# Credit Risk Models

## 11. PDE approach to counterparty credit modeling

Andrew Lesniewski

Baruch College
New York

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# Outline

# Counterparty credit risk

- We will present a unified approach to counterparty credit risk modeling based on risk neutral valuation.
- The objective is to derive a PDE which is a generalization of the Black-Scholes PDE.
- We follow closely the presentation by Burgard and Kjaer in their original paper [1]. A more comprehensive, textbook presentation can be found in [2].

# Counterparty credit risk

- We consider an OTC derivative transaction on an asset $S$ between a bank B and a client C, both of which may default.
- We assume that the price of the asset $S$ is not affected by a default of either B or C.
- We also assume that $S$ follows a Markov process with generator $\mathcal{A}_t$. For example, if $S$ is modeled as a lognormal process

$$\frac{dS(t)}{S(t)} = \mu(t)\,dt + \sigma(t)\,dW(t)\,,$$

then

$$\mathcal{A}_t = \frac{1}{2}\,\sigma(t)^2\,x^2\,\frac{\partial^2}{\partial x^2} + \mu(t)\,x\,\frac{\partial}{\partial x}\,.$$

# Counterparty credit risk

- Let $\widehat{V} = \widehat{V}(t, S, J_B, J_C)$ denote the counterparty credit risky value of the contract, and let $V = V(t, S)$ denote the value of the derivative between two parties that cannot default.
- Here, $J_B$ and $J_C$ are counting processes, so that $J_B = 0$ means no default, while $J_B = 1$ means default of B (and analogously for $J_C$).
- At default of either the counterparty or the seller, the value of the derivative (to the seller) is determined by means of a mark-to-market rule $M = \widehat{V}$ or $M = V$.

# Notation

- In the following, we will use the notation:
  (i) $r \triangleq$ risk free rate on the default free zero coupon bond $P$ maturing at $T$.
  (ii) $r_B \triangleq$ yield on the zero recovery zero coupon bond of $P_B$ of B, $r_C \triangleq$ yield on the recovery zero coupon bond $P_C$ of C. These bonds mature at $T$ (unless they default prior to that date).
  (iii) $\lambda_B \triangleq r_B - r$, $\lambda_C \triangleq r_C - r$.
  (iv) $r_F \triangleq$ bank's funding rate for borrowed cash on seller's derivatives replication cash account. Note that

$$r_F \triangleq \begin{cases} r, & \text{if the derivative can be used as collateral,} \\ r + (1 - R_B)\lambda_B, & \text{otherwise.} \end{cases}$$

  (v) $s_F \triangleq r_F - r$.
  (vi) $R_B \triangleq$ recovery rate of B, $R_C \triangleq$ recovery rate of C.

# Counterparty credit risk

- We assume that the dynamics of the assets $P$, $P_B$, $P_C$, and $S$ are given by the following SDEs:

$$\begin{aligned}\frac{dP(t)}{P(t)} &= r(t)\,dt,\\ \frac{dP_B(t)}{P_B(t)} &= r_B(t)\,dt - dJ_B(t),\\ \frac{dP_C(t)}{P_C(t)} &= r_C(t)\,dt - dJ_C(t),\\ \frac{dS(t)}{S(t)} &= \mu(t)\,dt + \sigma(t)\,dW(t),\end{aligned}$$

  under the physical measure P. Here, $r(t)$, $r_B(t)$, $r_C(t)$, and $\sigma(t)$ are deterministic functions, and $J_B(t)$ and $J_C(t)$ are counting processes taking values in $\{0, 1\}$.
- Assume that B and C enter in an OTC derivative on $S$ with a payoff $H(S)$ at time $T$. Thus $H > 0$ means that B receives cash (or an asset) at time $T$ from the counterparty C.

# Counterparty credit risk

- When one of the parties B or C defaults, the MtM of the derivative determines the close out (or claim) on the position. However, the precise nature of this depends on the contractual details and the mechanism by which the MtM is determined.
- The 2002 ISDA Master Agreement specifies that the derivative contract will return to the surviving party the recovery value of its positive mark-to-market value (from the view of the surviving counterparty) just prior to default, whereas the full MtM has to be paid to the defaulting party if the MtM value is negative (from the view of the surviving counterparty).
- The Master Agreement specifies a dealer poll mechanism to establish the MtM to the seller $M(t)$ at default, without referring to the names of the counterparties involved in the derivative transaction.

# Counterparty credit risk

- In this case, one would expect $M(t)$ to be close to $V(t)$, even though it is unclear whether the dealers in the poll may or may not include their funding costs in the derivatives price.
- In other cases, not following the Master Agreement, there may be other mechanisms. Hence we will derive the PDE for the general case $M(t)$ and consider the two special cases: $M(t) = \widehat{V}(t)$ and $M(t) = V(t)$
- We have the following boundary conditions in case of a default:

$$\widehat{V}(t, S, 1, 0) = M^{+}(t) + R_B M^{-}(t), \text{ if } \tau_B < \tau_C,$$
$$\widehat{V}(t, S, 0, 1) = R_C M^{+}(t) + M^{-}(t), \text{ if } \tau_C < \tau_B.$$

# Dynamics of the model

- We follow the Black-Scholes framework and construct a self-financing portfolio which replicates the risk of the derivative.
- Specifically, the replicating portfolio $\Pi$ set up by B consists of $\delta(t)$ units of $S$, $\alpha_B(t)$ units of $P_B$, $\alpha_C(t)$ units of $P_C$, and $\beta(t)$ units of cash, such that the portfolio value at time $t$ matches the value of the derivative contract, i.e.

$$\widehat{V}(t) + \Pi(t) = 0,$$

or

$$\begin{aligned} -\widehat{V}(t) &= \Pi(t) \\ &= \delta(t) S(t) + \alpha_B(t) P_B(t) + \alpha_C(t) P_C(t) + \beta(t). \end{aligned}$$

# Dynamics of the model

- We note that when $\widehat{V} \geq 0$, the bank will incur a loss in case if C defaults. To hedge this potential loss, we need $\alpha_C \leq 0$. Since $P_C$ is a zero recovery bond, this $\lambda_C$ is the default intensity of C.
- On the other hand, if $\widehat{V} < 0$ the bank will benefit from its own default, and so we expect that $\alpha_B \geq 0$. The proceeds from the sale of $P_B$ are invested at the risk free rate $r$.

# Self-financing condition

- Assuming that the portfolio Π is self-financing means that

$$-d\widehat{V}(t) = \delta(t)\, dS(t) + \alpha_B(t)\, dP_B(t) + \alpha_C(t)\, dP_C(t) + d\beta(t)\,.$$

- The change in cash $d\beta(t)$ is decomposed into its components

$$d\beta(t) = d\beta_S(t) + d\beta_F(t) + d\beta_C(t)\,.$$

- Note that the change in the cash account occurs before rebalancing of the portfolio. The self-financing condition ensures that after the time *dt* the rebalancing can happen only at zero overall cost.

# Self-financing condition

- $d\beta_S(t)$: The position in $S$ generates dividend income of $\delta(t)\gamma_S(t)\,dt$ and a financing cost of $-\delta(t)\,q_S(t)\,S(t)\,dt$, where $q_S(t)$ is related to the repo rate of $S$. Consequently,

$$d\beta_S(t) = \delta(t)\,(\gamma_S(t) - q_S(t))S(t)\,dt.$$

- $d\beta_C(t)$: Furthermore, B shorts the counterparty bond through a repurchase agreement and, assuming zero haircut, incurs financing costs of

$$d\beta_C(t) = -\alpha_C(t)\,r(t)\,P_C(t)\,dt.$$

# Self-financing condition

- $d\beta_F(t)$: Any surplus cash held by the bank after its own bonds have been purchased must earn the risk free rate $r$ in order not to introduce any further credit risk. When borrowing money, B needs to pay the funding rate $r_F$.
- We will consider two cases:
  (i) The derivative itself can be used as collateral for the required funding (assume no haircut) and so $r_F = r$.
  (ii) The derivative cannot be used as collateral, in which case the funding rate is the yield of the unsecured bank bond with recovery $R_B$: i.e. $r_F = r + (1 - R_B)\lambda_B$ (in practice this is a more realistic assumption).
- In either case we have:

$$
\begin{aligned}
d\beta_F(t) &= r(t)\big(-\widehat{V}(t) - \alpha_B(t)P_B(t)\big)^+ dt + r_F(t)\big(-\widehat{V}(t) - \alpha_B(t)P_B(t)\big)^- dt \\
&= -r(t)\big(\widehat{V}(t) + \alpha_B(t)P_B(t)\big)dt + s_F(t)\big(-\widehat{V}(t) - \alpha_B(t)P_B(t)\big)^- dt.
\end{aligned}
$$

# Self-financing condition

- In summary, the change in the cash account is given by

$$d\beta = \delta(\gamma_S - q_S)Sdt + \big(-r(\widehat{V} + \alpha_B P_B) + s_F(-\widehat{V} - \alpha_B P_B)^-\big)dt - \alpha_C r P_C dt.$$

- As a consequence, we obtain the equation

$$\begin{aligned} -d\widehat{V}(t) &= \delta dS + \alpha_B dP_B + \alpha_C dP_C + d\beta \\ &= \delta dS + \alpha_B P_B(rdt - dJ_B) + \alpha_C P_C(rdt - dJ_C) \\ &\quad + \big(\delta(\gamma_S - q_S)S - r(\widehat{V} + \alpha_B P_B) + s_F(-\widehat{V} - \alpha_B P_B)^- - \alpha_C r P_C\big)dt \\ &= \big(-r\widehat{V} + s_F(-\widehat{V} - \alpha_B P_B)^- + (\gamma_S - q_S)\delta S + \lambda_B \alpha_B P_B + \lambda_C \alpha_C P_C\big)dt \\ &\quad + \delta dS - \alpha_B P_B dJ_B - \alpha_C P_C dJ_C, \end{aligned}$$

where we have used $\lambda_B = r_B - r$, and $\lambda_C = r_C - r$.

# Risk neutral valuation

- Recall now Ito's formula for jump diffusions (Lecture Notes #2):

$$d\widehat{V} = \frac{\partial \widehat{V}}{\partial t}\,dt + \frac{\partial \widehat{V}}{\partial S}\,dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \widehat{V}}{\partial S^2}\,dt + \Delta \widehat{V}_B dJ_B + \Delta \widehat{V}_C dJ_C,$$

where the jump amounts are given by

$$\begin{aligned}\Delta \widehat{V}_B &= \widehat{V}(t,S,1,0) - \widehat{V}(t,S,0,0),\\ \Delta \widehat{V}_C &= \widehat{V}(t,S,0,1) - \widehat{V}(t,S,0,0).\end{aligned}$$

# Risk neutral valuation

- Comparing this with the formula above and choosing

$$
\begin{aligned}
\delta &= -\frac{\partial \widehat{V}}{\partial S}\,,\\
\alpha_B &= \frac{\Delta \widehat{V}_B}{P_B}\\
&= -\frac{\widehat{V} - (M^+ + R_B M^-)}{P_B}\,,\\
\alpha_C &= \frac{\Delta \widehat{V}_C}{P_C}\\
&= -\frac{\widehat{V} - (M^- + R_C M^+)}{P_C}\,,
\end{aligned}
$$

we eliminate all risky terms in the portfolio.

# Risk neutral valuation

- The cash account evolution can thus be written as

$$d\beta_F(t) = (-r(t)\, R_B M^- - r_F(t)\, M^+)dt,$$

so the amount of cash deposited by the bank at the risk-free rate equals $-R_B M^-$ and the amount borrowed at the funding rate $r_F$ equals $-M^+$.

- We can can write the equation for $\widehat{V}$ along with the terminal condition as

$$\frac{\partial \widehat{V}}{\partial t} + \mathcal{A}_t \widehat{V} - r\widehat{V} = s_F(\widehat{V} + \Delta\widehat{V}_B)^+ - \lambda_B \Delta\widehat{V}_B - \lambda_C \Delta\widehat{V}_C,$$
$$\widehat{V}(T, S) = H(S),$$

where

$$\mathcal{A}_t \widehat{V} \triangleq \frac{1}{2}\,\sigma^2 S^2\, \frac{\partial^2 \widehat{V}}{\partial S^2} + (q_S - \gamma_S) S\, \frac{\partial \widehat{V}}{\partial S}\,.$$

# Risk neutral valuation

- Inserting the explicit expressions for $\Delta\widehat{V}_B$ and $\Delta\widehat{V}_C$, we finally obtain the fundamental equation:

$$\begin{aligned}\frac{\partial\widehat{V}}{\partial t}+\mathcal{A}_t\widehat{V}-r\widehat{V}&=(\lambda_B+\lambda_C)\widehat{V}+s_F M^+-\lambda_B(R_B M^-+M^+)-\lambda_C(R_C M^++M^-),\\ \widehat{V}(T,S)&=H(S),\end{aligned}$$

  where we have used $(\widehat{V}+\Delta\widehat{V}_B)^+=(M^++R_B M^-)^+=M^+$.
- This is to be contrasted with the standard Black-Scholes equation:

$$\begin{aligned}\frac{\partial V}{\partial t}+\mathcal{A}_t V-rV&=0,\\ V(T,S)&=H(S).\end{aligned}$$

# Risk neutral valuation

- We can interpret the new terms on the right hand side as follows:
  (i) The first term is the additional growth rate that B requires on the risky asset $\widehat{V}$ to compensate for the risk that its own default or C's default will terminate the derivative contract.
  (ii) The second term is the additional funding cost for negative values of the cash account of the hedging strategy.
  (iii) The third term is the adjustment in growth rate that B can accept because of the cash flow occurring at own default.
  (iv) The fourth term is the adjustment in growth rate that B can accept because of the cash flow occurring at C's default.

# Risk neutral valuation

- The first, third and fourth terms are related to counterparty risk, while the second term represents the funding cost.
- It follows that the PDE for a so called *extinguisher trade*, in which it is agreed that no party gets anything at default, is obtained by removing terms three and four from the fundamental equation.
- Let us now consider the following special cases:
  (i) $M(t,S) = \widehat{V}(t,S,0,0)$ and $r_F = r$.
  (ii) $M(t,S) = \widehat{V}t,S,0,0)$ and $r_F = r + s_F$.
  (iii) $M(t,S) = V(t,S)$ and $r_F = r$.
  (iv) $M(t,S) = V(t,S)$ and $r_F = r + s_F$.

# Feynman-Kac formula

- Before proceeding, let us take a pause and briefly review the Feynman-Kac formula.
- Consider first the backward Kolmogorov equation:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\,\sigma(t,x)^2\,\frac{\partial^2 f}{\partial x^2} + \mu(t,x)\,\frac{\partial f}{\partial x} = 0,$$
$$f(T,x) = h(x),$$

associated with the Ito process

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t).$$

- The Feynman-Kac formula allows us to represent the solution to this terminal value problem in terms of the expected value of a function of the process $X$.

# Feynman-Kac formula

- In order to derive this representation, we apply Ito's lemma to the process $f(t, X(t))$:

$$df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\,\sigma(t, X(t))^2\,\frac{\partial^2 f}{\partial x^2} + \mu(t, X(t))\,\frac{\partial f}{\partial x}\right)dt + \sigma(t, X(t))\,\frac{\partial f}{\partial x}\,dW(t)\,.$$

- Assume now that $f$ is a solution to the backward Kolmogorov equation; then the coefficient of $dt$ in the formula above is zero. Integrating from $t$ to $T$ and taking the expected value conditioned on $t$ yields:

$$\mathrm{E}_t[f(T, X(T))] - f(t, x) = 0,$$

  where $x$ denotes the value of $X$ at time $t$.
- In other words,

$$f(t, x) = \mathrm{E}_t[h(X(T))].$$

  This is the simplest case of the Feynman-Kac formula.

# Feynman-Kac formula

- It is not hard to extend the Feynman-Kac formula to the Black-Scholes PDE:

$$\begin{aligned}\frac{\partial f}{\partial t} + \frac{1}{2}\,\sigma(t,x)^2\,\frac{\partial^2 f}{\partial x^2} + \mu(t,x)\,\frac{\partial f}{\partial x} &= r\,(t)\,f,\\ f(T,x) &= h(x),\end{aligned}$$

  where $r\,(t)$ is a deterministic (riskless) interest rate.
- To this end, we substitute

$$f(t,x) = e^{\int_0^t r(s)ds} g(t,x).$$

- Carrying out the derivatives, we verify that $g$ satisfies the BKE:

$$\begin{aligned}\frac{\partial g}{\partial t} + \frac{1}{2}\,\sigma(t,x)^2\,\frac{\partial^2 g}{\partial x^2} + \mu(t,x)\,\frac{\partial g}{\partial x} &= 0,\\ g(T,x) &= e^{-\int_0^T r(s)ds} h(x).\end{aligned}$$

# Feynman-Kac formula

- Applying the just established Feynman-Kac formula to the problem above, we find that

$$
\begin{aligned}
f(t,x) &= e^{\int_0^t r(s)ds}\mathrm{E}_t\left[e^{-\int_0^T r(s)ds}h(X(T))\right] \\
&= \mathrm{E}_t\left[e^{-\int_t^T r(s)ds}h(X(T))\right].
\end{aligned}
$$

- This formula represents the solution to the Black-Scholes equation in terms of the discounted conditional expected value of the payoff function.
- Finally, let us add a further level of complexity by considering the terminal value problem for the following *inhomogeneous* Black-Scholes PDE:

$$
\frac{\partial f}{\partial t} + \frac{1}{2}\,\sigma(t,x)^2\,\frac{\partial^2 f}{\partial x^2} + \mu(t,x)\,\frac{\partial f}{\partial x} - r(t)\,f = \eta,
$$

where $\eta(t,x)$ is a known “source” function.

# Feynman-Kac formula

- Observe that the function

$$\psi(t)=-\mathrm{E}_t\Big[\int_t^T e^{-\int_t^u r(s)ds}\eta(u,X(u))du\Big]$$

is a (particular) solution to the inhomogeneous equation, satisfying $\psi(T)=0$.

- Since the solution to the inhomogeneous equation can always be written as the solution to the homogeneous equation plus a particular solution to the inhomogeneous equation, we find that

$$\begin{aligned} f(t,x) &= \mathrm{E}_t\big[e^{-\int_t^T r(s)ds}h(X(T))\big]+\psi(t)\\ &= \mathrm{E}_t\Big[e^{-\int_t^T r(s)ds}h(X(T))-\int_t^T e^{-\int_t^u r(s)ds}\eta(u,X(u))du\Big], \end{aligned}$$

with $X(t)=x$.

- This is the Feynman-Kac formula for the inhomogeneous Black-Scholes equation.
- It is straightforward to generalize this derivation to the multi-factor case.
- We are now going back to the fundamental PDE.

# Using $\widehat{V}$ as MtM

- This is conceptually the simplest case: if the defaulting party is in the money with respect to the derivative contract, there is no additional effect on the P&L at the time of default. Likewise, if the surviving party is in the money on the derivative contract, then its loss is simply $(1-R)\widehat{V}$.
- The fundamental PDE reads now:

$$\frac{\partial \widehat{V}}{\partial t} + \mathcal{A}_t \widehat{V} - r\widehat{V} = (1-R_B)\lambda_B \widehat{V}^- + (1-R_C)\lambda_C \widehat{V}^+ + s_F \widehat{V}^+,$$
$$\widehat{V}(T,S) = H(S).$$

- The hedge ratios are given by

$$\alpha_B = -\frac{(1-R_B)\widehat{V}^-}{P_B},$$
$$\alpha_C = -\frac{(1-R_C)\widehat{V}^+}{P_C}.$$

Consequently, $\alpha_B \geq 0$, and $\alpha_C \leq 0$, and the replication strategy generates enough cash (namely, $-\widehat{V}^-$) for B to purchase back its own bonds.

# Using $\widehat{V}$ as MtM

- We now write $\widehat{V} = V + Y$, where $Y$ denotes the CVA, to obtain the following PDE:

$$\frac{\partial Y}{\partial t} + \mathcal{A}_t Y - rY = (1 - R_B)\lambda_B (V + Y)^- + (1 - R_C)\lambda_C (V + Y)^+ + s_F (V + Y)^+,$$
$$Y(T, S) = 0.$$

- From the Feynman-Kac formula for the inhomogeneous Black-Scholes equation (we are assuming deterministic rates) we find that

$$\begin{aligned} Y(t, S) = &-(1 - R_B) \int_t^T \lambda_B(u)\, P(t, u) \mathrm{E}_t\big[\big(V(u, S(u)) + Y(u, S(u))\big)^-\big] du \\ &- (1 - R_C) \int_t^T \lambda_C(u)\, P(t, u) \mathrm{E}_t\big[\big(V(u, S(u)) + Y(u, S(u))\big)^+\big] du \\ &- \int_t^T s_F(u)\, P(t, u) \mathrm{E}_t\big[\big(V(u, S(u)) + Y(u, S(u))\big)^+\big] du, \end{aligned}$$

where $P(t, u) = e^{-\int_t^u r(s) ds}$ is the riskless discount factor.

# Using $\widehat{V}$ as MtM

- Note that the first term on the right hand side is absent as a consequence of $Y(T) = 0$.
- Also note that, as $Y$ occurs on both sides of the expression above, the Feynman-Kac formula is not a solution to the fundamental PDE but rather a way of writing it as an equivalent integral equation.
- We can thus compute the CVA by computing $V$ and then solving either the nonlinear PDE or the integral equation above.
- Let us study two examples, namely where $\widehat{V}$ corresponds to bonds of the seller or the counterparty, where those the bonds are either without and with recovery.

# Example: B sells its bond

- Consider first the case of B selling its no-recovery bond to C. Then $\widehat{V} = \widehat{V}^- P_B$, and $R_B = 0$. With deterministic rates and credit spreads,

$$\frac{\partial \widehat{V}}{\partial t} = (r + \lambda_B)\widehat{V},$$
$$\widehat{V}(T, S) = -1,$$

  and so (as expected)

$$\widehat{V}(t) = -\exp\Big(-\int_t^T (r(s) + \lambda_B(s))ds\Big).$$

- Assuming that the bond has a recovery rate $R_B$,

$$\frac{\partial \widehat{V}}{\partial t} = (r + (1 - R_B)\lambda_B)\widehat{V},$$
$$\widehat{V}(T, S) = -1,$$

  and

$$\widehat{V}(t) = -\exp\Big(-\int_t^T (r(s) + (1 - R_B)\lambda_B(s))ds\Big).$$

# Example: B buys C's bond

- Let us now consider the case of B buying $P_C$ from C without recovery. Then

$$\frac{\partial \widehat{V}}{\partial t} = (r_F + \lambda_C)\widehat{V},$$
$$\widehat{V}(T, S) = 1.$$

Assuming that $r_F = r$ (i.e. B can use the asset a collateral), we find that

$$\widehat{V}(t) = \exp\Big(-\int_t^T (r(s) + \lambda_C(s))ds\Big).$$

- Assuming that the bond has a recovery rate $R_C$, we find that

$$\widehat{V}(t) = \exp\Big(-\int_t^T (r(s) + (1 - R_C)\lambda_B(s))ds\Big).$$

# Using $\widehat{V}$ as MtM, $r_F = r$

- Assume now that the case of a derivative contract with $r_F = r$. Then,

$$\frac{\partial \widehat{V}}{\partial t} + \mathcal{A}_t \widehat{V} - r\widehat{V} = (1 - R_B)\lambda_B \widehat{V}^- + (1 - R_C)\lambda_C \widehat{V}^+,$$
$$\widehat{V}(T, S) = H(S).$$

- If $\widehat{V} \leq 0$, then

$$Y(t, S) = -V(t, S) \int_t^T (1 - R_B)\lambda_B(u) \exp\Big(-\int_t^u (1 - R_B)\lambda_B(s)\, ds)\Big) du.$$

- If $\widehat{V} \geq 0$, then

$$Y(t, S) = -V(t, S) \int_t^T (1 - R_C)\lambda_C(u) \exp\Big(-\int_t^u (1 - R_C)\lambda_C(s)\, ds)\Big) du.$$

- Note that if $\widehat{V} \leq 0$, then the CVA depends only on the credit of the bank; if $\widehat{V} \geq 0$, then it depends only on the credit of the client.

# Using $\widehat{V}$ as MtM, $r_F = r + (1 - R_B)\lambda_B$

- If the contract cannot be posted as collateral, i.e. $r_F = r + (1 - R_B)\lambda_B$, then

$$\frac{\partial \widehat{V}}{\partial t} + \mathcal{A}_t \widehat{V} - r\widehat{V} = (1 - R_B)\lambda_B \widehat{V}^- + \big((1 - R_B)\lambda_B + (1 - R_C)\lambda_C\big)\widehat{V}^+,$$
$$\widehat{V}(T, S) = H(S).$$

- If $\widehat{V} \le 0$, then again

$$Y(t, S) = -V(t, S)\int_t^T (1 - R_B)\lambda_B(u) \exp\Big(-\int_t^u ((1 - R_B)\lambda_B(s)\, ds)\Big)\, du.$$

- If $\widehat{V} \ge 0$, then

$$Y(t, S) = -V(t, S)\int_t^T (1 - R_B)k(u) \exp\Big(-\int_t^u k(s)\, ds)\Big)\, du,$$

where $k \triangleq (1 - R_B)\lambda_B + (1 - R_C)\lambda_C$.
- Note that when B buys an option from C, then it faces an additional funding spread $s_F = (1 - R_B)\lambda_B$.

# Using $V$ as MtM

- We now consider the case where payments in case of default are based on $V$, and so $M(t,S) = V(t,S)$. Then

$$\frac{\partial \widehat{V}}{\partial t} + \mathcal{A}_t \widehat{V} - (r + \lambda_B + \lambda_C)\widehat{V} = -(R_B\lambda_B + \lambda_C)V^- - (\lambda_B + R_C\lambda_C)V^+ + s_F V^+,$$
$$\widehat{V}(T,S) = H(S),$$

  Note that this is a linear PDE with a source term on the right hand side.
- We write $\widehat{V} = V + Y$, and express the hedge ratios as

$$\alpha_B = \frac{Y + (1 - R_B)\widehat{V}^-}{P_B},$$
$$\alpha_C = \frac{Y + (1 - R_C)\widehat{V}^+}{P_C}.$$

# Using $V$ as MtM

- As before, we now write $\widehat{V} = V + Y$, to obtain:

$$\frac{\partial Y}{\partial t} + \mathcal{A}_t Y - (r + \lambda_B + \lambda_C) Y = (1 - R_B)\lambda_B V^- + (1 - R_C)\lambda_C V^+ + s_F V^+,$$
$$Y(T, S) = 0.$$

- From the Feynman-Kac formula for the inhomogeneous Blck-Scholes equation, we get

$$\begin{aligned} Y(t, S) = &-(1 - R_B) \int_t^T \lambda_B(u)\,\mathcal{P}(t, u)\mathrm{E}_t\big[V(u, S(u))^-\big]\,du \\ &- (1 - R_C) \int_t^T \lambda_C(u)\,\mathcal{P}(t, u)\mathrm{E}_t\big[V(u, S(u))^+\big]\,du \\ &- \int_t^T s_F(u)\,\mathcal{P}(t, u)\mathrm{E}_t\big[V(u, S(u))^+\big]\,du, \end{aligned}$$

where $\mathcal{P}(t, u)$ denotes discounting on the risky rate $r + \lambda_B + \lambda_C$.

# Using $V$ as MtM

- The CVA can be calculated by using $V(t, S)$ as a known source term when solving the fundamental PDE or computing the integrals in the Feynman-Kac representation. This is in contrast to the case of $M = \widehat{V}$, in which the the right hand side depends nonlinearly on $\widehat{V}$.
- In the case where we can use the derivative as collateral for the funding of the cash account, i.e. $s_F = 0$, the last term vanishes and the equation reduces to the BCVA derived earlier in the course.
- The bilateral benefit in this case does not come from any gains attributable to worsening of B's credit. Instead, it comes from the ability to use the cash generated by the hedging strategy and buy back its own bonds, thus generating an excess return of $(1 - R_B)\lambda_B$.

# Using $V$ as MtM

- In practice, however, we cannot use the derivative as collateral and the above equation describes a consistent adjustment to the price of the derivative for bilateral counterparty risk and funding cost.
- In particular, if the funding spread corresponds to that of the unsecured B bond, i.e. $s_F = (1 - R_B)\lambda_B$ , we may express $Y$ as

$$
\begin{aligned}
Y(t, S) &= -(1 - R_B) \int_t^T \lambda_B(u)\, \mathcal{P}(t, u) \mathrm{E}_t\big[V(u, S(u))\big]\, du \\
&= -(1 - R_C) \int_t^T \lambda_C(u)\, \mathcal{P}(t, u) \mathrm{E}_t\big[V(u, S(u))^+\big]\, du.
\end{aligned}
$$

- The first term in this equation contains the bilateral asset described above and the funding liability arising from the fact that the higher rate $r_F = (1 - R_B)\lambda_B$ is paid when borrowing for the hedging strategy's cash account.

# Example: counterparty risk of a call option

We can explicitly calculate the counterparty credit risky derivative value $\widehat{V}$ for a call option (bought by B) in a number of explicit examples. Assuming constant credit intensities $\lambda_B$ and $\lambda_C$, we have:

- If $M = \widehat{V}$ and $s_F = 0$, then

$$Y(t,S) = -\Big(1 - \exp\big(-(1-R_C)\lambda_C(T-t)\big)\Big)V(t,S).$$

- If $M = \widehat{V}$ and $s_F = (1-R_B)\lambda_B$, then

$$Y(t,S) = -\Big(1 - \exp\big(-((1-R_B)\lambda_B + (1-R_C)\lambda_C)(T-t)\big)\Big)V(t,S).$$

- If $M = V$ and $s_F = 0$, then

$$Y(t,S) = -\frac{(1-R_C)\lambda_C\big(1 - \exp(-(\lambda_B+\lambda_C)(T-t))\big)}{\lambda_B+\lambda_C}\,V(t,S).$$

- If $M = V$ and $s_F = (1-R_B)\lambda_B$, then

$$Y(t,S) = -\frac{\big((1-R_B)\lambda_B + (1-R_C)\lambda_C\big)\big(1 - \exp(-(\lambda_B+\lambda_C)(T-t))\big)}{\lambda_B+\lambda_C}\,V(t,S).$$

# References


- Burgard, C., and Kjaer, M.: PDE representations of derivatives with bilateral counterparty risk and funding costs (2012).
- Green, A.: *XVA: Credit, Funding and Capital Valuation Adjustments*, Wiley (2016).