On the Class of Elliptical Distributions and their Applications to the Theory of Portfolio Choice

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# On the Class of Elliptical Distributions and their Applications to the Theory of Portfolio Choice

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**ABSTRACT**

It is shown that the class of elliptical distributions extend the Tobin [14] separation theorem, Bawa's [2] rules of ordering uncertain prospects, Ross's [12] mutual fund separation theorems, and the results of the CAPM to non-normal distributions, which are not necessarily stable. Further, the mean-covariance matrix framework is generalized to a mean-characteristic matrix framework in which the characteristic matrix is the basis for a spread or risk measure, and a generalized equilibrium pricing equation is arrived at. The implications to empirical testing of the CAPM and modeling the empirical distribution of speculative prices are discussed.

THE PURPOSE OF THIS paper is to describe the class of elliptical distributions and delineate their relevance to portfolio theory and its empirical applications. The class of elliptical distributions contains the multivariate normal (multinormal, henceforth) distribution as a special case; as well as many non-normal multivariate distributions including the multivariate Cauchy, the multivariate exponential, a multivariate elliptical analog of Student's $t$-distribution (the multivariate $t$, henceforth[1]), and non-normal variance mixtures of multinormal distributions. We show that some of the general characteristics of all members in the class of elliptical distributions make them admissible candidates for generalizing the theory of portfolio choice and equilibrium in capital asset markets, as well as attractive alternatives for use in empirical studies. More specifically, elliptical distributions are characterized by two parameters and extend the Tobin's [14] separation theorem, Bawa's [2] rules of ordering uncertain prospects, Ross's [13] mutual fund separation theorems, and the results of the CAPM to non-normal distributions, which are not necessarily stable.

The definitions and properties of elliptical distributions are in Section I. The main results are in Section II and the importance of elliptical distributions in empirical works is discussed in Section III.

## I. Elliptical Distributions (E-D), Definitions, and some Properties

There are several equivalent characterizational definitions of random variables which follow E-D:[2] let $\Delta$ be a fixed $p$-component vector and $\Omega$ a $(p \times p)$ positive

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[1] This is the so called Dunnett-Sobel multivariate $t$; it is also called the central multivariate $t$; see Johnson and Kotz [9] p. 134.

[2] This section relies on Kelker [10], Devlin et al., [6], and the references therein.

definite symmetric matrix. Then, a $p$-component random vector $X = (X_1, X_2, \cdots, X_p)'$ is said to be distributed elliptically, $X \sim e\,(\Delta, \Omega)$:

*Definition (a)* if and only if the characteristic function of $X$ is of the form: $E\{\text{EXP}(it'X)\} \equiv c_X(t) = \Psi(t'\Omega t)\text{EXP}(it'\Delta)$; ($E$ denotes the expectation operator, $i = \sqrt{-1}$, and a prime denotes transpose, throughout). Thus, $c_X(t)$ depends only on the quadratic form $t'\Omega t$.

*Definition (b)* if and only if for all nonzero $p$-component scalar vectors $\alpha$, all the univariate random variables $\alpha'X$ such that $\text{VAR}(\alpha'X)$ is constant, follow the same distribution.

*Definition (c)* if $X$ has a density function $f_X(x)$ say, then $f_X(x)$ can be expressed in the form of $f_X(x) = c_p\,|\,\Omega\,|^{-1/2}\xi((x - \Delta)'\Omega^{-1}(x - \Delta))$, for some function $\xi$, where $\xi$ is independent of $p$. Thus, $f_X(x)$ is only a function of the quadratic form $(x - \Delta)'\Omega^{-1}(x - \Delta)$, which is positive by definition.

For example, the multinormal density is given by $n_X(x) = (2\pi)^{-p/2}\,|\,\Omega\,|^{-1/2}\text{EXP}[-\tfrac{1}{2}(x - \Delta)'\Omega^{-1}(x - \Delta)]$.[3] Let $c_p^n \equiv (2\pi)^{-p/2}$ and $\xi(\cdot)$ be the exponential function $\text{EXP}(\cdot)$, then $n_X(x)$ is elliptical by definition ($c$). The same holds for the multivariate $t$ with $n$ degrees of freedom:

$$t_X(x) = c_p^t\,|\,\Omega\,|^{-1/2}\left[1 + \frac{1}{n}\,(x - \Delta)'\Omega^{-1}(x - \Delta)\right]^{-(n+p)/2},$$

where $c_p^t$ is the constant of integration and $\xi(\cdot)$ is the negative $(n + p)/2$ power function. Finally, the characteristic functions of symmetric stable distributions are given by $C_S(s) = \text{EXP}[-\,(\sum_{j=1}^{p} s_j^2)^{\alpha/2}]$; $0 < \alpha \leq 2$. $C_S(s)$ clearly fits Definition (a) with $\Omega = 1$ and $\Delta = 0$; thus the symmetric stable distributions are elliptical.

Several remarks are appropriate at this point. First, it should be noted that in general $\Delta$ need not necessarily be a vector of expectations and $\Omega$ need not necessarily be a covariance matrix (see P. 3 below). Second, observe that Definition (b) is similar to the definition of stable distributions. However, from Definitions (a) and (c) it is clear that there are non-normal members of the class of E-D which admit an arbitrary number of moments and therefore, are not stable.[4] Third, observe that Definitions (a) and (c) imply that different distributions in the E-D class may possess the same parameters $\Delta$ and $\Omega$. Fourth, it can be shown that the density of an elliptically distributed random vector can always be presented as a nondegenerate variance mixture of (at least two) normal densities. Finally, most elliptical distributions possess density functions and when the latter exist, their contours are of the same shape as the multinormal density; namely, elliptical. However, these densities are more flexible than the normal density in that they allow longer or shorter tails. Thus, the usual heavy (fat) tails of most members of the class of elliptical distributions make them

[3] For all the examples in this paragraph see Johnson and Kotz [9].

[4] For the definition and mathematical development of properties of stable distributions, see Feller [8]. A complete characterization of the overlap and nonoverlap which exists between stable and elliptical distributions is beyond the scope of this work, and for the most part, is open for future research.

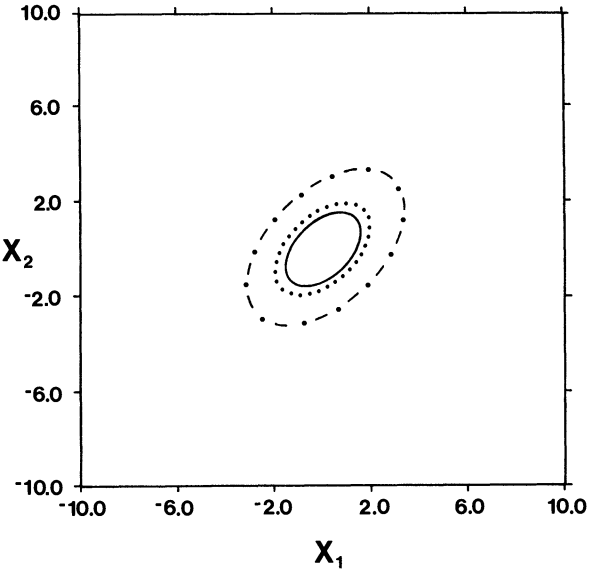


**Figure 1.** Plot of 0.70 probability contours of bivariate normal (———), bivariate $t_3$ (. . . . . .), and bivariate Cauchy (–·–·–·). In all three cases $\Delta = (0, 0)'$ and $\Omega = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}$.

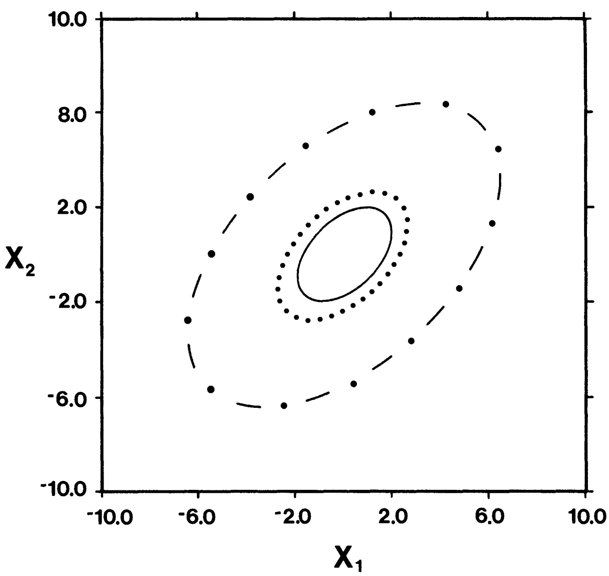


**Figure 2.** Plot of 0.85 probability contours of bivariate normal (———), bivariate $t_3$ (. . . . . .), and bivariate Cauchy (–·–·–·). In all three cases $\Delta = (0, 0)'$ and $\Omega = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}$.

natural candidates in modeling the empirical distribution of speculative prices (returns). Figures 1 and 2 may provide an intuitive feel for elliptical distributions. In all the three cases, $X = (X_1 X_2)'$, $\Delta = (0, 0)'$, and $\Omega = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}$. The graphs clearly indicate the "fat" tails of the bivariate $t$ with three degrees of freedom, $t_3$, and Cauchy distributions relative to the bivariate normal.[5]

The properties which lend the class of E-D to portfolio analysis are listed below:

(*P.1*) if $X \sim e(\Delta, \Omega)$, then for any given $(mxp)$ matrix $D$ of rank $m (m \leq p)$ the random vector $Y = DX$ is also elliptically distributed with characteristic matrix $D\Omega D'$; i.e., $Y \sim e(D\Delta, D\Omega D')$. It follows that any linear combination of the components of $X$ is elliptical. In addition, any subset of $x$'s $(X_1, X_2, \cdots, X_q)$: $q < p$ is also elliptical.

(*P.2*) Regression is linear in this class of elliptical distributions. Formally, let $X \sim e(\Delta, \Omega)$, $X = (X_1', X_2')'$ such that $X_1$ is a $q$-component random vector $(q < p)$. Let $\Delta_1$, $\Delta_2$, $\Omega_{11}$, $\Omega_{21}$, and $\Omega_{22}$ be the appropriate partitions of $\Delta$ and $\Omega$ such that $\Omega_{11}$ is a $(qxq)$ matrix and $\Omega_{22}$ is positive definite. Then, if the conditional mean of $X_1$ given $X_2 = x_2$ exists, it is given by $E(X_1 \mid X_2 = x_2) = \Delta_1 + \Omega_{12}\Omega_{22}^{-1}(x_2 - \Delta_2)$. In particular

(*P.2*a) If $X_1$ and $X_2$ are uncorrelated, then $E(X_1 \mid X_2 = x_2) = 0 = E(X_1)$. In words $X_1$ and $X_2$ are semi-independent or independent in the conditional sense if they are uncorrelated.

(*P.3*) Suppose an E-D possess $k$ moments. If $k \geq 1$, then the mean of the distribution is $\Delta$. If $k \geq 2$, then the covariance matrix $\Sigma$ exists and $\Sigma = \gamma\Omega$ where $\gamma$ is a nonnegative scalar parameter which is independent on $\Delta$ and $\Omega$. Observe that if $\Delta = 0$ and $\Omega = I_{(p)}$, then $\gamma$ is the variance of the univariate marginal distribution. Further, from the properties of the characteristic function $\Psi$, it follows that $\gamma = 2\Psi'(0)$, where $\Psi'(0) \equiv \partial\Psi(t'\Omega t)\partial t \mid_{t=0}$.

## II. Elliptical Distributions and Portfolio Theory

In this section, we show that E-D generalize the mean-variance analysis to non-normal distributions. In particular, the Tobin and Ross separation theorems hold for risk averse investors, provided that prices follow an E-D; Bawa's optimal rules of ordering uncertain prospects hold for E-D since these distributions are location-scale and finally, the CAPM results hold with a little change in interpretation.

### *A. Tobin's Separation Theorem*

Tobin [14] proved the first separation theorem that stated the conditions under which the optimal choice of risky investments (for any risk averse investor) is independent of the investor's wealth. Tobin recognized that separation holds for any stochastic return generating process if investors' utility functions are quadratic in wealth; or, for any utility function if the stochastic return generating process is multinormal. It is well known that Tobin's ([14], Sec. 3.3) conjecture

[5] In general, these probability contours are obtained as follows: let the $p$-dimensional random vector $X \sim e(0, \Omega)$. The probability contours are the loci of all the points $X$ that satisfy $\Pr\{X'\Omega^{-1}X \leq r^2\} = \beta$, for a fixed $\beta$; $0 < \beta < 1$. To solve this equation, we set $p = 2$ and employed the density function of $X'\Omega^{-1}X$ provided by Kelker [10] p. 429.

that separation holds for any two parameter family of distributions is incorrect, (see e.g., Samuelson [13] or Feldstein [7]), since not all these distributions are closed under linear transformation. However, we now show that it holds true for elliptical distributions.[6]

Property (P.1) clearly states that any linear transformation of an elliptical random vector is also elliptical; the form may be different but it still is an E-D. Thus, if $X \sim e(\Delta; \Omega)$ is the vector of random returns of risky securities and $D$ is the row vector $D \equiv \Pi' = (\Pi_1, \cdots, \Pi_p)$ of proportion holdings in the risky securities, then $Y = \Pi'X$ is the univariate random return of the portfolio which according to (P.1) follows a unidimensional elliptical distribution with parameters $\Pi'\Delta$ and $\Pi'\Omega\Pi$. It is well known that (P.1) holds for a multinormal random vector; however, we see that it also holds for all other members of the class of elliptical distributions. Moreover, since the risk measure $\Pi'\Omega\Pi$ is positively proportional to the variance, ranking of portfolios by risk averse investors retain their ordering. Thus, we have just shown that:

*Proposition 1:* Any elliptically distributed random variable with a finite mean vector satisfies the distributional conditions of Tobin's separation theorem.

Observe that Proposition 1 holds regardless of whether or not the distribution of $Y$ is the same as the E-D of $X$ since the former is an E-D which depends only on the parameters $\Pi'\Delta$ and $\Pi'\Omega\Pi$.

### B. Bawa's Optimal Rules of Ordering Uncertain Prospects

Bawa [2] has analyzed the relationship between various degrees of stochastic dominance and investors ranking of uncertain returns. In particular, he discussed the importance of the family of distributions which are characterized by location and scale parameters (Bawa [2] Section 5). First, we extend Bawa's univariate definition of location-scale distributions to the multivariate case as follows: let $X$ be a random vector whose distribution depends on the parameters (vector) $\Delta$ and (positive definite matrix) $\Omega$. Let the univariate $Z$ be defined for some positive scalar $k$ and any fixed vector $\alpha$: $Z = [\alpha'X - E(\alpha'X)]/[k\sqrt{\alpha'\Omega\alpha}]$, assume that the density of $Z$ exists and denote it by $f\{z, \alpha\}$.

*Definition (d).* The random vector $X$ is said to possess a location-scale distribution if and only if $f\{z, \alpha\}$ does not depend on $\alpha$.

We now show that location-scale distributions and elliptical distributions are synonymous when densities exist.

*Proposition 2:* Let the density of $X$ exist. Then the random vector $X$ possesses a scale-location-parameter distribution iff $X$ has an elliptical distribution.

*Proof* (necessity): Let $f\{z, \alpha\} = f\{z\}$. Then, for all $\alpha: \alpha \neq 0$

$$f\{\alpha'X\} = (k\sqrt{\alpha'\Omega\alpha})^{-1} \cdot f\left\{\frac{\alpha'X - E(\alpha'X)}{k\sqrt{\alpha'\Omega\alpha}}\right\}$$

$$= c_p(\alpha'\Omega\alpha)^{-1/2} \cdot f\{\alpha'(X - \Delta) \cdot (\alpha'\Omega\alpha)^{-1} \cdot (X - \Delta)'\alpha\},$$

[6] We note that the two examples provided by Agnew [1] are elliptical.

which implies that $f$ is elliptical by Definition (c). (Sufficiency) Let $X \sim e\,(\Delta; \Omega)$ with a density given by Definition (c). The distribution of $\alpha' X$, say $f$, can be transformed as above to a function of $z$ alone, with $f$ independent of $\alpha$. Notice that $k = +\sqrt{\gamma}$ of property (P.3).

Two remarks are in place: (1) the existence of the density is only a minor restriction since most E-D possess densities (see Kelker [10] p. 421). (2) When the covariance matrix exists $k(\alpha'\Omega\alpha)^{1/2}$ is the standard deviation. This property together with the fact that $f\{z\}$ is independent of $\alpha$ is sufficient to characterize elliptical distributions. This is tantamount to observing that the class of E-D in itself is the multivariate extension of the univariate location-scale-parameters distributions discussed by Bawa [2].

### *C. Ross's Mutual Fund Separation Theorems*

Ross [12] has characterized the class of distributions that generalize the mean-variance model and provided conditions under which one, two and $k$ fund separability hold. Distributions which satisfy these conditions are called the separating distributions. For example,

*Theorem 3:* (Ross [12] p. 261). A vector of asset returns, $X$, exhibits one fund separation iff $\exists$ random variables $Y$ and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)'$ and a scalar vector $\alpha$: (i) $X_i = Y + \varepsilon_i \quad i = 1, \cdots, n$; (ii) $E(\varepsilon_i \mid Y) = 0$ a.e.; and (iii) $\alpha'\varepsilon = 0; \quad \alpha'\mathbf{1} = 1$.

($\mathbf{1} = (1, \cdots, 1)'$ a vector of ones of the appropriate order). We now prove

*Proposition 4:* A vector of returns $X$ exhibits one fund separation if $X - EX$ has an elliptical distribution.

*Proof:* Define a scalar vector $\alpha$: $\alpha'\mathbf{1} = 1$ and the random vector $\varepsilon$: $\varepsilon = (I - \mathbf{1}\alpha')(X - EX)$. Clearly $\alpha'\varepsilon = 0$. Moreover, by rearranging terms $X - EX = \mathbf{1}\alpha'(X - EX) + \varepsilon$. Thus conditions (i) and (iii) of Theorem 3 hold. To show that condition (ii) holds define the random variable $Y$: $Y = \alpha'(X - EX)$, then by (P.2) $E(\varepsilon \mid Y) = \Omega_{\varepsilon Y}(Y - EY)/(V_Y)$ where $\Omega_{\varepsilon Y}$ and $V_Y$ are (obvious) partitions of $\Omega$. But since $\Omega_{\varepsilon Y} = \mathrm{cov}[(I - \mathbf{1}\alpha')(X - EX), \alpha'(X - EX)]$ we have that $E(\varepsilon \mid Y) = (I - \mathbf{1}\alpha')\Omega\alpha = \Omega A - \mathbf{1}\alpha'\Omega\alpha$. Let $\alpha = \Omega^{-1}\mathbf{1}/\mathbf{1}'\Omega^{-1}\mathbf{1}$ then $E(\varepsilon \mid Y) = 0$.[7]

Stronger results can be proved with the inclusion of a risk free asset.

*Proposition 5:* Two fund separation holds for elliptical distributions.

*Proposition 6:* $k$ fund separation holds for elliptical distributions.

The proofs of these propositions are extensions of the above proof in two and $k$ dimensions and are provided in Owen and Rabinovitch [11].

### *D. Elliptical Distributions and Equilibrium in the Capital Asset Market*

The current theory of capital asset pricing and its empirical tests rely on the assumption that prices (or returns) are multinormal random variables. Thus, the valuation results of the CAPM emphasize the role of the mean and covariance in determining equilibrium prices and portfolios. However, following the discus-

[7] Note that the $\alpha$ we chose is the well known vector of weights of the minimum variance portfolio.

sions in Sections II.B and II.C, we now derive an equilibrium capital asset pricing model which constitutes a generalization of the mean-variance CAPM. Let securities' prices $P$ follow some E-D such that all investors possess homogeneous expectations regarding the parameters $\Delta$ and $\Omega$, and retain all the other assumptions underlying the CAPM. Then investor $i$'s utility function $U_i = U_i(\Pi'\Delta, \Pi'\Omega\Pi)$, where now $\Pi$ is the vector of the individual holdings (i.e., $\Pi_{ij}$ is the number of shares of stock $j$ held by individual $i$.) The generalization is in two directions. First, $\Omega$ need not be a covariance matrix but its components are some measure of co-spread and spread of $P$ around $\Delta$. As Bawa points out (Bawa [2], p. 117) "the scale parameter (and not the variance) is the natural measure of dispersion . . . ;" thus $\Pi'\Omega\Pi$ is a measure of risk that suits the model.[8] Second, the generalized CAPM does not require investors' full agreement upon the specific generating elliptical distribution. Let investor $i$ assume that prices of risky securities follow some E-D, $e_i(\Delta, \Omega)$. Then his expected utility of end-of-period wealth $(W_{1i})$, given initial Capital $w_{0i}$ and a price per share of the risk free asset $p_0$, is given by

$$E(U_i\{W_{1i}\}) = E(U_i\{p_0 w_{0i} + \Pi'(P - p_0 P_e)\}) = u_i\{p_0 w_{0i} + \Pi'\Delta, \Pi'\Omega\Pi\}$$

The different E-D for different investors just changes the form of the functions $u_i$, but not the fact that $u_i$ depends on $\Pi$ through $\Pi'\Delta$ and $\Pi'\Omega\Pi$, which are the same for all investors. For example, investor $i$ may assume a multivariate $t$-distribution with $(\Delta, \Omega)$ and one degree of freedom and investor $j$ a multinormal $(\Delta, \Omega)$ distribution; both $u_i$ and $u_j$ depend on $\Pi'\Delta$ and $\Pi'\Omega\Pi$ which are the parameters of an elliptical distribution, and are the same for both investors.

If we now set the familiar maximum expected utility model under the individual's budget constraint and apply the market clearing condition, we obtain the equilibrium pricing equation and the optimal (equilibrium) individual holdings in this generalized CAPM: $P_e^0 = (\Delta - \lambda_1\Omega Q)/(1 + R_F)$, and $X_i^* = a_i Q$, where $Q$ is the vector representing the number of outstanding shares of risky securities, $R_F$ is the risk free rate, and $a_i$ is a measure of the individual risk tolerance. We see that equilibrium prices remain a linear function of the parameters $\Delta$ and $\Omega$ and equilibrium portfolios do not change relative to the CAPM. However, $\Omega Q$ is not necessarily a covariance. If the covariance matrix exists, then we have $p_e^1 = (\Delta - (\lambda_1/\gamma)\Sigma Q)/(1 + R_F)$, by (P.3). Thus, we conclude that the results of the CAPM are quite robust under the generalized setting suggested here. The coefficient $\lambda_1/\gamma$ can be interpreted as the market price of risk but the market portfolio remains unchanged.[9]

## III. Concluding Remarks

We have shown that the class of elliptical distributions generalizes the distributional framework of portfolio theory to non-normal and nonstable distributions. This class extends Tobin's separation theorem to a broad class of two parameter

[8] Ross ([12], p. 271) also emphasizes that the two fund separation property does not specify a unique spread function as a measure of risk.

[9] Brown [4] discusses a similar phenomenon. Notice, however, that while the market portfolio remains unchanged, the market price of risk $\lambda_1/\gamma$ depends on the specific elliptical distribution chosen for the model. We have assumed here that all investors hold the same E-D.

distributions that are also location-scale-parameter distributions in the sense of Bawa, and are part of the class of separating distributions in the sense of Ross. Finally, employing elliptical distributions for prices (returns) of risky assets permits generalizing the mean-covariance matrix CAPM to a mean-characteristic matrix asset pricing model.

In closing, we emphasize the potential usefulness of E-D in modeling the empirical distribution of speculative prices (returns). On an empirical basis, Clark [5] showed that a subordinated lognormal normal process may fit the data better than a stable distribution, based on goodness of fit tests. A conclusion in the same spirit was reached by Blattberg and Gonedes [3] who showed that a unidimensional Student's $t$-distribution may outperform a stable distribution in explaining the empirical distribution of speculative returns. Since the shape of elliptical densities is flexible and allows for fat tails, this class provides a variety of possible multivariate models for speculative prices (returns). Of course, there is the problem of sparsity of general multivariate non-normal fitting methods and tables. Their development and the testing of such models is open for future research. Fitting an elliptical distribution to the empirical distribution of prices of risky assets will permit testing the CAPM pricing equation with a non-normal distribution.

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