Credit Risk Models

7. Portfolio credit derivatives

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Outline

CDOs

- *Collateralized debt obligations* (CDOs) are securities whose cash flows are linked to an underlying portfolio of credit risky assets such as bonds, loans, tranches of MBS deals, or CDs (referred to as the *collateral*).
- The issued CDOs are typically divided into a number of *tranches*. Using a mechanism of *structural subordination*, the coupon and principal payments on the different securities in the pool of collateral are paid according to a set of rules known as the *waterfall*.
- The waterfall, described in the prospectus of the CDO maybe very complex (few hundred pages...).
- The tranches of a CDO are rated by the rating agencies (S&P, Moody's, Fitch,...).

Single tranche synthetic CDOs

- A category of unfunded portfolio credit derivatives are single tranche synthetic CDOs (STCDOs).
- Consider a basket of CDS on N distinct names, each of which is associated with LGD amounts of l_i, $i = 1, \ldots, N$. If the notional of the i-th CDS is $\mathcal{N}_i$, then $l_i = \mathcal{N}_i(1 - R_i)$, where R_i is the recovery rate associated with name i. The total initial notional in the pool is $\mathcal{N}_{tot} = \sum_{i=1}^{N} \mathcal{N}_i$. As before, we define the total portfolio loss by $L(t) = \sum_{i=1}^{N} l_i 1_{\tau_i < t}$.
- The total portfolio loss is funneled through tranches, each of which is characterized by an interval $[a, d]$; here a is the *attachment point* of the tranche, and d is the *detachment point* of the tranche. Usually the percentage attachment and detachment $a/\mathcal{N}_{tot}$ and $d/\mathcal{N}_{tot}$ are quoted.
- In a synthetic CDO swap referencing a tranche $[a, d]$, the protection leg of the swap pays out all portfolio losses that take place inside the interval $[a, d]$. The fixed leg pays a pre-agreed premium (upfront fee and / or running coupon) on the notional that is left in the tranche after accounting for payouts on the protection leg.

Single tranche synthetic CDOs

- Originally, all STCDO swaps were of the "bespoke" type, i.e. based on customized portfolios of CDSs. With every STCDO different from the other, price discovery was difficult, and the dealers marked their positions to the empirical correlations.
- Since 2003 or so there has been a market for standardized synthetic CDOs linked to the credit indices discussed in Lecture Notes #3 (CDX, iTraxx, etc.).
- For example, the following tranches on the CDX NA IG index are quoted:
  - $[0\%, 3\%]$ (equity),
  - $[3\%, 7\%]$ (junior mezzanine),
  - $[7\%, 10\%]$ (senior mezzanine),
  - $[10\%, 15\%]$ (senior),
  - $[15\%, 30\%]$ (super senior).
- With these tranches traded relatively liquidly, break-even spread of index STCDOs became visible.

Single tranche synthetic CDOs

- Let us formulate this mathematically. The amount of the tranche that has been eaten away by the portfolio losses over the time interval $[0, t]$ are given by:

$$L_{[a,d]}(t) = (L(t) - a)^+ - (L(t) - d)^+ .$$

- Notice that $L_{[a,d]} = 0$ until $L \geq a$. Following that, $L_{[a,d]}$ grows linearly with L. After L hits d, the tranche is completely wiped away.
- The process for the random variable $L_{[a,d]}(t)$ is a discrete jump process, starting when the discrete jump process $L(t)$ reaches a and stopping when $L(t)$ exceeds d.
- The protection leg of a synthetic CDO pays out these jumps in $L_{[a,d]}(t)$, at the times they occur, provided that they occur prior to the maturity T. In other words, the fraction of the portfolio losses within the interval $[a, d]$, that occur over the time period $[0, T]$, are paid out at the times the losses occur.

Single tranche synthetic CDOs

- The cash flow over the time interval $[t, t + dt]$ is
$$dL_{[a,d]}(t) = L_{[a,d]}(t + dt) - L_{[a,d]}(t),$$
where $t \leq T$.
- The minimum amount that can be paid out on the on the protection leg is of course 0, the maximum amount is $d - a$ (total wipe-out of the tranch).
- Let us now turn to the premium leg of the swap. The total outstanding tranche notional $\mathcal{N}_{[a,d]}(t)$ at time t is given by
$$\mathcal{N}_{[a,d]}(t) = (d - a) - L_{[a,d]}(t).$$

- At time 0, $\mathcal{N}_{[a,d]}(0) = d - a$, but as losses occur within the tranche, the notional amortizes to zero.
- Coupon payments are scheduled at times $T_1, \ldots, T_M$, where $T_M = T$. The total coupon payment at time T_i will be based on the average tranche notional in effect on $[T_{i-1}, T_i]$.
- To a good approximation the premium leg coupon payment at time T_i is

$$cpn(T_i) = \delta_i C\big(\mathcal{N}_{[a,d]}(T_i) + \mathcal{N}_{[a,d]}(T_{i-1})\big)/2,$$

  where δ_i is the day count fraction.
- The maximum coupon is $\delta_i C(d - a)$, the minimum coupon is 0.

Some remarks about CDOs

- A CDO tranch with attachment point $a = 0$ is called the *equity tranche*. Equity tranches have no “buffer” below them and incur losses as soon as the underlying portfolio starts experiencing losses.
- The most risky tranches of a CDO with low values of a and d are referred to as *junior tranches*. Safer tranches with high values of a and d are called *senior tranches*. Tranches in between are known as *mezzanine tranches*.
- For a CDO swap, the value of the coupon C that renders the value of the transaction 0 is known as the *break-even spread*.
- During the financial crisis of 2007-2008, there has been a lot of losses experienced by both cash and synthetic CDOs. Particularly hard hit were CDOs backed by portfolios of mortgage backed securities and / or ABS CDSs.
- Significantly, in many cases entire CDOs, including the AAA rated senior tranches (typically purchased by portfolio managers for investment purposes) were entirely wiped out.

Break-even spread

- The price of a CDO swap is given by

$$
\begin{aligned}
V_{[a,d]}(0) &= \mathrm{E}\Big[\int_0^T e^{-\int_0^t r(s)ds} dL_{[a,d]}(t)\Big] - C\mathrm{E}\Big[\sum_{i=1}^{M} \delta_i \big(\mathcal{N}_{[a,d]}(T_i) + \mathcal{N}_{[a,d]}(T_{i-1})\big)/2\Big] \\
&\triangleq \mathrm{E}\Big[\int_0^T e^{-\int_0^t r(s)ds} dL_{[a,d]}(t)\Big] - C\mathcal{A}_{[a,d]}(0)
\end{aligned}
$$

  where we have introduced a risky annuity $\mathcal{A}_{[a,d]}(t)$ associated with the tranche.
- The break-even spread is

$$
C_0 = \frac{\mathrm{E}\Big[\int_0^T e^{-\int_0^t r(s)ds} dL_{[a,d]}(t)\Big]}{\mathcal{A}_{[a,d]}(0)} .
$$

- We shall consider ways to compute this, but first let us consider how the value, or the break-even spread, of a CDO tranche depends on default co-dependence in the portfolio.

Break-even spread

- First, consider a senior tranche. For a senior tranche to suffer a loss, a very significant number of firms must default. This, in general, is more likely in a setting where default co-dependence is high. As a consequence, high co-dependence implies a high break-even spread (and a high value of the floating leg) for a senior tranche.
- For an equity (or junior) tranche, high co-dependence implies (as we have seen) an increased likelihood of having zero losses in the portfolio, and thereby a reduced probability of experiencing losses in the equity tranche. As a consequence, high co-dependence implies a low break-even spread (and a low value of the floating leg) for an equity tranche.
- Mezzanine tranches being in-between junior and senior tranches have typically relatively little sensitivity to the default co-dependence structure of the portfolio.

Gaussian copula model

- We now turn to the Gaussian copula model of CDO tranches. We shall apply the copula C_ρ constructed in Lecture Notes #6 to default times τ_i of the N individual names in the underlying portfolio.
- This approach is popular for a variety of reasons, most notably for its simplicity.
- Among the key drawbacks of Gaussian copula is the fact that, as discussed in the previous lecture, it exhibits no tail dependence. As a consequence, it does a poor job capturing tail events in which simultaneous defaults of multiple names occur.
- Gaussian copula is closely related to Merton's structural model discussed in Lecture Notes #1.
- Specifically, recall that default in Merton's model occurs over the time interval $[0, T]$ if the firm's value $V(T)$ drops below its debt threshold level F.

Gaussian copula model

- Firm's $V(T)$ is assumed lognormal with drift r,

$$V(T) = V_0 e^{rT} e^{\sigma_V Z - \frac{\sigma_V^2 T}{2}}$$

  where $Z \sim N(0,1)$.
- As a result,

$$\begin{aligned} \mathsf{P}(\tau \leq T) &= \mathsf{P}\Big(V_0 e^{rT} e^{\sigma_V \sqrt{T} Z - \frac{\sigma_V^2 T}{2}} \leq F\Big) \\ &= \mathsf{P}(Z \leq H^*(T)), \end{aligned}$$

  where

$$H^*(T) \triangleq \frac{\log \frac{F}{V_0} - rT + \frac{\sigma_V^2 T}{2}}{\sigma_V \sqrt{T}}.$$

Gaussian copula model

- On the other hand, in the Gaussian copula, we have for name i:

$$\begin{aligned}\mathsf{P}(\tau_i \leq T) &= \mathsf{P}(F_i^{-1}(N(Z_i)) \leq T)\\ &= \mathsf{P}(Z_i \leq H_i(T)),\end{aligned}$$

where

$$\begin{aligned}H_i(T) &\triangleq N^{-1}(F_i(T))\\ &= N^{-1}(\mathsf{P}(\tau_i \leq T)).\end{aligned}$$

Gaussian copula model

- While the barriers $H^*(T)$ and $H_i(T)$ are defined in different manners (one model uses balance sheet information, the other uses the default probabilities implied from the CDS market), the structure of the models are similar.
- Furthermore, we can interpret the correlation matrix ρ in the Gaussian copula as follows. Its elements can be interpreted as the correlations between assets of the different firms in the basket.
- As a proxy for asset correlations one can use equity correlations, so that empirical estimates of ρ are based on stock return correlations.

Gaussian copula model: Monte Carlo simulation

- The marginal default time distributions are given by $\mathsf{P}(\tau_i \leq t) = 1 - S_i(t) \triangleq Q_i(t)$, where $S_i(t)$ is the survival probability for name i, implied from quoted CDS prices.
- Given all the marginal distributions, we construct the joint default time distribution as

$$\mathsf{P}(\tau_1 \leq t_1, \ldots, \tau_N \leq t_N) = C_\rho(Q_1(t_1), \ldots, Q_N(t_N)).$$

- Since using this expression to explicit calculations is infeasible, we resort to Monte Carlo simulations.

Gaussian copula model: Monte Carlo simulation

- We apply the algorithm for simulating Gaussian copula presented in Lecture Notes #6 and proceed as follows.
  - *Step 1*. Given a simulated N-dimensional sample $U_1, \ldots, U_N$, we set $\tau_1 = Q_1^{-1}(U_1), \ldots, \tau_1 = Q_N^{-1}(U_N)$. This is a simulated scenario for random default times consistent with the Gaussian copula.
  - *Step 2*. Given this scenario, generate the corresponding cash flows for the premium and protection legs of the CDO.
  - *Step 3*. Discount these cash flows to calculate the single scenario estimates $\widehat{V}_{prem}$ and $\widehat{V}_{prot}$.
  - *Step 4*. Repeat Steps 1 - 3 n times, and calculate the averages:

$$\widehat{V}^{av}_{prem} = \frac{1}{n} \sum_{i=1}^{n} \widehat{V}^{i}_{prem},$$

$$\widehat{V}^{av}_{prot} = \frac{1}{n} \sum_{i=1}^{n} \widehat{V}^{i}_{prot}.$$

- The value $\widehat{V}^{av}_{prot} - \widehat{V}^{av}_{prem}$ is the Monte Carlo estimate of the price of the STCDO.

Student t copula model

- The main reason for using a Student t copula is to fatten up loss tails relative to the Gaussian copula. In particular, we know from Lecture Notes #6 that the Student t copula has upper tail dependence.
- The lower the number of degrees ν is, the fatter the tail of the portfolio loss distribution.
- Using the language of Merton's model, we can also interpret the Student t distribution as representing asset returns with fat tails.
- Choice of ν can be based on empirical tails of equity returns.
- Monte Carlo simulations using the Student t copula follows the same outline as in the case of the Gaussian copula.

1-factor Gaussian copula model

- The basis for a Gaussian copula is a N-dimensional Gaussian variable Z with a (symmetric) correlation matrix ρ so that $\mathrm{Corr}(Z_i, Z_j) = \rho_{ij}$
- This leaves us with a model with $\binom{N}{2} = N(N-1)/2$ calibratable parameters. For $N = 125$, this means that 7750 parameters require calibration, a clearly rather daunting task.
- Instead, let us introduce a Gaussian systematic factor S (i.e. $S \sim N(0,1)$), common to all names and set, for all $i = 1, \ldots, N$,

$$Z_i = \beta_i S + \sqrt{1-\beta_i^2}\, \varepsilon_i,$$

  where $|\beta_i| < 1$. The idiosyncratic residuals ε_i are also Gaussian and are assumed to be independent from each other,

$$\mathrm{Corr}(\varepsilon_i, \varepsilon_j) = 0, \text{ for } i \neq j$$

1-factor Gaussian copula model

- The factor model is a special case of the Gaussian copula, as all dependence between the N names is driven by their dependence on a single common factor S. The only calibratable parameters are the N *factor loadings* β_i.
- There is a clear parallel between the 1-factor Gaussian copula model and the CAPM model. It is natural to think of S as a systematic market variable, reflecting overall economic conditions. We can generalize this model to a multi-factor model, assuming that S is multi-dimensional.

1-factor Gaussian copula model

- In the above specification,

$$\operatorname{Corr}(Z_i, Z_j) = \beta_i \beta_j, \text{ for } i \neq j.$$

  The correlation matrix ρ has thus a factor form.
- Typically, correlations that are not precisely of factor form. The 1-factor model is thus applicable if the correlation matrix can be reasonably closely approximated by a correlation matrix in a factor form.
- One (small) advantage of the factor model is that it does not require Cholesky's decomposition in Monte Carlo simulations. We simply draw S and ε_i, $i = 1, \ldots, N$ from $N(0,1)$ independently, and then compute all the Z_i's.

Pricing a CDO tranche

- Above we discussed tranche pricing by means of Monte Carlo simulations, using simulated samples of default times.
- In general, however, to price a standard CDO tranche, the knowledge the time 0 marginal distribution of the portfolio loss $L(t)$ is sufficient; we do not necessarily need to know the default times of the individual names.
- To see this, consider the price of the protection leg on a tranche $[a, d]$. Assuming non-stochastic interest rates, we have

$$\begin{aligned} V_{prot}(0) &= \mathrm{E}\Big[\int_0^T P_0(t)\, dL_{[a,d]}(t)\Big] \\ &= \int_0^T P_0(t)\, \mathrm{E}[dL_{[a,d]}(t)]. \end{aligned}$$

Pricing a CDO tranche

- We approximate this integral by a sum:

$$
\begin{aligned}
V_{prot} &= \sum_{i=1}^{M} P_0((T_{i-1} + T_i)/2) \mathsf{E}\big[L_{[a,d]}(T_i) - L_{[a,d]}(T_{i-1})\big] \\
&= \sum_{i=1}^{M} P_0((T_{i-1} + T_i)/2) \big(\mathsf{E}\big[L_{[a,d]}(T_i)\big] - \mathsf{E}\big[L_{[a,d]}(T_{i-1})\big]\big),
\end{aligned}
$$

  where the dates T_i, $i = 1, \ldots, M$, are chosen to coincide with the premium leg coupon payment dates.
- If we know the full distribution of $L(t)$, computation of $\mathsf{E}\big[L_{[a,d]}(T_i)\big]$ is easy, since

$$
L_{[a,d]}(T_j) = (L(T_j) - a)^+ - (L(T_j) - d)^+,
$$

  and we can compute $\mathsf{E}[(L(T_i) - a)^+]$ and $\mathsf{E}[(L(T_i) - d)^+]$ by summing over the discrete distribution of $L(T_i)$.
- The same is true for the premium leg of a CDO swap: if we know the distribution of $L(T_i)$ for all dates T_i, we can price it.

LHP model

- Let us now turn to non-Monte Carlo methods of pricing of a CDO tranch.
- We first consider a toy model, namely the large homogenous portfolio (LHP) model. Within the LHP model, the portfolio loss distribution can be computed in an almost explicit form.
- Namely, consider a time horizon T, and assume that all N names in the basket have identical default probabilities $Q(T)$ and identical LGDs l. We assume a 1-factor Gaussian copula model with identical factor loadings for all names, namely $\beta_i = \beta$, for all $i = 1, \ldots, N$.
- In particular, this means, that all off-diagonal correlations are equal

$$\rho_{ij} = \beta^2, \text{ for all } i \neq j.$$

- The quantity of interest is the conditional default probability $Q(T|s)$ at time T given that $S = s$.

LHP model

- Recall that probability of default in the Gaussian copula is given by

$$
\begin{aligned}
\mathrm{P}(\tau_i \le T) &= \mathrm{P}(Z_i \le H_i(T)),\\
H_i(T) &= N^{-1}(Q_i(T)).
\end{aligned}
$$

- Therefore,

$$
\begin{aligned}
Q(T|s) &= \mathrm{P}(\tau_i \le T|S = s)\\
&= \mathrm{P}(Z_i \le H_i(T)|S = s)\\
&= \mathrm{P}\big(\beta S + \sqrt{1-\beta^2}\varepsilon_i \le H_i(T)|S = s\big)\\
&= \mathrm{P}\big(\sqrt{1-\beta^2}\varepsilon_i \le H_i(T) - \beta s\big)\\
&= \mathrm{P}\Big(\varepsilon_i \le (H_i(T) - \beta s)/\sqrt{1-\beta^2}\Big)\\
&= N\Big((H_i(T) - \beta s)/\sqrt{1-\beta^2}\Big).
\end{aligned}
$$

LHP model

- Conditional on $S = s$ all defaults of the N names are completely independent, and they all have the same conditional default probability $Q(T|s)$.
- As a consequence, the conditional number of defaults D follows a Bernoulli distribution:

$$\mathsf{P}(D = k|S = s) = \binom{N}{k} Q(T|s)^k (1 - Q(T|s))^{N-k}.$$

- Since the loss per name is a constant l, we can rewrite the above formula as

$$\mathsf{P}(L_N(T) = kl|S = s) = \binom{N}{k} Q(T|s)^k (1 - Q(T|s))^{N-k}.$$

This is the distribution of the portfolio loss conditional on $S = s$.

LHP model

- The unconditional loss is given by total probability formula:

$$\begin{aligned} \mathrm{P}(L_N(T) = kl) &= \int_{-\infty}^{\infty} \mathrm{P}(L_N(T) = kl | S = s) d\mathrm{P}(S = s) \\ &= \frac{1}{\sqrt{2\pi}} \binom{N}{k} \int_{-\infty}^{\infty} Q(T|s)^k (1 - Q(T|s))^{N-k} e^{-s^2/2} ds. \end{aligned}$$

- Evaluation of this integral requires straightforward numerical integration (for example, Simpson's rule or Gauss-Hermite quadrature).

LHP model

- Consider now the limit of $N \to \infty$. Let

$$l_N \triangleq \frac{L(T)}{N}$$

denote the number of defaulting names per portfolio size.

- From the properties of Bernoulli distribution, we find that

$$\begin{aligned} \mathsf{E}[l_N|s] &= \frac{l}{N} NQ(T|s) \\ &= lQ(T|s) \\ \mathrm{Var}[l_N|s] &= \frac{l^2}{N^2} NQ(T|s)(1 - Q(T|s)) \\ &= \frac{1}{N} l^2 Q(T|s)(1 - Q(T|s)). \end{aligned}$$

- As a consequence,

$$\lim_{N\to\infty} \mathrm{Var}[l_N|s] = 0.$$

LHP model

- Let $l_\infty \triangleq \lim_{N\to\infty} l_N$. Then, explicitly,

$$l_\infty = lQ(T|s)$$

is a non-random value.

- For the total (unconditional) probability distribution we have, for any $x \in [0,1]$,

$$\begin{aligned}
\mathsf{P}(l_N \le xl) &= \mathsf{P}(Q(T|S) \le x) \\
&= \mathsf{P}\Big(\frac{H(T) - \beta S}{\sqrt{1-\beta^2}} \le N^{-1}(x)\Big) \\
&= \mathsf{P}(S \ge K(x)),
\end{aligned}$$

where

$$K(x) \triangleq \frac{H(T) - \sqrt{1-\beta^2}N^{-1}(x)}{\beta}.$$

LHP model

- As a consequence,

$$\begin{aligned}\mathrm{P}(l_N \leq xl) &= 1 - N(K(x)) \\ &= N(-K(x)).\end{aligned}$$

- We can write the above formula explicitly as

$$\mathrm{P}(l_N \leq xl) = N\Big(\frac{\sqrt{1-\beta^2}N^{-1}(x) - N^{-1}(Q(T))}{\beta}\Big).$$

  This result was obtained in the 1987 by Vasicek.
- While Vasicek's formula was derived for large homogeneous portfolios, it often provides a reasonably accurate approximation for loss distributions of finite portfolios, and can be used as a "rule of thumb".
- It can be used for non-homogenous portfolios, if we use average default probabilities and recovery rates for each of the names.

Recursion approach to pricing CDO tranches

- Now we would like to relax the assumption that all the names in the portfolio have identical losses and default probabilities. We shall discuss a practical approach that most banks actually use in computing loss distributions.
- Let
$$Q_i(T) = \mathsf{P}(\tau_i \leq T)$$
denote the default probability of name i on the time horizon T.
- We also allow the LGDs to be name specific. For reasons that that will become clear below, we represent all LGDs in terms of integer multiples of a minimum loss unit U.
- In other words, the loss given default of name i is
$$l_i = x_i U,$$
where $x_i = 0, 1, 2, \ldots$.
- If all LGDs are identical (which may be the case if one uses the market standard 40% recovery for each name), we can set U equal to the common LGD, and $x_i = 1$ for all names.

Recursion approach to pricing CDO tranches

- For each name we define conditional probability distribution:

$$\begin{aligned} Q_i(T|s) &= \mathrm{P}(\tau_i \leq T|S = s) \\ &= N\Big(\frac{N^{-1}(Q_i(T)) - \beta_i s}{\sqrt{1-\beta_i^2}}\Big), \end{aligned}$$

  where $i = 1, \ldots, N$.
- Conditional on $S = s$, all defaults are independent from each other. This fact is the key input in the construction of the total conditional loss distribution.
- We use a recursive procedure, where we add the names to the portfolio one after the other. Let $L^j(T|s)$ denote the portfolio loss over the time horizon T with names $i = 1, \ldots, j$ already in the portfolio, conditional on $S = s$.
- By construction, $L^j(T|s)$ takes only discrete values

$$0, U, 2U, \ldots, \Big(\sum_{i=1}^{j} x_i\Big)U.$$

Recursion approach to pricing CDO tranches

- Denote
$$P^j(T, x|s) = \mathrm{P}(L^j(T|s) \leq xU),$$
and proceed by induction. We can start with $P^0(T, x|s) = 1_{x=0}$ and in each recursion step add one name until we have reached $P^N(T, x|s)$, for all $x \in [0, x_{\max}]$, where $x_{\max} \triangleq \sum_{i=1}^N x_i$.
- Assuming that $P^j(T, x|s)$ is known for all x, we use
$$P^{j+1}(T, x|s) = P^j(T, x - x_{j+1}|s) Q_{j+1}(T|s) + P^j(T, x|s)(1 - Q_{j+1}(T|s))$$
in order to add name $j + 1$.
- Once we have reached $j = N$, we can construct the full portfolio loss distribution by numerically integrating over s:
$$\mathrm{P}(L(T) = xU) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} P^N(T, x|s) e^{-s^2/2} ds.$$
- In applications, we usually do not need to construct the total loss distribution explicitly. As we will see below, it is sufficient to have the conditional distribution.

Recursion approach to pricing CDO tranches

- Let us now apply the recursion algorithm above to the pricing of an STCDO spanning a tranche $[a, d]$, and paying a fixed coupon C on payment dates $T_j, j = 1, \ldots, M$.
- We compute the values of both legs of the CDO swap by numerically evaluating the integrals

$$V_{prot}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V_{prot}(0|s) e^{-s^2/2} ds,$$
$$V_{prem}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V_{prem}(0|s) e^{-s^2/2} ds$$

of the respective conditional values $V_{prot}(0|s)$ and $V_{prem}(0|s)$.
- Numerical integration algorithms required above (such as Gauss-Hermite quadrature or Simpson's rule) require evaluating $V_{prot}(0|s)$ and $V_{prem}(0|s)$ on a finite number of values of s.

Recursion approach to pricing CDO tranches

- For each value of s needed in the numerical integral above, we proceed in the following steps:
  1. Use the recursion algorithm to find $P^N(T, x|s)$, for all T_j and $x \in [0, x_{\max}]$.
  2. For all dates T_j, compute expected tranche loss values:

$$\mathrm{E}\left[L_{[a,d]}(T_j)|s\right] = \sum_{x=0}^{x_{\max}} \left((xU - a)^+ - (xU - d)^+\right) P^N(T, x|s)$$

  3. Compute the values of $V_{prot}(0|s)$ and $V_{prem}(0|s)$
- Note that the computational cost of this algorithm is $O(x_{\max}^2 M)$,where M is the number of coupon payment dates.
- For efficiency it thus good to choose $x_{\max}$ low, which is obtained by using a large minimum loss unit U. A large loss unit may introduce large rounding errors, and we face a precision / efficiency trade-off.
- For typical STCDOs, and assuming carefull computer code implementation, this pricing algorithm takes a fraction of a second to run.

Risk sensitivities of CDO tranches

- To hedge credit spread exposure, we would like to measure the impact on a STCDO swap price from making changes to the credit spread curve of the individual firms in the basket.
- A simple, but computationally very costly, way of doing this is to (i) perturb the spreads and reprice the entire instrument, and (ii) calculate the difference between the perturbed and base prices. This is too slow to be feasible in practice, as the number of shocks required may be in the 100s (or even 1000s). We shall discuss a faster method.

Risk sensitivities of CDO tranches

- Let us consider name i, and assume that its intensity curve (as seen now) is $\lambda_i(t)$. We are interested in calculating the sensitivity of the CDO swap (from the perspective of, say, protection buyer) to the intensity curve. To this end, we replace $\lambda_i(t)$ by $\lambda_i(t) + \varepsilon$, and evaluate the derivative

$$\frac{d}{d\varepsilon}\left(V_{prot}(0) - V_{prem}(0)\right)\Big|_{\varepsilon=0}.$$

- By the chain rule, we write this as

$$\frac{d}{d\varepsilon}\left(V_{prot}(0) - V_{prem}(0)\right)\Big|_{\varepsilon=0} = \sum_{j=1}^{M}\frac{d}{dQ_i(T_j)}\left(V_{prot}(0) - V_{prem}(0)\right)\frac{dQ_i(T_j)}{d\varepsilon}\Big|_{\varepsilon=0}.$$

- Notice that

$$\frac{dQ_i(T_j)}{d\varepsilon}\Big|_{\varepsilon=0} = T_j(1 - Q_i(T_j)).$$

Risk sensitivities of CDO tranches

- We now need to evaluate the terms occurring in $dV_{prot}(0)/dQ_i(T_j)$ and $dV_{prem}(0)/dQ_i(T_j)$.
- From the recursion algorithm above, this, in turn, requires us to compute terms of the form
$$\frac{d}{dQ_i(T_j)}\,\mathrm{E}[L_{[a,d]}(T_j)],$$
as $Q_i(T_j)$ does not affect $\mathrm{E}[L_{[a,d]}(T_k)]$, for $k \neq j$.
- Now, using the chain rule, we find that
$$\begin{aligned}\frac{d}{dQ_i(T_j)}\,\mathrm{E}[L_{[a,d]}(T_j)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d}{dQ_i(T_j)}\,\mathrm{E}[L_{[a,d]}(T_j)|s]e^{-s^2/2}ds\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{dQ_i(T_j|s)}{dQ_i(T_j)}\,\frac{d}{dQ_i(T_j|s)}\,\mathrm{E}[L_{[a,d]}(T_j)|s]e^{-s^2/2}ds.\end{aligned}$$
- In this expression, we can find $dQ_i(T_j|s)/dQ_i(T_j)$ analytically from :
$$Q_i(T|s) = N\Big(\frac{N^{-1}(Q_i(T)) - \beta_i s}{\sqrt{1-\beta_i^2}}\Big).$$

Risk sensitivities of CDO tranches

- We then have to evaluate of expressions of the form

$$\frac{d}{dQ_i(T_j)}\,\mathrm{E}[L_{[a,d]}(T_j)|s] = \frac{d}{dQ_i(T_j)}\sum_{x=0}^{x_{\max}}\left((xU-a)^+ - (xU-d)^+\right)P^N(T_j,x|s)$$
$$= \sum_{x=0}^{x_{\max}}\left((xU-a)^+ - (xU-d)^+\right)\frac{d}{dQ_i(T_j)}\,P^N(T_j,x|s).$$

- We have thus ultimately reduced the problem to evaluating of $dP^N(T_j,x|s)/dQ_i(T_j)$, for all x and s.
- For this, let $P^N_{-i}(T_j,x|s)$ denote the conditional distribution of the portfolio loss with name i removed from the portfolio
- From the loss recursion algorithm,

$$P^N(T,x|s) = P^N_{-i}(T_j,x-x_i|s)Q_i(T_j|s) + P^N_{-i}(T,x|s)(1-Q_i(T_j|s)),$$

which implies that

$$\frac{d}{dQ_i(T_j)}\,P^N(T_j,x|s) = P^N_{-i}(T_j,x-x_i|s) - P^N_{-i}(T,x|s).$$

Risk sensitivities of CDO tranches

- We are left with one final hurdle to clear: efficient computation of $P^N_{-i}(T_j, x|s)$, for all x. But here we can use the recursion directly, and write

$$P^N_{-i}(T_j, x|s) = \frac{P^N(T, x|s) - P^N_{-i}(T, x - x_i|s)Q_i(T_j|s)}{1 - Q_i(T_j|s)} ,$$

  for $x = 0, 1, \ldots, \sum_{j=0}^N x_j - x_i$.
- This leads to a recursion in x, that can be started from $x = 0$, in which case

$$P^N_{-i}(T_j, 0|s) = \frac{P^N(T, 0|s)}{1 - Q_i(T_j|s)} .$$

- We are nearly done, except for a computational efficiency point. The equation above performs best if $Q_i(T_j|s) < 1/2$. If this is not the case, we start the recursion from the top, at $x = \sum_{j=0}^N x_j - x_i$, and iterate down in x.

Risk management of CDO tranches

- For a typical STCDO, a dealer computes the sensitivities with respect to all individual CDSs used as inputs into the construction of the survival curves for all N names in the portfolio. This can amount to more than a thousand individual spread deltas per STCDO.
- These deltas are used to calculate a hedge constructed out of individual CDSs and CDS indices, to neutralize the sensitivity to credit spreads.
- Furthermore, a dealer computes the sensitivities with respect to the assumed LGDs, and with respect to the effects of the default that each of the N names has on the spot. These numbers are used to hedge against the effects of outright defaults in the underlying portfolio.
- The instruments are also subject to stress tests based on fixed scenarios, and some measures for spread gamma are computed. The large number of risk measures that need to be computed requires an efficient underlying pricing model. This is the primary reason why the 1-factor Gaussian copula and the recursion algorithm have become market standards.
- However, the Gaussian copula model does not match observable market prices of the tranches well. We will next discuss the effects of this.

Base correlation

- It became apparent that the standard index tranches did not trade according to a simple Gaussian copula. Equity tranche break-even spreads were too high, as were the break-even spreads of senior tranches. On the other hand, mezzanine tranches had break-even spreads that were too low, relative to a Gaussian copula.
- The way the market represents this phenomenon is through implied Gaussian copula correlations.
- One type of implied correlation is called *compound correlation*. For a given tranche $[a, d]$ with a known market price, we back out (numerically) the flat correlation ρ that would make a Gaussian copula match the market price of the tranche.
- Mathematically, compound correlation is an ill defined concept. For example, mezzanine tranches, which are quite insensitive to correlation, may not have a compound correlation at all. Alternatively, sometimes a mezzanine tranche has several compound correlations.
- The problem stems from the fact that tranches are call spreads on the portfolio loss distribution, and are not monotone functions of the correlation.

Base correlation

- We can get around this by only considering equity tranches, i.e. tranches where the attachment point is 0. The break-even spread of equity tranches are decreasing in Gaussian copula correlation parameter, facilitating the computation of implied correlation.
- We thus define a curve $\rho_{base}(x)$ defined as the correlation required to correctly price an equity tranche covering $[0, x]$.
- This correlation function, and the curve it generates, is known as the *base correlation*.
- Only one equity tranche is traded in the market, namely the $[0, 3\%]$ tranche. This is not a problem, as any tranche $[a, d]$ can be priced by subtracting the price of a tranche $[0, a]$ from the price of a tranche $[0, d]$. Consequently, if we know the break-even spread of a $[0, 3\%]$ tranche and of a $[3\%, 7\%]$ tranche, we can compute the break-even spread of the $[0, 7\%]$.
- Equivalently, if we want to price some tranche $[a, d]$ from a base correlation curve, we first look up $\rho_{base}(a)$ and price the tranche $[0, a]$. Then we look up $\rho_{base}(d)$ and price the tranche $[0, d]$. The price of the tranche $[a, d]$ is then obtained by subtraction.

Base correlation

- The market is interpolating, extrapolating, and massaging the correlation skew when pricing regular CDO tranches.
- Some care must be taken in this process. Not all interpolation schemes are arbitrage free (and not all base correlation skews are allowed). Also, there are arbitrage free prices that have no implied base correlation skew.
- From the base correlation curve, it is possible to back out the market implied loss distribution. We can do this by differentiating tranche prices twice with respect to the detachment level. This is similar to the replication method we discussed in the context of CMDSs.
- Exact form of loss distribution is very dependent on interpolation of market quotes, but qualitatively we have a “bang-bang” regime tendency in the market implied loss distribution: either very few defaults will take place (low correlation) or very many defaults will take place (high correlation).
- The base correlation methodology is not a model, but only an interpolation mechanism.

Base correlation

- There are a number of models that try to explain the base correlation smile.
- Most of these models are of the factor type. That is, conditional on some factor Z, we are given conditional default probabilities in some form

$$\mathrm{P}(\tau_i \leq T | Z = z) = f_i(T|z), \text{ where } i = 1, \ldots, N.$$

- The form of the f_i is typically motivated by economic considerations, as is the distribution of the Zs (which may not be Gaussian, but can contain jumps and other complications).
- The primary application of such models are for non-standard STCDOs; for regular STCDOs, the base correlation approach is market standard.

Final comments

- During the recent credit crash, the market price for senior tranche risk has occasionally reached near-panic levels (e.g. 50 bps for 60-100% tranches). Ordinary factor models cannot handle this.
- What is required are models that allow recovery to be random variables, to ensure that all losses (including a 100% loss) are reachable.
- A heuristic approach involves marking different recovery rates for different tranches (to be interpreted as the average recovery rate to be experienced when the tranches suffer a loss).
- Another (possibly more consistent) approach is to extend factor models to allow recovery rates to be functions of the systematic variable, $R_i = R_i(Z)$. These functions should be increasing in Z, since we want low recovery rates when there is a systemic crash (which happens when Z is very low).

References

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