Lecture 5: Exotic Markets

Modeling and Marketing Making in Foreign Exchange

Exotic Products in FX

- The vanilla market is very liquid in G7 FX
  - Bid/ask spreads are low, traded notionals large
  - Dynamic vega hedging costs for exotics are also therefore low
- Vanilla exotics
  - Mostly barrier-style derivatives
  - Knockout options, one touches, range binaries
  - Traded mostly by vanilla market makers
- Complex exotics
  - Single-asset exotics, volatility products, multi-asset exotics
  - Traded mostly by separate exotic traders

Trading Desk Structure

Two standard models for how exotic and vanilla trading desks interact on the street

- One joint book
  - (Almost) all exotic trades are booked alongside the vanilla trades and managed as a package
  - Exotic traders make prices on complex exotics, vanilla traders make prices on vanilla exotics **and** manage all portfolio risk
- Separate books
  - Exotic traders price exotic trades (usually both vanilla & complex) and also manage the portfolio risk for the exotics
  - Trade vanilla options as hedges against the vanilla books

Trading Desk Structure

Desk Structure	Pros	Cons
One joint book	• No gaming of prices between vanilla and exotic traders • Expert risk managers manage all the risk	• Vanilla traders have a tough time with exotic risks • Hard to separate out PNL performance of just the exotics
Separate books	• Easy to track PNL performance of exotics • Exotic traders understand exotic risks between than vanilla traders	• Vanilla traders have a captive market for hedges and price accordingly • Exotic traders often focused more on pricing than ongoing risk management

Vanilla Structures

- European-exercise products that have a payout that is more complex than a vanilla option

- European digitals
  - Pays 1 unit of currency if spot on an expiration date is past a barrier

- European knockouts
  - Vanilla that knocks against a spot fixing on its expiration date
  - Only makes sense for “in-the-money” barrier levels
    - Where the option is ITM when the barrier is hit

Vanilla Exotics

- Called “vanilla” because they are so liquid
  - In many cases bid/ask spreads are comparable to vanilla options
- Knockout options
  - Vanilla option whose price goes to zero if a barrier is touched at any point before expiration
  - Also knock-in versions
- One touch binary options
  - Get 1 unit of currency if a spot touches a barrier before expiration
- Range binary options
  - Get 1 unit of currency if spot does not leave a range by expiration

Vanilla Exotics

- Barriers are mostly continuous, not once-per-day discrete
  - No fixings – lots of dealer discretion on whether spot traded through a barrier

- Dealers are often incentivized to avoid having a barrier be touched, or for a barrier to be touched
  - Mostly because it's very hard to hedge out specific barrier risk
    - PNL performance around the barrier when time to expiration is small can be largely unhedgeable
  - Sometimes see "weird" spot action around levels that correspond to big expiring barriers in the market
    - Spot desks often want to see information about the barrier options that the exotics traders have put on, or hear about trades that were quoted but missed (and maybe traded somewhere else)

Single Asset Exotics

- More complex barrier derivatives
  - Window barrier options
  - Knock-in knock-out options (knock into a knockout)

- Target range accrual notes
  - Bond with coupons linked to FX rates (forward, option, barrier payoffs)
  - If total accumulated coupon amount reaches a target, the principal is repaid immediately, otherwise not until the final settlement date

- Discrete knockout or “fadeout” options
  - Knocks (or fades/accumulates notional) against daily fixes

Volatility Products

- Volatility swap
  - Calculate realized volatility (from daily fixings) over some period
  - Swap that against a pre-agreed strike
- Variance swap
  - Calculate realized volatility squared over some period
  - Swap that against a pre-agreed strike (squared)
- Forward volatility agreement (FVA)
  - Option starting on some date in the future (and expiring after that)
  - Strike set based on spot on the start date
- Countdown options: expires when realized variance hits target
- Vol knockouts: vanillas that knock against realized vol

Multi-Asset Exotics (FX)

- Basket options
  - Option to buy or sell a basket of currencies for a fixed price
  - Sometimes physically settled into the actual currency exchange
  - Sometimes cash settled based on fixes
  - Can incorporate knockouts, either against the basket spot or against an individual FX spot

- Best-of options
  - Defined by a package of vanilla options
  - At expiry, owner gets the most valuable of those options
  - Also worst-of variation

Cross-Asset Exotics

- Derivatives that depend on FX and another asset
  - Basket options with FX, equities, commodities, etc
  - “Dual digitals” that pay if FX and an equity are both above trigger levels at expiration
    - Or FX and an interest rate swap fixing

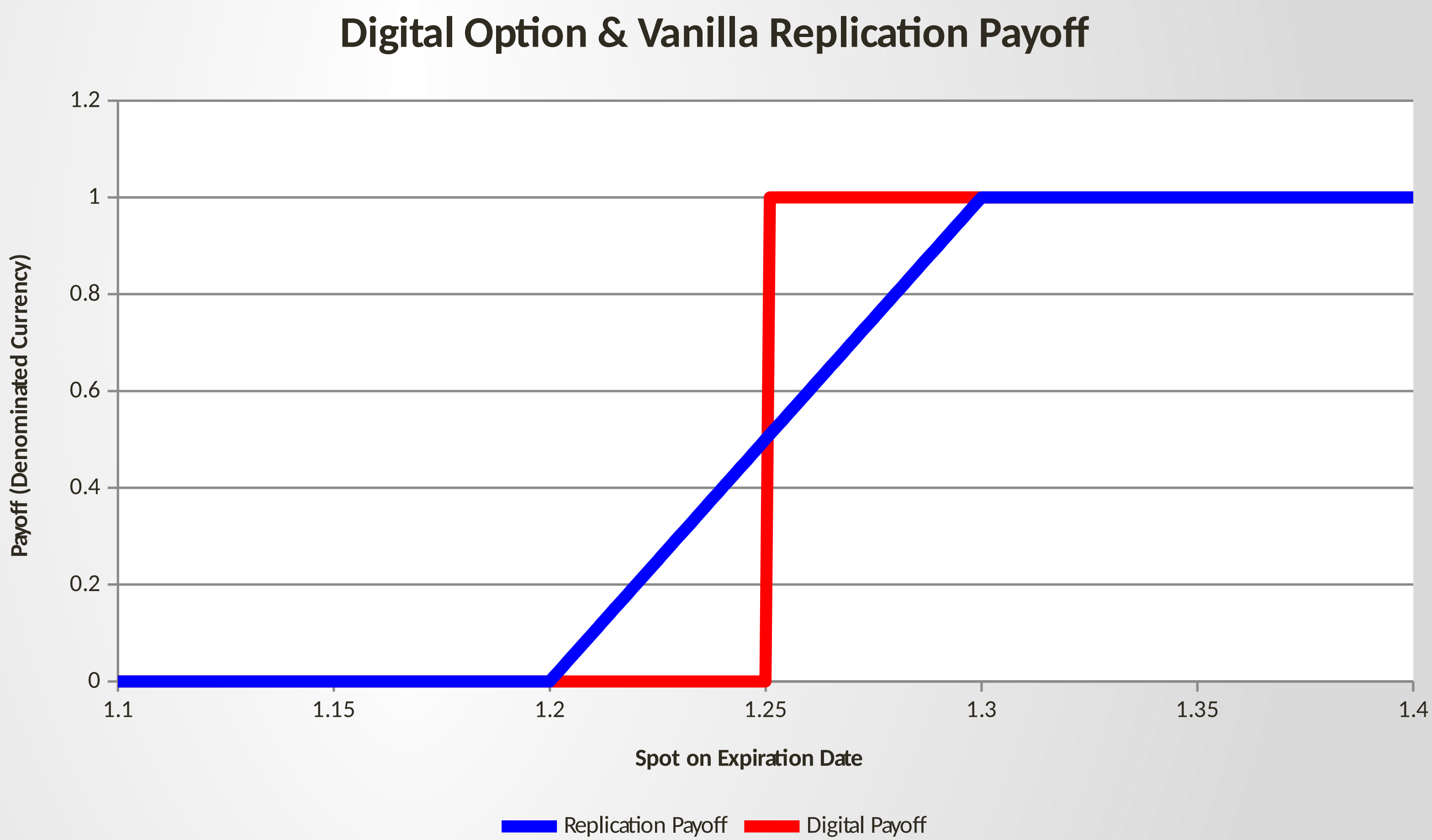
- Generally structured as a separate business outside the main FX business
  - Trades hedges with the FX desk
  - Manages risk separately

European Digital Pricing

- Replication with vanillas, so no exotic model required, just volatility interpolation

- Replicate a digital call (strike K) with a call option spread
  - Buy N units of a call struck at K-1/(2N)
  - Sell N units of a call struck at K+1/(2N)
  - Limit N->infinity, gives the digital payoff

European Digital Pricing



European Digital Pricing

- In the limit as N->infinity, the European digital price approaches the -1 * slope of call price with respect to strike

$$D(K) = -\frac{dC(K)}{dK}$$

$$= -\frac{\partial C_{BS}(K,\sigma(K))}{\partial K} - \frac{\partial C_{BS}(K,\sigma(K))}{\partial \sigma}\frac{d\sigma}{dK}$$

European Digital Bid/Ask

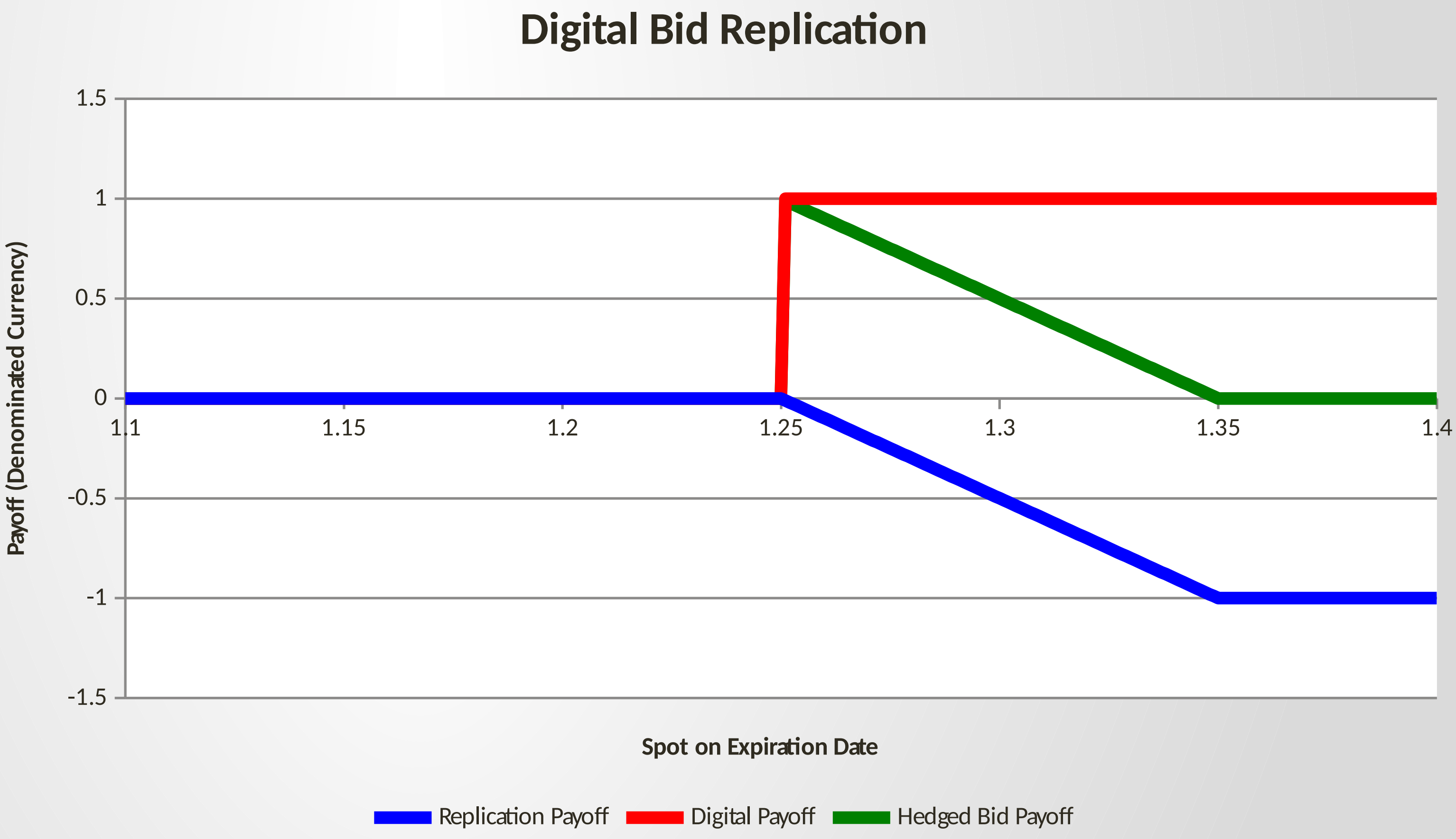
- In practice you cannot trade N->infinity!
  - Some maximum practical notional of vanillas, or minimum strike interval, determines the limit

- This leaves a digital market maker with fundamentally unhedgeable risk

- Constructing super-replicating portfolios is how we determine bid/ask spreads for digitals
  - In addition to charging bid/ask on implied volatility itself

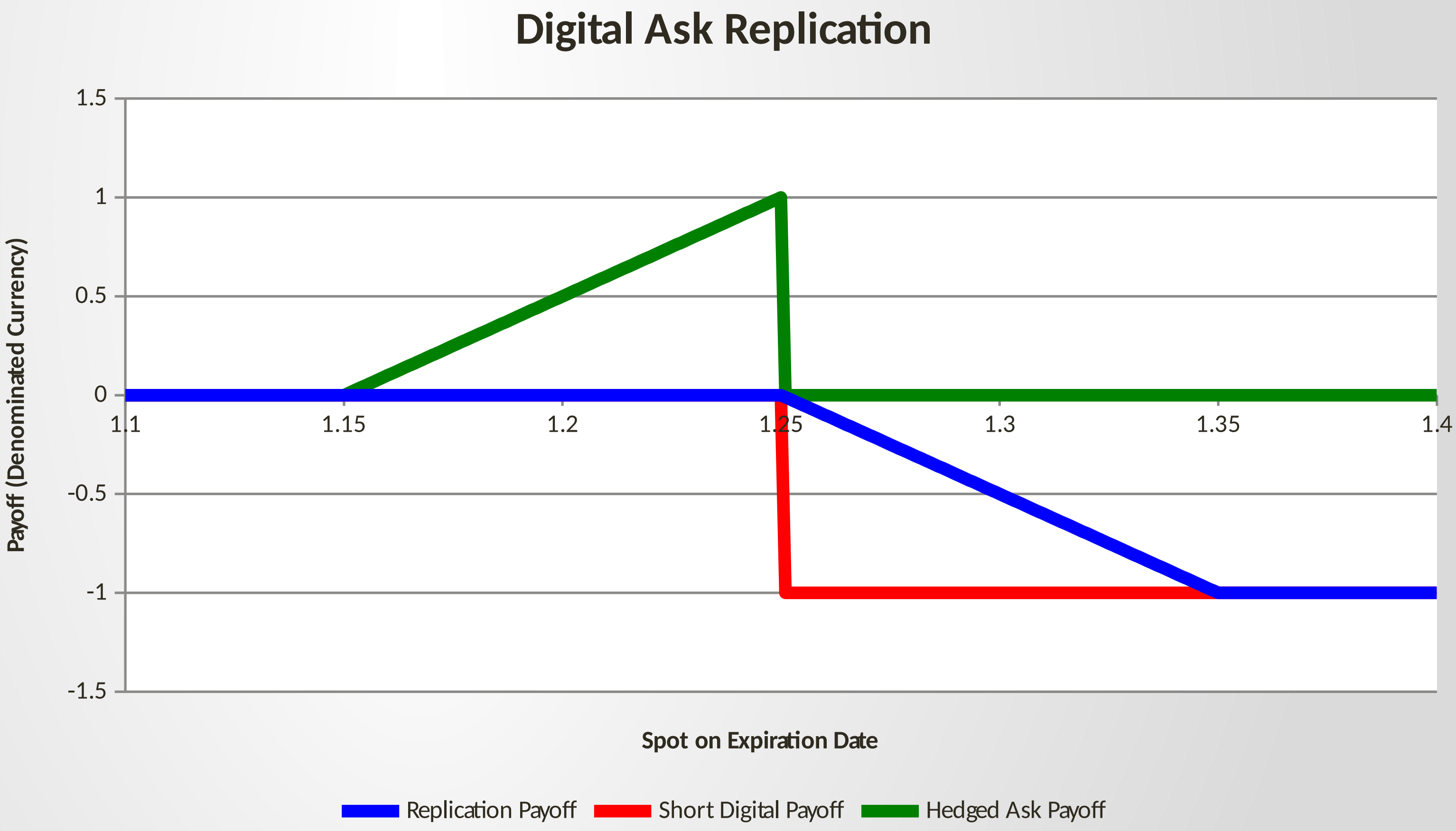
European Digital Bid/Ask

- Bid price: price at which one would buy the digital



European Digital Bid/Ask

- Ask price: price at which one would sell the digital



OTM Knockout Pricing

- OTM (out-of-the-money) knockouts are ones where the option is OTM when the barrier is hit
  - Down and out call/up and out put
- Very tight market for OTM knockouts: spreads=vanilla spreads
- There is no static replication for an OTM knockout, so a model is required to estimate dynamic hedging costs
- What market dynamics are OTM knockouts sensitive to that a model should be designed to pick up?
  - Risk reversal beta is the most important dynamic for knockouts!

OTM Knockout Hedging

- Consider a down-and-out call option
  - Strike K, barrier B<K, expiration T

- Look at an approximate replication:
  - Long 1 unit of a call with same expiration T, same strike K
  - Short 1 unit of put with same expiration T, strike $K' = B^2/K$

- If spot never touches the barrier by expiration, the vanilla call replicates the knockout payoff

- If spot touches the barrier, the knockout price goes to zero
  - Replication price is close to zero too as call & put offset

OTM Knockout Hedging

- If spot touches the barrier, the knockout price really is zero, but the replication portfolio's price is only approximately zero
  - Need to do some dynamic hedging in that case!
  - Need to unwind the replication portfolio if the barrier is touched

- The cost of unwinding the replication if spot touches the barrier depends on implied volatilities in that state

- But the replication at that point looks like a risk reversal
  - Vega (to ATM vol) is very low
  - BF risk (to moves in smile) is very low
  - RR risk is the only real risk

OTM Knockout Hedging

- The cost of unwinding the replication depends mostly on the level of the risk reversal (skew) when spot has moved to the barrier
- This is basically the risk reversal beta: how much risk reversal moves as spot moves from the current spot down to the barrier
  - Not exactly, because it's not an instantaneous move in spot: you care what happens to risk reversal as spot moves over some extended time
  - But pretty close!
- Don't care about what happens to ATM vol or smile

One Touch Pricing

- A one touch pays 1 unit of a denominated currency if spot touches a barrier at any point before its expiration date
  - Sometimes called an American digital
- There is no static replication for one touches
  - Unlike European digitals
- But like OTM knockouts, there is a pretty decent semi-static hedge that you can use to figure out what market dynamics are important
  - Fortunately it's the same as OTM knockouts: risk reversal beta

One Touch Hedging

- Like with OTM knockouts, we want to find a replication that is a static hedge if the barrier isn't breached, and is an approximate replication if the barrier is touched
  - Only approximate, so there will be some model-dependent cost of unwinding the hedge if the barrier is touched

- Replication is 2 units of a European digital with the same strike and expiration
  - Spot never touches the barrier, one touch and digital both expire worthless
  - Spot touches the barrier, one touch worth 100%, European digital worth approximately 50%
    - 2xEuropean digital is close to one touch payoff

One Touch Hedging

- The price of a European digital with strike is approximately ATM depends mostly on skew
  - Black-Scholes digital price is close to 50%
  - Second term depends on slope of implied vol with respect to strike

- So again, the cost of unwinding the replication depends mostly on the level of risk reversal when spot has moved to the barrier
  - Risk reversal beta

- And it doesn't depend much on what happens to ATM vol or smile

Models for Risk Reversal Beta

- One approach: model stochastic spot/vol correlation

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sqrt{v(t)}\, dz_S$$

$$dv(t) = \beta(\bar{v} - v(t))dt + \alpha\sqrt{v(t)}\, dz_v$$

$$E\left[dz_S dz_v \right] = \rho(t) dt$$

$$d\rho(t) = \gamma(\bar{\rho} - \rho(t))dt + \varepsilon\sqrt{1-\rho^2(t)}\, dz_\rho$$

$$E\left[dz_S dz_\rho \right] = \rho_{cS} dt$$

$$E\left[dz_v dz_\rho \right] = \rho_{cv} dt$$

Models for Risk Reversal Beta

- More common approach due to computational efficiency: local vol/stochastic vol mixture models
- Local vol limit: lots of risk reversal beta
- Stochastic vol limit: very little risk reversal beta
- Introduce a parameter that interpolates somehow between those two limits
  - That parameter controls the model's risk reversal beta

Local Vol/Stoch Vol Mixture

- Standard formulation:

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma(S,t)\sqrt{v}\, dz_S$$

$$dv = \beta(1 - v(t))dt + \alpha\sqrt{v}\, dz_v$$

$$E\left[dz_S dz_v\right] = \rho\, dt$$

- v(t): stochastic bit of volatility
- $\sigma(S,t)$: local volatility bit of volatility, calibrated to vanillas
- α is effectively the mixture parameter

Local Volatility Model

- Two ways of generating local volatilities in the pure local volatility limit

- Dupire volatility
  - Generate non-parametric local volatility function to match all vanilla options (as a function of strike and time to expiration)
  - Often quite unstable in practice but closed-form function in terms of (interpolated) implied volatilities

- Parametric local volatility
  - Choose some parametric form, numerically match it to a discrete set of strikes on a discrete set of expiration dates
  - More stable but numerical fit

Heston Stochastic Volatility

- The other limit is (almost) the Heston stochastic volatility model

$$dx(t) = \left(\mu - \frac{v(t)}{2} \right) dt + \sqrt{v(t)}\, dz_S$$

$$dv(t) = \beta(\bar{v} - v(t))dt + \alpha\sqrt{v(t)}\, dz_v$$

$$E\left[dz_S dz_v \right] = \rho\, dt$$

- Switched to x(t) = ln(S(t)), where S(t)=spot
- Model parameters are v(0), $\bar{v}$ α, β, and ρ
  - Initial and mean volatilities, along with mean reversion, control ATM term structure
  - α controls smile, ρ controls skew

Heston Stochastic Volatility

- Let's solve vanilla pricing in the Heston model because the technique is beautiful and useful in many places
- Start by writing down the characteristic function of ending log(spot) at some future time T, which is basically the Fourier transform of the PDF

$$f(x_t, v_t; \theta, T) = E\left[e^{j\theta x_T}\right]$$

- x_t, v_t are the current values (at current time t) of x=log spot and v=instantaneous volatility squared
- θ is the Fourier variable and T is the future time we're looking at

Heston Stochastic Volatility

- Apply Ito's Lemma to f and note that it is a martingale, so all the dt terms must sum to zero

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2 + \frac{\partial f}{\partial v}dv + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}(dv)^2 + \frac{\partial^2 f}{\partial x \partial v}dx\,dv$$

$$df = dt\left[\frac{\partial f}{\partial t} + \left(\mu - \frac{v}{2}\right)\frac{\partial f}{\partial x} + \frac{v}{2}\frac{\partial^2 f}{\partial x^2} + \beta(\bar{v} - v)\frac{\partial f}{\partial v} + \frac{\alpha^2 v}{2}\frac{\partial^2 f}{\partial v^2} + \rho\alpha v\frac{\partial^2 f}{\partial x \partial v}\right]$$

$$+\frac{\partial f}{\partial x}\sqrt{v}\,dz_S + \frac{\partial f}{\partial v}\alpha\sqrt{v}\,dz_V$$

Heston Stochastic Volatility

- Write down the PDE that f must satisfy

$$\frac{\partial f}{\partial t}+\left(\mu-\frac{v}{2}\right)\frac{\partial f}{\partial x}+\frac{v}{2}\frac{\partial^2 f}{\partial x^2}+\beta(\overline{v}-v)\frac{\partial f}{\partial v}+\frac{\alpha^2 v}{2}\frac{\partial^2 f}{\partial v^2}+\rho\alpha v\frac{\partial^2 f}{\partial x\partial v}=0$$

- Also need an initial condition to define f(T=t), which we get from the definition of f

$$f(T=t)=e^{i\theta x_t}$$

- The initial condition depends only on x_t, not at all on v_t

Heston Stochastic Volatility

- All the factors in front of the derivatives in that PDE are constants or are linear in v

- Guess a functional form for f and see if that helps:

$$f(x_t, v_t; \theta, T) = e^{i\theta x_t + A(t) + B(t)v}$$

- A and B here are functions only of time t, not of x or v
- Now we can plug this form back into the PDE and see whether there exist forms for A and B that satisfy the PDE and its initial condition

Heston Stochastic Volatility

- When you do this, and realize that the PDE has to solve for all values of v, you reduce the PDE to two simultaneous ordinary differential equations

$$\dot{A} = \mu i\theta + \beta\overline{v}B$$

$$\dot{B} = -\frac{1}{2}(i\theta + \theta^2) - \beta B + \frac{\alpha^2 B^2}{2} + \rho\alpha i\theta B$$

$$A = A(\tau), B = B(\tau), \tau = T - t$$

$$A(0) = B(0) = 0$$

Heston Stochastic Volatility

- To get vanilla prices you could inverse-FT (one numerical integration) and then integrate over the PDF to get a vanilla price (a second numerical integration)

- Turns out there's a beautiful way to reduce that two-dimensional numerical integration to just one numerical integration
  - Significant computational savings
  - Plus more robust

Heston Stochastic Volatility

- Call price from characteristic function of ending x=ln(spot)

$$C(K) = F - \frac{K}{2} - \frac{K}{\pi}\mathrm{Re}\left[\int_{\theta=0}^{\infty} \frac{f(\theta)e^{-i\theta\ln\left(\frac{K}{S}\right)}}{\theta^2 + i\theta}\,d\theta\right]$$

- C(K) = call option price for strike=K (no discounting included!)
- F = forward to settlement time T, S = current spot
- f(θ) = characteristic function we just derived