BARUCH, MFE

MTH 9876 Assignment One

Zhou, ShengQuan

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1 Loss Given Default

Loss given default (LGD) is a measure of the severity of a default, and is defined as the fraction of the face value of a bond that is not repaid to the investor. In other words,

$$\mathrm{LGD} = (1-R) \times F.$$

(i) Find the probability distribution of LGD in Merton's model.
Solution: Given the occurrence of a default at time T, the amount paid to the investor is the value of the firm V_T. Thus, the fraction of the face value of a bond that is not repaid to the investor is

$$\mathrm{LGD} \triangleq F(1-R) = (F - V_T)^+ ,$$

By definition, $0 \le \mathrm{LGD} < F$. Let $0 \le x < F$,

$$\begin{aligned}
\mathbb{P}(\mathrm{LGD} < x) &= \mathbb{P}(F - V_T \le x | V_T < F)\, \mathbb{P}(V_T < F) + \mathbb{P}(V_T \ge F) \\
&= \mathbb{P}(F - x \le V_T < F) + \mathbb{P}(V_T \ge F) \\
&= \mathbb{P}(V_T \ge F - x).
\end{aligned}$$

Invoking the well-known formula for the risk-neutral survival probability with face value F,

$$\mathbb{P}(V_T \ge F) = N(d_-)$$

where

$$d_- \triangleq \frac{1}{\sigma_V \sqrt{T}} \left[\log \frac{V_0}{F} + \left(r - \frac{1}{2}\sigma_V^2 \right) T \right],$$

the cumulative distribution function of LGD can be written as

$$\mathbb{P}(\mathrm{LGD} < x) = N(d_-(x)),$$

where $d_-(x)$ is obtained by replacing F by $F - x$ in d_-, such that $d_-(0) = d_-$:

$$d_-(x) \triangleq \frac{1}{\sigma_V \sqrt{T}} \left[\log \frac{V_0}{F - x} + \left(r - \frac{1}{2}\sigma_V^2 \right) T \right].$$

The density function is obtained by differentiating $\mathbb{P}(\mathrm{LGD} < x)$ with respect to x, for $0 \le x < F$,

$$\begin{aligned}
f_{\mathrm{LGD}}(x) &= \frac{\partial}{\partial x} \mathbb{P}(\mathrm{LGD} < x) \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_-(x)^2}{2}} \frac{\partial d_-(x)}{\partial x} + N(d_-)\, \delta(x) \\
&= \frac{e^{-\frac{d_-(x)^2}{2}}}{\sigma_V \sqrt{2\pi T}(F - x)} + N(d_-)\, \delta(x).
\end{aligned}$$

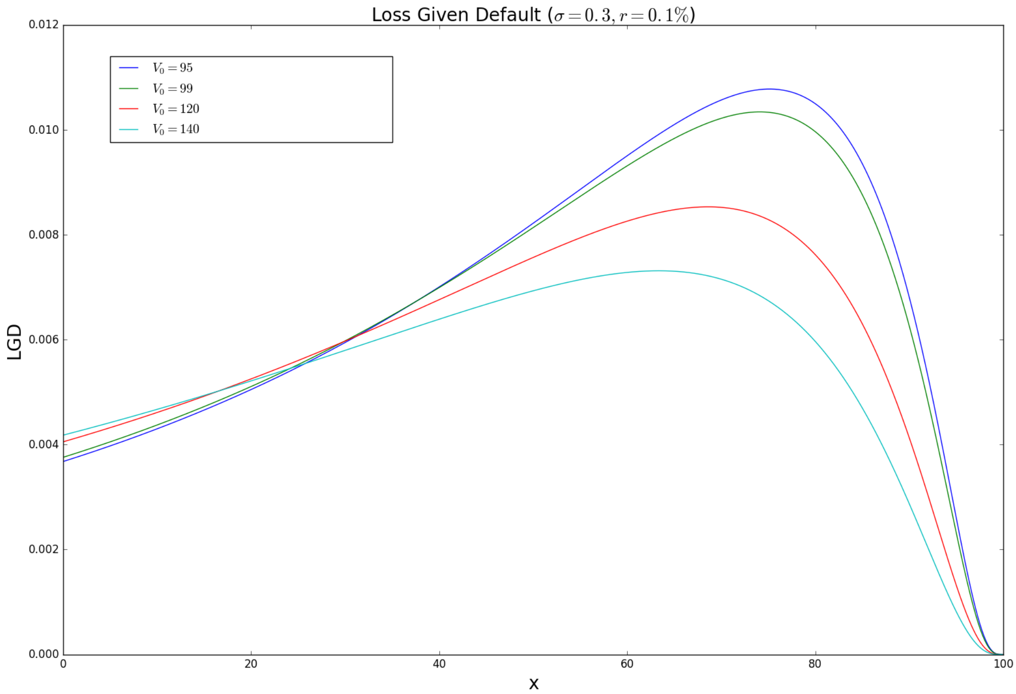
For $x < 0$, we have $f_{\mathrm{LGD}}(x) = 0$. Notice that there is a Dirac-δ component in the probability density function because at $x = 0$, the cumulative distribution function is discontinuous:

$$\mathbb{P}(\mathrm{LGD} < 0) = 0,$$
$$\mathbb{P}(\mathrm{LGD} \le 0) = N(d_-).$$

The above discontinuity is caused by the fact that the finite probability $N(d_-)$ of the equity value ending up with $V_T > F$ contributes to the isolated point of zero LGD.

(ii) Implement the formula for the density function and plot it for $T = 10$ and $V_0 = 95, 99, 120, 140$ (as usual, $F = 100$).

Solution: We choose $\sigma_V = 30\%$ and $r = 0.1\%$. The singular components corresponding to the Dirac-δ located at $x = 0$ are not depicted in graph. This example indicates that, given default, the most probable recovery rate falls in the approximate range of $R^* \in [20\%, 40\%]$.



2 Recovery Value in Black-Cox Model

Find a closed form expression for the recovery value $B^{\text{rec}}(t,T)$ in the Black and Cox model by evaluating

$$B^{\text{rec}}(t,T) = -\int_t^T e^{-r(s-t)} H(s) \frac{\partial}{\partial s} \Phi(-d(t), -d(t), -b, s-t)\, ds,$$

where $d(t) = \frac{1}{\sigma_V} \log \frac{H(t)}{V_0}$, $H(t) = H_0 e^{at}$, $b = \frac{r - \frac{1}{2}\sigma_V^2 - a}{\sigma_V}$ and $\Phi(\alpha, \beta, b, t) = N\left(\frac{\alpha - bt}{\sqrt{t}}\right) - e^{2b\beta} N\left(\frac{\alpha - 2\beta - bt}{\sqrt{t}}\right)$.
Solution: Given $H(t) = H_0 e^{at}$, we can write $d(t) = \frac{1}{\sigma_V}\left(\log \frac{H_0}{V_0} + at\right)$. Let $u = s - t$ in

$$\Phi(-d(t), -d(t), -b, s-t) = N\left(\frac{-d(t) + b(s-t)}{\sqrt{s-t}}\right) - e^{2bd(t)} N\left(\frac{d(t) + b(s-t)}{\sqrt{s-t}}\right),$$

we write

$$\begin{aligned}
B^{\text{rec}}(t,T) &= -H_0 e^{at} \int_0^{T-t} e^{-(a-r)u} \frac{\partial}{\partial u} \Phi(-d(t), -d(t), -b, u)\, du \\
&= -H_0 e^{at} \int_0^{T-t} e^{-(a-r)u} \left[\frac{\partial}{\partial u} N\left(\frac{-d(t) + bu}{\sqrt{u}}\right) - e^{2bd(t)} \frac{\partial}{\partial u} N\left(\frac{d(t) + bu}{\sqrt{u}}\right)\right] du \\
&= -H_0 e^{at} \left[\int_0^{T-t} e^{-(a-r)u} \frac{\partial}{\partial u} N\left(\frac{-d(t) + bu}{\sqrt{u}}\right) du - e^{2bd(t)} \int_0^{T-t} e^{-(a-r)u} \frac{\partial}{\partial u} N\left(\frac{d(t) + bu}{\sqrt{u}}\right) du\right].
\end{aligned}$$

Denote $\tau = T - t$ and $c = a - r$, we compute

$$\begin{aligned}
&\int_0^\tau e^{-cu} \frac{\partial}{\partial u} N\left(\frac{bu - d}{\sqrt{u}}\right) du \\
&= \frac{1}{2\sqrt{2\pi}} \int_0^\tau e^{-cu} e^{-\frac{(bu-d)^2}{2u}} \left(\frac{b}{\sqrt{u}} + \frac{d}{\sqrt{u^3}}\right) du \\
&= \frac{1}{2\sqrt{2\pi}} \int_0^\tau e^{-\frac{2cu^2 + (bu-d)^2}{2u}} \left(\frac{b}{\sqrt{u}} + \frac{d}{\sqrt{u^3}}\right) du \\
&= \frac{1}{2\sqrt{2\pi}} \int_0^\tau e^{-\frac{\left(2c + b^2\right)u^2 - 2bdu + d^2}{2u}} \left(\frac{b}{\sqrt{u}} + \frac{d}{\sqrt{u^3}}\right) du \\
&= \frac{1}{2\sqrt{2\pi}} \int_0^\tau e^{-\frac{g^2u^2 - 2bdu + d^2}{2u}} \left(\frac{b}{\sqrt{u}} + \frac{d}{\sqrt{u^3}}\right) du
\end{aligned}$$

Here denote: $g \triangleq \sqrt{2c + b^2}$

$$\begin{aligned}
&= \frac{e^{(b\pm g)d}}{2\sqrt{2\pi}} \int_0^\tau e^{-\frac{(gu\pm d)^2}{2u}} \left(\frac{b}{\sqrt{u}} + \frac{d}{\sqrt{u^3}}\right) du \\
&= \frac{e^{(b\pm g)d}}{4\sqrt{2\pi}} \int_0^\tau e^{-\frac{(gu\pm d)^2}{2u}} \left[\left(\frac{b}{g} + 1\right)\left(\frac{g}{\sqrt{u}} + \frac{d}{\sqrt{u^3}}\right) + \left(\frac{b}{g} - 1\right)\left(\frac{g}{\sqrt{u}} - \frac{d}{\sqrt{u^3}}\right)\right] du \\
&= \frac{e^{(b-g)d}}{4\sqrt{2\pi}} \left(\frac{b}{g} + 1\right) \int_0^\tau e^{-\frac{(gu-d)^2}{2u}} \left(\frac{g}{\sqrt{u}} + \frac{d}{\sqrt{u^3}}\right) du + \frac{e^{(b+g)d}}{4\sqrt{2\pi}} \left(\frac{b}{g} - 1\right) \int_0^\tau e^{-\frac{(gu+d)^2}{2u}} \left(\frac{g}{\sqrt{u}} - \frac{d}{\sqrt{u^3}}\right) du \\
&= \frac{(b+g)e^{(b-g)d}}{2g} \int_0^\tau dN\left(\frac{gu - d}{\sqrt{u}}\right) + \frac{(b-g)e^{(b+g)d}}{2g} \int_0^\tau dN\left(\frac{gu + d}{\sqrt{u}}\right)
\end{aligned}$$

$$= \frac{(b+g)e^{(b-g)d}}{2g} N\left(\frac{gu-d}{\sqrt{u}}\right)\Bigg|_0^\tau + \frac{(b-g)e^{(b+g)d}}{2g} N\left(\frac{gu+d}{\sqrt{u}}\right)\Bigg|_0^\tau.$$

Thus, if $d > 0$,

$$\int_0^\tau e^{-cu} \frac{\partial}{\partial u} N\left(\frac{bu-d}{\sqrt{u}}\right) du = \frac{(b+g)e^{(b-g)d}}{2g} N\left(\frac{g\tau-d}{\sqrt{\tau}}\right) - \frac{(b-g)e^{(b+g)d}}{2g} N\left(-\frac{g\tau+d}{\sqrt{\tau}}\right)$$
$$\int_0^\tau e^{-cu} \frac{\partial}{\partial u} N\left(\frac{bu+d}{\sqrt{u}}\right) du = \frac{(b-g)e^{-(b+g)d}}{2g} N\left(\frac{g\tau-d}{\sqrt{\tau}}\right) - \frac{(b+g)e^{-(b-g)d}}{2g} N\left(-\frac{g\tau+d}{\sqrt{\tau}}\right).$$

Back to the evaluation of $B^{\text{rec}}(t,T)$, assuming $b^2+2(a-r)>0$. If $d(t) = \frac{1}{\sigma_V}\left(\log\frac{H_0}{V_0} + at\right) > 0$,

$$\begin{aligned}
B^{\text{rec}}(t,T) &= -H_0 e^{at}\left[\frac{(b+g)e^{(b-g)d(t)}}{2g} N\left(\frac{g\tau-d(t)}{\sqrt{\tau}}\right) - \frac{(b-g)e^{(b+g)d(t)}}{2g} N\left(-\frac{g\tau+d(t)}{\sqrt{\tau}}\right)\right] \\
&\quad + H_0 e^{at+2bd(t)}\left[\frac{(b-g)e^{-(b+g)d(t)}}{2g} N\left(\frac{g\tau-d(t)}{\sqrt{\tau}}\right) - \frac{(b+g)e^{-(b-g)d(t)}}{2g} N\left(-\frac{g\tau+d(t)}{\sqrt{\tau}}\right)\right] \\
&= -H_0 e^{at}\left[\frac{(b+g)e^{(b-g)d(t)}}{2g} N\left(\frac{g\tau-d(t)}{\sqrt{\tau}}\right) - \frac{(b-g)e^{(b+g)d(t)}}{2g} N\left(-\frac{g\tau+d(t)}{\sqrt{\tau}}\right)\right] \\
&\quad + H_0 e^{at}\left[\frac{(b-g)e^{(b-g)d(t)}}{2g} N\left(\frac{g\tau-d(t)}{\sqrt{\tau}}\right) - \frac{(b+g)e^{(b+g)d(t)}}{2g} N\left(-\frac{g\tau+d(t)}{\sqrt{\tau}}\right)\right] \\
&= -H_0 e^{at}\left[e^{(b-g)d(t)} N\left(\frac{g(T-t)-d(t)}{\sqrt{T-t}}\right) + e^{(b+g)d(t)} N\left(-\frac{g(T-t)+d(t)}{\sqrt{T-t}}\right)\right] < 0.
\end{aligned}$$

If instead $d(t) = \frac{1}{\sigma_V}\left(\log\frac{H_0}{V_0} + at\right) < 0$, which is the more relevant case because $H(T) < F$,

$$\begin{aligned}
B^{\text{rec}}(t,T) &= -H_0 e^{at}\left[\frac{(b-g)e^{(b+g)d(t)}}{2g} N\left(\frac{g\tau+d(t)}{\sqrt{\tau}}\right) - \frac{(b+g)e^{(b-g)d(t)}}{2g} N\left(-\frac{g\tau-d(t)}{\sqrt{\tau}}\right)\right] \\
&\quad + H_0 e^{at+2bd(t)}\left[\frac{(b+g)e^{-(b-g)d(t)}}{2g} N\left(\frac{g\tau+d(t)}{\sqrt{\tau}}\right) - \frac{(b-g)e^{-(b+g)d(t)}}{2g} N\left(-\frac{g\tau-d(t)}{\sqrt{\tau}}\right)\right] \\
&= -H_0 e^{at}\left[\frac{(b-g)e^{(b+g)d(t)}}{2g} N\left(\frac{g\tau+d(t)}{\sqrt{\tau}}\right) - \frac{(b+g)e^{(b-g)d(t)}}{2g} N\left(-\frac{g\tau-d(t)}{\sqrt{\tau}}\right)\right] \\
&\quad + H_0 e^{at}\left[\frac{(b+g)e^{(b+g)d(t)}}{2g} N\left(\frac{g\tau+d(t)}{\sqrt{\tau}}\right) - \frac{(b-g)e^{(b-g)d(t)}}{2g} N\left(-\frac{g\tau-d(t)}{\sqrt{\tau}}\right)\right] \\
&= H_0 e^{at}\left[e^{(b-g)d(t)} N\left(-\frac{g(T-t)-d(t)}{\sqrt{T-t}}\right) + e^{(b+g)d(t)} N\left(\frac{g(T-t)u+d(t)}{\sqrt{T-t}}\right)\right] > 0.
\end{aligned}$$

where $g \triangleq \sqrt{b^2+2(a-r)}$.

3 Cox Process with Normal Intensity

Consider the Cox model where the intensity follows the normal process

$$d\lambda(t) = \theta dt + \sigma dW(t),$$

with constant θ and $\lambda(t=0) = \lambda_0$.
(i) Calculate the survival probability $S(0,t)$ for this Cox process.
Solution: Integrate the normal process, $\lambda(t) = \lambda_0 + \theta t + \sigma W(t)$. Integrate again with respect to time,

$$\int_0^t \lambda(s)ds = \lambda_0 t + \frac{1}{2}\theta t^2 + \sigma \int_0^t W(s)ds \sim N\left(\lambda_0 t + \frac{1}{2}\theta t^2, \frac{1}{3}\sigma^2 t^3\right),$$

where the stochastic version of Fubini's thoerem is used to compute the probability distribution of the integral $\int_0^t W(s)ds$. The survival probability

$$\begin{aligned} S(0,t) &\triangleq \mathbb{E}\left[e^{-\int_0^t \lambda(s)ds}\right] \\ &= \mathbb{E}\left[e^{-\lambda_0 t - \frac{1}{2}\theta t^2 - \sigma \int_0^t W(s)ds}\right] \\ &= e^{-\lambda_0 t - \frac{1}{2}\theta t^2 + \frac{1}{6}\sigma^2 t^3}, \end{aligned}$$

where we have used $\mathbb{E}\left[e^{-X}\right] = e^{-\mathbb{E}[X] + \frac{1}{2}\mathrm{Var}[X]}$ for a normal variable X.

(ii) Design and implement an algorithm for simulating events from this Cox process.

```
import math
import numpy as np
print("-------------------------")
print("Simulation of Cox Process")
print("-------------------------")
# input parameters
lambda0 = 0.03 # initial intensity
theta = 0.005 # intensity drift
sigma = 0.008 # intensity diffusion
t = 10.0 # time horizon
ns = 1000000 # number of samples
dt = 0.001 # time step width
timer = np.array([1,2,3,4,5,6,7,8,9,10]) # time points to record counts

# derived parameters
n = int(t/dt) # time step number
dt_root = math.sqrt(dt) # square root of time step
theta_dt = theta*dt
indexer = (timer/dt).astype(int);
m = len(timer)
stats = np.zeros(m)
```

```python
# simulation
for k in range(ns):
    timesteps = np.zeros(n+1)
    lambdas = np.zeros(n+1)
    lambdas[0] = lambda0
    counts = np.zeros(n+1)
    for i in range(n):
        # march forward in time
        timesteps[i+1] = timesteps[i] + dt
        # generate intensity process
        normal = np.random.normal(0, dt_root)
        lambdas[i+1] = lambdas[i] + theta_dt + sigma*normal
        # generate counting process
        lambda_dt = lambdas[i+1]*dt
        uniform = np.random.random()
        if uniform <= lambda_dt: counts[i+1] = counts[i]+1
        else: counts[i+1] = counts[i]

    # generate statistics
    for i in range(m):
        if counts[indexer[i]] > 0:
            stats[i] += 1
```

(iii) Verify the correctness of your algorithm by comparing the estimated survival probabilities with the closed form formula. Use $\lambda_0 = 0.03$, $\theta = 0.005$, $\sigma = 0.008$, and a range of values of t, say $t = 1, 2, \cdots, 10$.

Solution: We sample a number $N_s = 1{,}000{,}000$ of $\lambda(t)_{0\le t\le T}$ paths and simulate the counting process conditioned on each $\lambda(t)$ path. The width of each time step is chosen to be $\Delta t = 0.001$. The results are reported in the following table. The column "Hit Events" records the number of sample paths that end up with a default, and the vice versa for column "Miss Events". The estimated survival rates (ratio of Miss Events and N_s) tabulated in the column "$S(0,t)$ Estimate" match the theoretical values of the survival rate up to the third significant digits. Using a coarser time step $\Delta t = 0.01$ yields the same level of accuracy with $1{,}000{,}000$ sample paths. This indicates that the cause of the error is primarily statistical $\sim \frac{1}{\sqrt{N_s}} \approx 0.001$.

t	$S(0,t)$ Theoretical	Hit Events	Miss Events	$S(0,t)$ Estimate	Approximation Error
1	0.968033	31788	968212	0.968212	0.0185%
2	0.932473	67308	932692	0.932692	0.0234%
3	0.893855	106099	893901	0.893901	0.0051%
4	0.852726	147220	852780	0.852780	0.0063%
5	0.809639	190362	809638	0.809638	-0.0001%
6	0.76514	234901	765099	0.765099	-0.0054%
7	0.719757	280106	719894	0.719894	0.0190%
8	0.673991	325597	674403	0.674403	0.0611%
9	0.628308	371545	628455	0.628455	0.0234%
10	0.583137	416500	583500	0.583500	0.0623%