November 25, 2015

MTH 9831 Solutions to Quiz 9.

(1) (5 points) Assume the BSM model with $r \neq 0$. Find explicitly the portfolio value process $X(t)$, $0 \leq t \leq T$, such that

$$X(T) = \frac{1}{T} \int_0^T S(t)\,dt - S(T).$$

Solution. Let $X(t) = \gamma(t)S(t) + (X(t) - \gamma(t)S(t))$ be a self-financing portfolio, where $\gamma(t)$ is a differentiable process. We have

$$\begin{aligned} d(e^{-rt}X(t)) &= e^{-rt}\left(-rX(t)dt + \gamma(t)dS(t) + r(X(t) - \gamma(t)S(t))dt\right) \\ &= e^{-rt}(\gamma(t)dS(t) - r\gamma(t)S(t)dt) \quad (1) \end{aligned}$$

On the other hand,

$$d(e^{-rt}\gamma(t)S(t)) = e^{-rt}\left(-r\gamma(t)S(t)dt + \gamma(t)dS(t) + S(t)d\gamma(t) + d[S,\gamma](t)\right).$$

Since $\gamma(t)$ is assumed to be regular, $d[S,\gamma](t) = 0$. This gives us the following expression for the right-hand side of (1)

$$e^{-rt}(\gamma(t)dS(t) - r\gamma(t)S(t)dt) = d(e^{-rt}\gamma(t)S(t)) - e^{-rt}S(t)d\gamma(t).$$

Thus,

$$d(e^{-rt}X(t)) = d(e^{-rt}\gamma(t)S(t)) - e^{-rt}S(t)d\gamma(t).$$

Integrating from 0 to T and multiplying by e^{rT} throughout we get

$$X(T) = e^{rT}(X(0) - \gamma(0)S(0)) + \gamma(T)S(T) - \int_0^T e^{r(T-t)}S(t)\gamma'(t)dt.$$

To match the required value $X(T)$ we need

$$X(0) = \gamma(0)S(0), \quad \gamma(T) = -1,$$
$$-e^{r(T-t)}S(t)\gamma'(t) = \frac{1}{T}S(t) \quad \Rightarrow \quad \gamma'(t) = -\frac{1}{T}e^{-r(T-t)}.$$

Integrating the last equation from t to T we get

$$\gamma(T) - \gamma(t) = -\frac{1}{T}\int_t^T e^{-r(T-u)}du = -\frac{1}{rT}(1 - e^{-r(T-t)}).$$

Using the condition $\gamma(T) = -1$ we conclude that

$$\gamma(t) = -1 + \frac{1}{rT}(1 - e^{-r(T-t)}).$$

This strategy and the fact that $X(0) = \gamma(0)S(0)$ guarantee the required value of $X(T)$.

(2) (2 points) What is the general solution of the equation $v(x) - xv'(x) - 2x^2v''(x) = 0$ on $(0, \infty)$?

Solution. In financial applications we are interested in a solution defined on the positive semi-axis. We look for a solution in the form $v(x) = x^\alpha$, $x > 0$, for some unknown constant α. Substituting $v(x) = x^\alpha$ into the equation we obtain

$$x^\alpha - \alpha x^\alpha - 2\alpha(\alpha - 1)x^\alpha = 0.$$

For this equation to hold for all $x > 0$ we should have $1 - \alpha - 2\alpha(\alpha - 1) = 0$. This quadratic equation has 2 solutions, $\alpha = 1$ and $\alpha = -1/2$. Since functions x and $x^{-1/2}$ are linearly independent and the space of solutions of our equation is 2 dimensional, every solution of this ODE on $(0, \infty)$ should be of the form

$$v(x) = Ax^{-1/2} + Bx,$$

where A and B are arbitrary constants.

(3) (4=1+1.5+1.5 points) Let $\mu \in \mathbb{R}$, $m > 0$, $X(t) = \mu t + B(t)$ and $\tau_m = \min\{t \geq 0 : X(t) = m\}$. Theorem 8.3.2 states that for all $\lambda > 0$

$$\mathbb{E}(e^{-\lambda\tau_m}) = e^{-m(-\mu+\sqrt{\mu^2+2\lambda})}. \tag{2}$$

(a) Use the above formula to compute $\mathbb{P}(\tau_m < \infty)$.

(b) For which $\sigma > 0$ the process $e^{\sigma X(t) - \lambda t}$ is a martingale?

(c) Assume for simplicity that we are allowed to apply the optional stopping theorem with τ_m instead of $t \wedge \tau_m$. Then how how would you derive (2) from (b) in just a couple of lines?

Solution. (a) We have

$$\mathbb{E}(e^{-\lambda\tau_m}) = \mathbb{E}(e^{-\lambda\tau_m} 1_{\{\tau_m<\infty\}}) + \mathbb{E}(e^{-\lambda\tau_m} 1_{\{\tau_m=\infty\}}) = \mathbb{E}(e^{-\lambda\tau_m} 1_{\{\tau_m<\infty\}}).$$

Note that $0 \leq e^{-\lambda\tau_m} 1_{\{\tau_m<\infty\}}$ is monotone in λ and converges to $1_{\{\tau_m<\infty\}}$ as $\lambda \downarrow 0$. By the above equality and the monotone convergence theorem,

$$\lim_{\lambda\to0} \mathbb{E}(e^{-\lambda\tau_m}) = \lim_{\lambda\to0} \mathbb{E}(e^{-\lambda\tau_m} 1_{\{\tau_m<\infty\}}) = \mathbb{P}(\tau_m < \infty).$$

We have

$$\mathbb{P}(\tau_m < \infty) = \lim_{\lambda\to0} \mathbb{E}(e^{-\lambda\tau_m}) = \lim_{\lambda\to0} e^{-m(-\mu+\sqrt{\mu^2+2\lambda})} = e^{-m(-\mu+|\mu|)}.$$

Therefore,

$$\mathbb{P}(\tau_m < \infty) = e^{-m(-\mu+|\mu|)} = \begin{cases} 1, & \text{if } \mu \geq 0; \\ e^{-2m|\mu|} < 1, & \text{if } \mu < 0. \end{cases}$$

(b) We know that for every $\sigma > 0$ the process $e^{\sigma B(t) - \sigma^2 t/2}$ is a martingale. Therefore

$$e^{\sigma X(t)-\lambda t} = e^{\sigma(B(t)+\mu t)-\lambda t} = e^{\sigma B(t)+(\sigma\mu-\lambda)t}$$

is a martingale if $\sigma\mu - \lambda = -\sigma^2/2$. Solving the quadratic equation for σ we get

$$\sigma = -\mu \pm \sqrt{\mu^2 + 2\lambda}.$$

Since we need $\sigma > 0$, the answer is $\sigma = \sqrt{\mu^2 + 2\lambda} - \mu$.

(c) Let $\sigma = \sqrt{\mu^2 + 2\lambda} - \mu$. By part (b), $e^{\sigma X(t)-\lambda t}$ is a martingale. If we could use the optional stopping theorem directly for τ_m we would get

$$\mathbb{E}(e^{\sigma X(\tau_m)-\lambda\tau_m}) = e^{\sigma X(0)} = 1.$$

We want to say that $X(\tau_m) = m$ and, thus, $\mathbb{E}(e^{\sigma m-\lambda\tau_m}) = 1$, which is the same as $\mathbb{E}(e^{-\lambda\tau_m}) = e^{-\sigma m}$, where $\sigma = \sqrt{\mu^2 + 2\lambda} - \mu$, and we are done.

This is indeed the main idea of the proof. But I can not refrain from a comment. It is not right to say that $X(\tau_m) = m$ if $\tau_m = \infty$ (or apply the optional stopping theorem, to begin with). When $\mu \geq 0$ we know from (b) that $\mathbb{P}(\tau_m = \infty) = 0$ and can easily ignore a set of measure 0. But when $\mu < 0$ the above reasoning fails, since $\mathbb{P}(\tau_m = \infty) > 0$

and we neither can say that $X(\tau_m) = m$ nor ignore this set. But there is a remedy for this situation. We know that $X(t) < m$ for all t on the set $\{\tau_m = \infty\}$, and it is reasonable to say that $e^{\sigma X(\tau_m) - \lambda\tau_m} = 0$ on this set. Thus,

$$\mathbb{E}(e^{\sigma X(\tau_m)-\lambda\tau_m}) = \mathbb{E}(e^{\sigma X(\tau_m)-\lambda\tau_m} 1_{\{\tau_m<\infty\}}) = \mathbb{E}(e^{\sigma m-\lambda\tau_m} 1_{\{\tau_m<\infty\}}) = 1.$$

Dividing the latter equality by $e^{\sigma m}$ we get $\mathbb{E}(e^{-\lambda\tau_m} 1_{\{\tau_m<\infty\}}) = e^{-\sigma m}$. Remembering again that $e^{-\lambda\tau_m} = 0$ when $\tau_m = \infty$ we arrive at the formula

$$\mathbb{E}(e^{-\lambda\tau_m}) = \mathbb{E}(e^{-\lambda\tau_m} 1_{\{\tau_m<\infty\}}) = e^{-\sigma m} = e^{-m(-\mu+\sqrt{\mu^2+2\lambda})}.$$

Conclusion: even though the main idea is simple but, once you start implementing it, you run into all sorts of questions which are fully answered only through a rigorous proof.[1]

[1]As you know, the honest derivation requires to consider bounded stopping times $\tau_m \wedge t$ (to which the optional stopping theorem is always applicable) and then show that extra terms vanish when we let $t \to \infty$. Refer to Theorem 8.3.2 for full details.