Solutions to MTH 9831 Fall 2015 HW #7
Baruch College MFE Program

Please note that these are *possible* solutions, and that there are often many ways to approach and solve a problem or prove a theorem.

Problem 1 (PDE with terminal condition) $g(t,x)$ satisfies the PDE with terminal condition

$$g_t(t,x) + \frac{1}{2}\sigma^2 g_{xx}(t,x) = rg(t,x);\ g(T,x) = x^4.$$

To apply discounted Feynman-Kac via the SDE

$$dX(t) = \beta(t,X(t))dt + \gamma(t,X(t))dW(t),$$

we have $\beta(t,x) = 0$, $\gamma(t,x) = \sigma$, $h(x) = x^4$ satisfying the PDE with terminal condition

$$g_t(t,x) + \beta(t,x)g_x(t,x) + \frac{1}{2}\gamma^2(t,x)g_{xx}(t,x) = rg(t,x);\ g(T,x) = h(x).$$

By discounted Feynman-Kac, then,

$$g(t,x) = e^{-r(T-t)}E^{t,x}h(X(T))$$

satisfies the PDE and boundary condition above. We solve for $X(t)$ and evaluate the expected value:

$$\begin{aligned}
dX(t) &= \sigma dW(t) \implies X(t) = \sigma W(t) \implies W(t) = \tfrac{X(t)}{\sigma} \\
\implies g(t,x) &= e^{-r(T-t)}E(h(X(T))\,|\,X(t)=x) \\
&= e^{-r(T-t)}E(X^4(T))\,|\,\sigma W(t)=x) \\
&= e^{-r(T-t)}E(X^4(T))\,|\,W(t)=\tfrac{x}{\sigma}) \\
&= e^{-r(T-t)}\sigma^4 E\left(W^4(T)\,|\,W(t)=\tfrac{x}{\sigma}\right).
\end{aligned}$$

Checking the terminal condition $g(T,x) = x^4$, we see that

$$g(T,x) = e^{-r(T-T)}\sigma^4 E\left(W^4(T)\middle|W(T) = \frac{x}{\sigma}\right) = x^4,$$

so our final answer is

$$g(t,x) = e^{-r(T-t)}\sigma^4 E\left(W^4(T)\,\middle|\,W(t) = \frac{x}{\sigma}\right),$$

which can be simplified tremendously: we decompose $W(T)$ into time-t-measurable and independent-of-time-t pieces. The odd powers of independent terms drop out (odd moments of Brownian motion are 0), kurtosis, etc. give our calculations: using $W(T) - W(t) \sim \sqrt{T-t}Z$, $Z \sim N(0,1)$, and knowing the moments $E(Z) = E(Z^3) = 0$, $E(Z^2) = 1$, $E(Z^4) = 3$,

$$\begin{aligned}
g(t,x) &= e^{-r(T-t)}\sigma^4 E(W^4(T)|W(t)=\tfrac{x}{\sigma}) \\
&= e^{-r(T-t)}\sigma^4 E([W(T)-W(t)+W(t)]^4|W(t)=\tfrac{x}{\sigma}) \\
&= e^{-r(T-t)}\sigma^4 E([\sqrt{T-t}Z + x]^4) \\
&= e^{-r(T-t)}[\sigma^4(T-t)^2E(Z^4) + 4(T-t)^{3/2}E(Z^3) + 6(T-t)E(Z^2) \\
&\quad +4\sqrt{T-t}E(Z) + x^4] \\
&= e^{-r(T-t)}\left[\sigma^4\left(3(T-t)^2 + 6(T-t)\left(\tfrac{x}{\sigma}\right)^2 + \left(\tfrac{x}{\sigma}\right)^4\right)\right] \\
&= e^{-r(T-t)}[3\sigma^4(T-t)^2 + 6\sigma^2(T-t)x^2 + x^4].
\end{aligned}$$

To finish the check, take the partials and match the PDE:

$$g(t,x) = e^{-r(T-t)} j(t,x)$$

$$g_t(t,x) + \frac{1}{2}\sigma^2 g_{xx}(t,x) = rg(t,x) + e^{-r(T-t)}\left[j_t(t,x) + \frac{1}{2}j_{xx}(t,x)\right];$$

$$j_t(t,x) + \frac{1}{2}\sigma^2 j_{xx}(t,x) = -6\sigma^4(T-t)\sigma^2 x^2 - \frac{1}{2}\sigma^2(12\sigma^2(T-t) + 12\sigma^2 x^2) = 0.$$

Problem 2 (Shreve Vol. 2, #6.1, p. 282) We are given $X(t) = x$,

$$\begin{aligned}
dX(u) &= (a(u) + b(u)X(u))du + (\gamma(u) + \sigma(u)X(u))dW(u) \\
a(u), b(u), \gamma(u), \sigma(u) &\in \mathcal{F}(u) \\
Z(u) &= \exp\left\{\int_t^u \sigma(v)dW(v) + \int_t^u \left(b(v) - \tfrac{1}{2}\sigma^2(v)\right)dv\right\} \\
Y(u) &= x + \int_t^u \tfrac{a(v)-\sigma(v)\gamma(v)}{Z(v)}dv + \int_t^u \tfrac{\gamma(v)}{Z(v)}dW(v).
\end{aligned}$$

(i) $Z(t) = 1$ is easy: $Z(t) = e^0 = 1$ since the integrals start at t.

$$\begin{aligned}
dZ(u) &= Z(u)\left[\sigma(u)dW(u) + \left(b(u) - \tfrac{1}{2}\sigma^2(u)\right)du + \tfrac{1}{2}\sigma^2(u)du\right] \\
&= Z(u)\left[\sigma(u)dW(u) + b(u)du\right].
\end{aligned}$$

(a) isn't needed; it's already done, since we know what $Z(u)$ is.
(b) Let $Y(t) = x$; what we are given shows that

$$dY(u) = k(u)du + l(u)dW(u)$$

is solved by

$$k(u) = \frac{a(u) - \sigma(u)\gamma(u)}{Z(u)},\ \ l(u) = \frac{\gamma(u)}{Z(u)}.$$

(ii) $X(u) = Y(u)Z(u)$ satisfies $dX(u)$ written above: we're given

$$X(t) = Y(t)Z(t) = x,$$

$$dY(u) = \frac{a(u) - \sigma(u)\gamma(u)}{Z(u)}du + \frac{\gamma(u)}{Z(u)}dW(u),\ \ u \geq t,$$

and by Itô's product rule, it's clear that

$$\begin{aligned}
d(Y(u)Z(u)) &= Y(u)dZ(u) + Z(u)dY(u) + dY(u)dZ(u) \\
&= Y(u)Z(u)\left[\sigma(u)dW(u) + b(u)du\right] + (a(u) - \sigma(u)\gamma(u))\,du + \gamma(u)dW(u) \\
&\quad + \gamma(u)\sigma(u)du \\
&= (a(u) + b(u)X(u))du + (\gamma(u) + \sigma(u)X(u))dW(u) = dX(u).
\end{aligned}$$

Therefore, the solution is

$$X(u) = Y(u)Z(u) = Z(u)\left[x + \int_t^u \frac{a(v) - \sigma(v)\gamma(v)}{Z(v)}dv + \int_t^u \frac{\gamma(v)}{Z(v)}dW(v)\right].$$

Problem 3 (which line is non-trivial?) Our original equation is

$$g(t,x) = E^{t,x}[h(X(T))] + \int_t^T E^{t,x}[f(X(u))]du$$

subject to the SDE

$$dX(u) = b(u, X(u))du + \sigma(u, X(u))dB(u), \ X(t) = x.$$

The changes occurring from line to line:

- 1) to 2) moves "initial conditions" to "conditioning", which is allowed by stochastic Fubini's theorem in the integral term.
- 2) to 3) moves from fixed initial point $X(t) = x$ to a random one, changing the deterministic function $g(t,x)$ into the stochastic process $g(t, X(t))$.
- 3) to 4) moves from conditioning on only $X(t)$ to the entire history from times $[0,t]$ - this now takes into account the past, not only time t, but works because, as Corollary 6.3.2, p 267 states, solutions to SDEs are Markov processes. This means that the conditional expectation only looks at time t, not the entire past. (This is the line that we would most likely say is non-trivial and requires the most justification.) As I stated in the TA session, this is where we stop thinking about the Markov property of $g(t, X(t))$ and start thinking about what its associated martingale is. To do that, we need the entire history to measure on, not just the current time.
- 4) to 5) decomposes the integral $\int_0^T f(X(u))du = \int_0^t f(X(u))du + \int_t^T f(X(u))du$ into $\mathcal{F}(t)$-measurable and independent terms.
- 5) to 6) substitutes the boundary condition $g(T,x) = h(x)$ at $X(T) = x$.

This last line shows us that the martingale in question is

$$G(t, X(t)) := g(t, X(t)) + \int_0^t f(X(u))du = E\left[g(T, X(T)) + \int_0^T f(X(u))du \,\middle|\, \mathcal{F}(t) \right].$$

Note that if $f(x) \equiv 0$, then we have our usual Feynman-Kac Theorem situation, and the PDE would be

$$g_t(t,x) + b(t,x)g_x(t,x) + \frac{1}{2}\sigma^2(t,x)g_{xx}(t,x) = 0; \ g(T,x) = h(x).$$

We take the differential of our martingale $G(t,x)$; its dt term must be equal to 0. This dt term is hence the PDE satisfied by our original equation; it will be in terms of f, g, and h.

$$\begin{aligned}
dG(t, X(t)) &= dg(t, X(t)) + f(X(t))dt \\
&= g_t(t, X(t))dt + g_x(t, X(t))dX(t) + \tfrac{1}{2}g_{xx}(t, X(t))dX(t)dX(t) + f(X(t))dt \\
&= g_t dt + b g_x dt + \sigma g_x dB(t) + \tfrac{1}{2}\sigma^2 g_{xx} dt + f dt \\
&= \left[g_t + b g_x + \tfrac{1}{2}\sigma^2 g_{xx} + f\right] dt + \sigma g_x dB(t).
\end{aligned}$$

The PDE and boundary condition are, therefore,

$$g_t(t,x) + b(t,x)g_x(t,x) + \frac{1}{2}\sigma^2(t,x)g_{xx}(t,x) = -f(x); \ g(T,x) = h(x).$$

Problem 4 (from PDE to SDE)

(a) $\mathcal{A} = (2-x)\frac{\partial}{\partial x} + 2x^2\frac{\partial^2}{\partial x^2}$ comes from the diffusion process

$$dX(t) = \beta(X(t))dt + \gamma(X(t))dW(t)$$

where $f(t, X(t)$ is a martingale, and so

$$df(t,X(t)) = f_t dt + f_x dX(t) + \frac{1}{2}f_{xx}d[X,X](t) = \left(f_t + \beta f_x + \frac{1}{2}\gamma^2 f_{xx}\right)dt + \gamma f_x dW(t);$$
$$f_t + \beta f_x + \frac{1}{2}\gamma^2 f_{xx} = 0.$$

We know the generator fits the form

$$\mathcal{A} = \beta(x)\frac{\partial}{\partial x} + \frac{1}{2}\gamma^2(x)\frac{\partial^2}{\partial x^2}$$

which gives us $\beta(x) = 2-x$ and $\gamma(x) = 2x$. Thus, the SDE is

$$dX(t) = (2 - X(t))dt + 2X(t)dW(t).$$

We can solve this by applying Problem 2 with $a(u) = 2$, $b(u) = -1$, $\gamma(u) = 0$, and $\sigma(u) = 2$.

(b) The two-dimensional SDE from generator

$$\mathcal{A} = \frac{1+x^2}{2}\frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial x\partial y} + \frac{1}{2}\frac{\partial^2}{\partial y^2} + 2x\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x}$$

is of the form

$$d\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} \beta_1(X(t),Y(t)) \\ \beta_2(X(t),Y(t)) \end{pmatrix} dt + \begin{pmatrix} \gamma_{11}(X(t),Y(t)) & \gamma_{12}(X(t),Y(t)) \\ \gamma_{21}(X(t),Y(t)) & \gamma_{22}(X(t),Y(t)) \end{pmatrix}\begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix},$$

where $W(t) = (W_1(t), W_2(t))$ is a standard 2-dimensional Brownian motion. The generator gives us the unique first derivative drift functions

$$\beta_1(x,y) = 2y,\ \beta_2(x,y) = 2x$$

but the second derivative diffusion functions are not unique; the generator gives

$$\frac{1}{2}\begin{pmatrix} \gamma_{11}(x,y) & \gamma_{12}(x,y) \\ \gamma_{21}(x,y) & \gamma_{22}(x,y) \end{pmatrix}\begin{pmatrix} \gamma_{11}(x,y) & \gamma_{12}(x,y) \\ \gamma_{21}(x,y) & \gamma_{22}(x,y) \end{pmatrix}^T = \frac{1}{2}\begin{pmatrix} 1+x^2 & x \\ x & 1 \end{pmatrix}$$

which yields infinitely many solution for $\gamma_{ij}(x,y)$. Two reasonably solvable solutions are

$$\begin{pmatrix} \gamma_{11}(x,y) & \gamma_{12}(x,y) \\ \gamma_{21}(x,y) & \gamma_{22}(x,y) \end{pmatrix} = \begin{pmatrix} \sqrt{1+x^2} & 0 \\ \frac{x}{\sqrt{1+x^2}} & \frac{1}{\sqrt{1+x^2}} \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

yielding the SDE systems

$$dX(t) = 2Y(t)dt + \sqrt{1+X^2(t)}dW_1(t)$$
$$dY(t) = 2X(t)dt + \frac{X(t)}{\sqrt{1+X^2(t)}}dW_1(t) + \frac{1}{\sqrt{1+X^2(t)}}dW_2(t)$$

and

$$dX(t) = 2Y(t)dt + dW_1(t) + X(t)dW_2(t)$$
$$dY(t) = 2X(t)dt + dW_2(t).$$

Problem 5 (F-K + Girsanov)

The Feynman-Kac theorem states that the PDE with terminal condition

$$g_t(u,x) + \beta(u,x)g_x(u,x) + \frac{1}{2}\gamma^2(u,x)g_{xx}(u,x) = 0, \; g(T,x) = h(x), \; t \le u \le T$$

is associated with the SDE

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u), \; X(t) = x,$$

where W is a standard Brownian motion on the measure P (using E for expected values). (This setup is just like in Problems 3 and 4, except here we're only ranging time over $u \in [t,T]$.) Here we are given $\beta(u,x) = \theta(x)$ and $\gamma(u,x) \equiv 1$, so the SDE reduces to

$$dX(u) = \theta(X(u))du + dW(u), \; X(t) = x.$$

At this point, think about the structure of this SDE while considering Girsanov's theorem: the function $\theta(X(u))$ is an adapted function of u. Therefore, we can shift $X(u)$ in space and time in the following way, via Girsanov, and end up with a new Brownian motion on a new measure. Define

$$\tilde{B}(u) := X(u+t) - x, \; 0 \le u \le T-t.$$

Then, integrating the $dX(u)$ SDE, we get the translation

$$\begin{aligned}
&dX(u) = \theta(X(u))du + dW(u), \; X(t) = x, \; t \le u \le T \\
\implies &X(u) - x = \int_t^u \theta(X(v))dv + [W(u) - W(t)] \\
\implies &\tilde{B}(u) \overset{(d)}{=} \int_0^u \theta(x + \tilde{B}(v))dv + W(u), \; 0 \le u \le T-t;
\end{aligned}$$

where $\tilde{B}(u)$ is a standard Brownian motion on the measure $\tilde{P}$ ($\tilde{E}$). (We say "equal in distribution" (d) in the last equality because I've swapped out $W(u+t)$ for an increment. Since we're only taking an expected value, it's ok to use this.) How do we shift between measures? We need the Radon-Nikodym derivative $\frac{d\tilde{P}}{dP} = Z(T-t)$ from the process

$$\begin{aligned}
Z(u) &= \exp\left(-\int_0^u \theta(\tilde{B}(v) + x)dW(v) - \frac{1}{2}\int_0^u \theta^2(\tilde{B}(v) + x)dv \right), \; 0 \le u \le T-t \\
&= \exp\left(-\int_0^u \theta(\tilde{B}(v) + x)d\tilde{B}(v) + \frac{1}{2}\int_0^u \theta^2(\tilde{B}(v) + x)dv \right)
\end{aligned}$$

since $-dW(u) = -d\tilde{B}(v) + \theta(x + \tilde{B}(v))dv$. Finally, this yields the F-K solution for $g(t,x)$, with a change of measure from E with $W(t) = x$ to $\tilde{E}$ with $\tilde{B}(0) = 0$, and at the end a change back to E since we're talking about a standard Brownian motion inside an expectation, so the notation no longer matters:

$$\begin{aligned}
g(t,x) &= E^{t,x}(h(X(T)) = \tilde{E}^{0,0}(Z^{-1}(T-t)h(x + \tilde{B}(T-t))) \\
&= \tilde{E}^{0,0}\left(\exp\left(-\int_0^{T-t} \theta(\tilde{B}(v) + x)d\tilde{B}(v) + \frac{1}{2}\int_0^{T-t} \theta^2(\tilde{B}(v) + x)dv \right) h(x + \tilde{B}(T-t)) \right) \\
&= E^{t,x}\left(\exp\left(-\int_t^T \theta(W(v))dW(v) + \frac{1}{2}\int_t^u \theta^2(W(v))dv \right) h(W(T)) \right).
\end{aligned}$$

Problem 6 (Exercise 6.10: implying the volatility surface)

This is merely walking through some regular calculus integration-by-parts calculations. No need to reproduce a solution here.