Interest Rate Models

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Outline

1. Options on LIBOR based instruments
2. Valuations of LIBOR optionsx
3. Local volatility models

Options on LIBOR based instruments

- We have learned how to construct the current snapshot of the interest rate market. This will serve as the initial value for the stochastic process that models the curve dynamics.
- The next step is to construct the volatility cube. The presence of the third dimension is a consequence of the term structure. Notice that this was not the case in the equity market where one can use the volatility surface.
- The volatility cube is built out of implied volatilities of observed options from the market.

Caps and floors

- Caps and floors are baskets of European calls (caplets) and puts (floorlets) on LIBOR forward rates.
- Consider the 10 year spot starting cap with strike 2.5%.
- It consists of 39 caplets. The expiration for each of them is 3 months.
- The payoff of each caplet is

$$\text{Payoff} = \max\{\text{current LIBOR} - 2.5\%,\ 0\} \cdot \delta.$$

- δ is the day count fraction (act/360, modified next business day convention)
- The very first period is excluded because that LIBOR is known.

Caps and floors

- Forward starting instruments are also traded.
- 1 year $\times$5 years ("1 by 5") cap struck at 2.5% consists of 16 caplets the first of which matures one year from now.
- The last caplet matures 5 years and 9 months from now.
- The first period is included in the contract, since the corresponding LIBOR fixing is not known today. The maturity of $m \times n$ cap is n years.
- Floors are analogously defined. The payoff of each floorlet is

$$\text{Payoff} = \max\{\text{strike} - \text{current LIBOR},\ 0\} \cdot \delta.$$

Eurodollar options

- *Eurodollar options* are standardized contracts traded on Merc. They are American calls and puts on Eurodollar futures with short expirations.
- The options on the first eight quarterly Eurodollar futures and the options on the first two serial futures are listed.
- The expirations coincide with the maturity dates of the underlying futures.

strike	call	put
98.875	0.4450	0.0050
99.000	0.3225	0.0075
99.125	0.2075	0.0175
99.250	0.1125	0.0475
99.375	0.0425	0.1025
99.500	0.0175	0.2025

Expiration: Jun 2016
Price of the underlying: 99.325
Quotes from February 2016

Eurodollar options

- *Midcurve options* are American calls and puts on longer dated Eurodollar futures with expirations between three months and one year.
- The expirations do not coincide with the expirations of the futures, which mature in one, two, or four years.

Swaptions

- European swaptions are European calls and puts on interest rate swaps. They are called *receivers* and *payers*, respectively.
- A holder of the payer swaption has the right to pay the fixed coupon on a swap. A holder of the receiver swaption has the right to receive the fixed on a swap.
- Swaptions are traded over the counter.
- A 2.5% 1Y $\rightarrow$ 5Y ("1 into 5") receiver swaption gives the right to the holder to receive 2.5% on a 5 year swap starting in 1 year. The holder may exercise the swaption on the 1 year anniversary of today in which case they enter into a receiver swap that starts two days after.
- The total maturity of $m \rightarrow n$ swaption is $m + n$ years.

Black's model

- Black's model is an adaptation of the standard Black-Scholes model adapted to forward underlying assets.
- Assume that the forward rate (such as LIBOR forward or a forward swap rate) $F(t)$ is a driftless geometric Brownian motion
$$dF(t) = \sigma F(t)\, dW(t).$$
- We are assuming that we have chosen a numeraire $\mathcal{N}$ such that F is tradable asset in units of $\mathcal{N}$. The process $F(t)$ is a martingale with respect to the underlying probability distribution $\mathbb{Q}$.

Black's model

- The process F can be expressed as

$$F(t) = F_0 e^{-\frac{1}{2}\sigma^2 t + \sigma W(t)}.$$

- Assume that T is fixed and that $\mathcal{N}$ is normalized to have $\mathcal{N}(T) = 1$.
- The price of European call with strike K and expiration T is

$$\begin{aligned} P^{\text{call}} &= P^{\text{call}}(T, K, F_0, \sigma) = \mathcal{N}(0)\mathbb{E}^{\mathbb{Q}}\left[\max\left\{F(T) - K, 0\right\}\right] \\ &= \frac{\mathcal{N}(0)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \max\left\{F_0 e^{-\frac{1}{2}\sigma^2 + \sigma\sqrt{T}z} - K, 0\right\} e^{-\frac{1}{2}z^2}\, dz \end{aligned}$$

Black's model

- Therefore

$$\begin{aligned} P^{\text{call}} &= \mathcal{N}(0)\left[F_0 N(d_+) - KN(d_-)\right] \\ &\equiv \mathcal{N}(0) B_{\text{call}}(T, K, F_0, \sigma), \end{aligned}$$

Where $d_\pm = \dfrac{\ln \frac{F_0}{K} \pm \frac{\sigma^2}{2} T}{\sigma\sqrt{T}}$ and $N(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt.$

- Similarly,

$$\begin{aligned} P^{\text{put}} &= \mathcal{N}(0)\left[KN(-d_-) - F_0 N(-d_+)\right] \\ &\equiv \mathcal{N}(0) B_{\text{put}}(T, K, F_0, \sigma). \end{aligned}$$

Valuation of caps and floors

- A cap is a basket of LIBOR forward rates.
- The OIS forward rate for the period $[S, T]$ has the value

$$\begin{aligned} F(t, S, T) &= \frac{1}{\delta}\left(\frac{P(t, t, S)}{P(t, t, T)} - 1\right) \\ &= \frac{1}{\delta} \cdot \frac{P(t, t, S) - P(t, t, T)}{P(t, t, T)}. \end{aligned}$$

- Notice that $F(t, S, T)$ is a tradable asset if the numeraire is the zero coupon bond with maturity T: Buy $\frac{1}{\delta}$ face value zero coupon bond with maturity S and sell $\frac{1}{\delta}$ zero coupon bonds with maturity T.
- Therefore OIS forward rate is a martingale corresponding to the T-forward measure denoted by $\mathbb{Q}_T$.

Valuation of caps and floors

- The LIBOR forward spanning the same accrual period satisfies

$$L(t, S, T) = F(t, S, T) + B(t, S, T).$$

- We will assume in these lectures that the LIBOR/OIS spread B is deterministic.
- There are no liquidly traded assets based on this spread, hence we cannot calibrate the models with stochastic spreads.

$$L(t, S, T) = \frac{1}{\delta} \cdot \frac{P(t, t, S) - P(t, t, T) + \delta B(t, S, T) P(t, T, T)}{P(t, t, T)}.$$

- Thus, LIBOR forward rate is also a martingale with respect to the T-forward measure $\mathbb{Q}_T$.

Valuation of caps and floors

- The price of the caplet is the price of the call option on $L(S,T)$ and it is given by

$$P^{\text{caplet}}(T,K,L_0,\sigma) \;=\; \delta P_0(0,T) B_{\text{call}}(S,K,L_0,\sigma).$$

  $L_0 = L(0,S,T)$ is the today's value of the forward.
- The cap is a basket of caplets, hence its value is

$$P^{\text{cap}} = \sum_{j=1}^{n} \delta_j P_0(0,T_j) B_{\text{call}}(T_{j-1},K,L_0(T_{j-1},T_j),\sigma).$$

- δ_j is the day count factor that corresponds to the time interval $[T_{j-1},T_j]$. $L_0(T_{j-1},T_j)$ is the LIBOR rate for that period.
- The date T_{j-1} is has to be adjusted to accurately reflect the expiration of the option (2 business days before the start of the accrual period).

Valuation of caps and floors

- Similarly, the value of the floor is

$$P^{\text{floor}} = \sum_{j=1}^{n} \delta_j P_0(0, T_j) B_{\text{put}}(T_{j-1}, K, L_0(T_{j-1}, T_j), \sigma).$$

- At the money cap (ATM) is the one for which the put and call have the same value.
- Put-call parity for the caps and floors is

$$\begin{aligned} P^{\text{floor}} - P^{\text{cap}} &= \sum_{j=1}^{n} \delta_j P_0(0, T_j) \mathbb{E}^{\mathbb{Q}_j} \left[(K - L_j)^+ - (L_j - K)^- \right] \\ &= \sum_{j=1}^{n} \delta_j P_0(0, T_j) \mathbb{E}^{\mathbb{Q}_j} \left[K - L_j \right]. \end{aligned}$$

Valuation of caps and floors

- Since L_j is a martingale with respect to the T_j-forward measure we obtain $\mathbb{E}^{\mathbb{Q}_j}[L_j] = L_0(T_{j-1}, T_j)$. Thus

$$P^{\text{floor}} - P^{\text{cap}} = K \sum_{j=1}^{n} \delta_j P_0(0, T_j) - \sum_{j=1}^{n} \delta_j P_0(0, T_j) L_0(T_{j-1}, T_j).$$

- The right-hand side of the last equation is the present value of the swap receiving K on the quarterly, act/360 basis.
- Thus the ATM rate is the break even rate of the above swap.

Valuation of swaptions

- Consider the swap that settles at T_0 and matures at T. The break-even forward rate is

$$S(t, T_0, T) \;=\; \frac{\sum_{j=1}^{n_f} \delta_j L_j P(t, T_{\text{val}}, T_j^f)}{A(t, T_{\text{val}}, T_0, T)}.$$

  $T_{\text{val}} \leq T_0$ is the valuation date of the swap. The value of the rate does not depend on the choice of the valuation date.

- Therefore

$$S(t, T_0, T) \;=\; \frac{\sum_{j=1}^{n_f} \delta_j F_j P(t, t, T_j^f) + \sum_{j=1}^{n_f} \delta_j B_j P(t, t, T_j^f)}{A(t, t, T_0, T)}.$$

  The first summation is telescopic since $\delta_j F_j P(t, t, T_j^f) = P(t, t, T_{j-1}^f) - P(t, t, T_j^f)$.

Valuation of swaptions

- We obtain

$$S(t, T_0, T) = \frac{P(t, t, T_0) - P(t, t, T) + \sum_{j=1}^{n_f} \delta_j B_j P(t, t, T_j^f)}{A(t, t, T_0, T)}.$$

- The forward annuity function $A(t, t, T_0, T)$ is the time t present value of an annuity that pays \$1 coupons on dates $T_1^c, \ldots, T_{n_c}^c$.
- Since we assumed that B_j are deterministic, we conclude that $S(t, T_0, T)$ is a tradable asset in units of $A(t, t, T_0, T)$.
- Therefore the forward swap rate is a martingale with respect to the EMM associated with $A(t, t, T_0, T)$. This EMM is called *swap measure*.

Valuation of swaptions

- Recall that the holder of the reciever swaption has the right to receive the fixed on a swap.
- Therefore the payoff of the receiver swaption is the same as the payoff of the put option derived on $S(t, T_0, T)$ as underlying. Hence the price can be calculated as

$$P^{\text{rec}} = A_0(T_0, T) B_{\text{put}}(T_0, K, S_0, \sigma).$$

- In an analogous way we obtain

$$P^{\text{pay}} = A_0(T_0, T) B_{\text{call}}(T_0, K, S_0, \sigma).$$

- Here $A_0(T_0, T) = A(0, 0, T_0, T)$ and S_0 is the today's value of the forward swap $S_0(T_0, T)$.

Valuation of swaptions

- Put-call parity for the swaptions is

$$
\begin{aligned}
P^{\text{rec}} - P^{\text{pay}} &= \text{PV of the swap paying } K \\
&\quad \text{on the semi-annual, 30/360 basis.}
\end{aligned}
$$

Beyond Black's model

- Black's model assumes that σ is independent of T, K, and F_0.
- This is not observed in the market.
- Implied volatilities exhibit:
  (i) *Term structure:* At the money volatility depends on the option expiration.
  (ii) *Smile (skew):* For a given expiration, the implied volatilities depend on the option strike.
- Not much progress is made in modeling the term structure. We will focus on modeling the volatility smile.

Local volatility models

- Option pricing is done using risk neutral probabilities (EMMs).
- One way to obtain them is to simplify the model and end up with something like Black's model.
- There is another way to obtain the risk-neutral probability without the assumption that the volatility is constant. This other method uses the market data for available liquid options.
- We start with the following observation:

$$\frac{d}{dK}(S-K)^+ = -\theta(S-K),$$

where $\theta(x) = 1_{x>0}$.

Local volatility models

- Now we have

$$\frac{d^2}{dK^2}(S-K)^+ \;=\; -\frac{d}{dK}\theta(S-K) = \delta(S-K).$$

- Therefore the price of a security with payoff V satisfies

$$\frac{d^2}{dK^2}V(0) \;=\; \int \delta(S-K)\, d\mathbb{Q}_T.$$

- The last integral can be easily expressed in terms of the pdf of $\mathbb{Q}_T$. Recall that $\mathbb{Q}_T$ is the forward swap rate. Let us denote by $g_T(S, S_0)$ its distribution function. Then we get

$$\frac{d^2}{dK^2}V(0) = g_T(K, S_0).$$

Local volatility models

- The terminal probability distribution can be obtained from the option prices.
- This will allow us to model underlying process to match the market implied volatilities.
- Local volatility models are specified in the form

  $$dF(t) = C(t, F(t))dW(t).$$

  The quantity $C(t, F)$ is called effective instantaneous volatility.
- Popular models are:
  (i) Normal model;
  (ii) Shifted lognormal model;
  (iii) CEV model.

Normal model

- In the normal model C is assumed to be constant and F satisfies

$$dF(t) = \sigma dW(t).$$

- σ is called the *normal volatility*.
- The solution is $F(t) = F_0 + \sigma W(t)$.
- A drawback of the model is that F may become negative.
- The prices of calls and puts are given by

$$\begin{aligned} P^{\text{call}}(T, K, F_0, \sigma) &= \mathcal{N}(0) B_{\text{n}}^{\text{call}}(T, K, F_0, \sigma), \\ P^{\text{put}}(T, K, F_0, \sigma) &= \mathcal{N}(0) B_{\text{n}}^{\text{put}}(T, K, F_0, \sigma). \end{aligned}$$

Normal model

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where

$$B_{\text{n}}^{\text{call}}(T,K,F_0,\sigma) = \sigma\sqrt{T}\left(d_+N(d_+) + N'(d_-)\right), \quad (1)$$

$$B_{\text{n}}^{\text{put}}(T,K,F_0,\sigma) = \sigma\sqrt{T}\left(d_-N(d_-) + N'(d_-)\right), \quad (2)$$

$$d_\pm = \pm\frac{F_0 - K}{\sigma\sqrt{T}}.$$

- Homework: Prove this!

Normal model

- In some cases implied volatilies are quoted in terms of normal model.
- Many traders think in terms of normal model as it seems to capture the dynamics better than the lognormal model.
- Combining normal and log normal model gives us shifted lognormal model.

Shifted lognormal model

- The dynamics is

$$dF(t) = (\sigma_0 + \sigma_1 F(t))\, dW(t).$$

- Let us denote by $B_{\mathrm{sln}}^{\mathrm{call}}$ and $B_{\mathrm{sln}}^{\mathrm{put}}$ the functions for which

$$\begin{aligned} P^{\mathrm{call}}(T, K, F_0, \sigma_0, \sigma_1) &= \mathcal{N}(0) B_{\mathrm{sln}}^{\mathrm{call}}\left(T, K, F_0, \sigma_0, \sigma_1\right), && (3) \\ P^{\mathrm{put}}(T, K, F_0, \sigma_0, \sigma_1) &= \mathcal{N}(0) B_{\mathrm{sln}}^{\mathrm{put}}\left(T, K, F_0, \sigma_0, \sigma_1\right). && (4) \end{aligned}$$

Shifted lognormal model

- Prove the following for the homework:

$$\begin{aligned} B_{\text{sln}}^{\text{call}}(T, K, F_0, \sigma_0, \sigma_1) &= \left(F_0 + \frac{\sigma_0}{\sigma_1}\right) N(d_+) \\ &\quad - \left(K + \frac{\sigma_0}{\sigma_1}\right) N(d_-) \qquad (5) \\ B_{\text{sln}}^{\text{put}}(T, K, F_0, \sigma_0, \sigma_1) &= -\left(F_0 + \frac{\sigma_0}{\sigma_1}\right) N(-d_+) \\ &\quad + \left(K + \frac{\sigma_0}{\sigma_1}\right) N(-d_-), \qquad (6) \end{aligned}$$

$$\text{where } d_\pm = \frac{\log \frac{\sigma_1 F_0 + \sigma_0}{\sigma_1 K + \sigma_0} \pm \frac{\sigma_1^2 T}{2}}{\sigma_1 \sqrt{T}}.$$

CEV model

- CEV stands for *Constant Elasticity of Variance*.
- The dynamics of the CEV model is

$$dF(t) = \sigma F^{\beta}(t)\, dW(t), \quad \text{for } \beta < 1.$$

- Note that β is allowed to be negative.
- In order for this to be well defined F must be prevented from being negative. This is achieved by placing a boundary condition.
  (i) Dirichlet (absorbing): $F|_0 = 0$. Solutions exist for all β.
  (ii) Neumann (reflecting): $F'|_0 = 0$. Solutions exist for $\frac{1}{2} \leq \beta < 1$.

CEV model

- If we denote by $B(t, f)$ the price of the derived security at time t under the condition that the price of the underlying is f, the requirement that B is a martingale can be read as

$$\frac{\partial}{\partial t} B(t, f) + \frac{1}{2}\sigma^2 f^{2\beta} \frac{\partial^2}{\partial f^2} B(t, f) = 0.$$

- Together with the appropriate boundary condition and the terminal condition $B(T, f) = (f - K)^+$ (in the case of calls) we obtain the backward Kolmogorov equation.

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$$\begin{aligned} P^{\text{call}}(T, K, F_0, \sigma) &= \mathcal{N}(0) B^{\text{call}}_{\text{CEV}}(T, K, F_0, \sigma), \qquad (7) \\ P^{\text{put}}(T, K, F_0, \sigma) &= \mathcal{N}(0) B^{\text{put}}_{\text{CEV}}(T, K, F_0, \sigma). \qquad (8) \end{aligned}$$

CEV model

- The functions $B_{\text{sln}}^{\text{call}}$ and $B_{\text{sln}}^{\text{put}}$ are the time zero values of the solutions to the former terminal value problems.
- In order to express them analytically we will use the cumulative function for the non-central χ^2 distribution:

$$\begin{aligned}
\chi^2(x; r, \lambda) &= \int_0^x \rho(y; r, \lambda)\, dy \\
\rho(x; r, \lambda) &= \frac{1}{2}\left(\frac{x}{\lambda}\right)^{\frac{r-2}{4}} e^{-\frac{x+\lambda}{2}} I_{\frac{r-2}{2}}\left(\sqrt{\lambda x}\right),
\end{aligned}$$

where $I_\mu(y)$ is a modified Bessel function of the first kind given by

$$I_\mu(y) = (y/2)^\mu \sum_{j=0}^{\infty} \frac{(y^2/4)^j}{j!\Gamma(\mu + j + 1)}.$$

CEV model

- Define $\nu = \frac{1}{2(1-\beta)}$. Then $\nu \geq \frac{1}{2}$.
- The formulas for the calls and puts in the CEV model under the Dirichlet's boundary condition are:

$$\begin{aligned} B_{\mathrm{CEV}}^{\mathrm{call}} &= F_0 \left(1 - \chi^2 \left(\frac{4\nu^2 K^{\frac{1}{\nu}}}{\sigma^2 T}; 2\nu + 2, \frac{4\nu^2 F_0^{\frac{1}{\nu}}}{\sigma^2 T} \right) \right) \\ &\quad - K \chi^2 \left(\frac{4\nu^2 F_0^{\frac{1}{\nu}}}{\sigma^2 T}; 2\nu, \frac{4\nu^2 K^{\frac{1}{\nu}}}{\sigma^2 T} \right) \end{aligned} \tag{9}$$

CEV model

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$$B^{\text{put}}_{\text{CEV}} = F_0\chi^2\left(\frac{4\nu^2K^{\frac{1}{\nu}}}{\sigma^2T};2\nu+2,\frac{4\nu^2F_0^{\frac{1}{\nu}}}{\sigma^2T}\right) -K\left(1-\chi^2\left(\frac{4\nu^2F_0^{\frac{1}{\nu}}}{\sigma^2T};2\nu,\frac{4\nu^2K^{\frac{1}{\nu}}}{\sigma^2T}\right)\right) \quad (10)$$

- From these formulas one can obtain the terminal probability density

$$g_T(F,F_0) = \frac{4\nu F_0F^{\frac{1}{\nu}-2}}{\sigma^2T}\rho\left(\frac{4\nu^2F^{\frac{1}{\nu}}}{\sigma^2T};2\nu+2,\frac{4\nu^2F_0^{\frac{1}{\nu}}}{\sigma^2T}\right).$$

CEV model

- There is a nonzero probability of absorption at 0 since the density function integrates to something less than 1.

$$\int_0^{+\infty} g_T(F, F_0)\, dF = 1 - \frac{1}{\Gamma(\nu)} \Gamma\left(\nu, \frac{2\nu^2 F_0^{\frac{1}{\nu}}}{\sigma^2 T}\right),$$

$$\text{where } \Gamma(\nu, x) = \int_x^{+\infty} t^{\nu-1} e^{-t}\, dt.$$

The last function is called *complementary incomplete gamma function*.

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