Sampling, Replication, and z-transforms

PREFACE: A discrete-time signal is modeled as a sampled version of an underlying continuous-time signal. The consequence of sampling is replication of the signal spectrum. The sample-replication duality is symmetric in that a replicated signal has a sampled spectrum. This duality is explored in the first part of this homework.

The second part asks you to find the z-transform of the impulse responses that we have been studying in this class, along with the gain and first moments as computed via transform techniques.

PREPARATION: The impulse-comb impulse response is required below. Add to your collection of impulse-response generating functions by writing a function that returns a comb:

1. **Impulse Comb:** (N_period, N_window) Implements an impulse comb

$$h_{\text{comb}}[n] \equiv \sum_{k=-\infty}^{\infty} \delta\left(n - kN_p\right), \quad n \in [0, N_{\text{window}} - 1] \tag{1}$$

   Concretely, $h_{\text{comb}}[n]$ values are a vector of zeros except at locations kN_p, $k \in \mathbb{Z}$.

PROBLEMS:

1. **Sampling in Time:** On a computer it is a bit odd to sample a time series because all time series are necessarily discrete. Here we will build (discrete) impulse responses and then sample them at a longer period. For this part use N_window = 1024 and N_period = 4.

   (a) Ema impulse response: Generate h_{ema} with $N_{\text{eff}} = 128$.

   (b) Box impulse response: Generate h_{box} with $N_{\text{box}} = N_{\text{eff}}/\left(1 - e^{-1}\right)$.

   (c) Comb impulse response: Generate h_{comb} with $N_p = 4$ (that is, N_period = 4).

   (d) Sampled ema: Compute $h_{\text{ema}-\text{sampled}} \equiv h_{\text{comb}} \times h_{\text{ema}}$.

   (e) Sampled box: Compute $h_{\text{box}-\text{sampled}} \equiv h_{\text{comb}} \times h_{\text{box}}$.

   (f) Ema and Box amplitude spectra: Compute $H_{\text{ema}}(\omega) \equiv \mathcal{F}\left(h_{\text{ema}}[n]\right)$ and keep the magnitude $|H_{\text{ema}}(\omega)|$. Likewise, compute $|H_{\text{box}}(\omega)|$, $|H_{\text{ema}-\text{sampled}}(\omega)|$ and $|H_{\text{box}-\text{sampled}}(\omega)|$.

   (g) Spectral comparison: Referring to Eq. (3.40) of my lecture notes[1] there is a $2\pi/T$ coefficient to deal with. For the "continuous-time" responses $h_*[n]$ the period is $T_{ct} = 1/1024$, whereas for the sampled responses the period is $T_s = 4/1024$. We wish to compare $\sum_{n=-\infty}^{\infty} X\left(\omega - \frac{2\pi n}{T}\right)$

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for the "continuous" and sampled results. Multiplying through, the fair comparison is

$$T_{ct} H_{\text{comb}}(\omega; T_{ct}) * X(\omega) \sim T_s H_{\text{comb}}(\omega; T_s) * X(\omega) \tag{2}$$

(h) Plots: On one plot overlay h_{ema} and $h_{\text{ema}-\text{sampled}}$. Make a similar overlay for h_{box} and $h_{\text{box}-\text{sampled}}$ on another plot. On a third plot overlay $|H_{\text{ema}}(\omega)|$ with $T_s\,|H_{\text{ema}-\text{sampled}}(\omega)|$. Repeat for the box spectra on a fourth plot.

(i) Remark on the sample period and the period of spectral replications.

2. **Replication in Time:** The last problem showed that sampling in one domain (time) induces replication in the other (frequency). Here the reverse is explored: replication in one domain (time) induces sampling in the other (frequency). For this work select `N_period = N_window / 4`.

   (a) Generate h_{ema} with $N_{\text{eff}} = 32$ and h_{box} with $N_{\text{box}} = N_{\text{eff}}/(1 - e^{-1})$.

   (b) Generate h_{comb} with the new period, $N_p = 256$.

   (c) Replicate these impulse responses by convolving with the comb. Unlike previous work, directly compute

   $$\text{cand} = \text{conv}\left(h_{\text{comb}}, h_*\right)$$

   then

   $$h_{*-\text{repl}} = \text{cand}(1 : \text{length}(h_*))$$

   That is, do not subtract the first sample and then add it back in.

   (d) Compute the Fourier transforms $|H_{\text{ema}}(\omega)|$, $|H_{\text{box}}(\omega)|$, $|H_{\text{ema}-\text{repl}}(\omega)|$ and $|H_{\text{box}-\text{repl}}(\omega)|$.

   (e) Note that following the argument in part (1g), fair comparison of the spectra requires the replicated spectra to be multiplied by T_s^{-1}, not T_s.

   (f) Plot $h_{\text{ema}-\text{repl}}$ and $h_{\text{box}-\text{repl}}$ on two figures. Overlay $|H_{\text{ema}}(\omega)|$ and $T_s^{-1}\,|H_{\text{ema}-\text{repl}}(\omega)|$ on another figure, and do the same with the box spectra.

   (g) Comment on the duality between sampling and replication.

3. ***z*-transforms:** The following are exercises to familiarize you with the z-transform representation of most of the impulse responses we have focused on to date. While it would be best for you to derive the transforms explicitly, there is value in simply knowing where to look. The excellent wiki page for z-transforms is http://en.wikipedia.org/wiki/Z−transform.

   Give the z-transform $H(z)$ of each impulse response. When the gain and/or first-moment are asked for, compute the answer in terms of the z-transform. Recall that the gain and first moment in terms of the z-transform are

$$g \equiv H(z = +1) \tag{3}$$

   and

$$M_1 \equiv \frac{d}{d\zeta}\, H(\zeta)\Big|_{\zeta=+1} \tag{4}$$

   where $\zeta \equiv z^{-1}$.

   (a) Delayed Impulse: $h[n] = \delta[n - N]$.

   (b) Box: $h_{\text{box}}[n] = N_{\text{box}}^{-1}\,(u[n] - \delta[n - N_{\text{box}}] * u[n])$. Compute the gain and first moment.

   (c) Unit Step: $h[n] = u[n]$. Compute the gain.

   (d) Ema: $h_{\text{ema}}[n] = (1 - p)p^n u[n]$. Compute the gain and first moment. Denote the first moment as M_1. Compute the decay rate p in terms of M_1.

   (e) Ema-Poly1: $h_{\text{poly}}[n] = (1 - p)^2(n + 1)p^n u[n]$. Compute the gain and first moment. Compute p in terms of M_1.

   (f) Lifted Macd-Poly:

$$h_{\text{lift}}[n] = (2N_{\text{eff}}/3)^{-1}\, u[n] * (h_{\text{ema}}(N_{\text{eff}}/3)[n] - h_{\text{poly}}(N_{\text{eff}})[n])$$

   Compute the gain. If you have access to Mathematica, compute the first moment in terms of N_{eff}, where $p_{\text{ema}} = (N_{\text{eff}}/3)/(N_{\text{eff}}/3 + 1)$ and $p_{\text{poly}} = (N_{\text{eff}}/2)/(N_{\text{eff}}/2 + 1)$. The answer is very compact.

   (g) Macd: $h_{\text{macd}}[n] = h_{\text{ema}}(N_{\text{eff}\,+})[n] - h_{\text{ema}}(N_{\text{eff}\,-})[n]$. Compute the gain.

   (h) Macd-M1: $h^{(M_1)}_{\text{macd}}[n] = \delta[n - N_{\text{eff}\,+}] - \delta[n - N_{\text{eff}\,-}]$. Compute the gain.