BARUCH, MFE

MTH 9878 Assignment Three *

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* We agree for our work to be posted on the forum.

1 NORMAL MODEL

In the normal model, the forward rate $F(t)$ satisfies the following SDE

$$dF(t) = \sigma dW(t), \tag{1.1}$$

where σ is called the *normal volatility*. The solution is

$$F(t) = F_0 + \sigma W(t) \sim F_0 + \sigma\sqrt{T}Z, \tag{1.2}$$

where Z is standard normal. Find the price of calls and puts.
Solution: The price of European call with strike K and expiration T is

$$\begin{aligned}
P^{\text{call}} &= P^{\text{call}}(T,K,F_0,\sigma) \\
&= \mathcal{N}(0)\mathbb{E}^{\mathbb{Q}}\left[(F(T)-K)^+\right] \\
&= \mathcal{N}(0)\int_{-\infty}^{+\infty}(F_0+\sigma\sqrt{T}z-K)^+ e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}} \\
&= \mathcal{N}(0)\sigma\sqrt{T}\int_{-\frac{F_0-K}{\sigma\sqrt{T}}}^{+\infty}\left(z+\frac{F_0-K}{\sigma\sqrt{T}}\right)e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}} \\
\left(d_\pm = \pm\frac{F_0-K}{\sigma\sqrt{T}}\right) \quad &= \mathcal{N}(0)\sigma\sqrt{T}\int_{-d_+}^{+\infty}(z+d_+)\,e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}} \\
&= \mathcal{N}(0)\sigma\sqrt{T}\left[d_+\int_{-d_+}^{+\infty}e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}+\int_{-d_+}^{+\infty}e^{-\frac{z^2}{2}}\frac{zdz}{\sqrt{2\pi}}\right] \\
&= \mathcal{N}(0)\sigma\sqrt{T}\left[d_+\int_{-\infty}^{d_+}e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}+\frac{1}{\sqrt{2\pi}}\int_{-d_+}^{+\infty}e^{-\frac{z^2}{2}}d\left(\frac{z^2}{2}\right)\right] \\
&= \mathcal{N}(0)\sigma\sqrt{T}\left[d_+N(d_+)-\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\Big|_{-d_+}^{+\infty}\right] \\
&= \mathcal{N}(0)\sigma\sqrt{T}\left[d_+N(d_+)+\frac{1}{\sqrt{2\pi}}e^{-\frac{d_+^2}{2}}\right] \\
&= \mathcal{N}(0)\sigma\sqrt{T}\left[d_+N(d_+)+N'(d_+)\right] \\
&\triangleq \mathcal{N}(0)B_{\text{n}}^{\text{call}}(T,K,F_0,\sigma),
\end{aligned} \tag{1.3}$$

where $B_{\text{n}}^{\text{call}}(T,K,F_0,\sigma) \triangleq \sigma\sqrt{T}\left[d_+N(d_+)+N'(d_+)\right]$. In Lecture 3 Equation (1), the formula is written as $B_n^{\text{call}}(T,K,F_0,\sigma) = \sigma\sqrt{T}\left[d_+N(d_+)+N'(d_-)\right]$, which is the same as what we wrote in Equation 1.3 because $d_+^2 = d_-^2$ and $N'(d_+) = N'(d_-)$.

We could do a similar integral to compute the put price:

$$\begin{aligned}
P^{\text{put}} &= P^{\text{put}}(T,K,F_0,\sigma) \\
&= \mathcal{N}(0)\mathbb{E}^{\mathbb{Q}}\left[(K-F(T))^+\right] \\
&= \mathcal{N}(0)\int_{-\infty}^{+\infty}(K-F_0-\sigma\sqrt{T}z)^+ e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{N}(0)\sigma\sqrt{T}\int_{-\infty}^{\frac{K-F_0}{\sigma\sqrt{T}}}\left(\frac{K-F_0}{\sigma\sqrt{T}}-z\right)e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}\\
\left(d_\pm = \pm\frac{F_0-K}{\sigma\sqrt{T}}\right) \quad &= \mathcal{N}(0)\sigma\sqrt{T}\int_{-\infty}^{d_-}(d_- - z)\,e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}\\
&= \mathcal{N}(0)\sigma\sqrt{T}\left[d_-\int_{-\infty}^{d_-}e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}-\int_{-\infty}^{d_-}e^{-\frac{z^2}{2}}\frac{zdz}{\sqrt{2\pi}}\right]\\
&= \mathcal{N}(0)\sigma\sqrt{T}\left[d_-\int_{-\infty}^{d_-}e^{-\frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}}-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-}e^{-\frac{z^2}{2}}d\left(\frac{z^2}{2}\right)\right]\\
&= \mathcal{N}(0)\sigma\sqrt{T}\left[d_-N(d_-)+\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\Big|_{-\infty}^{d_-}\right]\\
&= \mathcal{N}(0)\sigma\sqrt{T}\left[d_-N(d_-)+\frac{1}{\sqrt{2\pi}}e^{-\frac{d_-^2}{2}}\right]\\
&= \mathcal{N}(0)\sigma\sqrt{T}\left[d_-N(d_-)+N'(d_-)\right]\\
&\triangleq \mathcal{N}(0)B_{\mathrm{n}}^{\mathrm{put}}(T,K,F_0,\sigma), \qquad (1.4)
\end{aligned}
$$

where $B_{\mathrm{n}}^{\mathrm{put}}(T,K,F_0,\sigma) \triangleq \sigma\sqrt{T}\left[d_-N(d_-)+N'(d_-)\right]$.

The put-call parity still holds

$$
\begin{aligned}
P^{\mathrm{call}} - P^{\mathrm{put}} &= \mathcal{N}(0)\sigma\sqrt{T}\,[d_+N(d_+)-d_-N(d_-)]\\
(d_- = -d_+) \quad &= \mathcal{N}(0)\sigma\sqrt{T}d_+\left[\underbrace{N(d_+)+N(-d_+)}_{=1}\right]\\
&= F_0 - K.
\end{aligned}
$$

2 Shifted Lognormal Model

The dynamics of shifted lognormal model is

$$dF(t) = (\sigma_0 + \sigma_1 F(t))dW(t). \tag{2.1}$$

Notice that $\tilde{F}(t) \triangleq \sigma_0 + \sigma_1 F(t)$ satisfies the SDE for a regular lognormal process with volatility σ_1 and without shift

$$d\tilde{F}(t) = d(\sigma_0 + \sigma_1 F(t)) = \sigma_1 dF(t) = \sigma_1(\sigma_0 + \sigma_1 F(t))dW(t) = \sigma_1 \tilde{F}(t)dW(t), \tag{2.2}$$

the solution to which is well known

$$\tilde{F}(t) = \tilde{F}_0 e^{-\frac{1}{2}\sigma_1^2 t + \sigma_1 W(t)}, \tag{2.3}$$

where $\tilde{F}_0 = \sigma_0 + \sigma_1 F_0$. Representing the terminal payoff of European calls and puts in terms of $\tilde{F}(T)$ instead of $F(T)$

$$\text{Call Payoff} = (F(T) - K)^+ = \left(\frac{\tilde{F}(T) - \sigma_0}{\sigma_1} - K\right)^+ = \frac{1}{\sigma_1}\left(\tilde{F}(T) - \sigma_0 - \sigma_1 K\right)^+ \triangleq \frac{1}{\sigma_1}\left(\tilde{F}(T) - \tilde{K}\right)^+,$$
$$\text{Put Payoff} = (K - F(T))^+ = \left(K - \frac{\tilde{F}(T) - \sigma_0}{\sigma_1}\right)^+ = \frac{1}{\sigma_1}\left(\sigma_0 + \sigma_1 K - \tilde{F}(T)\right)^+ \triangleq \frac{1}{\sigma_1}\left(\tilde{K} - \tilde{F}(T)\right)^+,$$

we notice that this is equivalent to a regular lognormal model without drift (or Black's model according to lecture notes) with strike $\tilde{K} \triangleq \sigma_0 + \sigma_1 K$, volatility σ_1, and scaled by a constant factor σ_1^{-1}. Because the derivation of Black-Scholes formula is so familiar, I am not going to re-derive it. Invoking the results for classical Black-Scholes formula for zero interest rate and dividend rate directly

$$P^{\text{call}} = +S_0 N(+d_+) - KN(+d_-),$$
$$P^{\text{put}} = -S_0 N(-d_+) + KN(-d_-),$$

where

$$d_\pm = \frac{\log\frac{S_0}{K} \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}},$$

we can simply write down the results for the shifted lognormal process by making the following re-definition

$$d_\pm = \frac{\log\frac{\tilde{F}_0}{\tilde{K}} \pm \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}} = \frac{\log\frac{\sigma_0+\sigma_1 F_0}{\sigma_0+\sigma_1 K} \pm \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}},$$

and the call price and the put price are

$$P^{\text{call}} = P^{\text{call}}(T, K, F_0, \sigma) = \mathcal{N}(0) B^{\text{call}}_{\text{sln}}(T, K, F_0, \sigma),$$
$$P^{\text{put}} = P^{\text{put}}(T, K, F_0, \sigma) = \mathcal{N}(0) B^{\text{put}}_{\text{sln}}(T, K, F_0, \sigma).$$

where

$$B^{\text{call}}_{\text{sln}}(T, K, F_0, \sigma) = \frac{1}{\sigma_1}\left[\tilde{F}_0 N(+d_+) - \tilde{K}N(+d_-)\right] = \left(F_0 + \frac{\sigma_0}{\sigma_1}\right)N(+d_+) - \left(K + \frac{\sigma_0}{\sigma_1}\right)N(+d_-),$$
$$B^{\text{put}}_{\text{sln}}(T, K, F_0, \sigma) = \frac{1}{\sigma_1}\left[\tilde{K}N(-d_-) - \tilde{F}_0 N(-d_+)\right] = \left(K + \frac{\sigma_0}{\sigma_1}\right)N(-d_-) - \left(F_0 + \frac{\sigma_0}{\sigma_1}\right)N(-d_+).$$

3 Integral Asymptotics of Laplace

Assume that f is integrable function with bounded first derivative on $\mathbb{R}_+$ and that ϕ is a positive convex function that attains its minimum at $u_0 \in (0, +\infty)$. Assume that there exists a real number $\delta > 0$ such that $\delta < \phi''$ and $\phi''' < \frac{1}{\delta}$. Prove that there exists a real number $\theta > 0$ such that

$$\lim_{\epsilon \to 0^+} \frac{\int_0^{+\infty} f(u) e^{-\frac{\phi(u)}{\epsilon}} du - f(u_0) e^{-\frac{\phi(u_0)}{\epsilon}} \sqrt{\frac{2\pi\epsilon}{\phi''(u_0)}}}{\epsilon^\theta} = 0. \tag{3.1}$$

Note: Firstly, this problem is probably not formulated properly. Because of the factor $e^{-\frac{\phi(u_0)}{\epsilon}}$ in the leading approximation term

$$f(u_0) e^{-\frac{\phi(u_0)}{\epsilon}} \sqrt{\frac{2\pi\epsilon}{\phi''(u_0)}} \sim O\left(\sqrt{\epsilon} e^{-\frac{\phi(u_0)}{\epsilon}}\right),$$

the order estimation of the error term ϵ^θ is actually larger in magnitude than the leading approximation itself

$$\epsilon^\theta \gg O\left(\sqrt{\epsilon} e^{-\frac{\phi(u_0)}{\epsilon}}\right)$$

for any $\theta \in \mathbb{R}$ in the sense that

$$\lim_{\epsilon \to 0^+} \frac{\sqrt{\epsilon} e^{-\frac{\phi(u_0)}{\epsilon}}}{\epsilon^\theta} = \lim_{\epsilon \to 0^+} \epsilon^{\frac{1}{2}-\theta} e^{-\frac{\phi(u_0)}{\epsilon}} = 0.$$

It does not make sense to compare the error of the asymptotic approximation

$$\int_0^{+\infty} f(u) e^{-\frac{\phi(u)}{\epsilon}} du - f(u_0) e^{-\frac{\phi(u_0)}{\epsilon}} \sqrt{\frac{2\pi\epsilon}{\phi''(u_0)}}$$

with a term ϵ^θ larger than the leading approximation itself. The problem is more properly formulated to prove the existence of a real number $\theta > 0$ such that

$$\lim_{\epsilon \to 0^+} \frac{\int_0^{+\infty} f(u) e^{-\frac{\phi(u)}{\epsilon}} du - f(u_0) e^{-\frac{\phi(u_0)}{\epsilon}} \sqrt{\frac{2\pi\epsilon}{\phi''(u_0)}}}{e^{-\frac{\phi(u_0)}{\epsilon}} \epsilon^\theta} = 0,$$

by including the factor $e^{-\frac{\phi(u_0)}{\epsilon}}$ in the error estimation.

Proof: Divide the integration region into three intervals:

$$(0, +\infty) = (0, u_0 - \Delta] \cup (u_0 - \Delta, u_0 + \Delta) \cup [u_0 + \Delta, +\infty),$$

we write the integral as a sum of three parts

$$I \triangleq \int_0^{+\infty} f(u) e^{-\frac{\phi(u)}{\epsilon}} du = \underbrace{\int_{u_0-\Delta}^{u_0+\Delta} f(u) e^{-\frac{\phi(u)}{\epsilon}} du}_{\triangleq I_1} + \underbrace{\int_0^{u_0-\Delta} f(u) e^{-\frac{\phi(u)}{\epsilon}} du}_{\triangleq I_2} + \underbrace{\int_{u_0+\Delta}^{+\infty} f(u) e^{-\frac{\phi(u)}{\epsilon}} du}_{\triangleq I_3}.$$

We first evaluate the order of the terms I_2 and I_3. Because the minimum of $\phi(u)$ falls in $(u_0-\Delta, u_0+\Delta)$, we know that $\phi'(u)\neq 0$ in the intervals $(0, u_0-\Delta]$ and $[u_0+\Delta, +\infty)$. Invoking the integration by parts,

$$
\begin{aligned}
I_2 &= \int_0^{u_0-\Delta} f(u) e^{-\frac{\phi(u)}{\epsilon}} du \\
(\phi'(u)\neq 0) \quad &= \int_0^{u_0-\Delta} \frac{f(u)}{\phi'(u)} e^{-\frac{\phi(u)}{\epsilon}} d\phi(u) \\
&= -\epsilon \int_0^{u_0-\Delta} \frac{f(u)}{\phi'(u)} d e^{-\frac{\phi(u)}{\epsilon}} \\
&= -\epsilon \frac{f(u)}{\phi'(u)} e^{-\frac{\phi(u)}{\epsilon}} \Big|_0^{u_0-\Delta} + \epsilon \int_0^{u_0-\Delta} \left[\frac{f(u)}{\phi'(u)}\right]' e^{-\frac{\phi(u)}{\epsilon}} du \\
&= \underbrace{\epsilon \left[\frac{f(0)}{\phi'(0)} e^{-\frac{\phi(0)}{\epsilon}} - \frac{f(u_0-\Delta)}{\phi'(u_0-\Delta)} e^{-\frac{\phi(u_0-\Delta)}{\epsilon}}\right]}_{\sim O\left(\epsilon e^{-\frac{c_2}{\epsilon}}\right)} + \underbrace{\epsilon \int_0^{u_0-\Delta} \left[\frac{f(u)}{\phi'(u)}\right]' \frac{1}{\phi'(u)} e^{-\frac{\phi(u)}{\epsilon}} d\phi(u)}_{\triangleq r(\epsilon)},
\end{aligned}
$$

where $c_2 = \min\big(\phi(0), \phi(u_0-\Delta)\big)$. The magnitude of the remainder term

$$
\begin{aligned}
|r(\epsilon)| &= \left|\epsilon \int_0^{u_0-\Delta} \left[\frac{f(u)}{\phi'(u)}\right]' \frac{1}{\phi'(u)} e^{-\frac{\phi(u)}{\epsilon}} d\phi(u)\right| \\
&\le M\epsilon \left|\int_0^{u_0-\Delta} e^{-\frac{\phi(u)}{\epsilon}} d\phi(u)\right|, \quad M \triangleq \max_{0<u\le u_0-\Delta} \left|\left[\frac{f(u)}{\phi'(u)}\right]' \frac{1}{\phi'(u)}\right| < \infty \\
&\le M\epsilon^2 \left|e^{-\frac{\phi(u_0-\Delta)}{\epsilon}} - e^{-\frac{\phi(0)}{\epsilon}}\right| \\
&\sim O\left(\epsilon^2 e^{-\frac{c_2}{\epsilon}}\right)
\end{aligned}
$$

is a higher order small quantity compared to the boundary term. Thus,

$$
I_2 \sim O\left(\epsilon e^{-\frac{c_2}{\epsilon}}\right),
$$

Similarly,

$$
I_3 \sim O\left(\epsilon e^{-\frac{c_3}{\epsilon}}\right),
$$

where $c_3 = \min\big(\phi(u_0+\Delta), \phi(\infty)\big)$. Notice that both I_2 and I_3 are transcendentally smaller than a power function ϵ^θ, so we will ignore these two terms when evaluating the integral

$$
I = I_1 + I_2 + I_3 \sim I_1 + O\left(\epsilon e^{-\frac{\min(c_2,c_3)}{\epsilon}}\right). \tag{3.2}
$$

Next, we proceed to evaluate the term I_1. According to Taylor's theorem, there exists $\theta_1, \theta_2 \in (-1, +1)$ (dependent on u) such that

$$
I_1 = \int_{u_0-\Delta}^{u_0+\Delta} f(u) e^{-\frac{\phi(u)}{\epsilon}} du
$$

$$
\begin{aligned}
&= e^{-\frac{\phi(u_0)}{\epsilon}} \int_{u_0-\Delta}^{u_0+\Delta} \left[f(u_0) + f'(u_0+\theta_1\Delta)(u-u_0)\right] e^{-\frac{\phi''(u_0+\theta_2\Delta)(u-u_0)^2}{2\epsilon}} du \\
&= e^{-\frac{\phi(u_0)}{\epsilon}} \int_{-\Delta}^{+\Delta} \left[f(u_0) + f'(u_0+\theta_1\Delta)v\right] e^{-\frac{\phi''(u_0+\theta_2\Delta)v^2}{2\epsilon}} dv.
\end{aligned}
$$

Because of the continuity of $f(u)$ and $\phi''(u)$, there exists constant Δ_f and Δ_ϕ such that

$$
\begin{aligned}
&f(u_0) - \Delta_f < f(u_0+\theta_1\Delta) < f(u_0) + \Delta_f, \\
&\phi''(u_0) - \Delta_\phi < \phi''(u_0+\theta_2\Delta) < \phi''(u_0) + \Delta_\phi.
\end{aligned}
$$

Thus, we can put lower and upper bounds on the integral I_1:

$$
e^{-\frac{\phi(u_0)}{\epsilon}} \left(f(u_0) - \Delta_f\right) \int_{-\Delta}^{+\Delta} e^{-\frac{\left(\phi''(u_0)+\Delta_\phi\right)v^2}{2\epsilon}} dv < I_1 < e^{-\frac{\phi(u_0)}{\epsilon}} \left(f(u_0) + \Delta_f\right) \int_{-\Delta}^{+\Delta} e^{-\frac{\left(\phi''(u_0)-\Delta_\phi\right)v^2}{2\epsilon}} dv.
$$

Notice that

$$
\begin{aligned}
\int_{-\Delta}^{+\Delta} e^{-\frac{\left(\phi''(u_0)+\Delta_\phi\right)v^2}{2\epsilon}} dv &= \int_{-\infty}^{+\infty} e^{-\frac{\left(\phi''(u_0)+\Delta_\phi\right)v^2}{2\epsilon}} dv - \int_{+\Delta}^{+\infty} e^{-\frac{\left(\phi''(u_0)+\Delta_\phi\right)v^2}{2\epsilon}} dv - \int_{-\infty}^{-\Delta} e^{-\frac{\left(\phi''(u_0)+\Delta_\phi\right)v^2}{2\epsilon}} dv \\
&= \int_{-\infty}^{+\infty} e^{-\frac{\left(\phi''(u_0)+\Delta_\phi\right)v^2}{2\epsilon}} dv - 2\int_{+\Delta}^{+\infty} e^{-\frac{\left(\phi''(u_0)+\Delta_\phi\right)v^2}{2\epsilon}} dv \\
&= \sqrt{\frac{2\pi\epsilon}{\phi''(u_0)+\Delta_\phi}} - \sqrt{\frac{2\pi\epsilon}{\phi''(u_0)+\Delta_\phi}} \underbrace{\operatorname{erfc}\left(\Delta\sqrt{\frac{\phi''(u_0)+\Delta_\phi}{2\epsilon}}\right)}_{\leq \exp\left(-\frac{\phi''(u_0)+\Delta_\phi}{2\epsilon}\Delta^2\right)} \\
&= \sqrt{\frac{2\pi\epsilon}{\phi''(u_0)+\Delta_\phi}} \times \left(1 + O\left(e^{-\frac{c_+}{\epsilon}}\right)\right).
\end{aligned}
$$

Here, we used the property of the complementary error function $\operatorname{erfc}(x) \leq e^{-x^2}$. Similarly,

$$
\int_{-\Delta}^{+\Delta} e^{-\frac{\left(\phi''(u_0)-\Delta_\phi\right)v^2}{2\epsilon}} dv = \sqrt{\frac{2\pi\epsilon}{\phi''(u_0)-\Delta_\phi}} \times \left(1 + O\left(e^{-\frac{c_-}{\epsilon}}\right)\right),
$$

where $c_\pm = \frac{1}{2}\Delta^2(\phi''(u_0) \pm \Delta_\phi)$. Again, the terms $O\left(e^{-\frac{c_\pm}{\epsilon}}\right)$ are transcendentally smaller than $O(1)$. This allows us to expand the integration region from $(-\Delta, +\Delta)$ to $(-\infty, +\infty)$ and expand $f(u)$ and $\phi(u)$ in the neighborhood of u_0 to all orders

$$
\begin{aligned}
I_1 &= e^{-\frac{\phi(u_0)}{\epsilon}} \int_{-\infty}^{+\infty} \left[f(u_0) + f'(u_0+\theta_1\Delta)v\right] e^{-\frac{\phi''(u_0+\theta_2\Delta)v^2}{2\epsilon}} dv \\
&= e^{-\frac{\phi(u_0)}{\epsilon}} \int_{-\infty}^{+\infty} \left[f(u_0) + f'(u_0)v + \frac{1}{2}f''(u_0)v^2 + \cdots\right] e^{-\frac{\phi''(u_0)v^2}{2\epsilon} + \frac{\phi'''(u_0)v^3}{6\epsilon} + \cdots} dv \\
&= e^{-\frac{\phi(u_0)}{\epsilon}} \int_{-\infty}^{+\infty} \left[f(u_0) + f'(u_0)v + \frac{1}{2}f''(u_0)v^2 + \cdots\right] \times \left[1 + \frac{\phi'''(u_0)v^3}{6\epsilon} + \cdots\right] e^{-\frac{\phi''(u_0)v^2}{2\epsilon}} dv.
\end{aligned}
\tag{3.3}
$$

where the finite condition $\phi''' < \frac{1}{\delta}$ is assumed. At this stage, we have a series of Gaussian integrals to compute for even n

$$\int_{-\infty}^{+\infty} du u^n e^{-\frac{au^2}{2}} = \begin{cases} 0, & n \text{ odd} \\ \sqrt{\frac{2\pi}{a^{n+1}}}(n-1)(n-3)\cdots(3)(1), & n \text{ even.} \end{cases}$$

Collecting terms of the same order in ϵ in Equation 3.3, we get

$$I = f(u_0)e^{-\frac{\phi(u_0)}{\epsilon}}\sqrt{\frac{2\pi\epsilon}{\phi''(u_0)}} \times [1 + O(\epsilon)],$$

wher we have used Equation 3.2 that our original integral I differs from I_1 by transcendentally small terms in ϵ. Thus,

$$I - f(u_0)e^{-\frac{\phi(u_0)}{\epsilon}}\sqrt{\frac{2\pi\epsilon}{\phi''(u_0)}} \sim O\left(\epsilon^{\frac{3}{2}} e^{-\frac{\phi(u_0)}{\epsilon}}\right).$$

In other words, for $\theta < \frac{3}{2}$, we have

$$\lim_{\epsilon \to 0^+} \frac{I - f(u_0)e^{-\frac{\phi(u_0)}{\epsilon}}\sqrt{\frac{2\pi\epsilon}{\phi''(u_0)}}}{e^{-\frac{\phi(u_0)}{\epsilon}}\epsilon^{\theta}} = 0,$$

where $I = \int_0^{+\infty} f(u)e^{-\frac{\phi(u)}{\epsilon}}du$ is the Laplace integral under consideration.

4 SABR ATM Implied Volatility

The asymptotic formulas for the implied normal and lognormal volatilities $\sigma_{\rm n}(T, K, F_0, \sigma_0, \alpha, \beta, \rho)$ and $\sigma_{\ln}(T, K, F_0, \sigma_0, \alpha, \beta, \rho)$, respectively, in the SABR model, as given in Lecture Notes 4, contain singularities of type $\frac{0}{0}$ for the at-the-money strikes. Calculate the SABR ATM Implied Volatilities.

Solution: For normal volatilities,

$$\sigma_{\rm n} = \alpha \frac{F_0 - K}{D(\zeta)} \left(1 + O(\epsilon)\right),$$

where

$$\begin{aligned}
\zeta &= \frac{\alpha}{\sigma_0(1-\beta)} \left(F_0^{1-\beta} - K^{1-\beta}\right), \\
D(\zeta) &= \log \frac{\sqrt{1 - 2\rho\zeta + \zeta^2} + \zeta - \rho}{1-\rho}.
\end{aligned}$$

In the ATM limit, let $F_0 - K = \delta \to 0$, we know $\zeta \to 0$. We first examine the behavior of $D(\zeta)$ in this limit,

$$\begin{aligned}
D(\zeta) &= \log \frac{\sqrt{1 - 2\rho\zeta + \zeta^2} + \zeta - \rho}{1-\rho} \\
&= \log\left(\frac{1 - \rho\zeta + \zeta - \rho}{1-\rho} + O\left(\zeta^2\right)\right) \\
&= \log\left(\frac{(1-\rho)(1+\zeta)}{1-\rho} + O\left(\zeta^2\right)\right) \\
&= \log\left(1 + \zeta + O\left(\zeta^2\right)\right) \\
&= \zeta + O\left(\zeta^2\right).
\end{aligned}$$

This means, in the expansion ζ in terms of δ, we only need to expand to the first order:

$$\begin{aligned}
\zeta &= \frac{\alpha}{\sigma_0(1-\beta)} \left(F_0^{1-\beta} - K^{1-\beta}\right) \\
&= \frac{\alpha}{\sigma_0(1-\beta)} \left((K+\delta)^{1-\beta} - K^{1-\beta}\right) \\
&= \frac{\alpha K^{1-\beta}}{\sigma_0(1-\beta)} \left(\left(1 + \frac{\delta}{K}\right)^{1-\beta} - 1\right) \\
&= \frac{\alpha K^{1-\beta}}{\sigma_0(1-\beta)} \left(1 + (1-\beta)\frac{\delta}{K} + O\left(\delta^2\right) - 1\right) \\
&= \frac{\alpha K^{-\beta}}{\sigma_0}\delta + O\left(\delta^2\right) \\
\Rightarrow D(\zeta) &= \frac{\alpha K^{-\beta}}{\sigma_0}\delta + O\left(\delta^2\right).
\end{aligned}$$

Thus, in the ATM limit $\delta \to 0$,

$$\begin{aligned}
\sigma_{\mathrm{n}} &= \alpha \frac{F_0 - K}{D(\zeta)} \left(1 + O(\epsilon)\right) \\
&= \alpha \frac{\delta}{\frac{\alpha K^{-\beta}}{\sigma_0}\delta + O\left(\delta^2\right)} \left(1 + O(\epsilon)\right) \\
&= \sigma_0 K^{\beta} \left(1 + O(\epsilon)\right) + O\left(\delta\right).
\end{aligned}$$

In other words,

$$\boxed{\sigma_{\mathrm{n}}^{\mathrm{ATM}} = \sigma_0 K^{\beta} \left(1 + O(\epsilon)\right).}$$

For lognormal volatilities,

$$\begin{aligned}
\sigma_{\mathrm{ln}} &= \alpha \frac{\log F_0 - \log K}{D(\zeta)} \left(1 + O(\epsilon)\right) \\
&= \alpha \frac{\log(K + \delta) - \log K}{D(\zeta)} \left(1 + O(\epsilon)\right) \\
&= \alpha \frac{\log\left(1 + \frac{\delta}{K}\right)}{D(\zeta)} \left(1 + O(\epsilon)\right) \\
&= \alpha \frac{\frac{\delta}{K} + O\left(\delta^2\right)}{D(\zeta)} \left(1 + O(\epsilon)\right) \\
&= \alpha \frac{\frac{\delta}{K} + O\left(\delta^2\right)}{\frac{\alpha K^{-\beta}}{\sigma_0}\delta + O\left(\delta^2\right)} \left(1 + O(\epsilon)\right) \\
&= \sigma_0 K^{\beta - 1} \left(1 + O(\epsilon)\right) + O\left(\delta\right).
\end{aligned}$$

In other words,

$$\boxed{\sigma_{\mathrm{ln}}^{\mathrm{ATM}} = \sigma_0 K^{\beta - 1} \left(1 + O(\epsilon)\right).}$$