dW(t) = A\,dB(t)$ and $W(t)$ is a standard 2-dimensional Brownian motion.

(b) Show formally by using basic matrix operations that with your choice of A you indeed have $dW(t)(dW(t))^T = I_2 dt$ (I_2 is the 2×2 identity matrix).

Solution. (a) If $dB(t) = A^{-1}\,dW(t)$ then

$$dB(t)(dB(t))^T = A^{-1} \underbrace{dW(t)(dW(t))^T}_{=I_2 dt} (A^{-1})^T = A^{-1}(A^{-1})^T dt.$$

Therefore, $A^{-1} = \begin{pmatrix} 1 & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0 \\ -1/2 & \sqrt{5}/2 \end{pmatrix}$.

(b) Let $A = \begin{pmatrix} 1 & 0 \\ -1/2 & \sqrt{5}/2 \end{pmatrix}$. Then

$$dW(t)(dW(t))^T = A\,dB(t)(dB(t))^T A^T = A(A^{-1}(A^{-1})^T dt)A^T = I_2 dt.$$

(2) (3 points) Let $(B_1(t), B_2(t))$, $t \geq 0$, be a standard two-dimensional Brownian motion and $R^2(t) = B_1^2(t) + B_2^2(t)$. What is the distribution of $R^2(t)$?

Solution. $(R^2(t))_{t \geq 0}$ is a two-dimensional squared Bessel process.[1] Let Z_1 and Z_2 be two independent standard normal random variables. Then for every $t > 0$ and $x > 0$

$$\begin{aligned} F_{R^2(t)}(x) &= P(R^2(t) \leq x) = P(Z_1^2 + Z_2^2 \leq x/t) \\ &= \frac{1}{2\pi} \int_0^{\sqrt{x/t}} \int_0^{2\pi} e^{-r^2/2} r\,dr d\theta = \int_0^{\sqrt{x/t}} e^{-r^2/2} r\,dr. \end{aligned}$$

Differentiating with respect to x we get the density

$$f_{R^2(t)}(x) = F'_{R^2(t)}(x) = \frac{1}{2t} e^{-x/(2t)}, \quad x > 0.$$

We conclude that $R^2(t)$ has exponential distribution with parameter $1/(2t)$. Since $R^2(0) = 0$ (non-random), the distribution at $t = 0$ is δ_0 (degenerate).

(3) (4 points) At time t the portfolio consists of $\Delta(t)$ shares of stock and $\Gamma(t)$ shares of MMA. Its value is $X(t) = \Delta(t)S(t) + \Gamma(t)M(t)$.

(a) Use multi-dimensional Itô's formula to write $dX(t)$.

(b) Assume that $M(t) = e^{rt}$. What equation should be satisfied if we want the answer in (a) to coincide with the following formula:

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt?$$

Hint: I am asking you to derive the self-financing condition.

[1] At home you were supposed to find the distribution of an n-dimensional squared Bessel process. This is a special case.

Solution. (a) By Itô's product rule,

$$dX(t) = \Delta(t)dS(t) + S(t)d\Delta(t) + d[S,\Delta]_t + \Gamma(t)dM(t) + M(t)d\Gamma(t) + d[M,\Gamma]_t.$$

(b) We want to have $dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))\,dt$. Since $X(t) = \Delta(t)S(t) + \Gamma(t)M(t)$, we can replace $X(t) - \Delta(t)S(t)$ with $M(t)\Gamma(t)$. Moreover, $dM(t) = rM(t)dt$. Therefore,

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))\,dt = \Delta(t)dS(t) + rM(t)\Gamma(t)dt = \Delta(t)dS(t) + \Gamma(t)dM(t).$$

Comparing with part (a) we see that the sum of the following four terms should be 0:

$$S(t)d\Delta(t) + d[S,\Delta]_t + M(t)d\Gamma(t) + d[M,\Gamma]_t = 0.$$

This is the self-fincancing condition.