Homework 1

Problem 1.1. Prove that an agent is risk-averse, ie. inequality

$$E[u(w + \tilde{z})] \leq u(w + E[\tilde{z}])$$

holds for all w and $\tilde{z}$, if and only if $u()$ is concave.

Problem 1.2. Show that when $u(w) = -\exp(-aw)/a$ and $\tilde{w}$ is normally distributed with mean μ and variance σ^2, the Arrow-Pratt approximation is exact and indeed,

$$E[u(\tilde{w})] = u\left(\mu - \frac{1}{2}a\sigma^2\right)$$

Problem 1.3. Show that the following three conditions are equivalent:
(a) Agent v is more risk-averse than agent u,
(b) For all w, $A_v(w) \geq A_u(w)$.
(c) Function v is a concave transformation of function u, meaning:

$$\exists\, \phi \text{ with } \phi' > 0 \text{ and } \phi'' < 0 \text{ such that } v(w) = \phi(u(w))$$

Problem 1.4. Consider a function $v(.)$ such that $v(x) = a + bu(x)$ for all x, for some pair of scalars a and b, where $b > 0$. Show that a decision-maker with utility function $v()$ makes the same decisions and has the same certainty-equivalents as a decision maker with utility function $u()$.

Problem 1.5. Suppose you are a fund-of-funds manager with investments in n different hedge funds for some $n \geq 2$. Let r_i denote the annualized return of the i-th fund. Suppose that

$$r_i = \beta r_M + \epsilon_i, \quad \text{var}(\epsilon_i) = \sigma_i^2$$

where r_M denotes the return of the market portfolio (approximated by the S&P 500 in the US) with variance σ_M^2. Suppose that ϵ_i and ϵ_j are independent random variables if $i \neq j$, and that ϵ_i is independent from r_M for all $i = 1, \ldots, n$. Suppose that your fund-of-funds has invested $h_i > 0$ dollars in the i-th hedge fund, so their profit/loss is

$$\pi = h'r = \sum_i h_i r_i.$$

Throughout the following, assume $h = (1/n, 1/n, \ldots, 1/n) \in \mathbb{R}^n$ for simplicity, ie. the fund-of-funds has one unit of capital evenly distributed across its constituents.

(a) Calculate $\mathbb{E}[h'r]$ and $\mathbb{V}[h'r]$. Note that $\mathbb{V}[h'r]$ can be expressed as $\mathbb{V}(h'r) = f(\beta, \sigma_M^2) + g(\sigma_1^2, \ldots, \sigma_n^2)$; find functions $f()$ and $g()$ explicitly.

(b) Take $\beta = 0.5$ and $\sigma_M = 0.2$. Assume that each constituent fund has an annualized volatility target of 10% and all $\sigma_i \approx 0.03$. The "fraction of variance explained by the market" for the fund-of-funds is defined to be $f/(f+g)$. Numerically compute and plot this fraction as a function of n for $n = 2 \ldots 30$.

(c) Take the same assumptions as (b). Further assume that each ϵ_i has a Sharpe ratio of 1.5, so that $\mathbb{E}[\epsilon_i] = 1.5 \cdot \sigma_i$, and the market's expected annual return is $\mathbb{E}[r_M] = 0.07$. The fund-of-funds charges a fee of 0.01 on capital. Numerically compute and plot the Sharpe ratio, $\mathbb{E}[h'r - 0.01]/\sqrt{\mathbb{V}[h'r]}$ as a function of n for $n = 2 \ldots 30$. How does this change if the Sharpe ratio of ϵ_i is 2.0 rather than 1.5?

(d) If the fund-of-funds could simply invest in a single fund with the same properties as the others except that this fund has $\beta = 0$ and $\sigma_i = 0.1$, would that be better or worse, in terms of Sharpe ratio, than the above scenario?