# REFRESHER SEMINAR READING NOTES

SHENGQUAN ZHOU*

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## ABSTRACT

In this reading notes based on the Primer [1], we briefly recapitulate the basic formalism of mathematical finance. The material is presented in the order of general principles, model-independent results, specific models and applications.

1 A Primer for the Mathematics of Financial Engineering, Second Edition, Dan Stefanica.

* *Commodities R&D, Bloomberg L. P., New York, U.S.A.*

# Part I. Concepts

## 1 PORTFOLIO VALUE

**Definition 1** (The Value of a Portfolio)**.** The value of a portfolio which has long positions in one unit of securities $X_1, X_2, \cdots, X_N$ and short positions in one unit of securities $Y_1, Y_2, \cdots, Y_M$ at time $t$ is

$$V(t) = \sum_{i=1}^{N} X_i - \sum_{j=1}^{M} Y_j, \tag{1}$$

where the securities $X$ and $Y$ can be assets, stocks, or options, etc.

**Definition 2** (The Cash Position to Set Up a Portfolio)**.** The cash position needed to set up a portfolio which has long positions in one unit of securities $X_1, X_2, \cdots, X_N$ and short positions in one unit of securities $Y_1, Y_2, \cdots, Y_M$ at time $t$ is

$$M(t) = -\sum_{i=1}^{N} X_i + \sum_{j=1}^{M} Y_j, \tag{2}$$

where the securities $X$ and $Y$ can be assets, stocks, or options, etc.

It is important to clarify that the value of a portfolio is equal to the cash flow generated if the portfolio is liquidated, and not to the cash flow needed to set up the portfolio. As time progresses, the cash position grows at risk-free interest rate while the portfolio value grows with the market.

## 2 RATE OF RETURN

Denote by $V(t)$ the value of a portfolio (or of a single asset) at time $t$.

**Definition 3** (Percentage Return)**.** The *percentage return* of a portfolio between time $t$ and time $t' > t$ is

$$R_{t \to t'} \triangleq \frac{\Delta V}{V} = \frac{V(t') - V(t)}{V(t)}. \tag{3}$$

The *rate of return* between times $t$ and $t'$ of a portfolio with value $V(t)$ at time $t$ is equal to

$$r_{t \to t'} \triangleq \frac{1}{\Delta t}\frac{\Delta V}{V} = \frac{1}{(t'-t)}\frac{V(t') - V(t)}{V(t)}. \tag{4}$$

The percentage return is also called the arithmetic return. This is the type of return most commonly implied when one refers to the return of the portfolio, without further specification.

If the value of the portfolio is guaranteed to be positive, e.g., if the portfolio is made of long positions in stocks, call options, and put options, then the log return of the portfolio can be defined.

**Definition 4** (Log Return)**.** The *log return* of a portfolio between time $t$ and time $t' > t$ is

$$L_{t \to t'} \triangleq \log \frac{V(t')}{V(t)}. \tag{5}$$

The corresponding log rate of return of the portfolio is

$$l_{t \to t'} \triangleq \frac{1}{t'-t} \log \frac{V(t')}{V(t)} = \frac{\log V(t') - \log V(t)}{t'-t}. \tag{6}$$

The log rate of return of the portfolio is also called the continuously compounded rate of return of the portfolio, as

$$V(t') = V(t) e^{l_{t \to t'}(t'-t)}. \tag{7}$$

From Definition 4 and 4, it is easy to see that

$$L = \log \frac{V(t')}{V(t)} = \log \left( 1 + \frac{V(t') - V(t)}{V(t)} \right) = \log(1+R). \tag{8}$$

It is important to note that percentage return is greater than log return, i.e.

$$L = \log(1+R) \leqslant R, \quad \forall R > -1. \tag{9}$$

For small returns, e.g., when the difference between the times $t$ and $t'$ is very small, it follows that the percentage and log returns are similar, i.e.

$$L = \log(1+R) \approx R - \frac{1}{2} R^2. \tag{10}$$

Log returns are well suited for computing individual asset returns, since they are additive over consecutive time intervals. Let $t < t' < t''$, we have

$$L_{t \to t''} = L_{t \to t'} + L_{t' \to t''}. \tag{11}$$

Percentage returns do not have this property. Instead,

$$R_{t \to t''} = R_{t \to t'} + R_{t' \to t''} + R_{t \to t'} R_{t' \to t''}. \tag{12}$$

Percentage returns are well suited for computing portfolio returns, since the percentage return of a portfolio made of different assets is the weighted average of the percentage returns of the assets. Denote the value of the portfolio at time $t$ by $\Pi(t) = \sum_i V_i(t)$, the weighting factors are given by

$$w_i = \frac{V_i(t)}{\Pi(t)}. \tag{13}$$

The percentage return of the portfolio between time $t$ and $t' > t$ can be written as

$$\begin{aligned} R_{t \to t'}(\Pi) &= \frac{\Pi(t') - \Pi(t)}{\Pi(t)} \\ &= \frac{\sum_i [V_i(t') - V_i(t)]}{\Pi(t)} \\ &= \frac{\sum_i R_{t \to t'}(V_i) V_i(t)}{\Pi(t)} \\ &= \sum_i w_i R_{t \to t'}(V_i). \end{aligned} \tag{14}$$

The log return of the portfolio does not have this property. Instead,

$$\begin{aligned} L_{t\to t'}(\Pi) &= \log\frac{\Pi(t')}{\Pi(t)} \\ &= \log\sum_i w_i \frac{V_i(t')}{V_i(t)} \\ &\geqslant \sum_i w_i \log\frac{V_i(t')}{V_i(t)} = \sum_i w_i L_{t\to t'}(V_i). \end{aligned} \tag{15}$$

Here we used the Jensen's inequality

$$\log(\lambda x + (1-\lambda)y) \geqslant \lambda\log x + (1-\lambda)\log y. \tag{16}$$

## 3 SHORT SELLING

To explain the concept of *short selling*, consider the case of equity options, i.e., options where the underlying asset is a stock. Selling short one share of stock is done by borrowing the share through a broker and then selling the share on the market. Part of the cash is deposited with the broker in a margin account as collateral (usually, 50% of the sale price), while the rest is deposited in a brokerage account. The margin account must be settled when a margin call is issued, which happens when the price of the shorted asset appreciates beyond a certain level. Cash must then be added to the margin account to reach the level of 50% of the amount needed to close the short, i.e., to by one share of stock at the current price of the asset, or the short is closed by the broker on your behalf. The cash from the brokerage account can be invested freely, while the cash from the margin account earns interest at a fixed rate, but cannot be invested otherwise. The short is closed by buying the share at a later time on the arket and returning it to the original owner via the broker (the owner rarely knows that the asset was borrowed and sold short). Also, the short could be closed by the broker at any time, if no other shares are available for borrowing and the original owner places a sell order.

We will not consider here these or other issues, such as margin calls, the liquidity of the market and the availability of shares for short selling, transaction costs, and the impossibility of taking the exact position required for a perfect hedge.

## 4 ARBITRAGE-FREE PRICING

An arbitrage opportunity is an investment opportunity that is guaranteed to earn money without any risk involved. More specifically, we define *arbitrage* as a trading strategy that begins with no money, has zero probability of losing money, and has a positive probability of making money. An essential feature of an efficient market is that if a trading strategy can turn nothing into something, then it must also run the risk of loss. Otherwise, there would be an *arbitrage*.

In an arbitrage-free market, we can infer relationships between the prices of various securities, based on the following principle:

**Theorem 4.1** (The Law of One Price)**.** *If two portfolios are guaranteed to have the same value at a future time* $t' > t$ *regardless of the state of the market at time*

*$t$, then they must have the same value at time $t'$. If one portfolio is guaranteed to be more valuable (or less valuable) than another portfolio at a future time $t > t'$ regardless of the state of the market at time $t$, then that portfolio is more valuable (or less valuable) then the other at time $t'$ as well.*

**Corollary 4.1.1.** *If the value of a portfolio is guaranteed to be equal to zero at a future time $t > t'$ regardless of the state of the market at time $t$, then the value of the portfolio at time $t'$ must have been zero as well.*

**Example 4.1.1.** *If, at time $t$, an asset becomes worthless, i.e. if $S(t) = 0$, then the price of the asset will never be above zero again. Otherwise, an arbitrage opportunity arises: buy the asset at no cost at time $t$, and sell it for risk-free profit as soon as its value is above zero.*

# Part II. Statics and Kinematics

## 5 INTEREST RATES AND BONDS

The primary objects of investigation in this section are *zero coupon bonds*, also known as *pure discount bonds* of various maturities.

### 5.1 Zero Rates, Instantaneous Rates and Forward Rates

**Definition 5** (Zero Rate)**.** The *zero rate* $r(0,t)$ between time zero and time $t$ is the rate of return of a cash deposit made at time zero and maturing at time $t$. If specified for all values of $t$, then $r(0,t)$ is called the *zero rate curve*.

Assume continuous compounding, the value at time $t$ of $B(0)$ units of cash at time zero is

$$B(t) = B(0)e^{r(0,t)t}. \tag{17}$$

Conversely, the value at time zero of $B(t)$ units of cash at time $t$ is

$$B(0) = B(t)e^{-r(0,t)t}. \tag{18}$$

**Definition 6** (Instantaneous Rate)**.** The instantaneous rate $r(t)$ at time $t$ is the rate of return of deposits made at time $t$ and maturing at time $t + dt$, where $dt$ is infinitesimal, i.e.,

$$r(t) = \lim_{dt \to 0} \frac{1}{dt} \frac{B(t + dt) - B(t)}{B(t)} = \frac{B'(t)}{B(t)}. \tag{19}$$

From Equation 19, we find that

$$B(t) = B(0)e^{\int_0^t r(t')dt'}, \forall t > 0, \tag{20}$$

or equivalently,

$$B(0) = B(t)e^{-\int_0^t r(t')dt'}, \forall t > 0, \tag{21}$$

where the term $e^{-\int_0^t r(t')dt'}$ is called the *discount factor*.

From Equation 17 and 20, it follows that

$$r(0,t) = \frac{1}{t}\int_0^t r(t')dt'. \tag{22}$$

In other words, the zero rate at time $t$ is the average of the instantaneous rates over the time interval $[0,t]$. Conversely, the instantaneous rate is uniquely determined if the zero rate curve is known:

$$r(t) = r(0,t) + t\frac{dr(0,t)}{dt}. \tag{23}$$

**Example 5.0.2.** *A common assumption made when pricing derivative securities with relatively short maturities is that the risk-free rates are constants over the life of the derivative security. When the assumption that interest rates are constant is made, if either one of the zero rate or the instantaneous rate is constant and equal to $r$, then the other rate is also constant and equal to $r$, i.e.,*

$$r(0,t) = r, \forall t > 0 \Leftrightarrow r(t) = r, \forall t > 0. \tag{24}$$

*The above equivalence can be verified by direct computation from Equation 22 and 23.*

**Lemma 5.0.1.** *If the value of a portfolio $V(t)$ at time $t$ in the future is independent of the state of the market at time $t$, then*

$$V(t') = V(t)e^{-r(t-t')} \tag{25}$$

*where $t' < t$ and $r$ is the constant risk free rate.*

*Proof.* Consider a portfolio made of $V(t)e^{-r(t-t')}$ cash at time $t'$. The value of this portfolio at time $t$ is $e^{r(t-t')} \times V(t)e^{-r(t-t')} = V(t)$. From Theorem 4.1, we conclude that $V(t') = V(t)e^{-r(t-t')}$. □

**Definition 7** (Forward Rate)**.** The *forward rate* of return $r(0;t',t)$ between $t$ and $t'$ is the constant rate of return, as seen at time zero, of a deposit that will be made at time $t'$ and will mature at time $t > t'$.

An arbitrage-free value for the forward rate in terms of the zero rate curve can be found using the Law of One Price, Theorem 4.1, as follows. Consider two different strategies for investing one unit of cash at time zero.

STRATEGY 1 At time zero, deposit one unit of cash until time $t'$ with interest rate $r(0,t')$. Then, at $t'$, deposit the proceeds until time $t$, at the forward rate $r(0;t',t)$, which was locked in at time zero. The value of the deposit at time $t'$ is $V_1(t') = e^{r(0,t')t'}$. At time $t$, the value is

$$V_1(t) = V_1(t')e^{r(0;t',t)(t-t')} = e^{r(0,t')t'+r(0;t',t)(t-t')}. \tag{26}$$

STRATEGY 2 Deposit one unit of cash until time $t$ with interest rate $r(0,t)$. At time $t$, the value of the deposit is

$$V_2(t) = e^{r(0,t)t}. \tag{27}$$

Both investment strategies are risk-free, and the cash amount invested at time zero is the same for both strategies. From the Law of One Price, Theorem 4.1, if follows that $V_1(t) = V_2(t)$ and

$$\begin{aligned} & r(0,t')t' + r(0;t',t)(t-t') = r(0,t)t \\ \Rightarrow \quad & r(0;t',t) = \frac{r(0,t)t - r(0,t')t'}{t-t'}. \end{aligned} \tag{28}$$

By direct computation, one can verify

$$\begin{aligned}
\lim_{t\to t'^+} r(0;t',t) &= \lim_{t\to t'^+} \frac{r(0,t)t - r(0,t')t'}{t-t'} \\
&= \lim_{t\to t'^+} \frac{(r(0,t)t - r(0,t)t') + (r(0,t)t' - r(0,t')t')}{t-t'} \\
&= \lim_{t\to t'^+} \left( r(0,t) + t'\frac{r(0,t)-r(0,t')}{t-t'} \right) \\
&= r(0,t) + t\frac{dr(0,t)}{dt} \\
&= r(t). \qquad (29)
\end{aligned}$$

## 5.2 Discrete Interest Compounding

Assume that interest is compounded $m$ times every year, and let $r_m(0,t)$ be the corresponding zero rate curve. The amount $B_m(t)$ that accumulates at time $t$ from an amount $B(0)$ at time zero by compounding interest $m$ times a year is given by

$$B_m(t) = B(0)\left(1 + \frac{r_m(0,t)\tau}{m}\right)^{\frac{mt}{\tau}}, \qquad (30)$$

where $\tau$ is the time period over which the rate of return is quoted. It is conventional to choose $\tau$ to be one year. Thus,

$$B_m(t) = B(0)\left(1 + \frac{r_m(0,t)}{m}\right)^{mt}, \qquad (31)$$

where time is in units of $\tau$ and rates are in units of $\tau^{-1}$. The corresponding discount factor is

$$\left(1 + \frac{r_m(0,t)}{m}\right)^{-mt}. \qquad (32)$$

The most common types of discretely compounded interest are:

Annually compounded interest

$$B_1(t) = B(0)(1 + r_1(0,t))^t; \qquad (33)$$

Semi-annually compounded interest

$$B_2(t) = B(0)\left(1 + \frac{r_2(0,t)}{2}\right)^{2t}; \qquad (34)$$

Quarterly compounded interest

$$B_4(t) = B(0)\left(1 + \frac{r_4(0,t)}{4}\right)^{4t}; \qquad (35)$$

Monthly compounded interest

$$B_{12}(t) = B(0)\left(1 + \frac{r_{12}(0,t)}{12}\right)^{12t}. \qquad (36)$$

It is very important to understand that all discretely compounded interest rates $r_m(0,t)$ are approximations of the same rate of return over one year, irrespective of the compounding frequency $m$.

If a zero rate curve corresponding to any type of compounding is known, then all the zero rate curves corresponding to any other type of compounding are also known for no-arbitrage reasons - the future value of an initial cash amount must be the same regardless of how interest is compounded. For example, the following relationship between the continuously compounded zero rate curve $r(0,t)$ and the zero rate curve $r_m(0,t)$ corresponding to interest compounded $m$ times every year can be derived from Equation 17 and 30:

$$\left(1+\frac{r_m(0,t)\tau}{m}\right)^{\frac{mt}{\tau}} = e^{r(0,t)t} = e^{r(0,t)\tau\times\left(\frac{t}{\tau}\right)}. \tag{37}$$

Thus,

$$r(0,t) = \frac{m}{\tau}\log\left(1+\frac{r_m(0,t)\tau}{m}\right), \tag{38}$$

or conversely,

$$r_m(0,t) = \frac{m}{\tau}\left(e^{\frac{r(0,t)\tau}{m}}-1\right). \tag{39}$$

Finally we note that compounding interest continuously is the limiting case of discretely compounded interest, if interest is compounded very frequently:

$$\lim_{m\to\infty} r_m(0,t) = r(0,t). \tag{40}$$

### 5.3 Bond Pricing, Yield, Duration and Convexity

**Definition 8** (Coupon Bond)**.** A *coupon bond* with maturity $t$ is a contract which pays the *face value* of the bond at a certain time $t$ in the future called the *maturity* of the bond and makes other payments called *coupons* at predetermined times in the future by the issuer of the bond to the buyer of the bond.

The price of a bond is equal to the sum of all the future cash flows discounted to the present by the risk-free zero rates. Let $B$ be the value of a bond with future cash flows $c_i$ to be paid to the holder of the bond at time $t_i$ and $r(0,t_i)$ be the continuously compounded zero rate corresponding to $t_i$, $\forall i = 1,\cdots,n$. Therefore,

$$B = \sum_{i=1}^{n} c_i e^{-r(0,t_i)t_i} = \sum_{i=1}^{n} e^{-\int_0^{t_i} r(t')dt'}, \tag{41}$$

where $r(t)$ denotes the instantaneous rates. It is conventional to assume the face value of a bond is equal to 100, i.e. $F = 100$.

<u>ANNUAL COUPON BOND</u> with face value $F$, coupon rate $C$ and maturity $T$ pays the holder of the bond a coupon payment $CF$ every year, except at maturity. The final payment at maturity is the face value of the bond plus one coupon payment, i.e. $(1+C)F$.

<u>SEMI-ANNUAL COUPON BOND</u> with face value $F$, coupon rate $C$ and maturity $T$ pays the holder of the bond a coupon payment $\frac{1}{2}CF$ every six months, except at maturity. The final payment at maturity is the face value of the bond plus one coupon payment, i.e. $(1+\frac{1}{2}C)F$.

QUARTER COUPON BOND with face value $F$, coupon rate $C$ and maturity $T$ pays the holder of the bond a coupon payment $\frac{1}{4}CF$ every three months, except at maturity. The final payment at maturity is the face value of the bond plus one coupon payment, i.e. $(1+\frac{1}{4}C)F$.

**Definition 9** (Bond Yield)**.** The *yield* of a bond is the internal rate of return of the bond, i.e., the constant rate at which the sum of the discounted future cash flows of the bond is equal to the price of the bond. If $B$ is the price of a bond with cash flows $c_i$ at times $t_i$, $i = 1, \cdots, n$, and if $y$ is the yield of the bond, then

$$B \triangleq \sum_{i=1}^{n} c_i e^{-y t_i}. \tag{42}$$

Note that the price of a bond is a decreasing function of the yield:

$$\frac{\partial B}{\partial y} = -\sum_{i=1}^{n} c_i t_i e^{-y t_i} < 0. \tag{43}$$

**Definition 10** (Par Yield)**.** *Par yield* is the coupon rate that makes the value of the bond equal to its face value. In other words, par yield is the value of $C$ of the coupon rate such that $B = F$.

**Definition 11** (Bond Duration)**.** The modified *duration* $D$ of a bond is the rate of change of the price of the bond with respect to the yield of the bond, normalized by the price of the bond, and with the opposite sign, i.e.,

$$D \triangleq -\frac{1}{B}\frac{\partial B}{\partial y}. \tag{44}$$

Here, the price $B$ of the bond is considered a function of the yield $y$ of the bond, i.e. $B = B(y)$.

From Equation 42, it follows that

$$D = \frac{1}{B}\sum_{i=1}^{n} c_i t_i e^{-y t_i}. \tag{45}$$

**Definition 12** (Macaulay Duration)**.** The *Macaulay duration* of a bond is the weighted average of the cash flows times, with weights equal to the value of the corresponding cash flow discounted with respect to the yield of the bond, i.e.,

$$D_{\mathrm{Mac}} = \frac{1}{B}\sum_{i=1}^{n} c_i t_i \mathrm{Disc}(t_i, y), \tag{46}$$

where $\mathrm{Disc}(t_i, y)$ is the discount factor corresponding to time $t_i$, computed with respect to the yield of the bond.

The Macaulay duration and the modified duration of a bond have the same value if interest is compounded continuously. This is not the case if interest is compounded discretely, when the modified duration of a bond is smaller than its Macaulay duration.

Let $\Delta y$ be the change in the yield of the bond, the discretized version of 44 can be written as

$$\frac{\Delta B}{B} \approx -\Delta y D. \tag{47}$$

In other words, the return $\Delta B/B$ of the bond can be approximated by the duration of the bond multiplied by the parallel shift in the yield curve, with opposite sign. This is a useful result, since a small, parallel shift in the yield curve generates a change of approximately equal size in the yield of a bond.

**Definition 13** (Bond Convexity)**.** The convexity $C$ of a bond with price $B$ and yield $y$ is

$$C = \frac{1}{B}\frac{\partial^2 B}{\partial y^2}. \tag{48}$$

Using Equation 42, it is easy to see that

$$C = \frac{1}{B}\sum_{i=1}^{n} c_i t_i^2 e^{-y t_i}. \tag{49}$$

Note that the following approximation is more accurate than Equation 47:

$$\frac{\Delta B}{B} \approx -D\Delta y + \frac{1}{2}C(\Delta y)^2. \tag{50}$$

Formula 50 can be used to justify a fact well known in bond trading: for two bonds with the same duration, the bond with higher convexity provides a higher return for small changes in the yield curve. To see this formally,

$$\begin{aligned} \frac{\Delta B_1}{B_1} &\approx -D_1\Delta y + \frac{1}{2}C_1(\Delta y)^2 = -D\Delta y + \frac{1}{2}C_1(\Delta y)^2, \\ \frac{\Delta B_2}{B_2} &\approx -D_2\Delta y + \frac{1}{2}C_2(\Delta y)^2 = -D\Delta y + \frac{1}{2}C_2(\Delta y)^2. \end{aligned} \tag{51}$$

Since $C_1 \geqslant C_2$ and $\delta y^2 \geqslant 0$, we conclude that regardless of whether the yield goes up or down,

$$\frac{\Delta B_1}{B_1} \geqslant \frac{\Delta B_2}{B_2}.$$

In other words, the bond with higher convexity provides a higher return, and is the better investment, all other things being considered equal.

## 5.4 Parallel Shifts in the Yield Curve

A frequent assumption made when analyzing bond portfolios is that small, parallel shifts in the zero rate curve $r(0,t)$ result in identical changes in the yield of every bond in the portfolio.

**Lemma 5.0.2.** *If the zero rate curve experiences a small parallel shift of size $\delta r$, the corresponding change $\delta y$ in the yield of a bond is approximately equal to the parallel shift in the zero rates, i.e., $\delta y \approx \delta r$.*

*Proof.* To clarify why it is possible to make this assumption, consider a bond with future cash flows $c_i$ at times $t_i$, $i = 1, \cdots, n$. Let $B$ be the value of the bond and let $y$ be the yield of the bond. From Equation 41 and Equation 42,

$$B = \sum_{i=1}^{n} c_i e^{-r(0,t_i)t_i} = \sum_{i=1}^{n} c_i e^{-y t_i}, \tag{52}$$

where $r(0,t)$ is the continuously compounded zero rate curve. Upon an infinitesimal parallel shift in the zero rate curve $r(0,t) \to r(0,t) + \delta r$.

$$\delta r \sum_{i=1}^{n} t_i c_i e^{-r(0,t_i)t_i} = \delta y \sum_{i=1}^{n} t_i c_i e^{-y t_i}. \tag{53}$$

Now make the following approximation on duration,

$$\sum_{i=1}^{n} t_i c_i e^{-r(0,t_i)t_i} \approx \sum_{i=1}^{n} t_i c_i e^{-y t_i}, \tag{54}$$

we conclude that $\delta y \approx \delta r$. $\square$

## 5.5 Dollar Duration, Dollar Convexity and Bond Portfolio

The bond duration and convexity are not well suited for analyzing bond portfolios since they are derivatives relative to bond price and thus non-additive, i.e. the duration and the convexity of a portfolio made of positions in different bonds are not equal to the sum of the durations or of the convexities, respectively. Dollar duration and dollar convexity are additive and can be used to measure the sensitivity of bond portfolios with respect to parallel changes in the zero rate curve.

**Definition 14.** The *dollar duration* of a bond measures the sensitivity of the bond price with respect to small changes of the bond yield:

$$D_{\$} = -\frac{\partial B}{\partial y}. \tag{55}$$

The *dollar convexity* of a bond measures the sensitivity of the dollar duration of a bond with respect to small changes of the bond yield:

$$C_{\$} = \frac{\partial^2 B}{\partial y^2} = -\frac{\partial D_{\$}}{\partial y}. \tag{56}$$

A similar relationship as Formula 50 exists for dollar duration and dollar convexity,

$$\Delta B \approx -D_{\$}\Delta y + \frac{1}{2} C_{\$}(\Delta y)^2. \tag{57}$$

Consider a bond portfolio made of (either long or short) positions in $p$ different bonds, and let $B_j$, $j = 1, \cdots, p$, be the (positive or negative) values of these bond positions. Denote by

$$V = \sum_{j=1}^{p} B_j, \tag{58}$$

the value of the bond portfolio.

**Definition 15.** The dollar duration $D_{\$}(V)$ of a bond portfolio is the sum of the dollar durations of the underlying bond positions, i.e.,

$$D_{\$}(V) = \sum_{j=1}^{p} D_{\$}(B_j). \tag{59}$$

The dollar convexity $C_{\$}(V)$ of a bond portfolio is the sum of the dollar convexities of the underlying bond positions, i.e.,

$$C_{\$}(V) = \sum_{j=1}^{p} C_{\$}(B_j). \tag{60}$$

Consider a bond portfolio with value $V$ and denote by $D_{\$}(V)$ and $C_{\$}(V)$ the dollar duration and the dollar convexity of the portfolio. If the zero rate curve experiences a small parallel shift of size $\delta r$, the corresponding change $\Delta V$ in the value of the bond portfolio can be approximated as follows

$$\Delta V \approx -D_{\$}(V)\delta r + \frac{1}{2} C_{\$}(V)\delta r^2. \tag{61}$$

This results from the linearity of Equation 59 and Equation 60.

**Definition 16** (DV01)**.** The DV01 of a bond measures the change in the value of the bond for a decrease of one basis point (i.e., of $0.01\%$, or equivalently, of $0.0001$) in the yield of the bond, and is equal to the dollar duration of the bond divided by $10000$, i.e.

$$\text{DV01} \triangleq \frac{D_\$}{10000}, \tag{62}$$

the change $\Delta B$ in the value of a bond corresponding to a change $\Delta y = -0.0001$ in the value of the bond yield approximated as follows:

$$\Delta B \approx -D_\$ \Delta y = D_\$ \times 0.0001 \triangleq \text{DV01}. \tag{63}$$

**Definition 17.** The DV01 of a bond portfolio measures the change in the value of the portfolio for a downward parallel shift of the zero rate curve of one basis point, and is equal to the dollar duration of the bond portfolio divided by $10000$, i.e.

$$\text{DV01}(V) = \frac{D_\$(V)}{10000}. \tag{64}$$

Note that, the duration/convexity of a single bond involves the sensitivity with respect to changes of bond yield, while the duration/convexity of a bond portfolio involves the sensitivity with respect to parallel shift of the zero rate curve. The distinctions are related through the results on parallel shifts in the yield curve derived in Section 5.4.

To reduce the sensitivity of a bond portfolio with respect to small changes in the zero rate curve, it is desirable to have a portfolio with small dollar duration and dollar convexity. The process of obtaining a portfolio with zero dollar duration and dollar convexity is called *portfolio immunization*, and can be done by taking positions in other bonds available on the market.

Let $V$ be the value of a portfolio with dollar duration $D_\$$ and dollar convexity $C_\$$. Take positions of sizes $B_1$ and $B_2$, respectively, in two bonds with duration and convexity $D_i$ and $C_i$, $i = 1, 2$. The value of the new portfolio is

$$\Pi = V + B_1 + B_2. \tag{65}$$

Choose $B_1$ and $B_2$ such that the dollar duration and dollar convexity of the portfolio $\Pi$ are equal to zero, i.e.

$$\begin{aligned} D_\$ &= D_\$(V) + D_\$(B_1) + D_\$(B_2) = D_\$(V) + B_1 D_1 + B_2 D_2 = 0, \\ C_\$ &= C_\$(V) + C_\$(B_1) + C_\$(B_2) = C_\$(V) + B_1 C_1 + B_2 C_2 = 0. \end{aligned}$$

The resulting system for $B_1$ and $B_2$, i.e.

$$\begin{aligned} B_1 D_1 + B_2 D_2 &= -D_\$(V), \\ B_1 C_1 + B_2 C_2 &= -C_\$(V). \end{aligned}$$

has a unique solution if and only if

$$\frac{D_1}{C_1} \neq \frac{D_2}{C_2}. \tag{66}$$

Portfolio immunization is similar, in theory, to hedging portfolios of derivative securities with respect to Greeks, described in Section 7.4.

### 5.6 Bond with Discrete Interest Compounding

We derive formulas for bond pricing, yield, duration, and convexity for discretely compounded interest. Assume that interest is compounded $m$ times a year and let $r_m(0,t)$ be the corresponding zero rate curve. From Equation 32, the price $B$ of a bond with future cash flows $c_i$ at times $t_i$, $i = 1, \cdots, n$ is

$$B = \sum_{i=1}^{n} c_i \left(1 + \frac{r_m(0,t_i)}{m}\right)^{-mt_i}, \tag{67}$$

where time is in units of $\tau$ and rate is in units of $\tau^{-1}$ and $\tau$ is chosen to be a time period of one year. The yield $y$ of the bond, compounded $m$ times a year, is the internal rate of return of the bond. In other words,

$$B = \sum_{i=1}^{n} c_i \left(1 + \frac{y}{m}\right)^{-mt_i}. \tag{68}$$

By direct computation, the modified duration and the convexity

$$D = \frac{1}{B} \sum_{i=1}^{n} c_i t_i \left(1 + \frac{y}{m}\right)^{-mt_i - 1}, \tag{69}$$

$$C = \frac{1}{B} \sum_{i=1}^{n} c_i t_i \left(t_i + \frac{1}{m}\right) \left(1 + \frac{y}{m}\right)^{-mt_i - 2}. \tag{70}$$

It is interesting to note the difference between continuous and discrete compounding which occurs when looking at the Macaulay duration of a bond. If the bond yield is compounded discretely, e.g. $m$ times a year, the discount factors are

$$\mathrm{Disc}(t_i, y) = \left(1 + \frac{y}{m}\right)^{-mt_i}. \tag{71}$$

From the definition of Macaulay duration, Equation 46,

$$D_{\mathrm{Mac}} = \frac{1}{B} \sum_{i=1}^{n} c_i t_i \left(1 + \frac{y}{m}\right)^{-mt_i}. \tag{72}$$

Compare Equation 69 and Equation 72, we obtain that

$$D = D_{\mathrm{Mac}} \left(1 + \frac{y}{m}\right)^{-1}. \tag{73}$$

In other words, he modified duration of a bond is small than the Macaulay duration of the bond, for discretely compounded interest, unlike for continuously compounded interest, where the modified duration and the Macaulay duration of a bond are equal.

### 5.7 Zero Coupon Bond

**Definition 18** (Zero Coupon Bond)**.** A *zero coupon bond* with face value $F$ and maturity $T$ is a contract which guarantees the holder the face value of the bond to be paid at maturity. and provides no other cash flows. In other words, a zero coupon bond is a bond with coupon rate equal to zero.

Let $F$ be the face value of a zero coupon bond with maturity $T$. If interest is compounded continuously, the bond pricing formula Equation 41 becomes

$$B = Fe^{-r(0,T)T}, \tag{74}$$

where $B$ is the price of the bond at time zero and $r(0,T)$ is the zero rate corresponding to time $T$. If the instantaneous interest rate $r(t)$ is given, the bond pricing can be written as

$$B = Fe^{-\int_0^T r(t')dt'}. \tag{75}$$

From the definition of bond yield,

$$B = Fe^{-yT}, \tag{76}$$

and Equation 74, we conclude that $y = r(0,T)$, i.e., the yield of a zero coupon bond is the same as the zero rate corresponding to the maturity of the bond. This explains why the zero rate curve $r(0,t)$ is also called the yield curve.

By direct computation, one can verify that the duration of a zero coupon bond

$$D = T, \tag{77}$$

and the convexity of a zero coupon bond

$$C = T^2. \tag{78}$$

If interest is compounded discretely, let $r_m(0,t)$ be the zero rate curve corresponding to interest compounded $m$ times a year. Let $B$ and $y$ be the price and the yield of a zero coupon bond with face value $F$ and maturity $T$. From Equation 67,

$$B = F\left(1 + \frac{r_m(0,T)}{m}\right)^{-mT}; \tag{79}$$

from Equation 68,

$$B = F\left(1 + \frac{y}{m}\right)^{-mT}. \tag{80}$$

From Equation 79 and Equation 80, we conclude that $y = r_m(0,T)$, i.e., the yield of the zero coupon bond is the same as the zero rate corresponding to the maturity of the bond.

By direct computation, we find the duration and convexity of a zero coupon bond,

$$D_{\text{Mac}} = T; \tag{81}$$

$$D = T\left(1 + \frac{y}{m}\right)^{-1}; \tag{82}$$

$$C = T^2\left(1 + \frac{1}{mT}\right)\left(1 + \frac{y}{m}\right)^{-2}. \tag{83}$$

## 5.8 Bootstrapping for Finding Zero Rate Curves

Recall that the zero rate $r(0,r)$ between time zero and time $t$ is the rate of return of a cash deposit made at time zero and maturing at time $t$. Here we assume that interest is compounded continuously. Also recall from Equation 41 that the price $B$ of a bond with future cash flows $c_i$ to be paid to the holder of the bond on dates $t_i$, $i = 1, \cdots, n$ is given by

$$B = \sum_{i=1}^{n} c_i e^{-r(0,t_i)t_i}. \tag{84}$$

If the prices of several different bonds are known, then we are provided with information on the zero rates corresponding to each date when a bond makes a payment. Since this is a discrete and finite set of data, the general form of the zero rate curve $r(0,t)$ for all times $t$ smaller than the largest maturity of any of the bonds cannot be determined uniquely.

The bootstrapping method provides a simplified framework in which the zero rate curve can be uniquely determined from a given set of bond prices, assuming zero credit risk, i.e. that the bonds never default and requiring that the zero rate is linear between any two consecutive bond maturities. Specifically, if one is given the prices of a coupon-paying bonds of each maturity $t_1 < \cdots < t_n$,

$$B_j = \sum_{i=1}^{n} c_{j,i} e^{-r(0,t_i)t_i}, \quad j = 1, \cdots, m, \tag{85}$$

one can solve the following triangular system for the discount factor $e^{-r(0,t_i)t_i}$ recursively

$$\begin{pmatrix} c_{1,1} & & & \\ c_{2,1} & c_{2,2} & & \\ \vdots & \vdots & \ddots & \\ c_{m,1} & c_{m,2} & \cdots & c_{m,n} \end{pmatrix} \begin{pmatrix} \mathrm{Disc}(t_1) \\ \mathrm{Disc}(t_2) \\ \vdots \\ \mathrm{Disc}(t_n) \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix}, \tag{86}$$

where $\mathrm{Disc}(t_i) = e^{-r(0,t_i)t_i}$.

## 6 FORWARD CONTRACTS

**Definition 19** (Forward Contract)**.** A *forward contract* with *delivery price* $K$ obligates its holder to buy one unit of the asset at expiration time $T$ in exchange for payment $K$.

To value a forward contract, consider a forward contract with delivery price $K$ that will expire at time $T$. Let $F(t)$ be the value of the forward contract at time $t$, consider a portfolio made of the following assets:

- long one forward contract;
- short one unit of the underlying asset;

The value of the portfolio at time $t$ is $V(t) = F(t) - S(t)$. At maturity $T$, the forward contract is exercised and the short position is closed: we pay $K$ for one unit of the underlying asset and return it to the lender. The value of the portfolio at time $T$ is $V(T) = -K$. Since $V(T)$ is a cash amount independent of the price $S(T)$ of the underlying asset at maturity, we find from Lemma 5.0.1 that

$$V(t) = V(T)e^{-r(T-t)} = -Ke^{-r(T-t)}. \tag{87}$$

Thus, the value at time $t$ of a forward contract with delivery price $K$ and maturity $T$ is

$$F(t) = S(t) - Ke^{-r(T-t)}. \tag{88}$$

If the underlying asset pays dividends continuously at rate $q$, the value of the forward contract is

$$F(t) = S(t)e^{-q(T-t)} - Ke^{-r(T-t)}. \tag{89}$$

**Definition 20** (Forward Price)**.** The *forward price* of an asset at time $t$ is defined to be the value of delivery price $K$ of the forward contract at time $t$ to have value zero. From Equation 88, the forward price $F = S(0)e^{(r-q)T}$.

# 7 EUROPEAN CALL AND PUT OPTIONS

**Definition 21** (Call Option)**.** A *call option* on an underlying asset is a contract between two parties which gives the buyer of the option the right, but not the obligation, to **buy** from the seller of the option one unit of the asset at a predetermined time $T$ in the future, called the *maturity* of the option, for a predetermined price $K$, called the *strike* of the option. For this right, the buyer of the option pays $C(t)$ at time $t < T$ to the seller of the option.

**Definition 22** (Put Option)**.** A *put option* on an underlying asset is a contract between two parties which gives the buyer of the option the right, but not the obligation, to **sell** to the seller of the option one unit of the asset at a predetermined time $T$ in the future, called the *maturity* of the option, for a predetermined price $K$, called the *strike* of the option. For this right, the buyer of the option pays $P(t)$ at time $t < T$ to the seller of the option.

The payoff of a call option at maturity is

$$C(T) = (S(T) - K)^+ = \begin{cases} S(T) - K, & \text{if } S(T) > K; \\ 0, & \text{otherwise.} \end{cases} \tag{90}$$

The payoff of a put option at maturity is

$$C(T) = (K - S(T))^+ = \begin{cases} K - S(T), & \text{if } S(T) < K; \\ 0, & \text{otherwise.} \end{cases} \tag{91}$$

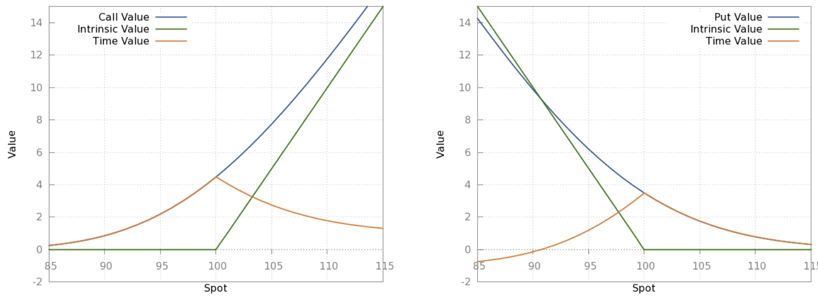
Reference to Figure 1 on the following page. The options described above are plain vanilla European options. An option which can be exercised at any time prior to maturity is called an American option. Two parties exist in an option contract: the buyer of the option and the seller of the option. The buyer of the option is long the option (i.e., has a long position in the option) and the seller of the option is short the option (i.e., has a short position in the option).

**Definition 23.** Let $S(t)$ be the price of the underlying asset at time $t$. A call option is *in the money* (ITM), *at the money* (ATM), or *out of the money* (OTM) at time $t$ if $S(t) > K$, $S(t) = K$, or $S(t) < K$, respectively. A put option is *in the money* (ITM), *at the money* (ATM), or *out of the money* (OTM) at time $t$ if $S(t) < K$, $S(t) = K$, or $S(t) > K$, respectively.

**Example 7.0.3.** *How much are plain vanilla European options worth if the value of the underlying asset is zero?*

*Solution.* According to Example 4.1.1, at maturity, the spot price of the underlying asset will be zero, i.e., $S(T) = 0$. Then

$$\begin{aligned} C(T) &= (S(T) - K)^+ = 0, \\ P(T) &= (K - S(T))^+ = K. \end{aligned}$$




**(a)** European call option. **(b)** European put option.

**Figure 1:** Payoff of European call and put options as functions of the price of underlying asset.

From Lemma 5.0.1, we conclude that

$$C(t) = 0, \tag{92}$$

$$P(t) = Ke^{-r\tau}, \tag{93}$$

where $\tau = T - t$ is the time to maturity. □

## 7.1 The Put-Call Parity

**Theorem 7.1** (Put-Call Parity). *Let $C(t)$ and $P(t)$ be the values at time $t$ of the European call and put option, respectively, which maturity $T$ and strike $K$, on the same non-dividend paying asset with spot price $S(t)$. The put-call parity states that*

$$P(t) + S(t) - C(t) = Ke^{-r(T-t)}. \tag{94}$$

*If the underlying asset pays dividends continuously at the rate $q$, the put-call parity has the form*

$$P(t) + S(t)e^{-q(T-t)} - C(t) = Ke^{-r(T-t)}. \tag{95}$$

*Proof.* For simplicity, we restrict our attention to non-dividend paying assets. Consider a portfolio made of the following assets:

- long one put option;
- long one unit of the underlying asset;
- short one call option.

The value of the portfolio at time $t$ is

$$V(t) = P(t) + S(t) - C(t). \tag{96}$$

Regardless of the value $S(T)$ of the underlying asset at the maturity of the option

$$V(T) = (K - S(T))^+ + S(T) - (S(T) - K)^+ \equiv K. \tag{97}$$

From Lemma 5.0.1,

$$V(t) = P(t) + S(t) - C(t) = K^{-r(T-t)}. \tag{98}$$

Note that the put-call parity is model-independent. □

Being long a call and short a put with the same strike $K$ and maturity $T$ is equivalent to being long a forward contract with delivery price $K$ and maturity $T$. To see this, the value at time $T$ of a long call and short put position is

$$C(T) - P(T) = (K - S(T))^+ - (S(T) - K)^+ = S(T) - K \triangleq F(T). \quad (99)$$

From Theorem 4.1,

$$C(t) - P(t) = F(t). \quad (100)$$

The same result follows from Theorem 214 and Eq.88,

$$C(t) - P(t) = S(t) - Ke^{-r(T-t)} = F(t). \quad (101)$$

If the put-call parity is violated, a *theoretical* arbitrage is present. Consider, for simplicity, a non-dividend paying asset, and let $C$ and $P$ be the values of a call and a put option, respectively, on this asset, with the same strike $K$ and the same maturity $T$.

CASE 1 Assume that

$$P(t) + S(t) - C(t) > Ke^{-r(T-t)}. \quad (102)$$

To arbitrage this, a trader would *buy low* and *sell high*. More precisely, to realize the arbitrage, set up the following portfolio at time $t$:

- sell one put option: $-P(t)$;
- sell short one unit of the underlying asset: $-S(t)$;
- buy one call option: $+C(t)$;
- cash position: $P(t) + S(t) - C(t)$.

At time $T$, buy one share for the market price $S(T)$ to close the short sale. The cash position grows at the risk-free rate $r$ to $[P(t) + S(t) - C(t)]e^{r(T-t)}$ at time $T$. Therefore, the value of the portfolio at time $T$ is

$$\begin{aligned} V(T) &= -P(T) - S(T) + C(T) + [P(t) + S(t) - C(t)]e^{r(T-t)} \\ &= -K + [P(t) + S(t) - C(t)]e^{r(T-t)} \\ &= [P(t) + S(t) - C(t) - Ke^{-r(T-t)}]e^{r(T-t)} \\ &> 0. \end{aligned} \quad (103)$$

In other words, we set up a portfolio at time $t$ with cost zero, i.e., with $V(t) = 0$, whose value $V(T)$ at time $T$ is positive for any state of the market. According to Theorem 4.1, this creates an arbitrage.

CASE 2 Assume that

$$P(t) + S(t) - C(t) < Ke^{-r(T-t)}. \quad (104)$$

To arbitrage this, a trader would *buy low* and *sell high*. More precisely, to realize the arbitrage, set up the following portfolio at time $t$:

- long one put option: $+P(t)$;
- long one unit of the underlying asset: $+S(t)$;
- short one call option: $-C(t)$;
- cash position: $-P(t) - S(t) + C(t)$.

At time $T$, the cash position grows at the risk-free rate $r$ to $[-P(t) - S(t) + C(t)]e^{r(T-t)}$ at time $T$. Therefore, the value of the portfolio at time $T$ is

$$\begin{aligned} V(T) &= P(T) + S(T) - C(T) + [-P(t) - S(t) + C(t)]e^{r(T-t)} \\ &= K + [-P(t) - S(t) + C(t)]e^{r(T-t)} \\ &= [-P(t) - S(t) + C(t) + Ke^{-r(T-t)}]e^{r(T-t)} \\ &> 0. \end{aligned} \tag{105}$$

As before, we set up a portfolio at time $t$ with cost zero, i.e., with $V(t) = 0$, whose value $V(T)$ at time $T$ is positive for any state of the market. According to Theorem 4.1, this creates an arbitrage.

## 7.2 No-Arbitrage Conditions

**Theorem 7.2.** *Call options on the same underlying asset and with the same maturity, with strikes $K_1 < K_2 < K_3$, are trading for $C_1$, $C_2$ and $C_3$, respectively (no bid-ask spread), with $C_1 > C_2 > C_3$. Find necessary and sufficient conditions on the prices $C_1$, $C_2$ and $C_3$ such that no-arbitrage exists corresponding to a portfolio made of positions in the three options.*

See Fabio Mercurio's paper *No-arbitrage Conditions For a Finite Options System*.

**Theorem 7.3.** *Assume that interest rates are constant and equal to $r$. Show that, unless the price of a call option $C$ or a put option $P$ with strike $K$ and maturity $T$ on a non-dividend paying asset with spot price $S$ satisfies the inequality*

$$Se^{-qT} - Ke^{-rT} \leqslant C \leqslant Se^{-qT}, \tag{106}$$

$$Ke^{-rT} - Se^{-qT} \leqslant P \leqslant Ke^{-rT}. \tag{107}$$

*arbitrage opportunities arise.*

## 7.3 The Greeks

Let $V$ be the value of a portfolio of derivative securities on one underlying asset, by which we mean, the portfolio could simply be, for example, the underlying asset, or a plain vanilla option on the underlying asset. The rates of change of the value of the portfolio with respect to various parameters are important for hedging purposes. These rates of change are called the *Greeks*, and are denoted by symbols from Greek alphabet.

- Delta ($\Delta$) is the rate of change of the value of the portfolio with respect to the spot price $S$ of the underlying asset:

$$\Delta = \frac{\partial V}{\partial S}; \tag{108}$$

- Gamma ($\Gamma$) is the rate of change of the Delta of the portfolio with respect to the spot price $S$ of the underlying asset, i.e., the second partial derivative of the value of the portfolio with respect to $S$:

$$\Gamma = \frac{\partial^2 V}{\partial S^2}; \tag{109}$$

- Theta ($\Theta$) is the rate of change of the value of the portfolio with respect to time (not with respect to maturity $T$):

$$\Theta = \frac{\partial V}{\partial t}; \tag{110}$$

- Rho ($\rho$) is the rate of change of the value of the portfolio with respect to the risk-free interest rate:

$$\rho = \frac{\partial V}{\partial r}; \tag{111}$$

- Vega is the rate of change of the value of the portfolio with respect to the volatility $\sigma$ of the underlying asset:

$$\nu = \frac{\partial V}{\partial \sigma}. \tag{112}$$

### 7.4 $\Delta$-hedging and $\Gamma$-hedging

Assume you are long a call option. If the price of the underlying asset declines, the value of the call decreases and the long call position loses money. To protect against a downturn in the price of he underlying asset, i.e., to hedge, you sell short $\Delta$ units of the underlying asset. You now have a portfolio consisting of a long position in one call option and a short position in $\Delta$ units of the underlying asset. The goal is to choose $\Delta$ in such a way that the value of the portfolio is not sensitive to small changes in the price of the underlying asset.

If $\Pi$ is the value of the portfolio, then

$$\Pi(S) = C(S) - \Delta S. \tag{113}$$

Assume that the spot price $S$ of the underlying asset undergoes a small change to $S + dS$, i.e., $dS \ll S$. The change in the value of the portfolio is

$$\begin{aligned} \Pi(S + dS) - \Pi(S) &= [C(S + dS) - \Delta(S + dS)] - [C(S) - \Delta S] \\ &= C(S + dS) - C(S) - \Delta dS \\ &\approx 0, \end{aligned} \tag{114}$$

where we look for $\Delta$ such that the value of the portfolio is insensitive to small changes in the price of the underlying asset and find that

$$\Delta \approx \frac{C(S + dS) - C(S)}{dS} = \frac{\partial C}{\partial S}, \tag{115}$$

which is the same as $\Delta(C)$, the Delta of a call option defined in Definition 108 as the rate of change of the value of the call with respect to changes in the price of the underlying asset.

A similar argument on hedging a long position in a portfolio with price $V$ made of derivative securities on a single underlying asset shows that the appropriate $\Delta$-hedging position to be taken in the asset is

$$\Delta = \frac{\partial V}{\partial S} \tag{116}$$

where $S$ is the spot price of the asset. The value of the new portfolio $\Pi = V - \Delta S$ will be insensitive to small changes in the price of the underlying asset.

Recall that the correct hedging position for a long call is short $\Delta$ shares. Therefore, the hedging position for a short call is long $\Delta$ shares. Similarly, a long put position is hedged by going long $\Delta$ shares: as the spot price of the underlying asset goes up, a long put position loses value while a long asset position will gain value, thus offsetting the loss on the long put. A short

**Table 1:** $\Delta$-hedge positions

| Positions | | |
|---|---|---|
| Option Position | Hedge Position | Sign of Option Delta |
| Long Call | Short Asset | $+$ |
| Short Call | Long Asset | $-$ |
| Long Put | Long Asset | $-$ |
| Short Put | Short Asset | $+$ |

put is hedged by going short $\Delta$ shares. These facts are summarized in the table below: Note that a portfolio is Delta-neutral only over a short period of time. As the price of the underlying asset changes, the portfolio might become unbalanced, i.e., not Delta-neutral. In this case, the hedge needs to be rebalanced, i.e., units of the underlying asset must be bought or sold in order to make the portfolio Delta-neutral again. Deciding when to rebalance the portfolio is a compromise between rebalancing the hedge often (which keeps the portfolio close to Delta-neutral, and therefore reduces the risk of losses if the price of the underlying asset changes rapidly, but increases the trading costs), and rebalancing only when the hedge becomes inaccurate enough (which reduces trading costs but increases the risk of having to manage a portflio that is close to, but not exactly Delta-neutral, most of the time).

To achieve a better hedge of the portfolio, i.e., to have a portfolio that is even less sensitive to small changes in the price of the underlying asset than a Delta-neutral portfolio, we look for a portfolio that is both Delta-neutral and Gamma-neutral, i.e. such that

$$\Delta(\Pi) = \frac{\partial \Pi}{\partial S} = 0 \quad \text{and} \quad \Gamma(\Pi) = \frac{\partial^2 \Pi}{\partial S^2} = \frac{\partial \Delta(\Pi)}{\partial S} = 0, \tag{117}$$

where the Gamma of the portfolio is denoted by $\Gamma(\Gamma)$. Therefore, if a portfolio is Gamma-neutral, its Delta will be rather insensitive to small changes in the price of the underlying asset. In particular, if the portfolio is both Delta-neutral and Gamma-neutral, then the Delta of the portfolio will change significantly only if large changes in the price of the underlying asset occur. The need to rebalance the hedge for such a portfolio occurs less than in the case of a portfolio that is Delta-neutral but not Gamma-neutral. This saves trading costs and the portfolios are, generally speaking, better hedged most of the time.

To obtain a Delta- and Gamma-neutral portfolio, starting with a portfolio $V$ made of derivative securities over a single underlying asset, we need to be able to trade in two different derivative securities on the underlying asset, e.g., the asset itself and an option (call or put, with any maturity) on the asset. Let $\Delta(V)$ and $\Gamma(V)$ be the Delta and Gamma of the portfolio, respectively. Assume that we can take long and short positions of arbitrary size in two derivative securities with values $D_1$ and $D_2$ on the same underlying asset. The value $\Pi$ of a portfolio made of the initial portfolio and positions $x_1$ in $D_1$ and $x_2$ in $D_2$, respectively, is

$$\Pi = V + x_1 D_1 + x_2 D_2. \tag{118}$$

This portfolio is Delta- and Gamma-neutral, i.e. $\Delta(\Pi) = 0$ and $\Gamma(\Pi) = 0$, if and only if

$$\begin{aligned} \Delta(\Pi) &= \Delta(V) + x_1\Delta(D_1) + x_2\Delta(D_2) = 0, && (119) \\ \Gamma(\Pi) &= \Gamma(V) + x_1\Gamma(D_1) + x_2\Gamma(D_2) = 0. && (120) \end{aligned}$$

Note that this system has a unique solution as long as $D_1$ and $D_2$ do not have the same Delta to Gamma ratio, i.e., if

$$\frac{\Delta(D_1)}{\Gamma(D_1)} \neq \frac{\Delta(D_2)}{\Gamma(D_2)}. \tag{121}$$

## 7.5 European Barrier Options

An exotic option is any option that is not a plain vanilla. Some exotic options are path-dependent: the value of the option depends not only on the value of the underlying asset at maturity, but also on the path followed by the price of the asset between the inception of the option and maturity. It rarely happens to have closed formulas for pricing exotic options. However, European barrier options are path-dependent options for which closed form pricing formulas exist.

**Definition 24** (European Barrier Option)**.** A *barrier option* is different from a plain vanilla option due to the existence of a *barrier* $B$: the option either expires worthless (*knock-out* options), or becomes a plain vanilla option (*knock-in* options) if the price of the underlying asset hits the barrier before maturity. Depending on whether the barrier must be hit from below or from above in order to be triggered, an option is called *up* or *down*. Every option can be either a *call* or a *put* option.

There are, at the first count, at least eight different types of options:

$$\begin{Bmatrix} \text{UP} \\ \text{DOWN} \end{Bmatrix} \times \begin{Bmatrix} \text{IN} \\ \text{OUT} \end{Bmatrix} \times \begin{Bmatrix} \text{PUT} \\ \text{CALL} \end{Bmatrix}. \tag{122}$$

This should be read as follows: choose one entry in each column, e.g., up, out, call. This option is an up-and-out call, which expires worthless if the price of the underlying asset hits the barrier $B$ from below, or has the same payoff at maturity as a call option with strike $K$ otherwise.

The position of the strike $K$ relative to the barrier $B$ is also important. If the spot price $S(0)$ of the underlying asset at time zero is such that the barrier is already triggered, then the option is either worth zero, or it is equivalent to a plain vanilla option. Therefore, we are interested in the case when the barrier is not triggered already at time zero. There are sixteen different barrier options to price, corresponding to

$$\begin{Bmatrix} \text{UP} \\ \text{DOWN} \end{Bmatrix} \times \begin{Bmatrix} \text{IN} \\ \text{OUT} \end{Bmatrix} \times \begin{Bmatrix} \text{PUT} \\ \text{CALL} \end{Bmatrix} \times \begin{Bmatrix} B < K \\ B \geqslant K \end{Bmatrix}. \tag{123}$$

Several barrier options can be priced by using simple no-arbitrage arguments, either in an absolute way, or relative to other barrier options. For example, long positions in one up-and-out and one up-and-in option with all other parameters, i.e., barrier, maturity, and strike of the underlying option, being the same is equivalent to a long position in the corresponding plain vanilla option. To see this, note that, for every possible path for the price of the underlying asset, the final payoff of the plain vanilla option and of a portfolio made of the up-and-out and the up-and-in options are the same:

- If the barrier is hit before the expiry of the options, then the up-and-out option expires worthless, while the up-and-in option becomes a plain vanilla option. Thus, the payoff of the portfolio at maturity will be the same as the payoff of a plain vanilla option.
- If the price of the underlying asset never reaches the barrier, then the up-and-in option is never knocked in, and will expire worthless, while the up-and-out option is never knocked out, and it will have the same payoff at maturity as a plain vanilla option. The payoff at maturity of the portfolio made the barrier options will be the same as the payoff of a plain vanilla option.

Similarly, a portfolio made of a down-and-out and a down-and-in option is equivalent to a plain vanilla option. We only have to find the value of either the in or the out option.

The following knock-out barrier options have value zero:

- up-and-out call with $S_0 < B \leqslant K$;
- down-and-out put with $S_0 > B > K$,

since the barrier must be triggered, and therefore the option will be knocked out, in order for the underlying plain vanilla option to expire in the money.

Using the in-out duality, we conclude that the following knock-in options have the same value as the corresponding plain vanilla options:

- up-and-in call with $S_0 < B \leqslant K$;
- down-and-in put with $S_0 > B > K$.

For all the options listed above, the barrier must be triggered in order for the option to expire in the money. Therefore, it is enough to price the following barrier options:

- up-and-in call with $B > K$;
- up-and-in put with $B > K$;
- up-and-in put with $B < K$;
- down-and-in call with $B > K$;
- down-and-in call with $B < K$;
- down-and-in put with $B < K$.

The corresponding knock-out options can be derived from the in-out duality.

# 8 AMERICAN CALL AND PUT OPTIONS

**Definition 25.** An option which can be exercised at any time prior to maturity is called an *American option*.

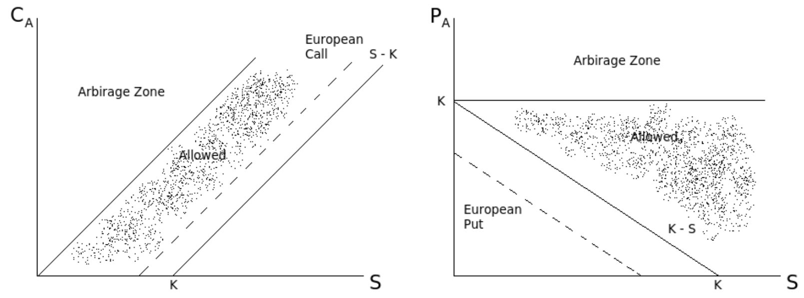
Most traded stock options and futures are of American type while most index options are of European type. The central issue is when to exercise? From the holder point of view, the goal is to maximize the holder's profit.

## 8.1 General Relations Case I. No Diviends

### THE CALL OPTIONS

**Lemma 8.0.1.** $C_A(0) \geqslant (S_0 - K)^+$.

*Proof.* Consider two scenario where $S_0 > K$ or $S_0 \leqslant K$:




**(a)** American options.

**Figure 2:** Payoff of European call and put options as functions of the price of underlying asset.

- Optionality: $C_A(0) \geqslant 0$;
- Assume $S_0 > K$, if $C_A(0) < S_0 - K$, by the option at $C_A(0)$, then exercise immediately. This leads to a profit $S_0 - K$ and the net profit $S_0 - K - C_A(0) > 0$, which gives rise to an arbitrage opportunity.

Hence, the no-arbitrage argument yields $C_A(0) \geqslant (S_0 - K)^+$. □

**Lemma 8.0.2.** $C_A(0) \leqslant S_0$.

*Proof.* If $C_A(0) > S_0$, buy $S_0$ and sell $C_A(0)$ and yield a net profit $> 0$ at $t = 0$. Because the possession of the stock can always allow the deliverance of the stock to cover the exercise if exercised, then we are guaranteed to have a positive future profit. Hence an arbitrage opportunity. □

**Lemma 8.0.3.** $C_A(0) \geqslant C_E(0) \geqslant (S_0 - e^{-rTK})^+$, *where* $C_E$ *is the European call option with the same maturity* $T$ *and strike* $K$.

#### THE PUT OPTIONS

**Lemma 8.0.4.** $P_A(0) \geqslant (K - S_0)^+$.

*Proof.* If $P_A(0) < K - S_0$, buy $P_A$ and exercise immediately, yielding, then, the total cash flow $-P_A(0) + (K - S_0) > 0$, giving rise to an arbitrage. □

**Lemma 8.0.5.** $P_A(0) \leqslant K$.

*Proof.* □

**Lemma 8.0.6.** $P_A(0) \geqslant P_E(0)$.

Note that, for a put, the profit is bounded by $K$. This fact limits the benefit from waiting to exercise and its financial consequence is that one may exercise early if $S_0$ is very small.

### 8.2 The Put-Call Parity for American Options

**Theorem 8.1** (Put-Call Parity for American Options). *Let* $C_A(t)$ *and* $P_A(t)$ *be the values at time* $t$ *of the American call and put option, respectively, which maturity* $T$ *and strike* $K$, *on the same non-dividend paying asset with spot price* $S(t)$. *The put-call parity states that*

$$S(t) - K \leqslant C_A - P_A \leqslant S(t) - Ke^{-r(T-t)}. \tag{124}$$

*Proof.* For the upper bound,

$$\begin{aligned} P_A(t) &\geqslant P_E(t) \\ &= C_E(t) - S(t) + Ke^{-r(T-t)} \\ &= C_A(t) - S(t) + Ke^{-r(T-t)} \\ &\Rightarrow C_A(t) - P_A(t) \leqslant S(t) - Ke^{-r(T-t)}. \end{aligned} \tag{125}$$

For the lower bound, consider the portfolio:

- long one call;
- short one put;
- short the stock;
- hold $K$ dollars in cash

with position $C_A(t) - P_A(t) - S(t) + K$. Here the call is never exercised early. If the put is exercised early at $t' > t$, our position is

$$C_A(t') - (K - S(t')) - S(t') + Ke^{r(t'-t)} = C_A(t') + Ke^{r(t'-t)} - K \geqslant 0, \tag{126}$$

that is, liquidated with net positive profit. On the other hand, if the put is not exercised early, at maturity $T$, we have

$$(S(T) - K)^+ - (K - S(T))^+ - S(T) + Ke^{r(T-t)} = Ke^{r(T-t)} - K \geqslant 0. \tag{127}$$

Therefore, the payoff of the portfolio is non-negative irrespective of whether $S(T) > K$ or $S(T) < K$. Thus, the present value ofthe portfolio is also non-negative:

$$C_A(t) - P_A(t) - S(t) + K \geqslant 0 \Rightarrow C_A(t) - P_A(t) \geqslant S(t) - K. \tag{128}$$

Combining Equation 125 and 128, we proved Equation 124. □

## 8.3 The Optimality of Early Exercise

**Theorem 8.2** (Optimality of Early Exercise)**.** *American call options on non-dividend paying assets are never optimal to exercise, i.e.* $C_A(0) = C_E(0)$, *provided that interest rates are positive.*

*Proof.* Instead of exercising the call, entering into a static hedge by shorting one unit of the underlying asset is guaranteed to be more profitable. To see this, let $T$ be the maturity of the option and assume that, at time $t < T$, the call option is in the money, i.e., $S(t) > K$. For the long call position, consider two different strategies:

First Strategy Exercise the call at time $t$ and deposit the premium $S(t) - K$ at the constant risk-free rate $r$ until time $T$. The value of the resulting portfolio at time $T$ is

$$V_1(T) = (S(t) - K)e^{r(T-t)}. \tag{129}$$

Second Strategy Do not exercise the call; short one unit of the underlying asset at time $t$, and wait until maturity $T$ without exercising the American option. At time $t$, the portfolio will consist of

- long one American call;
- short one unit of the underlying asset;

- cash $S(t)$.

This strategy is a static hedge strategy. The value of this portfolio at time $T$ depends on whether the call expires in the money or not.

If $S(T) > K$, the call is exercised at time $T$, an amount $K$ is paid for one unit of the underlying asset, which is used to close the short position. The cash position of $S(t)$ at time $t$ earns interest at the constant risk free rate $r$. Its value at time $T$ is $S(t)e^{r(T-t)}$. The value of the portfolio will be

$$V_{2,1}(T) = S(t)e^{r(T-t)} - K. \tag{130}$$

If $S(T) \leqslant K$, then the call expires worthless. One unit of the underlying asset is bought at time $T$, for the price $S(T)$, to close the short position. The value of the portfolio will be

$$V_{2,2}(T) = S(t)e^{r(T-t)} - S(T) \geqslant S(t)e^{r(T-t)} - K = V_{2,1}(T). \tag{131}$$

We conclude that, if the second strategy is used, the value of the portfolio at time $T$ is guaranteed to be at least

$$\begin{aligned} V_2(T) &\geqslant \min(V_{2,1}(T), V_{2,2}(T)) \\ &= S(t)e^{r(T-t)} - K \\ &> (S(t) - K)e^{r(T-t)} \\ &= V_1(T) \end{aligned}$$

provided that interest rates are positive, i.e., $r > 0$. The value of the second portfolio is higher than the value of the portfolio if the option is exercised at time $t < T$. In other words, it is never optimal to exercise early an American call option on a non-dividend paying asset. Shorting the underlying asset and holding the American option until maturity will earn a greater payoff at maturity than exercising the call option early. The intuition behind is that consider paying $K$ to get a stock now versus paying $K$ to get a stock later, one gets the interest on $K$, therefore, the difference is $Ke^{r\tau} - K$ if one waits. □

Note that Theorem 8.2 is not true for American put options, nor for American calls on dividend paying assets.

When shorting a dividend paying asset, you are responsible for paying the dividends to the lender of the asset. Therefore, if the static hedge strategy is employed, the value of the portfolio at time $T$ will be reduced by the amount of the dividends paid. Then, if the call option expires in the money, all we can say is that

$$V_{2,1}(T) < S(t)e^{r(T-t)} - K. \tag{132}$$

If the underlying asset pays dividends continuously at the rate $q$, it can be shown that

$$V_{2,1}(T) = S(t)e^{(r-q)(T-t)} - K, \tag{133}$$

which no longer means that $V_{2,1}(T) > (S(t) - K)e^{r(T-t)} = V_1(T)$. In other words, the static hedging strategy of shorting the underlying asset is no longer guaranteed to be worth more at time $T$ than the strategy of exercising the call at time $t$.

For an American put option which is in the money at time $t$, i.e., $S(t) < K$, exercising the option and holding the cash until maturity $T$ generates the following payoff at time $T$:

$$V_3(T) = (K - S(t))e^{r(T-t)}. \tag{134}$$

The static hedging strategy for the long put position is to buy one unit of the underlying asset (paying $S(t)$ cash to do so). At time $t$, the portfolio will consist of

- long one American put;
- long one unit of the underlying asset;
- $-S(t)$ in cash.

In this case, the interest earned by the cash borrowed to buy the asset reduces the value of the portfolio. This is similar to the role played by dividends for the static hedging of American call options.

If the put expires in the money, i.e., $S(T) < K$, then it is exercised, and the share will be sold for $K$. The value of the portfolio in this case is

$$V_{4,1}(T) = K - S(t)e^{r(T-t)}. \tag{135}$$

If the put expires worthless, i.e., $S(T) \geqslant K$, the value of the portfolio is

$$V_{4,2}(T) = S(T) - S(t)e^{r(T-t)} \geqslant K - S(t)e^{r(T-t)} = V_{4,1}(T). \tag{136}$$

Thus,

$$\begin{aligned} V_4(T) &\geqslant \min(V_{4,1}(T), V_{4,2}(T)) \\ &= K - S(t)e^{r(T-t)} \\ &\leqslant (K - S(t))e^{r(T-t)} \\ &= V_3(T). \end{aligned} \tag{137}$$

For early exercise not to be optimal for the American put, the value $V_4(T)$ of the static hedging portfolio at time $T$ should be greater than $V_3(T)$, for any value $S(T)$ of the underlying asset at maturity. However, $V_4(T)$ may be greater or less than $V_3(T)$ from Equation 137 depending on market condition at maturity. We conclude that the argument that worked for the American call option does not work for put options.

### 8.4 General Relations Case II. Diviends

#### THE CALL OPTION

# Part III. Dynamics

## 9 THE LOGNORMAL MODEL

To price derivative securities, we need to make assumptions on the evolution of the price of the underlying assets. A simple assumption is that the price of the asset has lognormal distribution.

To justify the lognormal assumpation, note that the rate of return of an asset over an infinitesimal time period $\delta t$ between times $t$ and $t + \delta t$ is

$$\frac{S(t+\delta t) - S(t)}{\delta t \times S(t)}. \tag{138}$$

If the rate of return were constant, i.e., if there existed a parameter $\mu$ such that

$$\frac{S(t+\delta t) - S(t)}{\delta t \times S(t)} = \mu, \tag{139}$$

then the price $S(t)$ of the underlying asset would be an exponential function satisfying the ODE

$$\frac{dS}{dt} = \mu S, \quad \forall 0 < t < T, \tag{140}$$

with initial condition at $t = 0$ given by $S(0)$, the price of the asset at time zero. The solution of this ODE is $S(t) = S(0)e^{\mu t}$. Since market data shows that asset prices do not grow exponentially, the constant rate of return assumption cannot be correct.

A more realistic model is to assume that the rate of return has a random oscillation around its mean. This oscillation is assumed to be normally distributed, with mean zero and standard deviation on the order of $1/\sqrt{\delta t}$ because any order of magnitude other than $1/\sqrt{\delta t}$ would render the asset price either infinitely large or infinitely small. In other words, we consider the following model for $S(t)$:

$$\frac{S(t+\delta t) - S(t)}{\delta t \times S(t)} = \mu + \frac{\sigma}{\sqrt{\delta t}} Z. \tag{141}$$

The parameters $\mu$ and $\sigma$ are called the drift and the volatility of te underlying asset, respectively, and represent the expected value and the standard deviation of the returns of the asset. Based on Equation 141, we require $S(t)$ to satisfy the following stochastic differential equation (SDE):

$$dS = \mu S\,dt + \sigma S\,dW, \tag{142}$$

with initial condition at $t = 0$ given by $S(0)$. Here $W(t), t \geqslant 0$, is a Weiner process. If the asset pays dividends continuously at rate $q$, the SDE Equation 142 becomes

$$dS = (\mu - q) S\,dt + \sigma S\,dW, \tag{143}$$

where the drift coefficient $(\mu - q)$ is commonly denoted by $\alpha$. The solution to the above SDE Equation 143 for $t \geqslant 0$ is the lognormal model

$$\log\left(\frac{S(t_2)}{S(t_1)}\right) = \left(\alpha - \frac{\sigma^2}{2}\right)(t_2 - t_1) + \sigma\left(W(t_2) - W(t_1)\right), \tag{144}$$

or equivalently

$$S(t) = S(0) e^{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W(t)} \tag{145}$$

by direct computation.

## 9.1 An European Option that Expires in the Money

Consider a plain vanilla European call option with strike $K$ and maturity $T$ on an underlying asset with spot price having a lognormal distribution with drift $\mu$ and volatility $\sigma$, and paying dividend continuously at rate $q$. Assume that the risk-free interest rate are constant and equal to $r$ until the maturity of the option.

We compute the probability that the call option will expire in the money and will be exercised, which is equal to the probability that the spot price at maturity is higher than the strike price, i.e. to $\mathbb{P}(S(T) > K)$.

$$
\begin{aligned}
\mathbb{P}(S(T) > K) &= \mathbb{P}\left(\frac{S(T)}{S(0)} > \frac{K}{S(0)}\right) \\
&\overset{S_0 \leftarrow S(0)}{=} \mathbb{P}\left(\log\frac{S(T)}{S_0} > \log\frac{K}{S_0}\right) \\
&\overset{\text{Eq.144}}{=} \mathbb{P}\left(\left(\mu - q - \frac{1}{2}\sigma^2\right)T + \sigma W(T) > \log\frac{K}{S_0}\right) \\
&= \mathbb{P}\left(W(T) > \frac{\log\frac{K}{S_0} - \left(\mu - q - \frac{1}{2}\sigma^2\right)T}{\sigma}\right) \\
&= \mathbb{P}\left(W(1) > -\underbrace{\frac{\log\frac{S_0}{K} + \left(\mu - q - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}}_{\triangleq d_-(\mu)}\right) \\
&= \mathbb{P}\left(W(1) > -d_{-,\mu}\right) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-d_{-,\mu}}^{+\infty} e^{-\frac{x^2}{2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+d_{-,\mu}} e^{-\frac{x^2}{2}}\,dx = N(d_{-,\mu}),
\end{aligned}
$$

where we defined

$$
d_{\pm,\mu} = \frac{1}{\sigma\sqrt{\tau}}\left[\log\left(\frac{S}{K}\right) + \left(\mu - q \pm \frac{\sigma^2}{2}\right)\tau\right], \tag{146}
$$

and $N(z)$ is the cumulative distribution of the standard normal variable

$$
N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{x^2}{2}}\,dx. \tag{147}
$$

By definition, the probability that a plain vanilla European put option expires in the money is euqal to

$$
\mathbb{P}(S(T) < K) = 1 - \mathbb{P}(S(T) > K) = 1 - N(d_{-,\mu}) = N(-d_{-,\mu}). \tag{148}
$$

# 10 BLACK-SCHOLES MODEL

The Black-Scholes formulas give the price of plain vanilla European call and put options, under the assumption that the price of the underlying asset has lognormal distribution. In the Black-Scholes formulas, the price of a plain vanilla European option depends on the following parameters:

- $S$, the spot price of the underlying asset at time $t$;
- $T$, the maturity of the option; the time to maturity is $\tau = T - t$;
- $K$, the strike price of the option;
- $r$, the risk-free interest rate, assumed constant over the option's life;
- $\sigma$, the volatility of the underlying asset;
- $q$, the dividend rate of the underlying asset, paid at a continuous rate.

## 10.1 The Black-Scholes PDE

If the underlying asset follows a lognormal process, the value $V(S,t)$ of a plain vanilla European option satisfies a partial differential equation, i.e. an equation involving partial derivatives of $V$ with respect to the spot price $S$ of the underlying asset and the time $t$. This partial differential equation is called the Black-Scholes PDE.

To write the quadratic Taylor expansion of $V(S,t)$ around the point $(S,t)$:

$$dV(S,t) \approx \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} dS^2 + \frac{1}{2}\frac{\partial^2 V}{\partial t^2} dt^2 + \frac{\partial^2 V}{\partial S \partial t} dS\,dt. \tag{149}$$

For the discretized evolution model of the price of the underlying asset to converge to a lognormal model, the following condition must be satisfied in the sense of integration:

$$dS^2 \approx \sigma^2 S^2 dt. \tag{150}$$

We ignore all the terms of order larger than $dt$ from Equation 149, we find that

$$dV \approx \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt. \tag{151}$$

To hedge a long position in the derivative security, a short position in $\Delta$ units of the underlying asset must be taken. If $\Pi$ is the value of the resulting portfolio, then $\Pi = V - \Delta S$. Note that if the value of the underlying asset changes from $S$ to $S + dS$, then the value of the portfolio changes as follows:

$$d\Pi = dV - \Delta dS - q\Delta S\,dt \tag{152}$$

if the underlying asset paid dividends continuously at the rate $q$. To avoid arbitrage opportunities, the value of this portfolio must grow at the same rate as the risk free interest rate $r$, i.e.,

$$d\Pi = r\Pi\,dt, \tag{153}$$

or

$$\frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt - \Delta dS - q\Delta S\,dt = r(V - \Delta S)\,dt. \tag{154}$$

We reproduce the Delta-hedge described in Section 7.4

$$\Delta = \frac{\partial V}{\partial S} \tag{155}$$

and derived the Black-Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \quad \forall S > 0, \quad 0 < t < T. \tag{156}$$

Black-Scholes PDE is a linear, second order, parabolic partial differential equation backward in time with non-constant coefficients.

**Example 10.0.1.** *The value $V(S,T)$ of the option at maturity is equal to the payoff of the option at maturity, i.e.,*

$$\begin{aligned} V(S,T) &= (S-K)^+, \quad \textit{for call options;} \quad (157) \\ V(S,T) &= (K-S)^+, \quad \textit{for put options.} \quad (158) \end{aligned}$$

*The solution to Black-Scholes PDE with boundary conditions Equation 157 is the Black-Scholes value of a call/put option.*

**Example 10.0.2.** *Note that many financial instruments other than the plain vanilla European options also satisfy the Black-Scholes PDE with various kinds of boundary conditions. For example, the value* $V(S,T) = S$ *of the non-dividend paying underlying asset also satisfies the Black-Scholes PDE, since*

$$\frac{\partial V}{\partial t} = 0, \quad \frac{\partial V}{\partial S} = 1, \quad \frac{\partial^2 V}{\partial S^2} = 0. \tag{159}$$

The change of variable is as follows

$$V(S,t) = e^{-ax-b\tau}u(x,\tau), \tag{160}$$

where

$$\begin{aligned} x &= \log\left(\frac{S}{K}\right), \\ \tau &= \frac{1}{2}\sigma^2(T-t). \end{aligned}$$

reduces the Black-Scholes PDE, Equation 156, to the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{161}$$

for which a closed form solution is known, if the constants $a$ and $b$ are chosen to be

$$\begin{aligned} a &= \frac{r-q}{\sigma^2} - \frac{1}{2}, \\ b &= \left(\frac{r-q}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2q}{\sigma^2}. \end{aligned}$$

The boundary conditions 157 is accordingly transformed into initial conditions

$$\begin{aligned} u(x,0) &= Ke^{ax}(e^x - 1)^+, && \text{for call options;} \\ u(x,0) &= Ke^{ax}(1 - e^x)^+, && \text{for put options.} \end{aligned}$$

Solving the heat equation with the above initial conditions leads to the closed form formula of the Black-Scholes formula described in Section 10.3.

## 10.2 Financial Interpretation of the Black-Scholes PDE

To find the financial meaning of the terms from Black-Scholes PDE 156, we rewrite it using the Delta and Gamma of the option described in Section 7.4 as

$$\Theta \triangleq \frac{\partial V}{\partial t} = r(V - \Delta S) - \frac{1}{2}\sigma^2 S^2 \Gamma + q\Delta S. \tag{162}$$

A plain vanilla European option with price $V$ can be replicated dynamically using a portfolio made of a long position in $\Delta$ units of the underlying asset and a cash position of size $V - \Delta S$. Then, formula 162 can be interpreted as follows. The change $dV$ in the price of the option between times $t$ and $t + dt$

- the interest $r(V - \Delta S)dt$ realized on the cash position;
- the replication cost $\frac{1}{2}\sigma^2 S^2 \Gamma dt$, also called slippage;
- the dividend gain $q\Delta S dt$ of the $\Delta$ position in the underlying asset.

Rewrite Equation 162 in the form

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r(V - \Delta S) + q\Delta S. \tag{163}$$

Typically, the right hand side of Equation 163 is small compared to the terms $\Theta$ and $\sigma^2 S^2 \Gamma$. This justifies the following commonly used approximation:

$$1 + \frac{1}{2}\sigma^2 S^2 \frac{\Gamma}{\Theta} \approx 0. \tag{164}$$

Note that, if the above approximation holds, then $\Theta$ and $\Gamma$ have different signs.

Plain vanilla European call and put options have positive $\Gamma$, since $\Delta$ is an increasing function of the price of the underlying asset. This means that, in general, $\Theta$ is negative: as the option moves closer to maturity its value decrease (all other parameters such as the spot price being assumed not to change), since the opportunity for a larger payoff at maturity, if the option is in-the-money, or for the option to expire in the money, if the option is out-of-the-money, decreases.

We note that one instance when $\Theta$ and $\Gamma$ do not have opposite signs is for deep in-the-money European put options, when both $\Theta$ and $\Gamma$ are positive. The approximation 164 is more reliable for deep out-of-the-money options (corresponding to small values of $S/K$) and deteriorates for deep in-the-money options (corresponding to the large values of $S/K$). Also, the approximation is more accurate if the underlying asset pays dividends.

## 10.3 The Black-Scholes Formula

Let $C(S,t)$ be the value at time $t$ of a call option with strike $K$ and maturity $T$, and let $P(S,t)$ be the value at time $t$ of a put option with strike $K$ and maturity $T$. Then the Black-Scholes formulas for European call and put options are

$$C(S,t) = +Se^{-q\tau}N(+d_+) - Ke^{-r\tau}N(+d_-), \tag{165}$$
$$P(S,t) = -Se^{-q\tau}N(-d_+) + Ke^{-r\tau}N(-d_-), \tag{166}$$

where $\tau = T - t$ is the time to maturity and

$$d_\pm = \frac{1}{\sigma\sqrt{\tau}}\left[\log\left(\frac{S}{K}\right) + \left(r - q \pm \frac{\sigma^2}{2}\right)\tau\right]. \tag{167}$$

Note that $d_+ - d_- = \sigma\sqrt{\tau}$. Note that $d_\pm$ defined above is very simillar to $d_{\pm,\mu}$ defined in Equation 146 with the drift of the underlying asset set to the constant risk-free rate, i.e. $\mu = r$. The financial interpretation of $N(d_-)$ is the risk-neutral probability that the call option expires in the money.

The put-call parity can be verified

$$\begin{aligned} C - P &= Se^{-q\tau}\left(N(+d_+) + N(-d_+)\right) - Ke^{-r\tau}\left(N(+d_-) + N(-d_-)\right) \\ &= Se^{-q\tau} - Ke^{-r\tau} \quad \text{note: } N(a) + N(-a) = 1 \\ &= F. \end{aligned} \tag{168}$$

See Equation 89.

**Example 10.0.3.** *It is worth noting that closed formulas for the values of barrier options, introduced in Section 7.5, can be evaluated in terms of Black-Scholes formula. For example, the value of the down-and-out call with* $B < K$ *is*

$$V(S,t) = C(S,t) - \left(\frac{B}{S}\right)^{2a} C\left(\frac{B^2}{K}, t\right), \tag{169}$$

*where the constant* $a = (r-q)/\sigma^2 - 1/2$ *is also seen in Equation 162. Here,* $C(S,t)$ *is the value at time* $t$ *of a plain vanilla call with strike* $K$ *on the same underlying asset, and* $C(B^2/S,t)$ *is the Black-Scholes value at time* $t$ *of a plain vanilla call with strike* $K$ *and at maturity* $T$ *on an asset having spot price* $B^2/S$ *and the same volatility as the underlying.*

*Note that, as expected,* $V(S,t) < C(S,t)$*. Also,* $V(S,t) \approx C(S,t)$ *as the barrier goes to zero, when the probability of hitting the barrier and knocking the option out also goes to zero.*

## 10.4 Risk-Neutral Derivation of Black-Scholes

**Definition 26.** *Risk-neutral valuation* refers to valuing derivative securities as discounted expected values of their payoffs at maturity, under the assumption that the underlying asset has lognormal distribution with drift equal to the constant risk-free rate.

Risk-neutral valuation can be used for derivative securities which can be perfectly hedged dynamically using cash and units of the underlying asset. Plain vanilla European options, as well as European options with other payoffs at maturity can be priced using risk-neutrality. Risk-neutral valuation cannot be used for path-dependent options such as American options, barrier options, and Asian options, since these options cannot be perfectly hedged dynamically.

The value of a derivative security with payoff $V(T)$ at maturity $T$ given by risk-neutral valuation is

$$V(0) = e^{-rT}\tilde{\mathbb{E}}\left[V(T)\right], \tag{170}$$

where $\tilde{\mathbb{E}}[\cdots]$ denotes risk-neutral expectation value with respect to $S(T)$ given by Equation 145 for $\mu = r$, i.e., $\alpha = r - q$:

$$S(t) = S(0)e^{\left(r-q-\frac{\sigma^2}{2}\right)t+\sigma W(t)}. \tag{171}$$

Applying Equation 170 to plain vanilla European options, we find that

$$C(0) = e^{-rT}\tilde{\mathbb{E}}\left[(S(T)-K)^+\right], \tag{172}$$

$$P(0) = e^{-rT}\tilde{\mathbb{E}}\left[(K-S(T))^+\right], \tag{173}$$

where the expected value is computed with respect to $S(T)$ given by Equation 171. We derive the Black-Scholes formula for call options.

$$\begin{aligned}
C(0) &= e^{-rT}\tilde{\mathbb{E}}\left[(S(T)-K)^+\right] \\
&= e^{-rT}\int_{-\infty}^{+\infty} dx \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}}\left[S(0)e^{\left(r-q-\frac{1}{2}\sigma^2\right)T+\sigma x}-K\right]^+ \\
&\overset{S_0\leftarrow S(0)}{=} \frac{e^{-rT}}{\sqrt{2\pi T}}\int_{-\infty}^{+\infty} dx e^{-\frac{x^2}{2T}}\left[S_0 e^{\left(r-q-\frac{1}{2}\sigma^2\right)T+\sigma x}-K\right]^+ \\
&= \frac{e^{-rT}}{\sqrt{2\pi T}}\int_{-\infty}^{+\infty} dx e^{-\frac{x^2}{2T}}\left[S_0 e^{\left(r-q-\frac{1}{2}\sigma^2\right)T+\sigma x}-K\right]^+ \\
&= \frac{e^{-rT}}{\sqrt{2\pi T}}\int_{\frac{1}{\sigma}\left[\log\left(\frac{K}{S_0}\right)-\left(r-q-\frac{1}{2}\sigma^2\right)T\right]}^{+\infty} dx e^{-\frac{x^2}{2T}}\left[S_0 e^{\left(r-q-\frac{1}{2}\sigma^2\right)T+\sigma x}-K\right] \\
&\overset{x\to\sqrt{T}x}{=} \frac{e^{-rT}}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma\sqrt{T}}\left[\log\left(\frac{S_0}{K}\right)+\left(r-q-\frac{1}{2}\sigma^2\right)T\right]}^{+\infty} dx e^{-\frac{x^2}{2}}\left[S_0 e^{\left(r-q-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}x}-K\right] \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-d_-}^{+\infty} dx e^{-\frac{x^2}{2}}\left[S_0 e^{\left(r-q-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}x}-K\right] \\
&= \frac{e^{-rT}S_0}{\sqrt{2\pi}}\int_{-d_-}^{+\infty} dx e^{-\frac{x^2}{2}+\sigma\sqrt{T}x+\left(r-q-\frac{1}{2}\sigma^2\right)T} - \frac{e^{-rT}K}{\sqrt{2\pi}}\int_{-d_-}^{+\infty} dx e^{-\frac{x^2}{2}} \\
&= \frac{e^{-qT}S_0}{\sqrt{2\pi}}\int_{-d_-}^{+\infty} dx e^{-\frac{1}{2}\left(x^2-2\sigma\sqrt{T}x+\sigma^2 T\right)} - Ke^{-rT}N(d_-) \\
&= \frac{e^{-qT}S_0}{\sqrt{2\pi}}\int_{-d_-}^{+\infty} dx e^{-\frac{1}{2}\left(x-\sigma\sqrt{T}\right)^2} - Ke^{-rT}N(d_-) \\
&= \frac{e^{-qT}S_0}{\sqrt{2\pi}}\int_{-d_- -\sigma\sqrt{T}}^{+\infty} dx e^{-\frac{1}{2}x^2} - Ke^{-rT}N(d_-) \\
&= \frac{e^{-qT}S_0}{\sqrt{2\pi}}\int_{-d_+}^{+\infty} dx e^{-\frac{1}{2}x^2} - Ke^{-rT}N(d_-) \\
&= S_0 e^{-qT}N(d_+) - Ke^{-rT}N(d_-).
\end{aligned}$$

This is exactly the Black-Scholes formula Equation 165 obtained by setting $t = 0$.

**Example 10.0.4.** *Find the value of an option with payoff $(\sqrt{S(T)}-K)^+$ at maturity, if the underlying asset pays no dividends and has lognormal distribution.*

*Solution.* Using risk-neutral valuation, we evaluation the following expression:

$$
\begin{aligned}
V(0) &= e^{-rT}\tilde{\mathbb{E}}\left[(\sqrt{S(T)}-K)^{+}\right] \\
&= \frac{e^{-rT}}{\sqrt{2\pi T}}\int_{-\infty}^{+\infty} dx e^{-\frac{x^2}{2T}}\left[\sqrt{S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma x}}-K\right]^{+} \\
&= \frac{e^{-rT}}{\sqrt{2\pi T}}\int_{\frac{1}{\sigma}\left[\log\left(\frac{K^2}{S_0}\right)-(r-\frac{1}{2}\sigma^2)T\right]}^{+\infty} dx e^{-\frac{x^2}{2T}}\left[\sqrt{S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma x}}-K\right] \\
&\overset{x\to\sqrt{T}x}{=} \frac{e^{-rT}}{\sqrt{2\pi}}\int_{\frac{-1}{\sigma\sqrt{T}}\left[\log\left(\frac{S_0}{K^2}\right)+(r-\frac{1}{2}\sigma^2)T\right]}^{+\infty} dx e^{-\frac{x^2}{2}}\left[\sqrt{S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x}}-K\right] \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-a}^{+\infty} dx e^{-\frac{x^2}{2}}\left[\sqrt{S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x}}-K\right] \\
&= \sqrt{\frac{S_0}{2\pi}}\int_{-a}^{+\infty} dx e^{-\frac{x^2}{2}-\frac{1}{2}(r+\frac{1}{2}\sigma^2)T+\frac{1}{2}\sigma\sqrt{T}x} - Ke^{-rT}N(a) \\
&= \sqrt{\frac{S_0}{2\pi}}e^{-\frac{1}{2}(r+\frac{1}{4}\sigma^2)T}\int_{-a}^{+\infty} dx e^{-\frac{1}{2}(x-\frac{1}{2}\sigma\sqrt{T})^2} - Ke^{-rT}N(a) \\
&= \sqrt{\frac{S_0}{2\pi}}e^{-\frac{1}{2}(r+\frac{1}{4}\sigma^2)T}\int_{-a-\frac{1}{2}\sigma\sqrt{T}}^{+\infty} dx e^{-\frac{1}{2}x^2} - Ke^{-rT}N(a) \\
&= \sqrt{S_0}e^{-\frac{1}{2}(r+\frac{1}{4}\sigma^2)T}N\left(a+\frac{1}{2}\sigma\sqrt{T}\right) - Ke^{-rT}N(a), \qquad (174)
\end{aligned}
$$

where we have defined a constant

$$
a \triangleq \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{S_0}{K^2}\right)+\left(r-\frac{1}{2}\sigma^2\right)T\right]. \qquad (175)
$$

□

**Example 10.0.5.** *Find the value of an option with payoff* $[(S(T)-K)^{+}]^2$ *at maturity, if the underlying asset pays no dividends and has lognormal distribution.*

*Solution.* Using risk-neutral valuation, we evaluation the following expression:

$$
\begin{aligned}
V(0) &= e^{-rT}\tilde{\mathbb{E}}\left[\left\{(S(T)-K)^{+}\right\}^2\right] \\
&= \frac{e^{-rT}}{\sqrt{2\pi T}}\int_{-\infty}^{+\infty} dx e^{-\frac{x^2}{2T}}\left[\left(S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma x}-K\right)^{+}\right]^2 \\
&\overset{x\to\sqrt{T}x}{=} \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-d_-}^{+\infty} dx e^{-\frac{x^2}{2}}\left[S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x}-K\right]^2 \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-d_-}^{+\infty} dx e^{-\frac{x^2}{2}}\left[S_0^2 e^{(2r-\sigma^2)T+2\sigma\sqrt{T}x}-2KS_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x}+K^2\right] \\
&= S_0^2 e^{(r+\sigma^2)T}N(d_-+2\sigma\sqrt{T}) - 2KS_0N(d_+) + K^2e^{-rT}N(d_-). \qquad (176)
\end{aligned}
$$

□

**Example 10.0.6.** *Find the value of an option with payoff* $S(T)^{-1}$ *at maturity, if the underlying asset pays dividends continuously at rate* $q$ *and has lognormal distribution.*

*Solution.* Using risk-neutral valuation, we evaluation the following expression:

$$\begin{aligned}
V(0) &= e^{-rT}\tilde{\mathbb{E}}\left[S(T)^{-1}\right] \\
&= \frac{e^{-rT}S_0^{-1}}{\sqrt{2\pi T}}\int_{-\infty}^{+\infty} dx e^{-\frac{x^2}{2T}} e^{-\left(r-q-\frac{1}{2}\sigma^2\right)T-\sigma x} \\
&\overset{x\to\sqrt{T}x}{=} \frac{e^{-rT}S_0^{-1}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dx e^{-\frac{x^2}{2}-\sigma\sqrt{T}x-\left(r-q-\frac{1}{2}\sigma^2\right)T} \\
&= \frac{e^{-(2r-q)T}S_0^{-1}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}(x+\sigma\sqrt{T})^2} \\
&= e^{-(2r-q)T}S_0^{-1}. \qquad (177)
\end{aligned}$$

□

**Example 10.0.7.** *Find the value of an option with payoff*

$$\left[\frac{1}{K}-\frac{1}{S(T)}\right]^+ \qquad (178)$$

*at maturity, if the underlying asset pays dividends continuously at rate* $q$ *and has lognormal distribution.*

*Solution.* Using risk-neutral valuation, we evaluation the following expression:

$$\begin{aligned}
V(0) &= e^{-rT}\tilde{\mathbb{E}}\left[\left(\frac{1}{K}-\frac{1}{S(T)}\right)^+\right] \\
&= \frac{e^{-rT}}{\sqrt{2\pi T}}\int_{-\infty}^{+\infty} dx e^{-\frac{x^2}{2T}}\left(\frac{1}{K}-\frac{1}{S(T)}\right)^+ \\
&\overset{x\to\sqrt{T}x}{=} \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-d_-}^{+\infty} dx e^{-\frac{x^2}{2}}\left(\frac{1}{K}-\frac{1}{S(T)}\right) \\
&= \frac{e^{-rT}}{K}N(d_-) - \frac{e^{-rT}}{S_0\sqrt{2\pi}}\int_{-d_-}^{+\infty} dx e^{-\frac{x^2}{2}} e^{-\left(r-q-\frac{1}{2}\sigma^2\right)T-\sigma\sqrt{T}x} \\
&= \frac{e^{-rT}}{K}N(d_-) - \frac{e^{-(2r-q)T}}{S_0\sqrt{2\pi}}\int_{-d_-}^{+\infty} dx e^{-\frac{1}{2}(x+\sigma\sqrt{T})^2} \\
&= \frac{e^{-rT}}{K}N(d_-) - \frac{e^{-(2r-q)T}}{S_0\sqrt{2\pi}}\int_{-d_-+\sigma\sqrt{T}}^{+\infty} dx e^{-\frac{x^2}{2}} \\
&= \frac{e^{-rT}}{K}N(d_-) - \frac{e^{-(2r-q)T}}{S_0}N(d_- - \sigma\sqrt{T}). \qquad (179)
\end{aligned}$$

□

## 10.5 Greeks Computations

It is interesting to note that the formulae for the Greeks in Black-Scholes model are simpler than expected, which results from the following identity:

**Lemma 10.0.1.** $Se^{-q\tau}N'(d_+) = Ke^{-r\tau}N'(d_-)$.

*Proof.* Recall that $N(z)$ is the cumulative distribution of the standard normal variable, defined in Equation 147, we find that

$$N'(d_+) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_+^2}{2}}; \tag{180}$$

$$N'(d_-) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_-^2}{2}}. \tag{181}$$

Direct computation gives the following identity from Equation 167:

$$\begin{aligned} & d_+^2 - d_-^2 = 2\left[\log\left(\frac{S}{K}\right) + (r-q)\tau\right] \\ \Rightarrow \quad & -\frac{d_+^2}{2} + \log S - q\tau = -\frac{d_-^2}{2} + \log K - r\tau \end{aligned} \tag{182}$$

Exponentiate both sides of the above equation, we have proved Lemma 10.0.1. □

We return to the computation of Greeks for a European call option. From Definition 108,

$$\begin{aligned} \Delta(C) &= \frac{\partial C}{\partial S} \\ &= e^{-q\tau}N(d_+) + \underbrace{Se^{-q\tau}N'(d_+)\frac{\partial d_+}{\partial S} - Ke^{-r\tau}N'(d_-)\frac{\partial d_-}{\partial S}}_{=0, \quad \because \frac{\partial d_+}{\partial S} = \frac{\partial d_-}{\partial S} = \frac{1}{\dots}} \\ &= e^{-q\tau}N(d_+). \end{aligned} \tag{183}$$

From Definition 109,

$$\begin{aligned} \Gamma(C) &= \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Theta}{\partial S} \\ &= e^{-q\tau}N'(d_+)\frac{\partial d_+}{\partial S} \\ &= \frac{e^{-q\tau}N'(d_+)}{\sigma S\sqrt{\tau}}. \end{aligned} \tag{184}$$

From Definition 110,

$$\begin{aligned} \Theta(C) &= \frac{\partial C}{\partial t} = -\frac{\partial C}{\partial \tau} \\ &= qSe^{-q\tau}N(d_+) - rKe^{-r\tau}N(d_-) \\ &\quad + \underbrace{Se^{-q\tau}N'(d_+)\frac{\partial d_+}{\partial t} - Ke^{-r\tau}N'(d_-)\frac{\partial d_-}{\partial t}}_{= -Se^{-q\tau}N'(d_+)\frac{\sigma}{2\sqrt{\tau}} \quad \because \frac{\partial(d_+ - d_-)}{\partial t} = -\frac{\sigma}{2\sqrt{\tau}}} \\ &= qSe^{-q\tau}N(d_+) - rKe^{-r\tau}N(d_-) - Se^{-q\tau}N'(d_+)\frac{\sigma}{2\sqrt{\tau}}. \end{aligned}$$

From Definition 111,

$$\begin{aligned} \rho(C) &= \frac{\partial C}{\partial r} \\ &= K\tau e^{-r\tau}N(d_-) + \underbrace{Se^{-q\tau}N'(d_+)\frac{\partial d_+}{\partial r} - Ke^{-r\tau}N'(d_-)\frac{\partial d_-}{\partial r}}_{=0, \quad \because \frac{\partial d_+}{\partial r} = \frac{\partial d_-}{\partial r} = \sigma\sqrt{\tau}.} \\ &= K\tau e^{-r\tau}N(d_-). \end{aligned} \tag{185}$$

From Definition 112,

$$\begin{aligned} \nu(C) = &= \frac{\partial C}{\partial \sigma} \\ &= Se^{-q\tau}N'(d_+)\frac{\partial d_+}{\partial \sigma} - Ke^{-r\tau}N'(d_-)\frac{\partial d_-}{\partial \sigma} \\ &= Se^{-q\tau}N'(d_+)\sqrt{\tau} \quad \because \frac{\partial(d_+ - d_-)}{\partial \sigma} = \sqrt{\tau} \end{aligned} \tag{186}$$

Note that, once the formulas for the Greeks corresponding to a call option are derived, the put-call parity can be used to obtain the corresponding Greeks for the put option. For American and exotic options, closed formulas for the Greeks are rarely available, a notable exception being European barrier options.

## 10.6 Implied Volatility

The only parameter needed in Black-Scholes formulas which is not directly observable in the market is the volatility $\sigma$ of the underlying asset. The risk-free rate $r$ and the continuous dividend yield $q$ of the asset can be estimated from market data; the maturity date $T$ and the strike $K$ of the option, as well as the spot price $S$ of the underlying asset are known when a price for the option is quoted.

**Definition 27.** The *implied volatility* $\tilde{\sigma}$ is the value of the volatility parameter $\sigma$ that makes the Black-Scholes value of the option equal to the traded price of the option.

Denote by $C_{BS}(S, K, T, \sigma, r, q)$ the Black-Scholes value of a call option with strike $K$ and maturity $T$ on an underlying asset with spot price $S$ paying dividends continuously at the rate $q$, if interest rates are constant and equal to $r$. Let $C$ be the market price of a call with the same parameters except the volatility. The implied volatility $\tilde{\sigma}$ corresponding to the price $C$ is, by definition, the solution to

$$C_{BS}(S, K, T, \tilde{\sigma}, r, q) = C. \tag{187}$$

The implied volatility can also be derived from the market price of a put option price $P$ by solving

$$P_{BS}(S, K, T, \tilde{\sigma}, r, q) = P. \tag{188}$$

where $P_{BS}(S, K, T, \sigma, r, q)$ is the Black-Scholes value of a put option. Note that the Black-Scholes values of both call and put options are strictly increasing as functions of volatility (Equation 186)

$$\frac{\partial C_{BS}}{\partial \sigma} = \frac{\partial P_{BS}}{\partial \sigma} > 0. \tag{189}$$

Therefore, if a solution $\tilde{\sigma}$ for Equation 187 or Equation 188 exists, it is unique. From a computational perspective, Equation 187 and Equation 188 are one-dimensional nonlinear equations and can be solved very efficiently using Newton's method.

**Theorem 10.1.** *The implied volatilities corresponding to Black-Scholes put and call options with the same strike and maturity on the same underlying asset are equal.*

*Proof.* Denote by $\tilde{\sigma}_P$ and $\tilde{\sigma}_C$ the implied volatilities corresponding to a put option with price $P$ and to a call option with price $C$, respectively. Both options have strike $K$ and maturity $T$ and are written on the same underlying asset. In other words,

$$\begin{aligned} P_{BS}(\tilde{\sigma}_P) &= P, && (190)\\ C_{BS}(\tilde{\sigma}_C) &= C. && (191)\end{aligned}$$

For no-arbitrage, the option values $P$ and $C$ must satisfy the Put-Call parity, i.e.

$$P + Se^{-q\tau} - C = Ke^{-r\tau}. \tag{192}$$

From Equation 190, we find that

$$P_{BS}(\tilde{\sigma}_P) + Se^{-q\tau} - C_{BS}(\tilde{\sigma}_C) = Ke^{-r\tau}. \tag{193}$$

Recall that the Black-Scholes values of put and call options satisfy the Put-Call parity, i.e.,

$$P_{BS}(\sigma) + Se^{-q\tau} - C_{BS}(\sigma) = Ke^{-r\tau}. \tag{194}$$

for any value $\sigma > 0$ of the volatility. Let $\sigma = \tilde{\sigma}_C$, then

$$P_{BS}(\tilde{\sigma}_C) + Se^{-q\tau} - C_{BS}(\tilde{\sigma}_C) = Ke^{-r\tau}. \tag{195}$$

From Equation 193 and Equation 195, it follows that

$$P_{BS}(\tilde{\sigma}_P) = P_{BS}(\tilde{\sigma}_C). \tag{196}$$

Recall from Equation 189 that the Black-Scholes value of a put option is a strictly increasing function of volatility. Therefore, we obtain that

$$\tilde{\sigma}_P = \tilde{\sigma}_C \tag{197}$$

and conclude that the implied volatilities corresponding to put and call options with the same strike and maturity on the same asset must be equal. □

## 10.7 Approximations of the ATM Black-Scholes Formula

In this section, we derive approximations of the ATM (at-the-money) Black-Scholes formula under the conditions

$$r\tau \sim q\tau \ll \sigma\sqrt{\tau} \ll 1. \tag{198}$$

As a consequence, the argument of the cumulative standard normal function appearing in ATM ($S = K$) Black-Scholes formula is small

$$d_{\pm} = \frac{1}{\sigma\sqrt{\tau}}\left[\log\left(\frac{S}{K}\right) + \left(r - q \pm \frac{\sigma^2}{2}\right)\tau\right] = \frac{(r-q)\tau}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau} \ll 1, \tag{199}$$

and the following approximation for the cumulative standard normal can be employed

$$N(x) \approx \frac{1}{2} + \frac{x}{\sqrt{2\pi}}, \quad x \ll 1. \tag{200}$$

To analyze the order of magnitude of small items in

$$\frac{C}{S} \approx \underbrace{\frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}}}_{O(\epsilon)} + \underbrace{\frac{(r-q)\tau}{2}}_{O(\epsilon^{\alpha})} - \underbrace{\frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}} \times \frac{(r+q)\tau}{2}}_{O(\epsilon^{\alpha+1})} + \frac{1}{\sqrt{2\pi}}\underbrace{\frac{[(r-q)\tau]^2}{\sigma\sqrt{\tau}}}_{O(\epsilon^{2\alpha-1})}, \tag{201}$$

we assume that $\sigma\sqrt{\tau} \sim \epsilon$ and $r\tau \sim q\tau \sim \epsilon^{\alpha}$, where $\alpha > 1$ because of the assumption 198. From the order of magnitude comparison between the last two items, we conclude that, by keeping items of order up to $\min(\epsilon^{\alpha+1}, \epsilon^{2\alpha-1})$:

$$\frac{C}{S} \approx \begin{cases} \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}} + \frac{(r-q)\tau}{2} - \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}} \times \frac{(r+q)\tau}{2}, & \text{if } 1 < \alpha < 2; \\ \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}} + \frac{(r-q)\tau}{2} + \frac{1}{\sqrt{2\pi}} \frac{[(r-q)\tau]^2}{\sigma\sqrt{\tau}}, & \text{if } \alpha > 2. \end{cases} \tag{202}$$

In the usual case as discussed in the Primer, we assume $1 < \alpha < 2$

$$\frac{C}{S} \approx \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}} + \frac{(r-q)\tau}{2} - \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}} \times \frac{(r+q)\tau}{2}, \tag{203}$$

$$\frac{P}{S} \approx \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}} - \frac{(r-q)\tau}{2} - \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}} \times \frac{(r+q)\tau}{2}. \tag{204}$$

The approximate implied volatility can be read off immediately

$$\tilde{\sigma} \approx \sqrt{\frac{2\pi}{\tau}} \times \frac{\frac{C}{S} - \frac{(r-q)\tau}{2}}{1 - \frac{(r+q)\tau}{2}}. \tag{205}$$

Note that Equations 203 cannot be used to derive approximate formula for $\Delta$ and $\Gamma$ because the Equations 203 are derived under the ATM condition. Instead, Equation 183 and Equation 184 need to invoked directly. For simplicity, we assume $q = r = 0$, the Black-Scholes Greeks can be approximated as follows:

$$\begin{aligned}
\Delta(C) &\approx \frac{1}{2} + \frac{\sigma\sqrt{\tau}}{2\sqrt{2\pi}} \approx \frac{1}{2} + 0.2\sigma\sqrt{\tau}, \\
\Delta(P) &\approx -\frac{1}{2} + \frac{\sigma\sqrt{\tau}}{2\sqrt{2\pi}} \approx -\frac{1}{2} + 0.2\sigma\sqrt{\tau}, \\
\Gamma(C) = \Gamma(P) &\approx \frac{1}{\sqrt{2\pi\tau}\sigma S}\left(1 - \frac{1}{8}\sigma\tau^2\right) \approx \frac{0.4}{S\sigma\sqrt{\tau}}\left(1 - \frac{1}{8}\sigma\tau^2\right), \\
\nu(C) = \nu(P) &\approx S\sqrt{\frac{\tau}{2\pi}}\left(1 - \frac{1}{8}\sigma\tau^2\right) \approx 0.4S\sqrt{\tau}\left(1 - \frac{1}{8}\sigma\tau^2\right), \\
\Theta(C) = \Theta(P) &\approx -\frac{S\sigma}{2\sqrt{2\pi\tau}}\left(1 - \frac{1}{8}\sigma\tau^2\right) \approx \frac{0.2S\sigma}{\tau}\left(1 - \frac{1}{8}\sigma\tau^2\right), \\
\rho(C) &\approx \frac{S\tau}{2}\left(1 - \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}}\right) \approx S\tau\left(\frac{1}{2} - 0.2\sigma\sqrt{\tau}\right), \\
\rho(P) &\approx -\frac{S\tau}{2}\left(1 + \frac{\sigma\sqrt{\tau}}{\sqrt{2\pi}}\right) \approx -S\tau\left(\frac{1}{2} + 0.2\sigma\sqrt{\tau}\right).
\end{aligned}$$

Similar but more complicated formulae exist for $r, q \neq 0$.

Finally, we derive a theoretical estimate for the approximation error of formula 203 for the ATM call on a non-dividend paying asset under zero interest rates. In other words, we would like to make the condition 198 on the volatility precise.

**Theorem 10.2.** *Consider an at-the-money call option on a non-dividend paying asset, and assume zero interest rates. Let $C_{BS}$ be the Black-Scholes value of the call, and let $\bar{C}$ be the approximate value of the call option given by Equation 203. The relative error given by the approximation formula 203 for ATM calls is less than 1%, i.e.,*

$$\frac{|\bar{C} - C_{BS}|}{C_{BS}} \leqslant 0.01, \tag{206}$$

*provided that $\sigma\sqrt{T} \leqslant 0.4874$.*

*Proof.* We begin by taking the difference between $C_{BS}$ and $\bar{C}$, assuming that $r = q = 0$ and $S = K$ so that $d_{\pm} = \pm\frac{1}{2}\sigma\sqrt{\tau}$:

$$\begin{aligned}
\bar{C} - C_{BS} &= \sigma S\sqrt{\frac{\tau}{2\pi}} - S\left[N(d_+) - N(d_-)\right] \\
&= \sigma S\sqrt{\frac{\tau}{2\pi}} - S\left[2N\left(\frac{1}{2}\sigma\sqrt{\tau}\right) - 1\right] \\
&= S\left[1 + \sigma\sqrt{\frac{\tau}{2\pi}} - 2N\left(\frac{1}{2}\sigma\sqrt{\tau}\right)\right].
\end{aligned}$$

Now we invoke the integral form of the Taylor approximation error

$$f(x) - \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) = \int_{x_0}^{x} \frac{(x-x')^n}{n!} f^{(n+1)}(x')\,dx'. \tag{207}$$

for $N(x)$ when $n = 1$,

$$N(x) - \frac{1}{2} - \frac{x}{\sqrt{2\pi}} = -\frac{1}{\sqrt{2\pi}} \int_0^x (x - x')x' e^{-\frac{x'^2}{2}}\,dx'. \tag{208}$$

Thus,

$$\bar{C} - C_{BS} = S\sqrt{\frac{2}{\pi}} \int_0^{\frac{1}{2}\sigma\sqrt{\tau}} \left(\frac{1}{2}\sigma\sqrt{\tau} - x'\right) x' e^{-\frac{x'^2}{2}}\,dx', \tag{209}$$

and the relative error

$$\begin{aligned}
\frac{|\bar{C} - C_{BS}|}{C_{BS}} &= \frac{\bar{C} - C_{BS}}{C_{BS}} = \frac{\bar{C} - C_{BS}}{\bar{C}} \times \frac{\bar{C}}{C_{BS}} \\
&= \frac{\bar{C} - C_{BS}}{\bar{C}} \times \frac{1}{1 - \frac{\bar{C} - C_{BS}}{\bar{C}}} \leqslant \frac{1}{100} \Rightarrow \frac{\bar{C} - C_{BS}}{\bar{C}} \leqslant \frac{1}{101},
\end{aligned}$$

or

$$\frac{2}{\sigma\sqrt{\tau}} \int_0^{\frac{1}{2}\sigma\sqrt{\tau}} \left(\frac{1}{2}\sigma\sqrt{\tau} - x'\right) x' e^{-\frac{x'^2}{2}}\,dx' \leqslant \frac{1}{101}. \tag{210}$$

To derive an upper bound the left hand side of Equation 210,

$$\begin{aligned}
\frac{2}{\sigma\sqrt{\tau}} \int_0^{\frac{1}{2}\sigma\sqrt{\tau}} \left(\frac{1}{2}\sigma\sqrt{\tau} - x'\right) x' e^{-\frac{x'^2}{2}}\,dx' &< \frac{2}{\sigma\sqrt{\tau}} \int_0^{\frac{1}{2}\sigma\sqrt{\tau}} \left(\frac{1}{2}\sigma\sqrt{\tau} - x'\right) x'\,dx' \\
&= \frac{1}{24}(\sigma\sqrt{\tau})^2.
\end{aligned}$$

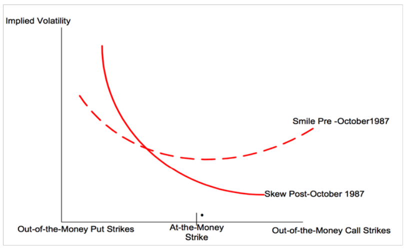
Thus, a sufficient condition for the inequality 210 to hold is

$$\sigma\sqrt{\tau} \leqslant \sqrt{\frac{24}{101}} \approx 0.4874. \tag{211}$$

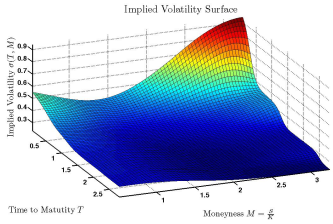
$\square$

# Part IV. Portfolio Optimization

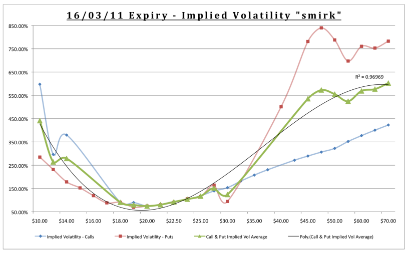
Reference to Table 1 on page 22.



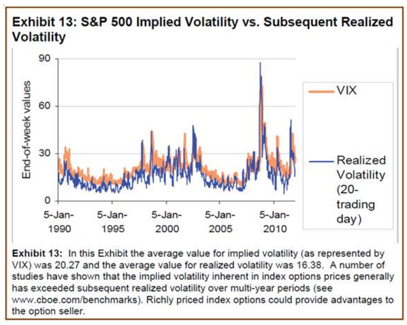

**(a)** Implied Volatility




**(b)** Volatility Surface




**(c)** Volatility Smirk




**(d)** Realized Volatility

**Figure 3:** Graphs on Volatility.

### 10.8 Figure Composed of Subfigures

Reference the figure composed of multiple subfigures as Figure 3. Reference one of the subfigures as Figure 3b.

# Part V.
# Appendix

## 11 EQUAL-SPACING NUMERICAL INTEGRATION

In this section, we derive the numerical algorithms for integrations over a closed interval on real line with equal-spacing discretization. The algorithms are characterized by an order of magnitude estimate of leading error in terms of discretization spacing and, for sufficiently smooth functions, are refined by higher-order approximations.

### 11.1 Euler–Maclaurin Formula

We want an approximation that is useful for smooth functions $f(x)$, and a smooth function is one for which the higher derivatives tend to be small. Therefore, if we can extract more terms in a way that leaves a remainder term that depends only on high derivatives of the function, then we have made progress. This can be accomplished by successively integrating by parts, each time differentiating $f(x)$ and integrating the function that multiplies it. We can define a set of functions

$$V_0(x) = 1, \quad V_m(x) = \int dx V_{m-1}(x), \quad \forall m > 0 \tag{212}$$

Equation 212 is not quite well-defined, however, because each indefinite integral is defined only up to an arbitrary constant of integration. Regardless of how these constants of integration are chosen, however, one can successively integrate by parts

$$\begin{aligned}
\int_{-1}^{+1} dx f(x) &= \underbrace{\int_{-1}^{+1} dx V_0(x) f(x)}_{\int_{-1}^{+1} dV_1(x) f(x)} \\
&= [V_1(1)f(1) - V_1(-1)f(-1)] - \underbrace{\int_{-1}^{+1} dx V_1(x) f'(x)}_{\int_{-1}^{+1} dV_2(x) f'(x)} \\
&= [V_1(1)f(1) - V_1(-1)f(-1)] \\
&\quad - \left[V_2(1)f'(1) - V_2(-1)f'(-1)\right] + \underbrace{\int_{-1}^{+1} dx V_2(x) f''(x)}_{\int_{-1}^{+1} dV_3(x) f''(x)} \\
&= \cdots \\
&= \sum_{k=1}^{2m} (-1)^{k+1} \left[V_k(1) f^{(k-1)}(1) - V_k(-1) f^{(k-1)}(-1)\right] \\
&\quad + \int_{-1}^{+1} dV_{2m+1}(x) f^{(2m)}(x),
\end{aligned} \tag{213}$$

where $f^{(k)}(x)$ denotes the $k$th derivative of $f(x)$ with respect to $x$.

Equation 213 is valid for any choice of integration constants in Equation 212, so we can seek a choice that simplifies the result. Note that $V_1(x) = x$ can be chosen to be odd under $x \leftrightarrow -x$. We can therefore choose the integration constants so that

$$V_m(-x) = \begin{cases} V_m(x), & \text{if } m \text{ is even;} \\ -V_m(x), & \text{if } m \text{ is odd.} \end{cases} \tag{214}$$

This even/odd requirement uniquely fixes the integration constant in Equation 212 when $n$ is odd, because the sum of an odd function and a non-zero constant would no longer be odd. We are still free, however, to choose the integration constants when $n$ is even. When $n$ is even in Equation 212, we choose the integration constant so that

$$\int_{-1}^{+1} dx V_m(x) = 0, \tag{215}$$

which is always true when $m$ is odd. Thus, Equation 215 is true for all $m > 0$. The $V_m$'s are now uniquely defined. In our construction we used the antisymmetry property in Equation 214 to fix the constant of integration for odd $m$, and the vanishing of the integral in Equation 215 to fix the integration constant for even $m$. The functions $V_m(x)$ can therefore be defined succinctly by

$$V_0(x) = 1, \quad V_m(x) = \int dx V_{m-1}(x), \tag{216a}$$

$$\int_{-1}^{+1} dx V_m(x) = 0, \quad \forall m > 0. \tag{216b}$$

It then follows that

$$V_m(1) - V_m(-1) = \int_{-1}^{+1} dx V_{m-1}(x) = 0, \quad \forall m > 1 \tag{217}$$

If $n$ is odd then Equation 217 implies that

$$V_m(1) = V_m(-1) = 0, \quad \forall \text{ odd } m > 1 \tag{218}$$

Using the even/odd parity of $V_m(x)$, Equation 213 can be simplified to

$$\begin{aligned}\int_{-1}^{+1} dx f(x) &= \sum_{k=1}^{2m} V_k(1)\left[(-1)^{k+1} f^{(k-1)}(1) + f^{(k-1)}(-1)\right] \\ &\quad + \int_{-1}^{+1} dV_{2m+1}(x) f^{(2m)}(x) \\ &= f(1) + f(-1) - \sum_{k=1}^{m} V_{2k}(1)\left[f^{(2k-1)}(1) - f^{(2k-1)}(-1)\right] \\ &\quad + \int_{-1}^{+1} dV_{2m+1}(x) f^{(2m)}(x).\end{aligned} \tag{219}$$

To apply Equation 219 to an arbitrary interval $a < x < a + h$, one needs only to change variables

$$x \leftarrow a + \frac{h}{2}(x+1). \tag{220}$$

Rewriting Equation 219 in terms of the transformed variable, while dividing the whole equation by 2 for convenience, one has

$$\begin{aligned}\int_{a}^{a+h} dx f(x) &= \frac{h}{2}[f(a) + f(a+h)] \\ &\quad - \sum_{k=1}^{m}\left(\frac{h}{2}\right)^{2k} V_{2k}(1)\left[f^{(2k-1)}(a+h) - f^{(2k-1)}(a)\right] \\ &\quad + \left(\frac{h}{2}\right)^{2m}\int_{a}^{a+h} dV_{2m+1}\left(\frac{2}{h}(x-a) - 1\right) f^{(2m)}(x)\end{aligned} \tag{221}$$

The Euler-MacLaurin formula can now be completed by extending Equation 221 to an interval $a < x < b$, divided into $n$ steps of size $h = (b-a)/n$. Adding an expression of the form Equation 221 for each interval of size $h$, one has

$$\begin{aligned}\int_{a}^{b} dx f(x) &= h\left[\frac{1}{2}f(a) + \underbrace{f(a+h) + \cdots + f(b-h)}_{(n-1)\text{ interior points}} + \frac{1}{2}f(b)\right] \\ &\quad - \sum_{k=1}^{m}\left(\frac{h}{2}\right)^{2k} V_{2k}(1)\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right]\end{aligned} \tag{222}$$

For all practical purposes, we have neglected the final term in Equation 222, often called the remainder term. One normally does not evaluate this term, but one wants to use it to argue that the remainder is small.

## 11.2 Midpoint Rule

Divide an interval $a < x < b$ into $m$ steps of size $h = (b-a)/n$, the midpoint rule of approximating an integral is

$$\int_a^b dx f(x) \approx h \sum_{i=1}^{n} f(x_{i-1/2}) \triangleq M_n, \tag{223}$$

where the integrand is sampled at midpoints of intervals

$$x_{i-1/2} \equiv a + \left(i - \frac{1}{2}\right) h, \quad i = 1, \cdots, n. \tag{224}$$

Note that there is an equal number of $n$ midpoints as the number of intervals and each midpoint is assigned a unit weight. The initial estimation with a single interval is

$$M_1 = hf\left(\frac{a+b}{2}\right). \tag{225}$$

## 11.3 Trapezoidal Rule

Divide an interval $a < x < b$ into $m$ steps of size $h = (b-a)/n$, the trapezoidal rule of approximating an integral is

$$\int_a^b dx f(x) \approx h \left[\frac{1}{2} f(a) + \underbrace{f(a+h) + \cdots + f(b-h)}_{(n-1)\text{ interior points}} + \frac{1}{2} f(b)\right] \triangleq T_n, \tag{226}$$

where the integrand is sampled at boundary points of intervals

$$x_i = a + ih, \quad i = 0, \cdots, n. \tag{227}$$

while the two boundary points of the whole range $[a, b]$ are assigned half weight and the interior points are assigned unit weight. Note that there are $n+1$ boundary points for a number of $n$ intervals. The initial estimation with a single interval is

$$T_1 = \frac{h}{2}\left[f(a) + f(b)\right]. \tag{228}$$

A useful observation to relate midpoint rule and trapezoidal rule can be made when we double the number of steps dividing the interval $[a, b]$:

$$2T_{2n} = T_n + M_n, \quad \forall n \geqslant 1. \tag{229}$$

*Proof.* Denote the step width of the $n$-step discretization scheme by $h = (b-a)/n$, then the step width of the $2n$-step scheme is $h/2$. From the definition of trapezoidal rule, Equation 226:

$$\begin{aligned}
2T_{2n} &= 2 \times \frac{h}{2}\left[\frac{1}{2}f(a) + \sum_{i=1}^{2n-1} f(x_i) + \frac{1}{2}f(b)\right], \quad x_i = a + i \times \frac{h}{2} \\
&= h\left[\frac{1}{2}f(a) + \underbrace{\sum_{i=1}^{n} f(x_{2i-1})}_{\text{odd terms}} + \underbrace{\sum_{i=1}^{n-1} f(x_{2i})}_{\text{even terms}} + \frac{1}{2}f(b)\right] \\
&= h\left[\underbrace{\frac{1}{2}f(a) + \sum_{i=1}^{n-1} f(a+ih) + \frac{1}{2}f(b)}_{\text{Trapezoidal}} + \underbrace{\sum_{i=1}^{n} f\left(a + \left(i - \frac{1}{2}\right)h\right)}_{\text{Midpoints}}\right] \\
&= T_n + M_n. \qquad (230)
\end{aligned}$$

□

According to Euler-Maclaurin, Equation 222,

$$I = T_n + Ch^2 + \cdots \qquad (231)$$

where I denotes the exact value of the integral, the leading error term for trapezoidal rule is on the order of $h^2$, and $C = -V_2(1)[f'(b) - f'(a)]/4$. From the recursion relation, Equation 229,

$$M_n = 2T_{2n} - T_n = 2\left(I - \frac{1}{4}Ch^2\right) - \left(I - Ch^2\right) + \cdots = I + \frac{1}{2}Ch^2 + \cdots \qquad (232)$$

Thus,

$$I = M_n - \frac{1}{2}Ch^2 + \cdots, \qquad (233)$$

which indicates that the leading error term of midpoint rule is also on the order of $h^2$ and is roughly half the size of the leading error term of trapezoidal rule.

## 11.4 Simpson's Rule

Simpson's rule is a higher-order scheme relative to trapezoidal rule or midpoint rule, which aims at reducing the order of magnitude of the leading error term given by trapezoidal rule. Simpson's rue can be most easily derived by invoking Euler-MacLaurin, Equation 222, to the 4th order of step width $h$,

$$\begin{aligned}
I &= T_n + Ch^2 + Dh^4 \cdots \\
&= T_{2n} + \frac{1}{4}Ch^2 + \frac{1}{16}Dh^4 \cdots,
\end{aligned}$$

and perform a linear combination of $T_n$ and $T_{2n}$ to eliminate the 2nd order term of step width

$$I = \frac{4T_{2n} - T_n}{3} - \frac{1}{4}Dh^4 + \cdots \triangleq S_n - \frac{1}{4}Dh^4 + \cdots \qquad (234)$$

where $D = -V_4(1)[f'''(b) - f'''(a)]/16$ and the Simpson's rule is defined as

$$S_n = \frac{4T_{2n} - T_n}{3}. \tag{235}$$

The leading error term of Simpson's rule is on the order of $h^4$, which is more accurate an approximation compared to trapezoidal rule or midpoint rule if the integrand function is sufficiently smooth since the coefficient $D$ is dependent on the third order derivative of the integrand function at boundary points.

## 11.5 Romberg Integration

Romberg integration scheme is a generalization of Simpson's rule to even higher orders. For example, to derive a rule with leading error term on the order $h^6$, we use Equation 234 applied to step width equal to $h$ and $h/2$

$$\begin{aligned} I &= S_n - \frac{1}{4}Dh^4 + O(h^6) + \cdots \\ &= S_{2n} - \frac{1}{64}Dh^4 + O(h^6) + \cdots, \end{aligned}$$

and perform a linear combination of $S_n$ and $S_{2n}$ to eliminate the 4th order term of step width

$$I = \frac{16S_{2n} - S_n}{15} + O(h^6). \tag{236}$$

Thus we have derived a 6th order rule based on the 4th Simpson's rule — a Romberg rule of 6th order

$$R_n^{(6)} = \frac{16S_{2n} - S_n}{15}. \tag{237}$$

Note that Romberg's rules always have an even order, where Simpson's rule is a special case of 4th order $S_n = R_n^{(4)}$.

To apply the above reasoning to higher and higher orders with respect to step width, we need to generalize our notations. Denote an approximation scheme by $A_{mn}$, where the closed interval of integration is divided into $2^n$ steps and the leading error term is $(2m+2)^{\text{th}}$ order of the step width. We choose the lowest order approximation to the trapezoidal rule and define the recursion relation

$$\begin{aligned} A_{0n} &\triangleq T_{2^n}, \\ A_{mn} &= \frac{4^m A_{m-1,n} - A_{m-1,n-1}}{4^m - 1}, \quad m \leqslant n. \end{aligned} \tag{238}$$

From Equation 238, we recover the Simpson's rule, Equation 235,

$$R_{2^n}^{(4)} = A_{1,n+1}, \quad n \geqslant 0, \tag{239}$$

and the Romberg's rule of 3rd order, Equation 237,

$$R_{2^n}^{(6)} = A_{2,n+2}, \quad n \geqslant 0. \tag{240}$$

For sufficiently smooth integrand function, the Romberg's rule of 10th order $R_{2^n}^{(10)} = A_{4,n+4}$ is commonly used and implemented in Numerical

Recipes. Finally, we have constructed a triangular hierarchy of approximations

$$\begin{array}{llll} A_{00} = T_1 & & & \\ A_{01} = T_2 & A_{11} = S_1 & & \\ A_{02} = T_4 & A_{12} = S_2 & A_{22} & \\ A_{03} = T_8 & A_{13} = S_4 & A_{23} & A_{33} \\ \vdots & \vdots & \vdots & \vdots \end{array} \tag{241}$$

The hierarchy refines the approximation in the vertical direction by adding more granular discretization steps and in the horizontal direction by constructing higher order rules. The iteration of refining the integration is usually measured by the difference between successive approximation and is usually stopped when the difference falls below a tolerance threshold $\delta$:

$$|A_{mn} - A_{m-1,n}| < \delta. \tag{242}$$