September 16, 2015

MTH 9831. Solutions to Quiz 2.

(1) (3 points) Let $(B(t))_{t\geq 0}$ be a standard Brownian motion and $\tau_3 = \inf\{t \geq 0 : B(t) = 3\}$. Write a formula for computing $P(\tau_3 > 5)$.

Solution. This is a consequence of the reflection principle.

$$P(\tau_3 > 5) = P(\max_{s\in[0,5]} B(s) < 3) = 1 - 2P(B(5) > 3)$$
$$= 1 - 2P(Z > 3/\sqrt{5}) = 2N(3/\sqrt{5}) - 1,$$

where Z is a standard normal random variable and $N(x)$ is its CDF.

(2) (3 points) Let $Y(t) = B(t) + t$, $t \geq 0$. For which $\sigma \in \mathbb{R}$ the process $e^{\sigma Y(t)-4t}$ is a martingale (with respect to the natural filtration of the Brownian motion)?

Solution. For every $\sigma \in \mathbb{R}$ the process $e^{\sigma B(t)-\sigma^2 t/2}$, $t \geq 0$, is a martingale with respect to the above mentioned filtration.[1] The process which we want to be a martingale is $e^{\sigma B(t)-(4-\sigma)t}$, $t \geq 0$. All we need is to solve the equation $4 - \sigma = \sigma^2/2$. Thus, $\sigma = 2$ or $\sigma = -4$.[2]

(3) (4 points) Let $u(t,x)$ be the solution to the terminal value problem

$$u_t + \frac{1}{2}u_{xx} = 0, \quad (t,x) \in [0,1) \times \mathbb{R}, \quad u(1,x) = |x|, \quad x \in \mathbb{R}.$$

Write

(a) the stochastic representation of $u(t,x)$ (in the form of expectation);
(b) the explicit integral formula for $u(t,x)$.

Solution.

(a) $u(t,x) = E(|B(1-t)| \,|\, B(0) = x) = E(|x + \tilde{B}(1-t)| \,|\tilde{B}(0) = 0)$, where $\tilde{B}$ is a standard Brownian motion.
(b) $u(t,x) = E(|x + \sqrt{1-t}\,Z|) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} |x + y\sqrt{1-t}| e^{-y^2/2}\,dy.$

- *Extra credit (2 points):* Find explicitly $u(1/3, 0)$.

$$u(1/3,0) = \frac{\sqrt{2/3}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |y| e^{-y^2/2}\,dy = \frac{2}{\sqrt{3\pi}} \int_0^{\infty} y e^{-y^2/2}\,dy = \frac{2}{\sqrt{3\pi}}.$$

[1] In the notes I stated it first just for $\sigma > 0$ but, since $-B(t)$ is a Brownian motion, the fact holds for all $\sigma \in \mathbb{R}$ (for $\sigma = 0$ also as every constant process is a martingale). I have made the correction.

[2] Notice that in applications of the exponential martingale (see Exercise 3.7 in the textbook, for example) we have to choose the right one, so that we can pass to the limit as $t \to \infty$ after using the optional stopping theorem for the martingale $M(\tau \wedge t)$.