HW #4.3 - z-transforms

(a) $h[n] = \delta[n-N]$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n} = \sum_{n=-\infty}^{+\infty} \delta[n-N] z^{-n} = z^{-N} \Rightarrow g = 1, \quad M_1 = N.$$

(b) Box: $h[n] = \frac{1}{N_{box}}\left(u[n] - \delta[n-N_{box}] * u[n]\right)$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n} = \sum_{n=-\infty}^{+\infty} \frac{1}{N_{box}}\left(u[n] - u[n-N_{box}]\right) z^{-n}$$

$$= \frac{1}{N_{box}}\left(\sum_{n=-\infty}^{+\infty} u[n] z^{-n} - \sum_{n=-\infty}^{+\infty} u[n-N_{box}] z^{-n}\right)$$

$$= \frac{1}{N_{box}}\left(\sum_{n=0}^{+\infty} z^{-n} - \sum_{n=N_{box}}^{+\infty} z^{-n}\right)$$

$$= \frac{1}{N_{box}} \cdot \frac{1-z^{-N_{box}}}{1-z^{-1}}$$

$$\therefore g = H(z=1) = \frac{1}{N_{box}} \lim_{z\to 1} \frac{1-z^{-N_{box}}}{1-z^{-1}} = \frac{1}{N_{box}} \lim_{z\to 1} \frac{N_{box} z^{N_{box}-1}}{N_{box} z^{N_{box}-1} - (N_{box}-1) z^{N_{box}-2}} = \frac{N_{box}}{N_{box}}$$

$$= 1$$

$$M_1 = \frac{d}{dz^{-1}} H(z)\Big|_{z=1} \overset{\zeta = z^{-1}}{=} \frac{1}{N_{box}} \frac{d}{d\zeta}\left(\frac{\zeta^N - 1}{\zeta - 1}\right)\Big|_{\zeta=1} = \frac{N_{box}-1}{2}.$$

(c) $h[n] = u[n]$

Similar to (b): $H(z) = \sum_{n=-\infty}^{+\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}$

$$g = H(z=1) \to \infty.$$

$$M_1 \to \infty.$$

(d) $h[n] = (1-p)p^n u[n]$

$$H(z) = \sum_{n=-\infty}^{+\infty} (1-p)p^n u[n] z^{-n} = \sum_{n=0}^{\infty} (1-p) p^n z^{-n}$$

$$= (1-p)\sum_{n=0}^{\infty} \left(\frac{z}{p}\right)^{-n} = (1-p)\frac{1}{1-\left(\frac{z}{p}\right)^{-1}} = \frac{1-p}{1-pz^{-1}}$$

$\therefore\ g = H(z=1) = 1$

$$M_1 = \frac{\partial}{\partial z}\left(\frac{1-p}{1-pz}\right)_{z=1} = (1-p)\cdot\frac{p}{(1-pz)^2}\Bigg|_{z=1} = \frac{p}{1-p}$$

$\Rightarrow p = \frac{M_1}{M_1+1}$: express parameter p
in terms of physical observable M_1.

(e) $h[n] = (1-p)^2(n+1)p^n u[n]$

$$H(z) = \sum_{n=-\infty}^{+\infty} (1-p)^2(n+1)p^n u[n] z^{-n} = \sum_{n=0}^{\infty} (1-p)^2 (n+1) p^n z^{-n}$$

$$= (1-p)^2 \sum_{n=0}^{\infty} (n+1)\left(\frac{z}{p}\right)^{-n}$$

$$= (1-p)^2\left[\sum_{n=0}^{\infty} n\left(\frac{z}{p}\right)^{-n} + \frac{1}{1-pz^{-1}}\right]$$

$$= (1-p)^2\left[\frac{\partial}{\partial\left(\frac{p}{z}\right)}\underbrace{\sum_{n=0}^{\infty}\left(\frac{p}{z}\right)^{n}}_{=\frac{1}{1-pz^{-1}}} \times \left(+\frac{p}{z}\right) + \frac{1}{1-pz^{-1}}\right]$$

$$= (1-p)^2\left[+\left(\frac{p}{z}\right)\cdot\frac{1}{(1-pz^{-1})^2} + \frac{1}{1-pz^{-1}}\right]$$

$$= \frac{(1-p)^2}{(1-pz^{-1})^2} \Rightarrow g = H(z=1) = 1$$

$$M_1 = \frac{\partial}{\partial z}\left(\frac{(1-p)^2}{(1-pz)^2}\right)\Bigg|_{z=1} = \frac{2p}{1-p}$$

$$\Rightarrow p = \frac{\frac{1}{2}M_1}{\frac{1}{2}M_1+1}.$$

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(f). $h[n] = \left(\frac{2N_{eff}}{3}\right)^{-1} u[n] * \left(h_{ema}\left(\frac{N_{eff}}{3}\right)[n] - h_{poly}(N_{eff})[n] \right)$.

- $u[n] * h_{ema}[n] = \sum_{m=-\infty}^{n} h_{ema}[m] = \sum_{m=-\infty}^{n} (1-p)p^m u[m]$

$= \sum_{m=0}^{n} (1-p)p^m = (1-p)\frac{1-p^n}{1-p} = (1-p^n), \quad n \geq 0$

- $u[n] * h_{poly}[n] = \sum_{m=0}^{n} (1-p')^2 (m+1) p'^m$

$= (n-1)p'^{n+1} - np'^n + p' + (1-p')(1-p'^n)$

$= np'^{n+1} - (n+1)p'^n + 1 \quad , n \geq 0$

$\therefore h[n] = \left(\frac{2N_{eff}}{3}\right)^{-1} \left[\cancel{1} - p^n - np'^{n+1} + (n+1)p'^n \cancel{-1} \right] \cdot u[n]$

$= \left(\frac{2N_{eff}}{3}\right)^{-1} \left[(n+1)p'^n - np'^{n+1} - p^n \right] \cdot u[n]$

$\therefore H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n}$

$= \left(\frac{2N_{eff}}{3}\right)^{-1} \sum_{n=0}^{\infty} \left[(n+1)p'^n - np'^{n+1} - p^n \right] z^{-n}$.

Here we have, $N_{eff}^{+} = \frac{1}{3} N_{eff}$, $N_{eff}^{-} = N_{eff}$

$p = \frac{N_{eff}^{+}}{N_{eff}^{+} + 1} = \frac{\frac{1}{3} N_{eff}}{\frac{1}{3} N_{eff} + 1}$, $p' = \frac{\frac{1}{2} N_{eff}^{-}}{\frac{1}{2} N_{eff}^{-} + 1} = \frac{\frac{1}{2} N_{eff}}{\frac{1}{2} N_{eff} + 1}$.

~from (d) ~from(e)

Computation gives.

$H(z) = \frac{1}{g(1-z^{-1})} \left(\frac{1-p}{1-pz^{-1}} - \frac{(1-p')^2}{(1-p'z^{-1})^2} \right)$

where $g = H(z=1) = \frac{2p'}{1-p'} - \frac{p}{1-p} = \frac{2}{3} N_{eff}$.

$M_1 = \frac{23}{24} N_{eff}$.

(g) $h[n] = h_{ema}^{(N_{eff}^{+})}[n] - h_{ema}^{(N_{eff}^{-})}[n]$, use the result from (d).

$$\Rightarrow H(z) = \frac{1-p_+}{1-p_+ z^{-1}} - \frac{1-p_-}{1-p_- z^{-1}} \quad , \text{ where } p_\pm = \frac{\frac{1}{3}N_{eff}^{\pm}}{\frac{1}{3}N_{eff}^{\pm}+1} .$$

$$g = H(z=1) = 1-1 = 0 .$$

(h). $h[n] = \delta[n - N_{eff}^{+}] - \delta[n - N_{eff}^{-}]$, use the result from (a)

$$\Rightarrow H(z) = z^{-N_{eff}^{+}} - z^{-N_{eff}^{-}}$$

$$g = H(z=1) = 1-1 = 0 .$$