Interest Rate Models

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Outline

Beyond local volatility models

- Normal, shifted lognormal, and CEV model are local volatility models. They improve Black's model but do not capture volatility smile phenomenon.
- Implied volatilities from of shorter dated maturities and tenors rise for higher strikes.
- Possible improvements:
  (i) *Stochastic volatility models*. The volatility parameter is assumed to be a stochastic process.
  (ii) *Jump difusion models*. These models allow for underlying assets to follow a broader class of stochastic processes such as Levy processes. This way the dynamics of assets involves jumps.
- We will focus on (i) only.

Dynamics of SABR

- SABR (stochastic $\alpha\beta\rho$) is an extension of CEV. Instead of being constant, the volatility follows a separate stochastic differential equation.
- $$\begin{aligned} dF(t) &= \sigma(t)C(t, F(t))\, dW(t) \\ d\sigma(t) &= \alpha\sigma(t)\, dZ(t). \end{aligned}$$
- $F(t)$ is a forward rate process which may represent a LIBOR forward, a forward swap rate, or crude oil forward. $\sigma(t)$ is stochastic volatility parameter. The correlation between W and Z is assumed to be constant,

$$\mathbb{E}\left[dW(t)dZ(t)\right] = \rho\, dt.$$

Dynamics of SABR

- The diffusion $C(t, F(t))$ is assumed to be of CEV type:

$$C(t, F(t)) = F(t)^{\beta}.$$

- We are assuming that a suitable numeraire is chosen for which F is a martingale. The constant parameter α is the volatility of the stochastic process $\sigma(t)$. The constant α is often called *vol of vol*.
- We will consider the initial value problem with

$$\begin{aligned} F(0) &= f_0 \\ \sigma(0) &= \sigma_0. \end{aligned}$$

- Just as was the case with CEV model, we need to impose the boundary conditions for $F = 0$. Usually we consider Dirichlet boundary conditions.

SABR PDE

- Our goal is to find the value of the option $P(t, K, F_0, \sigma_0)$ with strike K for given initial values F_0 and σ_0 of the underlying securities.
- We face an additional challenge of not knowing the final value of the volatility. Consider first the valuation function $B(t, K, x, \Sigma, y)$ in which x corresponds to the underlying forward, and y corresponds to the volatility. The terminal value problem becomes

$$B_t + \frac{1}{2}y^2 \left(x^{2\beta} B_{xx} + 2\rho\alpha x^{\beta} B_{xy} + \alpha^2 B_{yy} \right) = 0 \qquad (1)$$

$$B(T, K, x, \Sigma, y) = \mathrm{Payoff}(K, x) \cdot \delta(\Sigma - y) \qquad (2)$$

SABR PDE

- We obtain the price of the option as

$$P(t, K, F_0, \sigma_0) = \mathcal{N}(0) \int_0^{+\infty} B(T, K, F_0, \Sigma, y_0)\, d\Sigma. \qquad (3)$$

- No explicit solution is known to (1)–(3) except when $\beta = 0$. Even for $\beta = 0$ the explicit solution is too complex to be of any practical use.
- Numerical methods are also too difficult to implement in the general case. One first needs to solve the three dimensional pde and then to evaluate the integral.

SABR implied volatility

- The solutions can be found approximately by a method of perturbation expansion in parameter $\varepsilon = T\alpha^2$. This parameter happens to be small and the approximate solution is accurate.
- It turns out that this method is easy to implement numerically.
- The parameter α has a stable term structure as a function of the swaption maturity and the tenor of the underlying swap.
- The parameter α is highest for the short expirations and short tenors, and the lowest for long expirations and long tenors.
- Typically, α is a monotone decreasing function in both expirations and tenors and its range is usually between 0.2 and 2.

SABR implied volatility

- The following approach is taken for the lack of explicit solutions: We force valuation formulas to be of the form

$$\begin{aligned} P^{\text{call}}(t, K, F_0, \sigma_0) &= \mathcal{N}(0)\sigma_n\sqrt{T}\left(d_+N(d_+) + N'(d_+)\right) \\ P^{\text{put}}(t, K, F_0, \sigma_0) &= \mathcal{N}(0)\sigma_n\sqrt{T}\left(d_-N(d_-) + N'(d_-)\right) \\ \text{for} \qquad & d_\pm = \pm\frac{F_0 - K}{\sigma_n\sqrt{T}}. \end{aligned}$$

  (These formulas are valid for the normal model.)
- We will consider the example of a *normal SABR model* in which $\beta = 0$ and $\rho = 0$.
- In fact, the general SABR model can be reduced to normal SABR model by a change of parameters.

Normal SABR model

- Consider the associated Green's function given by:

$$
\begin{aligned}
G_t + \frac{1}{2}y^2\left(G_{xx} + \alpha^2 G_{yy}\right) &= 0 && (4)\\
G(t, K, x, \Sigma, y) &= \delta(x-K)\delta(y-\Sigma). && (5)
\end{aligned}
$$

- Then for the case of call option, B is given by

$$
B(t, K, x, \Sigma, y) = \int (x-X)^+ G(t, K, X, \Sigma, y)\, dX.
$$

- If we do time reversal (i.e. substitute $\tau = T - t$) then G satisfies the following initial value problem

$$
\begin{aligned}
-G_\tau + \frac{1}{2}y^2\left(G_{xx} + \alpha^2 G_{yy}\right) &= 0 && (6)\\
G(0, K, x, \Sigma, y) &= \delta(x-K)\delta(y-\Sigma). && (7)
\end{aligned}
$$

Normal SABR model

- The initial value problem is solved by McKean's formula:

$$G(\tau, K, x, \Sigma, y) = \frac{e^{-\frac{\alpha^2\tau}{8}}\sqrt{2}}{(2\pi\tau\alpha^2)^{\frac{3}{2}}} \int_d^{+\infty} \frac{u e^{-\frac{u^2}{2\alpha^2\tau}}}{\sqrt{\cosh u - \cosh d}}\, du. \quad (8)$$

- Here $d = d(x, y, K, \Sigma)$ is the geodesic distance function:

$$\cosh d(x, y, K, \Sigma) = 1 + \frac{\alpha^2(x-K)^2 + (y-\Sigma)^2}{2y\Sigma}.$$

- The integral in (8) can be evaluated using asymptotic expansion for $\alpha^2\tau \to 0$.

Normal SABR model

- Homework 3: If φ has a unique minimum at $u_0 \in (0, +\infty)$ and some other conditions, then

$$\int_0^{+\infty} f(u) e^{-\frac{\varphi(u)}{\varepsilon}}\, du = f(u_0) e^{-\frac{\varphi(u_0)}{\varepsilon}} \cdot \sqrt{\frac{2\pi\varepsilon}{\varphi''(u_0)}} + O(\varepsilon).$$

- We now obtain the following approximation:

$$G(\tau, K, x, \Sigma, y) = \frac{1}{2\pi\alpha^2\tau} \sqrt{\frac{d}{\sinh d}} e^{-\frac{d^2}{2\alpha^2\tau}} \left(1 + O\left(\alpha^2\tau\right)\right).$$

Normal SABR model

- *Step 1.* The marginal density is obtained by integrating G over the terminal volatility parameter Σ:

$$\begin{aligned} P(\tau, K, x, y) &= \int_0^{+\infty} G(\tau, K, x, \Sigma, y)\, d\Sigma \\ &= \frac{1}{2\pi\alpha^2\tau} \int_0^{+\infty} \sqrt{\frac{d}{\sinh d}} e^{-\frac{d^2}{2\alpha^2\tau}}\, d\Sigma + O\left(\alpha^2\tau\right). \end{aligned}$$

- *Step 2.* Apply the same method to the function $\varphi(\Sigma) = \frac{1}{2} d(x, y, K, \Sigma)^2$ that attains its minimum at

$$\Sigma_0 = y\sqrt{\zeta^2 - 2\rho\zeta + 1}, \text{ where } \zeta = \frac{\alpha(F_0 - K)}{\sigma_0}.$$

Normal SABR model

- *Step 3.* Denote by $D(\zeta)$ the value of $d(x, y, K, \Sigma)$ for $\Sigma = \Sigma_0$. This can be explicitly calculated to obtain

$$D(\zeta) = \log \frac{\sqrt{\zeta^2 - 2\rho\zeta + 1} + \zeta - \rho}{1 - \rho}.$$

Let us also introduce

$$I(\zeta) = \cosh D(\zeta) - \rho \sinh D(\zeta) = \sqrt{\zeta^2 - 2\rho\zeta + 1}.$$

Normal SABR model

- Then we obtain

$$P(\tau, K, x, y) = \frac{1}{\sqrt{2\pi\tau}} \cdot \frac{1}{y \cdot I^{\frac{3}{2}}(\zeta)} e^{-\frac{D^2}{2\alpha^2\tau}} \left(1 + O\left(\alpha^2\tau\right)\right).$$

- We can now obtain the implied normal volatility by comparing the above distribution density function with the one associated to the normal random variable. We obtain

$$\sigma_{\mathrm{n}}\left(T, K, F_0, \sigma_0, \alpha, \beta, \rho\right) = \alpha \frac{F_0 - K}{D(\zeta)} \left(1 + O(\varepsilon)\right). \tag{9}$$

SABR implied volatility

- The previous method can be generalized for SABR models that are not necessarily normal. The implied normal volatility can be approximated by the formula

$$\begin{aligned}
\sigma_{\mathsf{n}} &= \sigma_{\mathsf{n}}\left(T, K, F_0, \sigma_0, \alpha, \beta, \rho\right) \\
&= \alpha \frac{F_0 - K}{D(\zeta)} \left\{ 1 + \varepsilon \left[\frac{2\gamma_2 - \gamma_1^2}{24} \cdot \left(\frac{\sigma_0 C\left(F_{\mathrm{mid}}\right)}{\alpha} \right)^2 \right. \right. \\
&\quad \left. \left. + \frac{\rho\gamma_1}{4} \cdot \frac{\sigma_0 C\left(F_{\mathrm{mid}}\right)}{\alpha} + \frac{2 - 3\rho^2}{24} \right] + \cdots \right\} \qquad (10)
\end{aligned}$$

Here F_{mid} is a number between F_0 and K (such as $\frac{F_0+K}{2}$) and

$$\gamma_1 = \frac{C'\left(F_{\mathrm{mid}}\right)}{C\left(F_{\mathrm{mid}}\right)}, \qquad \gamma_2 = \frac{C''\left(F_{\mathrm{mid}}\right)}{C\left(F_{\mathrm{mid}}\right)}, \quad \text{and} \quad C(F) = F^{\beta}.$$

SABR implied volatility

- The distance function D in the previous formula is given by

$$D(\zeta) = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}+\zeta-\rho}{1-\rho}\right),$$

where

$$\zeta \quad = \quad \frac{\alpha}{\sigma_0}\int_K^{F_0}\frac{dx}{C(x)} = \frac{\alpha}{\sigma_0(1-\beta)}\left(F_0^{1-\beta}-K^{1-\beta}\right).$$

SABR implied volatility

- The implied lognormal volatility can be obtained using similar asymptotic formula

$$
\begin{aligned}
\sigma_{\ln} &= \sigma_{\ln}\left(T, K, F_0, \sigma_0, \alpha, \beta, \rho\right) \\
&= \alpha \frac{\log \frac{F_0}{K}}{D(\zeta)} \left\{ 1 + \varepsilon \left[\frac{2\gamma_2 - \gamma_1^2 + \frac{1}{F_{\mathrm{mid}}^2}}{24} \cdot \left(\frac{\sigma_0 C\left(F_{\mathrm{mid}}\right)}{\alpha} \right)^2 \right.\right. \\
&\left.\left. + \ \frac{\rho\gamma_1}{4} \cdot \frac{\sigma_0 C\left(F_{\mathrm{mid}}\right)}{\alpha} + \frac{2 - 3\rho^2}{24} \right] + \cdots \right\} \qquad (11)
\end{aligned}
$$

Arbitrage problem in volatility modeling

- SABR model is easy to implement because of the explicit formulas (10) and (11). Although the formulas are asymptotic and require $\alpha^2 T << 1$ they are often treated as exact.
- We may regard (10) and (11) as implied volatilities obtained from a new model that is an approximated by SABR model. However, that model would not be valid if there is arbitrage.
- The two key requirements that the model is arbitrage free are:

  (i) Put-call parity.
  (ii) The terminal probability density function implied by the call and put prices needs to be positive.

  The put call parity holds because we are using the same implied volatility for both puts and calls.

Arbitrage problem in volatility modeling

- We now analyze the second requirement.
- The put and call prices can be written as

$$\begin{aligned} P^{\text{call}}(T,K) &= \int (F-K)^+ g_T(F,K)\,dF \\ P^{\text{put}}(T,K) &= \int (K-F)^+ g_T(F,K)\,dF. \end{aligned}$$

  Here $g_T(F,K)$ is the risk-neutral probability density function at the exercise date.
- We know from lecture 3 that

$$g_T(K,K) = \frac{d^2}{dK^2} P^{\text{call}}(T,K) = \frac{d^2}{dK^2} P^{\text{put}}(T,K).$$

Arbitrage problem in volatility modeling

- Therefore the implied volatility formula (10) can represent an arbitrage free model only if

$$
\begin{aligned}
\frac{\partial^2}{\partial K^2} P^{\mathrm{call}}\left(T, K, F_0, \sigma_{\mathrm{n}}(K, \dots)\right) &= \\
\frac{\partial^2}{\partial K^2} P^{\mathrm{put}}\left(T, K, F_0, \sigma_{\mathrm{n}}(K, \dots)\right) &\geq 0
\end{aligned}
$$

- The above condition is often violated for very low strikes and long expirations.

Calibration of SABR

- For each option maturity and underlying we need to specify 4 parameters: σ_0, α, β, and ρ. We have to use marked implied volatilities for different strikes.
- There is a bit of redundancy between parameters β and ρ. Calibration is usually done by fixing one of these two parameters. One of the following approaches may be taken:
  (i) Fix $\beta \leq 1$, for example $\beta = 0.5$ and calibrate σ_0, α, and ρ. This choice of calibration works well under normal conditions. During the distress in the period 2008–2009 this calibration resulted in ρ closed to ± 1. This was usually avoided by choosing β close to 1 for short expiry options.
  (ii) Fix ρ and calibrate σ_0, α, and β.
- Calibration shows that α is higher for short expiry options and smaller as the expiration grows. This suggests that short termed options should include jump component.

Building volatility cube

- Implied swaption volatilities depend on:
  (i) Option expiration
  (ii) Tenor of the underlying swap
  (iii) Strike on the option.
- This three dimensional object is called *volatility cube*. Although swaption market is liquid, at a given moment it can happen that only a limited number of options are actively traded. The market only provides information for certain benchmark maturities (1 month, 3 months, 6 months, 1 year, ...), underlyings (1 year, 2 years, 5 years, ...), and strikes (ATM, ± 50 bp, ...).
- We need to interpolate for remaining maturities, tenors, and strikes so that we do not create arbitrage.

Building volatility cube

- December 2011 snapshot of ATM lognormal volatilities.

	1Y	2Y	3Y	4Y	5Y	7Y	10Y	15Y	20Y	30Y
1M	71.7%	68.2%	61.8%	56.0%	50.6%	49.7%	47.3%	44.4%	43.4%	43.0%
3M	73.6%	68.8%	61.1%	55.9%	51.5%	49.6%	47.2%	44.0%	42.8%	42.8%
6M	72.2%	70.0%	60.0%	55.6%	51.9%	49%	46.1%	42.9%	42.1%	41.9%
1Y	73.8%	65.5%	57.1%	52.9%	49.7%	46.6%	43.8%	40.8%	40.3%	40.0%
2Y	73.7%	59.0%	51.4%	47.2%	45.4%	42.0%	39.4%	37.0%	36.6%	36.4%
3Y	57.8%	49.0%	44.3%	41.8%	40.4%	37.9%	36.0%	34.3%	34.1%	33.4%
4Y	46.0%	41.6%	39.2%	37.8%	36.8%	35.1%	33.9%	32.7%	32.4%	32.1%
5Y	39.7%	37.8%	36.6%	25.6%	34.8%	33.5%	32.6%	31.7%	31.6%	31.3%
7Y	34.4%	33.4%	32.4%	31.7%	31.2%	30.7%	30.7%	30.0%	29.7%	29.2%
10Y	29.8%	29.4%	29.1%	28.8%	28.7%	28.8%	29.2%	28.2%	27.5%	26.8%

Building volatility cube

- ATM normal volatilities (in basis points).

	1Y	2Y	3Y	4Y	5Y	7Y	10Y	15Y	20Y	30Y
1M	48	48	51	58	65	85	101	11	114	116
3M	51	49	53	61	69	88	102	111	113	116
6M	51	52	56	66	75	91	103	110	112	115
1Y	52	56	63	74	82	94	104	108	110	111
2Y	72	76	83	88	94	99	103	104	104	104
3Y	91	93	95	98	100	101	103	102	101	99
4Y	101	101	101	102	102	102	103	200	98	97
5Y	105	104	104	104	104	103	102	99	97	95
7Y	103	102	101	100	100	99	98	94	92	89
10Y	98	97	96	95	94	93	92	87	84	81

Building volatility cube

- The corresponding strikes for ATM swaptions are given in the table below

	1Y	2Y	3Y	4Y	5Y	7Y	10Y	15Y	20Y	30Y
1M	0.67%	0.70%	0.82%	.104%	1.29%	1.72%	2.13%	2.49%	2.69%	2.71%
3M	0.69%	0.72%	0.87%	1.10%	1.35%	1.77%	2.18%	2.52%	2.64%	2.72%
1Y	0.71%	0.76%	0.94%	1.19%	1.45%	1.86%	2.25%	2.57%	2.67%	2.75%
2Y	0.72%	0.87%	1.13%	1.41%	1.66%	2.03%	2.38%	2.66%	2.74%	2.80%
3Y	1.64%	1.96%	2.21%	2.39%	2.52%	2.72%	2.91%	3.00%	3.00%	3.00%
4Y	2.27%	2.50%	2.64%	2.75%	2.84%	2.98%	3.09%	3.11%	3.09%	3.07%
5Y	2.72%	2.84%	2.92%	2.99%	3.06%	3.15%	3.21%	3.18%	3.14%	3.11%
7Y	3.08%	3.16%	3.21%	3.26%	3.29%	3.31%	3.29%	3.22%	3.17%	3.13%
10Y	3.40%	3.41%	3.41%	3.40%	3.38%	3.33%	3.26%	3.18%	3.14%	3.11%

Stripping cap volatility

- The market does not quote caplets and floorlets.
- Instead, caps and floors are quoted in terms of a single number. Therefore we can obtain a single volatility (known as *flat volatility*) implied by cap. When plugged in the valuation formula it reproduces the correct price of the cap.
- A single caplet can be a part of different caps and as such can have assigned different flat volatilities.
- The process of obtaining actual implied caplet volatility from market quotes of cap prices is called *stripping* cap volatility.
- Its result is a sequence of ATM caplet volatilities for maturities ranging from one day to 30 years. Convenient benchmarks are 3 months, 6 months, 9 months, etc.
- One can obtain data for Eurodollar options and OTC caps and floors.

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