November 18, 2015

MTH 9831 Solutions to Quiz 8.

(1) (3 points) (Solving a linear ODE with constant coefficients) Let b and c be arbitrary constants such that $b^2 - 4c > 0$.[1] For given fixed φ_0 and φ_1 find the solution of

$$\varphi''(t) + b\varphi'(t) + c\varphi(t) = 0,\ 0 \le t \le T,\ \varphi(T) = \varphi_0,\ \varphi'(T) = \varphi_1,$$

Solution. This is a second order differential equation with constant coefficients. The general solution is of the form $\varphi(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$, where λ_1 and λ_2 are the roots of the quadratic equation $x^2 + bx + c = 0$ (which gives $\lambda_{1,2} = (-b \pm \sqrt{b^2 - 4c})/2$), C_1 and C_2 are arbitrary constants. We need a particular solution, which satisfies the given conditions. Since the conditions are given not at $t = 0$ but at $t = T$, it is more convenient to write the general solution as

$$\varphi(t) = A_1 e^{\lambda_1 (t-T)} + A_2 e^{\lambda_2 (t-T)}.$$

Substituting the data we get $A_1 + A_2 = \varphi_0$ and $A_1\lambda_1 + A_2\lambda_2 = \varphi_1$. Thus

$$A_1 = \frac{\varphi_0\lambda_2 - \varphi_1}{\lambda_2 - \lambda_1}; \quad A_2 = \frac{\varphi_1 - \varphi_0\lambda_1}{\lambda_2 - \lambda_1}.$$

(2) (3 points) (Boundary conditions for option pricing) Assume Black-Scholes model. Prices of options (i) and (ii) satisfy $v_t(t,x) + rxv_x(t,x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = rv(t,x)$. For each option give (a) the region where this PDE should hold, (b) all boundary conditions (includes a terminal condition).

(i) A vanilla put option with strike K and expiration T.

(ii) A down-and-out put option with strike K, barrier $B < K$, and expiration T.

Solution. (i) The region is $(t,x) \in [0,T) \times [0,\infty)$. There are 3 conditions:

$$\begin{aligned} v(T,x) &= (K-x)_+, \quad x \ge 0 \text{ (terminal condition);}\\ v(t,0) &= Ke^{-r(T-t)}, \quad 0 \le t \le T \text{ (condition on the left boundary);}\\ \lim_{x\to\infty} v(t,x) &= 0, \quad 0 \le t \le T \text{ (condition at infinity).} \end{aligned}$$

The solution is continuous on $[0,T] \times [0,\infty)$.

(ii) The region is $(t,x) \in [0,T) \times [B,\infty)$. There are 3 conditions:

$$\begin{aligned} v(T,x) &= (K-x)_+, \quad x \ge B \text{ (terminal condition);}\\ v(t,B) &= 0, \quad 0 \le t < T \text{ (condition on the left boundary);}\\ \lim_{x\to\infty} v(t,x) &= 0, \quad 0 \le t \le T \text{ (condition at infinity).} \end{aligned}$$

The solution has a discontinuity at a point (T,B). (At this point the terminal condition applies, since the price only hits the barrier at the expiration, and the payoff is still $(K-B)$. Usually we think of a knock out of the option as a *crossing* of the barrier level. At expiration there is no time for this crossing to occur.)

(3) (4 points) (Hedging zero-strike Asian call) Assume Black-Scholes model and consider a zero-strike Asian call option, i.e. the option with the payoff

$$V(T) = \frac{1}{T}\int_0^T S(u)\,du.$$

[1] Useful "take home" questions: what is the form of the general solution when (1) $b^2 - 4c = 0$? (2) $b^2 - 4c < 0$?

Suppose that at time t we have $S(t) = x \geq 0$ and $\int_0^t S(u)\,du = y \geq 0$. Use the fact that $e^{-rt}S(t)$ is a martingale under $\tilde{\mathbb{P}}$ to compute

$$e^{-r(T-t)}\tilde{\mathbb{E}}\left(\frac{1}{T}\int_0^T S(u)\,du \,\middle|\, \mathcal{F}(t)\right).$$

Call your answer $v(t, x, y)$. Then find explicitly $\Delta(t)$ for all $0 \leq t \leq T$.

Solution. By the definition of $Y(t)$, Fubini's theorem, and the fact that $e^{-rt}S(t)$ is a martingale under $\tilde{\mathbb{P}}$ we have

$$\begin{aligned}
&\tilde{\mathbb{E}}\left(\int_0^T S(u)\,du \,\middle|\, \mathcal{F}(t)\right) = \tilde{\mathbb{E}}\left(\int_0^t S(u)\,du + \int_t^T S(u)\,du \,\middle|\, \mathcal{F}(t)\right) \\
&= Y(t) + \int_t^T \tilde{\mathbb{E}}\left(S(u) \,\middle|\, \mathcal{F}(t)\right)\,du = Y(t) + \int_t^T e^{ru}\tilde{\mathbb{E}}\left(e^{-ru}S(u) \,\middle|\, \mathcal{F}(t)\right)\,du \\
&= Y(t) + \int_t^T e^{ru}e^{-rt}S(t)\,du = Y(t) + S(t)\int_t^T e^{r(u-t)}\,du \\
&= Y(t) + S(t)\,\frac{1}{r}\left(e^{r(T-t)} - 1\right).
\end{aligned}$$

Multiplying by $e^{-r(T-t)}$ and dividing by T we get

$$v(t, S(t), Y(t)) = \frac{1}{T}\,e^{-r(T-t)}Y(t) + \frac{1}{rT}\left(1 - e^{-r(T-t)}\right)S(t).$$

Therefore,

$$v(t,x,y) = \frac{1}{T}\,e^{-r(T-t)}y + \frac{1}{rT}\left(1 - e^{-r(T-t)}\right)x, \quad v_x(t,x,y) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right),$$

and $\Delta(t) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right)$, $0 \leq t \leq T$, a non-random smooth function.