# Time Series Analysis

## 2. Non-stationary univariate time series

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# Outline

# Trend-stationary and unit root processes

- So far we have focused on time series that are stationary (or, more precisely, covariance-stationary).
- We have seen that a stationary time series in the $ARMA(p, q)$ family can be written in the moving average (MA) form:

$$\begin{aligned} X_t &= \mu + \varepsilon_t + \gamma_1 \varepsilon_{t-1} + \gamma_2 \varepsilon_{t-2} \ldots \\ &= \mu + \gamma(L)\varepsilon_t, \end{aligned} \tag{1}$$

  where $L$ is the lag operator, and where $\sum_{j=1}^{\infty} |\gamma_j| < \infty$.
- Stationary series are rather unusual in finance, and hence the need for developing models that capture the non-stationary nature of financial time series.
- There are various approaches to model non-stationarity. We will initially focus on two of them:
  - (i) Non-stationary process with a deterministic trend and stationary disturbances.
  - (ii) Non-stationary process with a unit root (non-stationary disturbances).

# Trend-stationary and unit root processes

- An example of the former type of a time series is the following process with *linear trend*:

$$X_t = \alpha + \delta t + \gamma(L)\varepsilon_t, \tag{2}$$

where $\alpha, \delta \in \mathbb{R}$. This amounts to replacing the constant mean $\mu$ of the stationary process (1) with a linear function.

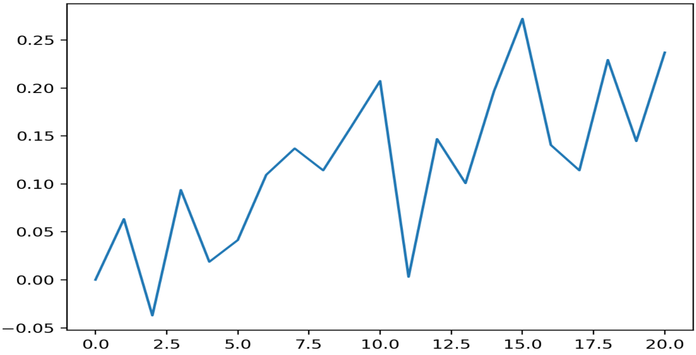
- The process behaves thus like a pure deterministic trend perturbed by a stationary random noise, and is referred to as a *trend-stationary* process.
- This is to be contrasted with the second type of non-stationarity mentioned above.
- Consider the following *unit root* process:

$$X_t = \delta + X_{t-1} + \gamma(L)\varepsilon_t. \tag{3}$$

Here the non-stationarity comes from the presence of the unit root $\beta = 1$ in the autoregressive part of the specification above.

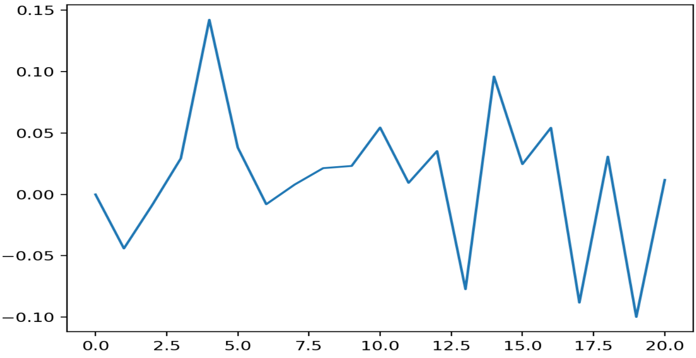
# Trend-stationary and unit root processes

- The graph below shows a simulated $AR(1)$ time series (with deterministic trend $\alpha + \delta t$) with the following choice of parameters: $\alpha = 0.0$, $\delta = 0.01$, $\beta = 0.3$, $\sigma = 0.05$.



# Trend-stationary and unit root processes

- The graph below shows a simulated unit root $AR(1)$ time series with the following choice of parameters: $\alpha = 0.0$, $\beta = 1.0$, $\sigma = 0.05$.



# Trend-stationary and unit root processes

- A unit root process exhibits purely stochastic, persisting trends that have their source in the non-stationarity of the shocks delivered to the system. Iterating (3), we can write it as

$$X_t = X_0 + \delta t + \gamma(L) \sum_{j=0}^{t} L^j \varepsilon_t. \quad (4)$$

- This representation shows explicitly that the variance of the random shock grows linearly in $t$.
- Processes of this form are also referred to as *integrated* of order 1 and are denoted by $I(1)$.
- This name is motivated by the following observation. The operator

$$\Delta = 1 - L \quad (5)$$

is called the first *difference operator*. One can think about $\Delta$ as a discretized time derivative.

# ARIMA models

- Using it, we can write (3) in the form:

$$\Delta X_t = \delta + \gamma(L)\varepsilon_t. \tag{6}$$

- Equation (4) is then the "integrated" version of this difference equation.
- More generally, integrated processes $I(d)$ of any integer order $d \geq 1$ are of the form:

$$\Delta^d X_t = \omega + \gamma(L)\varepsilon_t. \tag{7}$$

- For example, for $d = 2$, $(1 - L)^2 = 1 - 2L + L^2$, and an $I(2)$ process can be written as

$$X_t = \omega + 2X_{t-1} - X_{t-2} + \gamma(L)\varepsilon_t. \tag{8}$$

- We will now recast these ideas in terms of ARIMA models.

# ARIMA models

- Recall that an *ARMA*$(p, q)$ model can be written in the form:

$$\psi(L)X_t = \alpha + \varphi(L)\varepsilon_t, \tag{9}$$

where

$$\begin{aligned} \psi(z) &= 1 - \beta_1 z - \ldots - \beta_p z^p, \\ \varphi(z) &= 1 + \theta_1 z + \ldots + \theta_q z^q. \end{aligned} \tag{10}$$

Covariance-stationarity requires that the roots of $\psi(z)$ lie outside of the unit circle.

- This coincides with equation (1), if we set $\mu = \alpha/\psi(1)$ and $\gamma(L) = \psi(L)^{-1}\varphi(L)$.
- We will now assume that the characteristic polynomial has a unit root of degree $d > 0$, i.e. it is of the form $\psi_1(z)(1 - z)^d$, where $\psi_1(z)$ is a polynomial of degree $p$ with roots outside of the unit circle.

# ARIMA models

- This leads us to the concept of an *autoregressive integrated moving average (ARIMA) model.*
- An $ARIMA(p, d, q)$ model is specified as follows:

$$\begin{aligned} X_t &= \alpha + \delta t + U_t, \\ \psi(L)(1-L)^d U_t &= \varphi(L)\varepsilon_t. \end{aligned} \tag{11}$$

  Here, $p$ is the number of autoregressive lags (without the unit roots), $d$ is the order of integration (the order of the unit root), and $q$ is the number of the moving average lags (we are dropping the subscript 1 in $\psi$).
- Python implementation of $ARIMA(p, d, q)$ is in the package `statsmodels`.

# ARIMA models

- Examples of *ARIMA*$(p, d, q)$ models include:
  (i) *ARIMA*$(0, 0, 0)$. This is simply the white noise process:
  $$X_t = \varepsilon_t. \tag{12}$$
  (ii) *ARIMA*$(0, 1, 0)$. This is the random walk process:
  $$X_t = X_{t-1} + \varepsilon_t. \tag{13}$$
  (iii) *ARIMA*$(1, 0, 1)$. This is the *exponentially weighted moving average model* (EWMA):
  $$X_t = \lambda X_{t-1} + \varepsilon_t + (1 - \lambda)\varepsilon_{t-1}. \tag{14}$$
  (iv) *ARIMA*$(0, 2, 2)$. This is a general linear exponential smoothing model:
  $$X_t = 2X_{t-1} - X_{t-2} + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}. \tag{15}$$
  In this model both the level and the slope of the time series are smoothed using exponentially weighted moving averages.

# Forecasting non-stationary time series

- Forecasting a non-stationary time series uses the methodology explained in Lecture Notes #1. Consider first the trend stationary time series (2).
- A one-period forecast is given by

$$\begin{aligned} X^*_{t+1|1:t} &= \mathrm{E}_t(X_{t+1}) \\ &= \alpha + \delta(t+1) + \gamma_1\varepsilon_t + \gamma_2\varepsilon_{t-1} + \dots, \end{aligned} \tag{16}$$

  since $\mathrm{E}_t(\varepsilon_{t+1}) = 0$.
- Likewise, a $k$-period forecast is given by

$$\begin{aligned} X^*_{t+k|1:t} &= \mathrm{E}_t(X_{t+k}) \\ &= \alpha + \delta(t+k) + \gamma_k\varepsilon_t + \gamma_{k+1}\varepsilon_{t-1} + \dots, \end{aligned} \tag{17}$$

  since $\mathrm{E}_t(\varepsilon_{t+j}) = 0$, for all $j > 0$.

# Forecasting non-stationary time series

- Let us now estimate the magnitude of the forecast error. Its value is

$$\begin{aligned} X_{t+k} - X^*_{t+k|1:t} &= \alpha(t+k) + \delta + \varepsilon_{t+k} + \gamma_1\varepsilon_{t+k-1} + \gamma_2\varepsilon_{t+k-2} + \ldots \\ &\quad + \gamma_{k-1}\varepsilon_{t+1} + \gamma_k\varepsilon_t + \gamma_{k+1}\varepsilon_{t-1} + \ldots \\ &\quad - \big(\alpha + \delta(t+k) + \gamma_k\varepsilon_t + \gamma_{k+1}\varepsilon_{t-1} + \ldots\big) \\ &= \varepsilon_{t+k} + \gamma_1\varepsilon_{t+k-1} + \gamma_2\varepsilon_{t+k-2} + \ldots + \gamma_{k-1}\varepsilon_{t+1}. \end{aligned}$$

- The variance of the forecast error is

$$\mathsf{E}_t\big((X_{t+k} - X^*_{t+k|1:t})^2\big) = \sigma^2(1 + \gamma_1^2 + \ldots + \gamma_{k-1}^2). \tag{18}$$

- Note that the series on the RHS converges as the time horizon $k$ goes to infinity, and its limit is the variance of $\gamma(L)\varepsilon_t$.

# Forecasting non-stationary time series

- For the unit root process (3), the one-period forecast is

$$X^*_{t+1|1:t} = X_t + \delta + \gamma_1 \varepsilon_t + \gamma_2 \varepsilon_{t-1} + \ldots . \tag{19}$$

- More generally, the $k$-period forecast is

$$X^*_{t+k|1:t} = X_t + \delta k + (\gamma_1 + \ldots + \gamma_k)\varepsilon_t + (\gamma_2 + \ldots + \gamma_{k+1})\varepsilon_{t-1} + \ldots . \tag{20}$$

- It is easy to see that the forecast error is

$$\begin{aligned} X_{t+k} - X^*_{t+k|1:t} = \varepsilon_{t+k} + (1+\gamma_1)\varepsilon_{t+k-1} + (1+\gamma_1+\gamma_2)\varepsilon_{t+k-2} + \ldots \\ + (1+\gamma_1+\ldots+\gamma_{k-1})\varepsilon_{t+1} . \end{aligned} \tag{21}$$

# Forecasting non-stationary time series

- The variance of the forecast error is

$$\mathsf{E}_t\big((X_{t+k} - X^*_{t+k|1:t})^2\big) = \sigma^2(1 + (1+\gamma_1)^2 + \ldots + (1 + \gamma_1 + \ldots + \gamma_{k-1})^2). \quad (22)$$

- The quality of the forecast deteriorates significantly with the length of the forecasting horizon: this expression diverges linearly (proportionally to $k$), as $k \longrightarrow \infty$.
- In summary, for a trend-stationary process the forecast error remains bounded as the forecasting horizon increases.
- In contrast, for a unit root process, the forecast error increases (asymptotically) linearly, as the length of the horizon goes to infinity.

# Dickey-Fuller test

- It is of practical importance to determine whether a time series has a unit root. A number of statistical tests for detecting the presence of a unit root in a time series.
- The *Dickey-Fuller test* (DF) tests the null hypothesis $H_0$ of whether a unit root is present in an $AR(1)$ model against the alternative hypothesis $H_A$ of stationarity.
- In other words,
$$H_0 : \beta = 1 \text{ against } H_A : \beta < 1. \tag{23}$$
- Various versions of this test address different model specifications.

# Dickey-Fuller test

- Consider first the case of the $AR(1)$ time series with $\alpha = 0$, $\delta = 0$. As we saw in Lecture Notes #1, the MLE estimate of $\beta$ is given by

$$\widehat{\beta} = \frac{\sum_{t=0}^{T-1} x_t x_{t+1}}{\sum_{t=0}^{T-1} x_t^2}\,, \tag{24}$$

and hence the deviation of $\widehat{\beta}$ from 1 can be expressed as follows:

$$T(\widehat{\beta} - 1) = \frac{\frac{1}{T}\sum_{t=0}^{T-1} x_t \varepsilon_{t+1}}{\frac{1}{T^2}\sum_{t=0}^{T-1} x_t^2}\,. \tag{25}$$

- If the true value of $\beta$ is 1, then as $T \longrightarrow \infty$,

$$\begin{aligned}\frac{1}{T}\sum_{t=0}^{T-1} x_t \varepsilon_{t+1} &\longrightarrow \sigma^2 \int_0^1 W(s) dW(s)\\ &= \frac{1}{2}\sigma^2 (W(1)^2 - 1).\end{aligned}$$

# Dickey-Fuller test

- Similarly,

$$\frac{1}{T^2}\sum_{t=0}^{T-1} x_t^2 \longrightarrow \sigma^2 \int_0^1 W(s)^2 ds.$$

- As a result, under the null hypothesis $H_0$,

$$T(\widehat{\beta} - 1) \longrightarrow \frac{1}{2}\,\frac{W(1)^2 - 1}{\int_0^1 W(s)^2 ds}\,. \tag{26}$$

  The variable $T(\widehat{\beta} - 1)$ describes asymptotically the deviation of $\widehat{\beta}$ from 1. It is the ratio of a $\chi^2$-distributed variable and a non-standard random variable.

- The distribution of the random variable on the RHS of the expression above is not known in closed form.
- In practice, critical values of this random variables can be calculated for finite values of $T$ by means of Monte Carlo simulations for a number of benchmark levels.

# Dickey-Fuller test

- Alternatively, one can use the Dickey-Fuller $t$-stat:

$$t_\beta = \frac{\widehat{\beta} - 1}{\widehat{\sigma}_\beta}\,, \tag{27}$$

where

$$\widehat{\sigma}_\beta^2 = \widehat{\sigma}^2 + \sum_{t=1}^{T} x_{t-1}^2 \tag{28}$$

is the estimated variance for $\widehat{\beta} - 1$, and $\widehat{\sigma}^2$ is the estimated variance of the residuals.

- The limiting distribution of $t_\beta$ is not Gaussian. Reasoning as above, we see that its limit is explicitly given by

$$t_\beta \longrightarrow \frac{1}{2}\,\frac{W(1)^2 - 1}{\sqrt{\int_0^1 W(s)^2 ds}}\,. \tag{29}$$

# Dickey-Fuller test

- Next, we consider the same $AR(1)$ dynamics without drift, but we this time we include estimated $\alpha$ in the test. In this case, the large $T$ limits are:

$$\begin{aligned} T(\widehat{\beta}-1) &\longrightarrow \frac{\frac{1}{2}\left(W(1)^2-1\right)-W(1)\int_0^1 W(s)ds}{\int_0^1 W(s)^2ds-\left(\int_0^1 W(s)ds\right)^2}\,, \\ \sqrt{T}\widehat{\alpha} &\longrightarrow \sigma\,\frac{W(1)\int_0^1 W(s)^2ds-\frac{1}{2}\left(W(1)^2-1\right)}{\int_0^1 W(s)^2ds-\left(\int_0^1 W(s)ds\right)^2}\,. \end{aligned} \tag{30}$$

- Neither $T(\widehat{\beta}-1)$ nor $\sqrt{T}\widehat{\alpha}$ has normal distribution in the limit of large $T$.
- Their critical values, for finite $T$, can be found by means of Monte Carlo simulations.

# Dickey-Fuller test

- Alternatively, one can use the Dickey-Fuller $t$-stat defined in analogy to (27). Its $T \to \infty$ limit is

$$t_\beta \longrightarrow \frac{\frac{1}{2}\left(W(1)^2 - 1\right) - W(1)\int_0^1 W(s)ds}{\sqrt{\int_0^1 W(s)^2 ds - \left(\int_0^1 W(s)ds\right)^2}}\,. \tag{31}$$

- The limit distribution is non-Gaussian: the random variable on the right hand side of the expression above is the ratio of two non-standard variables.
- The technical details of the informal limit arguments can be found in Chapter 17 of Hamilton's book [1].
- Finite $T$ critical values are calculated by means of Monte Carlo simulations, and are available in software packages..
- There are a number of other cases (true process having a constant term, or a deterministic drift), for which DF tests have been developed (see, again [1]).

# Augmented Dickey-Fuller test

- A useful extension of the DF test is the *augmented Dickey-Fuller test* (ADF). It is designed to test the null hypothesis of the presence of a unit root in a general $AR(p)$ time series sample against the alternative hypothesis of stationarity.
- Let us discuss the simplest case of an $AR(p)$ process whose true dynamics does not have the constan term:

$$(1 - \beta_1 L - \ldots - \beta_p L_p) X_t = \varepsilon_t. \tag{32}$$

- Define

$$\begin{aligned} \beta &= \beta_1 + \ldots + \beta_p, \\ \gamma_j &= -(\beta_{j+1} + \ldots + \beta_p), \quad \text{for } j = 1, \ldots, p-1. \end{aligned} \tag{33}$$

- We can then write (32) in the form:

$$\big(1 - \beta L - (\gamma_1 L + \ldots + \gamma_{p-1} L^{p-1})(1 - L)\big) X_t = \varepsilon_t,$$

or equivalently,

$$X_t = \beta X_{t-1} + \gamma_1 \Delta X_{t-1} + \ldots + \gamma_{p-1} \Delta X_{t-p+1} + \varepsilon_t. \tag{34}$$

# Augmented Dickey-Fuller test

- Suppose now that $X_t$ has a single unit root. This means that $\psi(1) = 0$, and so $\beta = 1$, while other roots of $\psi(z)$ lie outside of the unit circle.
- Under the null hypothesis $\beta = 1$, the dynamics of $X_t$ can be thus written as

$$X_t = X_{t-1} + (1 - \gamma_1 L - \ldots - \gamma_{p-1} L^{p-1})^{-1} \varepsilon_t. \tag{35}$$

- This has almost the same form as the $AR(1)$ model without a constant term, except that the shocks $\varepsilon_t$ are replaced with more complex (but stationary) shocks $z_t = (1 - \gamma_1 L - \ldots - \gamma_{p-1} L^{p-1})^{-1} \varepsilon_t$.
- The calculations carried out above for $p = 1$ can be generalized to the general case. The augmented Dickey-Fuller test is a test on the $t$-stat $t_\beta$ defined in analogy to the univariate case.
- A thorough discussion of the ADF test is presented in Chapter 17 of [1].
- The test is implemented as the function `adfuller` in `statsmodels`.

# ARIMA models

- Unit root time series tend to be finicky, and the presence of a unit root may be a transient phenomenon.
- A common cause of breakdown of stationarity is a *structural break*: an unexpected shift in the time series, often caused by an exogenous shock.
- The presence of a structural break can create the appearance that the shocks have permanent, rather than transitory, effects. That may bias the conclusion of a test towards a unit root.
- Structural breaks can lead to model misspecification and, consequently, significant forecasting errors.

# ARFIMA models

- An *autoregressive fractionally integrated moving average model* $ARFIMA(p, d, q)$ is an extension of the $ARIMA(p, d, q)$ model to non-integer difference parameter $d$.
- The fractional power of $1 - L$ is defined as

$$\begin{aligned}(1-L)^d &= \sum_{k=0}^{\infty} \binom{d}{k} (-L)^k \\ &= \sum_{k=0}^{\infty} \frac{d(d-1)\ldots(d-k+1)}{k!} (-L)^k \\ &= 1 - dL + \frac{1}{2} d(d-1) L^2 + \ldots .\end{aligned} \tag{36}$$

- Note that, for fractional $d$, this is an power infinite series in $L$ (rather than a polynomial of order $d$). $ARFIMA(p, d, q)$ models are thus useful for modeling time series with *long memory*.

# ARFIMA models

- As an example, consider a simple model $ARFIMA(0, d, 0)$ specified by:

$$(1 - L)^d X_t = \varepsilon_t, \tag{37}$$

or

$$X_t = d\varepsilon_{t-1} - \frac{1}{2} d(d-1)\varepsilon_{t-2} + \ldots . \tag{38}$$

- There is no simple closed form expression for the autocorrelation function in this model. However, with a bit of an effort one can show that the autocorrelation function has a power law decay asymptotically as $j \longrightarrow \infty$,

$$\begin{aligned} R_j &\approx |j|^{2d-1} \\ &= |j|^{2H}, \end{aligned} \tag{39}$$

where the *Hurst exponent* $H$ is given by $H = d - 1/2$.
- This is similar to properties of the *fractional Brownian motion* $W_H(t)$.

# Description of cointegration

- Various economic / financial series exhibit the phenomenon of *cointegration*.
- Two time series $X_t$ and $Y_t$ are said to be *cointegrated* if
  (i) Each of the series is $I(1)$, i.e. each time series is unit root non-stationary.
  (ii) Some linear combination of $X_t$ and $Y_t$ is covariance-stationary.
- A vector $a$, such that the time series

$$a^{\mathrm{T}} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = a_1 X_t + a_2 Y_t \tag{40}$$

  is stationary, is called a *cointegrating vector*.
- It may also be useful to consider more than two time series. In this case, one can talk of one or more cointegrating vectors.
- From the economic perspective, cointegration between two time series means that there is a long-term equilibrium relationship between the two time series, even though they are exposed to random shocks during their time evolution.

# Cointegrated series

- As an example, we consider the following two time series:

$$\begin{aligned} X_t &= \alpha_1 + \gamma Y_t + \varepsilon_{t,1}, \\ Y_t &= \alpha_2 + Y_{t-1} + \varepsilon_{t,2}, \end{aligned} \tag{41}$$

where the innovations $\varepsilon_{t,1} \sim N(0, \sigma_1^2)$ and $\varepsilon_{t,2} \sim N(0, \sigma_2^2)$ are independent.

- Both of these series are $I(1)$. Indeed,

$$\begin{aligned} \Delta Y_t &= \alpha_2 + \varepsilon_{t,2}, \\ \Delta X_t &= \gamma \Delta Y_t + \varepsilon_{t,1} - \varepsilon_{t-1,1} \\ &= \gamma\alpha_2 + \gamma\varepsilon_{t,2} + \varepsilon_{t,1} - \varepsilon_{t-1,1}, \end{aligned}$$

and so both differenced processes $\Delta X_t$ and $\Delta Y_t$ are stationary.

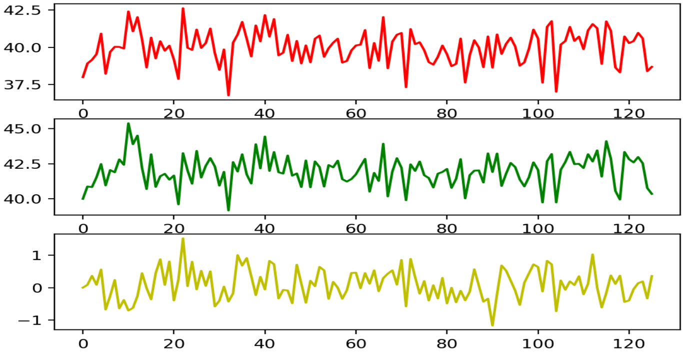
- The process

$$X_t - \gamma Y_t = a^{\mathrm{T}} \begin{pmatrix} X_t \\ Y_t \end{pmatrix}, \quad \text{where } a = \begin{pmatrix} 1 \\ -\gamma \end{pmatrix}, \tag{42}$$

is stationary, and thus the vector $a$ is a cointegrating vector for $X_t$ and $Y_t$.

# Cointegrated series

- The graph below shows a simulation of the system (41) with the following choice of parameters: $\alpha_1 = 0.1$, $\alpha_2 = 2.0$ $\gamma = 0.95$, $\sigma_1 = 0.5$, $\sigma_2 = 1.0$. The top graph shows $X_t$, the middle graph shows $Y_t$, and the bottom graph shows $X_t - \gamma Y_t$.



# Testing for cointegration

- We will now discuss statistical tests for cointegration. We continue assuming two univariate time series; we will address the general case in Lecture Notes #3.
- The test discussed below was originally developed by Engle and Granger, and it is based on regression techniques.
- Let $x_t$ and $y_t$ be the observations of the two time series, whose conintegration we want to test. We proceed as follows:
  (i) We first run the DF (or ADF) test for each of the series $x_t$ and $y_t$ is $I(1)$.
  (ii) Assuming, they both pass, we run the OLS regression $x = \gamma y + \alpha$, and form the vector $a$ as in formula (42).
  (iii) We now test whether $a$ is a cointegrating vector for $x_t$ and $y_t$. To this end, we first calcutate the "cointegrating residual process"

$$u_t = a^{\mathrm{T}} \begin{pmatrix} x_t \\ y_t \end{pmatrix} . \tag{43}$$

  (iv) We perform a unit root test on $u_t$ to determine whether it is $I(0)$ (i.e. stationary).

# Testing for cointegration

- Notice that step (ii) is skipped if the cointegrating vector $a$ is specified a priori.
- The null hypothesis $H_0$ in the Engle-Granger test is no cointegration, and the alternative is cointegration.
- There are two cases:
  (i) The proposed cointegrating vector $a$ is specified a priori, for exmple, by virtue of economic theory.
  (ii) The proposed cointegrating vector $a$ is estimated from the data by means of a linear regression. In this situation, several different cases may have to be considered, depending on whether the regression has no constant term, has a constant term, or has a deterministic trend.
- Tests for cointegration using a specified cointegrating vector $a$ are more powerful than tests relying on linear regression to estimate $a$.

# Stochastic volatility

- Another type of non-stationarity is present in models with stochastic volatility to which we shall now turn.
- In the specification of an $AR(1)$ time series model, it is assumed that the disturbances $\varepsilon_t$ are i.i.d. and $N(0, \sigma^2)$ distributed.
- Generally, this assumption is invalid in financial time series, as they typically exhibit periods of elevated and diminished volatility. This phenomenon is referred to as *stochastic volatility* or *heteroskedasticity*.
- The R package `stochvol` has a lot of useful code.

# EWMA model

- Conceptually, the simplest volatility model is the *exponentially weighted moving average* (EWMA) model. We have encountered this type of a model earlier in this lecture in a different context.
- It is specified as follows:

$$\begin{aligned}\varepsilon_t &= \sigma_t z_t,\\ \sigma_t^2 &= \lambda\sigma_{t-1}^2 + (1-\lambda)\varepsilon_{t-1}^2,\end{aligned} \tag{44}$$

  where $z_t \sim N(0,1)$, and where $0 < \lambda < 1$ is a parameter. Typically, $\lambda$ is close to 1, say $\lambda = 0.97$.
- The model is very easy to use. The innovations $\widehat{\varepsilon}_t$ are calculated from the observations $x_t$ of the underlying time series as $\widehat{\varepsilon}_t = x_t - x_{t-1}$. The scaling factor $\lambda$ can be found using MLE, or set to the preferred value.
- It is a good idea to "prime" the model by adding a period prior to $t = 0$ to let the model arise the impact of the initial value of the (unobserved) $\sigma$.

# ARCH models

- An *autoregressive conditional heteroskedasticity* (ARCH) model is specified as follows:

$$\begin{aligned} \varepsilon_t &= \sigma_t z_t \\ \sigma_t^2 &= \zeta_0 + \zeta_1 \varepsilon_{t-1}^2 + \ldots + \zeta_q \varepsilon_{t-q}^2, \end{aligned} \tag{45}$$

where, again, $z_t \sim N(0, 1)$ are i.i.d.. A models with this specification is denoted by $ARCH(q)$.
- The lagged terms on the RHS of the second equation describe the "moving average" character of the process.
- Notice that

$$\begin{aligned} \mathrm{E}(\varepsilon_t) &= \mathrm{E}(\mathrm{E}(\varepsilon_t | \varepsilon_{0:t-1})) \\ &= \mathrm{E}(\mathrm{E}(z_t)(\zeta_0 + \zeta_1 \varepsilon_{t-1}^2 + \ldots + \zeta_q \varepsilon_{t-q}^2)^{1/2}). \\ &= 0. \end{aligned} \tag{46}$$

# ARCH models

- The volatility process given by (45) is, in general, not covariance-stationary. It is easy to see that
$$E(\sigma_t^2) = \frac{\zeta_0}{1 - \zeta_1 - \ldots - \zeta_q}\,, \tag{47}$$
which is positive if $\zeta_1 + \ldots + \zeta_q < 1$. One can, in fact, show that this condition is necessary and sufficient for covariance-stationarity of the process.
- ARCH processes exhibit *leptokurtosis* (a.k.a. *fat tails*). Consider, for example, an *ARCH*(1) model:
$$\begin{aligned} \varepsilon_t &= \sigma_t z_t \\ \sigma_t^2 &= \zeta_0 + \zeta_1 \varepsilon_{t-1}^2, \end{aligned} \tag{48}$$
and calculate the 4-the moment of $\sigma_t$.
- Notice first that
$$\begin{aligned} E(\epsilon_t^4) &= E(\sigma_t^4)E(z_t^4) \\ &= 3E(\sigma_t^4) \end{aligned}$$

# ARCH models

- Next,

$$\begin{aligned}\mathrm{E}(\sigma_t^4) &= \zeta_0^2 + 2\zeta_0\zeta_1\mathrm{E}(\varepsilon_{t-1}^2) + \zeta_1^2\mathrm{E}(\varepsilon_{t-1}^4)\\ &= \zeta_0^2 + \frac{2\zeta_0^2\zeta_1}{1-\zeta_1} + 3\zeta_1^2\mathrm{E}(\sigma_{t-1}^4).\end{aligned}$$

- Since $\mathrm{E}(\sigma_t^4) = \mathrm{E}(\sigma_{t-1}^4)$, we infer that

$$\mathrm{E}(\sigma_t^4) = \frac{\zeta_0^2(1+\zeta_1)}{1-\zeta_1}\,, \tag{49}$$

and the kurtosis is of the disturbances in the $ARCH(1)$ model is equal to

$$\frac{\mathrm{E}(\varepsilon_t^4)}{\mathrm{E}(\varepsilon_t^2)^2} = 3\,\frac{1-\zeta_1^2}{1-3\zeta_1^2}\,. \tag{50}$$

# ARCH models

- Notice that the kurtosis is greater than 3 (and, in fact, it tends to infinity as $\zeta_1 \longrightarrow 1/\sqrt{3}$), and so the tails of the distribution are heavy.
- ARCH models can be estimated using MLE. We will not discuss it here; instead we will explain the method for the more general GARCH family of models below.
- ARCH models are not frequently used in practice, because they generally lead to unrealistic volatility structures. They are primarily of historical interest, as they were the first stochastic volatility models proposed in the econometric literature.

# GARCH models

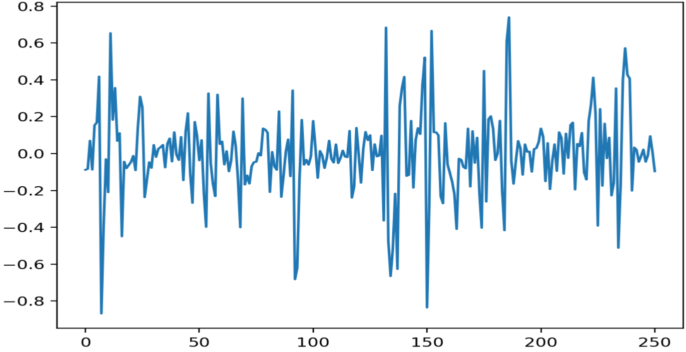
- A more general family of models is referred to as *generalized autoregressive conditional heteroskedasticity* (GARCH) models with the specification:

$$\begin{aligned} \varepsilon_t &= \sigma_t z_t, \\ \sigma_t^2 &= \kappa + \eta_1 \sigma_{t-1}^2 + \ldots + \eta_p \sigma_{t-p}^2 + \zeta_1 \varepsilon_{t-1}^2 + \ldots + \zeta_q \varepsilon_{t-q}^2. \end{aligned} \tag{51}$$

  A model with this specification is referred to as a *GARCH*$(p, q)$ model.
- Compare to *ARCH*$(q)$, *GARCH*$(p, q)$ models exhibit "autoregressive" terms, in addition to the "moving average" terms, in their specifications.
- These autoregressive terms render GARCH suitable for capturing the phenomenon of *volatility clustering*, i.e. prolonged periods of elevated or lower volatility.

# GARCH models

- The graph below shows a simulated *GARCH*$(1,1)$ time series with the following choice of parameters: $\kappa = 0.0$, $\eta = 0.7$, $\zeta = 0.1$.



# GARCH models

- Arguments similar to the ones in the case of ARCH models show that

$$\mathrm{E}(\varepsilon_t) = 0, \tag{52}$$

and

$$\mathrm{E}(\sigma_t^2) = \frac{\kappa}{1 - \sum_{i=1}^{\min(p,q)} (\eta_i + \zeta_i)} . \tag{53}$$

It is thus necessary for stationarity of the model that $\sum_{i=1}^{\min(p,q)} (\eta_i + \zeta_i) < 1$.

- Commonly used in finance is the *GARCH*$(1, 1)$ model with

$$\sigma_t^2 = \kappa + \eta \sigma_{t-1}^2 + \zeta \varepsilon_{t-1}^2 . \tag{54}$$

- For this model, $\mathrm{E}(\sigma_t^2) = \sigma^2$ is independent of $t$, and

$$\sigma^2 = \frac{\kappa}{1 - \eta - \zeta} . \tag{55}$$

# GARCH models

- Repeating the arguments leading to formula (50), we find that the kurtosis in the *GARCH*(1, 1) model is given by

$$\frac{\mathrm{E}(\varepsilon_t^4)}{\mathrm{E}(\varepsilon_t^2)^2} = 3\,\frac{1-(\zeta+\eta)^2}{1-(\zeta+\eta)^2-2\zeta^2}\,. \tag{56}$$

- The kurtosis is always greater than 3, and thus the model exhibits fat tails. Notice also that the fourth moment exists only if $(\zeta+\eta)^2+2\zeta^2<1$.
- As an example, consider a time series of observations $x_t$ with

$$\begin{aligned} X_t &= \alpha + \beta X_{t-1} + \varepsilon_t, \\ \varepsilon_t &= \sigma_t z_t, \\ \sigma_t^2 &= \kappa + \eta\sigma_{t-1}^2 + \zeta\varepsilon_{t-1}^2. \end{aligned} \tag{57}$$

- This is an *AR*(1) model with *GARCH*(1, 1)-style residuals.

# MLE for GARCH

- It is straightforward to write the likelihood function for a model with GARCH-style disturbances. As an example, consider the model introduced in (57).
- Consider a series of observations $x_0, x_1, \ldots, x_T$. Notice that the volatility parameters $\sigma_0, \sigma_1, \ldots, \sigma_T$ are not observed data.
- The implied i.i.d. normalized shocks are given by

$$\widehat{z}_t = \frac{x_t - \alpha - \beta x_{t-1}}{\sigma_t} \,. \tag{58}$$

- We adopt the conditional approach, in which the likelihood function is conditioned on $x_0$ and $\sigma_0 = 0$. The joint probability distribution function can be written as

$$\begin{aligned} f(x_{1:T}|x_0, \sigma_0, \theta) &= f(x_T|x_{0:T-1}, \sigma_0, \theta) f(x_{T-1}|x_{0:T-2}, \sigma_0, \theta) \ldots f(x_1|x_0, \sigma_0, \theta) \\ &= \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\Big( -\frac{\widehat{z}_t^2}{2} \Big). \end{aligned}$$

# MLE for GARCH

- The volatilities $\sigma_t$ are calculated recursively from (57):

$$\begin{aligned}\sigma_1^2 &= \kappa,\\ \sigma_2^2 &= \kappa + \eta\kappa + \zeta(x_1 - \alpha - \beta x_0)^2,\\ &\dots\\ \sigma_t^2 &= \kappa + \eta\sigma_{t-1}^2 + \zeta(x_{t-1} - \alpha - \beta x_{t-2})^2.\end{aligned}$$

- This leads to the following conditional LLF:

$$-\log\mathcal{L}(\theta|x_{0:T},\sigma_0) = \frac{1}{2}\sum_{t=1}^{T}\Big(\log\sigma_t^2 + \frac{(x_t - \alpha - \beta x_{t-1})^2}{\sigma_t^2}\Big) + const, \tag{59}$$

where $\theta$ denotes the collection of model parameters $(\alpha, \beta, \kappa, \eta, \zeta)$. Its minimum cannot be calculate in closed form but is straightforward to calculate numerically.

# Forecasting volatility with GARCH

- A one-period volatility forecast is given by

$$\begin{aligned}\sigma^{2*}_{t+1|1:t} &= \mathrm{E}_t(\sigma^2_{t+1}) \\ &= \kappa + \eta\sigma^2_t + \zeta\sigma^2_t \\ &= \sigma^2 + (\eta + \zeta)(\sigma^2_t - \sigma^2),\end{aligned} \tag{60}$$

where $\sigma^2$ is the expected variance given by (55).

- Continuing this process we find that the $k$-step forecast is given by

$$\sigma^{2*}_{t+k|1:t} = \sigma^2 + (\eta + \zeta)^k(\sigma^2_t - \sigma^2). \tag{61}$$

- From stationarity, $\eta + \zeta < 1$, and so in the limit of long forecasting horizon,

$$\sigma^{2*}_{t+k|1:t} \longrightarrow \sigma^2, \tag{62}$$

i.e. the forecast approaches the equilibrium value exponentially fast.

# Limitations of GARCH models

- GARCH is a stationary model. In reality, observed volatilities are non-stationary.
- Stationarity requires that the drift and all $\eta$'s and $\zeta$'s have to be positive, which limits the ways in which the disturbances impact the time evolution of volatility.
- The impact of the shocks $\varepsilon_t$ on the volatility process is symmetric: positive and negative shocks have the same impact. This is not the case in the equity markets, where negative shocks tend to elevate volatility, while positive shocks lower it.

# Extensions of the GARCH model

- Because of the limitations of the GARCH model, various (literally, hundreds of them) extensions have been proposed. Here are a few of them.
- An *integrated model IGARCH*$(1,1)$ is a unit root version of *GARCH*$(1,1)$ with $\eta + \zeta = 1$:
$$\sigma_t^2 = \kappa + \eta \sigma_{t-1}^2 + (1 - \eta)\varepsilon_{t-1}^2. \tag{63}$$
- More generally, *IGARCH*$(p, q)$ is specified by (51) with $\sum_{j=1}^{p} \eta_j + \sum_{j=1}^{q} \zeta_j = 1$.

# Extensions of the GARCH model

- The *exponential GARCH*$(p, q)$ *model* or *EGARCH*$(p, q)$ is defined by:

$$\log \sigma_t^2 = \kappa + \sum_{j=1}^{p} \eta_j \log \sigma_{t-1}^2 + \sum_{j=1}^{q} \zeta_j g(Z_{t-j}), \tag{64}$$

  where $g(Z) = \theta Z + \lambda(|Z| - \mathrm{E}(|Z|))$, and where $Z \sim N(0, 1)$.
- This choice of the structure of disturbances allows for asymmetric impact of $Z$ on the volatility discussed above.

# References

- Hamilton, J. D.: *Time Series Analysis*, Princeton University Press (1994).
- Tsay, R. S.: *Analysis of Financial Time Series*, Wiley (2010).