1. Firstly, let's look at the left hand side of (8):

$$Y_t = \psi(L)X_t = 1 - \sum_{r=1}^{p} c_r L^r$$

where L is the lag operator $LX_t = X_{t-1}$. Then its fourier transform can be written as:

$$S_Y(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} \Gamma_k^Y e^{-i\omega k}$$

$$\Gamma_k^Y = cov(y_t, y_{t+1})$$

For generality, let's define $\psi(L) = \sum_{r=-\infty}^{+\infty} a_r L^r$, then:

$$Y_t = \sum_{-\infty}^{\infty} a_r X_{t-r}$$

$$\begin{aligned}
\Gamma_k^Y &= \sum\sum a_r a_s cov(X_{t-r}, X_{t+k-s}) \\
&= \sum\sum a_r a_s \Gamma_{r+k-s}^X \\
&= \sum\sum a_r a_s \int f_X(\omega) e^{-i\omega(r+k-s)} d\omega \\
&= \int f_X(\omega) \sum\sum a_r a_s e^{-i\omega(r+k-s)} d\omega
\end{aligned}$$

Noting $\psi(z) = \sum a_s z^s$, then we have:

$$\begin{aligned}
\Gamma_k^Y &= \int f_X(\omega) \psi(e^{i\omega}) \psi(e^{-i\omega t}) e^{-i\omega k} d\omega \\
&= \int f_X(\omega) |\psi(e^{-i\omega})|^2 e^{-i\omega k} d\omega \\
&= \int f_Y(\omega) e^{-i\omega k} d\omega
\end{aligned}$$

In other words, we proved that $f_Y(\omega) = |\psi(e^{-i\omega})|^2 f_X(\omega)$. As a result, the eq. 8 becomes:

$$|\psi(e^{-i\omega})|^2 f_X(\omega) = \mathbb{F}(\alpha + \varphi(L)\varepsilon_t)$$

we already know that $\mathbb{F}(\varepsilon_t) = \frac{\sigma^2}{2\pi}$, Fourier transform of constant α is a δ function at 0, then:

$$\begin{aligned}
|\psi(e^{-i\omega})|^2 f_X(\omega) &= |\varphi(e^{-i\omega})|^2 \frac{\sigma^2}{2\pi} \\
f_X(\omega) &= \frac{\sigma^2}{2\pi} \frac{|\varphi(e^{-i\omega})|^2}{|\psi(e^{-i\omega})|^2}
\end{aligned}$$

for $\omega \neq 0$. If we factorize the polynomials of $\psi(z)$ and $\varphi(z)$:

$$f_X(\omega) = \frac{\sigma^2}{2\pi} \frac{|1-\mu_1 e^{-i\omega}|^2 |1-\mu_2 e^{-i\omega}|^2 ... |1-\mu_q e^{-i\omega}|^2}{|1-\lambda_1 e^{-i\omega}|^2 |1-\lambda_2 e^{-i\omega}|^2 ... |1-\lambda_p e^{-i\omega}|^2}$$

$$\begin{aligned}
|1-\mu e^{-i\omega}| &= \sqrt{(1-\mu\cos\omega)^2 + \mu^2 \sin^2\omega} \\
&= \sqrt{1 - 2\mu\cos\omega + \mu^2}
\end{aligned}$$

$$f_X(\omega) = \frac{\sigma^2}{2\pi} \frac{(1-2\mu_1\cos\omega + \mu_1^2)(1-2\mu_2\cos\omega+\mu_2^2)...(1-2\mu_q\cos\omega+\mu_q^2)}{(1-2\lambda_1\cos\omega+\lambda_1^2)(1-2\lambda_2\cos\omega+\lambda_2^2)...(1-2\lambda_p\cos\omega+\lambda_p^2)}$$

when $\omega = 0$, we need to add a constant term α which might be ignored in the slides.