Real Analysis and Probability (MTH 9831).
Final Examination

Instructions: Please **print** your name below. Solve any 6 problems. Points above 100 will be counted as extra credit. Indicate by a check in the comments column below up to 6 problems, for which you want to receive credit. Unchecked problems will not be graded. If you check more than 6 problems, the first 6 checked problems will be graded. **Good luck! And Happy Holiday Season!**

Student name:

Grade

Problem	Out of	Score	Comments
1	20		
2	20		
3	15		
4	15		
5	15		
6	15		
7	20		
8	15		
9	15		
10	20		
Total	170		required: 100

Problem 1. Suppose that the security's price follows the binomial model with parameters $u = 1.2$, $d = 1/1.2$, $r = 0.02$. The initial price of the security is \$60. Find the risk-neutral price of the derivative that expires at time 4 and pays \$3 if the price stays between 40 and 100 at all times before the expiration; otherwise it pays nothing.

Problem 2. Let $(N(t))_{t\geq 0}$ be a Poisson process with intensity λ. Let T_i be the first time when $N(t) = i$, $i \in \{1, 2\}$. Find

(a) $P(T_1 < s \,|\, N(t) = 2)$; (b) $E(T_2 \,|\, T_1)$.

Problem 3. Let (X_1, X_2) have the following joint moment generating function (λ, μ, ν are positive parameters)

$$M_{(X_1,X_2)}(t_1, t_2) = \exp\left(\lambda(e^{t_1} - 1) + \mu(e^{t_2} - 1) + \nu(e^{t_1+t_2} - 1)\right).$$

Determine the marginal distributions of X_1 and X_2. Are X_1 and X_2 are dependent or independent?

Problem 4. Let τ_1 be the first time the simple symmetric random walk hits 1. Show that τ_1 is a stopping time (with respect to the natural filtration of the random walk).

Problem 5. Consider a measurable space $(\mathbb{R}, \mathcal{B})$ and the following two probability measures on it: for every $A \in \mathcal{B}$

$$P(A) = \int_A \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx; \quad Q(A) = \int_A \frac{1}{\sqrt{2\pi}} e^{-(x+1)^2/2}\, dx.$$

Are these measures equivalent? If yes, then explain why and find $\frac{dQ}{dP}$. If no, then illustrate how the equivalence fails.

Problem 6. True or false? If X_1 and X_2 are two normal random variables and $\mathrm{Cov}(X_1, X_2) = 0$ then X_1 and X_2 are independent. Give either a proof or a counterexample.

Problem 7. Let $(X_n)_{n\geq 1}$ be an i.i.d. sequence of exponential random variables with parameter 1. Show that $X_n/n \to 0$ almost surely as $n \to \infty$.

Problem 8. Let $X_0, X_1, \ldots$ be i.i.d. random variables, and $P(X_0 = 1) = P(X_0 = -1) = 1/2$. Define the stochastic process $(S_n)_{n\geq 0}$ as follows

$$S_0 = 0, \quad S_{n+1} = \begin{cases} S_n + X_{n+1}, & \text{if } X_0 = 1 \\ S_n + 2X_{n+1}, & \text{if } X_0 = -1, \end{cases}$$

and let $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$, $n \geq 0$. Is this process a martingale? Is this process a Markov process? (Everything is with respect to $(\mathcal{F}_n)_{n\geq 0}$.)

Problem 9. Let τ_a be the first time when a Brownian motion hits level a, $a \neq 0$. Find the density of τ_a. Is τ_a integrable?

Problem 10. Let $(B(t))_{t\geq 0}$ be a Brownian motion.

(a) Show that $(B^2(t) - t)_{t\geq 0}$ is a martingale (with respect to the natural filtration).

(b) Let $a, b > 0$ and $\tau = \inf\{t \geq 0 : B(t) \in \{-a, b\}\}$. Prove that $E\tau = ab$.