Fourier Transforms and Spectra Analysis

PREFACE: In class and in my lecture notes there are several cases where two or three impulse responses of different shape are compared. For instance, Fig. 1.16 compares the smoothing generated by a box, an ema, and an integrated macd impulse response; and Fig. 1.17 compares the effect of a box differencer with an macd.

One of the essential ingredients for a fair comparison is to have the degree of smoothness matched, as best as possible, across the impulse responses under consideration.[1] The best way to do this is to match spectral bandwidth, which in turn requires the Fourier transform.

This homework will walk you through how to spectrally match several different filters.

PREPARATION:

You will need the price data that you generated from the last homework, along with the impulse response functions that you wrote. There are a few additional impulse response functions that you need to code for this homework. They are:

1. **Box-Differencer Function:** (N_box, N_window) Implements a box differencer impulse response as in Fig. 1.9. The positive arm is N_box long with amplitude $1/N_{\text{box}}$. The negative arm is the same but offset by $N_{\text{box}} + 1$. The gain is zero and gauge is unity.[2]

2. **Ema-Poly1 Function:** (N_eff, N_window) This impulse response is an ema with a polynomial coefficient. The impulse response is defined as

$$h_{\text{poly}}[n] \equiv (1-p)^2 (n+1) p^n u[n] \tag{1}$$

where the decay rate p is given by $p = (N_{\text{eff}}/2)/((N_{\text{eff}}/2) + 1)$.

3. **Integrated Macd-Poly Function:** (N_eff, N_window) First make the macd piece, defined as

$$h_{\text{macd-poly}}(N_{\text{eff}})[n] \equiv h_{\text{ema}}(N_{\text{eff}}/3)[n] - h_{\text{poly}}(N_{\text{eff}})[n] \tag{2}$$

Note that this macd−poly1 has zero gain and unit gauge. Next, make the integrated macd-poly piece, defined as

$$\begin{aligned} h_{\text{imacd}}(N_{\text{eff}})[n] &\equiv (2N_{\text{eff}}/3)^{-1}\, u[n] * h_{\text{macd-poly}}(N_{\text{eff}})[n] \\ &= (2N_{\text{eff}}/3)^{-1} \sum_{k=0}^{n} h_{\text{macd-poly}}(N_{\text{eff}})[k] \end{aligned} \tag{3}$$

This is just the cumulative sum of the difference. This response has unit gain.

[1] A good example of a wrong result, which I left in intentionally, is shown in Fig. 1.17, where the macd has imparted too much smoothing compared to the box differencer.

[2] Recall that gauge is a free variable, and is defined as the sum of the upper (or lower) arm of a zero-gain response.

For the macd part of the problems you will also have to change gauge. The macd gauge must be adjusted to establish a fair comparison with the box differencer, which has unit gauge without adjustment. You can use the snippet (written in matlab):

```
function h = switch_to_composite_unit_gauge(h_in)

% descrip: For zero-gain impulse functions, this function changes the gauge
%          so that the sum of positive values is unit, and therefore, by
%          symmetry, the negative arm has unit gain.

S = sum(h_in(h_in>0));
h = h_in / S;
```

The reason the box differencer does not require a gauge adjustment is because there is no overlap between the upper and lower arms before the substraction; after substraction the box differencer naturally has unit gauge.

PROBLEMS:

The following work has several steps, and these steps are basically repeated for three difference impulse-response comparisons. In all cases use $N_{\text{window}} = 1024$ and $N_{\text{box}} = 16$.

1. **Box and Ema Comparison:**
   (a) Indicative Responses: Set $N_{\text{eff}} = N_{\text{box}}/\left(1 - e^{-1}\right)$, compute $h_{\text{box}}[n]$ and $h_{\text{ema}}[n]$ and overlay these two responses on a plot.
   (b) Spectra: Using an FFT compute the Fourier transform of these impulse responses. Take the absolute value of the results, which throws away the phase information. On a separate graph overlay the two gain spectra.
   (c) Cumulative Amplitude Spectra: On a separate graph overlay the cumulative gain spectra of the two impulse responses. These cumulative spectra show how much energy is captured from D.C. frequency out to higher frequencies.
   (d) Mean-Squared Error: Write a function mse = calc_box_ema_spectra_mse(N_ema, N_box, N_window) that 1) generates h_{ema} and h_{box} given the input parameters; 2) takes the amplitude of the FFT for each response (the gain spectrum); 3) computes the cumulative sum of each gain spectrum; 4) computes and returns the MSE of the two cumulative gain spectra.
   (e) Minimize the MSE: Use a one-dimensional optimizer (fminbnd in matlab) to minimize the MSE with N_{eff} as the free parameter. Since the objective function is univariate but N_box, N_window are parameters of the MSE function you can define the function object

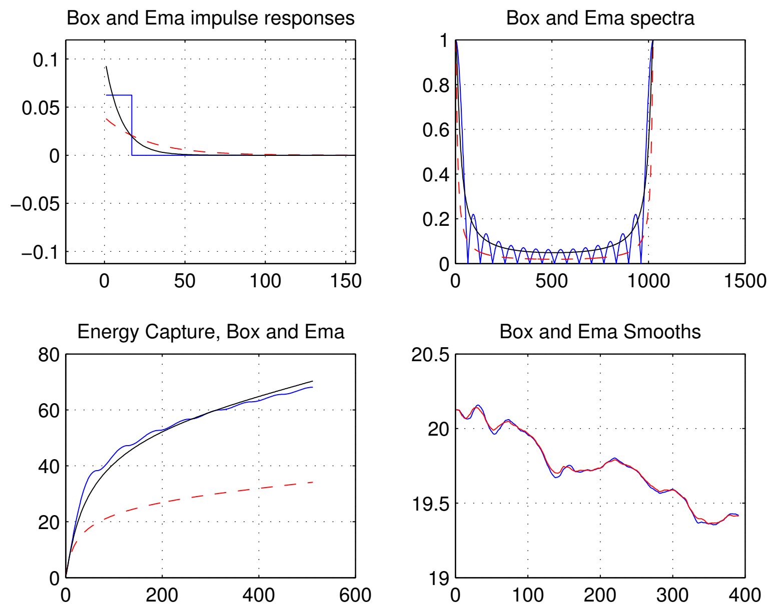

Figure 1: Example plots for box-ema comparison. Dashed line is the indicative starting point.

```
obj_ema = @(x)(calc_box_ema_spectra_mse(x, N_box, N_window));
```

and pass this to the optimizer. Compute the optimum N^*_{eff} with respect to spectral matching.

(f) Optimized Response: Using N^*_{eff} compute the ema impulse response, its gain spectrum and its cumulative gain spectrum. Overlay these results with those from tasks (1a-1c), respectively. Report the $N_{\text{eff}}/N_{\text{box}}$ ratio that you found.

(g) Apply to the Price Series: Convolve the box and spectrally-matched ema impulse responses with the price series you generated from last week. Overlay the results on a separate plot.

Figure 1 illustrates what I am looking for once all of the calculations are complete[3].

[3]Please note that in the picture of Energy Capture I have only plotted half of the spectrum. This is because and FFT spectrum is symmetric in its lower and upper halves, so all of the information is captured in one half or the other. This is a detail, compute and plot the full spectrum if you prefer, it will not change the result.

2. **Ema and Integrated Macd-Poly Comparison:**

   (a) Indicative Responses: Set Neff_ema=16, Neff_imacd=16, compute $h_{\text{ema}}[n]$ and $h_{\text{imacd}}[n]$, overlay these two responses on a plot.

   (b) Spectra: On a separate graph overlay the gain spectra.

   (c) Cumulative Amplitude Spectra: On a separate graph overlay the cumulative gain spectra.

   (d) Mean-Squared Error: Write a function mse = calc_imacd_spectra_mse(Neff_imacd, Neff_ema, N_window) that 1) generates h_{ema} and h_{imacd} given the input parameters; 2) takes the amplitude of the FFT for each response; 3) computes the cumulative sum of each gain spectrum; 4) computes and returns the MSE of the two cumulative gain spectra.

   (e) Minimize the MSE: Minimize the MSE with Neff_imacd as the free parameter. You can define a function object

   ```
   obj_int = @(x)(calc_imacd_spectra_mse(x, Neff_ema, N_window));
   ```

   and pass this to the optimizer. Compute the optimum N^*_{imacd} with respect to spectral matching.

   (f) Optimized Response: Using N^*_{imacd} compute the integrated macd-poly impulse response, its gain spectrum and its cumulative gain spectrum. Overlay these results with originals. Report the $N^*_{\text{imacd}}/N_{\text{eff}}$ ratio that you found.

   (g) Apply to the Price Series: Convolve the ema and spectrally-matched integrated macd-poly impulse responses with the price series you generated from last week. Overlay the results on a separate plot.

3. **Box-Difference and Macd Comparison:**

   For this problem, the ratio between $N_{\text{eff}\,+}$ and $N_{\text{eff}\,-}$ of the macd will be fixed to $N_{\text{eff}\,-}/N_{\text{eff}\,+} = 3$.

   (a) Indicative Responses: Set $N_{\text{eff}\,-} = N_{\text{box}}$. Compute $h_{\text{boxd}}[n]$ and $h_{\text{macd}}[n]$, overlay these two responses on a plot.

   (b) Spectra: On a separate graph overlay the gain spectra.

   (c) Cumulative Amplitude Spectra: On a separate graph overlay the cumulative gain spectra.

   (d) Mean-Squared Error: Write a function mse = calc_boxd_macd_spectra_mse (N_ema, N_box, N_window) that 1) generates h_{macd} and h_{boxd} given the input parameters; 2) takes the amplitude of the FFT for each response; 3) computes the cumulative sum of each gain spectrum; 4) computes and returns the MSE of the two cumulative gain spectra.

   (e) Minimize the MSE: Minimize the MSE with N_{eff} as the free parameter. You can define a function object

   ```
   obj_macd = @(x)(calc_boxd_macd_spectra_mse(x, N_box, N_window));
   ```

and pass this to the optimizer. Compute the optimum N_{eff}^* with respect to spectral matching.

(f) Optimized Response: Using N_{eff}^* compute the macd impulse response, its gain spectrum and its cumulative gain spectrum. Overlay these results with originals. Report the $N_{\text{box}}/N_{\text{eff}}^*$ ratio that you found.

(g) Apply to the Price Series: Convolve the box differencer and spectrally-matched macd impulse responses with the price series you generated from last week. Overlay the results on a separate plot.

The following problem focuses on the group-delay spectrum rather than the gain spectrum. This problem will be done in continuous time rather than discrete.

As a quick review of complex numbers in polar form, consider a complex number z. In cartesian and polar form we have

$$z = z_r + jz_i = |z|e^{\angle z}.$$

The magnitude and angle of z from its cartesian components is written

$$|z| = \sqrt{z_r^2 + z_i^2}, \quad \text{and} \quad \angle z = \tan^{-1}\left(z_i/z_r\right).$$

As a matter of detail, you need to recognize that a full 2π arc tangent function is required. In Matlab this is written `atan2(y, x)`. Additionally, the arc tangent is periodic on 2π, but we want the full phase (as best as possible). In Matlab you'll need the `unwrap(..)` function to unfold the resulting phase. This doesn't always work. The analytic expressions often given guidance as to where the phase needs to be unfolded.

4 **Phase spectra:** For the following continuous-time impulse responses, derive the analytic expression for the phase spectrum $\angle H(\omega)$. Use the Fourier transform tables from my lecture notes and/or those on Wikipedia.

$$h_1(t) = \delta(t-\tau) \tag{4}$$
$$h_2(t) = T^{-1}\left(u(t) - u(t-T)\right) \tag{5}$$
$$h_3(t) = \tau^{-1}e^{-t/\tau}u(t) \tag{6}$$
$$h_4(t) = \tau^{-2}\, te^{-t/\tau}u(t) \tag{7}$$
$$h_5(t) = (\tau_- - \tau_+)^{-1}\left(e^{-t/\tau_+} - e^{-t/\tau_-}\right)u(t) \tag{8}$$

(a) **Group-delay spectra:** For each phase spectrum $\angle H(\omega)$, compute the group-delay spectrum, defined as

$$\tau_{\text{grp}}(\omega) = -\frac{d}{d\omega}\angle H(\omega)$$

(b) **Average delay**: Report the DC value of the group-delay spectra that you computed above. Explain your results.

(c) **Plots:** Plot the phase and group-delay spectrum of each function. The customary way to plot spectra is to plot on a log-ω axis. Use the following parameters:

$$\begin{aligned} T &= 1 \\ \tau &= 1 \\ \tau_+ &= 1/2 \\ \tau_- &= 3/2 \\ \omega &\in [0.01, 100] \end{aligned}$$