# Emerging Markets and Inflation

## Lecture 5. Introduction to Inflation

Fall 2017
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# Agenda for Today

## Lecture 5. Introduction to Inflation

- Lecture is dedicated to the phenomenon of inflation
- Define and explain economics
- Use it to explain and understand mathematics behind

1. **Inflation Economics**
2. **Inflation-Linked Products and Markets**
3. **Bootstrapping Inflation Curve**

## Lecture 5. Introduction to Inflation

1. Inflation Economics
   - Definition of Inflation
   - Economic Theories of Inflation
   - Inflation Measures
2. Inflation-Linked Products and Markets
3. Bootstrapping Inflation Curve
4. Seasonality Effect

- **This will be a rare occasion of a deeper than usual economical study before diving into mathematics of the phenomenon**

- **Mathematical model of Inflation is fully founded on economics**

## Definition of Inflation

- **Barron's *Economics* [Wessels, 2006]: "*A rise in general level of prices*"**

- **Excellent book by Comley "*Inflation Matters*" [Comley, 2015] goes further**

- **Conceptually inflation is often perceived synonymously to *cost of living*, but we will see how all currently used inflation measures are not very well fit for that**

## Definition of Inflation (continued)

- Dictionary: ***a persistent increase in the level of consumer prices or a persistent decline in the purchasing power of money***

- **The above seems to be a different definition compared to the *purchasing power of money*. Or is it?**

- **Noticeable time evolution for these definitions!**

- **Compare to dictionary from 1970's: *an increase in the amount of fiduciary (paper or token) money issued beyond what is justified by the country's tangible resources***

- **Most definitions converge to easiest and common description of inflation referring to a measuring Index of some kind, but more on that later**

## Economic Theories of Inflation

**Let us follow already mentioned [Comley, 2015] and excellent book by Deacon, Derry and Mirfenderski, *Inflation-indexed Securities* [Deacon, 2004] for an overview of main economic theories of inflation**

## Money Supply Theory

- **One of the oldest definitions connects it to Inflation**
- **Simple picture of the world with limited supply of goods ➔ money inflow sparks prices increase through bidding**
- **Good example: Europe after discovery of Americas sustained sharp prices raise due to inflow of gold and silver into the world of still rather limited supply of goods and services**

## Money Supply Theory (continued)

- **Copernicus was the first one connecting prices to money supply:** ***P (Prices)*** $\propto$ ***M (Money Supply)***

- **In 1848, John Stewart Mill's *Quantity Theory of Money* expands the above introducing *Equation of Exchange* in stable economy:**

$$\boxed{MV = PQ} \tag{1}$$

**For a given period of time, Money Supply (*M)* scaled by time it is being used (*V)* is equal to Average prices (*P*) scaled by total value of goods (*Q*) generated during that time**

## Money Supply Theory (continued)

- See some evidence of Eq.(1) in history on Figure 1:

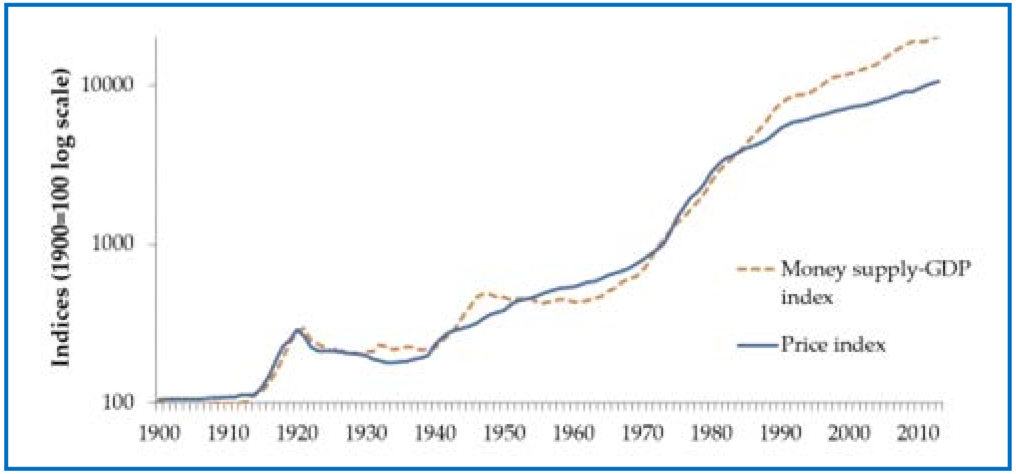


Figure 1. The link between UK prices and money supply (Comley, 2015)

- But there are issues: curves often diverge
- One explanation: money inflow going first into people's savings rather than economy directly. See last years on the chart
- Money supply diversion into assets, not retail prices

## Keynesian Theory

- **John Maynard Keynes proposes inflation theory as a combo of**
  - **Demand-pull factors**
  - **Inflation already built into the system**
  - **Cost-push factors:**

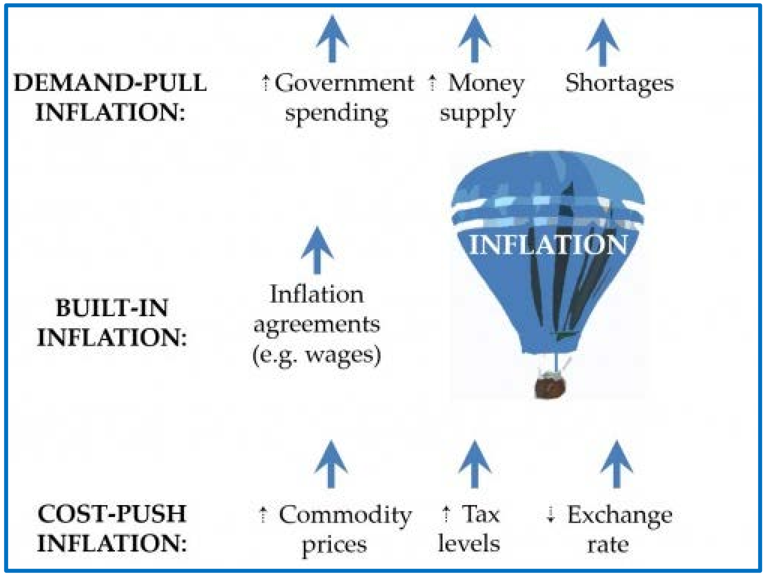


Figure 2. Keynesian view of inflation (Comley, 2015)

## Malthusian Theory

- **Rev. (Thomas) Robert Malthus proposed that *long term prices increase is mostly due to population increase***
- **Suggests population's rate of growth outpaces commodities supply**

## Joined picture

- **Three theories are not really competing**
- **Rather are addressing different time horizons with explanations**

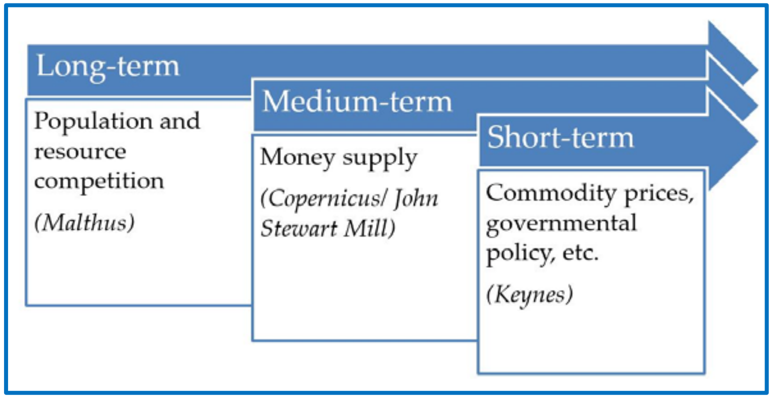


**Figure 3. Theories of inflation and their impact (Comley, 2015)**

## Inflation Measures

- **Define Inflation via level of prices, but how do we measure?**
- ***Consumer Price Indices (CPIs) are economic indicators constructed to measure the changes over time in the prices of consumer goods and services acquired, used or paid for by households*** **[J.P.Morgan, 2009]**
- **Usually CPI refers to well-defined weighted basket of goods and services**


Figure 4. Consumer Price Index as basket of goods and services

## Measures (continued)

- **CPI is rather young measure and started as a *Harmonized Index of Consumer Prices* (HICP)**
- **HICP was created to align inflation measures in pre-union Europe**
- **Countries still mostly stick to their original measures using HICP for comparison only**
- **Few other indices to mention:**
  - **RPI - Retail Prices Index**
  - **PPI - Producer Prices Index**
  - **GDP Deflator for actual calculated GDP**

## Measures Calculations

- Most use *Laspeyres approach*: compare the cost of fixed basket of goods in two periods based upon the quantities set at the start of period one

$$CPI_L(t_o, t) = \frac{\sum_i P_i(t) \cdot Q_i(t_o)}{\sum_i P_i(t_o) \cdot Q_i(t_o)}$$

Here $P_i(t)$ - price of item $i$ at time $t$

$Q_i(t)$ is quantity consumed of item $i$ at time 0 (base period)

- Alternative is *Paasche* that fixes the basket in the second period instead:

$$CPI_P(t_o, t) = \frac{\sum_i P_i(t) \cdot Q_i(t)}{\sum_i P_i(t_o) \cdot Q_i(t)}$$

## Measures Calculations (continued)

- **Using same basket at inception or at maturity is not optimal!**
- **Law of demand: *item's price increase leads to decrease in demand!***
- **Thus if consumer substitutes high weighted item with increased price by a cheaper one with lower weight ➔**
  - **Laspeyres overestimates the inflation**
  - **Paasche underestimates it**
  - ***Fisher* index was designed to compensates for that:**

$$\boxed{CPI_F = \sqrt{CPI_L \cdot CPI_P}}$$

## Inflation Measurement Issues

- **Index Complications:**
  - ❑ **Multitude of indices could be confusing**
  - ❑ **Prices and quantitates contributing to Index must be reliably and continuously observable**
  - ❑ **Substitution proxies are often used but their choices are tricky**
- **Chaining**
  - ❑ **Long time monitoring**
  - ❑ **Rebalance approaches**
  - ❑ **Substitutions and discontinuity**

## Inflation Measurement Issues (continued)

- **New items addition:**
  - **Development of new products and services is hard to account for**
  - **Changes in quality is hard to incorporate**
  - **Theory of *Hedonic Pricing*: mathematical model for prices of items depending on their features. Backfires estimating price decline based on features while street prices goes up**

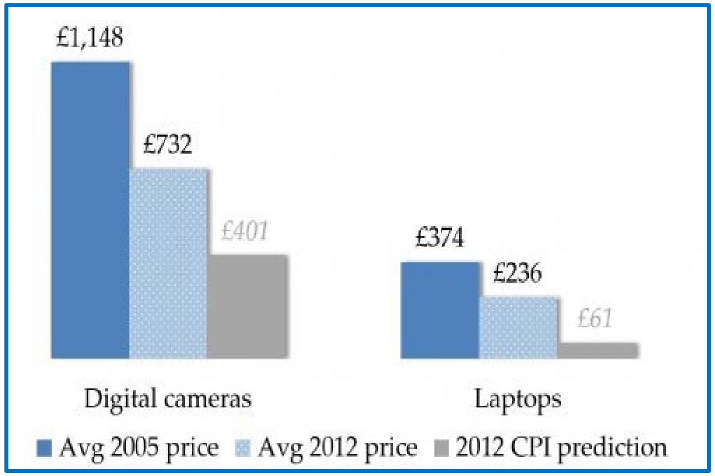


**Figure 5. Average prices in "Which?" 2005 vs 2012 compared with hedonic CPI prediction (Comley, 2015)**

## Lecture 5. Introduction to Inflation

1. Inflation Economics
2. **Inflation-Linked Products and Markets**
   - **Securities**
   - **Breakeven Inflation**
   - **Futures and Zero Coupon Swaps**
   - **Term-on-term Swaps**
   - **Inflation-linked derivatives market**
3. Bootstrapping Inflation Curve

# Inflation-Linked Products and Markets

- **Inflation present throughout financial markets whenever future cash payments are involved**
- **Thus only natural for the market to develop products and instruments helping participants to mitigate the risk**

## Securities

- **Naturally Inflation-linked with each cash flow linked to inflation**
- **Protecting investors against inflation**
- **First in 1742 when State of Massachusetts issues bills of public credit linked to cost of silver on London exchange**
- **But price of silver grows faster than general level of prices → so law was passed and inflation-linked weighted commodities basket was introduced in 1747**

## Securities (continued)

- **Inflation-indexed or *Capital Indexed Bond* (CIB) with notional and coupon adjusted by inflation:**

$$CF(t_i) = \begin{cases} (P + C_i) \cdot InflAdj(t_i) & , t_i = T \\ C_i \cdot InflAdj(t_i) & , t_i \neq T \end{cases}$$
$$InflAdj(t_i) \stackrel{def}{=} \frac{CPI(t_i - lag)}{CPI(t_o - lag)} \qquad (2)$$

***T*** **is bond's Maturity**

***P*** **is Notional or Principal of the bond**

***C*** **is coupon**

***lag*** **is period it takes to collect and publish inflation data. Usually is 1-3mth. Lag distorts protection quality, especially just before bond's maturity leaving principal repayment without protection**

## Securities (continued)

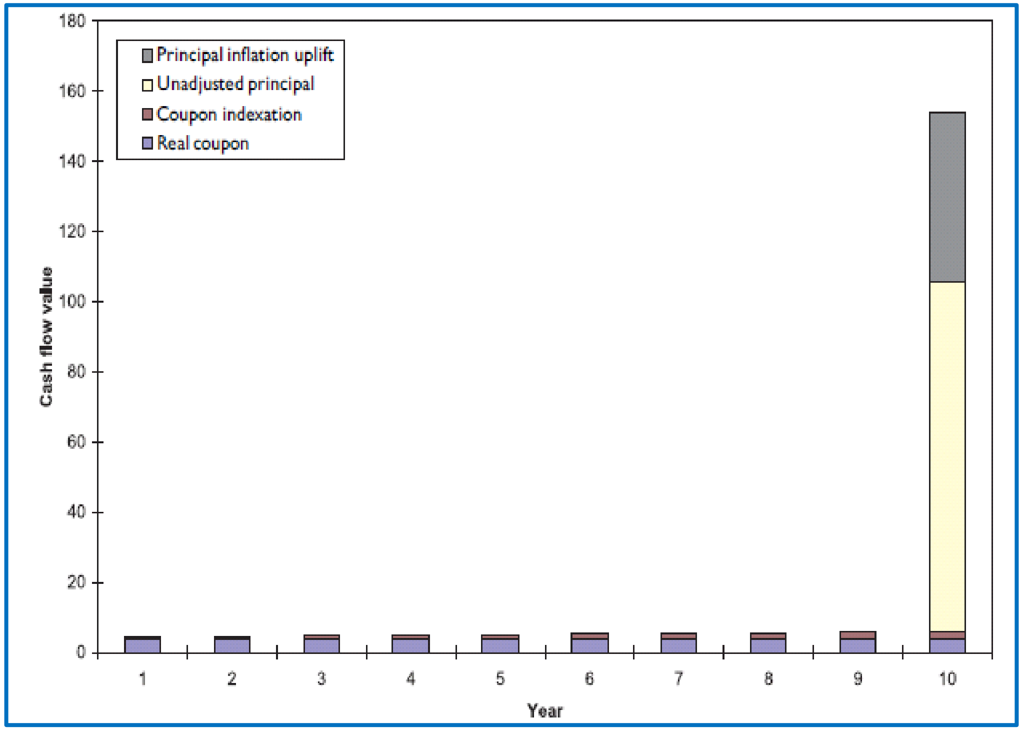


**Figure 6: Capital Indexed Bond (CIB) cash flow structure (Deacon, 2004)**

## Securities (continued)

- **Let us concentrate on CIB and look at yield to maturity in details:**
  - **Recall "regular" bond's yield definition. We call it *Nominal* here and express in exponential terms:**

$$PV_N(t,T) = P \cdot e^{-y_N(T-t)} + \sum_{t_i=t_1}^{T} C_i e^{-y_N(t_i-t)}$$

  - **Use similar expression for CIB to define *Real* yield-to-maturity**

$$PV_{CIB}(t,T) = P \cdot InflAdj(T) \cdot e^{-y_{CIB}(T-t)} + \sum_{t_i=t_1}^{T} C_i \cdot InflAdj(t_i) \cdot e^{-y_{CIB}(t_i-t)}$$
$$\equiv P \cdot e^{-y_R(T-t)} + \sum_{t_i=t_1}^{T} C_i \cdot e^{-y_R(t_i-t)}$$

## Securities (continued)

**HW1: Derive relationship between *Nominal* and *Real* yield of a simple semi-annual B/360 bond. Discuss definition of a break-even inflation based on Inflation Adjustment differential**

- **Continue via Eq. (2) separating Nominal and CIB parts:**

$$
\begin{aligned}
CF_N(t_i) &\overset{def}{=} \begin{cases} (P + C_i) & , t_i = T \\ C_i, & , t_i \neq T \end{cases} \\
CF_{CIB}(t_i) &= CF_N(t_i) \cdot InflAdj(t_i)
\end{aligned}
$$

- **Generalize for a bond's Dirty Price:**

$$
\begin{aligned}
DP_N(t) &= \sum_i CF_N(t_i) \cdot Z_N(t, t_i) \\
DP_{CIB}(t) &= \sum_i CF_{CIB}(t_i) \cdot Z_N(t, t_i) \\
&\equiv \sum_i CF_N(t_i) \cdot InflAdj(t_i) \cdot Z_N(t, t_i)
\end{aligned} \qquad (3)
$$

## Securities (continued)

- **Define now *Real* discounting and *Real* yield via Eq. (3):**

$$DP_{CIB}(t) \stackrel{def}{=} \sum_i CF_N(t_i) \cdot Z_R(t, t_i)$$
$$Z_R(t, t_i) \stackrel{def}{=} InflAdj(t_i) \cdot Z_N(t, t_i) \qquad (4)$$

**Break-Even Inflation:** **level of inflation required for an inflation-linked bond (CIB) to be of equal value to its *Nominal* equivalent**

**→ A bit theoretical definition since it assumes existence of such bonds of course**

## Break-Even Inflation (continued)

- **Decompose *nominal* and *real* yields more carefully:**

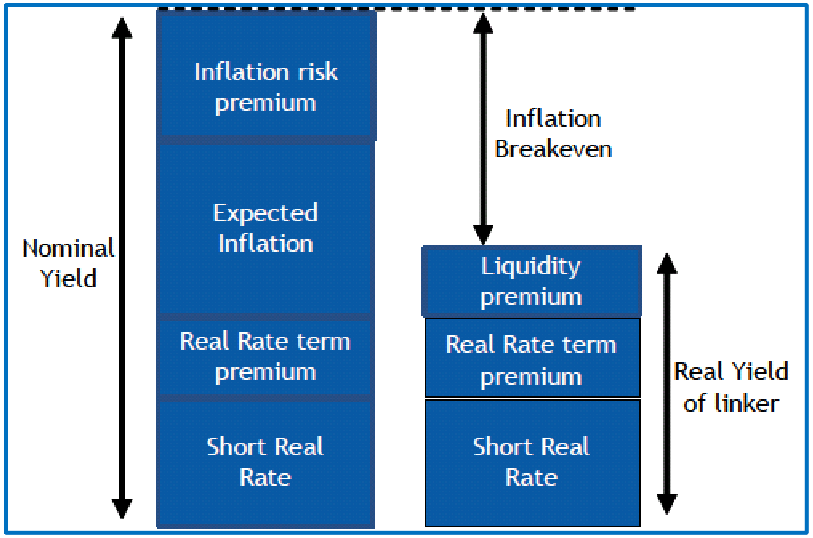


**Figure 7: What is Breakeven Inflation? (J.P.Morgan, Inflation-Linked Markets, 2009)**

- ***Inflation risk premium*: risk of increased inflation in the long end suggesting breakeven inflation overstating the *Expected Inflation***
- ***Liquidity premium* accounts for a lesser liquidity of inflation-linked bonds compared to their nominal counterparts**

## Breakeven Inflation

- The Fisher Equation via re-writing Eq. (4) with annualized yields and newly defined ***breakeven inflation*** ***bi***

$$\boxed{\begin{aligned} & Z_B(t,T) = (1+y)^{-\tau} \\ & (1+y_R)^{-\tau} = (1+bi)^{\tau} \cdot (1+y_N)^{-\tau} \\ & (1+y_N) = (1+y_R)\cdot(1+bi) \end{aligned}} \qquad (5)$$

- More details on differences between expected and breakeven inflation using [Kerkhof, 2005]:
  - Compounding effect due to stochastic nature is underestimating inflation if ignored. Effect is increasing in high vols and rates!

$$\boxed{\mathbf{E}\left[(1+bi(t,T))^{\tau}\right] \geq (1+\mathbf{E}[bi(t,T)])^{\tau}}$$

  - Inflation Convexity in classical sense of second order derivative
  - Inflation Risk Premium due to certainty of the offered protection

## Breakeven Inflation (continued)

- **Add all above mentioned components to the *Fisher Equation*:**

$$\boxed{(1+y_N)=(1+y_R)\cdot(1+\mathbf{E}[bi])\cdot(1+Cxty)\cdot(1+RiskPremium)}$$

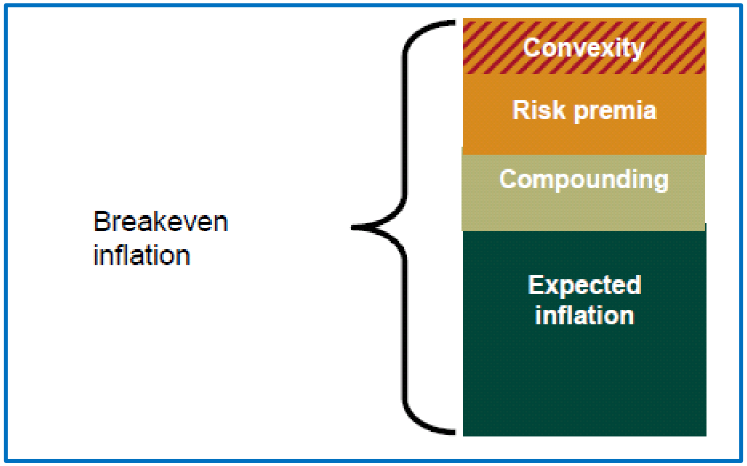


**Figure 8: Components of breakeven inflation (Kerkhof, 2005)**

- **Similar to our studies of IR instruments in Lecture 3, we start here by looking at instruments that are building inflation trading:**

## Futures and Zero Coupon Swaps

- **Inflation Futures indexing US CPI are trading on CME since 2004:**
  - **Speculate on the outright level of inflation**
  - **Hedge future levels of inflation**
  - **Mitigate counterparty risk**
  - **Resemble Eurodollar futures:**

$$\boxed{\Phi = 1 - \frac{1}{\delta}\left(\frac{CPI_T}{CPI_{T-\delta}} - 1\right)}$$

**with δ for the 3mth time interval**

## Zero Coupon Swaps

- One of the most liquidly traded inflation derivative
- Single cash flow exchange at maturity:
  - Swap payer receives annualized Fixed Rate coupon ZC at maturity
  - Swap Receiver gets the accrued inflation:

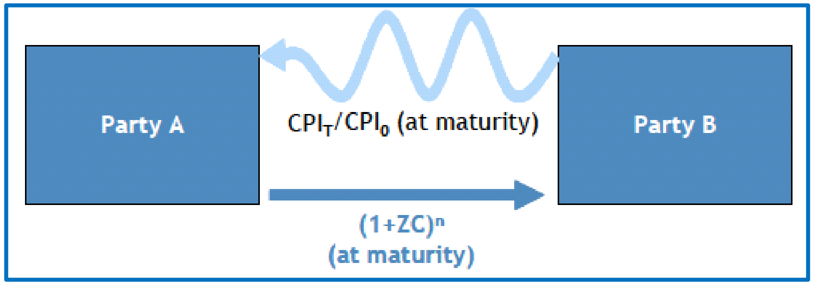


Figure 9: ZC Inflation Swap cashflows (J.P.Morgan, Inflation-Linked Markets, 2009)

## Zero Coupon Swaps (continued)

- **Accurate expression for *PV* of a Floating leg in Nominal risk-neutral economy:**

$$\boxed{PV_{ZCIS}(t,0,T)=\mathbf{E}_n\left[\frac{CPI_T}{CPI_o}\cdot\exp\left(-\int_t^T n(x)dx\right)\middle|F_t\right]} \qquad \textbf{(6)}$$

**With *n(x)* for stochastic nominal interest rates,**

**time of inception 0 and maturity *T*, with *t* as running time $0 \le t \le T$**

## Zero Coupon Swaps (continued)

- **Transform Eq.(6) now using Eq.(4) and definition of *Real* rates**

$$\mathbf{E}_n\left[CPI_T \cdot \exp\left(-\int_t^T n(x)dx\right)\middle| F_t\right] \equiv CPI_t \cdot Z_R(t,T)$$

$$PV_{ZCIS}(t,0,T) = \frac{CPI_t}{CPI_o} \cdot Z_R(t,T) \qquad (7)$$

- **Note model-independent expression for price of zero coupon inflation swap**

## Term-om-term Inflation-linked Swaps

- Generalize Zero coupon to multi-coupon:

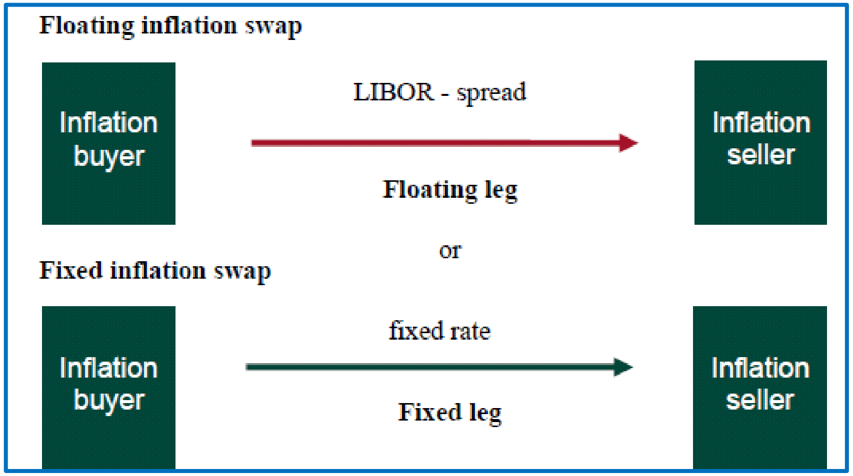


Figure 10: Fixed and Floating Inflation swaps (Kerkhof, 2005)

- Flavor when inflation adjustment is applied from inception resembles CIB bond and is not very interesting now
- So, we do not look at swaps with base date inflation adjustment
- Look at structures that rest inflation base every coupon instead

## Term-om-term Swaps (continued)

- **Year-on-year variety:**

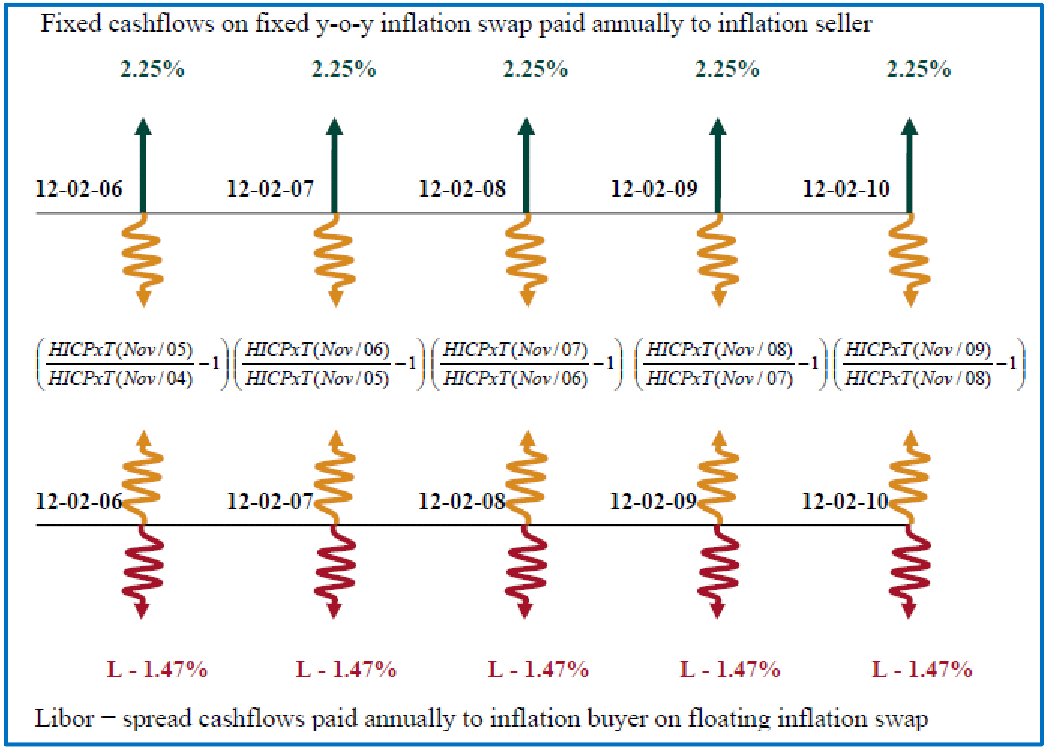


**Figure 11: Fixed and Floating Inflation swaps (Kerkhof, 2005)**

## Term-om-term Swaps (continued)

- **Let's repeat the valuation analysis as in Eq. (6) and (7) looking at one cash flow only:**

$$PV_{YoYIS}\left(t,0,T_{j-1},T_j\right)=\mathbf{E}_n\left[\left(\frac{CPI_{T_j}}{CPI_{T_{j-1}}}-1\right)\cdot\exp\left(-\int_t^{T_j}n(x)dx\right)\middle|F_t\right] \qquad (8)$$

- **More complex than before, but still can try to simplify re-arranging terms:**

$$\begin{aligned}PV_{YoYIS}\left(t,0,T_{j-1},T_j\right)&=\mathbf{E}_n\left[e^{-\int_t^{T_{j-1}}n(x)dx}\cdot\mathbf{E}_n\left\{e^{-\int_{T_{j-1}}^{T_j}n(x)dx}\cdot\left(\frac{CPI_{T_j}}{CPI_{T_{j-1}}}-1\right)\middle|F_{T_{j-1}}\right\}\middle|F_t\right]\\&=\mathbf{E}_n\left[\left\{Z_R\left(T_{j-1},T_j\right)-Z_N\left(T_{j-1},T_j\right)\right\}\cdot\exp\left(-\int_t^{T_{j-1}}n(x)dx\right)\middle|F_t\right]\end{aligned}$$

- **Not much simpler ➔ need a short rate model to proceed. Will come back to that later**

## Other Inflation Linear Products

- **Leave many out of our scope, but still mention for an overview:**
- **Inflation Forwards:**
  - More common in Emerging Markets and Latin America
  - Allows extracting medium and longer term inflation
  - Cover in more details next week
- **Inflation Asset Swap:**
  - Combination of Inflation-linked bond and inflation-linked swap
  - Isolates credit and liquidity component

## Other Inflation Volatility Products

- **Inflation Cap and Floor**:
  - Most common inflation vol product
  - Stems form inflation-linked bonds having embedded Floor at final redemption: long term and way out of the money, but still…
- **Inflation Swaption**:
  - Option to enter into an inflation-linked swap
- **LPI Swap**:
  - *Limited Price Index* swap flooring inflation adjustment at pre-agreed level
- **Inflation Spread Option**:
  - Pays spread between 2 inflation indices
  - Could be single Currency (Index) or multi

## Inflation-Linked Derivatives Markets

- **Who are its participants and what are their interests?**
- **Started to appear in 1990s in UK**
- **Is formed by natural Long's and Short's**
- **Supply (Inflation Payers):**
  - **Linked to entities receiving inflation connected cash flows**
  - **Sovereigns or Governments: issue inflation-linked debt to diversify inflation-linked revenues**
  - **Real Estate contracts (rents) indexed to inflation**
  - **Utility companies own big portion of corporate issuance of the inflation-linked debt**

## Inflation-Linked Derivatives Markets (continued)

- **Demand (Inflation Receivers):**
  - **Linked to entities making inflation connected payments**
  - **Pension Funds: most of pension schemes are inflation connected. Examples are Cost of Living in US or LPI's in UK**
  - **Insurance companies: similar are involved in selling inflation dedicated protection**

# Agenda for Today. Bootstrapping Inflation Curve

## Lecture 5. Introduction to Inflation

1. Inflation Economics

2. Inflation-Linked Products and Markets

3. **Bootstrapping Inflation Curve**

   - **Breakeven Inflation Curve**
   - **Inflation Curve via mean-Reverting Process**
   - **Forward Inflation via mean-Reverting Process**

# Bootstrapping Inflation Curve

- **Same as everywhere else and especially Interest Rates derivatives, pricing Inflation-Linked products requires a curve**
- **Same as with *Nominal* curves and products, curve must be bootstrapped from market instruments**
- **Will allow us to extract future inflation index values via forward inflation instantaneous rates**

## Breakeven Inflation Curve

- **Use directly observable breakeven inflation rates:**

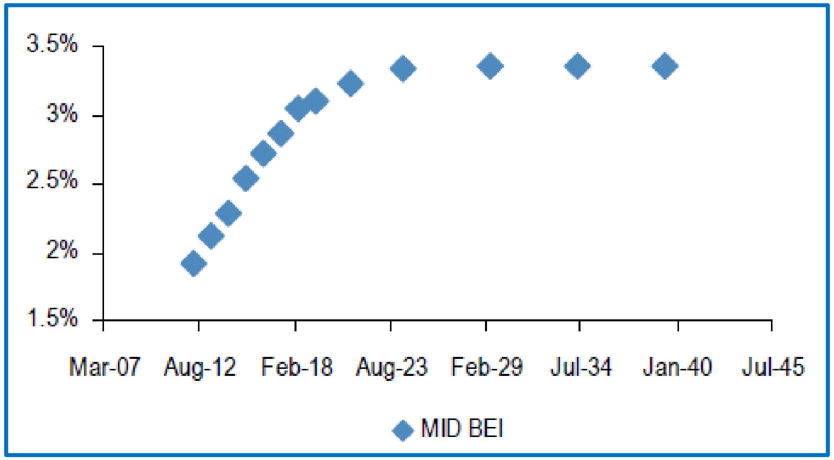


**Figure 12: UK RPI Zero Coupon Curve (J.P.Morgan, Inflation-Linked Markets, 2009)**

  - **Only available from 2 years and out**
  - **More info needed to shape the front end of the curve**
  - **Use economist's predictions and other subjective measures**

## Inflation Curve via Mean-Reverting process

- **Bootstrap inflation curve from inflation-adjusted bonds**

- **Assume mean-reverting process for *Real* rates to use later in forward inflation extraction**

- **Approximate via exponential rates in Eq.(7):**

$$Z_R(0,T) = \mathbf{E}\left[\exp\left\{-\int_o^T r(s)\,ds\right\}\right] \tag{9}$$

## Inflation Curve via Mean-Reverting process (continued)

- **Select mean-reverting O-U process for *real* rates:**

$$\boxed{dr(t) = \lambda(t)\left[\Theta(t) - r(t)\right]dt + \sigma(t)dW_t}$$

**With *r(t)* being short instantaneous real rates,**

**Mean reversion rate *λ(t)* and**

**Mean reversion level *Θ(t)***

- **Simplifying it for constants collapsing once again to Vasicek:**

$$\boxed{dr(t) = \lambda_r \cdot \left[\theta_r - r(t)\right]dt + \sigma_r \cdot dW_t}$$

## Inflation Curve via Mean-Reverting process (continued)

- **And integrate as usual to arrive at standard answer**

$$\boxed{r(t)=\theta+e^{-\lambda_r t}(r_o-\theta_r)+\sigma_r\int_o^t e^{-\lambda_r(t-s)}dW_s}$$

- **Next is to integrate for the zero coupon bond expression**

- **Can say that** $\int_o^t r(s)ds$ **is normally distributed with drift μ and variance *var***

$$\mu=\int_o^t\left[\theta_r+e^{-\lambda_r s}(r_o-\theta_r)\right]ds=r_o h(t,\lambda_r)+\lambda_r t-\lambda_r h(t,\lambda_r)$$

$$h(t,\lambda)\overset{def}{=}\frac{1-e^{-\lambda t}}{\lambda}$$

$$var=\left[\sigma_r\int_o^t\int_u^t e^{-\lambda_r(s-u)}ds\,dW_u\right]^2=\frac{\sigma_r^2}{\lambda_r^2}\left[t-2h(t,\lambda_r)+h(t,2\lambda_r)\right] \quad \textbf{(10)}$$

## Inflation Curve via Mean-Reverting process (continued)

- **Then price of Inflation-indexed Zero coupon bond from Eq.(9) simply is:**

$$Z_R(0,T) = \exp\left\{-\mu + \frac{var}{2}\right\} \tag{11}$$

- **Now we can decompose *N* coupon-paying bonds into set of zero coupons in order to apply Eq.(11)**

- **But need to upgrade to a piece-wise constant mean reversion**

$$\Theta(t) = \begin{cases} \theta_1, & 0 < t \le T_1 \\ \theta_2, & T_1 < t \le T_2 \\ \ldots & \\ \theta_N, & T_{N-1} < t \end{cases}$$

## Inflation Curve via Mean-Reverting process (continued)

- **This will lead us to a component-based expression for the drift from Eq.(10)**

$$\boxed{\mu = r_o h(t,\lambda_r) + \theta_N \left[ \frac{1}{\lambda_r}\left(e^{\lambda_r (t_{N-1}-t)} - 1\right) + t - t_{N-1} \right] + \sum_{i=1}^{N-1} \theta_i \left[ \frac{1}{\lambda_r}\left(e^{\lambda_r (t_{i-1}-t)} - e^{\lambda_r (t_i - t)}\right) + t_i - t_{i-1} \right]}$$

  - **Starting in order of increasing maturity**
  - **With collection of market or historically estimated parameters**
  - **Making assumption for the final redemption floor value in bonds**
  - **Dirty Price of the *i*-th bond with $M_i$ cash flows could be written:**

$$\boxed{DP_i = \frac{CPI_{Spot}}{CPI_{Base}} \left\{ \sum_{j=1}^{M_i} Z_r(0,T_{ij}) \cdot CF_{ij} + Z_r(0,T_{iM_i}) \right\} + Z_r(0,T_{iM_i}) Floor}$$

## Forward Inflation via Mean-Reverting process

- Let us now go back to term-on-term inflation swap and use our mean-reverting process and bootstrapped curve
- Rewrite Eq.(8) for forward inflation term with time period δ separating deterministic part from Convexity Adjustment under the *T*-forward measure:

$$\begin{aligned} FwdInfl_{\delta}(t,T) &= \mathbf{E}^{T}\left[\left(\frac{CPI_T}{CPI_{T-\delta}}-1\right)\middle| F_t\right] \\ &= \frac{CPI_T}{CPI_{T-\delta}}\cdot e^{ConvAdj}-1 \end{aligned} \tag{12}$$

- We also carefully separate Forward and Spot CPI similar to forward and short rate and write Eq.(3) in these new notations:

$$\begin{aligned} FCPI(t,T) &= \mathrm{E}^{T}\left[CPI(T)\middle| F_t\right] \\ &= CPI(t)\frac{Z_R(t,T)}{Z_N(t,T)} \end{aligned} \tag{13}$$

## Forward Inflation via Mean-Reverting process (continued)

- **Assume BGM-like dynamics for forward LIBOR and CPI:**

$$\frac{dFL(t,T-\delta,T)}{FL(t,T_{i-1},T_i)} = \sigma_L dW^T$$
$$\frac{dFCPI(t,T)}{FCPI(t,T-\sigma)} = \sigma_T dW^T$$

- **On the other hand expanding on Eq.(12) we can write:**

$$\frac{dFCPI(t,T)}{FCPI(t,T)} = \frac{dCPI(t)}{CPI(t)} + \frac{dZ_r(t,T)}{Z_r(t,T)} + \frac{dZ_n(t,T)}{Z_n(t,T)}$$

- **Or simply for the vols part:**

$$(\sigma_T)^2 = (\sigma_c)^2 + (\sigma_r)^2 + (\sigma_n)^2 + 2\rho_{r,c}\sigma_r\sigma_c - 2\rho_{n,c}\sigma_n\sigma_c - 2\rho_{n,r}\sigma_n\sigma_r$$

**Using $\sigma_c$ for vol of the spot CPI and collection of correlations $\rho$**

## Forward Inflation via Mean-Reverting process (continued)

- **Then explicitly recognizing CPI processes at inception and maturity of the period as separate and correlated dynamics and**
- **Defining $T_i = T - \delta$ and $T_f = T$ we can write connection between Brownian's**

$$\boxed{dW^{T_i} = dW^{T_f} - \frac{\delta L}{1+\delta L}\sigma_L dt}$$

- **And then naturally**

$$\boxed{\begin{array}{l} \dfrac{dFCPI(t,T_f)}{FCPI(t,T_f)} \propto \sigma_f dW^{T_f} \\ \dfrac{dFCPI(t,T_i)}{FCPI(t,T_i)} \propto \sigma_i dW^{T_i} = \sigma_i dW^{T_f} - \gamma dt; \qquad \gamma \overset{def}{=} \dfrac{\delta L}{1+\delta L}\rho_{L,T_i}\sigma_L\sigma_i \end{array}}$$

**Where L is forward rate of our time period and $\rho_{L,Ti}$ is correlation between $FCPI(t,T_i)$ and $L$**

## Forward Inflation via Mean-Reverting process (continued)

- **Then for ratio of CPI indices at start and end of the period**

$$\frac{FCPI(t,T_f)}{FCPI(s,T_i)} = \frac{FCPI(0,T_f)\cdot \exp\left\{-\frac{1}{2}\sigma_f^2 t + \sigma_f W^{T_f}(t)\right\}}{FCPI(0,T_i)\cdot \exp\left\{-\frac{1}{2}\sigma_i^2 s + \sigma_i W^{T_f}(s) - \int_o^s \gamma dx\right\}} \quad (14)$$

- **Re-write the above with some substitutions:**

$$\frac{FCPI(T_f,T_f)}{FCPI(T_i,T_i)} = \frac{FCPI(0,T_f)\cdot \exp\left\{-\frac{1}{2}\sigma_f^2 T_f + \sigma_f W^{T_f}(T_f)\right\}}{FCPI(0,T_i)\cdot \exp\left\{-\frac{1}{2}\sigma_i^2 T_i + \sigma_i W^{T_f}(T_i) - \int_o^{T_i} \gamma dx\right\}}$$

## Forward Inflation via Mean-Reverting process (continued

- **And finally for the inflation term combining all of the above and leaving some tedious calculations out**

- **Convexity is derived starting from**

$$FwdInfl_{\delta}(t,T) = \mathbf{E}^{T}\left[\left.\frac{FCPI(T_f,T_f)}{FCPI(T_i,T_i)}\right| F_t\right] - 1$$

- **And arriving at**

$$ConvAdj = \left[\rho_{L,T_i}\sigma_L\sigma_i\frac{\delta L}{1+\delta L} + \sigma_i\left(\sigma_i - \sigma_f\rho_{T_f,T_i}\right)\right](T_i - t)$$

**Where we introduce $\rho_{Tf,Ti}$ as correlation between *FCPI(t,T$_f$)* and *FCPI(t,T$_i$)***

**HW2: Estimate levels of this Convexity Adjustment for ranges of vols, correlations and maturities. Suggest validity limits where convexity could be safely ignored**

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