Mathematical Foundation of Convexity Correction

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Mathematical Foundation of Convexity Correction

Abstract

A broad class of exotic interest interest rate derivatives can be valued simply by adjusting the forward interest rate. This adjustment is known in the market as *convexity correction*. Various ad hoc rules are used to calculate the convexity correction for different products, many of them mutually inconsistent. In this paper we put convexity correction on a firm mathematical basis by showing that it can be interpreted as the side-effect of a change of probability measure. This provides us with a theoretically consistent framework to calculate convexity corrections. Using this framework we provide exact expressions for LIBOR in arrears, and diff swaps. Furthermore, we propose a simple method to calculate analytical approximations for general instances of convexity correction.

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1. Introduction

Many products that are actively traded in interest rate derivative markets have payoffs that only depend on a few interest rates which are only observed at one point in time. One could characterise such products as "exotic European" options. The value of these products is determined solely by the (joint) probability distribution of the relevant rates at this one point in time. This explains why this type of products has become particularly liquid in recent years. Examples of this type of products are LIBOR in arrears swaps, diff swaps and constant maturity swaps (also known as CMS swaps) and CMS caps.

All these products can be characterised by the fact that certain interest rates are paid at the "wrong" time and/or in the "wrong" currency. It turns out that the price of these products can be expressed as the discounted forward rate, where forward interest rate has to be adjusted to reflect the "incorrect" payment. This adjustment is known in the market as *convexity correction* or *convexity adjustment*.

Practitioners use various ad hoc rules to calculate convexity corrections for different products, often based on Taylor approximations. See, for example, Brotherton-Radcliffe and Iben (1993), Li and Raghavan (1996), Hunt and Pelsser (1998), or Benhamou (2000). For an overview of applications to various products, see Hull (2000, Ch. 19 and 20). However, many of these rules are theoretically inconsistent and cannot be used to derive convexity corrections for general products.

In this paper, we will put convexity correction on a firm mathematical basis by showing that it can be interpreted as the side-effect of a change of numeraire. This means that we will show how convexity correction can be understood mathematically as the expected value of an interest rate under a different probability measure than its own martingale measure. In this setting we derive an exact expression for LIBOR in arrears.[1] To analyse convexity correction across different currencies, we generalise the Change of Numeraire Theorem to a multi-currency setting. In this setting we derive an exact expression for diff swaps. Finally, we show how convexity corrections for general interest rate derivatives can be calculated analytically using the surprisingly simple but effective Linear Swap Rate Model of Hunt and Kennedy (2000).

[1] Some of the single currency results for LIBOR in arrears we derive in this paper, have also been derived independedently in a recent paper by Pugachevsky (2001). Pugachevsky also derives expressions for CMS rates, but he proposes a different approximation than is proposed in this paper.

The remainder of this paper is organised as follows. In Section 2 we show how the Change of Numeraire Theorem can be extended to a multi-currency setting. In Sections 3 and 4, we derive the theoretical framework to calculate convexity corrections for single and multi-currency products. Finally, we show in Sections 5 and 6 how convexity corrections for general interest rate payments and options on interest rate payments can be calculated using the approximation we propose.

2. Multi-Currency Change of Numeraire Theorem

In this section we review how, in an arbitrage-free economy, different probability measures can be used to value a given product. From this observation one can derive the well known Change of Numeraire Theorem. Then we extend this theorem to multi-currency economies. With these tools we are then in a position to put the concept of convexity correction on a form mathematical basis.

The well known Change of Numeraire Theorem due to Géman et al. (1995) shows how in an arbitrage-free economy an expectation under a probability measure $\boldsymbol{Q}^N$ generated by a numeraire N can be represented as an expectation under a probability measure $\boldsymbol{Q}^M$ generated by the numeraire M times the Radon-Nikodym derivative $d\boldsymbol{Q}^N/d\boldsymbol{Q}^M$ which is equal to the ratio of the numeraires N/M. For an expectation at time 0 of a random variable $H(T)$ at time T we have

$$\boldsymbol{E}^N\Big(H(T)\Big) = \boldsymbol{E}^M\Big(H(T)\frac{N(T)/N(0)}{M(T)/M(0)}\Big), \tag{1}$$

where $\boldsymbol{E}^N, \boldsymbol{E}^M$ denote expectations under the probability measures $\boldsymbol{Q}^N, \boldsymbol{Q}^M$ respectively.

Many of the products we are interested in will be multi-currency products. Hence, we will extend in this section the Change of Numeraire Theorem to include multi-currency economies. Suppose we have a domestic economy d and a foreign economy f together with the exchange rate $X^{(d/f)}(t)$ that expresses the value at time t of one unit of foreign currency in terms of domestic currency. This immediately implies the relation $X^{(f/d)} = 1/X^{(d/f)}$. Also we assume that this system of economies and exchange rates is arbitrage-free and complete. This implies that for a numeraire $N^{(d)}$ in the domestic economy there exists a unique martingale measure $\boldsymbol{Q}^{N,d}$ such that all $N^{(d)}$ rebased traded assets in the domestic economy become martingales. Note that we have two types of traded assets in the domestic economy: the domestic assets $Z^{(d)}$ and the foreign assets denominated in domestic currency which are given by $X^{(d/f)}Z^{(f)}$. All these assets are traded assets in the domestic economy and can be used as numeraires if their values are strictly positive.

Let us now consider two numeraires, one in the domestic economy, say $N^{(d)}$, and one in the foreign economy, say $M^{(f)}$. As the exchange rate is strictly positive, the domestic value of the foreign numeraire $X^{(d/f)}M^{(f)}$ is a valid numeraire in the domestic economy. Hence, there exists a unique martingale measure $\boldsymbol{Q}^{XM,d}$ such that all $X^{(d/f)}M^{(f)}$ rebased traded assets in the domestic economy become martingales. What is the relation between the probability measure $\boldsymbol{Q}^{XM,d}$ in the domestic economy and $\boldsymbol{Q}^{M,f}$ in the foreign economy? All $X^{(d/f)}M^{(f)}$ rebased traded assets in the domestic economy are martingales under $\boldsymbol{Q}^{XM,d}$. Hence, also the domestic value of the foreign traded assets: $X^{(d/f)}Z^{(f)}/X^{(d/f)}M^{(f)} = Z^{(f)}/M^{(f)}$ are martingales. But this implies (given the uniqueness of a probability measure for a particular numeraire) that $\boldsymbol{Q}^{XM,d}$ and $\boldsymbol{Q}^{M,f}$ are the same probability measure. So under the measure $\boldsymbol{Q}^{M,f}$ all $X^{(d/f)}M^{(f)}$ rebased traded assets are martingales in the domestic economy and all $M^{(f)}$ rebased traded assets are martingales in the foreign economy.

From the domestic economy perspective we have $N^{(d)}$ and $X^{(d/f)}M^{(f)}$ as domestic numeraires. Hence, we can apply the single currency Change of Numeraire Theorem which yields

$$\frac{d\boldsymbol{Q}^{N,d}}{d\boldsymbol{Q}^{M,f}} = \frac{N^{(d)}(T)}{X^{(d/f)}(T)M^{(f)}(T)} \frac{X^{(d/f)}(0)M^{(f)}(0)}{N^{(d)}(0)}. \tag{2}$$

On the other hand, from the foreign perspective we have $X^{(f/d)}N^{(d)}$ and $M^{(f)}$ as foreign numeraires. Hence, we can apply the single currency Change of Numeraire Theorem to this case as well which yields

$$\frac{d\boldsymbol{Q}^{N,d}}{d\boldsymbol{Q}^{M,f}} = \frac{X^{(f/d)}(T)N^{(d)}(T)}{M^{(f)}(T)} \frac{M^{(f)}(0)}{X^{(f/d)}(0)N^{(d)}(0)}. \tag{3}$$

Note that equation (2) and (3) are identical since $X^{(f/d)} = 1/X^{(d/f)}$. Hence, we have proven the[2]

[2] A similar result using discount bonds as numeraires is derived independently by Schlögl (2000) in the setting of multi-currency Market Models.

Multi-Currency Change of Numeraire Theorem. *Given an arbitrage-free system of economies* (d, f), *an exchange rate* X *and two numeraires* $N^{(d)}$ *and* $M^{(f)}$ *within the economies with the associated martingale measures* $\boldsymbol{Q}^{N,d}$ *and* $\boldsymbol{Q}^{M,f}$ *we have that*

$$\begin{aligned}\frac{d\boldsymbol{Q}^{N,d}}{d\boldsymbol{Q}^{M,f}} &= \frac{N^{(d)}(T)}{X^{(d/f)}(T)M^{(f)}(T)} \frac{X^{(d/f)}(0)M^{(f)}(0)}{N^{(d)}(0)} \\ &= \frac{X^{(f/d)}(T)N^{(d)}(T)}{M^{(f)}(T)} \frac{M^{(f)}(0)}{X^{(f/d)}(0)N^{(d)}(0)}.\end{aligned} \tag{4}$$

If we set $X \equiv 1$ the multi-currency Change of Numeraire Theorem reduces to the single currency case.

3. Convexity Correction for Single Currency

Given the Change of Numeraire Theorems, we are now in a position to investigate convexity correction. Convexity correction arises when an interest rate is paid out at the "wrong" time and/or in the "wrong" currency.

Suppose we are given a forward interest rate y with maturity T (which can be, for example, a forward LIBOR rate or a forward par swap rate) and a numeraire P such that the forward rate y is a martingale under the associated probability measure $\boldsymbol{Q}^P$. In the single currency case we can have a contract where the interest rate $y(T)$ is observed at T but paid at a later date $S \geq T$. If we denote the discount bond that matures at S by D_S, we have that the value of this contract at time T is given by $V(T) = y(T)D_S(T)$. Using D_S as the numeraire with the associated forward measure $\boldsymbol{Q}^S$ we can express the value of this contract at time 0 as

$$V(0) = D_S(0)\boldsymbol{E}^S\big(y(T)\big). \tag{5}$$

However, under the measure $\boldsymbol{Q}^S$ the process y is in general not a martingale. Expectation (5) can be expressed as $y(0)$ times a correction term. This correction term is known in the market as the *convexity correction* or *convexity adjustment*. Using (5) market participants calculate the value of the payoff as the discounted value of the convexity corrected payoff $\boldsymbol{E}^S(y)$. What remains to be done, is to find an expression for $\boldsymbol{E}^S(y)$.

We do know the process of y under its "own" martingale measure $\boldsymbol{Q}^P$. Using the Change of Numeraire Theorem, we can express the expectation $\boldsymbol{E}^S$ in terms of $\boldsymbol{E}^P$ as follows

$$\boldsymbol{E}^S\big(y(T)\big) = \boldsymbol{E}^P\left(y(T)\frac{d\boldsymbol{Q}^S}{d\boldsymbol{Q}^P}\right) = \boldsymbol{E}^P\left(y(T)\frac{D_S(T)}{P(T)}\frac{P(0)}{D_S(0)}\right) = \boldsymbol{E}^P\big(y(T)R(T)\big) \tag{6}$$

where R denotes the Radon-Nikodym derivative. Both y and R are martingales under the measure $\boldsymbol{Q}^P$. If we know the joint probability distribution of $y(T)$ and $R(T)$ the expectation can be calculated explicitly and we obtain an expression for the convexity correction.

One possible approach is to assume that both $y(T)$ and $R(T)$ have lognormal distributions. Hence, for $0 \leq t \leq T$ we assume that $y(t)$ follows the process $dy = \sigma_y y\, dW_y$ and $R(t)$ follows the process $dR = \sigma_R R\, dW_R$ under the measure $\boldsymbol{Q}^P$ with correlation $dW_y\, dW_R = \rho_{y,R}\, dt$. Under this assumption we can calculate (6) as

$$\boldsymbol{E}^S\big(y(T)\big) = \boldsymbol{E}^P\big(y(T)R(T)\big) = y(0)e^{\rho_{y,R}\sigma_y\sigma_R T}. \tag{7}$$

Note that the Radon-Nikodym derivative $R(t)$ is a ratio of traded assets whose values can be observed in the market. Hence, the instantaneous volatility and the correlation of the $R(t)$ process (which remain unaffected by a change of measure) can be estimated from historical data.

3.1. Example: LIBOR in Arrears

Let us consider an example where we can obtain an exact expression for a single currency convexity correction. In a normal LIBOR payment, the LIBOR interest rate L is observed at time T and paid at the end of the accrual period at time $T + \Delta$ as $\alpha L(T)$, where α denotes the daycount fraction for the time period $[T, T + \Delta]$. The forward LIBOR rate $L(t)$ is defined as

$$L(t) = \frac{1}{\alpha}\frac{D_T(t) - D_{T+\Delta}(t)}{D_{T+\Delta}(t)}, \tag{8}$$

where $D_T, D_{T+\Delta}$ denote discount factors that mature at times T and $T + \Delta$ respectively. Hence, if we choose $D_{T+\Delta}$ as the numeraire then under the associated martingale measure $\boldsymbol{Q}^{T+\Delta}$ the forward LIBOR rate L is a martingale.

In a LIBOR in arrears contract, the interest payment at time T is based on the rate $L(T)$. Hence, the LIBOR payment is fixed only at the end of the interest rate period $[T-\Delta, T]$. The payoff of a LIBOR in arrears payment at time T is therefore equal to $V^{\mathrm{LIA}}(T) = L(T)$. The value at time 0 of this payment is given by $V^{\mathrm{LIA}}(0) = D_T(0)\boldsymbol{E}^T\big(L(T)\big) = D_T(0)\tilde{L}$, where $\tilde{L}$ denotes the convexity corrected forward LIBOR rate to reflect the payment in arrears. To calculate the convexity correction we proceed as follows. From the Change of Numeraire Theorem we obtain that the Radon-Nikodym derivative for the change of measure $d\boldsymbol{Q}^{T+\Delta}/d\boldsymbol{Q}^T$ is given by the ratio of the numeraires

$$\frac{d\boldsymbol{Q}^T}{d\boldsymbol{Q}^{T+\Delta}} = \frac{D_T(T)}{D_{T+\Delta}(T)}\frac{D_{T+\Delta}(0)}{D_T(0)}. \tag{9}$$

Using the definition (8) we can rewrite the Radon-Nikodym derivative as

$$\frac{d\boldsymbol{Q}^T}{d\boldsymbol{Q}^{T+\Delta}} = \frac{1+\alpha L(T)}{1+\alpha L(0)}. \tag{10}$$

Using this expression we can calculate $\tilde{L}$ as

$$\begin{aligned}\tilde{L} = \boldsymbol{E}^T\big(L(T)\big) &= \boldsymbol{E}^{T+\Delta}\left(L(T)\frac{d\boldsymbol{Q}^T}{d\boldsymbol{Q}^{T+\Delta}}\right)\\ &= \frac{\boldsymbol{E}^{T+\Delta}\Big(L(T)\big(1+\alpha L(T)\big)\Big)}{1+\alpha L(0)}\\ &= \frac{L(0)+\alpha\boldsymbol{E}^{T+\Delta}\big(L(T)^2\big)}{1+\alpha L(0)}.\end{aligned} \tag{11}$$

The expression above is valid irrespective of the distribution of L. If we make the additional assumption that the forward LIBOR rate has a lognormal distribution under the measure $\boldsymbol{Q}^{T+\Delta}$ we have that $\boldsymbol{E}^{T+\Delta}\big(L(T)^2\big) = L(0)^2 e^{\sigma^2 T}$ and we obtain

$$\tilde{L} = L(0)\left(\frac{1+\alpha L(0)e^{\sigma^2 T}}{1+\alpha L(0)}\right) \tag{12}$$

where σ denotes the volatility of L.[3]

[3] Note that the LIBOR in arrears convexity correction approximation given in Hull (2000, p. 553) is to first order equal to the exact formula (12). The exact expressions for LIBOR in arrears have also been derived independently by Pugachevsky (2001).

4. Convexity Correction for Multi-Currency

Let us now consider the cross currency case. In this case we have a contract where a foreign interest rate $y^{(f)}(T)$ is observed at T but paid in domestic currency at a later date $S \geq T$. If we denote the domestic discount bond that matures at S by $D_S^{(d)}$, we have that the value of this contract in domestic terms at time T is given by $V^{(d)}(T) = y^{(f)}(T)D_S^{(d)}(T)$. Using $D_S^{(d)}$ as the numeraire and the associated the measure $\boldsymbol{Q}^{S,d}$ we can express the value of this contract at time 0 as

$$V^{(d)}(0) = D_S^{(d)}(0)\boldsymbol{E}^{S,d}\big(y^{(f)}(T)\big). \tag{13}$$

However, under the measure $\boldsymbol{Q}^{S,d}$ the process $y^{(f)}$ is in general not a martingale. We do know the process of $y^{(f)}$ under its "own" martingale measure $\boldsymbol{Q}^{P,f}$. Using the Change of Numeraire Theorem, we can express the expectation $\boldsymbol{E}^{S,d}$ in terms of $\boldsymbol{E}^{P,f}$ as follows

$$\boldsymbol{E}^{S,d}\big(y^{(f)}(T)\big) = \boldsymbol{E}^{P,f}\left(y^{(f)}(T)\frac{d\boldsymbol{Q}^{S,d}}{d\boldsymbol{Q}^{P,f}}\right) = \boldsymbol{E}^{P,f}\big(y^{(f)}(T)R^{(d/f)}(T)\big). \tag{14}$$

Just like in the single currency case the Radon-Nikodym derivative $R^{(d/f)}$ is a ratio of traded assets whose values can be observed in the market. Hence, the volatility and the correlation of the $R^{(d/f)}$ process (which remain unaffected by a change of measure) can be estimated from historical data. Another approach to evaluate (14) is to decompose $R^{(d/f)}$ as the forward exchange rate times $D_S^{(f)}/P^{(f)}$, which is the single currency Radon-Nikodym derivative of equation (6). This leads to the expression

$$\begin{aligned}
&\boldsymbol{E}^{S,d}\big(y^{(f)}(T)\big)\\
&\quad= \boldsymbol{E}^{P,f}\left(y^{(f)}(T)\frac{d\boldsymbol{Q}^{S,d}}{d\boldsymbol{Q}^{P,f}}\right)\\
&\quad= \boldsymbol{E}^{P,f}\left(y^{(f)}(T)\left(X^{(f/d)}(T)\frac{D_S^{(d)}(T)}{P^{(f)}(T)}\right)\right)\frac{P^{(f)}(0)}{X^{(f/d)}(0)D_S^{(d)}(0)}\\
&\quad= \boldsymbol{E}^{P,f}\left(y^{(f)}(T)\left(X^{(f/d)}(T)\frac{D_S^{(d)}(T)}{D_S^{(f)}(T)}\right)\frac{D_S^{(f)}(T)}{P^{(f)}(T)}\right)\frac{P^{(f)}(0)}{X^{(f/d)}(0)D_S^{(d)}(0)}\\
&\quad= \boldsymbol{E}^{P,f}\left(y^{(f)}(T)F_S^{(f/d)}(T)\frac{D_S^{(f)}(T)}{P^{(f)}(T)}\right)\frac{P^{(f)}(0)}{X^{(f/d)}(0)D_S^{(d)}(0)},
\end{aligned} \tag{15}$$

where F_S denotes the forward exchange rate for delivery at time S. The volatility of the forward exchange rate is quoted in the market as the implied volatility of a foreign exchange option with maturity S. The process F_S is not a martingale under the measure $\boldsymbol{Q}^{P,f}$, but the drift of the process can be determined from the fact that $D_S^{(f)}/P^{(f)}$ and $F_S D_S^{(f)}/P^{(f)}$ are martingales under $\boldsymbol{Q}^{P,f}$.

4.1. Example: Diffed LIBOR

For a *diffed* LIBOR contract (also known as quantoed LIBOR) a foreign LIBOR rate $L^{(f)}$ is observed at T and is paid in domestic currency at time $T + \Delta$, These diffed LIBOR payments are often paid in the form of a *differential swap* or diff swap. In a diff swap a floating interest rate in the foreign currency is exchanged against a floating rate in the domestic currency where both rates are applied to the same domestic notional principal. The domestic leg of a diff swap can be valued as a standard floating leg in the domestic currency. The foreign leg of the diff swap is a portfolio of diffed foreign LIBOR payments, and its value can be calculated as the sum of the individual diffed LIBOR payments.

For diffed LIBOR the expression (15) reduces to

$$\boldsymbol{E}^{T+\Delta,d}\big(L^{(f)}(T)\big) = \boldsymbol{E}^{T+\Delta,f}\left(L^{(f)}(T)F_{T+\Delta}^{(f/d)}(T)\right)\frac{1}{F_{T+\Delta}^{(f/d)}(0)}. \tag{16}$$

If we make the additional assumption that the forward LIBOR rate and the forward exchange rate have lognormal distributions under $\boldsymbol{Q}^{T+\Delta,f}$ then the convexity corrected foreign LIBOR rate $\tilde{L}_i^{(f)}$ given by

$$\tilde{L}_i^{(f)} = L_i^{(f)}(0)e^{\rho_{F,L}\sigma_F\sigma_L T}. \tag{17}$$

where the volatility of the forward exchange rate with delivery at time $T+\Delta$ is denoted by σ_F, σ_L denotes the volatility of $L^{(f)}$ and $\rho_{F,L}$ denotes the correlation between the forward (f/d) exchange rate and $L^{(f)}$. By the exchange rate (f/d) we mean the value of 1 unit of domestic currency in terms of foreign currency. For many practical applications, the correlation between the forward exchange rate and the forward LIBOR rate is approximated by the correlation between the spot exchange rate and the spot LIBOR rate.[4]

Using formula (15), one can also derive an exact expression for diffed LIBOR in arrears. This is left as an exercise to the reader.

[4] Note that, if we calculate the correlation between the forward (d/f) exchange rate and $L^{(f)}$, we must use $-\rho$ in our formula.

5. Simple Approximation Formula for Convexity Correction

Only for very special cases exact expressions for the convexity correction can be obtained. In these special cases the Radon-Nikodym derivative of the change of measure is equal to (a simple function of) the interest rate that determines the payoff. A point in case is LIBOR in arrears. In this section we want to propose a method to approximate convexity corrections that exploits the idea of making the Radon-Nikodym derivative a function of the payout rate.

5.1. Approximation for Single Currency

For derivative contracts with a payoff based on an interest rate y, the numeraire that makes the interest rate a martingale is always a portfolio of discounts bonds of the form $\sum_i \alpha_{i-1} D_i$. Let us denote this numeraire by P. In the general (single currency) case we have that the interest rate is observed at time T, and paid out at time $S \geq T$. Hence, the Radon-Nikodym derivative is of the form $R(T) = D_S(T)/P(T)$.

The Linear Swap Rate Model[5] (LSM) of Hunt and Kennedy (2000) provides us with a convenient way to approximate $R(T)$. In this model one approximates $D_S(T)/P(T)$ by the linear form $A + B_S y(T)$, where $A = (\sum_i \alpha_{i-1})^{-1}$ and $B_S = (D_S(0)/P(0) - A)/y(0)$. Note that A is a constant and B_S only depends on the maturity S. Using this approximation we can evaluate (6) explicitly as

$$\begin{aligned} \boldsymbol{E}^S\big(y(T)\big) &= \frac{P(0)}{D_S(0)}\boldsymbol{E}^P\left(y(T)\frac{D_S(T)}{P(T)}\right) \\ &= \frac{P(0)}{D_S(0)}\boldsymbol{E}^P\Big(y(T)\big(A + B_S y(T)\big)\Big) \\ &= y(0)\left(\frac{A + B_S y(0) e^{\sigma_y^2 T}}{A + B_S y(0)}\right). \end{aligned} \tag{18}$$

The linear approximation of the LSM does seem very crude at first, but can be justified by the following argument. Convexity corrections only become sizeable for large maturities. However, for large maturities the term structure almost moves in parallel. Hence, a change in the level of the long end of the curve is well described by the swap rate. Furthermore, for parallel moves in the curve, the *ratio* $D_S(T)/P(T)$ is closely approximated by a linear function of the swap rate, which is exactly what the LSM does.

[5] See the Appendix for a brief derivation of this model.

Table 1: Forward 20 year CMS rate $y(T)$ paid at $T + 1$.

T	σ_{LIBOR}	σ_{Swap}	LMM	StdErr	LSM
1	15.5%	11.8%	5.017%	0.001%	5.024%
2	17.5%	11.3%	5.032%	0.002%	5.045%
3	17.2%	10.8%	5.044%	0.002%	5.061%
4	16.6%	10.2%	5.053%	0.002%	5.073%
5	15.8%	9.7%	5.064%	0.002%	5.083%
6	15.1%	9.2%	5.067%	0.002%	5.090%
7	14.4%	8.7%	5.070%	0.002%	5.095%
8	13.8%	8.3%	5.073%	0.002%	5.099%
9	13.2%	8.0%	5.073%	0.002%	5.102%
10	12.7%	7.6%	5.074%	0.002%	5.103%
11	12.2%	7.3%	5.075%	0.002%	5.105%
12	11.8%	7.0%	5.075%	0.003%	5.106%
13	11.4%	6.8%	5.073%	0.003%	5.106%
14	11.0%	6.5%	5.072%	0.003%	5.107%
15	10.6%	6.3%	5.073%	0.003%	5.107%
16	10.3%	6.1%	5.075%	0.003%	5.107%
17	10.0%	5.9%	5.078%	0.003%	5.107%
18	9.8%	5.8%	5.078%	0.003%	5.106%
19	9.5%	5.6%	5.075%	0.003%	5.105%
20	9.3%	5.4%	5.074%	0.003%	5.104%

Simulations based on 500,000 runs in a 2-factor
LIBOR Market Model with volatility specification:
$\Lambda^1(T) = 0.256 \exp\{-0.145T\}$
$\Lambda^2(T) = 7.334 \exp\{-4.096T\}$
and correlation -0.744 between the factors.
Initial term-structure of LIBOR rates is flat at 5%.

Hence, exactly for long maturities the assumptions of the LSM become quite accurate. This leads to a good approximation of the convexity correction for long maturities.

5.2. Example: Constant Maturity Swap

A particularly liquid type of exotic European interest rate contract is a *Constant Maturity Swap* (or CMS). This is a swap where at every payment date a payment calculated from a swap rate y is exchanged for a fixed rate. Usually, these payments are treated as floating payments: at the preceding payment date T the swap rate $y(T)$ is observed, the actual payment is made at the next payment date S. Hence, each CMS payment consists of a swap rate y that is observed at time T and paid out (only once) at time $S \geq T$. Note, that the term and payment frequency of the swap rate may be different from the specifications of the CMS swap itself.

The forward swap rate $y(t)$ is defined as $(D_0(t) - D_N(t))/P(t)$, where D_0 has a maturity equal to the start date T of the swap, D_N has a maturity equal to the last payment date T_N of the swap and $P(t)$ is the PVBP of the swap given by $\sum_i \alpha_{i-1} D_i(t)$ where the D_i are the discount factors with maturity dates equal to the dates at which fixed payments are made in the swap. If we use the PVBP $P(t)$ as a numeraire, then the forward swap rate $y(t)$ is a martingale under the measure $\boldsymbol{Q}^P$. To calculate the value of a CMS payment at time 0 we use the convexity corrected swap rate $\tilde{y}$ given by (18).

Let us compare the convexity correction formula (18) with Monte Carlo simulations in a 2-factor LIBOR Market Model (LMM)[6]. In Table 1 we have summarised the results for the expectation $\boldsymbol{E}^{T+1}\big(y_{20}(T)\big)$ simulated in a 2-factor LMM with annual LIBOR rates. The volatility function of the LIBOR rates are given by $\Lambda^1(T) = 0.256 \exp\{-0.145T\}$ and $\Lambda^2(T) = 7.334 \exp\{-4.096T\}$. The correlation between the two factors is given by -0.744. Hence, we have a model with two negatively correlated factors, where the first factor has a low mean-reversion of 0.145 and the second factor has a high mean-reversion of 7.334. As shown in the second and third column of Table 1, we replicate with this volatility specification for the LMM the volatility hump of LIBOR rates and a declining volatility structure for 20-year swap rates which is consistent with empirical observations.

Using a Market Model with annual time-steps we have simulated the value of 20-year swap rates observed at time T which are paid time $T+1$ for $T = 1, \ldots, 20$. For this payoff scheme typically observed in CMS swaps, one cannot apply standard Taylor-series based convexity correction formulæ(as proposed in Hull (2000, Ch. 20)) without introducing additional correlations between the forward swap rate and the forward 1-year rate.

From Table 1 we see that the approximations calculated with the LSM formula (18) (see column "LSM") are very close to the Monte Carlo simulation in the 2-factor LMM (see column "LMM"). We do see that the LSM formula over-estimates the convexity correction somewhat. This effect can be attributed to the fact that the LSM is a one-factor model which tends to over-estimate the correlation between different interest rates. However, the error of the LSM formula is within one swaption vega.

5.3. Approximation for Multi-Currency

Let us consider the use of the Linear Swap Rate Model in the multi-currency case. In the foreign economy, we approximate $D_S^{(f)}/P^{(f)}$, by $A^{(f)} + B_S^{(f)} y^{(f)}$. First, we determine

[6] For further details on the implementation of a Monte Carlo simulation in a multi-factor LMM, we refer to Hull (2000, Ch. 22), or Pelsser (2000, Ch. 8).

the expected value of F from the fact that the Radon-Nikodym derivative $R^{(d/f)}$ is a martingale with expected value 1. This leads to

$$\boldsymbol{E}^{P,f}\big(F_S^{(f/d)}(T)\big) = F_S^{(f/d)}(0)\left(\frac{A^{(f)} + B_S^{(f)}y^{(f)}(0)}{A^{(f)} + B_S^{(f)}y^{(f)}(0)e^{\rho_{F,y}\sigma_F\sigma_y T}}\right) \tag{19}$$

where we have made the (market standard) assumption that F is a lognormal process, furthermore the F volatility is denoted by σ_F and $\rho_{F,y}$ denotes the correlation between F and y. We can now evaluate the cross currency expectation (15) as

$$\boldsymbol{E}^{S,d}\big(y^{(f)}(T)\big) = y^{(f)}(0)\left(e^{\rho_{F,y}\sigma_F\sigma_y T}\frac{A^{(f)} + B_S^{(f)}y^{(f)}(0)e^{(\sigma_y^2+\rho_{F,y}\sigma_F\sigma_y)T}}{A^{(f)} + B_S^{(f)}y^{(f)}(0)e^{\rho_{F,y}\sigma_F\sigma_y T}}\right). \tag{20}$$

Note that this expression depends on the F and the $y^{(f)}$ volatility which can both be observed in the market, also the correlation $\rho_{F,y}$ between the forward F/X process and the forward swap rate process enters the formula.

Note also that, if we set $\rho_{F,y} = 0$ or $\sigma_F = 0$, then the cross currency formula (20) reduces to the single currency formula (18).

6. Options on Convexity Corrected Rates

Not only are we interested in interest rates paid in different currencies and/or at different times, but also we want to value call and put options and digital options on these rates. Using the Change of Numeraire Theorem, we can proceed exactly as before.

Take for example a call option on a foreign interest rate observed at time T and paid in domestic currency at a later date $S \geq T$. The value of such a contract at time T is given by $V^{(d)}(T) = D_S^{(d)}(T)\max\{y^{(f)}(T) - K, 0\}$. We can value this payoff as

$$\begin{aligned} V^{(d)}(0) &= D_S^{(d)}(0)\boldsymbol{E}^{S,d}\big(\max\{y^{(f)}(T) - K, 0\}\big) &(21)\\ &= D_S^{(d)}(0)\boldsymbol{E}^{P,f}\big(\max\{y^{(f)}(T) - K, 0\}R^{(d/f)}(T)\big) &(22)\\ &= D_S^{(d)}(0)\boldsymbol{E}^{P,f}\big(\max\{y^{(f)}(T)R^{(d/f)}(T) - KR^{(d/f)}(T), 0\}\big). &(23)\end{aligned}$$

The last line can be interpreted as an exchange option between yR and KR. We know that y is a lognormal martingale under the measure $\boldsymbol{Q}^{P,f}$. In general the processes $y(t)R(t)$ and $KR(t)$ are not lognormal processes under the measure $\boldsymbol{Q}^{P,f}$. If we approximate the probability distributions of $y(T)R(T)$ and $KR(T)$ by lognormal distributions, the option value can then be evaluated by the Margrabe (1978) formula as follows:

$$\begin{aligned} V^{(d)}(0) &= D_S^{(d)}(0)\big(E_1 N(d_1) - E_2 N(d_2)\big),\\ E_1 &= \boldsymbol{E}^{P,f}\big(y^{(f)}(T)R^{(d/f)}(T)\big),\\ E_2 &= \boldsymbol{E}^{P,f}\big(KR^{(d/f)}(T)\big) = K,\\ d_{1,2} &= \frac{\log(E_1/E_2) \pm \frac{1}{2}\Sigma^2}{\Sigma},\\ \Sigma^2 &= \mathrm{Var}^{P,f}\left(\log\left(\frac{y^{(f)}(T)R^{(d/f)}(T)}{KR^{(d/f)}(T)}\right)\right)\\ &= \mathrm{Var}^{P,f}\big(\log y^{(f)}(T)\big) = \sigma_y^2 T. \end{aligned} \tag{24}$$

The expression for E_1 is the convexity corrected forward rate which can be denoted by $\tilde{y}^{(f)}$. Hence, we can simplify the expression above to obtain

$$\begin{aligned} V^{(d)}(0) &= D_S^{(d)}(0)\big(\tilde{y}^{(f)}N(d_1) - KN(d_2)\big),\\ d_{1,2} &= \frac{\log(\tilde{y}^{(f)}/K) \pm \frac{1}{2}\sigma_y^2 T}{\sigma_y\sqrt{T}}, \end{aligned} \tag{25}$$

which is the market standard method of valuing options on convexity corrected rates: apply the Black formula using the convexity corrected rate as the forward rate.

For the cases where we approximate the Radon-Nikodym derivative with the Linear Swap Rate Model, the approximation results in an expression for the Radon-Nikodym derivative which is not exactly lognormally distributed. However, the market standard method can still be used as a good approximation.

Appendix

In this appendix we derive the Linear Swap Rate Model (LSM). The LSM is due to Hunt and Kennedy (2000, Chapter 13). Let $y(t)$ be a forward par swap rate with swap start date T_0, payment dates $T_1, \ldots, T_N$, $P(t) = \sum_1^N \alpha_{i-1} D_i(t)$ the PVBP of the swap and $\boldsymbol{Q}^P$ the martingale measure associated with the numeraire P. The swap rate $y(t)$ is defined as $y(t) = \big(D_0(t) - D_N(t)\big)/P(t)$ and is therefore a martingale under $\boldsymbol{Q}^P$. Define $\hat{D}_S(t) = D_S(t)/P(t)$, then $\hat{D}_S(t)$ is also a martingale under the measure $\boldsymbol{Q}^P$.

In the LSM the assumption is made that $\hat{D}_S(T_0) = A + B_S y(T_0)$, where A is a constant and B_S is a deterministic function of S. Hence, it is assumed that the PVBP rebased discount factor $\hat{D}_S$ can be approximated by a linear expression in the swap rate y. Let us solve for the parameter A and the function B_S for $S \geq T_0$. We know that $\hat{D}_S$ should be a martingale. To check the martingale property we consider

$$\hat{D}_S(0) = \frac{D_S(0)}{P(0)} = \boldsymbol{E}^P\big(\hat{D}_S(T_0)\big) = \boldsymbol{E}^P\big(A + B_S y(T_0)\big) = A + B_S y(0).$$

The martingale property of $\hat{D}_S$ is ensured if we set

$$B_S = \frac{D_S(0)/P(0) - A}{y(0)}.$$

The parameter A can be determined as follows. Consider the identity $\sum_1^N \alpha_{i-1} \hat{D}_i(t) \equiv 1$. Expanding this expression yields

$$1 \quad \equiv \quad \underset{\displaystyle 1}{\underset{\displaystyle \|}{\left(A \sum_{i=1}^N \alpha_{i-1} \right)}} \quad + \quad \underset{\displaystyle 0}{\underset{\displaystyle \|}{\left(\sum_{i=1}^N \alpha_{i-1} B_{T_i} \right)}} \quad y(t).$$

Hence, we find $A = 1/\sum_1^N \alpha_{i-1}$. It is left as an exercise to the reader to check the condition on the B_{T_i}.

Note that, if y is a forward LIBOR rate L_i, then we have that $P(t) = \alpha D_{T+\Delta}(t)$. The LSM model now generates the following expression

$$\hat{D}_T(t) = \frac{D_T(t)}{\alpha_i D_{T+\Delta}(t)} = A + B_T L = \frac{1}{\alpha} + L(t),$$

which reflects exactly the definition of the forward LIBOR rate L_i.

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