MTH 9831 Lecture 10

Note Title

1. Asian options (continued: reduction of dimension through the change of numeraire.)
2. American options. Part 1. Perpetual American put.

1. BSM framework:

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{B}(t), \quad 0 \le t \le T.$$

$$dD(t) = -rD(t)dt \quad (D(t) = e^{-rt}, \ 0 \le t \le T)$$

$$d(D(t)S(t)) = \sigma D(t)S(t)d\tilde{B}(t), \quad 0 \le t \le T \qquad (1)$$

The time t value of the option is

$$V(t) = \tilde{\mathbb{E}}\left(e^{-r(T-t)}V(T) \,/\, \mathcal{F}(t)\right), \quad 0 \le t \le T.$$

($V(T)$ is the payoff), and

$$e^{-rt}V(t) = \tilde{\mathbb{E}}\left(e^{-rT}V(T) / \mathcal{F}(t)\right), \quad 0 \le t \le T,$$

is a $\tilde{\mathbb{P}}$-martingale.

We assume: $r > 0$ and $V(T) = \left(\frac{1}{c}\int_{T-c}^{T} S(t)dt - K\right)_+$

where c is a fixed constant, $0 < c \le T$.

The pricing will be done in 2 steps:

Step 1. Create a portfolio process $X(t)$ such that

$$X(T) = \frac{1}{c}\int_{T-c}^{T} S(t)\,dt - K. \qquad (2)$$

Step 2. Use 1 share of underlying as a unit, reduce the dimension, and set up a simpler PDE and boundary conditions than those in Lecture 9.

<u>Step 1</u>. Let $X(t) = \gamma(t)S(t) + (X(t) - \gamma(t)S(t))$, where $\gamma(t)$ is non-random and differentiable. Our goal is to find $\gamma(t)$ such that $X(T)$ matches (2).

$$d(e^{-rt}X(t)) = d(e^{-rt}\gamma(t)S(t) + e^{-rt}(X(t) - \gamma(t)S(t)))$$

$$= d(e^{-rt}\gamma(t)S(t)) - re^{-rt}(X(t) - \gamma(t)S(t))\,dt + e^{-rt}d(X(t) - \gamma(t)S(t))$$

$$= d(e^{-rt}\gamma(t)S(t)) - re^{-rt}(X(t) - \gamma(t)S(t))\,dt + e^{-rt}dX(t) - e^{-rt}d(\gamma(t)S(t)).$$

Recall that $dX(t) = \gamma(t)dS(t) + r(X(t) - \gamma(t)S(t))dt$ (under the self-financing condition)

Substituting this in the equation for $d(e^{-rt}X(t))$ we get

$$d(e^{-rt}X(t)) = d(e^{-rt}\gamma(t)S(t)) - e^{-rt}S(t)\,d\gamma(t) \quad (*)$$

Observe for future use that this and (1) imply

$$d(D(t)X(t)) = \gamma(t)\,\sigma\,D(t)S(t)\,d\tilde{B}(t) \qquad (3)$$

Integrate (*) from $T-c$ to T:

$$e^{-rT}X(T) - e^{-r(T-c)}X(T-c) = e^{-rT}\gamma(T)S(T) - e^{-r(T-c)}\gamma(T-c)S(T-c) - \int_{T-c}^{T} e^{-rt}S(t)\gamma'(t)\,dt.$$

$$X(T) = e^{rc}X(T-c) + \gamma(T)S(T) - e^{rc}\gamma(T-c)S(T-c) - \frac{1}{c}\int_{T-c}^{T} S(t)\,e^{-r(t-T)} \cdot c\,\gamma'(t)\,dt.$$

Compare with (2). To get a match we should set

$$-c\gamma'(t)\, e^{-r(t-T)} = 1 \;; \qquad T-c \le t \le T \qquad (4)$$

$$e^{rc} X(T-c) + \gamma(T) S(T) - e^{rc}\gamma(T-c) S(T-c) = -K \qquad (5)$$

From (4) we get

$$\gamma'(t) = -\frac{1}{c} e^{-r(T-t)}, \quad T-c \le t \le T,$$

$$\gamma(t) = \gamma(T-c) - \frac{1}{rc}\left(e^{-r(T-t)} - e^{-rc}\right). \qquad (6)$$

At time $T-c$ we know $X(T-c)$ and $S(T-c)$ but we can not deterministically control $S(T)$. Thus, the only way (5) can hold is when $\gamma(T) = 0$ and

$$e^{rc} X(T-c) - e^{rc}\gamma(T-c) S(T-c) = -K. \qquad (7)$$

From (6) :

$$0 = \gamma(T) = \gamma(T-c) - \frac{1}{rc}\left(1 - e^{-rc}\right)$$

$$\gamma(T-c) = \frac{1}{rc}\left(1 - e^{-rc}\right) \quad \text{and from (7)}$$

$$X(T-c) = \frac{1}{rc}\left(1 - e^{-rc}\right) S(T-c) - K e^{-rc}.$$

This corresponds to the following position at time $T-c$:

- long $\frac{1}{rc}\left(1 - e^{-rc}\right)$ shares of stock
- short $K e^{-rT}$ shares of MMA (at $e^{r(T-c)}$ per share).

What should we do at time 0 to ensure such position? We shall have this position if at time 0 we purchase

$$\frac{1}{rc}(1-e^{-rc}) \text{ shares by borrowing } e^{-rT}K$$

from a bank and just simply hold this portfolio up to time $T-c$ then we shall get what we want. Such portfolio at time 0 costs $X(0) = \frac{1}{rc}(1-e^{-rc})S(0) - e^{-rT}K$.

Our answer is : start with $X(0)$ in cash, make up a portfolio that has

$$\gamma(t) = \begin{cases} \frac{1}{rc}(1-e^{-rc}), & 0 \le t \le T-c \\ \frac{1}{rc}(1-e^{-r(T-t)}), & T-c \le t \le T \end{cases}$$

shares of stock. Then at time T we have (2)

Step 2. Now back to pricing. Under $\tilde{P}$

The price is $V(t) = \tilde{E}\left(e^{-r(T-t)}V(T) \mid \mathcal{F}(t)\right)$

$$= \tilde{E}\left(e^{-r(T-t)}(X(T))_+ \mid \mathcal{F}(t)\right)$$

Change of numeraire: define $Y(t) = \frac{X(t)}{S(t)}$.

Recall (1) and (3) :

$$d(D(t)X(t)) = \gamma(t)\sigma D(t)S(t)\,d\tilde{B}(t) = \frac{\gamma(t)\sigma}{Y(t)} D(t)X(t)\,d\tilde{B}(t)$$

$$d(D(t)S(t)) = \sigma D(t)S(t)\,d\tilde{B}(t).$$

By Theorem 3.3 of Lecture 7 we conclude that

$$dY(t) = Y(t)\left(\frac{\gamma(t)\sigma}{Y(t)} - \sigma\right)d\tilde{B}^S(t) = \sigma(\gamma(t) - Y(t))\,d\tilde{B}^S(t),$$

where $\tilde{B}^S(t) = \tilde{B}(t) - \sigma t$ is a BM under $\tilde{\mathbb{P}}^S$, and

$$\tilde{\mathbb{P}}^S(A) = \frac{1}{S(0)}\int_A D(T)S(T)\,d\tilde{\mathbb{P}} \quad \forall A \in \mathcal{F} = \mathcal{F}(T).$$

Since $Y(t)$, $0 \le t \le T$, satisfies $dY(t) = \sigma(\gamma(t) - Y(t))\,d\tilde{B}^S(t)$, it is a Markov process and a martingale under $\tilde{\mathbb{P}}^S$.

We have

$$V(t) = e^{rt}\tilde{\mathbb{E}}\left(e^{-rT}(X(T))_+ \mid \mathcal{F}(t)\right)$$

$$= e^{rt}\tilde{\mathbb{E}}\left(e^{-rT}S(T)\left(\frac{e^{-rT}X(T)}{e^{-rT}S(T)}\right)_+ \Big| \mathcal{F}(t)\right)$$

$$= e^{rt}S(0)\tilde{\mathbb{E}}\left(Z(T)\,(Y(T))_+ \mid \mathcal{F}(t)\right)$$

Lecture 5, Lemma 5.4
$$= e^{rt}S(0)Z(t)\tilde{\mathbb{E}}^S\left((Y(T))_+ \mid \mathcal{F}(t)\right)$$

$$= S(t)\tilde{\mathbb{E}}^S\left((Y(T))_+ \mid \mathcal{F}(t)\right).$$

Since Y is a Markov process, there is $g(t,y)$ such that $g(t, Y(t)) = \tilde{\mathbb{E}}^S((Y(T))_+ \mid \mathcal{F}(t))$, $0 \le t \le T$.

In particular, $g(T, Y(T)) = \tilde{\mathbb{E}}^S((Y(T))_+ \mid \mathcal{F}(T)) = (Y(T))_+$ and $g(T, y) = y_+$, $y \in \mathbb{R}$. (8)

(Recall that $Y(T) = \frac{X(T)}{S(T)}$, where $X(T)$ can be positive or negative.) As usual, by the tower property $g(t, Y(t))$ is a martingale under $\tilde{\mathbb{P}}^S$. Compute

$$dg(t,Y(t)) = \left(g_t(t,Y(t)) + \tfrac{1}{2}\sigma^2(\gamma(t)-Y(t))^2 g_{yy}(t,Y(t))\right)dt$$
$$+\sigma(\gamma(t)-Y(t))g_y(t,Y(t))\,d\tilde{B}^S(t).$$

We conclude that $g(t,y)$ has to solve

$$g_t(t,y) + \tfrac{1}{2}\sigma^2(\gamma(t)-y)^2 g_{yy}(t,y) = 0, \quad 0 \le t < T, y \in \mathbb{R}. \quad (9)$$

We have so far one boundary condition (8)
We need conditions at infinity to ensure uniqueness.
If $y \to -\infty$, then it means that $Y(t)$ is very negative, so the chances that $(Y(T))_+ > 0$ are going to 0. So in the limit $\tilde{\mathbb{E}}^S((Y(T))_+ \,|\, \mathcal{F}(t)) = 0$,

$$\text{and} \quad \lim_{y\to-\infty} g(t,y) = 0, \quad 0 \le t \le T. \quad (10)$$

If $y \to +\infty$ then the chances that $(Y(T))_+ > 0$ are going to 1, and in the limit

$$\tilde{\mathbb{E}}^S((Y(T))_+ \,|\, \mathcal{F}(t)) \simeq \tilde{\mathbb{E}}^S(Y(T)\,|\,\mathcal{F}(t)) = Y(t).$$

This gives

$$\lim_{y\to\infty} (g(t,y) - y) = 0, \quad 0 \le t \le T. \quad (11)$$

We have shown

<u>Theorem 1</u>. For $0 \le t \le T$, the price $V(t)$ at time t of the Asian call option paying

$$\left(\frac{1}{c}\int_{T-c}^{T} S(t)\,dt - K\right)_+ \quad \text{at time } T$$

is given by $\quad V(t) = S(t)\, g\left(t, \dfrac{X(t)}{S(t)}\right),$

where $g(t,y)$ satisfies (9) with boundary conditions (8), (10), (11), and $X(t)$ is a portfolio value process with $\gamma(t)$ given on p. 4.

Remark. Exactly the same steps can be used to price the Asian call with payoff

$$V(T) = \left(\frac{1}{m}\sum_{i=1}^{m} S(t_i) - K\right)_+, \quad 0<t_1<\ldots<t_m=T$$

2 Perpetual American put.

We shall need the following notions and facts:

1) Stopping time; first passage time.
   Fact: the first passage time of a continuous process is a stopping time, i.e. $\forall m \in \mathbb{R}$ $\tau_m := \inf\{t \geq 0 : X(t) = m\}$ is a stopping time with respect to the natural filtration of X.
   Proof (required reading): Shreve II, p. 341-2
2) Optional stopping theorem. Shreve II, p. 342.
3) Laplace transform of the first passage time for the drifted Brownian motion (see HW2 or Shreve II, Th. 8.3.2)

More precisely: Let $m>0$, $\mu \in \mathbb{R}$,
$X(t) = B(t) + \mu t$, and $\tau_m = \inf\{t \geq 0 ; X(t) = m\}$.
Then $\forall \lambda > 0$

$$\mathbb{E}\, e^{-\lambda \tau_m} = e^{m\mu - m\sqrt{\mu^2 + 2\lambda}} \qquad (12)$$

Assume the BSM model: under $\widetilde{\mathbb{P}}$ (risk-neutral)

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{B}(t),$$

$r, \sigma > 0$ (constant)

Consider a perpetual American put with strike K. Denote by $v_*(x)$ its price when $S(0) = x$.

Question: how to determine $v_*(x)$?

Notice that the decision to exercise the put at time t can only depend on the information up to time t. This tells us that the exercise time should be a stopping time. When is it optimal to exercise? When the risk-neutral expectation of the present value of the payoff is maximal, i.e. when τ (exercise time) is such that

$$\widetilde{\mathbb{E}}\left(e^{-\tau r}(K - S(\tau))\right) \text{ is maximized.}$$

Maximized over what? All possible stopping times. Thus we define

$$v_*(x) := \max_{\tau \in \mathcal{T}} \widetilde{\mathbb{E}}\left(e^{-\tau r}(K - S(\tau))\right)$$

where $\mathcal{T}$ is the set of all stopping times relative to the natural filtration generated by $(\widetilde{B}(t))_{t \geq 0}$.

This formula does not seem to help much, since we do not know how to maximize over all stopping times. How does one even start describing all possible stopping times?

This is where the "perpetuity" simplifies the problem. Since the option never expires, the decision to exercise should not depend on time to expiration (it is always ∞), it can only depend on the path. Since given $S(t)=x$ the future and the past of the process S are independent (Markov property), it seems reasonable to seek the optimal exercise time as the first time when the price falls to some level L_* ($<K$). If we look only at such stopping times then our problem is reduced to finding this optimal level L_*.

Let $\tau_L = \min\{t \geq 0 : S(t) = L\}$, $0 < L < K$.
and $v_L(S(0)) := \tilde{\mathbb{E}}\left(e^{-\tau_L r}(K - S(\tau_L))\right)$,
when $S(0) \geq L$

But $S(\tau_L) = L$, and for $0<L<K$ and $S(0) \geq L$

$$v_L(S(0)) = (K-L)\,\tilde{\mathbb{E}}\, e^{-\tau_L r}$$

If $S(0) < L$ then at time 0 the stock is already below L, and the option should be exercised immediately. Thus

$$v_L(S(0)) = \begin{cases} K - S(0) & \text{if } S(0) < L \\ (K-L)\,\tilde{\mathbb{E}}\, e^{-\tau_L r} & \text{if } S(0) \geq L. \end{cases}$$

When $S(0) = L$, $\tau_L = 0$, so

$$v_L(S(0)) = \begin{cases} K - S(0) & \text{if } S(0) \leq L \\ (K-L)\,\tilde{\mathbb{E}}\, e^{-\tau_L r} & \text{if } S(0) \geq L. \end{cases}$$

<u>Lemma 1</u>.

$$v_L(x) = \begin{cases} K - x & \text{if } x \in [0, L] \\ (K-L)\left(\dfrac{x}{L}\right)^{-\frac{2r}{\sigma^2}} & \text{if } x \geq L. \end{cases}$$

<u>Proof</u> We have already discussed the case $x \in [0, L]$. Assume now that $x > L$, i.e. $S(0) > L$. Then τ_L is the first time

$$S(t) = x\, e^{\sigma \tilde{B}(t) + (r - \frac{1}{2}\sigma^2) t}$$

reaches level L. So $S(t) = L$ iff

$$\sigma \tilde{B}(t) + (r - \tfrac{1}{2}\sigma^2) t = \log \frac{L}{x}$$

$$\underbrace{\tilde{B}(t) + \frac{r - \frac{1}{2}\sigma^2}{\sigma} t}_{\text{drifted BM}} = \underbrace{\frac{1}{\sigma} \log \frac{L}{x}}_{\text{level } m, \text{ but } m<0 \text{ as } x>L.}$$

This does not fit into the assumptions of (12). The change of sign will help:

$$\underbrace{-\tilde{B}(t)}_{\text{still BM}} + \underbrace{\left(\frac{1}{2}\sigma - \frac{r}{\sigma}\right)}_{\mu \in \mathbb{R}} t = \underbrace{\frac{1}{\sigma} \log \frac{x}{L}}_{= m > 0}$$

Using (12) with $\lambda = r$; $\mu = \frac{1}{2}\sigma - \frac{r}{\sigma}$; $m = \frac{1}{\sigma} \log \frac{x}{L}$ we get

$$\tilde{\mathbb{E}}\, e^{-\tau_L r} = e^{m\mu - m\sqrt{\mu^2 + 2r}}$$

$$\mu^2 + 2r = \frac{\sigma^2}{4} - r + \frac{r^2}{\sigma^2} + 2r = \left(\frac{\sigma}{2} + \frac{r}{\sigma}\right)^2$$

$$m\mu - m\sqrt{\mu^2 + 2r} = \frac{1}{\sigma} \log \frac{x}{L} \left(\frac{1}{2}\sigma - \frac{r}{\sigma} - \frac{\sigma}{2} - \frac{r}{\sigma}\right)$$

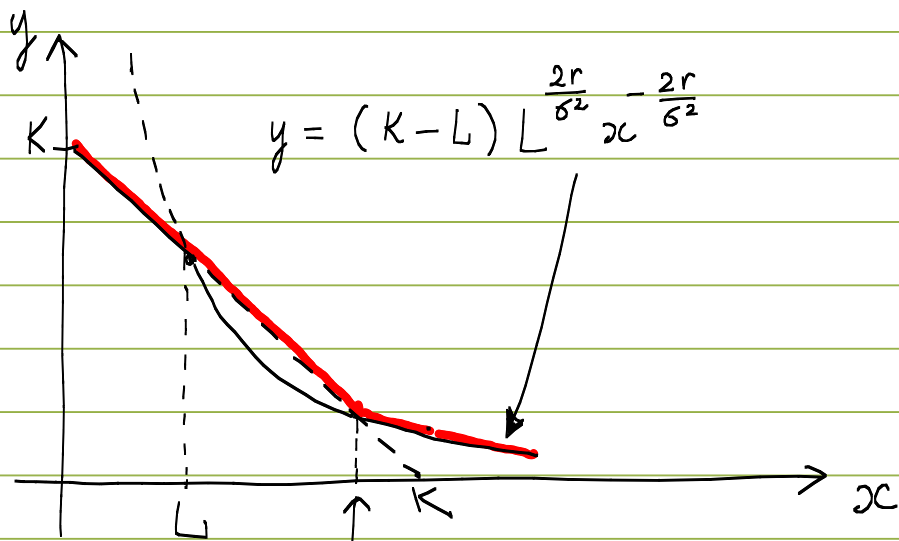
$$= -\frac{2r}{\sigma^2} \log \frac{x}{L}, \text{ and}$$

$$\tilde{\mathbb{E}}\, e^{-\tau_L r} = \left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}.$$

■

Now we have to find out which level L^* is optimal, i.e. for which L the function $v_L(x)$ is the largest. It could be that for different x we get different L^*, but, as we shall see, this does not happen.

$$v_L(x) = \begin{cases} K - x & \text{for } x \in [0, L] \\ (K-L)L^{\frac{2r}{\sigma^2}} x^{-\frac{2r}{\sigma^2}} & \text{for } x \geq L \end{cases}$$



Corresponds to another value, L', such that $(K-L)L^{2r/\sigma^2} = (K-L')L'^{2r/\sigma^2}$.

Notice that in our situation $L' > L$ and

$$v_{L'}(x) \geq v_L(x) \text{ for all } x > 0.$$

Set $g(L) = (K-L)L^{2r/\sigma^2}$ and maximize it over $L \in [0, K]$.

$$g'(L) = \frac{2r}{\sigma^2} K L^{\frac{2r}{\sigma^2}-1} - \left(\frac{2r}{\sigma^2}+1\right) L^{\frac{2r}{\sigma^2}}$$

$$= L^{\frac{2r}{\sigma^2}-1}\left(\frac{2r}{\sigma^2}K - \frac{2r+\sigma^2}{\sigma^2}L\right) = 0$$

$$L_* = \frac{2rK}{2r+\sigma^2} \in (0, K) \qquad (r, \sigma > 0)$$

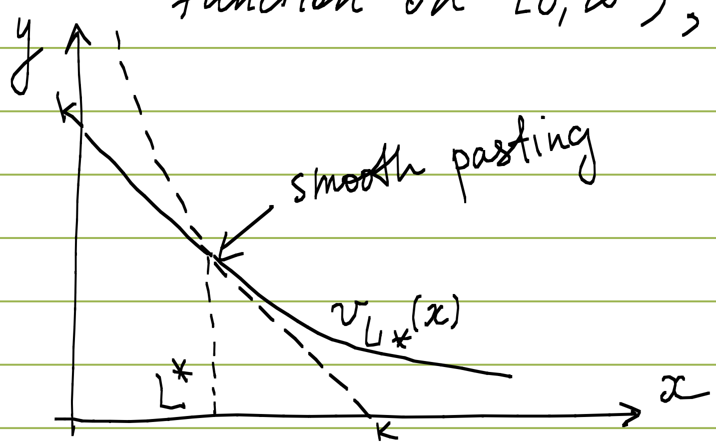
Since $g(0) = g(K) = 0$, $g(L) \geq 0$, and this is the only critical point, it is the point where $g(L)$ attains the absolute maximum on $[0, K]$. Putting this L_* in the equation for v_L we get

$$v_{L_*}(x) = \begin{cases} K - x, & x \in [0, L_*] \\ (K - L_*)\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}}, & x \geq L_* \end{cases}$$

$$v'_{L_*}(x) = \begin{cases} -1, & x \in [0, L_*] \\ -(K - L_*)\frac{2r}{\sigma^2}\left(\frac{1}{L^*}\right)\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}-1}, & x \geq L_* \end{cases}$$

$$v'_{L_*}(L_*+) = -\frac{(K - L_*)}{L_*}\frac{2r}{\sigma^2} = -\frac{K\,2r}{L^*\sigma^2} + \frac{2r}{\sigma^2} = -1$$

$v_{L_*}(x)$ is a continuously differentiable function on $[0, \infty)$, and $v'_{L_*}(L^*) = -1$.



It is obvious that $v''_{L_*}(x)$ can not be continuous and will have a jump at $x = L_*$.

$$v''_{L_*}(x) = \begin{cases} 0, & x \in [0, L_*] \\ (K - L_*)\frac{2r}{\sigma^2}\left(\frac{2r}{\sigma^2} + 1\right)\frac{1}{L_*^2}\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}-2}, & x \geq L_* \end{cases}$$

$v''_{L_*}(L_*+) > 0$.

For $x > L_*$ it can be verified by substitution that

$$r v_{L_*}(x) - r x v'_{L_*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L_*}(x)$$

$$= (K - L_*)\left(r + \frac{2r^2}{\sigma^2} - \frac{r(2r+\sigma^2)}{\sigma^2}\right)\left(\frac{x}{L_*}\right)^{-\frac{2r}{\sigma^2}} = 0 \qquad (13)$$

For $x \in [0, L_*)$

$$r v_{L_*}(x) - r x v'_{L_*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L_*}(x) \qquad (14)$$

$$= r(K - x) + rx = rK.$$

This implies that $v_{L_*}(x)$ satisfies "linear complementarity conditions"

- $v(x) \geq (K - x)_+ \quad \forall x \geq 0$ (15)
- $r v(x) - r x v'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \geq 0 \quad \forall x \geq 0$ (16)
- and for each $x \geq 0$, equality holds (17) in either (15) or (16).

At point $x = L_*$ we can use either $v''_{L_*}(L_*+)$ or $v''_{L_*}(L_*^-)$ in (16), the inequality still holds.

It turns out that $v_{L_*}(x)$ is the only bounded continuously differentiable (C^1) function on $[0,\infty)$ that satisfies (15)-(17). (This is Exercise 8.3 in Shreve II).

To conclude: analytically, the perpetual put price is the only bounded C^1 function which satisfies (15)-(17).