The Merton-Black-Scholes equation

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Abstract

We present no frills derivations of the Merton-Black-Scholes equation for dividend paying assets. Terms like *discount factor*, *risk-neutral* and *numéraire* are consciously avoided.

Contents

1 Continuous dividends

Let S_t be the time t price of a stock with volatility σ, mean return μ and dividend rate q.

$$dS_t = S_t(\mu dt + \sigma dW_t - q dt)$$

Let B_t be the price of a zero coupon bond maturing at time T:

$$\begin{aligned} dB_t &= rB_t dt \\ B_T &= 1 \end{aligned}$$

These are the simplest possible dynamics for S_t and B_t and the ones used the Merton-Black-Scholes formula.

A derivative is a contract that pays $f(S_T)$ for some fixed function f, usually $f(s) = (s-K)^+$ or $(K-s)^+$ (a call and a put option respectively) at time T.

Let V_t be the price of the derivative, so that $V_T = f(S_T)$[1]. Our goal is to find a portfolio of the stock and the bond whose value X_t matches V_t for all $t \in [0,T]$. Such a portfolio is called *replicating*.

Let Δ_t be the number of stocks, E_t is the number of bonds in the portfolio. Then we have the expression for X_t:

$$X_t = \Delta_t S_t + E_t B_t \tag{1}$$

[1]Note: it is not the case that $V_t = f(S_t)$, for $t \neq T$.

The portfolio must also be “self-financing” that is

$$dX_t = \Delta_t dS_t + E_t dB_t \tag{2}$$

This loosely means that the change in portfolio value is not due to capital in/outflows, but due to variation of the stock/bond price. Equation 2 be obtained as a limit of discrete time self-financing strategies, where the notion is more intuitive (see chapter 6 in [Bjö04]).

Unfortunately, equation 2 isn't completely adequate to describe our setting as there some capital inflows, namely the stock pays dividends. So we aren't just shifting our money from stock to bond and back, we actually get a continuous stream of dividend payments.

The correct equation to account for these inflows is:

$$dX_t = \Delta_t dS_t + E_t dB_t + \Delta_t q S_t dt \tag{3}$$

This is quite intuitive, once simplified:

$$\begin{align} dX_t &= \Delta_t dS_t + E_t dB_t + \Delta_t q S_t dt \tag{4}\\ &= \Delta_t S_t(\mu dt + \sigma dW_t - q dt) + E_t dB_t + \Delta_t q S_t dt \tag{5}\\ &= \Delta_t S_t(\mu dt + \sigma dW_t) + E_t dB_t \tag{6} \end{align}$$

Noting that the q terms cancel, we see that the dividend payments don't change the portfolio value. The stock loses $qS_t dt$ of its value per share, but we add that back to the portfolio in equation 3.[2]

Equations 1 and 6 above are readily combined into one, which we simplify and collect the dt and dW_t terms:

$$\begin{align} dX_t &= \Delta_t S_t(\mu dt + \sigma dW_t) + \frac{X_t - \Delta_t S_t}{B_t} dB_t \tag{7}\\ &= \Delta_t S_t(\mu dt + \sigma dW_t) + (X_t - \Delta_t S_t) r dt \tag{8}\\ &= \left[(\mu - r)\Delta_t S_t + r X_t\right] dt + \Delta_t S_t \sigma dW_t \tag{9} \end{align}$$

Our goal is to replicate the derivative, that is we want $X_t = V_t$. To do this, we assume that the price of the derivative can be expressed as a function of time and the price of the underlying stock:

$$V_t = c(t, S_t) \tag{10}$$

where c is a sufficiently regular two variable function. This is kind of an ansatz.

Assuming that (Δ_t, E_t) is indeed a replicating portfolio we have:

$$\begin{aligned} dX_t &= dV_t \\ X_T &= V_T \end{aligned}$$

We use Ito's lemma to compute dV_t and equate it with dX_t as found in equation 9:

$$dV_t = dc(t, S_t) \tag{11}$$

[2] This argument was taken from [Shr04].

$$= c_t(t,S_t)dt + c_s(t,S_t)dS_t + \frac{1}{2}c_{ss}(t,S_t)(dS_t)^2 \tag{12}$$

$$= c_t(t,S_t)dt + c_s(t,S_t)S_t(\mu dt + \sigma dW_t - q dt) + \frac{1}{2}c_{ss}(t,S_t)\sigma^2 S_t^2 dt \tag{13}$$

$$= \left[c_t(t,S_t) + (\mu - q)S_t c_s(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{ss}(t,S_t)\right] dt + \sigma S_t c_s(t,S_t)dW_t \tag{14}$$

To equate (9) and (14) it is sufficient to equate their dt and dW_t terms. Assuming the volatility and stock price are non-zero, we get from the dW_t terms:

$$\Delta_t = c_s(t,S_t) \tag{15}$$

This equation is know as the Δ-hedging rule. Equating the dt terms we get:

$$(\mu - r)\Delta_t S_t + rX_t = c_t(t,S_t) + (\mu - q)S_t c_s(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{ss}(t,S_t)$$

Swapping $c(t,S_t) = V_t$ for X_t and $c_s(t,S_t)$ for Δ_t we obtain:

$$(\mu - r)S_t c_s(t,S_t) + rc(t,S_t) = c_t(t,S_t) + (\mu - q)S_t c_s(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{ss}(t,S_t)$$

For c to satisfy this, it's sufficient that it satisfy the PDE:

$$(\mu - r)sc_s(t,s) + rc(t,s) = c_t(t,s) + (\mu - q)sc_s(t,s) + \frac{1}{2}\sigma^2 s^2 c_{ss}(t,s)$$

which simplifies to the familiar Merton-Black-Scholes PDE:

$$c_t(t,s) + (r - q)sc_s(t,s) + \frac{1}{2}\sigma^2 s^2 c_{ss}(t,s) - rc(t,s) = 0 \tag{16}$$

Having found a solution to (16) will ensure that $dX_t = dc(t,S_t) = dV_t$. This is not however sufficient for $X_t = c(t,S_t) = V_t$, for that we need to make sure they agree at least at one t:

$$X_T = c(T,S_T) = V_T = f(S_T)$$

The second and third equalities tell us that $c(T,s) = f(s)$ is a good choice for a terminal condition for equation 16.

Additional boundary data is often prescribed, however it is not mathematically necessary for a unique solution:

$$\begin{aligned} &c(t,0) = 0 \\ &c(t,S) \sim S \quad \text{as } S \to \infty \end{aligned}$$

For a detailed guide to solving the Merton-Black-Scholes PDE (with $q = 0$) see chapter 5 of [WHD95].

2 Discrete dividends

Suppose that the stock pays a single dividend, $q\%$ of the stock price at time $t = t_*$. The evolution of the stock price is modelled with geometric Brownian motion with a single down jump.

$$H_t = \begin{cases} 0 & \text{when } 0 \le t < t_* \\ 1 & \text{when } t_* \le t \le T \end{cases}$$

$$dS_t = S_{(t^-)}(\mu dt + \sigma dW_t - q dH_t)$$

To replicate the payoff of the derivative, we set up a portfolio that is self-financing except for the inflow of the dividend payment.

$$X_t = \Delta_t S_t + E_t B_t$$

$$\begin{aligned} dX_t &= \Delta_{(t^-)} dS_t + E_{(t^-)} dB_t + q\Delta_{(t^-)} S_{(t^-)} dH_t \\ &= \Delta_{(t^-)} S_{(t^-)}(\mu dt + \sigma dW_t - q dH_t) + E_{(t^-)} r B_t dt + q\Delta_{(t^-)} S_{(t^-)} dH_t \\ &= \Delta_{(t^-)} S_{(t^-)}(\mu dt + \sigma dW_t) + E_{(t^-)} r B_t dt \\ &= \Delta_t S_t(\mu dt + \sigma dW_t) + E_t r B_t dt \\ &= \Delta_t S_t(\mu dt + dW_t) + (X - \Delta_t S_t) r dt \\ &= [(\mu - r)\Delta_t S_t + r X_t] dt + \Delta_t S_t \sigma dW_t \end{aligned}$$

This result is identical to what happens in the continuous dividend case, as seen in equation 9.

As before, we define $c(t, s)$ such that $X_t = c(t, S_t)$ for all t. To apply Ito's lemma to $c(t, S_t)$ we first separate the continuous part of S_t:

$$S_t = S_0 + \int_0^t S_{(t^-)}(\mu dt + \sigma dW_t - q dH_t)$$

so its continuous part S_t^c is:

$$S_t^c = S_0 + \int_0^t S_{(t^-)}(\mu dt + \sigma dW_t)$$

Now we use Ito:

$$\begin{aligned} dc(t, S_t) &= c_t(t, S_t) dt + \left[c(t, S_t) - c(t^-, S_{t^-})\right] dH_t + c_s(t, S_t) dS_t^c + \frac{1}{2} c_{ss}(t, S_t) d[S^c, S^c]_t \\ &= [c_t(t, S_t) + \mu S_t c_s(t, S_t) + \frac{1}{2} c_{ss}(t, S_t) S_t^2 \sigma^2] dt \\ &\quad + c_s(t, S_t) S_t \sigma dW_t \\ &\quad + \left[c(t, S_t) - c(t^-, S_{t^-})\right] dH_t \end{aligned}$$

Equating the dX_t and $dc(t, S_t)$ we obtain:

$$(\mu - r)\Delta_t S_t + r X_t = c_t(t, S_t) + \mu S_t c_s(t, S_t) + \frac{1}{2} c_{ss}(t, S_t) S_t^2 \sigma^2 \qquad \text{equating } dt \tag{17}$$

$$\Delta_t S_t \sigma = c_s(t, S_t) S_t \sigma \qquad \text{equating } dW_t \tag{18}$$

$$0 = c(t_*, S_{t_*}) - c(t_*^-, S_{t_*^-}) \qquad \text{equating } dH_t \tag{19}$$

To ensure that equation 18 it is sufficient that

$$\Delta_t = c_s(t, S_t)$$

which is the usual delta-hedging rule.

Note that $S_{t_*} = (1-q)S_{t_*^-}$, letting us rewrite equation 19 as

$$c(t_*, (1-q)S_{t_*^-}) = c\left(t_*^-, S_{t^-}\right)$$

Therefore, for any value $S_{t_*} = s > 0$, we ask

$$c(t_*, (1-q)s) = c\left(t_*^-, s\right)$$

This shows that c tends to have a discontinuity at $t = t_*$.

Finally, from equating the drift terms, we got equation 17. Requiring that it holds for all $S_t = s$ (and plugging in $X_t = c(t, S_t)$,) $\Delta_t = c_s(t, S_t)$ we get:

$$(\mu - r)c_s(t,s)s + rc(t,s) = c_t(t,s) + \mu s c_s(t,s) + \frac{1}{2}c_{ss}(t,s)s^2\sigma^2$$

which, after some cancellation is the usual Merton-Black-Scholes PDE:

$$c_t(t,s) + rsc_s(t,s) + \frac{1}{2}\sigma^2 s^2 c_{ss}(t,s) - rc(t,s) = 0$$

The PDE is valid for all times except $t = t_*$. It can be solved in two steps:

1. Solve the terminal value problem corresponding to $c(T, s) = f(s)$, on the time interval $[t_*, T]$
2. Using the $c(t_*, s)$ obtained in the previous step, solve the terminal value problem $c(t_*^-, s) = c(t_*, (1-q)s)$ on the interval $[0, t_*^-]$.

The above framework can be readily extended to multiple proportional dividends over the lifetime of the option.

3 Reduction to the non-dividend case

As we have seen in the previous sections, continuous or discrete (proportional) dividend payments can be incorporated into the Merton-Black-Scholes framework. It is, however, sometimes more convenient to leave the framework alone, and modify the inputs (like S_t or K) to account for the dividend payments.

This is done by reinvesting the dividend payments in the stock. Namely, suppose you have a dividend process D_t and let A_t be the number of stocks you hold at time t.

$$dS_t = S_{(t^-)}(\mu dt + \sigma dW_t - dD_t) \tag{20}$$

Let us denote our "dividend-financed" portfolio by $\widetilde{S}$, so

$$\widetilde{S}_t = A_t S_t$$

Changes in its value are due to the stock price moving and the dividend stream:

$$d\widetilde{S}_t = A_{(t^-)}dS_t + A_{(t^-)}S_{(t^-)}dD_t$$

$$= A_{(t-)}S_{(t-)}(\mu dt + \sigma dW_t)$$
$$= \widetilde{S}_t(\mu dt + \sigma dW_t)$$

We can solve for $A_t = \frac{\widetilde{S}_t}{S_t}$ now by solving the SDEs for S_t and $\widetilde{S}_t$.

$$\widetilde{S}_t = \widetilde{S}_0 e^{(\mu-\sigma^2/2)t+\sigma W_t}$$

The solution of S_t depends on the form of D_t:

$$\begin{array}{rc|c} dDt = & qdt & qdH_t \\ S_t = & S_0 e^{(\mu-q-\sigma^2/2)t+\sigma W_t} & S_0 e^{(\mu-\sigma^2/2)t+\sigma W_t}(1-q)^{H_t} \end{array}$$

Therefore we get $A_t = \frac{\widetilde{S}_0}{S_0}e^{qt}$ for the continuous, and $A_t = \frac{\widetilde{S}_0}{S_0}(1-q)^{-H_t}$ for the discrete dividend case.

Choosing $A_0 = \frac{\widetilde{S}_0}{S_0}$ is a matter of taste, much like the choice of making B the bond maturing at time T or a money market account. Taking $A_0 = 1$ makes $\widetilde{S}$ into an asset where you start out with a single stock, and then reinvest the dividends it pays to buy more stock, as described above.

More interestingly, picking $A_0 = e^{-qT}$ or $(1-q)^{H_T}$, makes $\widetilde{S}_T = S_T$, so $\widetilde{S}$ corresponds to a *forward contract* with maturity T (strike 0), which A_t tells us how to replicate. We choose the latter approach here, as it makes the mathematics more convenient.

Now we can proceed with the usual replicating portfolio argument, replicating a derivative on S in terms of the bond B maturing at T, and $\widetilde{S}$.

Letting $\tilde{c}(t, \tilde{s})$ denote the price of the derivative at time t when $\widetilde{S}_t = s$, $\tilde{c}$ satisfies the Merton-Black-Scholes equation:

$$\tilde{c}_t(t, \tilde{s}) + r\tilde{s}c_{\tilde{s}}(t, \tilde{s}) + \frac{1}{2}\sigma^2\tilde{s}^2\tilde{c}_{\tilde{s}\tilde{s}}(t, \tilde{s}) - r\tilde{c}(t, \tilde{s}) = 0$$

for all times $0 < t < T$, with terminal condition:

$$\tilde{c}(T, \tilde{s}) = (K - \tilde{s})^+$$

for a put option, say.

The solution is of course the well-known Black-Scholes formula:

$$\tilde{c}(t, \tilde{s}) = N(-d_2)Ke^{-r(T-t)} - N(-d_1)\tilde{s}$$

where

$$d_1 = \frac{\ln\left(\frac{\tilde{s}}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{\ln\left(\frac{\tilde{s}}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

and N is the cumulative distribution function of the standard normal distribution.

The last step is to rewrite the above formula to be in terms of S_t rather than $\widetilde{S}_t$. Let $c(t, s)$ be the value of the option at time t when $S_t = s$. Then we have

$$c(t, S_t) = \tilde{c}(t, \widetilde{S}_t) = \tilde{c}(t, A_tS_t)$$

For continuous dividends we get:

$$c(t,s) = \tilde{c}\left(t, se^{-q(T-t)}\right)$$

and for discrete:

$$c(t,s) = \begin{cases} \tilde{c}\left(t, s(1-q)\right) & \text{when } 0 \leq t < t_* \\ \tilde{c}\left(t, s\right) & \text{when } t_* \leq t \leq T \end{cases}$$

(assuming that there is only one dividend).

In conclusion, we see that it is possible to price options on dividend paying asset by plugging the price of the forward contract into the non-dividend pricing formulas.

References

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[WHD95] Paul Wilmott, Sam Howison, and Jeff Dewynne. *The mathematics of financial derivatives: a student introduction.* Cambridge University Press, 1995.