MTH 9831 Fall 2015 HW #1 Solutions

Please note that these are *possible* solutions, and that there are often many ways to approach and solve a problem or prove a theorem.

**Problem 1** It should be easily recognized that

$$\phi_X(t) = \exp(2e^{it} - 2), \quad \phi_Y(t) = \left(\frac{3}{4}e^{it} + \frac{1}{4}\right)^{10}$$

are the characteristic functions of $X \sim \text{Poisson}(\lambda = 2)$ and $Y \sim B(n = 10, p = 3/4)$. Since $X$ and $Y$ are assumed independent, the requested values are easy to calculate:

$$(a) \qquad E(XY) = E(X)E(Y) = 2 \cdot 10 \cdot \frac{3}{4} = 15;$$

$$\begin{aligned}(b) \qquad P(XY = 0) &= P(\{X = 0\} \cup \{Y = 0\}) = P(X = 0) + P(Y = 0) - P(X = 0)P(Y = 0) \\ &= e^{-2} + \left(\frac{1}{4}\right)^{10} - e^{-2}\left(\frac{1}{4}\right)^{10};\end{aligned}$$

$$\begin{aligned}(c) \quad P(X + Y = 2) &= P(X = 0)P(Y = 2) + P(X = 1)P(Y = 1) + P(X = 2)P(Y = 0) \\ &= 45e^{-2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^8 + 20e^{-1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^9 + 2e^{-2}\left(\frac{1}{4}\right)^{10}.\end{aligned}$$

**Problem 2** Let $X$ and $Y$ be independent normal random variables, i.e. $X \sim N(\mu_X, \sigma)$, $Y \sim N(\mu_Y, \sigma)$. We prove that $X + Y$ and $X - Y$ are also independent by showing $\phi_{(X+Y,X-Y)}(u, v) = \phi_{X+Y}(u)\phi_{X-Y}(v)$ for $u, v \in \mathbb{R}$. Using the definition of the characteristic function, the independence of $X$ and $Y$ and that both $X$ and $Y$ are normally distributed, we get

$$\begin{aligned}\phi_{(X+Y,X-Y)}(u, v) &= E[e^{\mathrm{i}(X+Y)u+\mathrm{i}(X-Y)v}] = E[e^{\mathrm{i}(u+v)X+\mathrm{i}(u-v)Y}] = \phi_X(u + v)\phi_Y(u - v) \\ &= e^{\mathrm{i}u(\mu_X+\mu_Y)+\mathrm{i}v(\mu_X-\mu_Y)-\sigma^2(u^2+v^2)} = \phi_{X+Y}(u)\phi_{X-Y}(v).\end{aligned}$$

**Problem 3** $f_{B(t)}(x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$ is known. The conditional density of $B(t)$ given $B(\alpha t)$, $\alpha \in (0, 1)$ is easy to write down, since increments are independent, and $B(t) - B(\alpha t) \overset{(d)}{=} B((1 - \alpha)t)$:

$$f_{B(t)|B(\alpha t)}(y|x) = f_{B(t)-B(\alpha t)}(y - x) = f_{B((1-\alpha)t)}(y - x).$$

Thus, the conditional density of $B(\alpha t)$, $\alpha \in (0, 1)$, given $B(t) = y$, is, by Bayes' Law for densities,

$$\begin{aligned}f_{B(\alpha t)|B(t)}(x|y) &= \frac{f_{B(t)|B(\alpha t)}(y|x)f_{B(\alpha t)}(x)}{f_{B(t)}(y)} = \frac{f_{B((1-\alpha)t)}(y - x)f_{B(\alpha t)}(x)}{f_{B(t)}(y)} \\ &= \frac{\frac{1}{\sqrt{2\pi(1-\alpha)t}}e^{-\frac{(y-x)^2}{2(1-\alpha)t}}\frac{1}{\sqrt{2\pi\alpha t}}e^{-\frac{x^2}{2\alpha t}}}{\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}} = \ldots = \frac{1}{\sqrt{2\pi\alpha(1 - \alpha)t}}e^{-\frac{(x-\alpha y)^2}{2\alpha(1-\alpha)t}},\end{aligned}$$

where there's a square completion calculation that has been omitted here. This means that, conditioned on $B(t) = y$, $B(\alpha t) \sim N(\alpha y, \alpha(1 - \alpha)t)$.

**Problem 4** $B(t) = (B_1(t), B_2(t))$, $B_j(t)$ independent. The distance between $B(t)$ and the origin is $|B(t)| = \sqrt{B_1^2(t) + B_2^2(t)}$, which is the 2-dimensional Bessel process, with the

Rayleigh(t) distribution. For $x \geq 0$, and fixing $t > 0$, noting that $B_j(t) \sim \sqrt{t} Z_j$ for two independent standard normals, we have (with a change to polar coordinates)

$$\begin{aligned}
P(|B(t)| \leq x) &= P(Z_1^2 + Z_2^2 \leq x^2/t) \\
&= \frac{1}{2\pi} \iint_{\{(z_1,z_2):\, z_1^2+z_2^2 \leq x^2/t\}} e^{-(z_1^2+z_2^2)/2} dz_1 dz_2 \\
&= \int_0^{x/\sqrt{t}} \int_0^{2\pi} e^{-r^2/2} r\, dr\, d\theta \text{ (polar coordinates)} \\
&= 1 - e^{-\frac{x^2}{2t}}.
\end{aligned}$$

**Problem 5** Using $B(t)$ from #4, and $\rho \in [-1, 1]$, yes, $X(t) := \rho B_1(t) + \sqrt{1-\rho^2} B_2(t)$ is a Brownian motion. It starts at $X(0) = 0$, has continuous paths a.s., independent increments, and $Var(X(t)) = t$:

$$\begin{aligned}
X(t) - X(s) &= \rho(B_1(t) - B_1(s)) + \sqrt{1-\rho^2}(B_2(t) - B_2(s)) \perp X(s) = \rho B_1(s) + \sqrt{1-\rho^2} B_2(s) \\
Var(X(t)) &= \rho^2 Var(B_1(t)) + (1-\rho^2) Var(B_2(t)) = t \\
Cov(X(t), B_1(t)) &= E(X(t)B_1(t)) - E(X(t))E(B_1(t)) = E(\rho B_1^2(t) + \sqrt{1-\rho^2} B_1(t) B_2(t)) = \rho t.
\end{aligned}$$

**Problem 6** Which of these are Brownian motions?

(a) yes (the symmetry is obvious)

(b) yes

(c) no (this is the single RV $B(1)$, scaled deterministically as time moves forward)

(d) no (the increments are not independent - compare $X(2) - X(3/2) = B(4) - B(3) - B(2) + B(3/2)$ to $X(3/2) = B(3) - B(3/2)$)

(e) yes (for the interval $t \in [0, s]$)

**Problem 7** The joint distribution of BM and its max is, for $0 \leq a \leq x$, by the reflection principle,

$$\begin{aligned}
P(B^*(t) \geq a,\, B(t) \leq x) &= P(B^*(t) \geq a) - P(B^*(t) \geq a,\, B(t) > x) \\
&= 2P(B(t) \geq a) - P(B(t) > x) \\
&= 2(1 - N(a/\sqrt{t})) - (1 - N(x/\sqrt{t})) \\
&= 1 - 2N(a/\sqrt{t}) + N(x/\sqrt{t}).
\end{aligned}$$