From FDE to Impulse Response, and Critique of Polynomial-Ema Filters

<u>PREPARATION:</u> The first problem uses the methodology taught in class to convert a published FDE into an impulse response, from which we can measure the temporal scale of the FDE. The second problem introduces the *poly-ema* with which you will critically review the filters and decide which has the worst and the best performance.

<u>PROBLEMS:</u>

1. **Analysis of an AR(3) Equation:** On pg. 38 of R. Tsay's book (see my references) he calibrates an AR(3) model for quarterly real U.S. GNP. His expression is (adopting my notation)

$$y_n = 0.0047 + 0.35y_{n-1} + 0.18y_{n-2} - 0.14y_{n-3} + \epsilon_n \tag{1}$$

We are going to convert this FDE into its impulse response, reconcile a numerically computed impulse response with the analytically derived response, and plot the pole-zero diagram to analyze stability.

(a) Take the z-transform of (1). Report your expression.

Note the constant 0.0047 in the equation. Constants are to be treated as causal; recall that a constant over all time does not have a z-transform because the transform does not converge. Put another way, the constant just represents an intercept of y_n for $n = 0$.

For reference, your equation will have the form:

$$Q(z)Y(z) = \text{const}\ U(z) + \epsilon(z) \tag{2}$$

where $U(z)$ is the z-transform of the unit step (which is $\left(1 - z^{-1}\right)^{-1}$). Since z-transforms are linear we can treat the two components on the rhs separately.

(b) Find and report the roots of $Q(z)$[1]. Are any roots degenerate?

(c) Generate a pole-zero diagram plot of $Q^{-1}(z)$. Make sure you draw a unit circle so that the stability or lack thereof is clear.

[1] It is important to know how polynomials are represented in a math package. In Matlab the polynomial

$$p(x) = x^3 + a_2x^2 + a_1x + a_0$$

is represented as $[1\ a_2\ a_1\ a_0]$. For our work, our polynomials are in z^{-1}. To represent such polynomials in a matlab we have to write

$$1 - \phi_1z^{-1} - \phi_2z^{-2} - \phi_3z^{-3} = z^{-3}\left(z^3 - \phi_1z^2 - \phi_2z - \phi_3\right)$$

or, that is, $[-\phi_3\ -\phi_2\ -\phi_1\ 1] \rightarrow [1\ -\phi_1\ -\phi_2\ -\phi_3]$. This amounts to a left-right flip of the polynomial coefficient representation. You can (and should) always verify that your roots represent the polynomial you start with.

(d) Multiply through by $Q^{-1}(z)$. You will have an equation in the form

$$Y(z) = \text{const } H(z)U(z) + H(z)\epsilon(z)$$

Use partial-fraction expansion to write $H(z)$ in terms of simple polynomials of the roots of $Q(z)$. Don't report the numeric pole and residue values in your written result, use p_i, A_i and C_i as necessary. Of course you'll have these values in your code.

(e) Take the inverse z-transform of your $Y(z)$ expression. Write this as

$$\mathcal{Z}^{-1}\left(Y(z)\right) = \text{const } h[n] * u[n] + h[n] * \epsilon[n] \tag{3}$$

Report your expression for $h[n]$.

(f) For each pole of $Q(z)$ convert the root value to the first-moment N_{eff} of an appropriate impulse shape. Use $|p_i|$ if a root is complex.

(g) On an axis $n = [0 : 10N_{\text{eff}}]$ compute and plot $h[n]$, const $h[n] * u[n]$, and their sum.

(h) Following the method from the last homework, compute the impulse response of the AR(3) recursion equation. Set the initial conditions to zero. Denote this response $y[n]$. Plot this response, and overlay it with $y[n]$ of (3). (You might have to adjust the delay – I did.) The analytic and computed impulse responses should exactly match.

(i) Compute the gain in two ways: that from $H(z)$ and that from $h[n]$. Report your results.

(j) Comment on the impulse response Tsay has found using statistical methods. Do you believe this is a good model for out-of-sample updates? That is, can you propose one or more simple alternatives?

2. **Autocorrelation and Polynomial Emas:** Thus far in this class I have discussed at length filters from a temporal perspective, the impulse response, from a frequency perspective, the gain and group-delay spectra, and from a transfer function perspective, the pole-zero diagram. Moreover, through transform methods we can connect a causal impulse response with a finite-difference equation. The impulse response has applications for offline use, while the finite-difference equation is used online because of its fast update.

   However, until this lecture I have not presented criteria on which to compare one filter from another, to in effect rank them in terms of some quality, nor have I presented alternatives.

   Recall from class that I defined the continuous-time autocorrelation function (ACF) as

   $$R_x(\tau) = (x \star x)(\tau) \equiv \int_{\mathbb{R}} x(t)x(t+\tau)dt, \tag{4}$$

   where $\star$ is the correlation operator and τ is the lead/lag shift of one series with respect to the other.

   With this definition, the ACF of an input, R_x, is transformed by the system ACF (S-ACF) of the filter, κ_h, to produce an output ACF function according to

   $$R_y(\tau) = \kappa_h(\tau) * R_x(\tau), \tag{5}$$

   where the S-ACF is identified in terms of the impulse response according to

   $$\kappa_h(\tau) \equiv (h \star h)(\tau) = h(\tau) * h(-\tau). \tag{6}$$

   With this foundation we can use the autocorrelation length to rank the performance of various filters. We do need to make a fair comparison. To do so, I set the delay M_1 to be the same for all filters. With that, the filter that has the shorted autocorrelation length is the filter I prefer because my downsample stride is least, thereby preserving data. Specifically, I am interested in the value τ^* such that

   $$\kappa_h(\tau^*) = 5\%. \tag{7}$$

   In this problem you are going to work through the discrete-time *poly-ema* filter. I start with the transfer function without gain adjustment. After gain adjustment and the first-moment calculation, you will convert the transfer function to an impulse response, and also to a finite-difference equation. Next, you will numerically compute the (approximate) correlation lag required to bring the S-ACF to or less than 5%. Lastly, you will rank the filters in terms of quality.

   A *poly-ema* has an *order* which reflects the order of degeneracy of the pole. I denote the order number by m. The candidate transfer function that you will use is

$$H_{m,\mathrm{cand}}(z) = \frac{1}{(1-(p/m)z^{-1})^m}, \tag{8}$$

where p is the location of the pole on the real axis.

(a) **Stability:** What is the range of values of p such that $H_{m,\mathrm{cand}}(z)$ is BIBO stable?

(b) **Gain adjustment:** From the candidate transfer function, compute the gain g_m of $H_{m,\mathrm{cand}}(z)$. The gain adjustment g_m^{-1} is the adjustment required for the gain of $H_m(z)$ to be one. That is,

$$H_m(z) = g_m^{-1}\, H_{m,\mathrm{cand}}(z). \tag{9}$$

Report the gain adjusted transfer function $H_m(z)$.

(c) **First moment M_1:** From the transfer function $H_m(z)$, compute the first moment M_1 as a function of m. I recommend that you use ζ notation as detailed in Appendix A of my lecture notes. Report the first moments as a function of m.

(d) **Relative half width (RHW):** Again, from the transfer function $H_m(z)$, compute the RHW as a function of m. Recall that

$$\mathrm{RHW} \equiv \frac{1}{M_1}\sqrt{M_2 - M_1^2}. \tag{10}$$

Again I recommend that you use ζ notation. Be careful about the association of $H''(z)$ and $M_{1,2}$ for z-transform transfer functions (again, see Appendix A). Report the RHW's as a function of m.

(e) **Impulse response:** For $1 \le m \le 5$, and using z-transform tables, derive the impulse response $h_m[n]$ from $H_m(z)$. Report the impulse response expressions, one for each m. With

$$M_1 = 50, \quad N_{\mathrm{window}} = 1000, \tag{11}$$

plot $h_m[n]$, one for each m.

(f) **Numerical moments:** With $h_m[n]$ and parameter M_1 and N_{window} above, numerically compute M_0, M_1 and the RHW for each $h_m[n]$. For instance, the gain is

$$M_0(m) = \sum_{n=-\infty}^{\infty} h_m[n].$$

Tabulate and report your results.

(g) **Finite-difference equations:** From the transfer function $H_m(z)$, derive the finite difference equation for orders $1 \le m \le 5$. For each order, report your FDE.

(h) **FDE Validation:** With

$$M_1 = 50, \quad N_{\text{window}} = 1000, \tag{12}$$

like above, directly compute the impulse response of each of your FDEs. Plot the response and overlay them with the respective $h_m[n]$ to validate that you have correctly written the FDE.

(i) **Numerical calculation of the S-ACF:** Compute $\kappa_h(\ell)$ from the discrete-time impulse responses, $1 \leq m \leq 5$. This requires a bit of thought because the impulse response that is shifted by ℓ to the right must be zero padded to the left. The discrete-time S-ACF will read

$$\kappa_h(\ell) = \sum_{n=0}^{N} h_m[n] h_m[n-\ell]. \tag{13}$$

Recall that the S-ACF is symmetric, $\kappa_h(\ell) = \kappa_h(-\ell)$, so the delay by ℓ in (13) is equivalent to advancement.

Report that plots of your S-ACFs.

(j) **Find ℓ^*:** For each m, find ℓ^* such that

$$\kappa_h(\ell^*) \simeq 5\%.$$

Tabulate and report ℓ^* for each m.

(k) **Ranking:** Recall that ℓ^* is the downsample stride required to reduce the autocorrelation imparted by the filter down to the 5% level. With this as a selection criterion, which order m is the most favorable filter and which is the least. Explain your rationale.