INTEREST RATE MODELS

Homework assignment #3

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Problems

1. Derive formulas (27) - (30) of Lecture Notes #3 for the valuation of calls and puts in the shifted lognormal model $dF(t) = (\sigma_1 F(t) + \sigma_0) dW(t)$. Note that you do not need to start with Kolmogorov's backward equation. Instead, you can work with the analog of formula (3) of Lecture Notes #3 for the shifted lognormal model, and simply carry out the integration.

2. Consider an at the money option.

   (a) Show that the implied normal volatility σ_{n} can be expressed in terms of the implied lognormal volatility σ_{ln} as

   $$\sigma_{\mathrm{n}}(T, F_0, F_0, \sigma_{\mathrm{ln}}) = \sqrt{\frac{2\pi}{T}}\, F_0 \,\mathrm{erf}\Big(\frac{\sqrt{T}}{2\sqrt{2}}\,\sigma_{\mathrm{ln}}\Big), \tag{1}$$

   where

   $$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \tag{2}$$

   is the *error function*.

(b) Show that, for $\sigma_{\ln} T$ small, the following approximation holds:

$$\sigma_{\mathrm{n}} = F_0\, \sigma_{\ln} \Big(1 - \frac{1}{24}\, \sigma_{\ln}^2 T + \frac{1}{640}\, (\sigma_{\ln}^2 T)^2 + \dots \Big). \tag{3}$$

You can view this calculation as the simplest example of the type of analysis that is used in the derivation of the SABR asymptotic formula.

3. The asymptotic formulas for the implied normal and lognormal volatilities $\sigma_{\mathrm{n}}(T, K, F_0, \sigma_0, \alpha, \beta, \rho)$ and $\sigma_{\ln}(T, K, F_0, \sigma_0, \alpha, \beta, \rho)$, respectively, in the SABR model, as given in Lecture Notes #4, contain singularities of the type $\frac{0}{0}$ for the at the money strikes. Use l'Hopital's rule (or any other method of your choice) in order to calculate the at the money implied volatilities $\sigma_{\mathrm{n}}(T, F_0, F_0, \sigma_0, \alpha, \beta, \rho)$ and $\sigma_{\ln}(T, F_0, F_0, \sigma_0, \alpha, \beta, \rho)$. You will find the result very useful when implementing the SABR model in computer code.

This assignment is due on March 13