BARUCH, MFE

MTH 9878 Assignment Five *

He, Jing Qiu
Huang, Qing
Zhou, Minzhe
Zhou, ShengQuan

May 15, 2016

* We agree for our work to be posted on the forum.

1 HERMITE POLYNOMIALS: ORTHOGONALITY

Use induction to prove that for $n \neq m$,

$$\int_{-\infty}^{+\infty} H_n(x) H_m(x) e^{-x^2}\, dx = 0.$$

Proof: Induction is not really needed in this case. Without loss of generality, assume $n < m$; if $n > m$, we can simply switch the labels $n \leftrightarrow m$.

$$\begin{aligned}
\int_{-\infty}^{+\infty} H_n(x) H_m(x) e^{-x^2}\, dx &= \int_{-\infty}^{+\infty} H_n(x)(-)^m e^{x^2} \frac{d^m e^{-x^2}}{dx^m} e^{-x^2}\, dx \\
&= \int_{-\infty}^{+\infty} H_n(x)(-)^m \frac{d^m e^{-x^2}}{dx^m}\, dx \\
&= \int_{-\infty}^{+\infty} H_n(x)(-)^m d\left[\frac{d^{m-1} e^{-x^2}}{dx^{m-1}}\right] \\
\text{(Integration by parts)} &= \underbrace{\cancelto{0}{H_n(x)(-)^m \frac{d^{m-1} e^{-x^2}}{dx^{m-1}}\Big|_{-\infty}^{+\infty}}}_{\text{vanishing boundary conditions}} + \int_{-\infty}^{+\infty} dH_n(x)(-)^{m-1} \frac{d^{m-1} e^{-x^2}}{dx^{m-1}} \\
&= \int_{-\infty}^{+\infty} \frac{dH_n(x)}{dx}(-)^{m-1} \frac{d^{m-1} e^{-x^2}}{dx^{m-1}}\, dx \\
&= \cdots \text{(repeat integration by parts } n \text{ times)} \\
&= \int_{-\infty}^{+\infty} \underbrace{\frac{d^n H_n(x)}{dx^n}}_{=\text{constant because } H_n \text{ is order } n \text{ polynomial}} (-)^{m-n} \frac{d^{m-n} e^{-x^2}}{dx^{m-n}}\, dx \\
&= \frac{d^n H_n(x)}{dx^n} \int_{-\infty}^{+\infty} (-)^{m-n} \frac{d^{m-n} e^{-x^2}}{dx^{m-n}}\, dx \qquad (1.1) \\
\text{(Integration by parts on more time)} &= \frac{d^n H_n(x)}{dx^n} (-)^{m-n-1} \frac{d^{m-n-1} e^{-x^2}}{dx^{m-n-1}}\Big|_{-\infty}^{+\infty} \\
&= 0.
\end{aligned}$$

The last equality is true because of the assumption $m > n$ so that $m - n - 1 \geq 0$ and the vanishing boundary conditions for e^{-x^2} at $\pm\infty$ holds.

2 HERMITE POLYNOMIALS: NORMALIZATION

Prove that

$$\int_{-\infty}^{+\infty} h_n(x)h_m(x)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx = \delta_{mn},$$

where

$$h_n(x) = \frac{1}{\sqrt{2^n n!}}H_n\left(\frac{x}{\sqrt{2}}\right).$$

Proof: The orthogoanlity is implied by the previous problem, as for $n \neq m$,

$$\begin{aligned}
\int_{-\infty}^{+\infty} h_n(x)h_m(x)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx &\propto \int_{-\infty}^{+\infty} H_n\left(\frac{x}{\sqrt{2}}\right)H_m\left(\frac{x}{\sqrt{2}}\right)e^{-\frac{x^2}{2}}dx \\
(y \triangleq x/\sqrt{2}) &\propto \int_{-\infty}^{+\infty} H_n(y)H_m(y)e^{-y^2}dy \\
&= 0.
\end{aligned}$$

For the normalization part, we repeat the procedure done in the previous problem for $m = n$

$$\begin{aligned}
\int_{-\infty}^{+\infty} h_n(x)h_m(x)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx &= \frac{1}{2^n n!}\int_{-\infty}^{+\infty} H_n\left(\frac{x}{\sqrt{2}}\right)H_n\left(\frac{x}{\sqrt{2}}\right)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \\
&= \frac{1}{\sqrt{2\pi}2^n n!}\int_{-\infty}^{+\infty} H_n(y)H_n(y)e^{-y^2}dy \\
\text{(Eq.1.1)} \quad &= \frac{1}{2^n n!}\frac{d^n H_n(y)}{dy^n}\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-x^2}dx}_{=1} \\
&= \frac{1}{2^n n!}\frac{d^n H_n(y)}{dy^n} \\
\text{(use } H_n' = 2nH_{n-1}) \quad &= \frac{1}{2^n n!}\frac{d^n H_n(y)}{dy^n} \\
&= \frac{1}{2^n n!}\frac{d^{n-1} H_n'(y)}{dy^{n-1}} \\
&= \frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1} H_{n-1}(y)}{dy^{n-1}} \\
&= \cdots \\
\text{(repeat } k \text{ times)} \quad &= \frac{1}{2^{n-k}(n-k)!}\frac{d^{n-k} H_{n-k}(y)}{dy^{n-k}} \\
&= \cdots \\
&= H_0(y) \\
&= 1.
\end{aligned}$$

In summary,

$$\int_{-\infty}^{+\infty} h_n(x)h_m(x)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx = \delta_{mn}.$$

3 Hermite Polynomials: Conditional Expectation

Prove that

$$\mathbb{E}\left[h_n\left(\sqrt{\alpha}x+\sqrt{1-\alpha}Y\right)\right] \triangleq \int_{-\infty}^{+\infty} h_n\left(\sqrt{\alpha}x+\sqrt{1-\alpha}y\right)d\mu(y) = \alpha^{\frac{n}{2}}h_n(x).$$

Note: It's mentioned in class that this is not a homework problem.
Proof: Use the addition theorem:

$$\begin{aligned}
\int_{-\infty}^{+\infty} h_n\left(\sqrt{\alpha}x+\sqrt{1-\alpha}y\right)d\mu(y) &= \sum_{k=0}^{n}\sqrt{\binom{n}{k}}\alpha^{\frac{k}{2}}(1-\alpha)^{\frac{n-k}{2}}h_k(x)\int_{-\infty}^{+\infty} h_{n-k}(y)d\mu(y) \\
\text{(orthonormality)} \quad &= \sum_{k=0}^{n}\sqrt{\binom{n}{k}}\alpha^{\frac{k}{2}}(1-\alpha)^{\frac{n-k}{2}}h_k(x)\delta_{nk} \\
&= \alpha^{\frac{n}{2}}h_n(x).
\end{aligned}$$