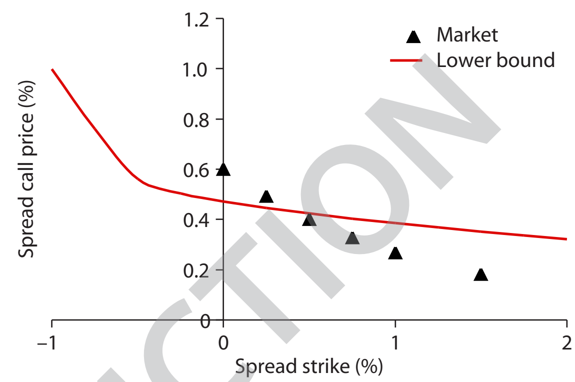
The CMS triangle arbitrage

A dislocation between options on constant maturity swap rates and spreads in 2009 led to a static arbitrage opportunity. Here, ***Paul McCloud*** *shows how this can be detected, and how a copula-based model strategy can exploit it*

For much of 2009 there was a static arbitrage in euro constant maturity swap (CMS) spread options, a consequence of the dislocation between the markets for options on CMS rates and CMS spreads. High volatility of volatility in the vanilla rates market pushed up the prices of long-dated CMS rate options, as these options are static replicated using the vanilla rates smile. In contrast, supply pressures kept the prices of CMS spread options suppressed. The requirement to maintain mark-to-market for CMS spread options led quants to consider models that do not directly reference the CMS rate marginals, choosing instead to model the CMS spread distribution directly. This allowed prices in these two markets to drift apart, with essentially independent models marked to increasingly disconnected markets. In some cases, the gap that opened up was sufficient to create an exploitable arbitrage between the markets.

This article considers the construction of the CMS triangle arbitrage, its mathematical limits and strategies for maximising the benefit. Throughout the discussion, the worked example corresponds to single-look 30-year/two-year euro CMS spread options with a 15-year expiry, based on market data from October 1, 2009. The prices for CMS rate options impose a no-arbitrage lower bound on the CMS spread option price. The lower bound is the price of a portfolio of CMS rate options whose payout is always less than the CMS spread option payout. In this example, the return from selling these CMS rate options exceeds the cost of buying the CMS spread option for strikes greater than 0.5%. The combined portfolio of CMS spread and rate options is guaranteed to have a positive payout, and so this trade generates an arbitrage.

1 Lower bound for the CMS spread call price



Copulas provide the natural framework for analysing the consistency relationships between a spread and its two constituent rates (see Nelsen, 1998, for an introduction), and this is the approach adopted here. The application of copulas to CMS spread option pricing has been considered previously, for example in Berrahoui (2004) and Benhamou & Croissant (2007). The no-arbitrage bounds for the spread option price are generated by the Fréchet-Hoeffding boundary copulas, which have been investigated in Cherubini & Luciano (2000).

The mathematics below applies equally to any spread options, and it is conceivable that similar arbitrages exist in other spread markets. As an example, the final section considers the application of the no-arbitrage bounds to the well-known triangle relationship between the foreign exchange options of currency triples. For more depth on the application of copulas to this and related problems in forex markets, see, for example, Bennett & Kennedy (2003).

The CMS triangle inequality

Spread option payouts satisfy a simple triangle inequality. For the two quantities A and B:

$$(A+B)^+ \le A^+ + B^+ \tag{1}$$

Two applications of this expression for the rates r_1 and r_2 with strikes k_1 and k_2 lead to the inequality:

$$(r_1-k_1)^+ - (r_2-k_2)^+ \le (s-l)^+ \le (r_1-k_1)^+ + (k_2-r_2)^+ \tag{2}$$

for the spread $s = r_1 - r_2$ with spread strike $l = k_1 - k_2$. The terms in this inequality are the payouts for rate and spread options, and the pairs of rate options in the lower and upper bounds define static hedges for the spread option.

The price of the spread option is constrained to lie between the hedge prices in arbitrage-free models. With strikes expressed as $k_1 = k + ½l$ and $k_2 = k - ½l$ in terms of the mid-strike $k = ½(k_1 + k_2)$ and the spread strike $l = k_1 - k_2$, the price constraints become:

$$\begin{aligned} \mathbb{E}\left[(s-l)^+\right] &\ge \mathbb{E}\left[\left(r_1-\left(k+\tfrac{1}{2}l\right)\right)^+\right] - \mathbb{E}\left[\left(r_2-\left(k-\tfrac{1}{2}l\right)\right)^+\right] \\ \mathbb{E}\left[(s-l)^+\right] &\le \mathbb{E}\left[\left(r_1-\left(k+\tfrac{1}{2}l\right)\right)^+\right] + \mathbb{E}\left[\left(\left(k-\tfrac{1}{2}l\right)-r_2\right)^+\right] \end{aligned} \tag{3}$$

where the expectation is in the pricing measure. The prices of the pairs of rate options on the right-hand side of the inequalities give bounds for the spread option price for any value of the mid-strike.

2 Prices for the lower and upper hedges of the CMS spread call

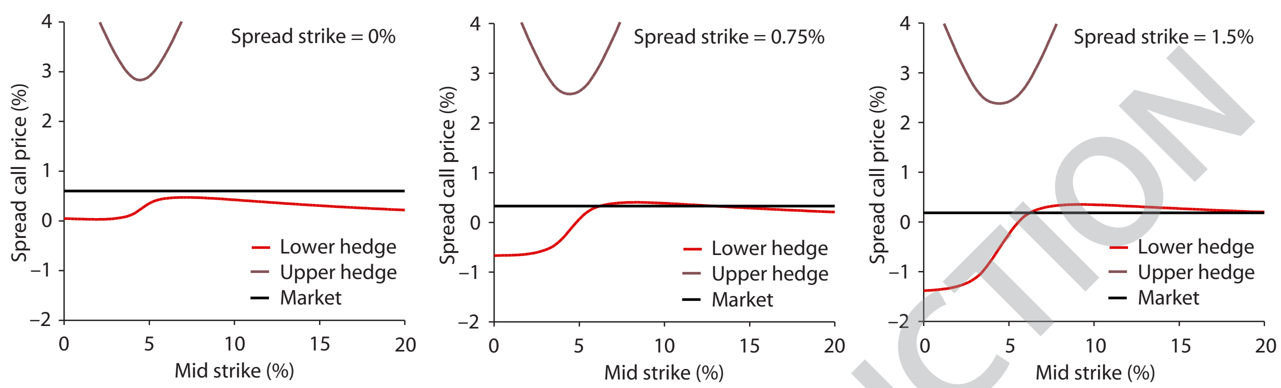


Figure 2 shows the hedge prices as functions of the mid-strike, for the example 30-year/two-year CMS spread call with strikes l = 0%, 0.75% and 1.5%. The prices for the CMS rate calls are determined by static replication with vanilla rate options, using conservative assumptions on the extrapolation for the tail component of the replication.

Comparing with the market, it can be seen from figure 2 that there is an arbitrage for the spread strike l = 1.5%, as the market price is lower than the peak lower hedge price in this case. The optimal mid-strike for exploiting the arbitrage occurs at the maximum of the lower hedge price, which is around k = 9.25%. The strategy for exploiting the arbitrage is remarkably simple, requiring a static hedge consisting of only two CMS rate calls:

- Buy a call on the 30-year/two-year CMS spread with strike 1.5%.
- Sell a call on the 30-year CMS rate with strike 10%.
- Buy a call on the two-year CMS rate with strike 8.5%.

Provided the CMS spread call and the two CMS rate calls at the specified strikes can be traded at these levels, the CMS triangle generates an unambiguous, model-independent, static arbitrage.

Note the qualitative difference in the graphs of the lower and upper hedge prices as functions of the mid-strike. While the upper hedge price has a convex shape with a unique global minimum, the lower hedge price can have multiple local minima and maxima. This observation complicates the identification of the optimal lower hedge, as the algorithm must identify the global maximum from the collection of local maxima. It will be demonstrated in the following that the true optimal lower bound for the spread option price incorporates all the turning points of the lower hedge price, minima as well as maxima, so that the global topology of the hedge price curve is relevant to the identification of the arbitrage opportunity.

Broadly speaking, the arbitrage opportunity arises when CMS spread options are cheap and CMS rate options are expensive. CMS rate options are priced using static replication with vanilla rate options, and the ambiguity in the tail replication feeds through to a considerable bid-offer spread for the options, which the arbitrage must exceed to be workable. As a real example, the market prices in August 2009 for 10-year/two-year euro CMS spread and rate option legs starting in five years and maturing in 20 years are shown in table A.

The price of the CMS triangle in this case is 1.03% (the market offer of the CMS spread option) minus 4.33% (the market bid of the first CMS rate option) plus 3.20% (the market offer of the second CMS rate option), giving a total price of –0.10%. In spite of the wide bid-offer spread for the 10-year CMS rate option, the CMS triangle still generates an arbitrage. Moreover, smart execution offers the possibility of significantly raising the value of the arbitrage. At the extreme, if the two-year CMS rate option is bought for 2.84% and the 10-year CMS rate option is sold for 6.58%, then the price of the CMS triangle becomes a whopping –2.71%. This value is purely due to the arbitrage, and does not include the additional value in the CMS spread option that results from the decorrelation of the CMS rates.

In practice, the CMS triangle is constructed using overlapping legs that generate an arbitrage only in aggregation, and there is a small basis risk in the slightly mismatched schedules for the legs. To avoid this risk materialising, the CMS triangle position is closed out when market moves eliminate the arbitrage.

Consistent rate and spread models

After some algebra, the spread option price can be written in two alternative ways:

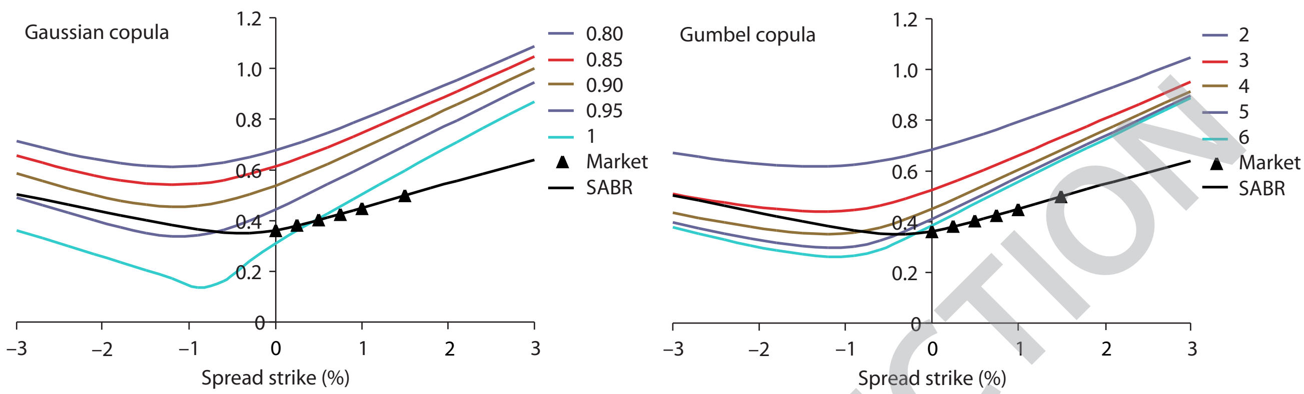
$$
\begin{aligned}
\mathbb{E}\left[(s-l)^+\right] = {} & \mathbb{E}\left[\left(r_1-\left(k+\tfrac{1}{2}l\right)\right)^+\right] - \mathbb{E}\left[\left(r_2-\left(k-\tfrac{1}{2}l\right)\right)^+\right] \\
& + \int_{\mu=-\infty}^{k} \mathbb{P}\left[\left(r_1>\mu+\tfrac{1}{2}l\right)\wedge\left(r_2<\mu-\tfrac{1}{2}l\right)\right]d\mu \\
& + \int_{\mu=k}^{\infty} \mathbb{P}\left[\left(r_1<\mu+\tfrac{1}{2}l\right)\wedge\left(r_2>\mu-\tfrac{1}{2}l\right)\right]d\mu
\end{aligned}
\qquad (4)
$$

$$
\begin{aligned}
\mathbb{E}\left[(s-l)^+\right] = {} & \mathbb{E}\left[\left(r_1-\left(k+\tfrac{1}{2}l\right)\right)^+\right] + \mathbb{E}\left[\left(\left(k-\tfrac{1}{2}l\right)-r_2\right)^+\right] \\
& - \int_{\mu=-\infty}^{k} \mathbb{P}\left[\left(r_1<\mu+\tfrac{1}{2}l\right)\wedge\left(r_2<\mu-\tfrac{1}{2}l\right)\right]d\mu \\
& - \int_{\mu=k}^{\infty} \mathbb{P}\left[\left(r_1>\mu+\tfrac{1}{2}l\right)\wedge\left(r_2>\mu-\tfrac{1}{2}l\right)\right]d\mu
\end{aligned}
\qquad (5)
$$

A. Prices in August 2009 for 10-year/two-year euro CMS spread and rate calls starting in five years and maturing in 20 years

Rate	Style	Strike	Start	Maturity	Bid	Offer
10y-2y	Call	1.5%	5y	20y		1.03%
10y	Call	10.0%	5y	20y	4.33%	6.58%
2y	Call	8.5%	5y	20y	2.84%	3.20%

3 CMS spread implied normal volatilities for different copula families



for any mid-strike k. The integrals are always positive, so these expressions verify the triangle inequality and furthermore demonstrate that the differences from the static hedges in the lower and upper bounds are given by tail integrals of the joint cumulative distribution function (CDF) for the two rates.

The triangle arbitrage occurs when the implied distribution for the spread is inconsistent with the implied distributions for the two rates. One way to ensure that the distributions are consistent is to generate them using a copula. The central observation of the copula approach is Sklar's theorem, which states that the joint distribution of the two rates is connected to the two marginal distributions by a copula function:

$$\mathbb{P}\left[\left(r_1 < k_1\right) \wedge \left(r_2 < k_2\right)\right] = C\left[\mathbb{P}\left[r_1 < k_1\right], \mathbb{P}\left[r_2 < k_2\right]\right] \tag{6}$$

This expression defines the joint CDF of the rates r_1 and r_2 in terms of their individual CDFs, using the copula function $C : [0, 1]^2 \to [0, 1]$. The marginal CDFs are the prices of rate digitals, and the copula weaves these marginals together to provide a joint CDF that can be used to price the spread option. The determination of the joint distribution for the two rates is then decoupled into the identification of the rate marginals, expressed as the market prices for rate options:

$$\begin{aligned}
f_1 &= \mathbb{E}\left[r_1\right] & f_2 &= \mathbb{E}\left[r_2\right] \\
c_1[k] &= \mathbb{E}\left[\left(r_1 - k\right)^+\right] & c_2[k] &= \mathbb{E}\left[\left(r_2 - k\right)^+\right] \\
p_1[k] &= \mathbb{E}\left[\left(k - r_1\right)^+\right] & p_2[k] &= \mathbb{E}\left[\left(k - r_2\right)^+\right] \\
u_1[k] &= \mathbb{P}\left[r_1 > k\right] & u_2[k] &= \mathbb{P}\left[r_2 > k\right] \\
d_1[k] &= \mathbb{P}\left[r_1 < k\right] & d_2[k] &= \mathbb{P}\left[r_2 < k\right]
\end{aligned} \tag{7}$$

and the specification of the joining copula.

Inserting the copula into the relationships above leads to:

$$\begin{aligned}
\mathbb{E}\left[(s-l)^+\right] = {} & c_1\left[k + \tfrac{1}{2}l\right] - c_2\left[k - \tfrac{1}{2}l\right] \\
& + \int_{\mu=-\infty}^{k} \left(d_2\left[\mu - \tfrac{1}{2}l\right] - C\left[d_1\left[\mu + \tfrac{1}{2}l\right], d_2\left[\mu - \tfrac{1}{2}l\right]\right]\right) d\mu \\
& + \int_{\mu=k}^{\infty} \left(d_1\left[\mu + \tfrac{1}{2}l\right] - C\left[d_1\left[\mu + \tfrac{1}{2}l\right], d_2\left[\mu - \tfrac{1}{2}l\right]\right]\right) d\mu
\end{aligned} \tag{8}$$

$$\begin{aligned}
\mathbb{E}\left[(s-l)^+\right] = {} & c_1\left[k + \tfrac{1}{2}l\right] + p_2\left[k - \tfrac{1}{2}l\right] \\
& - \int_{\mu=-\infty}^{k} C\left[d_1\left[\mu + \tfrac{1}{2}l\right], d_2\left[\mu - \tfrac{1}{2}l\right]\right] d\mu \\
& - \int_{\mu=k}^{\infty} \left(1 - d_1\left[\mu + \tfrac{1}{2}l\right] - d_2\left[\mu - \tfrac{1}{2}l\right]\right. \\
& \qquad \left. + C\left[d_1\left[\mu + \tfrac{1}{2}l\right], d_2\left[\mu - \tfrac{1}{2}l\right]\right]\right) d\mu
\end{aligned} \tag{9}$$

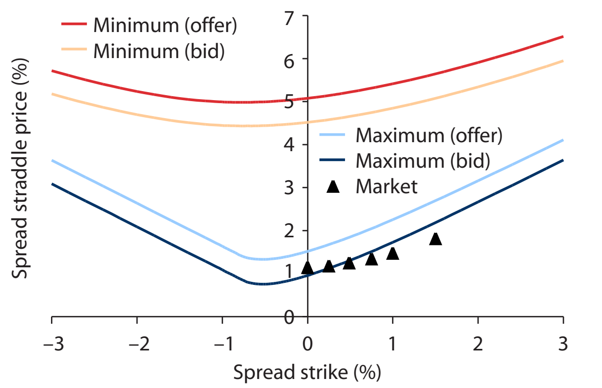
for any mid-strike k. Given the copula and the prices of rate options and digitals, these expressions are integrated using one-dimensional quadratures to determine consistent prices for the spread option. The span of consistent pricing models can then be explored by varying the copula.

Different families of copulas generate distinct smile profiles for spread options. Figure 3 demonstrates some results from this analysis for common types of copula. In each case, the results for a variety of parameter values in the copula family are compared with the market, with prices expressed as implied normal volatilities. While the span of the copulas used in figure 3 is hardly exhaustive, it is clear that the spread option market does not sit comfortably with the marginal rate option markets. Superposing the best-fit normal SABR smile through the market points, it is also apparent that standard parametric models for the spread option smile, while providing a good fit to the market, can be significantly inconsistent with the marginals.

Where the triangle inequality enables the identification of simple, static arbitrages in the spread option market, the copula allows the additional value over and above the proceeds of the arbitrage to be modelled consistently with the rate option markets. Varying the copula gives an idea of the range of prices that might be achieved by spread models that are consistent with the rate marginals. This result is not as clean as the simple triangle arbitrage of the previous section as it requires a complete understanding of the marginal distributions, which can only be achieved by using a model (such as SABR) to extrapolate the rate option markets. However, as will be seen below, the insight gained from the copula analysis informs the strategy for exploiting the arbitrage, and the copula provides a simple and efficient way to parameterise the spread option prices when marking the market to a model.

Bounds for the spread option price

Fortunately, the intuition from the copula analysis can be made precise, using a convenient property of copulas:



4 CMS spread straddle prices with the minimum and maximum copulas

$$C_-[\theta_1,\theta_2] \le C[\theta_1,\theta_2] \le C_+[\theta_1,\theta_2] \tag{10}$$

where:

$$\begin{aligned} C_-[\theta_1,\theta_2] &= \max[0,\theta_1+\theta_2-1] \\ C_+[\theta_1,\theta_2] &= \min[\theta_1,\theta_2] \end{aligned} \tag{11}$$

are the Fréchet-Hoeffding copula boundaries. Moreover, the two functions C_- and C_+ are themselves copulas, known as the minimum and maximum copulas, and the boundary property ensures that these copulas generate optimal bounds for the spread option price.

The optimal lower and upper bounds are demonstrated for the example 30-year/two-year CMS spread straddle in figure 4. The ambiguity in the marginal CMS rate distributions is represented by the gap between the bid and offer curves, which are generated using bounding assumptions for the tail replication of the CMS rate options that are calibrated to market prices. Note that, without smart execution, the best prices that can be achieved for the hedges are given by the outermost curves. The market crosses the lower bound at strike around $l = 0.5\%$, and the market generates an arbitrage for strikes higher than this crossover point.

The optimal bounds are evaluated by inserting the minimum and maximum copulas into the expressions of the previous section.

Optimal upper bound. The optimal upper bound is evaluated using the minimum copula:

$$\begin{aligned} \mathbb{E}\left[(s-l)^+\right] &\le c_1\left[k+\tfrac{1}{2}l\right]+p_2\left[k-\tfrac{1}{2}l\right] \\ &-\int_{\mu=-\infty}^{k}\max\left[0,d_1\left[\mu+\tfrac{1}{2}l\right]+d_2\left[\mu-\tfrac{1}{2}l\right]-1\right]d\mu \\ &-\int_{\mu=k}^{\infty}\Big(1-d_1\left[\mu+\tfrac{1}{2}l\right]-d_2\left[\mu-\tfrac{1}{2}l\right] \\ &\qquad+\max\left[0,d_1\left[\mu+\tfrac{1}{2}l\right]+d_2\left[\mu-\tfrac{1}{2}l\right]-1\right]\Big)d\mu \end{aligned} \tag{12}$$

In this case, a careful choice of the mid-strike k eliminates the two integrals. Define the optimal mid-strike $\bar{k}$ by:

$$d_1\left[\bar{k}+\tfrac{1}{2}l\right]+d_2\left[\bar{k}-\tfrac{1}{2}l\right]=1 \tag{13}$$

This expression defines a unique mid-strike when the rate CDFs are continuous and strictly increasing, which will be assumed in the following (this condition can easily be relaxed with only minor modification to the results that follow). The optimal upper bound is then:

$$\mathbb{E}\left[(s-l)^+\right] \le c_1\left[\bar{k}+\tfrac{1}{2}l\right]+p_2\left[\bar{k}-\tfrac{1}{2}l\right] \tag{14}$$

From the defining relationship, the optimal mid-strike $\bar{k}$ locates the global minimum of the upper hedge price:

$$\frac{\partial}{\partial k}\left(c_1\left[k+\tfrac{1}{2}l\right]+p_2\left[k-\tfrac{1}{2}l\right]\right)\bigg|_{k=\bar{k}}=0 \tag{15}$$

so that the optimal upper bound for the spread option price is given by the global minimum upper hedge price.

Optimal lower bound. The optimal lower bound is evaluated using the maximum copula:

$$\begin{aligned} \mathbb{E}\left[(s-l)^+\right] &\ge c_1\left[k+\tfrac{1}{2}l\right]-c_2\left[k-\tfrac{1}{2}l\right] \\ &+\int_{\mu=-\infty}^{k}\left(d_2\left[\mu-\tfrac{1}{2}l\right]-\min\left[d_1\left[\mu+\tfrac{1}{2}l\right],d_2\left[\mu-\tfrac{1}{2}l\right]\right]\right)d\mu \\ &+\int_{\mu=k}^{\infty}\left(d_1\left[\mu+\tfrac{1}{2}l\right]-\min\left[d_1\left[\mu+\tfrac{1}{2}l\right],d_2\left[\mu-\tfrac{1}{2}l\right]\right]\right)d\mu \end{aligned} \tag{16}$$

The evaluation of the integrals depends on the mid-strike domain $\mathcal{K}$ defined by:

$$(k\in\mathcal{K})=\left(d_1\left[k+\tfrac{1}{2}l\right]<d_2\left[k-\tfrac{1}{2}l\right]\right) \tag{17}$$

Using the mid-strike $k = \infty$, the optimal lower bound is evaluated as:

$$\mathbb{E}\left[(s-l)^+\right] \ge \int_{\mu\in\mathcal{K}}\left(d_2\left[\mu-\tfrac{1}{2}l\right]-d_1\left[\mu+\tfrac{1}{2}l\right]\right)d\mu \tag{18}$$

The mid-strike domain $\mathcal{K}$ is potentially unbounded in both directions and may be empty or consist of multiple disjoint intervals, $\mathcal{K} = \cup_{i=1}^{n}(k_-^i, k_+^i)$. The optimal lower bound is then:

$$\begin{aligned} &\mathbb{E}\left[(s-l)^+\right] \\ &\ge \sum_{i=1}^{n}\left(\left(c_1\left[k_-^i+\tfrac{1}{2}l\right]-c_2\left[k_-^i-\tfrac{1}{2}l\right]\right)-\left(c_1\left[k_+^i+\tfrac{1}{2}l\right]-c_2\left[k_+^i-\tfrac{1}{2}l\right]\right)\right) \end{aligned} \tag{19}$$

where the cases $k_-^1 = -\infty$ and $k_+^n = \infty$ are included using the well-defined limits $c_1[-\infty + ½l] - c_2[-\infty - ½l] = f_1 - f_2 - l$ and $c_1[\infty + ½l] - c_2[\infty - ½l] = 0$. From the defining relationship, the mid-strikes k_-^i locate the local maxima and the mid-strikes k_+^i locate the local minima of the lower hedge price:

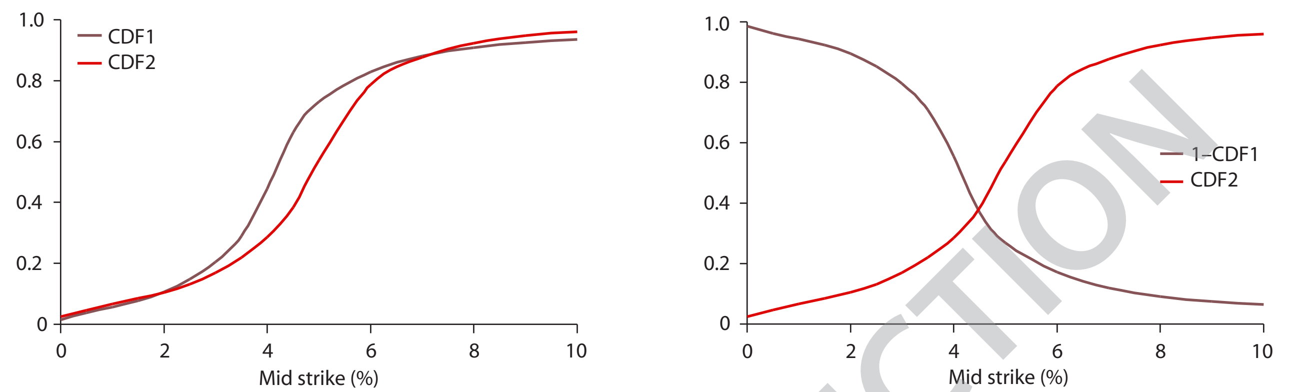
$$\frac{\partial}{\partial k}\left(c_1\left[k+\tfrac{1}{2}l\right]-c_2\left[k-\tfrac{1}{2}l\right]\right)\bigg|_{k=k_\pm^i}=0 \tag{20}$$

so that the optimal lower bound for the spread option price is given by the sum of the local maximum upper hedge prices minus the sum of the local minimum upper hedge prices.

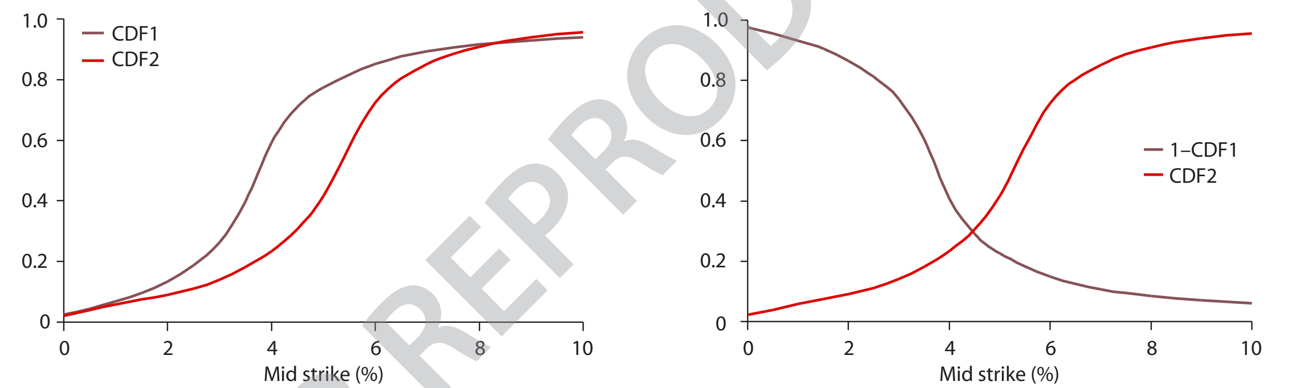
The optimal lower and upper bounds are attainable, using the maximum and minimum copulas respectively, and in general the spread option price generated by the copula C becomes:

$$\begin{aligned} &\mathbb{E}\left[(s-l)^+\right] \\ &= \sum_{i=1}^{n}\left(\left(c_1\left[k_-^i+\tfrac{1}{2}l\right]-c_2\left[k_-^i-\tfrac{1}{2}l\right]\right)-\left(c_1\left[k_+^i+\tfrac{1}{2}l\right]-c_2\left[k_+^i-\tfrac{1}{2}l\right]\right)\right) \\ &+\int_{\mu\in\mathcal{K}}\left(d_1\left[\mu+\tfrac{1}{2}l\right]-C\left[d_1\left[\mu+\tfrac{1}{2}l\right],d_2\left[\mu-\tfrac{1}{2}l\right]\right]\right)d\mu \\ &+\int_{\mu\notin\mathcal{K}}\left(d_2\left[\mu-\tfrac{1}{2}l\right]-C\left[d_1\left[\mu+\tfrac{1}{2}l\right],d_2\left[\mu-\tfrac{1}{2}l\right]\right]\right)d\mu \end{aligned} \tag{21}$$

5 CMS rate CDFs with spread strike 0%



6 CMS rate CDFs with spread strike 0.75%



$$\begin{aligned}\mathbb{E}\left[(s-l)^+\right] &= c_1\left[\bar{k}+\tfrac{1}{2}l\right]+p_2\left[\bar{k}-\tfrac{1}{2}l\right]\\ &\quad -\int_{\mu=-\infty}^{\bar{k}} C\left[d_1\left[\mu+\tfrac{1}{2}l\right],d_2\left[\mu-\tfrac{1}{2}l\right]\right]d\mu\\ &\quad -\int_{\mu=\bar{k}}^{\infty}\left(1-d_1\left[\mu+\tfrac{1}{2}l\right]-d_2\left[\mu-\tfrac{1}{2}l\right]\right.\\ &\qquad \left.+C\left[d_1\left[\mu+\tfrac{1}{2}l\right],d_2\left[\mu-\tfrac{1}{2}l\right]\right]\right)d\mu \end{aligned} \tag{22}$$

where the supplementary integrals are always positive. These expressions are useful for implementations, as they minimise the contributions from potentially inaccurate quadratures. The first expression is used when the correlation between the rates is positive, as in this case the spread option price is closest to the lower bound. Similarly, the second expression is used when the correlation between the rates is negative, as in this case the spread option price is closest to the upper bound.

The optimal bounds for the spread option price depend on the intersections of the marginal rate CDFs. Identifying these intersections enables the construction of an optimal portfolio of rate options that maximises the benefit from the triangle arbitrage, when it exists. Figures 5–7 show the CMS rate digital prices $CDF1 = d_1[k + ½l]$ and $CDF2 = d_2[k - ½l]$ for the spread strikes $l = 0\%$, 0.75% and 1.5% as functions of the mid-strike k.

For spread strike $l = 0\%$, the mid-strike domain is $\mathcal{K} = (-\infty, k_+^1) \cup (k_-^2, \infty)$ with $k_+^1 \approx 1.9\%$ and $k_-^2 \approx 7.2\%$, and the optimal mid-strike is $\bar{k} \approx 4.5\%$. The optimal bounds for the CMS spread option become:

$$\begin{aligned}\mathbb{E}\left[(s-l)^+\right] &\geq \left(c_1\left[k_-^2+\tfrac{1}{2}l\right]-c_2\left[k_-^2-\tfrac{1}{2}l\right]\right)\\ &\quad +\left(p_2\left[k_+^1-\tfrac{1}{2}l\right]-p_1\left[k_+^1+\tfrac{1}{2}l\right]\right)\\ \mathbb{E}\left[(s-l)^+\right] &\leq c_1\left[\bar{k}+\tfrac{1}{2}l\right]+p_2\left[\bar{k}-\tfrac{1}{2}l\right]\end{aligned} \tag{23}$$

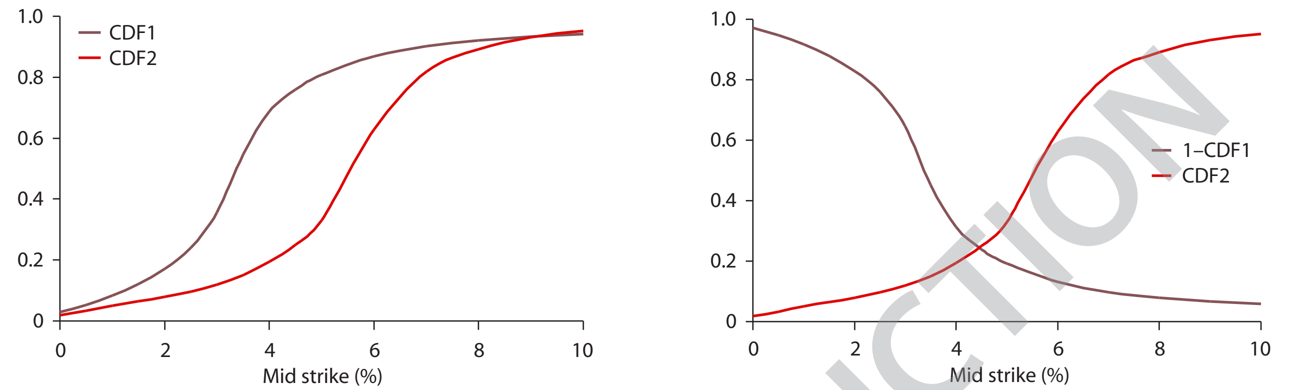
For spread strike $l = 0.75\%$, the mid-strike domain is $\mathcal{K} = (k_-^1, \infty)$ with $k_-^1 \approx 8.4\%$, and the optimal mid-strike is $\bar{k} \approx 4.6\%$. The optimal bounds for the CMS spread option become:

$$\begin{aligned}\mathbb{E}\left[(s-l)^+\right] &\geq c_1\left[k_-^1+\tfrac{1}{2}l\right]-c_2\left[k_-^1-\tfrac{1}{2}l\right]\\ \mathbb{E}\left[(s-l)^+\right] &\leq c_1\left[\bar{k}+\tfrac{1}{2}l\right]+p_2\left[\bar{k}-\tfrac{1}{2}l\right]\end{aligned} \tag{24}$$

For spread strike $l = 1.5\%$, the mid-strike domain is $\mathcal{K} = (k_-^1, \infty)$ with $k_-^1 \approx 9.2\%$, and the optimal mid-strike is $\bar{k} \approx 4.6\%$. The optimal bounds for the CMS spread option become:

$$\begin{aligned}\mathbb{E}\left[(s-l)^+\right] &\geq c_1\left[k_-^1+\tfrac{1}{2}l\right]-c_2\left[k_-^1-\tfrac{1}{2}l\right]\\ \mathbb{E}\left[(s-l)^+\right] &\leq c_1\left[\bar{k}+\tfrac{1}{2}l\right]+p_2\left[\bar{k}-\tfrac{1}{2}l\right]\end{aligned} \tag{25}$$

7 CMS rate CDFs with spread strike 1.5%



The FX triangle inequality

The triangle inequality is valid for the spread options between any three variables, and it is natural to consider whether there are other financial applications of this relationship. One such situation occurs in the consistency constraints between the smiles of forex options for three currencies. Consider the forex indexes x_1 and x_2, defined as the values of the currencies 1 and 2 in the domestic currency. The (undiscounted) prices for the forex options in the domestic currency are then:

$$\begin{aligned} f_1 &= \mathbb{E}[x_1] & f_2 &= \mathbb{E}[x_2] \\ c_1[k] &= \mathbb{E}\left[(x_1 - k)^+\right] & c_2[k] &= \mathbb{E}\left[(x_2 - k)^+\right] \\ p_1[k] &= \mathbb{E}\left[(k - x_1)^+\right] & p_2[k] &= \mathbb{E}\left[(k - x_2)^+\right] \\ u_1[k] &= \mathbb{P}[x_1 > k] & u_2[k] &= \mathbb{P}[x_2 > k] \\ d_1[k] &= \mathbb{P}[x_1 < k] & d_2[k] &= \mathbb{P}[x_2 < k] \end{aligned} \tag{26}$$

where the expectation is in the domestic pricing measure. The (undiscounted) price in currency 2 of the option to exchange one unit of currency 1 for l units of currency 2 is expressed as a spread option $\mathbb{E}[(x_1 - lx_2)^+]/f_2$ when evaluated using the domestic pricing measure. This price is constrained by:

$$\begin{aligned} &\sum_{i=1}^{n}\left(\left(c_1\left[k_-^i\right] - c_2\left[\frac{k_-^i}{l}\right]\right) - \left(c_1\left[k_+^i\right] - c_2\left[\frac{k_+^i}{l}\right]\right)\right) \\ &\le \mathbb{E}\left[(x_1 - lx_2)^+\right] \le c_1\left[\bar{k}\right] + p_2\left[\frac{\bar{k}}{l}\right] \end{aligned} \tag{27}$$

In this expression, $\mathcal{K} = \cup_{i=1}^{n}(k_-^i, k_+^i)$ is the mid-strike domain defined by:

$$(k \in \mathcal{K}) = \left(d_1[k] < d_2\left[\frac{k}{l}\right]\right) \tag{28}$$

and $\bar{k}$ is the optimal mid-strike that satisfies:

$$d_1\left[\bar{k}\right] + d_2\left[\frac{\bar{k}}{l}\right] = 1 \tag{29}$$

which is unique provided that the CDFs are strictly increasing. The two-sided inequality must be satisfied by the three forex smiles in order to avoid arbitrage. As with the CMS case, the forex arbitrage is identified by looking at the intersections of the CDFs of the two forex indexes, and is exploited by constructing a simple portfolio of forex options.

Conclusion

A consistent model must return a positive valuation for the CMS triangle, and so long as the arbitrage persists this model will necessarily be off-market for CMS spread options. A conservative model simply values the CMS triangle using the maximum copula. While this retains the proceeds of the arbitrage, it falls short of the true gain from the CMS triangle, which is only realised when the trade matures or the position is closed out. The research effort required to improve the CMS spread model should yield further rewards, as the arbitrage unwinds and CMS spread options find a new market level.

The CMS triangle arbitrage was a consequence of a dislocation between the CMS rate and spread option markets. Since the end of 2009, the unwinding of the CMS triangle arbitrage has led to a repricing of CMS spread options, which may be the catalyst for a paradigm shift in the modelling of CMS spreads. ■

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