Comments/Solutions, MTH 9831 Fall 2015 HW #11
Baruch College MFE Program

Please note that these are *possible* solutions, and that there are often many ways to approach and solve a problem or prove a theorem.

We start with $N(t) \sim \text{Poisson}(\lambda)$, $M(t) = N(t) - \lambda t$ a compensated Poisson process, $Q(t)$ a compound Poisson process with jump distribution $P(Y_1 = y_m) = p_m$, $m = 1, 2, ..., M$, and $M_Q(t) = Q(t) - \beta\lambda t$ (where $\beta = E(Y_1)$) is the compensated process for Q.

Problem 1 (quadratic variation of compound Poissons)

Letting $\Delta Q(s) = Q(s) - Q(s-)$, we have

$$\begin{aligned}
[Q,Q](t) &= \sum_{0<s\leq t} (\Delta Q(s))^2 = \sum_{j=1}^{N(t)} Y_j^2 \\
E([Q,Q](t)) &= E\left(\sum_{j=1}^{N(t)} Y_j^2\right) = E(N(t))E(Y_1^2) \qquad \text{(by Wald's identity)} \\
&= (\sigma^2 + \beta^2)\lambda t \qquad (\text{setting } \sigma^2 = Var(Y_1) = E(Y_1^2) - E(Y_1)^2) \\
[M_Q, M_Q](t) &= [M_Q^c, M_Q^c](t) + [J,J](t) \qquad (M_Q^c(t) = -\beta\lambda t,\ J(t) = Q(t)) \\
&= [Q,Q](t).
\end{aligned}$$

Problem 2 (Itô isometry with jumps, compensated Poisson)

The first part, showing that for ϕ a left-continuous square-integrable $M(t)$-adapted process,

$$E\left[\left(\int_0^t \phi(s)dM(s)\right)^2\right] = \lambda E\left[\int_0^t \phi^2(s)ds\right],$$

is a special case of Problem 3 with $Q(t) = N(t)$. Hence, we know that $\int_0^t \phi(s)dM(s)$ is a martingale starting at 0 since $M(t)$ is a martingale, and so

$$\begin{aligned}
Var\left(\int_0^t 2^{M(s-)}dM(s)\right) &= E\left[\left(\int_0^t 2^{M(s-)}dM(s)\right)^2\right] \\
&= \lambda E\left[\int_0^t 4^{M(s-)}ds\right] \\
&= \lambda E\left[\int_0^t e^{\ln(4)M(s-)}ds\right] = \lambda E\left[\int_0^t e^{\ln(4)(N(s-)-\lambda s)}ds\right].
\end{aligned}$$

At this point, we argue that the Lebesgue integrals $\int_0^t e^{\ln(4)(N(s-)-\lambda s)}ds = \int_0^t e^{\ln(4)(N(s)-\lambda s)}ds$ since $e^{\ln(4)N(s-)}$ and $e^{\ln(4)N(s)}$ only differ at finitely many points a.s., so their ds integrals match. We swap this out and use the MGF of $N(s)$, $E(e^{uN(s)}) = e^{\lambda s(e^u - 1)}$, to get

$$\begin{aligned}
Var\left(\int_0^t 2^{M(s-)}dM(s)\right) &= \lambda \int_0^t e^{\ln(4)(-\lambda s)}E(e^{\ln(4)N(s)})ds \\
&= \lambda \int_0^t e^{\ln(4)(-\lambda s)}e^{3\lambda s}ds \\
&= \lambda \int_0^t e^{(3-\ln(4))(\lambda s)}ds = \frac{e^{(3-\ln(4))\lambda t} - 1}{3 - \ln(4)}.
\end{aligned}$$

Problem 3 (Itô isometry with jumps, compound compensated Poisson)

We need to show that

$$E\left[\left(\int_0^t \phi(s)dM_Q(s)\right)^2\right] = \lambda E(Y_1^2)E\left[\int_0^t \phi^2(s)ds\right].$$

We'll start with the differential

$$X(t) = \int_0^t \phi(s)dM_Q(s),\ dX(t) = \phi(t)dM_Q(t),\ X(0) = 0.$$

Then differentiate $X^2(t)$:

$$\begin{aligned} &d(X^2(t)) = 2X(t)dX(t) + d[X,X](t) \\ \implies &X^2(t) = 2\int_0^t X(s)dX(s) + \int_0^t d[X,X](s) \quad = 2\int_0^t X(s-)dX(s) + \sum_{0<s\le t} \Delta X(s)^2 \\ &\qquad = 2\int_0^t X(s-)dX(s) + \sum_{j=1}^{N(t)} Y_j^2 \int_{S_{j-1}}^{S_j\wedge t} \phi^2(s)dN(s), \end{aligned}$$

where S_j is the time of the jth jump of N, $S_0 = 0$ (I'm being a little flaky on the notation here, but $S_j - S_{j-1} \sim \exp(\lambda)$ for each j, IID). Since the Y_j are IID, we'll be able to combine all these integrals into one integral in the expectation.

$M_Q(t)$ is a martingale, so $X(t)$ is a martingale, so $\int_0^t X(s-)dX(s)$ is a martingale since $X(s-)$ is left-continuous. Hence, since the Y_j are IID and independent of $N(t)$, and $N(t) = M(t) + \lambda t$ where $M(t)$ is a martingale,

$$\begin{aligned} E\left[X^2(t)\right] = E\left[\left(\int_0^t \phi(s)dM_Q(s)\right)^2\right] &= E\left[\sum_{j=1}^{N(t)} Y_j^2 \int_{S_{j-1}}^{S_j\wedge t} \phi^2(s)dN(s)\right] \\ &= E\left[Y_1^2\right]E\left[\int_0^t \phi^2(s)dN(s)\right] = \lambda E\left[Y_1^2\right]E\left[\int_0^t \phi^2(s)ds\right]. \end{aligned}$$

Problem 4 Shreve Vol. 2, Ch. 11 #11.6, p. 526 (BM and compound Poisson are independent)

We'll do the same thing as in #11.5 from the previous homework set: let

$$\begin{aligned} X(t) &= u_1W(t) - \frac{1}{2}u_1^2t + u_2Q(t) - \lambda t(\phi(u_2)-1) &&= X^c(t) + J(t); \\ X^c(t) &= u_1W(t) - \frac{1}{2}u_1^2t - \lambda t(\phi(u_2)-1); &&J(t) = u_2Q(t); \end{aligned}$$

where ϕ is the MGF of $Y(t)$, so that $Y(t) = e^{X(t)}$. We again show that $Y(t)$ is a martingale.

First, the continuous terms:

$$dX^c(t) = u_1dW_t + \left[-\frac{1}{2}u_1^2 - \lambda(\phi(u_2)-1)\right]dt; \qquad d[X^c,X^c](t) = u_1^2dt.$$

Then the jump terms:

$$\begin{aligned} \Delta X(t) &= \Delta J(t) = u_2\Delta Q(t) = u_2Y_{N(t)}\Delta N(t) \\ \Delta e^{X(t)} &= e^{X(t)} - e^{X(t-)} = e^{X(t-)}(e^{\Delta X(t)}-1) = e^{X(t-)}(e^{u_2Y_{N(t)}}-1)\Delta N(t). \end{aligned}$$

We'll consider the compound Poisson process $H(t) = \sum_{j=1}^{N(t)}(e^{u_2 Y_j} - 1)$, which looks just like the above, and so we combine via Itô to give us, for any $s \geq 0$,

$$\begin{aligned}
\Delta H(u) &= dH(u) = (e^{u_2 Y_{N(u)}} - 1)\Delta N(u) \\
H(t) &- \lambda(\phi(u_2) - 1)t \text{ is a martingale} \\
Y(t) = e^{X(t)} &= e^{X(s)} + \int_s^t e^{X(u-)} dX^c(u) + \frac{1}{2}\int_s^t e^{X(u-)} d[X^c, X^c](u) \\
&\quad + \sum_{s<u\leq t} e^{X(u-)}(e^{u_2 Y_{N(u)}} - 1)\Delta N(t) \\
&= e^{X(s)} + \int_s^t e^{X(u-)} u_1 dW_u - \int_s^t e^{X(u-)}\left[\frac{1}{2}u_1^2 + \lambda(\phi(u_2) - 1)\right] du \\
&\quad + \int_s^t e^{X(u-)} \frac{1}{2} u_1^2 du + \int_s^t e^{X(u-)} dH(u) \\
&= e^{X(s)} + \int_s^t e^{X(u-)} u_1 dW_u + \int_s^t e^{X(u-)} d\left[H(u) - \lambda(\phi(u_2) - 1)u\right].
\end{aligned}$$

Thus, $Y(t)$ is a martingale, so by the optional stopping theorem,

$$\begin{aligned}
&E[e^{u_1 N_1(t) + u_2 N_2(t) - \lambda_1 t(e^{u_1} - 1) - \lambda_2 t(e^{u_2} - 1)}] = E[Y(0)] = 1 \\
&\quad\implies E[e^{u_1 N_1(t) + u_2 N_2(t)}] = e^{\lambda_1 t(e^{u_1} - 1)} e^{\lambda_2 t(e^{u_2} - 1)} = E[e^{u_1 N_1(t)}]E[e^{u_2 N_2(t)}],
\end{aligned}$$

which implies that $N_1(t)$ and $N_2(t)$ are independent.

Problem 5 (change of measure for stock price, with jumps)

We are given the SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) + S(t-)dQ(t).$$

In the no-jumps case (where $y_m = 0$ for all m, for instance, so that $Q(t) = 0$ and $\beta = 0$), we merely change the measure via the market price of risk process $\theta = \theta(t) = \frac{\mu - r}{\sigma}$. Girsanov's theorem then says that, for fixed $T > 0$,

$$\begin{aligned}
\tilde{B}(t) &= B(t) + \int_0^t \theta(s)ds = B(t) + \left(\frac{\mu - r}{\sigma}\right)t \\
Z(t) &= \exp\left[-\int_0^t \theta(s)dB(s) - \frac{1}{2}\int_0^t \theta(s)^2 ds\right] = \exp\left[-\theta B(t) - \frac{1}{2}\theta^2 t\right], \ 0 \leq t \leq T
\end{aligned}$$

yields the $\tilde{P}$-martingale ($\tilde{P}(A) = \int_A Z(T)dP$ for $A \in \mathcal{F}(T)$)

$$\begin{aligned}
d(e^{-rt}S(t)) &= e^{-rt}\left(-rS(t)dt + dS(t)\right) \\
&= e^{-rt}\left((\mu - r)S(t)dt + \sigma S(t)dB(t)\right) \\
&= e^{-rt}\left((\mu - r)S(t)dt + \sigma S(t)d\tilde{B}(t) - \sigma\theta(t)dt\right) = e^{-rt}S(t)d\tilde{B}(t).
\end{aligned}$$

Rewriting the SDE in terms of the martingale $M_Q(t)$ instead of $Q(t)$ yields (where, on the continuous measure dt, we can swap $S(t)$ with $S(t-)$)

$$\begin{aligned}
dS(t) &= \mu S(t)dt + \sigma S(t)dB(t) + S(t-)d[Q(t) - \beta\lambda t + \beta\lambda t] \\
&= (\mu + \beta\lambda)S(t)dt + \sigma S(t)dB(t) + S(t-)dM_Q(t).
\end{aligned}$$

With jumps in play, we still need the dt term to be zero to make $e^{-rt}S(t)$ a $\tilde{P}$-martingale. To change measure, we can change the Poisson intensity from λ to $\tilde{\lambda}$, and change the probabilities of Y_1 to $\tilde{p}_m = \tilde{P}(Y_1 = y_m)$ so that $\tilde{\beta} = \tilde{E}(Y_1) = \sum_{m=1}^{M} \tilde{p}_m y_m$. However, we cannot change the y_m. We have

$$\begin{aligned}
\tilde{B}(t) &= B(t) + \int_0^t \theta(s)ds = B(t) + \theta t \\
\tilde{M}_Q(t) &= Q(t) - \tilde{\beta}\tilde{\lambda}t
\end{aligned}$$

with the constant θ to be determined, yielding the $\tilde{P}$-martingale

$$\begin{aligned}
d(e^{-rt}S(t)) &= e^{-rt}\left(-rS(t)dt + dS(t)\right) \\
&= e^{-rt}\left((\mu - r)S(t)dt + \sigma S(t)dB(t) + S(t-)dQ(t)\right) \\
&= e^{-rt}\left((\mu - r - \sigma\theta + \tilde{\beta}\tilde{\lambda})S(t)dt + \sigma S(t)d\tilde{B}(t) + S(t-)d\tilde{M}_Q(t)\right),
\end{aligned}$$

which is only a martingale if the dt term is zero, with

$$\theta = \frac{\mu - r + \tilde{\beta}\tilde{\lambda}}{\sigma}.$$

Now, defining $\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m$, where $\tilde{p}_m = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}$ is the change of measure on Y_1 (from $p_m = \frac{\lambda_m}{\lambda}$). Thus, our change of measure process changes $B(t)$ to $\tilde{B}(t)$ and $M_Q(t)$ to $\tilde{M}_Q(t)$:

$$Z(t) = \exp\left[-\theta B(t) - \frac{1}{2}\theta(s)^2 t + (\lambda - \tilde{\lambda})t\right] \prod_{j=1}^{N(t)} \frac{\tilde{\lambda}\tilde{p}(Y_j)}{\lambda p(Y_j)}$$

so that, for fixed $T > 0$, $\tilde{P}(A) = \int_A Z(T)dP$, $A \in \mathcal{F}(T)$ makes $e^{-rt}S(t)$ a $\tilde{P}$-martingale.

Problem 6 (price a BSM-Euro call with jumps)

We found the risk-neutral measure in #5 above, so we are granted the risk-neutral measure here: we are given the stock price model $S(t) = S^*(t)e^{Q(t)}$ with

$$S^*(t) = S(0)e^{\sigma B(t) + \mu t},\ \mu = r - \frac{\sigma^2}{2} - \lambda(E(e^{Y_1}) - 1);\ Q(t) = \sum_{j=1}^{N(t)} Y_j,\ Y_j \sim N(\mu_0, \sigma_0^2).$$

Thus, the time-zero cost of a European call option with strike price K and expiration t is simply the risk-neutral discounted expected value of the payoff function, as usual:

$$C(0) = e^{-rt}E[(S(t) - K)^+].$$

To calculate this expected value, we can condition on the number of jumps up to time t, i.e. on the event $\{N(t) = n\}$. If $N(t) = n$, then $Q(t)$ is a normal random variable, independent of $B(t)$, and so we can add all the normals together and scale to a standard normal Z to find when $S(t) > K$

probabilistically.

$$
\begin{aligned}
N(t)=n, S(t)>K &\implies \mu t+\sigma B(t)+Q(t)>\ln\left(\frac{K}{S(0)}\right) \\
&\implies \mu t+\left[\sigma B(t)+\sum_{j=1}^{N(t)=n} Y_j\right]>\ln\left(\frac{K}{S(0)}\right) && \text{(sum of n+1 indep. normals)} \\
&\implies \mu t+\mu_0 n+Z\sqrt{\sigma_0^2 n+\sigma^2 t}>\ln\left(\frac{K}{S(0)}\right) && (Z\sim N(0,1)) \\
&\implies Z>\frac{\ln\left(\frac{K}{S(0)}\right)-(\mu t+\mu_o n)}{\sqrt{\sigma_0^2 n+\sigma^2 t}} \\
&\implies Z' \overset{(d)}{=} -Z<\frac{\ln\left(\frac{S(0)}{K}\right)+(\mu t+\mu_o n)}{\sqrt{\sigma_0^2 n+\sigma^2 t}} =: d_-(n) && (Z'\sim N(0,1)).
\end{aligned}
$$

Thus, the price of the call option is

$$
\begin{aligned}
C(0) &= e^{-rt}E[(S(t)-K)^+] \\
&= e^{-rt}\sum_{n=0}^{\infty} E[(S(t)-K)^+ \mid N(t)=n]P(N(t)=n) \\
&= \frac{e^{-rt-\lambda}}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\int_{-\infty}^{d_-(n)}\left(S(0)e^{\mu t+\mu_0 n-x\sqrt{\sigma_0^2 n+\sigma^2 t}}-K\right)e^{-\frac{x^2}{2}}\,dx \\
&= e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left(S(0)N(d_+(n))-Ke^{-rt}N(d_-(n))\right),
\end{aligned}
$$

Note that setting $\lambda=0$ kills off all the jumps and reduces this to the $n=0$ case (using $0^0=1$), which is the usual BSM-Euro call price.