9831 Real Analysis and Probability

Midterm - Baruch MFE, Fall 2014

W_t denotes a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ in the following problems.

(1) Determine the mean and the variance of $\int_0^t W_s^2 dW_s$.

(2) Let $M_t = \mathbb{E}\left[e^{W_T} | \mathcal{F}_t\right]$, for $0 \le t \le T$. Find an adapted process θ such that $M_t = M_0 + \int_0^t \theta_s dW_s$.

(3) Determine the distribution of the minimum of $X_t = e^{\sigma W_t}$ over the interval $[0, T]$, $\sigma > 0$ is a constant.

(4) Solve the SDE

$$dX_t = \left(\frac{\sigma^2}{4} - \theta X_t\right) dt + \sigma\sqrt{X_t} dW_t, \quad X_0 = x,$$

where θ and σ are constants. Find the expected value of X_t.

(5) Assume zero interest rate and the price process S_t of an underlying in risk neutral measure is driven by

$$dS_t = (\alpha + \sigma S_t)dW_t.$$

(a) Solve the SDE with initial condition S_0.

(b) Calculate the price of ATM call.

(6) Let X_t satisfy

$$dX_t = \lambda t(m - X_t)dt + \sigma t dW_t,$$

where λ, m and σ are constants.

(a) Solve the SDE

(b) Determine the mean and variance of X_t.

(7) Consider the economy consisting of two assets S_t and N_t following respectively the SDEs

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dW_t, \quad S_0 = s;$$
$$\frac{dN_t}{N_t} = \mu_N dt + \sigma_N dW_t, \quad N_0 = n,$$

where μ_N, μ_S, σ_N, and σ_S are constants. Assume $\sigma_N \neq \sigma_S$.

(a) Use N_t as numéraire, determine the associated equivalent martingale measure $\widetilde{\mathbb{P}}$.

(b) Determine the price of the contingent claim which pays off $\max\{S_T, N_T\}$ at expiry T.

10pts each.

TOTAL: 100pts.