A Minimal Model for Nonlinear Market Impact

ShengQuan Zhou, Zhongkai He

Baruch MFE

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A fully consistent, minimal model for non-linear market impact

J. Donier,[1,2] J. Bonart,[1,3] I. Mastromatteo,[4] and J.-P. Bouchaud[1,3]

[1] *Capital Fund Management, 23-25 Rue de l'Université, 75007 Paris, France.*
[2] *Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie (Paris 6).*
[3] *CFM-Imperial Institute of Quantitative Finance, Department of Mathematics, Imperial College, 180 Queen's Gate, London SW7 2RH*
[4] *Centre de Mathématiques Appliquées, CNRS, UMR7641, Ecole Polytechnique, 91128 Palaiseau, France.*

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A Reaction-Diffusion Model of Latent Order Book

Two pivotal ideas

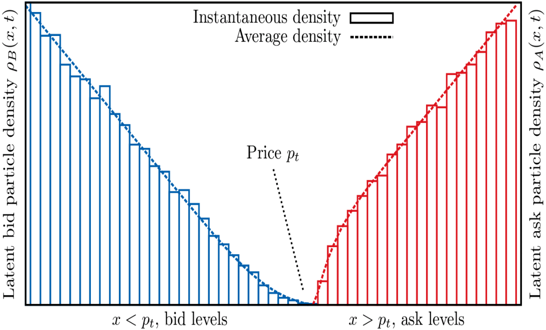
- Latent order book
- Reaction-Diffusion model

Order Book Dynamics (x : price)

- Buy volume density $\rho_B(x,t)$
- Sell volume density $\rho_A(x,t)$

Transaction Price p_t

$$\boxed{\rho_A(p_t,t) = \rho_B(p_t,t)}$$



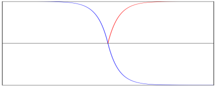
$$\frac{\partial \rho_B(x,t)}{\partial t} = -V_t \frac{\partial \rho_B(x,t)}{\partial x} + D\frac{\partial^2 \rho_B(x,t)}{\partial x^2} - \nu\rho_B(x,t) + \lambda\Theta(p_t - x) - \kappa R_{AB}(x,t)$$

$$\frac{\partial \rho_A(x,t)}{\partial t} = \underbrace{-V_t \frac{\partial \rho_A(x,t)}{\partial x} + D\frac{\partial^2 \rho_A(x,t)}{\partial x^2}}_{\substack{\text{Drift-Diffusion: price reassessments;} \\ V_t\sim\text{white noise, uninformed meta-orders}}} - \underbrace{\nu\rho_A(x,t)}_{\text{Cancellation}} + \underbrace{\lambda\Theta(x - p_t)}_{\text{Deposition}} - \underbrace{\kappa R_{AB}(x,t)}_{\text{Reaction: trades}}$$

Price Dynamics within a Locally Linear Order Book

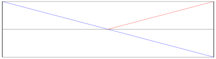
Define $\hat{p}_t \triangleq \int_0^t V_s ds$, $y \triangleq x - \hat{p}_t$, the net order density $\phi(x,t) \triangleq \rho_B(x,t) - \rho_A(x,t)$

$$\frac{\partial \phi(y,t)}{\partial t} = D\frac{\partial^2 \phi(y,t)}{\partial y^2} - \nu\phi(y,t) + \lambda \mathrm{sign}(p_t - \hat{p}_t - y)$$



Zoom into the *universal linear regime* $\nu \to 0$, $\lambda \to 0$, $J \triangleq D\left|\partial_y \phi_{\mathrm{s.t.}}\right|_{y=0} = \lambda\sqrt{D/\nu}$ fixed:

$$\phi_{\mathrm{s.t.}} = -\mathcal{L}y,$$



where $\mathcal{L} \triangleq J/D$ is the *latent liquidity* of the market. Add meta-order to the system:

$$\begin{cases} \frac{\partial \phi(y,t)}{\partial t} = D\frac{\partial^2 \phi(y,t)}{\partial y^2} + m_t \delta(y - y_t) \\ \partial_y \phi(y \to \pm\infty, t) = -\mathcal{L} \end{cases} \Rightarrow \phi(y,t) = -\mathcal{L}y + \int_0^t \frac{ds\, m_s}{\sqrt{4\pi(t-s)}} e^{-\frac{(y-y_s)^2}{4D(t-s)}}$$

where m_t is the signed trading intensity at time t. Transaction price $\phi(y_s, s) \equiv 0$

$$\boxed{y_t = \frac{1}{\mathcal{L}} \int_0^t \frac{ds\, m_s}{\sqrt{4\pi(t-s)}} e^{-\frac{(y_t-y_s)^2}{4D(t-s)}}}$$

This is the central equation of the paper: a self-consistent integral equation for $t > 0$.

Impact of Meta-Order and Impact Decay

For a meta-order of size Q executed at constant rate $m_t = \begin{cases} m_0 = Q/T, & t \in [0, T], \\ 0, & t > T. \end{cases}$

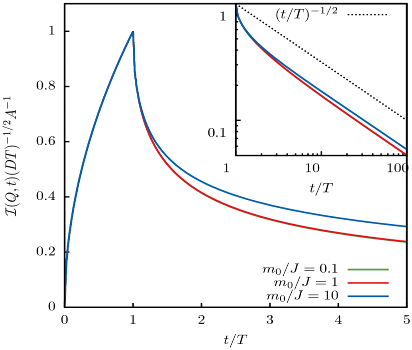
Market Impact $\mathcal{I}(Q,t) \triangleq \underbrace{\langle \epsilon \cdot (p_{t+T} - p_t) | Q \rangle}_{\text{Average over order signs } \epsilon} = \langle \epsilon \cdot (y_{t+T} - y_t) | Q \rangle \underbrace{+ \langle \epsilon \cdot (\hat{p}_{t+T} - \hat{p}_t) | Q \rangle}_{=0, \text{ uninformed meta-order}}.$

Solving the self-consistency equation for y_t yields the *impact profile*:

$$\frac{\mathcal{I}(Q,t)}{\mathcal{I}(Q)} = \begin{cases} \sqrt{\frac{t}{T}}, & t \in [0, T], \\ \underbrace{\sqrt{\frac{t}{T}} - \sqrt{\frac{t-T}{T}}}_{\text{power-law decay as } t \to \infty}, & t > T, \end{cases}$$

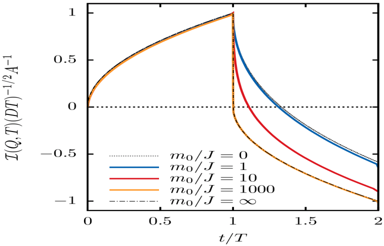
where the peak market impact gives precisely the celebrated *square-root impact law*:

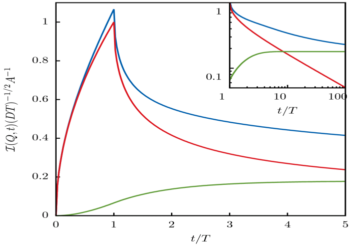
$$\mathcal{I}(Q) = \mathcal{I}(Q,T) \approx \begin{cases} \sqrt{\frac{m_0}{\pi J}} \times \sqrt{\frac{Q}{\mathcal{L}}}, & m_0 \ll J, \\ \sqrt{2} \times \sqrt{\frac{Q}{\mathcal{L}}}, & m_0 \gg J. \end{cases}$$



Reversions, Informational Impact & No Price Manipulation

- For a meta-order of size Q reverted after T, $m_t = \begin{cases} m_0 = Q/T, & t \in [0, T], \\ -m_0 = -Q/T, & t \in (T, 2T]. \end{cases}$
- Informative order flow m_t correlated with order book drift $V_{t'>t}$: *permanent impact*.





- Average cost of a closed price trajectory: $\int_0^T ds\, m_s = 0$.

$$\mathcal{C} = \int_0^T ds\, m_s y_s = \frac{1}{2}\int_0^T \int_0^T ds ds'\, M(s,s') m_s m_{s'}$$

where $M(s,s') = \frac{1}{\mathcal{L}\sqrt{4\pi|s-s'|}} e^{-\frac{(y_s - y_{s'})^2}{4D|s-s'|}}$ is a semi-positive definite kernel $\Rightarrow \mathcal{C} \geq 0$ for any execution schedule, i.e. price manipulation is impossible within the model of Locally Linear Order Book (LLOB).