MTH 9831 Assignment 1 (9/02 - 9/09).

Read Lecture 1. Some additional references for this material are:

1. J. Jacod, Ph. Protter, Probability Essentials, Chapter 16.
2. S. Shreve, Stochastic Calculus for Finance II, Chapter 3.
3. A. Etheridge, A Course in Financial Calculus, Chapter 3.

Rules of the game: When solving these problems you are allowed to use (with a proper reference) any statement from lecture notes, even if it was given without a proof. Citing a convenient theorem from some other source and deriving your solution from it is not allowed.

Solve:

(1) The characteristic functions of X and Y are given by

$$\varphi_X(t) = \exp(2e^{it} - 2), \quad \varphi_Y(t) = \left(\frac{3}{4}e^{it} + \frac{1}{4}\right)^{10}.$$

Assume that X and Y are independent and find

(a) $E(XY)$; (b) $P(XY = 0)$; (c) $P(X + Y = 2)$.

(2) Show that if X and Y are independent normal random variables with the same variance then $X + Y$ and $X - Y$ are also independent.

(3) Let $(B(t))_{t\geq 0}$ be a Brownian motion and $\alpha \in (0, 1)$. Find the conditional density of $B(\alpha t)$ given that $B(t) = y$. Clearly specify the type of the distribution and its parameters.

(4) Let $B(t) = (B_1(t), B_2(t))$, $t \geq 0$, be a two dimensional Brownian motion (assume that the coordinates are independent). Find the distribution of the distance from $B(t)$ to the origin.

(5) Let $B(t) = (B_1(t), B_2(t))$, $t \geq 0$, be a two dimensional Brownian motion (assume that the coordinates are independent), and $\rho \in [-1, 1]$. Is the process $X(t) = \rho B_1(t) + \sqrt{1 - \rho^2} B_2(t)$ a Brownian motion? What is the correlation between $X(t)$ and $B_1(t)$?

(6) Let $(B(t))_{t\geq 0}$ be a Brownian motion. Determine, which of the following processes are Brownian motions. Justify your answers.

(a) $(-B(t))_{t\geq 0}$;
(b) $(cB(t/c^2))_{t\geq 0}$, where $c > 0$ is a constant;
(c) $(\sqrt{t}B(1))_{t\geq 0}$;
(d) $(B(2t) - B(t))_{t\geq 0}$;
(e) $(B(s) - B(s - t))_{0\leq t\leq s}$, where s is fixed.

(7) Let $(B(t))_{t\geq 0}$ be a Brownian motion. Define $B^*(t) := \max_{0\leq s\leq t} B(s)$. For $0 \leq a \leq x$ calculate $P(B^*(t) \geq a,\, B(t) \leq x)$.