Emerging Markets and Inflation

Lecture 2. Linear Rates and FX Intro. Part 1: FX

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Agenda for Today

Lecture 2. Linear Rates and FX Intro. Part 1: FX

Linear FX modeling aspects important in Emerging Markets

1. **Introduction to Interest Rates and FX**

2. **Basis Interest Rates Concepts**

3. **Linear FX Instruments**

Introduction to Interest Rates and FX

- **Fixed Income modelling in Flow space**
- **Pay special attention to aspects important in Emerging Markets**
- **Start with basic IR instruments intro just enough to get us to**
- **Linear FX products. Vanilla instruments surprisingly require special attention in Emerging Markets. In particular:**
  - **Forward starting FX Forward and Convexity Adjustment in them**
  - **FX Future: does it also need Convexity Adjustment?**

Introduction to Interest Rates and FX

- **Distance from broad Lecture 1 intro to Emerging Markets**
- **Go back to fundamentals: look again at basics**
- **Start with Linear Fixed Income products as fundamental to all**
- **Define products contributing to Discounting Curves construction**
- **"…*world of cash flows independent of Equities or Commodities*" [Wilmott 2000]**
  - **Will come in the next Lecture and cover in more details**

Introduction to Interest Rates and FX

- **Money lending and changing continues to relations between Interest Rates and FX**
- **Introduce basic asset and derivatives traded**
- **Discuss modeling aspects ignored (rightfully so?) in Developed Markets, but necessary in Emerging Markets**
- **Special cases of Cash Settling products and practical aspects of Non-Deliverability**

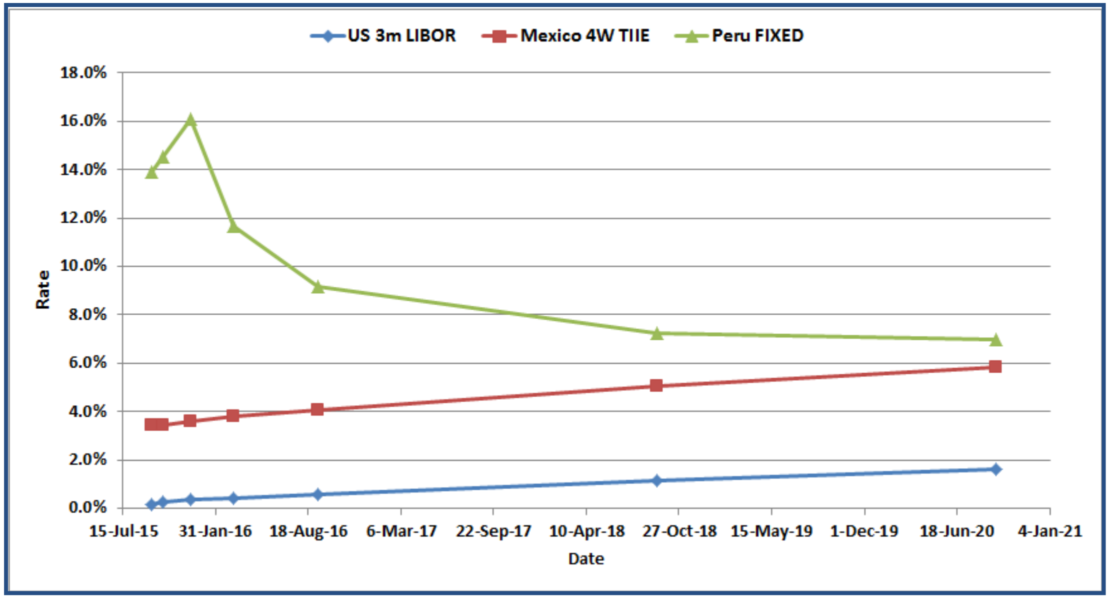
Lecture 2. Linear Rates and FX Intro. Part 1: FX

Linear FX modeling aspects important in Emerging Markets

1. Introduction to Interest Rates and FX
2. **Basis Interest Rates Concepts:**
   a) **Zero Coupon Bond, Discount Factor**
   b) **Deposit, Money Market, LIBOR rate**
3. Linear FX Instruments

Linear Interest Rates Instruments

- World of Cash Flows ➔ Future Cash Flows!
- Thus dependency on Interest Rates or
- Dependency on fundamental cost of lending and borrowing
- Present Value (price today) will require Discounting
- Discounting needs a curve ➔ Term-structure of Interest rates

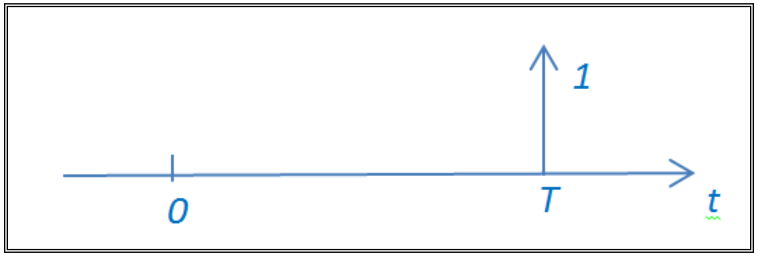


Emerging Markets and Inflation

Linear Interest Rates Instruments

- **Interest Rates change depending on Economics**
- **Depending on participants, lending and borrowing needs**
- **Fixed Income market trading**
  - **OTC (Over the Counter)**
  - **On Exchanges in Contracts limited in variety and maturity**
- **Let us now look at IR instruments needed to define a Discount Factor**

- **Zero Coupon Bond, Discount Factor**
  - **Known Fixed Cash Flow at maturity of the bond at time *T***
  - **Assume to be 1 for simplicity:**



  - **Price is by definition equals to the *Discount Factor*:**

$$\boxed{\begin{aligned} &Z(0,T) = DF(0,T) \\ &Z(t,t) = 1 \end{aligned}}$$

Linear Interest Rates Instruments

- **Deposit, Money Market and LIBOR rate**
  - **Deposit is a later return of an initial cash amount with pre-agreed interest and pre-agreed future date**
  - **Unit amount at *T* deposited at rate L with repayment at *T*+τ will have Future Value (*FV*)**

$$\boxed{FV(T,T+\tau)=1+\tau\cdot L}$$

  - **Discounted (Present) Value at time *T* in L-based economy is 1:**

$$1=(1+\tau\cdot L)\cdot Z(T,T+\tau)$$
$$Z(T,T+\tau)=\frac{1}{1+\tau\cdot L};\qquad \tau\cdot L=\frac{1}{Z(T,T+\tau)}-1 \tag{1}$$

  - **Example of L is LIBOR; τ is a time factor depending on Day Count, or ‘rule of counting days’**

Agenda for Today

Lecture 2. Linear Rates and FX Intro. Part 1: FX

Linear FX modeling aspects important in Emerging Markets

1. Introduction to Interest Rates and FX
2. Linear Interest Rates Instruments:
3. Linear FX Instruments
   a) FX Spot and Forward
   b) Non-Deliverable or Cash Settled FX Forward
   c) Forward Starting FX Forward
   d) Practical Comments on Convexity Adjustment
   e) FX Futures
   f) Convexity Adjustment in FX Futures. Practical Example

Linear FX Instruments

- **Personal bank account as first source of Interest Rates knowledge**
- **Personal International travel as a first source of FX knowledge**
- **FX as another fundamental asset class in Fixed Income**
- **Emerging Markets investment is always about border crossing, thus a very close connection between Interest Rate and FX in derivatives pricing and modelling**

Linear FX Instruments

FX Spot and Forward

- **Define *S* as FX Spot rate in currency *CCY* price of 1 US Dollar:**

$$\boxed{1^{USD} = S^{CCY}}$$

- **Define *F* as the forward amount at time *T* of the same**
- **Intuitive definition of FX exchange rate now and later**
- **Arbitrage free argument, consider two scenarios:**

A. **Invest \$1 at interest rate $r^{\$}$ for time period τ**

At the end of this period you will have:

$$\boxed{\$1 \cdot \left(1 + r^{\$} \cdot \tau\right)}$$

Emerging Markets and Inflation

FX Spot and Forward (continued)

- **Continue the arbitrage free argument, second scenario:**

  B. Convert $1 at today's exchange rate into currency CCY and invest the resulting amount S^{CCY} at interest rate r^{CCY} for time period τ

  At the end of this period you will have:

$$\boxed{S^{CCY} \cdot \left(1 + r^{CCY} \cdot \tau\right)}$$

  Convert this amount back into USD via future exchange rate F^{CCY} and end up with USD amount of

$$\boxed{\frac{S^{CCY} \cdot \left(1 + r^{CCY} \cdot \tau\right)}{F^{CCY}}}$$

FX Spot and Forward (continued)

- Results of scenarios A and B must be the same, so generalize assuming
  - Start of transaction at time *t*
  - Maturity of transaction at time *T=t+ τ*
  - Drop superscript CCY assuming notations for FX exchange rate from now on as price of some Foreign currency in USD

$$1+r^{\$}\cdot\tau=\frac{S(t,t)\cdot\left(1+r^{CCY}\cdot\tau\right)}{F(t,T)}$$

$$F(t,T)=S(t,t)\cdot\frac{\left(1+r^{CCY}\cdot\tau\right)}{\left(1+r^{\$}\cdot\tau\right)}$$

- Or re-writing it as per Eq. (1):

$$F(t,T)=S(t,t)\frac{Z^{USD}(t,T)}{Z^{CCY}(t,T)} \qquad (2)$$

FX Spot and Forward (continue)

- **Generalize concept of term investment and extend Eq.(1) from term-τ rate *L* to an annual rate *r* invested for *n* years**
- **Future value of our unit of investment then becomes $(1+r)^n$**
- **Shorten the annual term to m-times compounding:**

$$\boxed{\left(1+\frac{r}{m}\right)^{m\cdot n}}$$

- **Make *m* infinitely large and re-write Eq.(1) via exponential rates:**

$$\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{m\cdot n}=e^{r\cdot n}$$
$$Z(t,T)\propto e^{-r(T-t)}$$

FX Spot and Forward (continue)

- **FX Forward buys or sells asset at future time for pre-agreed price *K***
- ***Asset* here becomes an attribute of one of the currencies**
- ***Denominated* we call the other one**
- **Price of *Asset* is expressed in *Denominated*: \$1 costs ¥105**
- **Payoff of Future Value of FX Forward in 2 representations:**

1. **Payoff in Denominated, Quantity in Asset:**

$$\boxed{FV^D(t,T) = N^A \cdot \left[F(t,T) - K\right]}$$

2. **Payoff and Quantity in Denominated:** **(3)**

$$\boxed{FV^D(t,T) = N^D \cdot \left[1 - \frac{F(t,T)}{K}\right]}$$

- **FX Spot and Forward (continue)**
  - **FX Forward is always *Physically Settled***
  - **Means actual exchange of cash flows in two different currencies**
  - **Present Value under same no arbitrage condition:**

$$PV^D(t,T) = N^A \cdot \left[F(t,T) - K\right] \cdot Z^D(t,T)$$
$$PV^D(t,T) = N^D \cdot \left[1 - \frac{F(t,T)}{K}\right] \cdot Z^D(t,T) \qquad (4)$$

  - **And re-write Eq.(2) using *Asset* and *Denominated* notations:**

$$F(t,T) = S \cdot \frac{Z^A(t,T)}{Z^D(t,T)}$$

Linear FX Instruments

FX Spot and Forward (continue)

- **Introduce few FX timing concepts**
- ***Spot*: a date on which funds become physically available. Enter into transaction *Today*, actually execute it on *Spot***
- ***Spot Date rule* is how to get from *Today* to *Spot*. Usually 2 "good" business days in both currencies, but there are exceptions**
- **A 1 year FX Forward closed *Today* starts counting days on *Spot* and matures or expires or settles 1 year after *Spot* on *Expiry* or *Settlement* date**
- **Forward FX Exchange rate $F(t,T)$ used to settle the transaction will be the Spot FX exchange rate $S(T,T)$ observed on the market Spot Date rule number of days before the Expiry!**

Non-Deliverable or Cash Settled FX Forward (NDF)

- **Feature common and special for Emerging Markets**
- **Cash Settlement for FX products in Non-Deliverable currencies**
- **FX Forward physically settled exchanges funds in two currencies**
- **Same could be done via cash-settlement or netting of actual amount from Eq.(4) in one**
- **Extending Eq.(3) to an NDF and generalize to 3rd CCY settlement:**

1. **Quantity in Asset** $$\boxed{FV^S(T) = N^A \cdot \left[F(T) - K\right] \cdot F^S(T)}$$

(5)

2. **Quantity in Denominated** $$\boxed{FV^S(T) = N^D \cdot \left[1 - \frac{F(T)}{K}\right] \cdot F^S(T)}$$

Non-Deliverable or Cash Settled FX Forward (NDF) continue

- ***F(T)*** **is in default FX conventions of Denominated per Asset**
- ***$F^S(T)$*** **is FX Spot rate at T for Settlement Ccy per Denominated**
- **Simple two currencies case with Denominated as Non-Deliverable:**

1. **Quantity in Asset**

$$FV^A(T) = N^A \cdot \left[1 - \frac{K}{F(T)}\right]$$

(6)

2. **Quantity in Denominated**

$$FV^A(T) = N^D \cdot \left[\frac{1}{F(T)} - \frac{1}{K}\right]$$

Non-Deliverable or Cash Settled FX Forward

A bit of History and Economics per [Lipscomb, 2005]

- **In Emerging Markets NDFs are used to hedge or express view on currencies with limited access**
- ***Fixing Rate*: FX Exchange rate used to cash settle transaction sourced from a pre-agreed provider at a pre-agreed time**
- **Usually based on the same Ccy FX Spot Rate traded *onshore***
- ***Onshore* (compare to *Offshore*) describes purely local trading. Settles in local Ccy, driven by local funding rates and local central bank lending rules and regulations**
- **Offshore institutions trading onshore carry cross-border cash transfer or *Convertibility* risk. Hence possible difference in pricing**
- **NDF markets starting around 1990's in Ccys with expected regime change**
- **ISDA added NDF settlement to FX and currency option definitions in 1997**

Non-Deliverable or Cash Settled FX Forward (continue)

- **Look in more details at FX Forward physical settlement: receive K_p^{CCY} in exchange for \$1 at T_{Settle} with zero cost today**

$$\boxed{\begin{aligned} &K_p^{CCY} \cdot Z^{CCY}\left(T_{Spot}, T_{Settle}\right) = S \cdot Z^{USD}\left(T_{Spot}, T_{Settle}\right) \\ &K_p^{CCY} = S \cdot \frac{Z^{USD}\left(T_{Spot}, T_{Settle}\right)}{Z^{CCY}\left(T_{Spot}, T_{Settle}\right)} \end{aligned}} \quad (7)$$

- **On T_{Settle} we exchange cash via FX rate set on *(T_{Settle}- SpotDateRule)***

- **So receiving K_p^{CCY} in Denominated is equivalent to receive in Asset**

$$\boxed{\frac{K_p^{CCY}}{F\left(T_{Settle} - SpotDateRule\right)}} \quad (8)$$

Non-Deliverable or Cash Settled FX Forward (continue)

- **NDF Fixing rate is set on date T_{Fix}, but trade is settled after a specifically agreed offset we call *SettleDateOffset*:**

$$T_{Settle} = T_{Fix} + SettleDateOffset$$

- **Often *SettleDateOffset* = *SpotDateRule*, but not always. Generally:**

$$\begin{aligned}\left(FV\ of\ CCY\ Leg\right)^{USD} &= \frac{K_c^{\ CCY}}{F\left(T_{Fix}\right)} \\ &= \frac{K_c^{\ CCY}}{F\left(T_{Settle} - SettleDateOffset\right)}\end{aligned}$$

- **Compare to Eq.(8): prices of Physical and Cash Settled transaction are equal only if *SettleDateOffset* = *SpotDateRule***

Non-Deliverable or Cash Settled FX Forward (continue)

- In a more general case an expression for Fixed amount is

$$
\begin{aligned}
K_c &= K_p \cdot \frac{F\left(T_{Fix}\right)}{F\left(T_{Setle} - SettleDateOffset\right)} \\
&= K_p \cdot \frac{Z^{CCY}\left(T_{Fix} + SpotDateRule, T_{Settle}\right)}{Z^{USD}\left(T_{Fix} + SpotDateRule, T_{Settle}\right)} \\
&= S \cdot \frac{Z^{CCY}\left(T_{Spot}, T_{Fix} + SpotDateRule\right)}{Z^{USD}\left(T_{Spot}, T_{Fix} + SpotDateRule\right)}
\end{aligned} \qquad (9)
$$

- NDF is driven by ratio of Discount Factors from Spot Date to Fixing Date + SpotOffset, while physically settled forward is always driven by ratio of Discount Factors from Spot Date to the Settlement date

Non-Deliverable or Cash Settled FX Forward (continue)

- **PV of our NDF then will look like this**

$$\boxed{\begin{aligned} pv^{\$} &= \frac{\left[F\left(T_{Fix}\right)-K\right]}{K}\cdot Z^{CCY}\left(T_{Spot},T_{Fix}+SpotDateRule\right)\cdot Z^{\$}\left(T_{Fix}+SpotDateRule,T_{Settle}\right)\\ &= \frac{\left[F\left(T_{Fix}\right)-K\right]}{K}\cdot Z^{CCY}\left(T_{Spot},T_{Fix}+SpotDateRule\right)\times\\ &\quad\times Z^{\$}\left(T_{Fix}+SpotDateRule,T_{Fix}+SettleDateOffset\right)\end{aligned}} \quad (10)$$

- **Fixing Date hidden in transaction confo drives price of the transaction!**

HW1: **Estimate price and risk differences when evaluating non-standard settling NDFs as standard using market data provided. Use ARS-USD FX pair with market data supplied. Estimate present value correction for 5, 10 and 30 days of Settlement after the Fixing compared to the standard 2 days offset rule coinciding with Spot Date rule**

Linear FX Instruments

- **Forward Starting FX Forward**

Practical comments on Convexity Adjustment

- **Investor looking to a forward FX transaction at future date**

- **Strike is set at a later date as an offset to some reference: FX Spot**

- **Still linear FX, but now with *Convexity Adjustment*: a non-linear, vol dependent and / or model dependent correction to deterministic price. We will demonstrate and prove that a minute later!**

- **Intuitively vols, rates and correlations dependent**

- **Thus could be and usually is ignored in low rates and vols regimes, but cannot do so in Emerging Markets!**

Forward Starting FX Forward

Practical comments on Convexity Adjustment continued

- **Two main challenges in determining Convexity:**

1. **Choice of a Model (both for Developed and Emerging Markets):**
   - **Is to represent underlying dynamics, but not to be too complex**
   - **Most natural choices are simple 1-Factor models like Ho-Lee or Hall-White**
   - **Model does not have to coincide with main model used for underlying**

2. **Model Parameters calibration (In Emerging Markets):**
   - **Rates Vols, FX Vols and Correlations in markets that barely trade NDFs**
   - **Go back to subjects discussed in lecture 1: historical estimates of model parameters in Emerging Markets**

- **We only sketch derivation concentrating on qualitative results**

Forward Starting FX Forward

Convexity Adjustment via two approaches. Approach 1

- **Start from Eq.(6) for NDF with Quantity in Denominated:**

$$FV^{A}(T) = N^{D} \cdot \left[\frac{1}{F(T)} - \frac{1}{K} \right]$$

- **Add ΔS offset in FX pips (points) to FX Spot as a Strike setting rule:**

$$K = S(T_{StrikeSet}) + \Delta S$$

- **Note that substitution of future value of FX Spot in this expression with deterministic FX Forward does not work anymore:**

$$PV = \mathbf{E}\left[\frac{1}{S(T_{Expiry})} - \frac{1}{S(T_{StrikeSet}) + \Delta S} \right] \qquad (11)$$

$$\mathbf{E}\left[\frac{1}{S(T_{StrikeSet}) + \Delta S} \right] \propto \frac{1}{F(T_{StrikeSet})} \cdot \mathbf{E}\left[\frac{S(T_{StrikeSet})}{S(T_{StrikeSet}) + \Delta S} \right]$$

Forward Starting FX Forward

Convexity Adjustment via two approaches. Approach 1 continue

- **Continue from before we can show Convexity arising from**

$$\boxed{\mathbf{E}\left[\frac{S(T_{StrikeSet})}{S(T_{StrikeSet})+\Delta S}\right]}$$

- **Re-write it to reduce to**

$$\mathbf{E}\left[\frac{S(T_{StrikeSet})}{S(T_{StrikeSet})+\Delta S}\right]=\mathbf{E}\left[1-\frac{\Delta S}{S(T_{StrikeSet})+\Delta S}\right]$$

$$=1-\left(\frac{\Delta S}{F+\Delta S}\right)\cdot\mathbf{E}\left[\frac{1}{1+\left(1-\frac{\Delta S}{F+\Delta S}\sigma\sqrt{T_{StrikeSet}}\right)\cdot\mathbf{X}}\right]$$

- **Where σ is FX Vols and *X* is defined as**

$$\boxed{\begin{aligned}&\mathbf{E}[\mathbf{X}]=0\\&Var[\mathbf{X}]=1\end{aligned}}$$

Forward Starting FX Forward

Convexity Adjustment via two approaches. Approach 1 continue

- **Reduce some more**

$$\mathbf{E}\left[\frac{S\left(T_{StrikeSet}\right)}{S\left(T_{StrikeSet}\right)+\Delta S}\right]=1-\alpha\cdot\mathbf{E}\left[\frac{1}{1+\beta\cdot\mathbf{X}}\right]$$
$$\equiv 1-\frac{\Delta S}{F+\Delta S}\cdot CxtyAdj$$

where

$$CxtyAdj=\mathbf{E}\left[\frac{1}{1+\beta\cdot\mathbf{X}}\right]$$
$$\beta=\left(1-\alpha\right)\cdot\sigma\sqrt{T_{StrikeSet}}$$
$$\alpha=\frac{\Delta S}{F\left(T_{StrikeSet}\right)+\Delta S}$$

Forward Starting FX Forward

Convexity Adjustment via two approaches. Approach 1 continue

- **Use simple expansion to estimate:**

$$\boxed{\mathbf{E}\left[\frac{1}{1+\beta\cdot\mathbf{X}}\right]=1+\sum_{k=1}^{\infty}(-1)^k\cdot\beta^k\cdot\mathbf{E}[\mathbf{X}]}$$

- **Relaxing assumptions on *X* one can show**

$$\mathbf{E}\left[\frac{1}{1+\beta\cdot\mathbf{X}}\right]\propto\exp\left(\beta^2\right)$$

$$\mathbf{E}\left[\frac{S\left(T_{StrikeSet}\right)}{S\left(T_{StrikeSet}\right)+\Delta S}\right]=1-\alpha\cdot\mathbf{E}\left[\frac{1}{1+\beta\cdot\mathbf{X}}\right]$$

$$=1-\frac{\Delta S}{F\left(T_{StrikeSet}\right)+\Delta S}\exp\left(\frac{F^2\left(T_{StrikeSet}\right)}{\left(F\left(T_{StrikeSet}\right)+\Delta S\right)^2}\sigma^2T_{StrikeSet}\right)$$

HW2: Make some numerical estimates of the Convexity Adjustment in Forward Starting NDF using market data provided. Argue the validity limits

Forward Starting FX Forward

Convexity Adjustment via two approaches. Approach 2

- **FX Forward with Asset Notional paying in Denominated as in Eq.(4):**

$$\boxed{pv(t) = \mathbf{E}\left[S(T) - S\left(T_{StrikeSet}\right)\right] \cdot Z^D(t,T)}$$

- **In T-Forward Domestic measure:**

$$\boxed{pv(t) = \left[S \cdot \frac{Z^A(t,T)}{Z^D(t,T)} - S \cdot \frac{Z^A\left(t,T_{StrikeSet}\right)}{Z^D\left(t,T_{StrikeSet}\right)} \cdot CxtyAdj\left(t,T_{StrikeSet},T\right) \right] \cdot Z^D(t,T)}$$

- **Need a model now to derive the adjustment**

Forward Starting FX Forward

Convexity Adjustment via two approaches. Approach 2 continued

- **Use 3-Factor FX model that we will use many times in this course**

$$\begin{cases} \dfrac{dS}{S} = [r_d(t) - r_a(t)]dt + \sigma_S(t)dW_S \\ d\gamma_d(t) = [\theta_d(t) - \kappa_d \gamma_d(t)]dt + \sigma_d dW_d(t) \\ d\gamma_a(t) = [\theta_a(t) - \kappa_a \gamma_a(t) - \rho_{S,a}\sigma_S(t)\sigma_a]dt + \sigma_a dW_a \end{cases}$$

where

$$\begin{aligned} &\rho_{S,a} = Corr(dW_S, dW_a) \\ &\theta_i(t) = [1 - \exp(-2\kappa_i t)]\sigma_i^2 / 2\kappa_i, \quad i = d, a \\ &r_i(t) = f_i(0,t) + \gamma_i(t) \end{aligned}$$

- **Change *T-Forward* measure to $T_{StrikeSet}$*-Forward* to derive**

$$\mathbf{E}\left[S\left(T_{StrikeSet}\right)\right] = S \cdot \frac{Z^A\left(t, T_{StrikeSet}\right)}{Z^D\left(t, T_{StrikeSet}\right)} \cdot CxtyAdj\left(t, T_{StrikeSet}, T\right)$$

FX Future

- FX Forward exposes counter-parties to a possible other side default
- Future mimics Forward payoff, trades on Exchange minimizing counterparty risk via daily settlements of margin payments
- Collection of margins replicates Forward during the life of the trade
- Daily margin payment reduces FX Future at expiry to FX Spot trade
- Price of 1 lot (unit) of FX Future is

$$\boxed{pv = F(T) - C}$$

F(T) - expected FX Spot at Future's expiry

C – Exchange traded price of FX Future

- Note no discounting due to daily resetting

FX Future. Convexity Adjustment

- Daily cash from margins is reinvested and earns interest
- Every book on Math Finance mentions IR Future Convexity
- None mentions FX. Why? ➔
- Same as before: dependency on vols and rates allows to ignore it in low vols and rates environment
- Also DM mostly trade short dated FX Futs, while EM goes out to 5 years
- Is very important in Emerging Markets
- Look at examples of RUB and BRL FX markets and long dated FX Futures

FX Future. Convexity Adjustment continued

- A bit more formal derivation per [Vaillant, 1995]
- Simplify notations reducing to subscript. Price of a Forward at *T* and today at *0* in Denominated:

$$\boxed{\begin{aligned} FV_T &= S_T - K \\ PV_o &= \left[F_o - K\right] \cdot Z_o \end{aligned}}$$

- Price of a Future at Expiry is $\Phi_T = S_T$, or $FV_T = \Phi_T - K$
- Thus generally speaking $PV_o = f\left(Z_o, \Phi_o\right)$ for some function *f:*

$$\boxed{\left[F_o - K\right] \cdot Z_o = f\left(Z_o, \Phi_o\right)}$$

- Define $v_o = f\left(Z_o, \Phi_o\right)$ as amount difference in contract prices we received at inception, and engage in continuous trading strategy reinvesting proceeds into zero coupon bond of price *Z*

FX Future. Convexity Adjustment continued

- **Price of this portfolio π_t then behaves like**

$$d\pi_t = \theta_t d\Phi_t + \frac{\pi_t}{Z_t} dZ_t$$

Θ is a trading strategy we discuss later

- **With solution**

$$\pi_t = Z_t \left(\frac{v_o}{Z_o} + \int_o^t \hat{\theta}_t d\hat{\Phi}_t \right) \qquad \textbf{(13)}$$

- **With new process C_t defining the hatted processes:**

$$\hat{\Phi}_t \equiv \Phi_t / C_t$$

$$\hat{\theta}_t \equiv \theta_t \cdot C_t / Z_t$$

$$C_t \equiv \exp\left(\int_0^t \frac{1}{\Phi_s \cdot Z_s} d\langle \Phi, Z \rangle_s \right)$$

FX Future. Convexity Adjustment continued

- **We assume existence of v_o and strategy θ to ensure**

$$\pi_T = \pi_T(v_o, \theta) = [\Phi_T - K] \cdot Z_o$$

- **Combining this with Eq.(13) earlier gives us**

$$\frac{v_o}{Z_o} + \int_o^T \hat{\theta}_t d\hat{\Phi}_t = \Phi_T - K$$

- **Take expectation under measure where $\hat{\Phi}_t$ is martingale:**

$$v_o = (\mathbf{E}[\Phi_T] - K) \cdot Z_o$$

- **And for deterministic C_t**

$$\mathbf{E}[\Phi_T] = \mathbf{E}[\hat{\Phi}_T \cdot C_T] = C_T \cdot \mathbf{E}[\hat{\Phi}_T] = C_T \cdot \Phi_o$$
$$f(Z_o, \Phi_o) \equiv v_o = (C_T \cdot \Phi_o - K) \cdot Z_o$$

FX Future. Convexity Adjustment continued

- **Thus a generalized Convexity is**

$$
\begin{aligned}
&[F_o - K]\cdot Z_o = (C_T \cdot \Phi_o - K)\cdot Z_o \\
&F_o = C_T \cdot \Phi_o \\
&C_T = \exp\left(\int_0^T \frac{1}{\Phi_t \cdot Z_t} d\langle \Phi, Z\rangle_t\right)
\end{aligned}
\qquad (14)
$$

- **Make some assumptions for processes involved:**

$$
\begin{aligned}
&d\Phi_t = \mu_t \Phi_t dt + \sigma_\Phi \Phi_t dW_t \\
&Z_t \equiv \exp(-(T-t)\cdot R_t) \\
&dR_t = \gamma(R_\infty - R_t)dt + \sigma_R R_\infty dW''_t
\end{aligned}
\qquad (15)
$$

- **To reduce it in case of constant vols and correlations to**

$$
\begin{aligned}
C_T &= \exp\left[-R_\infty \int_o^T (T-t)\sigma_R \cdot \sigma_\Phi \cdot \rho dt\right] \\
&= \exp\left[-\sigma_R \cdot \sigma_\Phi \cdot \rho \cdot R_\infty \frac{T^2}{2}\right]
\end{aligned}
$$

FX Future. Convexity Adjustment continued

- **Turn to our FX Future decomposing it into Spot FX and Rates again. Proxy Future in Eq.(14) and (15) with Forward *F***

$$\boxed{F_t = S_t \frac{Z_t^{\ A}}{Z_t^{\ D}} = S_t \cdot \exp\left[\left(r_D - r_A\right)\cdot\left(T-t\right)\right]}$$

- **Assuming exponential rates form we can write for return**

$$\boxed{d\log F_t = d\log S_t + \left(T-t\right)\cdot\left(dr^D - dr^A\right)} \qquad \textbf{(16)}$$

- **Repeating Stochastic part of processes in Eq.(15) needed for covariance calculations in Eq.(14):**

$$\boxed{\begin{aligned} & dr^D \propto \sigma_D R_\infty^D dW^D \\ & dr^A \propto \sigma_A R_\infty^A dW^A \\ & dS_t \infty \sigma_S S_t dW^S \\ & dF_t \infty \sigma_\Phi F_t dW^\Phi \end{aligned}}$$

FX Future. Convexity Adjustment continued

- **Plugging that into Eq.(16) and writing just the stochastic part gives us**

$$\boxed{\sigma_\Phi dW^\Phi = \sigma_S dW^S + (T-t)\left[\sigma_D R_\infty^D dW^D - \sigma_A R_\infty^A dW^A\right]}$$

- **Reminding Convexity expression Eq.(14):**

$$\boxed{C_T = \exp\left(\int_0^T \frac{1}{\Phi_t \cdot Z^D{}_t} d\left\langle \Phi, Z^D \right\rangle_t\right)}$$

- **We could repeat derivation above to arrive at**

$$C_T = C_1 \cdot C_2 \cdot C_3$$

$$C_1 = \exp\left[-R_\infty^D \int_o^T (T-t)\sigma_D \sigma_S \rho_{D,S} dt\right]$$

$$C_2 = \exp\left[-\left(R_\infty^D\right)^2 \int_o^T (T-t)^2 (\sigma_D)^2 dt\right]$$

$$C_3 = \exp\left[-R_\infty^D \cdot R_\infty^A \int_o^T (T-t)^2 \sigma_D \sigma_A \rho_{D,A} dt\right]$$

- **FX Future. Convexity Adjustment continued**

- **And lastly assuming again constant vols and correlations**

$$-\log C_1 = \frac{T^2}{2}\sigma_s \cdot \sigma_D \cdot R_\infty^D \cdot \rho_{D,S}$$

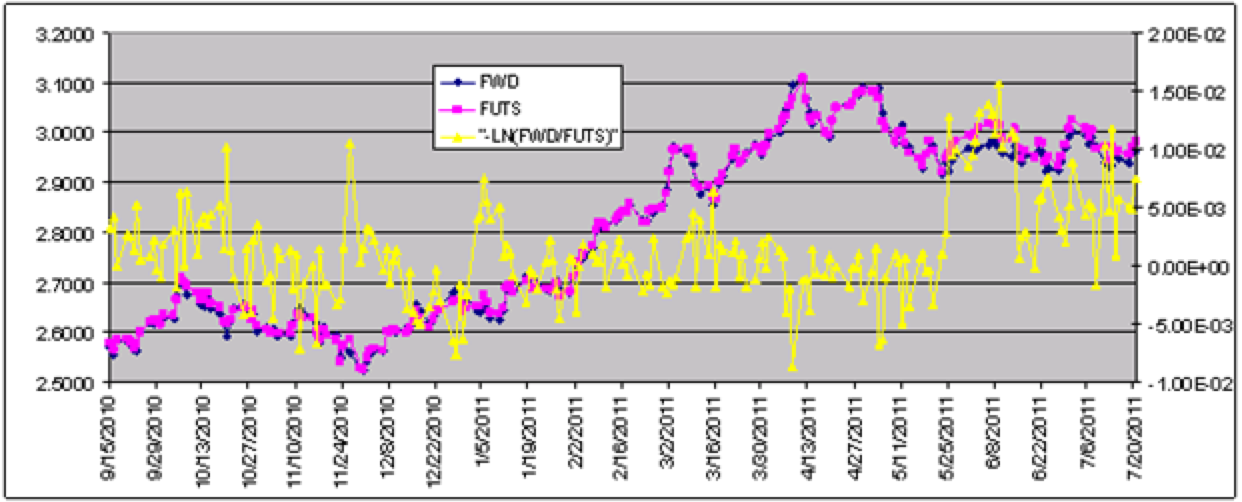
$$-\log C_2 = \frac{T^3}{3}\left(\sigma_D \cdot R_\infty^D\right)^2$$

$$\log C_3 = \frac{T^3}{3}\sigma_D \cdot R_\infty^D \cdot \sigma_A \cdot R_\infty^A \cdot \rho_{D,A}$$

FX Future. Convexity Adjustment

Practical Emerging Markets Example

- Convexity dependent on vols and rates and Expiry of Future contract
- Look at Sep15 USDRUB FX Future study performed in 2011
- Historical data observation from Sep 2010 to July 2011 show more expensive Future contract



FX Future. Convexity Adjustment

Practical Emerging Markets Example

- Recall Practical comments on Convexity: model params from History
- Practical side of Trading: does market recognize this Convexity?
- How and what can we hedge?
  - Not the IR vols Vega and correlations Delta
  - Sometimes not even the FX Vol Vega
  - But can hedge by FX Forward with size adjusted by Convexity

FFT: How much risk can we allow to bleed here? How much of a residual PnL will be lost due to this hedging inefficiency?

References

[Wilmott 2000] Wilmott, P. (2000). Quantitative Finance. John Wiley & Sons Ltd.

[Lipscomb, 2005] Lipscomb. (2005). An Overview of Non-Deliverable Foreign Exchange Forward Markets. Federal Reserve Bank of New York

[Vaillant, 1995] Vaillant, N. (1995). Convexity Adjustment between Futures and Forward Rates Using a Martingale Approach.

Homework data

1. **Estimate price and risk differences when evaluating non-standard settling NDFs as standard. Use ARS-USD FX pair with market data supplied. Estimate present value correction for 5, 10 and 30 days of Settlement after the Fixing compared to the standard 2 days offset rule coinciding with Spot Date rule.**

SpotDateRule, [days]	2		Days from Today	End Date	USD $r_{Spot\text{->}Settle}$	ARS $r_{Spot\text{->}Settle}$
			2	13-Sep-16		
Today	9-Sep-16		5	14-Sep-16	0.58%	19.67%
SpotDate	13-Sep-16		10	19-Sep-16	0.62%	19.67%
			30	9-Oct-16	0.75%	19.59%
			91	9-Dec-16	0.86%	19.45%
			181	9-Mar-17	0.91%	19.19%

2. **Make some numerical estimates of the Convexity Adjustment in Forward Starting NDF using market data provided. Argue the validity limits.**

FX Spot	3.75	
FX Multiplier	10,000	
Notional	$1,000,000	
	Strike Set	Expiry
Tau, [years]	1	2
FX Fwd points (PIPs)	4500	6800
USD Disc Factor	0.994	0.981
Strike Offset FX points (PIPs)	2300	
FX Fwd	4.20	4.43