MTH 9831. Solutions to Quiz 1.

(1) State one of the equivalent definitions of Brownian motion.

Please see Lecture 1.

(2) Let $(B(t))_{t\geq 0}$ be a standard Brownian motion. Given that $B(1) = x$ and $B(1/2) = y$, find the distribution of $B(3/4)$.

Solution.[1] When dealing with Gaussian vectors it is often much simpler and faster to use linear algebra than density calculations.

We need to find the conditional distribution of $B(3/4)$ given that $B(1/2) = y$ and $B(1) = x$. The key observation is that this distribution is the same as the one for $y + \widetilde{B}(1/4)$ given that $\widetilde{B}(1/2) = x - y$, where $\widetilde{B}(t)$, $t \geq 0$, is a standard Brownian motion.

Then we write (as was discussed in refresher and then reviewed and emphasized again in class)

$$\begin{aligned} \widetilde{B}(1/4) &= c\,\widetilde{B}(1/2) + W, \text{ where } W \text{ is independent from } \widetilde{B}(1/2). \\ \operatorname{Cov}(\widetilde{B}(1/2), \widetilde{B}(1/4)) &= c\operatorname{Var}(\widetilde{B}(1/2)) \;\Rightarrow\; c = 1/2. \end{aligned}$$

Computing variances in the first line we get

$$\begin{aligned} 1/4 &= c^2/2 + \operatorname{Var}(W) \;\Rightarrow\; \operatorname{Var}(W) = 1/8 \text{ and conclude that} \\ \widetilde{B}(1/4) &= 1/2\,\widetilde{B}(1/2) + W, \text{ where } W \sim N(0, 1/8). \end{aligned}$$

Since W is independent[2] from $\widetilde{B}(1/2)$, conditioning on $\widetilde{B}(1/2) = x - y$ does not change the distribution of W. Thus, we get that, conditionally on $\widetilde{B}(1/2) = x - y$,

$$\widetilde{B}(1/4) = (x - y)/2 + W.$$

From this we see that the distribution of $y + \widetilde{B}(1/4)$ is normal with mean $(x + y)/2$ and variance $1/8$.

Remark. Let's take a closer look at the key observation. The best way to do this, in fact, is to draw a picture. The following manipulations are based on the definition of Brownian motion.

$$\begin{aligned} B(3/4) \,|\, B(1/2) &= y,\; B(1) = x \\ B(1/2) + (B(3/4) - B(1/2)) \,|\, B(1/2) &= y,\; B(1) - B(1/2) = x - y \\ y + (B(3/4) - B(1/2)) \,|\, B(1) - B(1/2) &= x - y \quad (B(3/4) - B(1/2) \text{ is independent from } B(1/2)) \\ y + \widetilde{B}(1/4) \,|\, \widetilde{B}(1/2) &= x - y, \quad \text{where} \end{aligned}$$

$\widetilde{B}(t) := B(t + 1/2) - B(1/2)$, $t \geq 0$, is a standard Brownian motion.

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[1] The solution below is much more detailed than I expect to see in your quiz. A simple computation for problem 2 would suffice for a full credit.

[2] Since the joint distribution of W and $\widetilde{B}(1/2)$ is normal, independence follows from the fact that W and $B(1/2)$ are uncorrelated.