Problem 1. FDE Reprensentation

<u>note.</u> $z$-transforms were derived in $\boxed{\text{Homework \#4}}$.

a) Delayed impulses : $h[n] = \delta[n-N]$

$H(z) = z^{-N}$

choose $P(z) = z^{-N}$, $Q(z) = 1$

$\therefore Y(z) = z^{-N} X(z)$

$\therefore \boxed{y[n] = x[n-N]}$.

b) Unit Step : $h[n] = u[n]$, $H(z) = \frac{1}{1-z^{-1}}$

choose $P(z) = 1$, $Q(z) = 1 - z^{-1}$

$\therefore (1-z^{-1}) Y(z) = X(z)$

$\therefore y[n] - y[n-1] = x[n] \Rightarrow \boxed{y[n] = y[n-1] + x[n]}$.

c) Box : $h[n] = N^{-1}\left(u[n] - \delta[n-N] * u[n]\right) = N^{-1}\left(u[n] - u[n-N]\right)$

$H(z) = N^{-1} \frac{1-z^{-N}}{1-z^{-1}}$

choose $P(z) = 1 - z^{-N}$, $Q(z) = 1 - z^{-1}$

$\therefore (1-z^{-1}) Y(z) = \frac{1}{N}(1 - z^{-N}) X(z)$

$\therefore y[n] - y[n-1] = x[n] - x[n-N] \Rightarrow \boxed{y[n] = y[n-1] + \left(x[n] - x[n-N]\right)\frac{1}{N}}$

d) Ema : $h[n] = (1-p)p^n u[n]$, $H(z) = \frac{1-p}{1-pz^{-1}}$

choose $P(z) = 1-p$, $Q(z) = 1 - pz^{-1}$

$\therefore (1-pz^{-1}) Y(z) = (1-p) X(z)$

$\therefore y[n] - p y[n-1] = (1-p) x[n] \Rightarrow \boxed{y[n] = p y[n-1] + (1-p) x[n]}$

e) Ema-Poly1 : $h[n]=(1-p)^2(n+1)p^n u[n]$ , $H(z)=\dfrac{(1-p)^2}{(1-pz^{-1})^2}$

choose $P(z)=(1-p)^2$ , $Q(z)=(1-pz^{-1})^2=1-2pz^{-1}+p^2z^{-2}$

$\therefore (1-2pz^{-1}+p^2z^{-2})Y(z)=(1-p)^2X(z)$

$\therefore y[n]-2p\,y[n-1]+p^2y[n-2]=(1-p)^2x[n]$

$$\therefore \boxed{y[n]=2p\,y[n-1]-p^2y[n-2]+(1-p)^2x[n]}$$

d) Integrated Macd-Poly : ~~$h[n]=(1-p)^2$~~

$$h[n]=\left(\frac{2N}{3}\right)^{-1}u[n]*\left(h_{ema}\left(\tfrac{1}{3}N\right)[n]-h_{poly}(N)[n]\right)$$

$$H(z)=\frac{(2N/3)^{-1}}{1-z^{-1}}\left(\frac{1-p}{1-pz^{-1}}-\frac{(1-p')^2}{(1-p'z^{-1})^2}\right)=\left(\frac{2N}{3}\right)^{-1}\times\frac{(2p'-p'^2-p)-p'^2(1-p)z^{-1}}{1-(2p'+p)z^{-1}+p'(p'+2p)z^{-2}-p'^2pz^{-3}}$$

$\boxed{\text{note:}}$ $(1-p)(1-p'z^{-1})^2-(1-p')^2(1-pz^{-1})=(1-z^{-1})[2p'-p'^2-p-(p'^2-pp')z^{-1}]$

Choose: $P(z)=\left(\frac{2}{3}N\right)^{-1}\left[(2p'-p'^2-p)-p'^2(1-p)z^{-1}\right]$

$Q(z)=1-(2p'+p)z^{-1}+p'(p'+2p)z^{-2}-p'^2pz^{-3}$

$$\therefore \left[1-(2p'+p)z^{-1}+p'(p'+2p)z^{-2}-p'^2pz^{-3}\right]Y(z)$$
$$=\left(\tfrac{2}{3}N\right)^{-1}\left[(2p'-p'^2-p)-p'^2(1-p)z^{-1}\right]X(z)$$

$$\therefore y[n]-(2p'+p)y[n-1]+p'(p'+2p)y[n-2]-p'^2p\,y[n-3]$$
$$=\left(\tfrac{2}{3}N\right)^{-1}\left[(2p'-p'^2-p)x[n]-p'^2(1-p)x[n-1]\right]$$

$$\therefore \boxed{\begin{aligned}y[n]=&(2p'+p)y[n-1]-p'(p'+2p)y[n-2]+p'^2p\,y[n-3]\\&+\left(\tfrac{2}{3}N\right)^{-1}\left[(2p'-p'^2-p)x[n]-p'^2(1-p)x[n-1]\right]\end{aligned}}$$

g) Macd: $h[n]=h_{ema}(N_{eff}^{+})[n]-h_{ema}(N_{eff}^{-})[n]$

$$H(z)=\frac{1-p_+}{1-p_+z^{-1}}-\frac{1-p_-}{1-p_-z^{-1}},\quad \text{where } p_\pm=\frac{\frac{1}{3}N_{eff}^{\pm}}{\frac{1}{3}N_{eff}^{\pm}+1}.$$

$$=\frac{-(p_+-p_-)+(p_+-p_-)z^{-1}}{1-(p_++p_-)z^{-1}+p_+p_-z^{-2}}$$

choose $P(z)=-(p_+-p_-)+(p_+-p_-)z^{-1}$, $Q(z)=1-(p_++p_-)z^{-1}+p_+p_-z^{-2}$

$\therefore [1-(p_++p_-)z^{-1}+p_+p_-z^{-2}]Y(z)=-(p_+-p_-)(1-z^{-1})X(z)$

$\therefore y[n]-(p_++p_-)y[n-1]+p_+p_-y[n-2]=-(p_+-p_-)(x[n]-x[n-1])$

$$\therefore \boxed{y[n]=(p_++p_-)y[n-1]-p_+p_-y[n-2]-(p_+-p_-)(x[n]-x[n-1])}$$