MTH 9831. Solutions to Quiz 10.

(1) (3 points) Let $Q(t)$, $t \geq 0$, be a compound Poisson process, i.e. $Q(t) = \sum_{i=1}^{N(t)} Y_i$, where $N(t)$ is a Poisson process with intensity λ, $(Y_i)_{i\geq 1}$ are i.i.d. random variables independent of the process N. Set $\beta := E(Y_1)$, $\sigma^2 := \mathrm{Var}(Y_1)$. Then we know that $M_Q(t) := Q(t) - \beta\lambda t$ is a martingale (with respect to the natural filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ of the process Q). Compute $E(M_Q^2(t)\,|\,\mathcal{F}(s))$ and find out what needs to be subtracted from $M_Q^2(t)$ to make it into a martingale.

Solution. The approach is exactly the same as for Brownian motion. It works for all processes with independent increments.

$$
\begin{aligned}
E(M_Q^2(t)\,|\,\mathcal{F}(s)) &= E((M_Q(t) - M_Q(s) + M_Q(s))^2\,|\,\mathcal{F}(s)) \\
&= E((M_Q(t) - M_Q(s))^2 + 2M_Q(s)(M_Q(t) - M_Q(s)) + M_Q^2(s)\,|\,\mathcal{F}(s)) \\
&= \mathrm{Var}(M_Q(t) - M_Q(s)) + 2M_Q(s)E(M_Q(t) - M_Q(s)\,|\,\mathcal{F}(s)) + M_Q^2(s) \\
&= (\sigma^2 + \beta^2)\lambda(t-s) + M_Q^2(s).
\end{aligned}
$$

In the transition from the second to the third line we used the fact that $M_Q(t) - M_Q(s)$ is independent of $\mathcal{F}(s)$ and has expectation 0 as well as "taking-out-what's-known" property of conditional expectations. Moreover, $\mathrm{Var}(M_Q(t) - M_Q(s)) = \mathrm{Var}(Q(t) - Q(s)) = (\sigma^2 + \beta^2)\lambda(t-s)$.

We conclude that the process $M_Q^2(t) - (\sigma^2 + \beta^2)\lambda t$, $t \geq 0$, is a martingale.

(2) (3 points) Compute $\int_0^t M(s-)\,dM(s)$, where M is a compensated Poisson process.

Solution. The first guess would be that the answer has to do with $M^2(t)$. We apply Itô's formula to $M^2(t)$, i.e. we take $X(t) = M(t) = N(t) - \lambda t$ (so that $X^c(t) = -\lambda t$ and $[X^c, X^c](t) \equiv 0$) and $f(x) = x^2$. We get

$$
\begin{aligned}
M^2(t) &= 2\int_0^t M(s)\,dM^c(s) + \sum_{0<s\leq t} (M^2(s) - M^2(s-)) \\
&= 2\int_0^t M(s)\,dM^c(s) + \sum_{0<s\leq t} ((M(s-) + \Delta N(s))^2 - M^2(s-)) \\
&= 2\int_0^t M(s-)\,dM^c(s) + 2\sum_{0<s\leq t} M(s-)\Delta N(s) + \sum_{0<s\leq t} (\Delta N(s))^2 \\
&= 2\int_0^t M(s-)\,dM(s) + \sum_{0<s\leq t} (\Delta N(s))^2 \\
&= 2\int_0^t M(s-)\,dM(s) + \sum_{0<s\leq t} \Delta N(s) = 2\int_0^t M(s-)\,dM(s) + N(t).
\end{aligned}
$$

Therefore,

$$
\int_0^t M(s-)\,dM(s) = \frac{1}{2}\,(M^2(t) - N(t)).
$$

(3) (4 points) Let $N(t)$, $t \geq 0$, be a Poisson process with intensity 2. Apply Itô-Doeblin formula to show that the process $Z(t) = e^{-2t}2^{N(t)}$, $t \geq 0$ is a martingale.

Solution. Set $X(t) = -2t + (\ln 2)N(t)$. Then $X^c(t) = -2t$, $[X^c, X^c](t) \equiv 0$, and

$$
X(t) - X(t-) = (\ln 2)\Delta N(t).
$$

We shall apply Itô's formula to $f(X(t))$, where $f(x) = e^x$.

$$
\begin{aligned}
Z(t) = e^{X(t)} &= 1 + \int_0^t e^{X(s)}\, dX^c(s) + \sum_{0<s\le t} (e^{X(s)} - e^{X(s-)}) \\
&= 1 + \int_0^t e^{X(s-)}\, dX^c(s) + \sum_{0<s\le t} e^{X(s-)}(e^{X(s)-X(s-)} - 1) \\
&= 1 + \int_0^t e^{X(s-)}\, dX^c(s) + \sum_{0<s\le t} e^{X(s-)}(e^{(\ln 2)\Delta N(s)} - 1) \\
&= 1 + \int_0^t e^{X(s-)}\, dX^c(s) + \sum_{0<s\le t} e^{X(s-)}(2^{\Delta N(s)} - 1) \\
&= 1 + \int_0^t e^{X(s-)}\, dX^c(s) + \sum_{0<s\le t} e^{X(s-)}\Delta N(s) \\
&= 1 + \int_0^t e^{X(s-)}\, dX^c(s) + \int_0^t e^{X(s-)} dN(s) \\
&= 1 + \int_0^t e^{X(s-)}\, d(X^c(s) + N(s)) = 1 + \int_0^t e^{X(s-)}\, dM(s),
\end{aligned}
$$

where $M(s) = N(s) - 2t$, $t \ge 0$, is a martingale, since the intensity of N is equal to 2. We conclude that Z is a martingale (as a constant plus an integral of a left-continuous adapted process with respect to a martingale jump process).