Emerging Markets and Inflation

Lectures 6-7
Inflation in Latin America. Classic Inflation Modeling

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Agenda for Today

Lectures 6 &7: Inflation in Latin America. Classic Inflation Modeling

Phenomenon of Real Rate economy in Latin America and detailed derivation of Inflation math

6.1. Real Rates Inflation in Latin America

6.2. Inflation in Chile

6.3. Inflation in Brazil

6.4 Inflation in Colombia

7.1. 3-Factor Jarrow and Yildirim model of Inflation

7.2. Application of the 3-Factor Inflation model in Colombia

7.3. Pricing of Zero Coupon Inflation Swap (ZCIS)

7.4. Pricing of Year-on-year Inflation Swap (YYIIS)

Agenda for Today. Real Rates Inflation

Inflation in Latin America

1. **Real Rates Inflation in Latin America**
   - **Real Rate Currency**
   - **Inflation-Indexed bonds in Latin America**
   - **A little bit of history**
2. **Inflation in Chile**
3. **Inflation in Brazil**
4. **Inflation in Colombia**

Real Rate Inflation

- **Refresh some notations and definitions from Lecture 5**
- **Breakeven inflation as a differential between Nominal and Real rate**
- **Fisher equation in discount factors (or zero coupon bond prices):**

$$\boxed{Z_{Real}(t,T) \overset{def}{=} Z_{Nom}(t,T) \cdot InflAdj(t,T)} \qquad \textbf{(1)}$$

- **and rates:**

$$\boxed{\begin{aligned}&(1+r)^{-\tau} = (1+n)^{-\tau} \cdot (1+bi)^{\tau} \\ &(1+n) = (1+r) \cdot (1+bi)\end{aligned}}$$

- **Inflation adjustment factor is defined via CPI index ratio:**

$$\boxed{InflAdj(t,T) = \frac{CPI(T)}{CPI(t)}}$$

Emerging Markets and Inflation

Real Rate Inflation

- **Cash flows for a Capital Indexed Bond (CIB):**

$$CF_{CIB}(t_i) = \begin{cases} (P + C_i) \cdot InflAdj(t_o, t_i) & , t_i = T \\ C_i \cdot InflAdj(t_o, t_i) & , t_i \neq T \end{cases}$$

$$CF_{CIB}(t_i) \overset{def}{=} CF_{Nom}(t_i) \cdot InflAdj(t_o, t_i)$$

Here t_o is time of inflation index reset and

CF_{Nom} is *Nominal* cash flows of a bond expressed only via coupon rate

- **Then Dirty Price of a CIB connects *Nominal* and *Real* discounting:**

$$DP_{Nom}(t) = \sum_i CF_{Nom}(t_i) \cdot Z_{Nom}(t, t_i)$$

$$\begin{aligned} DP_{CIB}(t) &= \sum_i CF_{Nom}(t_i) \cdot InflAdj(t_i) \cdot Z_{Nom}(t, t_i) \\ &= \sum_i CF_{Nom}(t_i) \cdot Z_{Real}(t, t_i) \end{aligned} \qquad \textbf{(2)}$$

Real Rate Inflation

- **Inflation-indexed security expression only requires Real Rates**
- **No need for Inflation or Nominal rates ➔ bootstrapping of a Real curve**
- **Not enough in DM since Inflation curve is needed in pricing swaps**
- **But is sufficient in economy referencing Real Rates directly**
- **Good example: Latin America caused by high inflation in the past**
- **Real Rates trading gain momentum in late 20th century**

Real Rate Currency

- **What is missing in our Real Rate bond picture? The denomination!**

- **Start from CCY denominated bond and work through pricing. Expand on Eq. (2) using Q_{CCY} for CCY denominated Notional**

$$DP_{CIB}^{CCY}(t_o) = Q^{CCY} \cdot \sum_i CF_{Nom}(t_i) \cdot \frac{CPI(t_i)}{CPI(t_o)} \cdot Z_{Nom}(t_o, t_i)$$

- **For $t > t_o$ and $t_i > t$, and splitting known (realized) and unknown (forward) parts of CPI**

$$DP_{CIB}^{CCY}(t) = Q^{CCY} \cdot \sum_i CF_{Nom}(t_i) \cdot \frac{CPI(t)}{CPI(t_o)} \cdot \frac{CPI(t_i)}{CPI(t)} \cdot Z_{Nom}(t, t_i)$$

Real Rate Currency (continued)

- **Continue carefully expanding:**

$$
\begin{aligned}
DP_{CIB}^{CCY}(t) &= Q^{CCY}\cdot\sum_i CF_{Nom}(t_i)\cdot\frac{CPI(t)}{CPI(t_o)}\cdot\frac{CPI(t_i)}{CPI(t)}\cdot Z_{Nom}(t,t_i) \\
&= \left(Q^{CCY}\cdot\frac{CPI(t)}{CPI(t_o)}\right)\cdot\sum_i CF_{Nom}(t_i)\cdot\frac{CPI(t_i)}{CPI(t)}\cdot Z_{Nom}(t,t_i) \\
&= \left(Q^{CCY}\cdot\frac{CPI(t)}{CPI(t_o)}\right)\cdot\sum_i CF_{Nom}(t_i)\cdot Z_{Real}(t,t_i)
\end{aligned}
\qquad (3)
$$

- **We have arrived to the FX analogy defined by Eq. (3)**

- **CPI Index acting as equivalent FX rate**

Real Rate Currency

- **CPI Index acting as equivalent FX rate**
- **Recall FX definitions from earlier including non-deliverable part:**
  - ***Real* Rate Currency *RCCY* in country with *Nominal* Currency *CCY***
  - **CIB notional at inception is**

$$Q^{RCCY}(t_o) \stackrel{def}{=} \frac{Q^{CCY}(t_o)}{CPI(t_o)}$$

  - **CIB price via *RCCY* notional then**

$$DP_{CIB}^{CCY}(t) = Q^{RCCY} \cdot CPI(t) \cdot \sum_i CF_{Nom}(t_i) \cdot Z_{Real}(t, t_i)$$

Real Rate Currency

- **And same in RCCY**

$$DP_{CIB}^{RCCY}(t) = \frac{DP_{CIB}^{CCY}(t)}{CPI(t)}$$

$$DP_{CIB}^{RCCY}(t) = Q^{RCCY} \cdot \sum_i CF_{Nom}(t_i) \cdot Z_{Real}(t, t_i)$$

- **Thus a self-consistent *Real* Rate bond pricing in *RCCY* economy**

- **Nominal rates enter the picture via non-deliverability feature: products must settle in *CCY***

LatAm Linkers overview [J.P.Morgan 2009]

- **Quick overview paying attention to the Real Rate currency denomination and conventions**
- **Many use country inflation units for denomination**
- **Use this reference data later when take a closer look**
- **Argentina:**
  - **Instrument: CER bond (Treasury bond)**
  - **Inflation Index: CER (*coeficiente de estabilizacion de referencia*) is a CPI index**
  - **Cash-flow and MTM mechanism: Settled in ARS, principal in ARS adjusted by CER, coupons are a fixed percentage of the principal. No principal floor protection.**

LatAm Inflation-Indexed bonds

- **Brazil:**
  - **Instrument**: NTN-B bond (National Treasury Note, series B)
  - **Inflation Index**: IPCA is the Expanded Consumer Price Index
  - **Cash-flow and MTM mechanism**: Settled in BRL, principal adjusted by inflation, coupons are a fixed percentage of the principal. No principal floor protection

- **Chile:**
  - **Instrument**: BCU (Central Bank UF) and BTU (Central Bank Treasury) bonds
  - **Inflation Index**: UF (Unidad de Fomento) is a unit of account indexed to the CPI
  - **Cash-flow and MTM mechanism**: Settled in CLP, principal in UF units, coupons are a fixed percentage of the principal. No principal floor protection

LatAm Inflation-Indexed bonds

- **Colombia:**
  - **Instrument**: TES UVR (Treasury) bond
  - **Inflation Index**: UVR (Unidad de Valor Real). The UVR is a unit of account indexed to the CPI
  - **Cash-flow and MTM mechanism**: Settled in COP, principal in UVR units, coupons are a fixed percentage of the principal. No principal floor protection
- **Mexico:**
  - **Instrument**: UDI-Bonos (Treasury UDI bonds
  - **Inflation Index**: UDI is an accounting unit indexed to the CPI (INPC: National Consumer Price Index)
  - **Cash-flow and MTM mechanism**: Settled in MXN, principal in UDI units, fixed semiannual coupons as a percentage of the principal. No principal floor protection

LatAm Inflation-Indexed bonds

- **Peru:**
  - **Instrument**: **VAC (Treasury bond)**
  - **Inflation Index**: **VAC is a unit of account indexed to the CPI**
  - **Cash-flow and MTM mechanism**: **Settled in PEN, principal in VAC units, coupons are a fixed percentage of the principal. No principal floor protection**
- **Uruguay:**
  - **Instrument**: **UI bills, notes, and bonds**
  - **Inflation Index**: **UI (*Unidad Indexada*). The UI is an accounting unit indexed to inflation released monthly by the National Statistics Institute**
  - **Cash-flow and MTM mechanism**: **Settled in UYU. For domestic instruments, notional is expressed in UI**
- **Majority operate in Real Rate economy expressing in inflation units!**

A bit of history

- **Why inflation units in Latin America?**
- **Explosive inflation in second half of 20th century:**

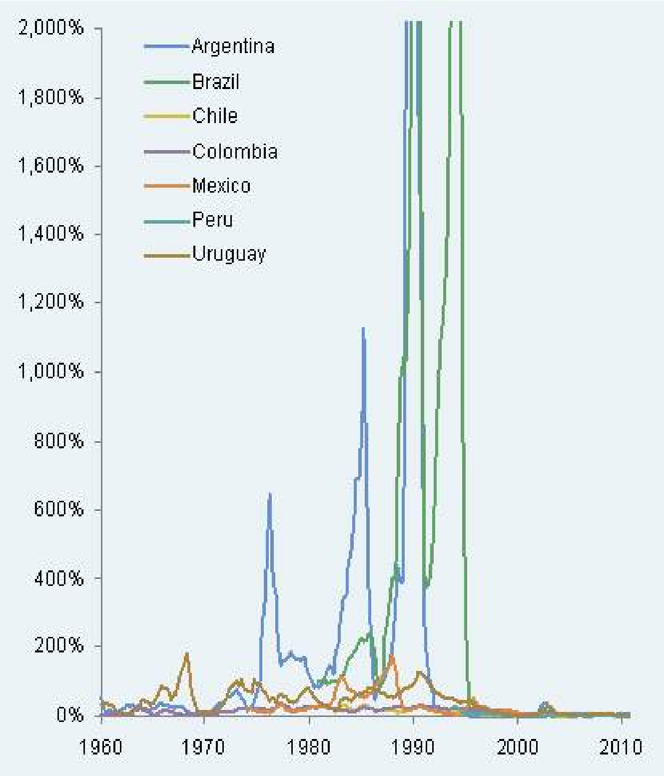

Figure 1. Inflation in Latin America since 1960 (J.P.Morgan 2011)

A bit of history (continued) from (J.P.Morgan 2011)

- **Long history of inflation and currency devaluation**
- **Well developed understanding of inflation phenomenon compared to other regions**
- **Local entities and governments have been using it for some time to guarantee stability**
- **Offers local corporates easier access to inflation market**
- **Inflation swaps market developed from Bonds market (as discussed earlier), but a *Real* Rates Swaps market!**

Agenda for Today. Inflation in Chile

Lecture 6. Inflation in Latin America

1. Real Rates Inflation in Latin America
2. **Inflation in Chile**
   - **Inflation Forwards**
3. Inflation in Brazil
4. Inflation in Colombia

Inflation in Chile

Chile

- **First Latin America country to adopt inflation-indexed UF units *Unidad de Fomento* in 1967**

- **Adopted not only in financial and business industry, but in consumer sector too:**
  - **Salary denominated in UF**
  - **Mortgage and insurance payments are denominated in UF**
  - **Cash settle UF transactions in *Nominal* CLP (Chilean Peso)**
  - **Every person knows CPI value for a day to withdraw cash from CLP account**

Inflation in Chile

Chile Nominal and Real products

- **Quoted equally representing country's financial landscape**

  A. **Bonds (Nominal and UF denominated)**

  B. **Security Forwards**

  C. **Interest Rate Swaps. Trades as Nominal (CLP) or Real (UF) Fixed against overnight floating rate known as *tasa promedio camara*, or just *camara***

  D. **Cross Currency Basis Swaps in all combinations: Nominal or Real Fixed or overnight Floating against US Libor Floating**

  E. **Nominal – Real CLP – UF Forwards also known as Inflation Forwards. We will spend more time looking at this product now**

Inflation Forwards

- **Future value of inflation CPI expressed as in Eq. (1)**

$$Z_{Real}(t,T) = Z_{Nom}(t,T) \cdot InflAdj(t,T)$$
$$CPI(T) = CPI(t)\frac{Z_{Real}(t,T)}{Z_{Nom}(t,T)}$$

- **And again compare to FX Forward expression:**

$$F(t,T) = S(t,t)\frac{Z^{Asset}(t,T)}{Z^{Denom}(t,T)}$$

So here *Real* Rate = Denominated and *Nominal* Rate = Asset

Inflation Forwards

- **Recall Non-Deliverable Forward with payoff in Denominated:**
  - **Quantity in Asset:** $FV^{CLP}(t,T) = N^{UF} \cdot [CPI(t,T) - K]$
  - **Quantity in Denominated:** $FV^{CLP}(t,T) = N^{CLP} \cdot \left[1 - \frac{CPI(t,T)}{K}\right]$ **(4)**
  - **And present value for both:**

$$\boxed{\begin{aligned} PV^{CLP}(t,T) &= N^{UF} \cdot [CPI(t,T) - K] \cdot Z^{CLP}(t,T) \\ PV^{CLP}(t,T) &= N^{CLP} \cdot \left[1 - \frac{CPI(t,T)}{K}\right] \cdot Z^{CLP}(t,T) \end{aligned}}$$

- **No mention of breakeven inflation and full FX replication**
- **Investor buys to bet on inflation levels**
- **Trades on monthly intervals ➔ excellent seasonality coverage compared to term-on-term inflation swaps**

Agenda for Today. Inflation in Brazil

Lecture 6. Inflation in Latin America

1. Real Rates Inflation in Latin America
2. Inflation in Chile
3. **Inflation in Brazil**
   - **Inflation-Indexed Bonds**
   - **Inflation-Indexed Swaps**
4. Inflation in Colombia

Inflation in Brazil

- **Stands alone across Latin America countries in treating inflation**
- **Despite high explosive inflation in the past never adopted inflation-indexed units**
- **Other than that, Real Rate bonds nature is similar to other countries**
- **Fixed Real Rate swap developed from Real Rate bonds trading**
- **Consider here first inflation-indexed bonds**
- **Then go to a rather complex world of Real Rate swaps and review models needed to price them**
- **Many Inflation Indices**

Inflation in Brazil

- **Many Inflation Indices:**
  - **IPCA: today's benchmark consumer price index**
  - **IGP-M**
- **Different by components, converge over time**

Inflation-Indexed Bonds. NTN-B

- **Most common Brazil inflation-indexed bond**
  - **Issuer: Brazilian National Treasury**
  - **Inflation Index: IPCA**
  - **Frequency, Rate type: semi-annual rate based on Business/252 day count**

Inflation-Indexed Bonds. NTN-B (continued)

- **Coupon and Interest calculations:**

  - **Cash Flows and Dirty price expression:**

$$CF_{NTN-B}(t_i) = \begin{cases} (C_i+1)^{\frac{6}{12}} & , t_i = T \\ (C_i+1)^{\frac{6}{12}} - 1 & , t_i \neq T \end{cases}$$

$$PU \equiv DP_{NTN-B}(t,T) = \sum_i \frac{CF_{NTN-B}(t_i)}{(1+y_R)^{\frac{BusDays(t,t_i)}{252}}} \qquad (5)$$

y_R - market quoted real rate yield to maturity defined via discounting in Eq. (5)

  - **No explicit Inflation adjustment in this expression**
  - ***PU* are price units equivalent to inflation-adjusted units!**
  - **Still requires cash settlement scaling factor as in Chile:**

Inflation-Indexed Swaps

- Two classes:

  1. Historically developed from bonds. Standard Real Rate swaps with bond conventions (we only consider this type here)

  2. Very distinct and Brazil specific referring to breakeven target inflation in economy with only *Real* and *Nominal* swaps trading

Inflation-Indexed Swaps. IPCA and IGP-M

- Replicates payoff of inflation-indexed bond

- Pay (receive) fixed coupon semi-annually scaled by zero-based inflation adjustment factor

Inflation-Indexed Swaps. IPCA and IGP-M (continued)

- **Onshore settling swap replicates bonds exactly**

- **Offshore settling still replicates bond's cash flows, but is cash settling (NDF)**

- **Present Value of the onshore swap is**

$$PV(t,0,T) = Q \cdot \mathbf{E}_n\left[\frac{CPI(T)}{CPI_o} \cdot Z_n(t,T) + \sum_i \frac{CPI(t_i)}{CPI_o} \cdot \left[(1+Cpn)^{\frac{1}{2}} - 1\right] \cdot Z_n(t,t_i) \middle| F_t \right]$$

- **As with bond, *Q* here is Notional in BRL and CPI is IPCA index at inception (t_o) and coupon payment days t_i**

- **Discounting and expectation in *Nominal* economy**

Inflation-Indexed Swaps. IPCA and IGP-M (continued)

- **Reduce payoff to *Real* rates only:**

$$\mathbf{E}_n\left[CPI(t,T)\cdot Z_n(t,T)\middle|F_t\right] = CPI(t)\cdot Z_r(t,T)$$

- **And re-write the pay-off to compare to bond's expression in Eq. (5)**

$$PV(t,0,T) = Q\cdot\frac{CPI(t)}{CPI_o}\cdot\left\{Z_r(t,T)+\sum_i\left[(1+Cpn)^{\frac{1}{2}}-1\right]\cdot Z_r(t,t_i)\right\}$$

Lecture 6. Inflation in Latin America

1. Real Rates Inflation in Latin America
2. Inflation in Chile
3. Inflation in Brazil
4. **Inflation in Colombia**
   - **UVR Rate and IPC Index**
   - **Term-on-Term Inflation pricing in Real Rates economy**

Inflation in Colombia

- **Adopted inflation targeting regime in 1999**
- **Works well operating via repo rate owned by central bank**

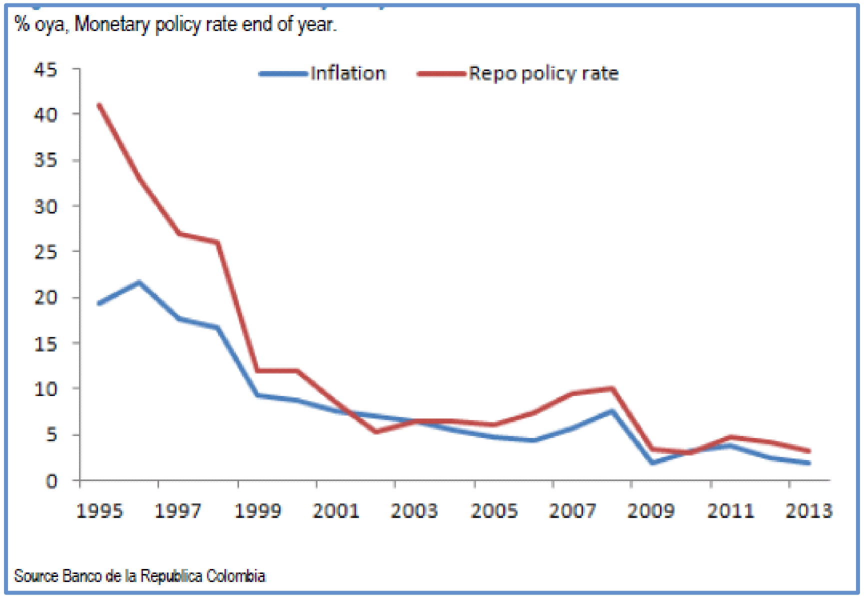

Figure 2: Inflation Rate vs. Monetary Policy Rate (J.P.Morgan, 2014)

- ***Real* Rate swaps trading paired with term-on-term inflation swaps**

UVR Rate and IPC Index

- **UVR is the unit of the Real Constant Value; reflects Purchasing Power of the Colombian Peso (COP) based on variation of CPI**

- **UVR is published on 15th of the month; need pro-rating otherwise**

$$UVR(t) = UVR(15, m) \cdot (1 + I(m-1))^{t/Dm}$$

Here *UVR(t)* - UVR index *t* days after the 15th of the month *m*. *1 < t < 31*

I(m-1)* - Monthly inflation % variation for month *(m-1)

Dm* - number of days in month *m

- **UVR via 2 rates and Spot value**

$$UVR(t,T) = UVR_{Spot} \frac{Z_R(t,T)}{Z_N(t,T)}$$

- **Actual Inflation-defining IPC index**

$$Inflation(YYYY) \stackrel{def}{=} \frac{IPC(31 \cdot Dec \cdot YYYY)}{IPC(31 \cdot Dec \cdot YYYY - 1)} = \frac{UVR(15 \cdot Feb \cdot YYYY + 1)}{UVR(15 \cdot Feb \cdot YYYY)}$$

Term-on-Term Inflation Swap in Real Rate Economy

- **Floating coupon:**

$$fv(T_i) = \left[1 + bi(T_{i-1}, T_i)\right]^{Act/365} - 1 \qquad \textbf{(6)}$$

Here *bi(*T_{i-1}*,* T_i*)* is realized breakeven inflation between coupon dates:

$$bi(T_{i-1}, T_i) \stackrel{def}{=} \frac{IPC_{T_i}}{IPC_{T_{i-1}}} - 1$$

- **We face now pay-off mixing Real Rates, Nominal Rates (for discounting) and actual Inflation**
- **Now we are ready to fully dive into 3-Factor model of Inflation**

7. Classic Inflation Modeling

- Continue analysis of inflation derivatives pricing and risk mgmt
- Follow 3-Factor model introduced earlier with rigorous derivation
- Consider two types of Market Models of BGM-type

7.1. 3-Factor Jarrow and Yildirim Model

7.2. Application in Colombia

7.3 Pricing of Zero Coupon Inflation-Indexed Swaps (ZCIIS)

7.4 Pricing of Year-on-Year Inflation-Indexed Swaps (YYIIS)

Agenda for Today. Jarrow - Yildirim

7. Classic Inflation Modeling

1. **3-Factor Jarrow and Yildirim Model**
   - **HJM and Short Rate refresher**
   - **Detailed JY derivation**
2. Application to Colombia market
3. Pricing of Zero Coupon Inflation-Indexed Swaps (ZCIIS)
4. Pricing of Year-on-Year Inflation-Indexed Swaps (YYIIS)

3-Factor Inflation Modeling. Jarrow - Yildirim

- **Main paper published in 2003 in Journal of Financial and Quantitative Analysis**

- **Fully exploit the foreign currency analogy assuming:**
  - ***Real* prices to correspond to Foreign prices**
  - ***Nominal* prices to correspond to Domestic prices**
  - **The CPI index corresponds to a spot FX exchange rate**

- **In this section we will follow main Jarrow - Yildirim paper, as well as excellent overviews from [Ahmad, 2008] and [Scardovi, 2011]**

- **Start with introducing notations:**

- **Start with introducing notations:**
  - **Subscript *r* - *Real* rates**
  - **Subscript *n* - *Nominal* rates**
  - **$Z_n(t,T)$ - time t USD price of a *Nominal* zero-coupon bond maturing at T**
  - **$I(t)$ - time t generalized CPI inflation index. Using FX standards, $I(t)$ is given in USD per CPI units**
  - **$Z_r(t,T)$ - time t CPI units price of a *Real* zero-coupon bond maturing at T**
  - **$f_k(t,T)$ - time t forward rates for date T where $k \in \{r, n\}$:**

$$\boxed{Z_k(t,T) = \exp\left\{-\int_t^T f_k(t,u)du\right\}} \qquad (7)$$

- **Continue with introducing notations:**
  - $r_k(t) = f_k(t,t)$ **- the time** t **short rate where** $k \in \{r, n\}$
  - $\beta_k(t)$ **- time** t **money market account value for** $k \in \{r, n\}$**:**

$$\boxed{\beta_k(t) = \exp\left\{\int_o^t r_k(s)ds\right\}} \qquad (8)$$

- **In FX analogy we define Real zero-coupon bond USD price:**

$$\boxed{Z_r^{\$}(t,T) = I(t)\cdot Z_r(t,T)} \qquad (9)$$

3-Factor Inflation Modeling. Jarrow - Yildirim

- **Define now the probability space (*Ω, F, P*):**
  - ***Ω*** **- the state space**
  - ***F*** **- set of possible events**
  - ***P*** **- statistical probability measure on (*Ω, F*)**

- **Operate in martingale measure *Q* where we recall relation between zero-coupon bond and stochastic Discount Factor:**

$$Z(t,T) = \mathbf{E}^Q\left[e^{-\int_t^T r(s)ds} Z(T,T) \middle| F_t \right] = \mathbf{E}^Q\left[D(t,T) | F_t \right] \qquad \textbf{(10)}$$

Emerging Markets and Inflation

Short Model, Forward Model and an HJM Refresher

- **HJM model for forward rate *f(t,T)* under martingale *Q*:**

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t) \\ f(0,T) &= f^*(0,T) \end{aligned} \qquad (11)$$

Where $f^*(0,T)$ are true observable market forward rates derived from observable zero-coupon bond prices via

$$f^*(t,T) = -\frac{\partial \log Z^*(t,T)}{\partial T} \qquad (12)$$

- **Condition to satisfy HJM drift, or for Eq. (7) and (10) to hold:**

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)' ds \qquad \forall t \le T \qquad (13)$$

Short Model, Forward Model and an HJM Refresher (cont'd)

- **Dynamics of zero-coupon bond *Z(t,T)* could be written as:**

$$\frac{dZ(t,T)}{Z(t,T)} = \mu_Z(t,T)dt + \Sigma(t,T)dW(t) \tag{14}$$

Where in line with dynamics of our forward rate *f(t,T)*:

$$\begin{cases} \mu_Z(t,T) = r(t) - \int_t^T \alpha(t,s)ds + \frac{1}{2}\left\|\Sigma(t,T)\right\|^2 \\ \Sigma(t,T) = -\int_t^T \sigma(t,s)ds \end{cases} \tag{15}$$

Short Model, Forward Model and an HJM Refresher (cont'd)

- **For short rate we can write:**

$$dr(t) = df(t,t) = \left[\alpha(t,t) + \frac{\partial}{\partial T} f(t,T)\Big|_{T=t} \right] dt + \sigma(t,t)\,dW(t)$$
$$r(t) = f(0,t) + \int_0^t \alpha(s,t)\,ds + \int_0^t \sigma(s,t)\,dW(s) \qquad (16)$$

- **Simplified generic short rate dynamics to extended Vasicek model:**

$$dr(t) = [\theta(t) - ar(t)]dt + \sigma dW(t) \qquad (17)$$

with *θ(t)* deterministic function of time chosen to fit observed term-structure of rates; *a* and *σ* are constants

Short Model, Forward Model and an HJM Refresher (cont'd)

- **With market zero-coupon bond prices *Z*(t,T)* and derived initial dynamics of forward rates *f*(0,T)* as per Eq.(12) we can write:**

$$\boxed{r(T)=\frac{1}{T}\int_0^T f^*(0,s)ds}$$

- **And the following expression for *θ(t)* exactly recovers the current term structure :**

$$\boxed{\theta(t)=\frac{\partial f^*(0,t)}{\partial t}+a\cdot f(0,t)+\frac{\sigma^2}{2a}\left(1-e^{-2at}\right)} \qquad \textbf{(18)}$$

- **Integrating Eq.(17):**

$$\boxed{r(t)=e^{-at}\left(r(0)+\int_0^t e^{as}\theta(s)ds+\int_0^t e^{as}\sigma dW(s)\right)}$$

Short Model, Forward Model and an HJM Refresher (cont'd)

- **Integrating it further towards price of zero-coupon bond:**

$$Z(t,T) = \mathbf{E}\left[e^{-\int_t^T r(s)ds} \middle| F_t \right]$$

- **We can write after careful integration:**

$$\begin{aligned} Z(t,T) &\equiv A(t,T)e^{-B(t,T)r(t)} \\ B(t,T) &= \frac{1}{a}\left[1-e^{-a(T-t)}\right] \\ A(t,T) &= \frac{Z(0,T)}{Z(0,t)}\exp\left\{ B(t,T)\cdot f(0,T) - \frac{\sigma^2}{4a}\left(1-e^{-2at}\right)B(t,T)^2 \right\} \end{aligned} \qquad (19)$$

Short Model, Forward Model and an HJM Refresher (cont'd)

- **Finally completing the loop back to HJM**
- **Dynamics of forward rate as in Eq.(11)**
- **And its drift condition as in Eq.(13)**
- **Converge under extended Vasicek form of the short rate:**

$$\boxed{df(t,T) = \alpha(t,T)dt + \sigma e^{-a(T-t)}dW(t)} \qquad (20)$$

Detailed JY Derivation

- **Go back to our probability space**
- **Brownians driving our 3-Factor model:**

$$
\boxed{
\begin{aligned}
&\left(W_n(t), W_n(t), W_n(t):\ t \in [0,T]\right)\\
&dW_n(t)\cdot dW_r(t) = \rho_{n,r}dt\\
&dW_n(t)\cdot dW_I(t) = \rho_{n,I}dt\\
&dW_r(t)\cdot dW_I(t) = \rho_{r,I}dt
\end{aligned}
}
$$

- **HJM representation of *Nominal* and *Real* rates:**

$$\boxed{df_n(t,T) = \alpha_n(t,T)dt + \sigma_n(t,T)dW_n(t)} \qquad \textbf{(21)}$$

$$\boxed{df_r(t,T) = \alpha_r(t,T)dt + \sigma_r(t,T)dW_r(t)} \qquad \textbf{(22)}$$

here $a_k(t,T)$ is assumed random while $\sigma_k(t,T)$ is deterministic, $k \in \{r, n\}$

Continue with detailed JY derivation…

- **Lognormal dynamics of the Inflation index:**

$$\boxed{\frac{dI(t)}{I(t)} = \mu_I(t)dt + \sigma_I(t)dW_I(t)} \qquad \textbf{(23)}$$

Similar to rates we assume $\mu_I(t,T)$ random while $\sigma_I(t,T)$ is deterministic

- **For *Nominal* rates in Eq.(21) HJM equations and Eq.(13) from earlier apply directly**

- **For *Real* and Inflation we need to pay more attention:**
  - **Explore market completeness**
  - **Arbitrage-free conditions**
  - **Under Q measure**

Continue with detailed JY derivation…

- **Under *Q* the following must be martingales:**

$$M_1(t,T) \stackrel{def}{=} \frac{Z_n(t,T)}{\beta_n(t)}, \quad M_2(t,T) \stackrel{def}{=} \frac{Z_r^{\$}(t,T)}{\beta_n(t)} \quad and \; M_3(t) \stackrel{def}{=} \frac{\beta_r^{\$}(t)}{\beta_n(t)} \quad where \;\; \beta_r^{\$}(t) \equiv I(t) \cdot \beta_r(t) \qquad \textbf{(24)}$$

- **Write that out starting from differential of *M*$_3$*(t)* keeping Ito in mind:**

$$\begin{aligned} dM_3(t) &= d\,\frac{I(t)\beta_r(t)}{\beta_n(t)} \\ &= -\frac{I(t)\beta_r(t)}{\beta_n^2(t)} d\beta_n(t) + \frac{d[I(t)\beta_r(t)]}{\beta_n(t)} \\ &= -\frac{I(t)\beta_r(t)}{\beta_n^2(t)} d\beta_n(t) + \frac{1}{\beta_n(t)}\{I(t)d\beta_r(t) + \beta_r(t)dI(t)\} \end{aligned}$$

Continue with detailed JY derivation…

- **Plug in CPI dynamics from Eq.(23) and Money Mkt acct β from Eq.(8):**

$$\frac{dM_3(t)}{M_3(t)} = \left[-r_n(t) + r_r(t) + \mu_I(t)\right]dt + \sigma_I(t)dW_I$$

- **Zero drift for *$M_3(t)$* to be martingale is the familiar Fisher Equation:**

$$\mu_I(t) = r_n(t) - r_r(t) \qquad (25)$$

- **Continue with *$M_2(t,T)$* where we will definitely need Ito:**

$$\begin{aligned} dM_2(t,T) &= d\,\frac{I(t)Z_r(t,T)}{\beta_n(t)} \\ &= -\frac{I(t)Z_r(t,T)}{\beta_n^2(t)}d\beta_n(t) + \frac{d\left(I(t)Z_r(t,T)\right)}{\beta_n(t)} \\ &= -\frac{I(t)Z_r(t,T)}{\beta_n^2(t)}r_n(t)\beta_n(t)dt + \frac{1}{\beta_n(t)}\left\{I(t)dZ_r(t,T) + Z_r(t,T)dI(t) + dI(t)dZ_r(t,T)\right\} \end{aligned}$$

Emerging Markets and Inflation

Continue with detailed JY derivation...

- **Plug in zero-coupon bond dynamics from Eq.(14) and (15), CPI dynamics from Eq.(23) and correlation cross-term from Ito:**

$$\frac{dM_2(t,T)}{M_2(t,T)} = \left[-r_n(t) + \mu_{Z_r}(t) + \mu_I(t) + \Sigma(t,T)\sigma_I(t)\rho_{r,I}\right]dt + \left[\Sigma(t,T)dW_r(t) + \sigma_I(t)dW_I(t)\right]$$

- **Explicitly and carefully write the drift term of that simplifying the Fisher Equation part:**

$$drift_{M_2} = -r_n(t) + \left(r_r(t) - \int_t^T \alpha_r(t,s)ds + \frac{1}{2}\left\| -\int_t^T \sigma_r(t,s)ds \right\|^2 \right) + \mu_I(t) - \sigma_I(t)\rho_{r,I}\int_t^T \sigma_r(t,s)ds$$

$$= -\int_t^T \alpha_r(t,s)ds + \frac{1}{2}\left\| -\int_t^T \sigma_r(t,s)ds \right\|^2 - \sigma_I(t)\rho_{r,I}\int_t^T \sigma_r(t,s)ds$$

Continue with detailed JY derivation…

- **Drift of $M_2(t,T)$ must be 0 to satisfy martingale condition under Q**
- **Differentiate it w.r.t T:**

$$\boxed{-\alpha_r(t,T)+\sigma_r(t,T)\int_t^T \sigma_r(t,s)ds-\sigma_r(t,T)\sigma_I(t)\rho_{r,I}=0}$$

- **And extract *Real* rate forward $f_r(t,T)$ drift $a_r(t,T)$ from it:**

$$\boxed{\alpha_r(t,T)=\sigma_r(t,T)\left[\int_t^T \sigma_r(t,s)ds-\sigma_I(t)\rho_{r,I}\right]} \qquad (26)$$

Continue with detailed JY derivation…

- Finally look at the first martingale $M_1(t,T)$ from Eq.(24)
- Discounted *nominal* zero-coupon bond price $Z_n(t,T)$ under Q must have same drift as the *nominal* bank account
- Write that recalling zero-coupon bond drift condition in Eq.(15)

$$r_n(t) - \int_t^T \alpha_n(t,s)ds + \frac{1}{2}\left\| \int_t^T \sigma_n(t,s)ds \right\|^2 = r_n(t)$$

- Which immediately reduces to:

$$\int_t^T \alpha_n(t,s)ds = \frac{1}{2}\left\| \int_t^T \sigma_n(t,s)ds \right\|^2$$

Continue with detailed JY derivation…

- **Differentiate it over *T* to arrive at expression for $\alpha_n(t,T)$:**

$$\alpha_n(t,T) = \sigma_n(t,T)\int_t^T \sigma_n(t,s)ds$$

- **Pairing the above with Eq.(14) and (15) we arrive at dynamics of $Z_n(t,T)$:**

$$\frac{dZ_n(t,T)}{Z_n(t,T)} = r_n(t)dt - \left(\int_t^T \sigma_n(t,s)ds\right)dW_n(t)$$

- **Next is dynamics of real zero-coupon bond in CPI units:**

$$\frac{dZ_r(t,T)}{Z_r(t,T)} = \left[r_r(t) - \int_t^T \alpha_r(t,s)ds + \frac{1}{2}\left(\int_t^T \sigma_r(t,s)ds\right)^2\right]dt - \left(\int_t^T \sigma_r(t,s)ds\right)dW_r(t) \qquad (27)$$

Continue with detailed JY derivation…

- **Recall $\alpha_r(t,T)$ from Eq.(26) for second term of the drift in the above:**

$$\int_t^T \alpha_r(t,s)ds = \int_t^T \left[\sigma_r(t,s) \int_s^t \sigma_r(t,u)du \right] ds - \int_t^T \sigma_r(t,s)\sigma_I(t)\rho_{r,I}\, ds$$

- **And integrate the first term by parts:**

$$\int_t^T \left[\sigma_r(t,s) \int_s^t \sigma_r(t,u)du \right] ds = \left(\int_s^t \sigma_r(t,u)du \right)^2 \Bigg|_{s=t}^{s=T} - \int_t^T \left(\int_s^t \sigma_r(t,u)du \right) \sigma_r(t,s)ds$$

$$\int_t^T \left[\sigma_r(t,s) \int_s^t \sigma_r(t,u)du \right] ds = \frac{1}{2} \left(\int_t^T \sigma_r(t,u)du \right)^2$$

Continue with detailed JY derivation…

- **Plug it now back into Eq.(27) for dynamics of $Z_r(t,T)$:**

$$\frac{dZ_r(t,T)}{Z_r(t,T)}=\left[r_r(t)+\sigma_I(t)\rho_{r,I}\int_t^T\sigma_r(t,s)ds\right]dt-\left(\int_t^T\sigma_r(t,s)ds\right)dW_r(t)$$

- **Use Eq.(9) for USD price of zero-coupon *Real* bond and apply Ito:**

$$\begin{aligned}dZ_r^{\$}(t,T)&=I(t)dZ_r(t,T)+Z_r(t,T)dI(t)+dI(t)dZ_r(t,T)\\&=Z_r^{\$}(t,T)\left\{\begin{aligned}&\left[r_r(t)+\sigma_I(t)\rho_{r,I}\int_t^T\sigma_r(t,s)ds\right]dt-\left(\int_t^T\sigma_r(t,s)ds\right)dW_r(t)\\&+\left[r_n(t)-r_r(t)\right]dt+\sigma_I(t)dW_I(t)-\sigma_I(t)\rho_{r,I}\left(\int_t^T\sigma_r(t,s)ds\right)dt\end{aligned}\right\}\end{aligned}$$

Continue with detailed JY derivation…

- **And simplify it to the final expression:**

$$\boxed{\frac{dZ_r^{\$}(t,T)}{Z_r^{\$}(t,T)} = r_n(t)dt - \left(\int_t^T \sigma_r(t,s)ds\right)dW_r(t) + \sigma_I(t)dW_I(t)}$$

- **We are done! Collect it all under martingale measure *Q*:**

Continue with detailed JY derivation...

$$\begin{cases} df_n(t,T) = \left(\sigma_n(t,T) \int_t^T \sigma_n(t,s)\,ds \right) dt + \sigma_n(t,T)\,dW_n(t) \\ df_r(t,T) = \sigma_r(t,T) \left(\int_t^T \sigma_r(t,s)\,ds - \sigma_I(t)\rho_{r,I} \right) dt + \sigma_r(t,T)\,dW_r(t) \\ \dfrac{dI(t)}{I(t)} = \mu_I(t)\,dt + \sigma_I(t)\,dW_I(t) \\ \dfrac{dZ_n(t,T)}{Z_n(t,T)} = r_n(t)\,dt - \left(\int_t^T \sigma_n(t,s)\,ds \right) dW_n(t) \\ \dfrac{dZ_r(t,T)}{Z_r(t,T)} = \left[r_r(t) + \sigma_I(t)\rho_{r,I} \int_t^T \sigma_r(t,s)\,ds \right] dt - \left(\int_t^T \sigma_r(t,s)\,ds \right) dW_r(t) \\ \dfrac{dZ_r^{\$}(t,T)}{Z_r^{\$}(t,T)} = r_n(t)\,dt - \left(\int_t^T \sigma_r(t,s)\,ds \right) dW_r(t) + \sigma_I(t)\,dW_I(t) \end{cases} \quad (28)$$

- ***Nominal*** **and** ***Real*** **rates and prices of the** ***Nominal*** **and** ***Real*** **zero-coupon bonds under the same martingale measure** ***Q***

Reduce JY via extended Vasicek

- **Simplify the above via Eq.(20) utilizing Extended Vasicek with vol**

$$\sigma_k(t,T)=\sigma_k e^{-\alpha_k(T-t)} \quad where \quad k\in\{n,r\} \quad and \quad \sigma_k = Const$$

- **Plug it into Eq.(28.1) for *Nominal* forward rate $f_n(t,T)$:**

$$df_n(t,T)=\left(\sigma_n e^{-a_n(T-t)}\int_t^T \sigma_n e^{-a_n(s-T)}ds\right)dt+\sigma_n e^{-a_n(T-t)}dW_n(t)$$
$$=\frac{\sigma_n^2}{\alpha_n}\left(-e^{-2a_n(T-t)}+e^{-\alpha_n(T-t)}\right)dt+\sigma_n e^{-a_n(T-t)}dW_n(t)$$

- **And integrate:**

$$f_n(t,T)=f_n^*(0,T)+\frac{\sigma_n^2}{a_n}\int_0^t\left(-e^{-2a_n(T-s)}+e^{-a_n(T-s)}\right)ds+\sigma_n\int_0^t e^{-a_n(T-s)}dW_n(s)$$

Reduce JY via extended Vasicek

- **Apply now short rate definition from Eq.(16):**

$$\begin{aligned} dr_n(t) &= \left(\left.\frac{\partial f_n^*(0,T)}{\partial T}\right|_{T=t} \right) dt + \frac{\sigma_n^2}{a_n}\left(\int_0^t \left[-e^{-2a_n(t-s)}(-2a_n) + e^{-a_n(t-s)}(-a_n) \right] ds \right) dt + \\ &+ \sigma_n dW_n(t) + \sigma_n \left(\int_0^t e^{-a_n(t-s)}(-a_n) dW_n(s) \right) dt \\ &= \left\{ \left.\frac{\partial f_n^*(0,T)}{\partial T}\right|_{T=t} - a_n r_n(t) - a_n \left[-f_n(0,t) - \frac{\sigma_n^2}{a_n} \int_0^t e^{-2a_n(t-s)} ds \right] \right\} dt + \sigma_n dW_n(t) \end{aligned}$$

- **To see it reducing to standard extended Vasicek form:**

$$\boxed{dr_n(t) = \left[\theta_n(t) - a_n r_n(t)\right] dt + \sigma_n dW_n(t)} \qquad (29)$$

Reduce JY via extended Vasicek

- **Following the same route for *Real* rates:**

$$dr_r(t) = \left[\theta_r(t) - a_r r_r(t) - \sigma_r \sigma_I \rho_{r,I}\right] dt + \sigma_r dW_r(t) \quad (30)$$

- **Recalling what $\theta_k(t)$ is from Eq.(18) under extended Vasicek:**

$$\theta_k(t) = \frac{\partial f_k(0,T)}{\partial T} + a_k f_k(0,t) + \frac{\sigma_k^2}{2a_k}\left(1 - e^{-2a_k t}\right), \quad k \in \{n, r\}$$

- **Note that moving from *Nominal* measure Q_n to *Real* Q_r is equivalent to change of numeraire from β_n to $\beta_r \cdot I$**

- **Easily leads us to:**

$$dW_r^{\beta_n}(t) = dW_r^{\beta_r I}(t) + \rho_{r,I}\sigma_I dt$$

Reduce JY via extended Vasicek

- **And allows to re-write Eq.(30) under Q_r:**

$$\boxed{dr_r(t) = [\theta_r(t) - a_r r_r(t)]dt + \sigma_r dW_r(t)}$$

- **So both *Nominal* and *Real* rates are normally distributed under their respective risk-neutral measures!**

- **So easy integration of CPI in Eq.(28.3) that is lognormal under Q_n:**

$$\boxed{I(T) = I(t)\exp\left\{\int_t^T [r_n(s) - r_r(s)]ds - \frac{1}{2}\sigma_I^2(T-t) + \sigma_I[W_I(T) - W_I(t)]\right\}}$$

Hedging Ratios

- **We are now done with derivation**
- **Last comments looking at some hedging ratios from our derivations:**

$$\frac{\partial\left[I(t)Z_r(t,T)\right]}{\partial r_n(t)} \equiv \frac{\partial Z_r^{\$}(t,T)}{\partial r_n(t)} = 0$$

$$\frac{\partial Z_n(t,T)}{\partial I(t)} = 0$$

$$\frac{\partial Z_n(t,T)}{\partial r_r(t)} = 0$$

- **(1): USD prices of *Real* Rate bonds do not depend on *Nominal* rates ➔ learned that before and thus validated *Real* Rate economy of LatAm**
- **(2): *Nominal* zero-coupon bond prices do not depend on inflation**
- **(3): *Nominal* zero-coupon bond prices do not depend on *Real* rates**

Final comments

- **Formulae allow us to price inflation-indexed derivatives in closed form**
- **Or at least via 3-F Monte-Carlo**
- **Model is tractable and allows closed form solution**
- **Only challenge is dynamics of *Real* rates that is neither observable, nor hedgeable!**
- **Try now some actual derivatives pricing applying all of the above…**

Term-on-Term Inflation Swap in Real Rate Economy

- **Recall Floating coupon for Colombia Inflation swap**

$$fv(T_i) = \left[1 + bi(T_{i-1}, T_i)\right]^{Act/365} - 1 \tag{6}$$

Here *bi(T* $_{i-1}$*, T* $_i$*)* is realized breakeven inflation between coupon dates:

$$bi(T_{i-1}, T_i) \overset{def}{=} \frac{IPC_{T_i}}{IPC_{T_{i-1}}} - 1$$

- **Present value of this in *Nominal* risk-neutral and then *T*-forward measures for cash flow at *T* for inflation period *T-δ* -> *T* with *t*< *T-δ* <*T*:**

$$\begin{aligned} pv(T) &= \mathbf{E}^{RN}\left[\left\{\left(\frac{IPC_T}{IPC_{T-\delta}}\right)^{\tau} - 1\right\} \cdot Z_N(t,T)\right] \\ &= Z_N(t,T) \cdot \mathbf{E}^{T}\left[\left(\frac{IPC_T}{IPC_{T-\delta}}\right)^{\tau} - 1\right] \end{aligned} \tag{31}$$

Term-on-Term Inflation Swap (continued)

- **Once again represent Eq.(31) via simplified expression and Convexity adjustment *C(T)***

$$\mathbf{E}^{T}\left[\left(\frac{IPC_{T}}{IPC_{T-\delta}}\right)^{\tau}\right]=\left(\frac{\mathbf{E}^{T}\left[IPC_{T}\right]}{\mathbf{E}^{T}\left[IPC_{T-\delta}\right]}\right)^{\tau}\cdot e^{\tau\cdot C(T)} \qquad (32)$$

  - **C(T) comes from expectation split**
  - **Change of measure in the denominator** $\mathbf{E}^{T}\left[\frac{IPC_{T}}{IPC_{T-\delta}}\right] \Rightarrow \frac{\mathbf{E}^{T}\left[IPC_{T}\right]}{\mathbf{E}^{T}\left[IPC_{T-\delta}\right]}$
  - **And concavity of the power function when $\tau < 1$**

- **How do we solve the above?**

3-Factor Inflation Model

- **We are to solve breakeven inflation expression**
- **Only using (modeling) market observables: *Nominal* rates, *Real* rates and CPI index dynamics**
- **Recall the FX analogy observed earlier**
- **Start with normal dynamics for rates and lognormal for CPI (reasonable?) in HJM mean-reverting framework we used earlier as in Eq.(29) and (30):**

$$\boxed{\begin{aligned} dn(t) &= [\theta_n(t) - \alpha_n n(t)]dt + \sigma_n dW_n(t) \\ dr(t) &= [\theta_r(t) - \alpha_r n(t) - \rho_{r,I}\sigma_r\sigma_I]dt + \sigma_r dW_r(t) \\ \frac{dI(t)}{I(t)} &= [n(t) - r(t)]dt + \sigma_I dW_I(t) \end{aligned}} \qquad (33)$$

3-Factor Inflation Model (continued)

- **Skip some steps covered earlier and write standard HJM**

$$\theta_n(t) = \frac{\partial f_n(0,t)}{\partial t} + \alpha_n f_n(0,t) + \frac{\sigma_n^2}{2\alpha_n}(1 - e^{-2\alpha_n t})$$

$$\theta_r(t) = \frac{\partial f_r(0,t)}{\partial t} + \alpha_r f_r(0,t) + \frac{\sigma_r^2}{2\alpha_r}(1 - e^{-2\alpha_r t})$$

- **Eq.(31) and (32) allow easy convexity derivation in T-forward measure. Re-write Eq.(34) for that**

$$dn(t) = \left[\theta_n(t) - \alpha_n n(t) - \sigma_n^2 B_n(t,T)\right]dt + \sigma_n dW_n(t)$$

$$dr(t) = \left[\theta_r(t) - \alpha_r n(t) - \rho_{r,I}\sigma_r\sigma_I - \rho_{n,r}\sigma_n\sigma_r B_n(t,T)\right]dt + \sigma_r dW_r(t) \qquad (34)$$

$$\frac{dI(t)}{I(t)} = \left[n(t) - r(t) - \rho_{n,I}\sigma_n\sigma_I B_n(t,T)\right]dt + \sigma_I dW_I(t)$$

3-Factor Inflation Model (continued)

- **Defined** $B_n(t,T) \overset{def}{=} \frac{1}{\alpha_n}(1-e^{-\alpha_n(T-t)})$
- **Go back to Eq.(32) and split Convexity into components (also simplifying $IPC_T = I_T$)**

$$\mathbf{E}^T\left[\left(\frac{I_T}{I_{T-\delta}}\right)^{\tau}\right] = \left(\mathbf{E}^T\left[\frac{I_T}{I_{T-\delta}}\right]\right)^{\tau}\cdot e^{\tau C_1(T)} = \left(\frac{\mathbf{E}^T[I_T]}{\mathbf{E}^{T-\delta}[I_{T-\delta}]}\cdot e^{C_2(T)}\right)^{\tau}\cdot e^{\tau\cdot C_1(T)}$$

- **For lognormal variable X:**

$$\mathbf{E}\left[X^{\tau}\right] = \left(\mathbf{E}[X]\right)^{\tau}\cdot\exp\left[-\tau(1-\tau)\frac{Var(\log X)}{2}\right]$$

- **Thus for $C_1(T)$:** $C_1(T) \overset{def}{=} -\frac{1}{2}(1-\tau)\cdot Var\left[\log\left(\frac{I_T}{I_{T-\delta}}\right)\right]$

3-Factor Inflation Model (continued)

- **Just few expressions to start on the integration path of index in Eq.(34):**

$$\log\left[\frac{I_T}{I_{T-\delta}}\right]=\int_{T-\delta}^{T}\left[n(t)-r(t)-\rho_{n,I}\sigma_n\sigma_I B_n(0,T)\right]dt-\frac{1}{2}\sigma_I^2\delta+\int_{T-\delta}^{T}\sigma_I dW_n^T(t)$$

- **So we can write the variance of the above as**

$$\begin{aligned}Var\left[\log\left(\frac{I_T}{I_{T-\delta}}\right)\right]&=Var\left[\int_{T-\delta}^{T}n(t)dt\right]+Var\left[\int_{T-\delta}^{T}r(t)dt\right]+\sigma_I^2\delta-2Cov\left[\int_{T-\delta}^{T}n(t)dt,\int_{T-\delta}^{T}r(t)dt\right]+\\&+2Cov\left[\int_{T-\delta}^{T}n(t)dt,\int_{T-\delta}^{T}\sigma_I dW_n^T(t)\right]-2Cov\left[\int_{T-\delta}^{T}r(t)dt,\int_{T-\delta}^{T}\sigma_I dW_n^T(t)\right]\end{aligned}$$

3-Factor Inflation Model (continued)

- **Variance of the normal process under HJM as we learned earlier:**

$$Var\left[\int_{T-\delta}^{T} x(t)dt\right] = \frac{\sigma_x^2}{\alpha_x^2}\left[\delta - B_x(T-\delta,T) - \frac{1}{2}\alpha_x e^{-2\alpha_x(T-\delta)} B_x^2(T-\delta,T)\right]$$

$$x = n, r$$

- **First convexity assuming small mean reversions**

$$Var\left[\log\left(\frac{I_T}{I_{T-\delta}}\right)\right] \propto \left(T\delta^2 - \frac{2}{3}\delta^3\right)\cdot\left(\sigma_n^2 + \sigma_r^2 - 2\rho_{n,r}\sigma_n\sigma_r\right) + \delta\sigma_I^2 + \delta^2\sigma_I\left(\rho_{n,I}\sigma_n - \rho_{r,I}\sigma_r\right)$$

$$C_1(T) = -\frac{1}{2}(1-\tau)\cdot\left\{\left(T\delta^2 - \frac{2}{3}\delta^3\right)\cdot\left(\sigma_n^2 + \sigma_r^2 - 2\rho_{n,r}\sigma_n\sigma_r\right) + \delta\sigma_I^2 + \delta^2\sigma_I\left(\rho_{n,I}\sigma_n - \rho_{r,I}\sigma_r\right)\right\}$$

3-Factor Inflation Model (continued)

- **Second convexity separately calculating under T-forward and (T-δ)-forward measures**

$$C_2(T) = \sigma_r B_r(T-\delta,T)\cdot\left[C_{22}(T) - C_{21}(T)\right]$$

$$C_{21}(T) \equiv \frac{\rho_{n,r}\sigma_n}{\alpha_n+\alpha_r}\left(1+\alpha_r B_n(0,T-\delta)\right)$$

$$C_{22}(T) \equiv B_r(T-\delta,T)\left[\rho_{r,I}\sigma_I - \frac{1}{2}\sigma_r B_r(T-\delta,T) + \frac{\rho_{n,r}\sigma_n}{\alpha_n+\alpha_r} + \alpha_r C_{21}(T)\right]$$

- **Note $C_2(T)$ going to zero at zero real rate vol**

3-Factor Inflation Model (continued)

- **To sum it up, 3-Factor model seems to be the most intuitive and tractable approach for inflation modeling**
- **This has been long recognized as the first approach in modeling inflation**
- **Only potential problem is lack of real rates trading in developed markets**
- **Let us now generalize J-Y approach and look at some alternative models**

7. Classic Inflation Modeling

7.1. 3-Factor Jarrow and Yildirim Model

7.2. Application in Colombia

7.3 Pricing of Zero Coupon Inflation-Indexed Swaps (ZCIIS)

7.4 Pricing of Year-on-Year Inflation-Indexed Swaps (YYIIS)

- **Derived model-independent expression for ZCIIS price in Lecture 5**
- **Quickly repeat that in a more formal setting**
- **Floating leg of a ZCIIS starting at time *t=0* and maturing at time *t=T* with *τ* representing day count fraction for the period *0->T* :**

$$fv_{ZCIIS}^{FLT}(t,T) = \left[\frac{I(T)}{I_o} - 1\right]$$

- **No-arbitrage PV under same old Q_n:**

$$pv_{ZCIIS}(t,T) = \mathbf{E}^{Q_n}\left\{ e^{-\int_t^T r_n(s)ds} \left[\frac{I(T)}{I_o} - 1\right] \middle| F_t \right\} \qquad (35)$$

ZCIIS Pricing

- **Recall USD price of *Real* zero-coupon bond changing from *Real* into *Nominal* measure:**

$$Z_r^{\$}(t,T) = I(t)\cdot Z_r(t,T) = I(t)\cdot \mathbf{E}^{Q_r}\left\{ e^{-\int_t^T r_r(s)ds} \middle| F_t \right\} = \mathbf{E}^{Q_n}\left\{ I(T)\cdot e^{-\int_t^T r_n(s)ds} \middle| F_t \right\}$$

- **And plug it into Eq.(25) to arrive at model-independent expression:**

$$pv_{ZCIIS}(t,T) = \left[\frac{I(t)}{I_o} Z_r(t,T) - Z_n(t,T) \right]$$

7. Classic Inflation Modeling

7.1. 3-Factor Jarrow and Yildirim Model

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7.4 Pricing of Year-on-Year Inflation-Indexed Swaps (YYIIS)

- **With Jarrow - Yildirim**
- **With First (Mercurio) Market Model**
- **With better Market Model**

YYIIS Pricing

- **Also noted need of model for YYIIS price in Lecture 5**
- **Now let's go through that more carefully**
- **Spaced coupons paying inflation' differential**
- **Index *i* for coupons maturing on T_i and paying inflation over $[T_{i-1}, T_i]$:**

$$fv_{YYIIS}^{FLT}(t, T_i) = \tau_i \cdot \left[\frac{I(T_i)}{I(T_{i-1})} - 1 \right]$$

Continue…

- **Under the same nominal Q_n measure price of our YYIIS:**

$$pv_{YYIIS}(t,T_i) = \tau_i \cdot \mathbf{E}^{Q_n}\left\{ e^{-\int_t^{T_i} r_n(s)ds}\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right] \middle| F_t \right\} \tag{36}$$

- **Split it into 2 consecutive parts:**

$$pv_{YYIIS}(t,T_i) = \tau_i \cdot \mathbf{E}^{Q_n}\left\{ e^{-\int_t^{T_{i-1}} r_n(s)ds} \mathbf{E}^{Q_n}\left[e^{-\int_{T_{i-1}}^{T_i} r_n(s)ds}\left(\frac{I(T_i)}{I(T_{i-1})} - 1\right) \middle| F_{T_{i-1}} \right] \middle| F_t \right\}$$

- **Inner expectation is identical to the price of ZCIIS at T_{i-1}:**

Continue...

$$pv_{YYIIS}(t,T_i)=\tau_i\cdot\mathbf{E}^{Q_n}\left\{e^{-\int_t^{T_{i-1}} r_n(s)ds}\left[Z_r(T_{i-1},T_i)-Z_n(T_{i-1},T_i)\right]\middle|F_t\right\}$$
$$=\tau_i\cdot\mathbf{E}^{Q_n}\left\{e^{-\int_t^{T_{i-1}} r_n(s)ds}\cdot Z_r(T_{i-1},T_i)\middle|F_t\right\}-\tau_i\cdot Z_n(t,T_i) \quad (37)$$

- **For deterministic Real rates it simply reduces to:**

$$pv_{YYIIS}(t,T_i)=\tau_i\cdot Z_r(T_{i-1},T_i)\cdot Z_n(t,T_{i-1})-\tau_i\cdot Z_n(t,T_i)$$
$$=\tau_i\cdot\frac{Z_r(t,T_i)}{Z_r(t,T_{i-1})}\cdot Z_n(t,T_{i-1})-\tau_i\cdot Z_n(t,T_i)$$

- **In generic case of stochastic rates we do need a model now!**

With Jarrow - Yildirim

- Re-write Eq.(37) under T_{i-1} forward measure in nominal space:

$$pv_{YYIIS}(t,T_i) = \tau_i \cdot Z_n(t,T_{i-1}) \cdot \mathbf{E}_{T_{i-1}}^{Q_n}\left[Z_r(T_{i-1},T_i) \middle| F_t\right] - \tau_i \cdot Z_n(t,T_i) \tag{38}$$

- Recall now *Real* zero-coupon bond dynamics from Eq.(19) and short rate dynamics from Eq.(30) under T_{i-1} forward measure with change of numeraire already done before:

$$dr_r(t) = \left[\theta_r(t) - a_r r_r(t) - \sigma_r \sigma_I \rho_{r,I} - \sigma_n \sigma_r \rho_{n,r} B_n(t,T_{i-1})\right]dt + \sigma_r dW_r^{T_{i-1}}(t)$$

- So we still see $r(T_{i-1})$ normally distributed under $Q_n^{T_{i-1}}$

- And $Z_r(T_{i-1},T_i)$ is lognormally distributed under the same measure

Continue with Jarrow - Yildirim…

- **This allows us to write price of YYIIS:**

$$pv_{YYIIS}(t,T_i)=\tau_i Z_n(t,T_{i-1})\frac{Z_r(t,T_i)}{Z_r(t,T_{i-1})}e^{C(t,T_i)}-\tau_i Z_n(t,T_i)$$

$$C(t,T_i)\equiv\sigma_r B_r(T_{i-1},T_i)\left[\begin{array}{l} B_r(t,T_{i-1})\left(\sigma_I\rho_{r,I}-\frac{1}{2}\sigma_r B_r(t,T_{i-1})+\frac{\sigma_n\rho_{n,r}}{a_n+a_r}\left(1+a_r B_n(t,T_{i-1})\right)\right) \\ -\frac{\sigma_n\rho_{n,r}}{a_n+a_r}B_n(t,T_{i-1}) \end{array}\right] \quad (39)$$

- **See two familiar things:**

1. **YYIIS price via deterministic prices of *Nominal* and *Real* zero-coupon bonds scaled by Correction. Same Convexity we saw in Colombia earlier!**
2. **Adjustment vanishes for deterministic *Real* rates: *C(t,T)=0* when $\sigma_r=0$**

YYIIS Pricing

Continue with Jarrow - Yildirim...

- **Price of the full YYIIS simply requires summation over coupons:**

$$pv_{YYIIS}(0,T) = \tau_1\left[Z_r(0,T_1) - Z_n(0,T_1)\right] + \sum_{i=2}^{M} \tau_i \left[Z_n(0,T_{i-1}) \frac{Z_r(0,T_i)}{Z_r(0,T_{i-1})} e^{C(0,T_i)} - Z_n(0,T_i) \right]$$

- **Solvable in closed form, but again is only convenient in markets with tradable *Nominal* and *Real* rates!**

- **Vols and correlations could be historically estimated, of course**

First Market Model

- **Recall expressions for Real and Nominal zero-coupon bond prices**
- **Write forward LIBOR rates per definition**

$$\boxed{L_k(t,T_i)=\frac{Z_k(t,T_{i-1})-Z_k(t,T_i)}{\tau_i Z_k(t,T_i)}, \quad where \quad k\in\{n,r\}} \qquad (40)$$

- **And re-write expectation from Eq.(38):**

$$\begin{aligned} Z_n(t,T_{i-1})\cdot \mathbf{E}^{Q_n}_{T_{i-1}}\left[Z_r(T_{i-1},T_i)\middle|F_t\right] &= Z_n(t,T_i)\cdot \mathbf{E}^{Q_n}_{T_i}\left[\frac{Z_r(T_{i-1},T_i)}{Z_n(T_{i-1},T_i)}\middle|F_t\right] \\ &= Z_n(t,T_i)\cdot \mathbf{E}^{Q_n}_{T_i}\left[\frac{1+\tau_i L_n(T_{i-1},T_i)}{1+\tau_i L_r(T_{i-1},T_i)}\middle|F_t\right] \end{aligned}$$

- **Could be evaluated in a forward LIBOR model (a BGM-type)**
- **But does not solve main limitation: dependency on *Real* rates info**

Better Market Model

- **Similarly to forward LIBOR defined in Eq.(40), define forward CPI**
- **Different notation for forward CPI *I(t,T)* expressed via spot CPI *I(t)*:**

$$\boxed{\mathbf{I}(t,T)=I(t)\frac{Z_r(t,T)}{Z_n(t,T)}}$$

- **Forward CPI martingale under nominal T_i-forward measure, so Eq.(36)**

$$\begin{aligned}pv_{YYIIS}(t,T_i)&=\tau_i Z_n(t,T_i)\cdot\mathbf{E}_{T_i}^{Q_n}\left\{\frac{I(T_i)}{I(T_{i-1})}-1\middle|F_t\right\}\\&=\tau_i Z_n(t,T_i)\cdot\mathbf{E}_{T_i}^{Q_n}\left\{\frac{\mathbf{I}(T_i,T_i)}{\mathbf{I}(T_{i-1},T_{i-1})}-1\middle|F_t\right\}\\&=\tau_i Z_n(t,T_i)\cdot\mathbf{E}_{T_i}^{Q_n}\left\{\frac{\mathbf{I}(T_{i-1},T_i)}{\mathbf{I}(T_{i-1},T_{i-1})}-1\middle|F_t\right\}\end{aligned} \quad (41)$$

Continue with better Market Model...

- So we also assume lognormal forward CPI with constant vol

$$\boxed{d\mathbf{I}(t,T_i) = \sigma_{I,i}\mathbf{I}(t,T_i)dW_i^I(t)} \qquad (42)$$

- Extend the above to *Real* and *Nominal* rates following similar Market Model with constant vols and correlations

$$\boxed{\begin{aligned} dL_n(t,T_i) &= \sigma_{n,i}L_n(t,T_i)dW_i^n(t) \\ dL_r(t,T_i) &= L_r(t,T_i)\left[-\rho_{I,r,i}\sigma_{I,i}\sigma_{r,i}dt + \sigma_{r,i}dW_i^n(t)\right] \end{aligned}}$$

- Eq.(42) holds for forward CPI at T_i under forward nominal measure $Q_n^{T_i}$ as wells as at T_{i-1} under $Q_n^{T_{i-1}}$. Change numeraire:

$$\boxed{d\mathbf{I}(t,T_{i-1}) = \mathbf{I}(t,T_{i-1})\sigma_{I,i-1}\left[-\frac{\tau_i\sigma_{n,i}L_n(t,T_i)}{1+\tau_iL_n(t,T_i)}\rho_{I,n,i}dt + dW_{i-1}^I(t)\right]}$$

Continue with better Market Model…

- Freeze the drift in the above at its current time-t value to ensure lognormality of $\mathbf{I}(T_{i-1}, T_{i-1})$ under $Q_n^{T_i}$ simplifies expectation in Eq.(41):

$$\boxed{\mathbf{E}_{T_i}^{Q_n}\left\{\frac{\mathbf{I}(T_{i-1}, T_i)}{\mathbf{I}(T_{i-1}, T_{i-1})} - 1 \middle| F_t\right\} = \frac{\mathbf{I}(t, T_i)}{\mathbf{I}(t, T_{i-1})} e^{D(t, T_i)}}$$

- So another Convexity-like correction factor

$$\boxed{D(t, T_{i-1}) = \sigma_{I,i-1}\left[\frac{\tau_i \sigma_{n,i} L_n(t, T_i)}{1 + \tau_i L_n(t, T_i)} \rho_{I,n,i} - \rho_{I,i}\sigma_{I,i} + \sigma_{I,i-1}\right](T_{i-1} - t)}$$

- And expression for price of one coupon of our YYIIS:

$$\begin{aligned} pv_{YYIIS}(t, T_i) &= \tau_i Z_n(t, T_i)\left[\frac{\mathbf{I}(t, T_i)}{\mathbf{I}(t, T_{i-1})} e^{D(t, T_i)} - 1\right] \\ &= \tau_i Z_n(t, T_i)\left[\frac{Z_n(t, T_{i-1}) \cdot Z_r(t, T_i)}{Z_n(t, T_i) \cdot Z_r(t, T_{i-1})} e^{D(t, T_i)} - 1\right] \end{aligned}$$

Continue with better Market Model…

- **And for the whole swap at time 0:**

$$pv_{YYIIS}(0,T) = \sum_{i=1}^{M} \tau_i Z_n(0,T_i)\left[\frac{\mathbf{I}(0,T_i)}{\mathbf{I}(0,T_{i-1})}e^{D(0,T_i)} - 1\right]$$
$$= \sum_{i=1}^{M} \tau_i Z_n(0,T_i)\left[\frac{1+\tau_i L_n(0,T_i)}{1+\tau_i L_r(0,T_i)}e^{D(0,T_i)} - 1\right]$$

- **Expression different from Eq.(39):**
  - ❑ **Depends only on vols and corrs of inflation and Nominal rates!**
  - ❑ **No Real rates mention makes it operation in observable quantitates and fully hedgeable!**

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