BARUCH, MFE

MTH 9878 Assignment One *

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* We agree for our work to be posted on the forum.

1 Forward Discount Factor

A *zero coupon bond* pays predefined principal amount at time T. The settlement time S is a time that satisfies $S < T$ and at which the payment to the issuer is made. We are interested in $P(t,S,T)$ of the forward zero coupon bond for valuation at time $t \leq S$, which is also called the *forward discount factor*. Common assumptions are, for $t < S < T$,

$$P(t,S,T) < 1,$$
$$\frac{\partial}{\partial T} P(t,S,T) < 0.$$

Theorem: If $P(t,S,T)$, $P(t,t,T)$, and $P(t,t,S)$ are known at time t, then

$$P(t,S,T) = \frac{P(t,t,T)}{P(t,t,S)}.$$

Proof: The contract with settlement time S and expiry T consists of two cash flows:

- $-P(t,S,T)$ at time S;
- $+1$ at time T.

If the amount $P(t,S,T)$ is appropriately chosen, then we need neither pay nor receive anything at time t, and thus the net value of the agreement is zero at time t. Discount the cash flows at time S and T back to time t, we get

$$1 \times P(t,t,T) - P(t,S,T) \times P(t,t,S) = 0.$$

In other words,

$$P(t,S,T) = \frac{P(t,t,T)}{P(t,t,S)}.$$

The <u>same</u> argument can be recapitulated in the following way:

- If an amount $P(t,S,T) \times P(t,t,S)$ is paid to the issuer at time t to purchase a zero coupon bond with settlement time t and expiry S, an amount $P(t,S,T)$ is received at time S; the amount $P(t,S,T)$ is then reinvested to purchase a zero coupon bond with settlement time S and expiry T, an amount 1 is received at time T.
- Equivalently, one can simply invest amount $P(t,t,T)$ at time t to purchase a zero coupon bond with settlement time t and expiry T directly and receive the principal amount 1 at time T.

The above two portfolios result in the same amount in receipt at time T irrespective of the market conditions, by the Law of One Price, they must have the same value at $t < T$. In other words,

$$P(t,S,T) \times P(t,t,S) = P(t,t,T) \Rightarrow P(t,S,T) = \frac{P(t,t,T)}{P(t,t,S)}.$$

TA's solution: Consider the following portfolio $V(u)$ with weights $w_1(u), w_2(u)$ and price processes $S_1(u), S_2(u)$ where

$$\begin{aligned} V(u) &= w_1(u)S_1(u) + w_2(u)S_2(u) \\ &= 1 \cdot P(t,u,T) - \frac{P(t,t,T)}{P(t,t,S)} \cdot P(t,u,S). \end{aligned}$$

Then $V(u)$ is self-financing because the weights are constant in u.

Now $V(t) = 0$, and

$$V(S) = P(t,S,T) - \frac{P(t,t,T)}{P(t,t,S)} \cdot \underbrace{P(t,S,S)}_{=1}.$$

If $P(t,S,T) > \frac{P(t,t,T)}{P(t,t,S)}$, then

$$\mathbb{P}\{V(S) \geq 0\} = \mathbb{P}\{V(S) > 0\} = 1,$$

so $V(u)$ is an arbitrage.

If $P(t,S,T) < \frac{P(t,t,T)}{P(t,t,S)}$, then

$$\mathbb{P}\{-V(S) \geq 0\} = \mathbb{P}\{-V(S) > 0\} = 1,$$

so $-V(u)$ is an arbitrage.

Thus, we are left with

$$P(t,S,T) = \frac{P(t,t,T)}{P(t,t,S)}.$$

2 OIS DISCOUNTING

Assume an interest rate model with a constant instantaneous OIS rate f_0, and a constant instantaneous LIBOR rate l_0. Consider a spot starting swap maturing T years from now, where T is an integer. Assume that the coupon dates on both fixed and floating legs are equally spaced with all day count fractions equal to 0.5 and 0.25, respectively. Neglect the 2 business day lag between LIBOR fixing and start of an accrual period.

(a) Find an explicit expression for the values of both legs of the swap.
Solution: Without loss of generality, we assume the notional principal to be one. For the fixed leg, the coupon payments are made semi-annually at time points $t = i \cdot \Delta t$, where $\Delta t = 1/2$ and $i = 1, \cdots, 2T$. Assume the fixed annual coupon rate is c, the value for the fixed leg is

$$V_{\text{fixed}} = \sum_{i=1}^{2T} \frac{c}{2} e^{-f_0 t_i} + e^{-f_0 T} = \sum_{i=1}^{2T} \frac{c}{2} e^{-\frac{i}{2} f_0} + e^{-f_0 T} = \frac{c}{2} \cdot \frac{1 - e^{-f_0 T}}{e^{\frac{1}{2} f_0} - 1} + e^{-f_0 T}.$$

For the floating leg, the coupon payments are made quarterly at time points $t = i \cdot \Delta t$, where $\Delta t = 1/4$ and $i = 1, \cdots, 4T$. The LIBOR forward rate with start S and maturity T is

$$L(t, S, T) = \frac{1}{\Delta t} \left(e^{l_0 (T - S)} - 1 \right)$$

and the coupon payment amount accrued over the day count fraction $\Delta t = 1/4$ is

$$L(t, t_i, t_i + \Delta t) \Delta t = e^{\frac{1}{4} l_0} - 1.$$

Thus,

$$V_{\text{float}} = \sum_{i=1}^{4T} \left(e^{\frac{1}{4} l_0} - 1 \right) e^{-f_0 t_i} + e^{-f_0 T} = \sum_{i=1}^{4T} \left(e^{\frac{1}{4} l_0} - 1 \right) e^{-\frac{i}{4} f_0} + e^{-f_0 T} = \left(e^{\frac{1}{4} l_0} - 1 \right) \cdot \frac{1 - e^{-f_0 T}}{e^{\frac{1}{4} f_0} - 1} + e^{-f_0 T}.$$

(b) Determine the par coupon for the swap.
Solution: The par coupon for the swap is determined by

$$V_{\text{fixed}} = V_{\text{float}} \Rightarrow c_{\text{par}} = 2 \left(e^{\frac{1}{4} l_0} - 1 \right) \cdot \frac{e^{\frac{1}{2} f_0} - 1}{e^{\frac{1}{4} f_0} - 1} = 2 \left(e^{\frac{1}{4} l_0} - 1 \right) \left(e^{\frac{1}{4} f_0} + 1 \right).$$

3 DERIVATIVES OF B-SPLINES

The recursive definition of B-splines is

$$B_k^{(d)}(t) = \frac{t - t_k}{t_{k+d} - t_k} B_k^{(d-1)}(t) + \frac{t_{k+d+1} - t}{t_{k+d+1} - t_{k+1}} B_{k+1}^{(d-1)}(t). \tag{3.1}$$

Prove that

$$\frac{d}{dt} B_k^{(d)}(t) = \frac{d}{t_{k+d} - t_k} B_k^{(d-1)}(t) - \frac{d}{t_{k+d+1} - t_{k+1}} B_{k+1}^{(d-1)}(t). \tag{3.2}$$

Proof: Use mathematical induction. At $d = 0$,

$$B_k^{(0)}(t) = \mathbb{1}_{[t_k, t_{k+1})}(t) = \begin{cases} 1, & t \in [t_k, t_{k+1}) \\ 0, & \text{otherwise} \end{cases}$$

The derivatives $\frac{d}{dt} B_k^{(0)}(t)$ are zero everywhere except at isolated points where the spline $B_k^{(0)}(t)$ is not continuous. For the base case $d = 0$, Equation 3.2 holds true no matter how we define $B_k^{(-1)}(t)$. Suppose Equation 3.2 is valid for degree d,

$$\begin{aligned}
\frac{d}{dt} B_k^{(d+1)}(t) &= \frac{d}{dt}\left(\frac{t - t_k}{t_{k+d+1} - t_k} B_k^{(d)}(t) + \frac{t_{k+d+2} - t}{t_{k+d+2} - t_{k+1}} B_{k+1}^{(d)}(t)\right) \\
&= \frac{1}{t_{k+d+1} - t_k} B_k^{(d)}(t) + \frac{t - t_k}{t_{k+d+1} - t_k} \frac{d}{dt} B_k^{(d)}(t) \\
&\quad - \frac{1}{t_{k+d+2} - t_{k+1}} B_{k+1}^{(d)}(t) + \frac{t_{k+d+2} - t}{t_{k+d+2} - t_{k+1}} \frac{d}{dt} B_{k+1}^{(d)}(t) \\
(Eq.3.2) \quad &= \frac{1}{t_{k+d+1} - t_k} B_k^{(d)}(t) + \frac{t - t_k}{t_{k+d+1} - t_k}\left(\frac{d}{t_{k+d} - t_k} B_k^{(d-1)}(t) - \frac{d}{t_{k+d+1} - t_{k+1}} B_{k+1}^{(d-1)}(t)\right) \\
&\quad - \frac{1}{t_{k+d+2} - t_{k+1}} B_{k+1}^{(d)}(t) + \frac{t_{k+d+2} - t}{t_{k+d+2} - t_{k+1}}\left(\frac{d}{t_{k+d+1} - t_{k+1}} B_{k+1}^{(d-1)}(t) - \frac{d}{t_{k+d+2} - t_{k+2}} B_{k+2}^{(d-1)}(t)\right) \\
&= \frac{1}{t_{k+d+1} - t_k} B_k^{(d)}(t) + \frac{d}{t_{k+d+1} - t_k}\left(\frac{t - t_k}{t_{k+d} - t_k} B_k^{(d-1)}(t) - \frac{t - t_k}{t_{k+d+1} - t_{k+1}} B_{k+1}^{(d-1)}(t)\right) \\
&\quad - \frac{1}{t_{k+d+2} - t_{k+1}} B_{k+1}^{(d)}(t) + \frac{d}{t_{k+d+2} - t_{k+1}}\left(\frac{t_{k+d+2} - t}{t_{k+d+1} - t_{k+1}} B_{k+1}^{(d-1)}(t) - \frac{t_{k+d+2} - t}{t_{k+d+2} - t_{k+2}} B_{k+2}^{(d-1)}(t)\right) \\
(Eq.3.1) \quad &= \frac{1}{t_{k+d+1} - t_k} B_k^{(d)}(t) + \frac{d}{t_{k+d+1} - t_k}\left(B_k^{(d)}(t) - \frac{t_{k+d+1} - t_k}{t_{k+d+1} - t_{k+1}} B_{k+1}^{(d-1)}(t)\right) \\
&\quad - \frac{1}{t_{k+d+2} - t_{k+1}} B_{k+1}^{(d)}(t) + \frac{d}{t_{k+d+2} - t_{k+1}}\left(\frac{t_{k+d+2} - t_{k+1}}{t_{k+d+1} - t_{k+1}} B_{k+1}^{(d-1)}(t) - B_{k+1}^{(d)}(t)\right) \\
&= \frac{d+1}{t_{k+d+1} - t_k} B_k^{(d)}(t) - \frac{d+1}{t_{k+d+2} - t_{k+1}} B_{k+1}^{(d)}(t) \\
&\quad \underbrace{- \frac{d}{\cancel{t_{k+d+1} - t_k}} \frac{\cancel{t_{k+d+1} - t_k}}{t_{k+d+1} - t_{k+1}} B_{k+1}^{(d-1)}(t) + \frac{d}{\cancel{t_{k+d+2} - t_{k+1}}} \frac{\cancel{t_{k+d+2} - t_{k+1}}}{t_{k+d+1} - t_{k+1}} B_{k+1}^{(d-1)}(t)}_{=0} \\
&= \frac{d+1}{t_{k+d+1} - t_k} B_k^{(d)}(t) - \frac{d+1}{t_{k+d+2} - t_{k+1}} B_{k+1}^{(d)}(t).
\end{aligned}$$

We have shown that Equation 3.2 holds for degree $d + 1$. By the principle of mathematical induction, Equation 3.2 is valid for all degrees d.