BARUCH, MFE

MTH 9876 Assignment One

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1 LOSS GIVEN DEFAULT

Loss given default (LGD) is a measure of the severity of a default, and is defined as the fraction of the face value of a bond that is not repaid to the investor. In other words,

$$\text{LGD} = (1 - R) \times F.$$

(i) Find the probability distribution of LGD in Merton's model.
Solution: Given the occurrence of a default at time T, the amount paid to the investor is the value of the firm V_T. Thus, the fraction of the face value of a bond that is not repaid to the investor is

$$\text{LGD} \triangleq F(1 - R) = (F - V_T)^+ ,$$

By definition, $0 \leq \text{LGD} < F$. Let $0 \leq x < F$,

$$\begin{aligned}
\mathbb{P}(\text{LGD} < x) &= \mathbb{P}(F - V_T \leq x | V_T < F)\, \mathbb{P}(V_T < F) + \mathbb{P}(V_T \geq F) \\
&= \mathbb{P}(F - x \leq V_T < F) + \mathbb{P}(V_T \geq F) \\
&= \mathbb{P}(V_T \geq F - x).
\end{aligned}$$

Invoking the well-known formula for the risk-neutral survival probability with face value F,

$$\mathbb{P}(V_T \geq F) = N(d_-)$$

where

$$d_- \triangleq \frac{1}{\sigma_V \sqrt{T}} \left[\log \frac{V_0}{F} + \left(r - \frac{1}{2}\sigma_V^2 \right) T \right],$$

the cumulative distribution function of LGD can be written as

$$\mathbb{P}(\text{LGD} < x) = N(d_-(x)),$$

where $d_-(x)$ is obtained by replacing F by $F - x$ in d_-, such that $d_-(0) = d_-$:

$$d_-(x) \triangleq \frac{1}{\sigma_V \sqrt{T}} \left[\log \frac{V_0}{F - x} + \left(r - \frac{1}{2}\sigma_V^2 \right) T \right].$$

The density function is obtained by differentiating $\mathbb{P}(\text{LGD} < x)$ with respect to x, for $0 \leq x < F$,

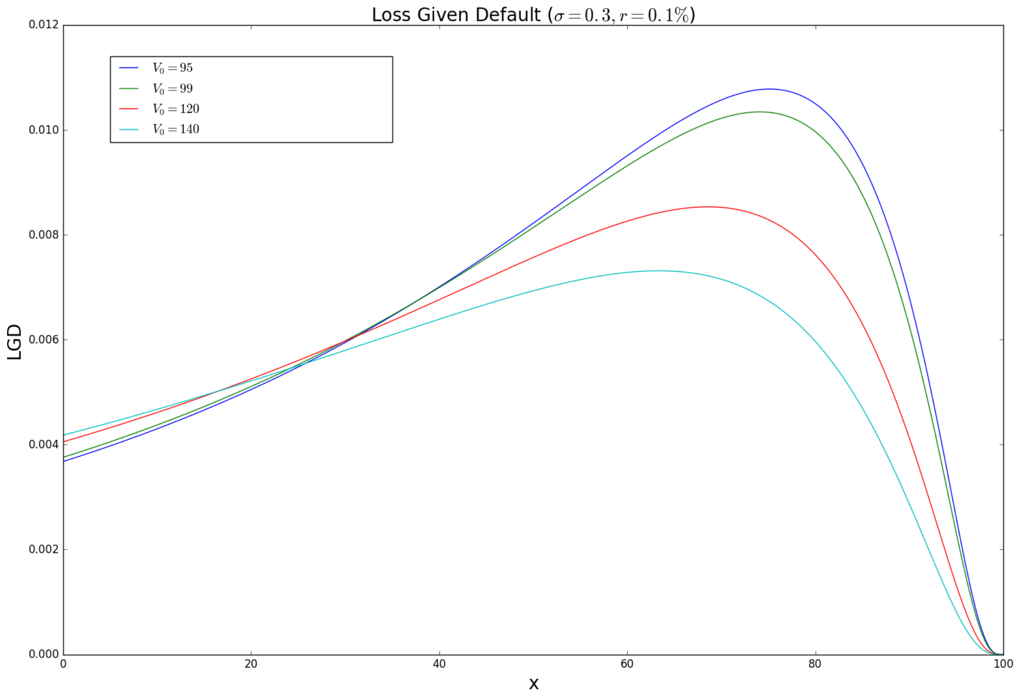
$$\begin{aligned}
f_{\text{LGD}}(x) &= \frac{\partial}{\partial x} \mathbb{P}(\text{LGD} < x) \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_-(x)^2}{2}} \frac{\partial d_-(x)}{\partial x} + N(d_-)\, \delta(x) \\
&= \frac{e^{-\frac{d_-(x)^2}{2}}}{\sigma_V \sqrt{2\pi T}(F - x)} + N(d_-)\, \delta(x).
\end{aligned}$$

For $x < 0$, we have $f_{\mathrm{LGD}}(x) = 0$. Notice that there is a Dirac-δ component in the probability density function because at $x = 0$, the cumulative distribution function is discontinuous:

$$\mathbb{P}\,(\mathrm{LGD} < 0) = 0,$$
$$\mathbb{P}\,(\mathrm{LGD} \le 0) = N(d_-).$$

The above discontinuity is caused by the fact that the finite probability $N(d_-)$ of the equity value ending up with $V_T > F$ contributes to the isolated point of zero LGD.

(ii) Implement the formula for the density function and plot it for $T = 10$ and $V_0 = 95, 99, 120, 140$ (as usual, $F = 100$).
Solution: We choose $\sigma_V = 30\%$ and $r = 0.1\%$. The singular components corresponding to the Dirac-δ located at $x = 0$ are not depicted in graph. This example indicates that, given default, the most probable recovery rate falls in the approximate range of $R^* \in [20\%, 40\%]$.



2 RECOVERY VALUE IN BLACK-COX MODEL

Find a closed form expression for the recovery value $B^{\text{rec}}(t,T)$ in the Black and Cox model by evaluating

$$B^{\text{rec}}(t,T) = -\int_t^T e^{-r(s-t)} H(s) \frac{\partial}{\partial s} \Phi(-d(t), -d(t), -b, s-t)\, ds,$$

where $d(t) = \frac{1}{\sigma_V} \log \frac{H(t)}{V_0}$, $H(t) = H_0 e^{at}$, $b = \frac{r - \frac{1}{2}\sigma_V^2 - a}{\sigma_V}$ and $\Phi(\alpha,\beta,b,t) = N\left(\frac{\alpha - bt}{\sqrt{t}}\right) - e^{2b\beta} N\left(\frac{\alpha - 2\beta - bt}{\sqrt{t}}\right)$.

Solution: Given $H(t) = H_0 e^{at}$, we can write $d(t) = \frac{1}{\sigma_V}\left(\log \frac{H_0}{V_0} + at\right)$. Let $u = s - t$ in

$$\Phi(-d(t), -d(t), -b, s-t) = N\left(\frac{-d(t) + b(s-t)}{\sqrt{s-t}}\right) - e^{2bd(t)} N\left(\frac{d(t) + b(s-t)}{\sqrt{s-t}}\right),$$

we write

$$\begin{aligned}
B^{\text{rec}}(t,T) &= -H_0 e^{at} \int_0^{T-t} e^{-(a-r)u} \frac{\partial}{\partial u} \Phi(-d(t), -d(t), -b, u)\, du \\
&= -H_0 e^{at} \int_0^{T-t} e^{-(a-r)u} \left[\frac{\partial}{\partial u} N\left(\frac{-d(t) + bu}{\sqrt{u}}\right) - e^{2bd(t)} \frac{\partial}{\partial u} N\left(\frac{d(t) + bu}{\sqrt{u}}\right)\right] du \\
&= -H_0 e^{at} \left[\int_0^{T-t} e^{-(a-r)u} \frac{\partial}{\partial u} N\left(\frac{-d(t) + bu}{\sqrt{u}}\right) du - e^{2bd(t)} \int_0^{T-t} e^{-(a-r)u} \frac{\partial}{\partial u} N\left(\frac{d(t) + bu}{\sqrt{u}}\right) du\right].
\end{aligned}$$

Denote $\tau = T - t$ and $c = a - r$, we compute

$$\begin{aligned}
&\int_0^\tau e^{-cu} \frac{\partial}{\partial u} N\left(\frac{bu - d}{\sqrt{u}}\right) du \\
&= \frac{1}{2\sqrt{2\pi}} \int_0^\tau e^{-cu} e^{-\frac{(bu-d)^2}{2u}} \left(\frac{b}{\sqrt{u}} + \frac{d}{\sqrt{u^3}}\right) du \\
&= \frac{1}{2\sqrt{2\pi}} \int_0^\tau e^{-\frac{2cu^2 + (bu-d)^2}{2u}} \left(\frac{b}{\sqrt{u}} + \frac{d}{\sqrt{u^3}}\right) du \\
&= \frac{1}{2\sqrt{2\pi}} \int_0^\tau e^{-\frac{\left(2c+b^2\right)u^2 - 2bdu + d^2}{2u}} \left(\frac{b}{\sqrt{u}} + \frac{d}{\sqrt{u^3}}\right) du \\
&= \frac{1}{2\sqrt{2\pi}} \int_0^\tau e^{-\frac{g^2u^2 - 2bdu + d^2}{2u}} \left(\frac{b}{\sqrt{u}} + \frac{d}{\sqrt{u^3}}\right) du
\end{aligned}$$

Here denote: $g \triangleq \sqrt{2c + b^2}$

$$\begin{aligned}
&= \frac{e^{(b\pm g)d}}{2\sqrt{2\pi}} \int_0^\tau e^{-\frac{(gu\pm d)^2}{2u}} \left(\frac{b}{\sqrt{u}} + \frac{d}{\sqrt{u^3}}\right) du \\
&= \frac{e^{(b\pm g)d}}{4\sqrt{2\pi}} \int_0^\tau e^{-\frac{(gu\pm d)^2}{2u}} \left[\left(\frac{b}{g}+1\right)\left(\frac{g}{\sqrt{u}} + \frac{d}{\sqrt{u^3}}\right) + \left(\frac{b}{g}-1\right)\left(\frac{g}{\sqrt{u}} - \frac{d}{\sqrt{u^3}}\right)\right] du \\
&= \frac{e^{(b-g)d}}{4\sqrt{2\pi}} \left(\frac{b}{g}+1\right) \int_0^\tau e^{-\frac{(gu-d)^2}{2u}} \left(\frac{g}{\sqrt{u}} + \frac{d}{\sqrt{u^3}}\right) du + \frac{e^{(b+g)d}}{4\sqrt{2\pi}} \left(\frac{b}{g}-1\right) \int_0^\tau e^{-\frac{(gu+d)^2}{2u}} \left(\frac{g}{\sqrt{u}} - \frac{d}{\sqrt{u^3}}\right) du \\
&= \frac{(b+g)e^{(b-g)d}}{2g} \int_0^\tau dN\left(\frac{gu-d}{\sqrt{u}}\right) + \frac{(b-g)e^{(b+g)d}}{2g} \int_0^\tau dN\left(\frac{gu+d}{\sqrt{u}}\right)
\end{aligned}$$

$$= \frac{(b+g)e^{(b-g)d}}{2g} N\left(\frac{gu-d}{\sqrt{u}}\right)\Bigg|_0^\tau + \frac{(b-g)e^{(b+g)d}}{2g} N\left(\frac{gu+d}{\sqrt{u}}\right)\Bigg|_0^\tau.$$

Thus, if $d > 0$,

$$\int_0^\tau e^{-cu}\frac{\partial}{\partial u} N\left(\frac{bu-d}{\sqrt{u}}\right) du = \frac{(b+g)e^{(b-g)d}}{2g} N\left(\frac{g\tau-d}{\sqrt{\tau}}\right) - \frac{(b-g)e^{(b+g)d}}{2g} N\left(-\frac{g\tau+d}{\sqrt{\tau}}\right)$$
$$\int_0^\tau e^{-cu}\frac{\partial}{\partial u} N\left(\frac{bu+d}{\sqrt{u}}\right) du = \frac{(b-g)e^{-(b+g)d}}{2g} N\left(\frac{g\tau-d}{\sqrt{\tau}}\right) - \frac{(b+g)e^{-(b-g)d}}{2g} N\left(-\frac{g\tau+d}{\sqrt{\tau}}\right).$$

Back to the evaluation of $B^{\text{rec}}(t,T)$, assuming $b^2+2(a-r)>0$. If $d(t) = \frac{1}{\sigma_V}\left(\log\frac{H_0}{V_0}+at\right) > 0$,

$$\begin{aligned}
B^{\text{rec}}(t,T) &= -H_0 e^{at}\left[\frac{(b+g)e^{(b-g)d(t)}}{2g} N\left(\frac{g\tau-d(t)}{\sqrt{\tau}}\right) - \frac{(b-g)e^{(b+g)d(t)}}{2g} N\left(-\frac{g\tau+d(t)}{\sqrt{\tau}}\right)\right] \\
&\quad + H_0 e^{at+2bd(t)}\left[\frac{(b-g)e^{-(b+g)d(t)}}{2g} N\left(\frac{g\tau-d(t)}{\sqrt{\tau}}\right) - \frac{(b+g)e^{-(b-g)d(t)}}{2g} N\left(-\frac{g\tau+d(t)}{\sqrt{\tau}}\right)\right] \\
&= -H_0 e^{at}\left[\frac{(b+g)e^{(b-g)d(t)}}{2g} N\left(\frac{g\tau-d(t)}{\sqrt{\tau}}\right) - \frac{(b-g)e^{(b+g)d(t)}}{2g} N\left(-\frac{g\tau+d(t)}{\sqrt{\tau}}\right)\right] \\
&\quad + H_0 e^{at}\left[\frac{(b-g)e^{(b-g)d(t)}}{2g} N\left(\frac{g\tau-d(t)}{\sqrt{\tau}}\right) - \frac{(b+g)e^{(b+g)d(t)}}{2g} N\left(-\frac{g\tau+d(t)}{\sqrt{\tau}}\right)\right] \\
&= -H_0 e^{at}\left[e^{(b-g)d(t)} N\left(\frac{g(T-t)-d(t)}{\sqrt{T-t}}\right) + e^{(b+g)d(t)} N\left(-\frac{g(T-t)+d(t)}{\sqrt{T-t}}\right)\right] < 0.
\end{aligned}$$

If instead $d(t) = \frac{1}{\sigma_V}\left(\log\frac{H_0}{V_0}+at\right) < 0$, which is the more relevant case because $H(T) < F$,

$$\begin{aligned}
B^{\text{rec}}(t,T) &= -H_0 e^{at}\left[\frac{(b-g)e^{(b+g)d(t)}}{2g} N\left(\frac{g\tau+d(t)}{\sqrt{\tau}}\right) - \frac{(b+g)e^{(b-g)d(t)}}{2g} N\left(-\frac{g\tau-d(t)}{\sqrt{\tau}}\right)\right] \\
&\quad + H_0 e^{at+2bd(t)}\left[\frac{(b+g)e^{-(b-g)d(t)}}{2g} N\left(\frac{g\tau+d(t)}{\sqrt{\tau}}\right) - \frac{(b-g)e^{-(b+g)d(t)}}{2g} N\left(-\frac{g\tau-d(t)}{\sqrt{\tau}}\right)\right] \\
&= -H_0 e^{at}\left[\frac{(b-g)e^{(b+g)d(t)}}{2g} N\left(\frac{g\tau+d(t)}{\sqrt{\tau}}\right) - \frac{(b+g)e^{(b-g)d(t)}}{2g} N\left(-\frac{g\tau-d(t)}{\sqrt{\tau}}\right)\right] \\
&\quad + H_0 e^{at}\left[\frac{(b+g)e^{(b+g)d(t)}}{2g} N\left(\frac{g\tau+d(t)}{\sqrt{\tau}}\right) - \frac{(b-g)e^{(b-g)d(t)}}{2g} N\left(-\frac{g\tau-d(t)}{\sqrt{\tau}}\right)\right] \\
&= H_0 e^{at}\left[e^{(b-g)d(t)} N\left(-\frac{g(T-t)-d(t)}{\sqrt{T-t}}\right) + e^{(b+g)d(t)} N\left(\frac{g(T-t)u+d(t)}{\sqrt{T-t}}\right)\right] > 0.
\end{aligned}$$

where $g \triangleq \sqrt{b^2+2(a-r)}$. The result is amazingly simple.

3 Cox Process with Normal Intensity

Consider the Cox model where the intensity follows the normal process

$$d\lambda(t) = \theta dt + \sigma dW(t),$$

with constant θ and $\lambda(t=0) = \lambda_0$.

(i) Calculate the survival probability $S(0,t)$ for this Cox process.

Solution: Integrate the normal process, $\lambda(t) = \lambda_0 + \theta t + \sigma W(t)$. Integrate again with respect to time,

$$\int_0^t \lambda(s)ds = \lambda_0 t + \frac{1}{2}\theta t^2 + \sigma \int_0^t W(s)ds \sim N\left(\lambda_0 t + \frac{1}{2}\theta t^2, \frac{1}{3}\sigma^2 t^3\right),$$

where the stochastic version of Fubini's thoerem is used to compute the probability distribution of the integral $\int_0^t W(s)ds$. The survival probability

$$\begin{aligned} S(0,t) &\triangleq \mathbb{E}\left[e^{-\int_0^t \lambda(s)ds}\right] \\ &= \mathbb{E}\left[e^{-\lambda_0 t - \frac{1}{2}\theta t^2 - \sigma\int_0^t W(s)ds}\right] \\ &= e^{-\lambda_0 t - \frac{1}{2}\theta t^2 + \frac{1}{6}\sigma^2 t^3}, \end{aligned}$$

where we have used $\mathbb{E}\left[e^{-X}\right] = e^{-\mathbb{E}[X] + \frac{1}{2}\mathrm{Var}[X]}$ for a normal variable X.

(ii) Design and implement an algorithm for simulating events from this Cox process.

```
import math
import numpy as np
print("-------------------------")
print("Simulation of Cox Process")
print("-------------------------")
# input parameters
lambda0 = 0.03 # initial intensity
theta = 0.005 # intensity drift
sigma = 0.008 # intensity diffusion
t = 10.0 # time horizon
ns = 1000000 # number of samples
dt = 0.001 # time step width
timer = np.array([1,2,3,4,5,6,7,8,9,10]) # time points to record counts

# derived parameters
n = int(t/dt) # time step number
dt_root = math.sqrt(dt) # square root of time step
theta_dt = theta*dt
indexer = (timer/dt).astype(int);
m = len(timer)
stats = np.zeros(m)
```

```python
# simulation
for k in range(ns):
    timesteps = np.zeros(n+1)
    lambdas = np.zeros(n+1)
    lambdas[0] = lambda0
    counts = np.zeros(n+1)
    for i in range(n):
        # march forward in time
        timesteps[i+1] = timesteps[i] + dt
        # generate intensity process
        normal = np.random.normal(0, dt_root)
        lambdas[i+1] = lambdas[i] + theta_dt + sigma*normal
        # generate counting process
        lambda_dt = lambdas[i+1]*dt
        uniform = np.random.random()
        if uniform <= lambda_dt: counts[i+1] = counts[i]+1
        else: counts[i+1] = counts[i]

    # generate statistics
    for i in range(m):
        if counts[indexer[i]] > 0:
            stats[i] += 1
```

(iii) Verify the correctness of your algorithm by comparing the estimated survival probabilities with the closed form formula. Use $\lambda_0 = 0.03$, $\theta = 0.005$, $\sigma = 0.008$, and a range of values of t, say $t = 1, 2, \cdots, 10$.

Solution: We sample 1,000,000 paths $\lambda(t)_{0\le t\le T}$ and simulate the counting process conditioned on each $\lambda(t)$ path. The width of each time step is chosen to be $\Delta t = 0.001$. The results are reported in the following table. The column "Miss Events" records the number of sample paths that do not end up with a default, and the vice versa for column "Hit Events". The estimated survival rates tabulated in the column "$S(0,t)$ Estimate" match the theoretical values of the survival rate up to the third significant digits, which reflects the choice of the time step width around 0.001. To improve the agreement, the time step width needs to be further refined.

t	$S(0,t)$ Theoretical	Miss Events	Hit Events	$S(0,t)$ Estimate	Approximation Error
1	0.968033	31788	968212	0.968212	0.0185%
2	0.932473	67308	932692	0.932692	0.0234%
3	0.893855	106099	893901	0.893901	0.0051%
4	0.852726	147220	852780	0.852780	0.0063%
5	0.809639	190362	809638	0.809638	-0.0001%
6	0.76514	234901	765099	0.765099	-0.0054%
7	0.719757	280106	719894	0.719894	0.0190%
8	0.673991	325597	674403	0.674403	0.0611%
9	0.628308	371545	628455	0.628455	0.0234%
10	0.583137	416500	583500	0.583500	0.0623%