

BARUCH, MFE

MTH 9876 Assignment Five

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1 A VaR MODEL FOR A PORTFOLIO OF EQUITIES

In this problem you will develop a VaR model for a portfolio of equities, based on historical simulations (this is essentially the industry standard methodology). It is somewhat data analysis intensive, so give yourself enough time to work on this problem.

(i) Download the historical daily prices since January 1, 2004 of the following names: AAPL, HAL, INTC, JPM, PFE, CAT, DE, MSFT, WMT, C. You can use, e.g., Yahoo Finance or Bloomberg as the source of data. Make sure to use the *adjusted* prices, this way you don't have to worry about dividends and/or splits. Consider a model portfolio that consists of 10,000 shares of each of these shares.

Solution: See attached spreadsheet *Problem1 VaR Model Equity Portfolio.xlsx* data tab.

(ii) Assuming the MPOR of 5 business days, calculate the time series of 5 day P&Ls of this portfolio, starting on November 1, 2006.

Solution: See attached spreadsheet *Problem1 VaR Model Equity Portfolio.xlsx* analysis tab.

(iii) Run the EWMA model with $\lambda = 0.97$ and $r_t = 5$ day portfolio P&L. Remember that you need an extra 6 month (or so) period to prime the model. Scale the time series of the portfolio P&L by σ_t in order to produce a normalized series (make sure that you use the correct units!).

Solution: See attached spreadsheet *Problem1 VaR Model Equity Portfolio.xlsx* analysis tab.

(iv) Assuming a rolling look-back period of 2 years (defined as 500 business days), calculate the portfolio VaR and CoVaR at the 99% level of confidence for each of the days starting on November 1, 2006.

Solution: See attached spreadsheet *Problem1 VaR Model Equity Portfolio.xlsx* analysis tab.

(v) Back test your VaR model for the period from November 1, 2006. Do a detailed statistical analysis of the VaR breaches, and link them to specific historic market events. Would using CoVaR instead of VaR mitigate the breaches? *Solution:* See attached spreadsheet *Problem1 VaR Model Equity Portfolio.xlsx* analysis tab.

2 A BASIC COMMODITY MODEL

Consider the commodity model discussed in class

$$\begin{aligned} S(t) &= \exp(f(t) + X(t)), \\ dX(t) &= \lambda(\theta - X(t))dt + \sigma dW(t), \end{aligned}$$

where $f(t)$ is a deterministic periodic function, and λ , θ , and θ are constant parameters. The process $S(t)$ is the spot price of the commodity (this could be a very basic model for crude oil).

(i) Show that

$$X(t) = X_0 e^{-\lambda t} + \theta(1 - e^{-\lambda t}) + \sigma \int_0^t e^{-\lambda(t-s)} dW(s)$$

is a solution to the SDE for $X(t)$ with the initial condition $X(0) = X_0$. Calculate the expected value $\mathbb{E}[X(t)]$ and variance $\text{Var}[X(t)]$ of $X(t)$.

Solution: Apply Itô's lemma to $e^{\lambda t} X(t)$,

$$\begin{aligned} d(e^{\lambda t} X(t)) &= \lambda e^{\lambda t} X(t) dt + e^{\lambda t} dX(t) \\ &= e^{\lambda t} (\lambda X(t) dt + dX(t)) \\ &= e^{\lambda t} [\lambda X(t) dt + \lambda(\theta - X(t)) dt + \sigma dW(t)] \\ &= e^{\lambda t} (\lambda \theta dt + \sigma dW(t)). \end{aligned}$$

Integrate from 0 to t ,

$$\begin{aligned} e^{\lambda t} X(t) - X(0) &= \theta \int_0^t e^{\lambda s} d(\lambda s) + \sigma \int_0^t e^{\lambda s} dW(s) \\ &= \theta(e^{\lambda t} - 1) + \sigma \int_0^t e^{\lambda s} dW(s). \end{aligned}$$

Re-arrange the terms, we get

$$X(t) = X_0 e^{-\lambda t} + \theta(1 - e^{-\lambda t}) + \sigma \int_0^t e^{-\lambda(t-s)} dW(s). \quad (2.1)$$

Given the martingality of the term $\sigma \int_0^t e^{-\lambda(t-s)} dW(s) \sim N(0, \sigma^2 \int_0^t e^{-2\lambda(t-s)} ds)$, we get

$$\begin{aligned} \mathbb{E}[X(t)] &= X_0 e^{-\lambda t} + \theta(1 - e^{-\lambda t}) \xrightarrow{t \rightarrow \infty} \theta, \\ \text{Var}[X(t)] &= \frac{\sigma^2}{2\lambda} (1 - e^{-2\lambda t}) \xrightarrow{t \rightarrow \infty} \frac{\sigma^2}{2\lambda}. \end{aligned}$$

(ii) Consider the forward price on oil for the delivery on date T . Its time t price, where $t < T$, is defined by

$$F_T(t) = \mathbb{E}_t[S(T)].$$

Find an explicit expression for $F_T(t)$ by calculating the expected value above.

Solution: Conditioned on information available up to time t ,

$$\begin{aligned} X(T) &= X(t)e^{-\lambda(T-t)} + \theta(1 - e^{-\lambda(T-t)}) + \sigma \int_t^T e^{-\lambda(T-s)} dW(s). \\ &\sim N\left(X(t)e^{-\lambda(T-t)} + \theta(1 - e^{-\lambda(T-t)}), \frac{\sigma^2}{2\lambda}(1 - e^{-2\lambda(T-t)})\right), \end{aligned}$$

where $S(t) = \exp(f(t) + X(t))$. Thus,

$$\begin{aligned} \mathbb{E}_t[\exp(X(T))] &= \exp\left(\mathbb{E}_t[X(T)] + \frac{1}{2}\text{Var}[X(T)]\right) \\ &= \exp\left(X(t)e^{-\lambda(T-t)} + \theta(1 - e^{-\lambda(T-t)}) + \frac{\sigma^2}{4\lambda}(1 - e^{-2\lambda(T-t)})\right), \end{aligned}$$

and

$$\begin{aligned} F_T(t) &= \mathbb{E}_t[S(T)] \\ &= \mathbb{E}_t[\exp(f(T) + X(T))] \\ &= \exp\left(f(T) + X(t)e^{-\lambda(T-t)} + \theta(1 - e^{-\lambda(T-t)}) + \frac{\sigma^2}{4\lambda}(1 - e^{-2\lambda(T-t)})\right) \\ &= S(t)e^{-\lambda(T-t)} \exp\left(f(T) - f(t)e^{-\lambda(T-t)} + \theta(1 - e^{-\lambda(T-t)}) + \frac{\sigma^2}{4\lambda}(1 - e^{-2\lambda(T-t)})\right). \end{aligned} \tag{2.2}$$

(iii) Consider the payer forward rate agreement (FRA) on crude oil on which the bank (B) agrees to buy crude at T at the price K . Find the risk neutral price $V_{\text{pay}}(t)$ of this contract.

Solution: The pay off of the payer forward rate agreement at time T is $S(T) - K$. The risk-neutral price of the FRA contract is equal to

$$V_{\text{pay}}(t) = e^{-r(T-t)} \mathbb{E}_t[S(T) - K] = e^{-r(T-t)} (\mathbb{E}_t[S(T)] - K),$$

where r is the risk-free zero rate over the time period $t \rightarrow T$ and $\mathbb{E}_t[S(T)]$ is obtained in the previous problem.

Note that this problem should not be reasoned in the same way as an equity forward contract, where the payoff can be replicated by a portfolio consisting of a long position of one share of the underlying asset $S(t)$ and a short position of $Ke^{-r(T-t)}$ cash at time t . According to the law of one price, the risk-neutral price of the forward contract is equal to the value of the replicating portfolio at time t ,

$$V_{\text{pay}}(t) = S(t) - Ke^{-r(T-t)}.$$

This above result based on the law of one price is in agreement with the fact that the risk-neutral expectation $e^{-r(T-t)} \mathbb{E}_t[S(T)] = S(t)$. Either approach assumes that there is no cost

associated with holding the underlying asset, e.g., an equity stock. For commodities, however, there is a convenience yield (cost, premium, etc.) associated with holding an underlying product and the equality $e^{-r(T-t)}\mathbb{E}_t[S(T)] = S(t)$ is no longer true. By buying a forward contract at time t , the buyer locks in the price of purchasing the commodity underlying asset at price $F_T(t)$ at time T . When settled at time t , the payoff of the FRA contract at time T is $F_T(t) - K$, effectively equivalent to that of a fixed income instrument.

(iv) Recall the stochastic intensity model introduced in Problem #3 of Assignment #1:

$$d\lambda_C(t) = \mu_C dt + \nu_C dZ_C(t),$$

where $Z_C(t)$ is a Brownian motion. Assume that this model describes the credit of the bank's counterparty (C).

Solution: OK. Recalled. ☺ The survival probability is

$$S(t, T) = e^{-\lambda_C(0)(T-t) - \frac{1}{2}\mu_C(T-t)^2 + \frac{1}{6}\nu_C^2(T-t)^3}. \quad (2.3)$$

(v) Assume that $W(t)$ and $Z_C(t)$ are uncorrelated (no wrong way risk). Assuming that the bank is default-free, calculate explicitly the CVA on the oil FRA (use the discretized CVA formula) and interpret the expression in option theoretic terms.

Solution: According to the discretized CVA formula without wrong way risk, the expected exposure and the counterparty default probability factorize:

$$\begin{aligned} \text{CVA}(t, T) &= (1 - \bar{R}) \sum_{i=1}^m \mathbb{E}\mathbb{E}_t(T_i, T) [Q(t, T_i) - Q(t, T_{i-1})] \\ &= (1 - \bar{R}) \sum_{i=1}^m \mathbb{E}\mathbb{E}_t(T_i, T) [S(t, T_{i-1}) - S(t, T_i)], \end{aligned}$$

where $T_0 = t$ and $T_m = T$. The discounted expected exposure

$$\begin{aligned} \mathbb{E}\mathbb{E}_t(T_i, T) &\triangleq \mathbb{E}^{\mathbb{Q}}[P(t, T_i)V(T_i, T)^+] \\ &= P(t, T_i)\mathbb{E}^{\mathbb{Q}}[V(T_i, T)^+], \end{aligned}$$

where we assumed that the discount factor $P(t, T_i) \perp V(T_i, T)$, and $V(T_i, T)$ is obtained in Part (iii):

$$\begin{aligned} V(T_i, T) &= V_{\text{pay}}(T_i) \\ &= P(T_i, T) (\mathbb{E}_{T_i}[S(T)] - K) \\ &= P(T_i, T) (F_T(T_i) - K), \end{aligned}$$

where $F_T(t)$ is obtained in Part (ii). Thus, write $P(t, T_i)P(T_i, T) = P(t, T)$,

$$\mathbb{E}\mathbb{E}_t(T_i, T) = P(t, T)\mathbb{E}^{\mathbb{Q}}[(F_T(T_i) - K)^+],$$

and

$$\text{CVA}(t, T) = (1 - \bar{R})P(t, T) \sum_{i=1}^m \mathbb{E}^{\mathbb{Q}}[(F_T(T_i) - K)^+] \cdot [S(t, T_{i-1}) - S(t, T_i)].$$

Notice that $\mathbb{E}^Q[(F_T(T_i) - K)^+]$ is the risk-neutral expectation of the payoff of a call option of an FRA contract settled at T_i to buy crude at T at strike price K . Thus, the CVA on the crude oil FRA can be interpreted as the present value of the unrecovered part of a sum of call options on FRA contracts, settled at intermediate times between $[t, T]$ and weighted by default probabilities for each time periods respectively. Effectively, the bank sells a series of call options to the counterparty to be able to default, at various times between $[t, T]$.

(vi) Assume now $W(t)$ and $Z_C(t)$ are correlated (the counterparty may be an oil company), i.e. $dW(t)dZ_C(t) = \rho dt$. Calculate the CVA under this assumption. What is the impact of the wrong way risk, depending on the sign of ρ ?

Solution: Similar to the Part (v) but the exposure and the counterparty default probability no longer factorize:

$$\text{CVA}(t, T) = (1 - \bar{R})P(t, T) \sum_{i=1}^m \mathbb{E}^Q[(F_T(T_i) - K)^+ \cdot (S(t, T_{i-1}) - S(t, T_i)) | \tau = T_i],$$

where $S(t, T)$ denotes a random variable instead of expectation value as in Eq. 2.3: $S(t, T) \triangleq \exp(-\int_t^T \lambda_C(s) ds)$. Thus, according to Eq. 2.2,

$$\begin{aligned} \text{CVA}(t, T) &= (1 - \bar{R})P(t, T) \\ &\times \sum_{i=1}^m \mathbb{E}^Q \left[\left(\exp \left(f(T) + X(T_i) e^{-\lambda(T-T_i)} + \theta(1 - e^{-\lambda(T-T_i)}) + \frac{\sigma^2}{4\lambda}(1 - e^{-2\lambda(T-T_i)}) \right) - K \right)^+ \right. \\ &\quad \left. \times \left(\exp \left(-\int_t^{T_{i-1}} \lambda_C(s) ds \right) - \exp \left(-\int_t^{T_i} \lambda_C(s) ds \right) \right) \middle| \tau = T_i \right], \end{aligned}$$

where according to Eq. 2.1,

$$\begin{aligned} X(T_i) e^{-\lambda(T-T_i)} &= X_0 e^{-\lambda T} + \theta(e^{-\lambda(T-T_i)} - e^{-\lambda T}) + \sigma \int_0^{T_i} e^{-\lambda(T-s)} dW(s) \\ \int_t^{T_i} \lambda_C(s) ds &= \lambda_0(T_i - t) + \frac{1}{2}\theta(T_i - t)^2 + \sigma \int_t^{T_i} Z_C(s) ds. \end{aligned}$$

Written in a fully explicit way,

$$\begin{aligned} \text{CVA}(t, T) &= (1 - \bar{R})P(t, T) \\ &\times \sum_{i=1}^m \mathbb{E}^Q \left[\left(\exp \left(\underbrace{f(T) + X_0 e^{-\lambda T} + \theta(1 - e^{-\lambda T}) + \frac{\sigma^2}{4\lambda}(1 - e^{-2\lambda(T-T_i)})}_{\text{deterministic}} + \sigma \int_0^{T_i} e^{-\lambda(T-s)} dW(s) \right) - K \right)^+ \right. \\ &\quad \left. \times \left(\exp \left(\underbrace{-\lambda_0(T_{i-1} - t) - \frac{1}{2}\theta(T_{i-1} - t)^2 - \sigma \int_t^{T_{i-1}} Z_C(s) ds}_{\text{deterministic}} \right) - \exp \left(\underbrace{-\lambda_0(T_i - t) - \frac{1}{2}\theta(T_i - t)^2 - \sigma \int_t^{T_i} Z_C(s) ds}_{\text{deterministic}} \right) \right) \middle| \tau = T_i \right], \end{aligned}$$

where $dW(t)dZ_C(t) = \rho dt$. The impact of the wrong way risk depending on the sign of ρ is unclear from the above expression only.

(vii) Assume that the bank's credit is given by a stochastic intensity model with the same specification but different parameters as in part (iv):

$$d\lambda_B(t) = \mu_B dt + \nu_B dZ_B(t).$$

Assume that the pairs of Brownian motions $W(t)$ and $Z_B(t)$, and $Z_C(t)$ and $Z_B(t)$ are independent. Derive an expression for BCVA on the oil payer FRA. Discuss the results (provide suitable graphs) for various ranges of the model parameters.

Solution: Assume there is no wrong way risk, the bilateral CVA formula

$$\begin{aligned} \text{BCVA}(t, T) &= (1 - \bar{R}_C) \sum_{i=1}^m \mathbb{E} \mathbb{E}_t(T_i, T) S_B(t, T_{i-1}) [Q_C(t, T_i) - Q_C(t, T_{i-1})] \\ &\quad + (1 - \bar{R}_B) \sum_{i=1}^m \mathbb{N} \mathbb{E}_t(T_i, T) S_C(t, T_{i-1}) [Q_B(t, T_i) - Q_B(t, T_{i-1})] \\ &= (1 - \bar{R}_C) \sum_{i=1}^m \mathbb{E}^{\mathbb{Q}} [P(t, T_i) V(T_i, T)^+] S_B(t, T_{i-1}) [Q_C(t, T_i) - Q_C(t, T_{i-1})] \\ &\quad + (1 - \bar{R}_B) \sum_{i=1}^m \mathbb{E}^{\mathbb{Q}} [P(t, T_i) V(T_i, T)^-] S_C(t, T_{i-1}) [Q_B(t, T_i) - Q_B(t, T_{i-1})] \\ &= (1 - \bar{R}_C) P(t, T) \sum_{i=1}^m \mathbb{E}^{\mathbb{Q}} [(F_T(T_i) - K)^+] S_B(t, T_{i-1}) [Q_C(t, T_i) - Q_C(t, T_{i-1})] \\ &\quad - (1 - \bar{R}_B) P(t, T) \sum_{i=1}^m \mathbb{E}^{\mathbb{Q}} [(K - F_T(T_i))^+] S_C(t, T_{i-1}) [Q_B(t, T_i) - Q_B(t, T_{i-1})], \end{aligned}$$

where $S_B(t, T)$, $S_C(t, T)$, $Q_B(t, T)$, and $Q_C(t, T)$ are defined accordingly.