Interest Rate Models

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Outline

1. Term structure modelling
2. Vasicek's model and its descendants
3. Application: ED / FRA convexity corrections
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Term structure modelling

- Instruments based on interest rates depend on a time interval and not only on one particular time in future.
- The appropriate models must describe a stochastic structure of the entire forward curve.
- There exist many term structure models.We will focus on the following two categories:
  - (i) *Short rate models* assume that the stochastic variable is the instantaneous forward rate. These were the earliest successful term structure models. The easiest to understand is a Vasicek's model.
  - (ii) *LIBOR market model* considers the entire forward curve as a stochastic state variable. These are descendants of the Heath - Jarrow - Morton (HJM) model and were developed in late 80s.

Term structure modelling

- In short rate models the instantaneous spot rate $r(t)$ is the basic state variable.
- In LIBOR/OIS framework this rate is defined as $r(t) = f(t, t)$, where $f(t, s)$ denotes the instantaneous discount OIS rate that we defined as

$$f(t, s) = -\frac{\partial}{\partial s} \log P(t, S, s).$$

- The dynamics of $r(t)$ is modeled by a number of Brownian motions. Depending on the number of these stochastic drivers, we categorize these models as one-, two-, or three-factor models. The SDEs are stated under the spot measure.

Vasicek's model

- In one factor models the SDE has the form

$$dr(t) = \mu(t, r(t))\, dt + \sigma(t, r(t))\, dW(t).$$

- Various choices for μ and σ have been studied.
- In multi-factor models $r(t)$ is represented as a sum of a deterministic component and several stochastic components. The factors are chosen in such a way that the obtained dynamics looks similar to the observed interest rate curve.
- The simplest is the Vasicek's model. Under the spot measure $\mathbb{Q}_0$ the model is give as

$$dr(t) = \lambda \left(\mu - r(t)\right)\, dt + \sigma dW(t), \quad r(0) = r_0.$$

- This is famous Ornstein-Uhlenbeck process.

Vasicek's model

- The SDE

$$dr(t) = \lambda\left(\mu - r(t)\right)\,dt + \sigma dW(t), \quad r(0) = r_0$$

can be simplified by multiplying both sides by $e^{\lambda t}$:

$$e^{\lambda t}dr(t) + \lambda e^{\lambda t}r(t)\,dt = \mu\lambda e^{\lambda t}\,dt + \sigma e^{\lambda t}\,dW(t).$$

- We now get

$$\begin{aligned} d\left(e^{\lambda t}r(t)\right) &= \mu\lambda e^{\lambda t}\,dt + \sigma e^{\lambda t}\,dW(t). \\ e^{\lambda t}r(t) &= C + \mu e^{\lambda t} + \sigma\int_0^t e^{\lambda s}\,dW(s). \end{aligned}$$

- From the condition $r(0) = r_0$ we obtain that $C = r_0 - \mu$.

Vasicek's model

- Thus in Vasicek's model we get

$$r(t) = \mu + (r_0 - \mu)\, e^{-\lambda t} + \sigma e^{-\lambda t} \int_0^t e^{\lambda s}\, dW(s).$$

- We can calculate the expected return and variance of $r(t)$.

$$\mathbb{E}\left[r(t)\right] = \mu + (r_0 - \mu)\, e^{-\lambda t} \qquad \operatorname{var}\left[r(t)\right] = \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda t}\right).$$

- As $t \to \infty$ we see that $r(t) \to \mu$ exponentially fast. This is called *mean reversion* of the short rate. The rate of mean reversion is λ and the time scale on which it happens is $\tau = \frac{1}{\lambda}$. Fluctuations are of the order of magnitude $\frac{\sigma}{\sqrt{2\lambda}}$.
- Strongly mean reverting processes have large λ which lead to small volatilities.

Shortcomings of Vasicek's model

- Impossible to fit the entire forward curve as the initial condition.
- Only one volatility parameter for calibration. Thus when the volatility structure is known the fitting is impossible.
- The model is one-factor.
- Rates may become negative with non-zero probability.
- Generalizations of the model can correct some of the above shortcomings.

One-factor Hull-White model

- We look at the following generalization in which the mean reversion is towards a time-dependent mean instead of a constant mean.

$$dr(t) \;=\; \left(\frac{d\mu(t)}{dt} + \lambda\left(\mu(t) - r(t)\right)\right) dt + \sigma(t)\, dW(t).$$

- Multiplying both sides by the integrating factor $e^{\lambda t}$ and rearranging gives us

$$\begin{aligned} d\left(e^{\lambda t} r(t)\right) &= d\left(e^{\lambda t}\mu(t)\right) + \sigma(t) e^{\lambda t}\, dW(t) \quad \text{and} \\ e^{\lambda t} r(t) &= C + e^{\lambda t}\mu(t) + \int_0^t \sigma(s) e^{\lambda s}\, dW(s). \end{aligned}$$

From $r(0) = r_0$ we obtain $C = r_0 - \mu(0)$.

One-factor Hull-White model

- The solution is

$$r(t) = r_0 e^{-\lambda t} + \mu(t) - \mu(0)e^{-\lambda t} + e^{-\lambda t} \int_0^t \sigma(s) e^{\lambda s} \, dW(s).$$

- We obtain the following expression for the expectation and variance of r.

$$\begin{aligned} \mathbb{E}\left[r(t)\right] &= r_0 e^{-\lambda t} + \mu(t) - \mu(0)e^{-\lambda t} \\ \mathrm{var}\left[r(t)\right] &= e^{-2\lambda t} \int_0^t \sigma^2(s) e^{2\lambda s} \, ds. \end{aligned}$$

- Since the solution is unique, we could write $r(t)$ as a solution starting from time s with the initial condition $r(s)$:

$$\begin{aligned} r(t) &= r(s)e^{-\lambda(t-s)} + \mu(t) - \mu(s)e^{-\lambda(t-s)} \\ &\quad + e^{-\lambda t} \int_s^t \sigma(u) e^{\lambda u} \, dW(u). \end{aligned}$$

One-factor Hull-White model

- We can think of $\mu(t)$ to be the current yield curve which is often denoted as $r_0(t)$. The rate $r(t)$ is random.
- After time s has passed we can calculate the expectation of $r(t)$ as

$$\mathbb{E}_s^{\mathbb{Q}_0}[r(t)] = r(s)e^{-\lambda(t-s)} + r_0(t) - r_0(s)e^{-\lambda(t-s)}.$$

- Since $r_0 = r(0) = r_0(0)$ we have that the solution to the Hull-White model can be written as

$$r(t) = r_0(t) + e^{-\lambda t}\int_0^t \sigma(s)e^{\lambda s}\,dW(s).$$

- The instantaneous 3 month LIBOR rate satisfies

$$r_{3\mathrm{ML}} = r(t) + b(t),$$

where $b(t)$ is basis between instantaneous LIBOR and OIS.
- We assume that $b(t)$ is deterministic.

Two-factor Hull-White model

- Instantaneous rate is now represented as

$$r(t) = r_0(t) + r_1(t) + r_2(t).$$

- $r_0(t)$ is the current rate and is deterministic. The two stochastic variables $r_1(t)$ and $r_2(t)$ control the levels (r_1) and the steepness (r_2) of the forward curve.
- We assume that their stochastic dynamics is

$$\begin{aligned} dr_1(t) &= -\lambda_1 r_1(t)\,dt + \sigma_1(t)\,dW_1(t) \\ dr_2(t) &= -\lambda_2 r_2(t)\,dt + \sigma_2(t)\,dW_2(t). \end{aligned}$$

- The two Brownian motions are correlated with

$$\mathbb{E}\left[W_1(t)W_2(t)\right] = \rho\,dt.$$

- ρ is usually negative and of order -0.9. The steepening of the curve tends to correlate negatively with the parallel shifts.

The zero-coupon bond in the Hull-White model

- The price of the zero-coupon bond is its expectation of the stochastic discount factor

$$P(t, T) = \mathbb{E}_t^{\mathbb{Q}_0} \left[e^{-\int_t^T r(u)\, du} \right].$$

- In Hull-White model this expectation can be expressed in a closed form.
- We start by the following obesrvation

$$\begin{aligned} r(u) &= r_0(u) + (r(t) - r_0(t))\, e^{-\lambda(u-t)} + e^{-\lambda u} \int_t^u \sigma(v) e^{\lambda v}\, dW(v) \\ \int_t^T r(u)\, du &= D + \int_t^T e^{-\lambda u} \int_t^u \sigma(v) e^{\lambda v}\, dW(v)\, du \\ &\quad \text{where } D = \frac{r(t) - r_0(t)}{\lambda} \left(1 - e^{-\lambda(T-t)}\right) + \int_t^T r_0(u)\, du. \end{aligned}$$

D is deterministic.

The zero-coupon bond in the Hull-White model

- Since D is deterministic we have

$$P(t,T) = e^{-D} \cdot \mathbb{E}_t^{\mathbb{Q}_0}\left[e^{-\int_t^T e^{-\lambda u}\int_t^u \sigma(v)e^{\lambda v}\,dW(v)\,du}\right].$$

- Changing the order of integration we first obtain

$$\int_t^T e^{-\lambda u}\int_t^u \sigma(v)e^{\lambda v}\,dW(v)\,du = \int_t^T \sigma(v)\frac{1-e^{-\lambda(T-v)}}{\lambda}\,dW(v). \tag{1}$$

- The last integral is of the form $\int_t^T \varphi(v)\,dW(v)$ where φ is deterministic.
- However, $\int_t^T \varphi(v)\,dW(v)$ is a normal random variable with mean 0 and variance $\int_t^T \varphi^2(v)\,dv$. Therefore

$$\mathbb{E}\left[e^{\int_0^t \varphi(v)\,dW(v)}\right] = e^{\frac{1}{2}\int_0^t \varphi^2(v)\,dv}.$$

The zero-coupon bond in the Hull-White model

- We obtain

$$
\begin{aligned}
P(t,T) &= A(t,T)e^{-h_\lambda(T-t)r(t)}, \quad \text{where} && (2)\\
A(t,T) &= e^{-\int_t^T r_0(u)\,du + h_\lambda(T-t)r_0(t) + \frac{1}{2}\int_t^T h_\lambda(T-v)^2\sigma^2(v)\,dv}, && (3)\\
h_\lambda(z) &= \frac{1-e^{-\lambda z}}{\lambda}. && (4)
\end{aligned}
$$

- The generalization to two-factor Hull-White model is:

$$
\begin{aligned}
P(t,T) &= A(t,T)e^{-h_{\lambda_1}(T-t)r_1(t) - h_{\lambda_2}(T-t)r_2(t)}, \quad \text{where}\\
A(t,T) &= e^{-\int_t^T r_0(u)\,du + h_{\lambda_1}(T-t)r_1(t) + h_{\lambda_2}(T-t)r_2(t) + \frac{1}{2}\int_t^T \Phi(v)\,dv},\\
\Phi(v) &= h^2_{\lambda_1}(T-v)\sigma_1^2(v) + 2\rho h_{\lambda_1}(T-v)\sigma_1(v)h_{\lambda_2}(T-v)\sigma_2(v)\\
&\quad + h^2_{\lambda_2}(T-v)\sigma_2^2(v).
\end{aligned}
$$

Options on zero-coupon bonds in the Hull-White model

- Consider a zero coupon bond with maturity T_{mat}.
- If $T < T_{\mathrm{mat}}$ then we can define a European call option with strike K and expiration T as a derived security with

$$\mathrm{Payoff} = \left(P(T, T_{\mathrm{mat}}) - K\right)^{+}.$$

- The price of the option can be explicitly calculated

$$\begin{aligned}
\mathrm{PV}_{\mathrm{call}} &= P_0\left(0, T_{\mathrm{mat}}\right) N(d_+) - K P_0(0, T) N(d_-) && (5) \\
d_{\pm} &= \frac{1}{\hat{\sigma}} \log \frac{P_0\left(0, T_{\mathrm{mat}}\right)}{P_0(0, T) K} \pm \frac{\hat{\sigma}}{2}, && (6) \\
\hat{\sigma} &= \sqrt{\int_0^T e^{-2\lambda(T-v)} \sigma^2(v)\, dv} \cdot h_{\lambda}\left(T_{\mathrm{mat}} - T\right). && (7)
\end{aligned}$$

Options on zero-coupon bonds in the Hull-White model

- Similarly, the price of the put option is

$$\mathrm{PV}_{\mathrm{put}} = KP_0(0,T)N(-d_-) - P_0\left(0, T_{\mathrm{mat}}\right)N(-d_+).$$

- Floorlets and caplets are puts and calls on FRAs. The above formulas can now be used as building blocks for prices of floors and caps in the Hull-White model.
- We cannot derive the closed formulas for swaptions because they are exotic instruments in the Hull-White model.

Calibration of the Hull-White model

- Before the model can be used for valuation purposes it has to be calibrated. All free parameters have to be assigned values so that the model replicates the prices of the selected liquid vanilla instruments.
- In the case of the Hull-White model the following has to be done:
  (i) Specifying the current forward curve $r_0(t)$. This is accomplished by choosing $r_0(t)$ to match the instantaneous OIS curve.
  (ii) Specifying the volatilities to match the selected options.

Calibration of the Hull-White model

- There is no calibration that would match the following prices at the same time: all caps, all floors, and all swaptions for all expirations. The main reasons for this are the volatility dynamics of the Hull-White model and the intrinsic smile. Also, there are too few parameters available for calibration.
- The following strategies are most commonly used:
  (i) Global optimization suitable for a portfolio that is being considered;
  (ii) Deal specific local calibration that is geared towards individual instrument.

Calibration of the Hull-White model

- We now describe the calibration for the instantaneous volatility function $\sigma(t)$.
- The function $\sigma(t)$ is build to be piecewise constant. The entire time period is divided into sub-intervals $[T_j, T_{j+1})$ and the function is set to be constant σ_j on each of them. The division is finer in the short end (around 3 monts) and coarser towards the far end.
- The choice of the parameters σ_j is made to minimize the objective function

$$\mathcal{L}(\sigma) = \frac{1}{2} \sum_{\text{all instruments}} \left(\sigma_n(\sigma) - \bar{\sigma}_n\right)^2 .$$

$\sigma_n(\sigma)$ and $\bar{\sigma}_n$ are model and market prices of all calibration instruments.

Calibration of the Hull-White model

- Local calibration is performed by selecting those particular instruments that have similar risk characteristics to the one that will be priced.
- Example: In Bermudan swaption modeling, one selects co-terminal swaptions of the same strike as calibrating instruments.
- Co-terminal swaptions are swaptions that correspond to co-terminal swaps. Co-terminal swaps are the ones that have the same final maturities, such as 1Y $\to$ 10Y, 2Y $\to$ 9Y, 3Y $\to$ 8Y, $\ldots$, 10Y $\to$ 1Y.
- Calibration of co-terminal swaptions is exact. The model will price the instruments exactly.

Pricing under the Hull-White model

- Prices of caps and floors can be calculated explicitly in the Hull-White model.
- However, the prices of the most instruments can not be calculated explicitly.
- Numerical techniques has to be used, and the two most popular ones are tree methods, and Monte Carlo methods.

Eurodollar/FRA convexity correction

- The currently observed LIBOR rate satisfies

$$
\begin{aligned}
L_0(T_1, T_2) &= \mathbb{E}^{\mathbb{Q}_{T_2}}\left[F(t, T_1, T_2)\right] + B_0(T_1, T_2) \\
&= \mathbb{E}^{\mathbb{Q}_{T_2}}\left[\frac{1}{\delta}\left(\frac{1}{P(t, T_1, T_2)} - 1\right)\right] + B_0(T_1, T_2).
\end{aligned}
$$

- We may choose $t = 0$ and obtain

$$
\begin{aligned}
L_0(T_1, T_2) &= \frac{1}{\delta}\left(\frac{1}{P_0(T_1, T_2)} - 1\right) + B_0(T_1, T_2) \\
&= \frac{1}{\delta}\left(\frac{P_0(0, T_1)}{P_0(0, T_2)} - 1\right) + B_0(T_1, T_2).
\end{aligned}
$$

- In the Hull-White model this rate can be easily calculated.

Eurodollar/FRA convexity correction

- The formulas (2–4) applied to $P_0(0, T_1)$ and $P_0(0, T_2)$ give:

$$\begin{aligned} L_0(T_1, T_2) &= \frac{1}{\delta}\Big(e^{\int_{T_1}^{T_2} r_0(u)\,du} \cdot e^{-\frac{1}{2}\int_0^{T_2} h_\lambda^2(T_2-u)\sigma^2(u)\,du} \\ &\quad \times e^{\frac{1}{2}\int_0^{T_1} h_\lambda^2(T_1-u)\sigma^2(u)\,du} - 1\Big) + B_0(T_1, T_2). \end{aligned}$$

- Last time we have seen that

$$\begin{aligned} L_0^{\mathrm{fut}}(T_1, T_2) &= \mathbb{E}^{\mathbb{Q}_0}\left[L(T_1, T_2)\right] \\ &= \frac{1}{\delta}\left(\mathbb{E}^{\mathbb{Q}_0}\left[e^{\int_{T_1}^{T_2} r(u)\,du}\right] - 1\right) + B_0(T_1, T_2). \end{aligned}$$

Eurodollar/FRA convexity correction

- Recall that $r(u) = r_0(u) + \int_0^u e^{-\lambda(u-v)}\sigma(v)\,dW(v)$.
- Change of order of integration leads to

$$\int_0^T \int_0^u e^{-\lambda(u-v)}\sigma(v)\,dW(v)\,du = \int_0^T h_\lambda(T-v)\sigma(v)\,dW(v).$$

- Therefore

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_0}\left[e^{\int_{T_1}^{T_2} r(u)\,du}\right] &= e^{\int_{T_1}^{T_2} r_0(u)\,du} \cdot \mathbb{E}^{\mathbb{Q}_0}\Big[e^{\int_0^{T_2} h_\lambda(T_2-v)\sigma(v)\,dW(v)} \\
&\quad \times e^{-\int_0^{T_1} h_\lambda(T_1-v)\sigma(v)\,dW(v)}\Big] \\
&= e^{\int_{T_1}^{T_2} r_0(u)\,du} \cdot \mathbb{E}^{\mathbb{Q}_0}\Big[e^{\int_{T_1}^{T_2} h_\lambda(T_2-v)\sigma(v)\,dW(v)} \\
&\quad \times e^{\int_0^{T_1}(h_\lambda(T_1-v)-h_\lambda(T_1-v))\sigma(v)\,dW(v)}\Big]
\end{aligned}$$

- The two factors in the last product are independent normal variables under $\mathbb{Q}_0$ whose expected values are 0.

Eurodollar/FRA convexity correction

- Thus

$$
\begin{aligned}
L_0^{\text{fut}}(T_1, T_2) &= \frac{1}{\delta}\Big(e^{\int_{T_1}^{T_2} r_0(u)\,du} \cdot e^{\frac{1}{2}\int_{T_1}^{T_2} h_\lambda^2(T_2 - v)\sigma^2(v)\,dv} \\
&\quad \times e^{\frac{1}{2}\int_0^{T_1}(h_\lambda(T_1 - v) - h_\lambda(T_1 - v))^2\sigma^2(v)\,dv} - 1\Big) + B_0(T_1, T_2)
\end{aligned}
$$

- Elementary algebraic manipulations now imply

$$
\begin{aligned}
\Delta_{\text{ED / FRA}} &= L_0^{\text{fut}}(T_1, T_2) - L_0(T_1, T_2) \\
&= \frac{1}{\delta}\left(1 + \delta\left(L_0(T_1, T_2) - B_0(T_1, T_2)\right)\right) \\
&\quad \times \left(e^{\int_0^{T_2} h_\lambda^2(T_2 - v)\sigma^2(v)\,dv - \int_0^{T_1} h_\lambda(T_2 - v)h_\lambda(T_1 - v)\sigma^2(v)\,dv} - 1\right).
\end{aligned}
$$

- The above formula can be approximated by using $e^x - 1 \approx x$ and neglecting the terms $\delta\left(L_0(T_1, T_2) - B_0(T_1, T_2)\right)$ and $\int_{T_1}^{T_2} h_\lambda^2(T_2 - v)\sigma^2(v)\,dv$.

Eurodollar/FRA convexity correction

- We get

$$\Delta_{\text{ED / FRA}} = \frac{1}{\delta}\int_0^{T_1} h_\lambda(T_2 - v)\,(h_\lambda(T_2 - v) - h_\lambda(T_1 - v))\,\sigma^2(v)\,dv.$$

- In the case of constant volatility:

$$\begin{aligned}\Delta_{\text{ED / FRA}} &= \frac{\sigma^2}{2\lambda^3\delta}\left(\left(1 - e^{-3\lambda T_1}\right)\left(1 - e^{-\lambda(T_2 - T_1)}\right)^2\right.\\ &\quad \left. + \left(1 - e^{-\lambda(T_2 - T_1)}\right)\left(1 - e^{-\lambda T1}\right)^2\right).\end{aligned}$$

- In two factor Hull-White model similar calculations give:

$$\begin{aligned}\Delta_{\text{ED / FRA}} &= \frac{1}{\delta}\sum_{1\le j,k\le 2}\rho_{jk}\int_0^{T_1} h_{\lambda_j}(T_2 - v)\\ &\quad \times \left(h_{\lambda_k}(T_2 - v) - h_{\lambda_j}(T_1 - v)\right)\sigma_j(v)\sigma_k(v)\,dv,\end{aligned}$$

  where $\rho_{11} = \rho_{22} = 1$ and $\rho_{12} = \rho_{21} = \rho$.
- The function h_λ is always non-negative and increasing. Thus the convexity adjustment is always positive.

Affine term structure models

- Affine term structure models are those for which the price of the zero-coupon bond has the form

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}.$$

- Hull-White model is an affine term structure model.
- In general, a model is affine if μ and σ^2 are affine functions and satisfy

$$\begin{aligned} \mu(t,x) &= a(t)x + b(t), \\ \sigma^2(t,x) &= c(t)x + d(t), \end{aligned}$$

where $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are deterministic.

Affine term structure models

- Then the functions A and B that are featured in the formula for $P(t, T)$ must satisfy:

$$
\begin{aligned}
\frac{dB(t,T)}{dt} + a(t)B(t,T) - \frac{1}{2}c(t)B^2(t,T) + 1 &= 0, \\
\frac{d \log A(t,T)}{dt} - b(t)B(t,T) + \frac{1}{2}d(t)B^2(t,T) &= 0, \\
B(T,T) = 0 \qquad \text{and} \qquad A(T,T) = 1.
\end{aligned}
$$

- The above equations are called Riccati's equations and, in general, cannot be solved in closed form.

The Cox-Ingersoll-Ross model

- The Cox-Ingersoll-Ross (CIR) model is the one in which $r(t)$ solves

$$
\begin{aligned}
dr(t) &= \lambda(\mu - r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t) \\
r(0) &= r_0,
\end{aligned}
$$

  under the risk-neutral measure.
- The condition $2\lambda\mu > \sigma^2$ guarantees that $r(t) > 0$ with probability 1.
- The CIR model is affine with

$$
\begin{aligned}
A(t,T) &= \left(\frac{2he^{\frac{(\lambda+h)(T-t)}{2}}}{2h + (\lambda+h)\left(e^{(T-t)h} - 1\right)} \right)^{\frac{2\lambda\mu}{\sigma^2}} \\
B(t,T) &= \frac{2\left(e^{(T-t)h} - 1\right)}{2h + (\lambda+h)\left(e^{(T-t)h} - 1\right)}
\end{aligned}
$$

The Cox-Ingersoll-Ross model

- The CIR model is not used in the industry because it cannot be fit to the initial term structure.
- The following extended CIR model can be fitted to the initial term structure of rates.

$$
\begin{aligned}
r(t) &= \theta(t) + x(t), \\
dx(t) &= \lambda(\mu - x(t))\,dt + \sigma\sqrt{x(t)}\,dW(t), \\
x(0) &= x_0.
\end{aligned}
$$

- Here $\theta(t)$ is deterministic and can be fitted to the initial term structure.
- Also, a two factor extension of CIR is used in industry.

Short rate models with stochastic volatility

- The following is an extension of Hull-White model with SABR type volatility.

$$\begin{aligned} r(t) &= r_0(t) + x(t), \\ dx(t) &= -\lambda x(t)\, dt + \sigma(t) v^{\beta}(t)\, dW(t), \\ dv(t) &= \alpha(t) v(t)\, dZ(t), \end{aligned}$$

  where $\sigma(t)$ and $\alpha(t)$ are deterministic and $x(0) = 0$, $v(0) = 1$.
- The correlation between Brownian motions is assumed to be ρ, i.e. $\mathbb{E}\left[dW(t) dZ(t)\right] = \rho\, dt$.

References

- AndersEn, L.; Piterbarg, V.: *Interest rate modeling*, Vol. 2, Atlantic Financial Press (2010)
- Brigo, D.; Mercurio, F: *Interest rate models – theory and practice*, Springer (2006)
- Hull, J: *Options, futures, and other derivatives*, Prentice Hall (2005)
- Lesniewski, A.: http://lesniewski.us/presentations.html