Department of Mathematics *May 23, 2013*

Probabiltiy and Stochastic Processes for Finance II (MTH 9862).

Final Examination.

Instructions: Please **print** your name below. Show all work and write legibly. Full credit corresponds to 100 points. **Good luck!**

Student name:

Grade

Problem	Out of	Score	Comments
1	20		
2	20		
3	20		6+7+7
4	20		
5	20		
6	20		
Total	120		

Problem 1. Let $B(t)$, $t \geq 0$, be a standard Brownian motion, $N(t)$, $t \geq 0$ be a Poisson process with intensity λ, and $M(t)$, $t \geq 0$ be a compensated Poisson process. Evaluate the following stochastic integrals:

(a) $\int_0^t B(s)e^{B(s)}dB(s)$;

(b) $\int_0^t e^{2B(s)-2s}\, dB(s)$;

(c) $\int_0^t N(s)\, dN(s)$;

(d) $\int_0^t M(s-)\, dM(s)$

Problem 2. Let $B(t)$, $t \geq 0$, be a standard Brownian motion, $Q(t)$, $t \geq 0$, be a compound Poisson process with intensity λ and jump distribution $P(Y_1 = y_m) = p_m$, $m \in \{1, 2, \ldots, M\}$ (both are adapted to the same filtration). Set

$$S(t) = S(0)e^{\mu t + \sigma B(t) + Q(t)},$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ are constants. Find

(a) $\operatorname{Var}(S(t))$;

(b) $\operatorname{Cov}(S(t), B(t))$;

(c) $\operatorname{Cov}(S(t), Q(t))$.

Problem 3. Let under $\mathbb{P}$ the process $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$. Let $\tilde{\lambda} \in (0, \infty)$.

(a) Find the measure $\tilde{\mathbb{P}}$ under which $\{N(t)\}_{0\leq t\leq T}$ is a Poisson process with intensity $\tilde{\lambda}$. Write explicitly the Radon-Nikodym derivative $Z(t)$, $0 \leq t \leq T$.

(b) Use the definition of a martingale and properties of Poisson process to show that $Z(t)$, $0 \leq t \leq T$, is a martingale.

(c) Apply Itô-Doeblin formula to show that $Z(t)$, $0 \leq t \leq T$, is a martingale.

Problem 4.

(a) Suppose that for $0 \leq t \leq u \leq T$

$$dX(u) = b(u, X(u))\, du + \sigma(u, X(u))\, dB(u), \quad X(t) = x.$$

Let $f(x)$ and $h(x)$ be given deterministic functions (such that all integrals below are well-defined). Find the PDE satisfied by

$$g(t, x) = E^{t,x}[h(X(T)] + \int_t^T E^{t,x}[f(X(u))]\, du.$$

(b) Solve explicitly the following terminal value problem:

$$g_t(t, x) + (1 - x)g_x(t, x) + 2g_{xx}(t, x) = 1, \quad g(T, x) = x^2.$$

Problem 5. Assume BSM model with a constant interest rate and no dividends. Let $S(t)$ be the price of the stock at time t. Define

$$Y(T) := \exp\left(\frac{1}{T}\int_0^T \ln S(t)\,dt\right).$$

Suppose that an Asian call option has payoff $(Y(T)-K)_+$ at time T. Find an explicit formula for the price of such an option at time 0.

Problem 6. Assume BSM model. An American cash-or-nothing option can be exercised at any time $t \geq 0$ (no expiration). If exercised at time t its payoff is

$$\begin{cases} 1, & \text{if } S(t) \leq K; \\ 0, & \text{if } S(t) > K. \end{cases}$$

(a) What is the optimal exercise strategy?

(b) What is the time 0 price of this option? You may use the fact that for $\mu \in \mathbb{R}$, $m > 0$, $X(t) = \mu t + B(t)$, $\tau_m = \inf\{t \geq 0 : X(t) = m\}$, and for all $\lambda > 0$

$$\mathbb{E}(e^{-\lambda \tau_m} 1_{\{\tau_m < \infty\}}) = e^{-m(-\mu + \sqrt{\mu^2 + 2\lambda})}.$$