8. Extension of the Option Model to Continuous Time

Limiting distribution of the binomial model

Recall that the terminal spot price of a binomial pricing model takes on values S_T of the form…

$$S_T = S_0 u^X d^{N-X}$$

…where X is a binomial random variable with N trials and success probability p_u. This means that the log-return is also a binomial random variable…

$$\ln\left(\frac{S_T}{S_0}\right) = X \ln u + (N - X)\ln d = N \ln d + X(\ln u - \ln d)$$

…that has been shifted and rescaled.

It is a commonplace of probability theory that binomial random variables, if appropriately rescaled, approach normal random variables for N large. Shreve, for example, shows this result and provides the outline of a proof for a random walk rescaled by the square root of N. Showing the same for the CRR parameterization in particular is somewhat more difficult.

We will simply assert, without any attempt at proof here, that the log-return under the binomial model is, at the limit of infinite N, normal. Our main concern is to determine the properties of this random variable and how it may be used in option pricing.

The lognormal distribution

We begin with a normal random variable X distributed with mean μ and standard deviation v; it is usually most convenient to express other normal random variables in terms of the standard normal random variable Z, with mean zero and a standard deviation of 1. Thus, we define the variable of interest Y:

$$Y = e^X = e^{\mu + vZ}$$

The cumulative distribution function of our lognormal random variable is simply:

$$F_Y(y) = P(Y < y) = P\left(e^{\mu + vZ} < y\right) = P(\mu + vZ < \ln y) = P\left(Z < \frac{\ln y - \mu}{v}\right)$$

$$F_Y(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln y - \mu}{v}} e^{-\frac{1}{2}z^2}\, dz = N\left(\frac{\ln y - \mu}{v}\right)$$

We use the definite integral of the standard normal distribution so often that we will usually abbreviate it $N(\bullet)$.

Its density function is thus:

$$f_Y(y) = F_Y'(y) = N'\left(\frac{\ln y - \mu}{v}\right)\frac{1}{vy}$$

$$f_Y(y) = \frac{1}{vy\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{v}\right)^2}$$

While this is good to know, we are first interested in determining its mean so that we can appropriately apply this model to asset pricing.

$$EY = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\mu + vz} e^{-\frac{1}{2}z^2}\, dz = e^{\mu} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(z^2 - 2vz\right)} dz$$

$$EY = e^{\mu + \frac{1}{2}v^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(z^2 - 2vz + v^2\right)} dz = e^{\mu + \frac{1}{2}v^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(z - v)^2}\, dz$$

$$EY = e^{\mu + \frac{1}{2}v^2}$$

From our previous calibration exercises with the binomial model, we know that this model must match the forward price in order for our expectation to be risk-neutral and thus avoid arbitrage:

$$ES_0 e^{\mu + vZ} = S_0 e^{(r-q)T}$$

$$S_0 e^{\mu + \frac{1}{2}v^2} = S_0 e^{(r-q)T}$$

$$\mu = (r - q)T - \frac{1}{2}v^2$$

The remaining question, then, is what we ought to choose as the parameter v.

One more discrete-time model

Suppose that, as before in our binomial model, we discretize the time to expiry T in N intervals of equal length $\Delta t = T / N$. In this case, we assume that the log-returns are iid normal with mean $(r - q)\Delta t - 1/2v_{\Delta t}^2$ and standard deviation $v_{\Delta t}$. That is, the terminal spot prices at each interval are…

$$S_{\Delta t} = S_0 Y_1 = S_0 e^{X_1} = S_0 e^{(r-q)\Delta t - \frac{1}{2}v_{\Delta t}^2 + v_{\Delta t} Z_1}$$

$$S_{2\Delta t} = S_0 Y_1 Y_2 = S_0 e^{X_1 + X_2} = S_0 e^{2\left[(r-q)\Delta t - \frac{1}{2}v_{\Delta t}^2\right] + v_{\Delta t}(Z_1 + Z_2)}$$

…and thus for arbitrary N we have the log-return:

$$\ln\left(\frac{S_{N\Delta t}}{S_0}\right) = (r - q)N\Delta t - \frac{1}{2}\left(v_{\Delta t}\sqrt{N}\right)^2 + v_{\Delta t} \sum_{i=1}^{N} Z_i$$

This sum of normal random variables is also normal, and since they are all independent, the variance of the sum is the sum of the variances—in this case, N. So we may replace this sum with another standard normal random variable….

$$\ln\left(\frac{S_{N\Delta t}}{S_0}\right) = (r - q)N\Delta t - \frac{1}{2}\left(v_{\Delta t}\sqrt{N}\right)^2 + v_{\Delta t}\sqrt{N} Z$$

This is an interesting result. Over multiple periods, the log-return has the same form with new standard deviation:

$$v_{N\Delta t} = v_{\Delta t}\sqrt{N}$$

Define the volatility σ as the standard deviation as above rescaled to a time of 1 year. Then for our interval Δt, if the year can be divided into an integer number of intervals of this size, it is clear that:

$$\sigma = v_{\Delta t}\sqrt{\frac{1}{\Delta t}}$$

$$v_{\Delta t} = \sigma\sqrt{\Delta t}$$

Even if we do not have an integer number of intervals, we may still calculate this quantity, which will then operate as desired in our discrete-time model even if 1 year is not a point on our time discretization.

Thus, to some maturity $T = N\Delta t$ in our discrete-time model, the log-return becomes:

$$\ln\left(\frac{S_{N\Delta t}}{S_0}\right) = (r-q)N\Delta t - \frac{1}{2}\left(\sigma\sqrt{N\Delta t}\right)^2 + \sigma\sqrt{N\Delta t}Z$$

$$\ln\left(\frac{S_T}{S_0}\right) = \left(r - q - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z$$

Note that this rescaling of the standard deviation of the log-return no longer contains any explicit dependence upon the size of the time step. In fact, the smoothness with which this model rescales makes one wonder whether it works only in discrete time.

Geometric Brownian motion

We note that, in the case of zero volatility, the above reduces to our familiar expression for a forward price:

$$\ln\left(\frac{S_T}{S_0}\right) = (r-q)T$$

If we assume for the moment that both r and q are constant, then this is a particular solution of the differential equation…

$$d(\ln S_t) = (r-q)dt$$

…over time T starting at $\ln S_0$. The question is whether there is some equivalent expression when the volatility is nonzero. The second term of our equation for the distribution of the log-return at arbitrary time T is evidently somewhat problematic.

As a notation, we define the random increment dW_t so that it satisfies the equation…

$$\int_0^T dW_t = \sqrt{T}Z$$

…and thus can render our statement of the distribution of the log-return in the differential form:

$$d(\ln S_t) = \left(r - q - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$$

The instantaneous increment dW_t generates what is known as a Wiener process, and its properties are largely inherited from the discrete-time model we posed above: The increments are independent and in some limiting sense still "normally" distributed. These increments evidently have mean zero, and from the definition above, we see that this integral also has variance:

$$Var\left[\int_0^T dW_t\right] = Var\left[\sqrt{T}Z\right] = T$$

That is, this process "accumulates" variance at a constant unit rate per year. The Wiener process is the basis of Brownian motion, and has a variety of interesting properties that you will study in other courses in great detail.

We are mainly interested in the notation and an introduction to the concept. This process—called geometric Brownian motion since it is the instantaneous *return* on the asset that is driven by the Wiener process—is central to continuous-time finance. The equation specifying its dynamics is usually given as…

$$\frac{dS}{S} = (r-q)dt + \sigma dW_t$$

…which is equivalent to our statement above concerning the evolution of the log-return, although we will require another tool to show that equivalence.

For now, it suffices to see that the discrete-time models we have contemplated so far can be extended into continuous time while retaining the same force they have previously had. In particular, it is important to understand that the dynamic delta-hedging argument we first encountered in the context of a binomial tree still effectively holds in the continuous-time analog, although of course it is physically impossible to actually implement the hedge.

Nevertheless, it is reassuring to know that we may now price a variety of payoffs under the assumption of normal log-returns, since we now know the proper parameters to use in specifying the distribution.

Closed-form valuation of a European call option

Using this information, we seek to price a call option on a single unit of an asset with the following specifications:

S_0: initial spot price
r: risk-free rate
q: dividend rate
T: time to expiry
σ: volatility

All parameters are constant in the presentation below, although generally the result we will see extends easily to the case where r, q, and σ are time-varying but nonrandom.

To price the call option, we take the discounted risk-neutral expectation of the payoff:

$$V_{call} = E^*\left[e^{-rT}\left(S_T - K\right)^+\right] = e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{S_T=K}^{+\infty}\left(S_0 e^{\left(r-q-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}z} - K\right)e^{-\frac{1}{2}z^2}dz$$

The lower limit of the integration is:

$$S_T = K$$

$$S_0 e^{\left(r-q-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}z} = K$$

$$\sigma\sqrt{T}z = \ln\left(\frac{K}{S_0}\right) - \left(r-q-\frac{1}{2}\sigma^2\right)T$$

$$z = -\frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right) + \left(r-q-\frac{1}{2}\sigma^2\right)T\right] = -d_2$$

We can then easily evaluate the second term:

$$Ke^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{+\infty}e^{-\frac{1}{2}z^2}dz = Ke^{-rT}\left(1-N(-d_2)\right) = Ke^{-rT}N(d_2)$$

The first term requires a bit more work:

$$e^{-rT}S_0 e^{\left(r-q-\frac{1}{2}\sigma^2\right)T}\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{+\infty}e^{\sigma\sqrt{T}z-\frac{1}{2}z^2}dz = S_0 e^{-qT}e^{-\frac{1}{2}\sigma^2 T}\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{+\infty}e^{-\frac{1}{2}\left(z^2-2\sigma\sqrt{T}z\right)}dz =$$

$$S_0 e^{-qT}e^{-\frac{1}{2}\sigma^2 T}e^{\frac{1}{2}\sigma^2 T}\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{+\infty}e^{-\frac{1}{2}\left(z^2-2\sigma\sqrt{T}z+\sigma^2 T\right)}dz = S_0 e^{-qT}\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{+\infty}e^{-\frac{1}{2}\left(z-\sigma\sqrt{T}\right)^2}dz$$

It’s useful here to make a change of variables…

$$w = z - \sigma\sqrt{T}$$

$$dw = dz$$

…making the lower limit of our integration:

$$-d_2 - \sigma\sqrt{T} = -\frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right) + \left(r-q-\frac{1}{2}\sigma^2\right)T\right] - \sigma\sqrt{T} =$$

$$-\frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right) + \left(r-q-\frac{1}{2}\sigma^2\right)T + \sigma^2 T\right] =$$

$$-\frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right) + \left(r-q+\frac{1}{2}\sigma^2\right)T\right] = -d_1$$

So we continue with the first term of our integral:

$$S_0 e^{-qT}\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{+\infty}e^{-\frac{1}{2}\left(z-\sigma\sqrt{T}\right)^2}dz = S_0 e^{-qT}\frac{1}{\sqrt{2\pi}}\int_{-d_1}^{+\infty}e^{-\frac{1}{2}w^2}dw =$$

$$S_0 e^{-qT}\left(1-N(-d_1)\right) = S_0 e^{-qT}N(d_1)$$

Thus, the final form of the equation is:

$$V_{call} = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

In our discussion of binomial trees, we indicated that the basis of our no-arbitrage pricing of these instruments hinged on the fact that they could be replicated by taking positions in the underlying and in a cash asset. Here, what we see for the European call is explicitly a recipe of this form.

Moreover, we observe that the second term evidently has a useful interpretation. The value Ke^{-rT} is clearly the present value of the strike; since this need only be paid in the event that the option finishes in the money, the full term appears to be a probability-weighted present value. In particular, the value $N(d_2)$ is evidently the (arbitrage-free) probability that the call option ends in the money, as inspection of the process we used to obtain this result shows.

Put-call parity and closed-form valuation of a European put option

We recall that the payoff of a long position in a European call option pays $(S_T - K)^+$, which we note is only the favorable portion of the payoff of a forward contract with strike K. Similarly, a short position in a European put option pays $-(K - S_T)^+$, which represents only the unfavorable portion of the payoff of a forward contract with the same strike.

Thus, at expiry we clearly may say…

$$S_T - K = V_{call,T} - V_{put,T}$$

…and by discounting and taking risk-neutral expectations, we obtain the result:

$$S_0 e^{-qT} - Ke^{-rT} = V_{call} - V_{put}$$

This relation, known as put-call parity, has many uses. One is to obtain an expression for the value of a European put…

$$V_{put} = V_{call} - S_0 e^{-qT} + Ke^{-rT}$$

$$V_{put} = -S_0 e^{-qT}\left(1 - N(d_1)\right) + Ke^{-rT}\left(1 - N(d_2)\right)$$

$$V_{put} = Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$$

…with d_1 and d_2 defined as above in our expression for a European call. Once again, we have a pricing formula that appears to spell out a recipe for replicating the option at this time, where in this case we are short the asset and long a cash flow at time T.

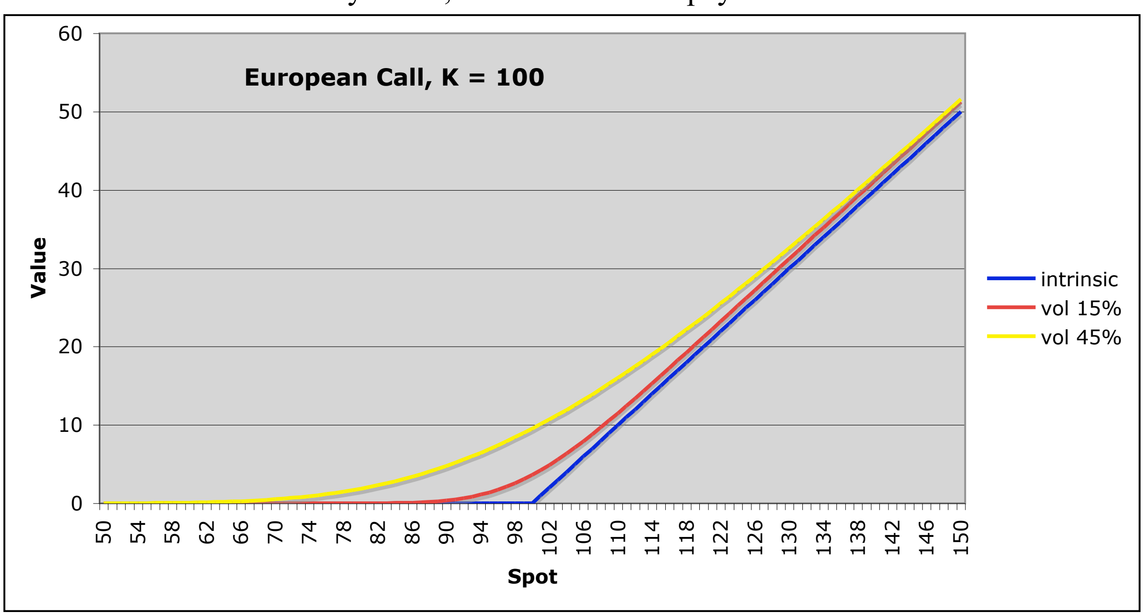
Intrinsic value and time value

It is fairly common for those new to the idea of options to think of their values only at expiry, or only in terms of what they would be worth if exercised immediately. In reality, exercise of options is comparatively rare. Most holders of options are less concerned with the exercise value of the option—or even the direction of the exposure, long or short. The main value to the holder of an option arises from the value it gains solely as a result of the volatility of the underlying asset.

In an effort to quantify this, the price of an option is often thought of as having two parts. Most obvious is the intrinsic value of the option, which is the value of its payoff if the option were to be exercised immediately. (Note that this concept applies equally well to American and European options, since what we are attempting here is merely a decomposition of the value.) The intrinsic value of an option that is out of the money (OTM) or at the money (ATM) is zero. For a call option, the intrinsic value if it is in the money (ITM) is $S_t - K$ for some time t. At expiry T, the intrinsic value of the option is exactly equal to its price.

At other times, however, the option's price differs from its intrinsic value; most often (though not always), the difference in values is positive. The difference is called the time value of the option, and this is in essence the component of the option's price that is attributable to the volatility of the underlying. For OTM options, the time value is equal to the entire value of the option. It is for this reason that OTM options, both calls and puts, are the most frequently traded: purchase of an ITM option requires some outlay to cover the intrinsic value of the option, and so in effect gives back a great deal of the leverage offered by these products. OTM options are a more efficient use of capital if the exposure sought is to the volatility of the underlying.

The graph below compares the intrinsic value, as well as the value at two different volatilities, of a European call struck at 100 with maturity 3 months. We assume that the risk-free rate to this maturity is 5%, and that the asset pays no dividends.



The effect of volatility on the time value of the option is clear here: as it decreases, the graph of the option price hugs the graph of the payoff ever more closely. Further, we note that the time value is greatest for options that are near the money, either in or out. At extremely low spot prices, the option has little time value because the probability of the option ending in the money, even at the higher volatility, is near zero. At high spot prices, the likelihood of the option ending out of the money becomes negligible, and the option ever more closely resembles a forward contract.

There is one interesting difference to note, however, between the OTM and ITM cases. At extremely low spot prices, the intrinsic value and the value at each of the two volatilities appear all to converge at zero. However, when the option is deep ITM, the values of the options at our two different volatilities seem to converge with one another, but neither seems to converge with the intrinsic value. That is, the ITM option appears always to retain at least some time value, even in the case where the volatility of the asset is largely unimportant.

Optimal exercise of American call options

The above observation for deep ITM call options when the underlying pays no dividends is directly relevant to the question of pricing American call options. These differ from European calls only in that the option can be converted to its intrinsic value in cash at any time if the holder wishes.

To begin with, it is clear that the addition of this exercise behavior to European call options cannot serve to lessen the option's value, providing the option is optimally exercised. Furthermore, it is clear that it is never optimal to exercise an OTM or ATM call. Thus, in an effort to determine what optimal exercise must be, we use the value of a European call option as a lower bound on the value of an ITM American call option and consider the time value…

$$V_{time,call} = V_{call} - (S_0 - K) = S_0 N(d_1) - Ke^{-rT} N(d_2) - (S_0 - K)$$

…recalling that we have stipulated that the underlying pays no dividends. Hopefully, this expression looks familiar from our discussion of put-call parity:

$$V_{time,call} = S_0 N(d_1) - Ke^{-rT} N(d_2) - \left(S_0 - Ke^{-rT}\right) + K\left(1 - e^{-rT}\right)$$

$$V_{time,call} = V_{put} + K\left(1 - e^{-rT}\right)$$

If we further stipulate that the interest rate is positive, then it is clear that the time value of our European call is positive, since both terms in the sum above must be positive. Thus, if given the option at any point to receive either the intrinsic value of the European call option or the call option itself, our choice should always be to take the call option.

Extending this reasoning to the American call, we may say that exercise should never be optimal in this case either, since we are offered the choice at each time between the intrinsic value and an option whose value is no less than the value of a European option, which is in turn always greater than the intrinsic value. Thus, subject to the constraint that interest rates are positive and the asset pays no dividends, we can conclude that the value of an American call option is equal to the value of a European option at all times, since it is never optimal to exercise the additional feature provided in the American call option, and therefore its value is zero.

At the limit of a call option that is very, very far in the money, what we hold is essentially a forward contract on the underlying. Seen this way, the choice of exercise boils down to this: We are ultimately agnostic as to whether we acquire the underlying asset now or later, since in expectation this asset grows at the risk-free rate; thus, our choice is either to exercise now and pay K now, or else to exercise at expiry and pay K then. Since the cost of the second course of action is less, this is the course of action we choose.

Furthermore, from the above we can see that the limiting time value of an infinitely ITM option is…

$$K\left(1-e^{-rT}\right)$$

…which is the distance we initially observed in our graph between the value to which a European call option converges at large spot and the intrinsic value of the option.

Extending this line of reasoning, we can likewise see why it might be optimal to exercise in the case when the asset pays dividends. If we again consider a forward contract to be an approximation of a deep ITM option and pretend that we are presented a choice at one time between this instrument and the intrinsic value, then the value of continuation is…

$$S_0e^{-qT}-Ke^{-rT}$$

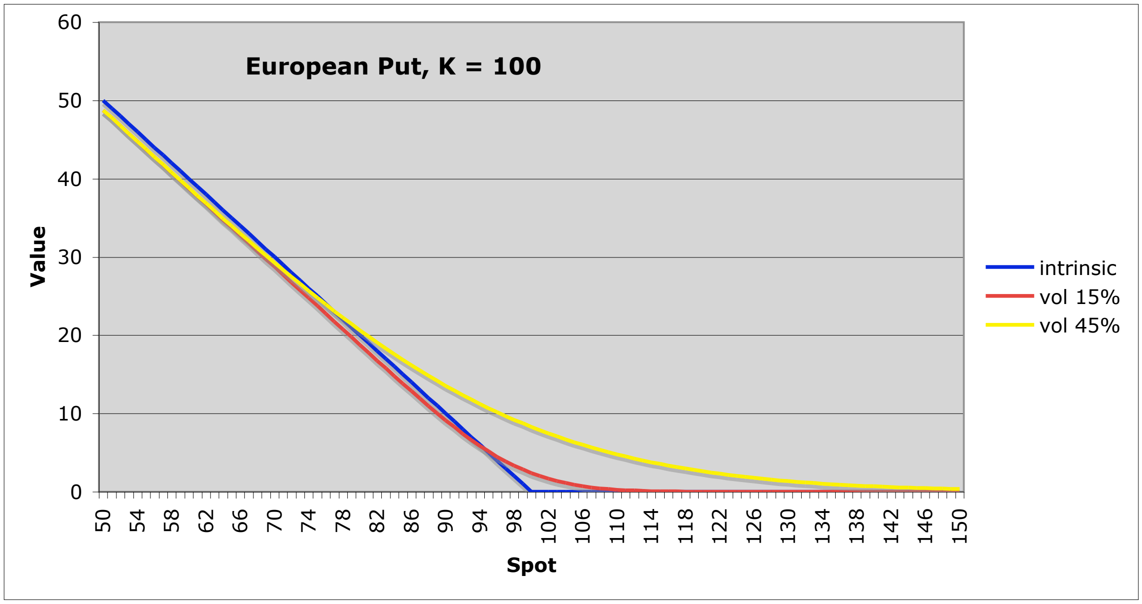
…whereas the value of exercise is:

$$S_0-K$$

If the value of the dividend paid is high enough, then it becomes worthwhile to incur the additional cost of paying K immediately to receive the dividends between now and expiry that we would otherwise forego. Obviously, the expressions above do not represent the value of an American call exactly, since the option to exercise exists at all times, rather than just once; still, it illustrates the idea without needing to resort to numerical methods, which would be required to answer the question more precisely.

The American put option

To begin with, consider a European put option with the same terms as the European call option we examined above. The corresponding graphs for this option are:



This case is clearly different from the call option. We see that the time value of a European put is negative below a certain threshold, and that this threshold occurs at a lower strike the greater the volatility, all else being equal.

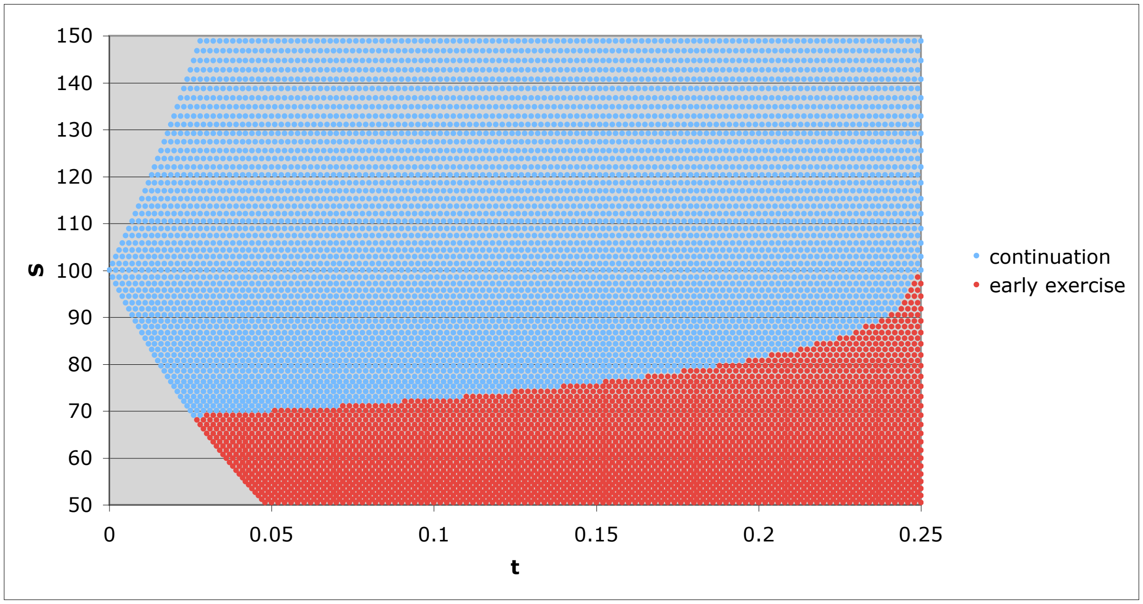
The negative time value in the limiting case of a far ITM option arises for the same reason we stated above: Since at the limit the European put resembles a short position in a forward contract struck at *K*, in this case delaying the receipt of the strike in cash is not advantageous to the holder.

There is no closed-form equation for an American put. Consider for a moment a Bermudan put—one that can only be exercised at defined times before expiry—with one exercise time $t < T$. At t, the continuation value is clearly the value of a European put, so at t, there is some threshold spot S_t* below which exercise is optimal. Thus, as seen at time zero, this option is in effect a payoff at t satisfying:

$$V_t = \begin{cases} K - S_t & S_t \leq S_t^* \\ V_{put}(S_t, T - t) & S_t > S_t^* \end{cases}$$

This payoff by itself is not particularly tractable, and becomes still less so when we wish to take its risk-neutral expected present value at time zero. When this idea is now extended to infinite possible exercise times, it becomes hopeless to approach the problem this way. The early exercise region has a complicated shape, and the details of this shape are affected by all of the input variables for the pricing function.

Fortunately, numerical methods are accurate and fast for this position. From them, we can also discern the shape of the early exercise region. The below shows the points of early exercise for an ATM American put with 45% vol:



There are closed-form approximate methods available for the pricing of American put options, the Barone-Adesi Whaley formula being a prominent example. This method treats the boundary of the early exercise region as being quadratic—evidently a reasonable assumption in the case above—and uses a numerical method to estimate the boundary point at pricing time.

Impressive as this approach is in terms of its ingenuity, fast numerical pricing methods combined with cheap computing power render it unnecessary. A good numerical method can return better prices quite quickly, and has the advantage that many alternate payoffs can be priced using the same approach.