Solutions to MTH 9831 Spring 2015 HW #5

Please note that these are *possible* solutions, and that there are often many ways to approach and solve a problem or prove a theorem.

Problem 1 Let $B(t) = (B_1(t), ..., B_n(t))$ be an n-dimensional Brownian motion, and define

$$R(t) := \sum_{i=1}^{n} B_i^2(t); \; dW(t) := \sum_{i=1}^{n} \frac{B_i(t)}{\sqrt{R(t)}} dB_i(t).$$

(a) $W(t)$ is a standard Brownian motion by checking Lévy's characterization:

- $W(t) = \int_0^t dW(u) = \sum_{i=1}^n \int_0^t \frac{B_i(u)}{\sqrt{R(u)}} dB_i(u)$ is a sum of stochastic integrals, each of which is continuous. Hence, $W(t)$ is continuous for every $t \geq 0$.
- $W(t)$ is a martingale since, as a sum of stochastic integrals, it is a sum of martingales.
- The quadratic variation of $W(t)$ is $[W, W](t) = t$, by the independence of B's coordinates:

$$d[W, W](t) = \sum_{i=1}^{n} \frac{B_i^2(t)}{R(t)} d[B_i, B_i](t) = \frac{\sum_{i=1}^n B_i^2(t)}{R(t)} dt = \frac{R(t)}{R(t)} dt = dt.$$

(b) To show that $R(t)$ satisfies the SDE

$$dR(t) = 2\sqrt{R(t)} dW9t) + n\, dt,$$

simply apply Itô's multidimensional formula to $R(t)$, remembering the independence of the coordinates of B:

$$\begin{aligned} dR(t) &= 2\sum_{i=1}^{n} B_i(t) dB_i(t) + \frac{2}{2}\sum_{i=1}^{n} d[B_i, B_i](t) = 2\sqrt{R(t)} \sum_{i=1}^{n} \frac{B_i(t)}{\sqrt{R(t)}} dB_i(t) + n\, dt \\ &= 2\sqrt{R(t)}\, dW(t) + n\, dt. \end{aligned}$$

(c) Again, just apply Itô to $X(t) = \sqrt{R(t)}$, using $dR(t)$ from (b) and the fact that $d[W, W](t) = dt$:

$$\begin{aligned} dX(t) &= \frac{1}{2\sqrt{R(t)}} dR(t) - \frac{1}{8R(t)\sqrt{R(t)}} d[R, R](t) \\ &= dW(t) + \frac{n\, dt}{2\sqrt{R(t)}} - \frac{dt}{2\sqrt{R(t)}} = dW(t) + \frac{n-1}{2X(t)} dt. \end{aligned}$$

Problem 2 (Shreve 4.13) Let $C(t)$ be the correlation matrix of $B(t) = (B_1(t), B_2(t))^T$:

$$C(t) = \begin{bmatrix} 1 & \rho(t) \\ \rho(t) & 1 \end{bmatrix}.$$

$C(t)$ is symmetric positive definite, and so has a Cholesky decomposition

$$C(t) = A(t)A^T(t) = \begin{bmatrix} 1 & 0 \\ \rho(t) & \sqrt{1-\rho^2(t)} \end{bmatrix} \begin{bmatrix} 1 & \rho(t) \\ 0 & \sqrt{1-\rho^2(t)} \end{bmatrix}.$$

Then we have the differential matrix equation $dB(t) = A(t)dW(t)$ corresponding to Shreve's definition of $B(t)$, and we want to show that $A^{-1}(t)dB(t) = dW(t)$ gives a two-dimensional BM $W(t)$.

Clearly,

$$A^{-1}(t)dB(t) = \begin{bmatrix} 1 & 0 \\ -\frac{\rho(t)}{\sqrt{1-\rho^2(t)}} & \frac{1}{\sqrt{1-\rho^2(t)}} \end{bmatrix} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix} = dW(t)$$

and we are given that $B_1(t) = W_1(t)$ is a Brownian motion. Therefore, we need to show two things: $W_2(t)$ is a Brownian motion, and W_1 and W_2 are independent. By Lévy's theorems on identifying Brownian motions, Thms. 4.6.4-5, pp. 168-9, all we need is to show

- $W_2(0) = 0$,
- $W_2(t)$ has continuous paths,
- $[W_2, W_2](t) = t$ for all $t \geq 0$,
- $[W_1, W_2](t) = 0$ for all $t \geq 0$.

We can rewrite W_2 in differential form:

$$dW_2(t) = \frac{-\rho(t)}{\sqrt{1-\rho^2(t)}} dB_1(t) + \frac{1}{\sqrt{1-\rho^2(t)}} dB_2(t).$$

$W_2(0) = 0$ because it is an Itô integral, and has continuous paths because $\rho(t)$ is continuous and the $B_j(t)$s have continuous paths. Lastly for W_2,

$$d[W_2, W_2](t) = \left[\frac{-\rho(t)}{\sqrt{1-\rho^2(t)}} dB_1(t) + \frac{1}{\sqrt{1-\rho^2(t)}} dB_2(t)\right]^2 = \frac{(\rho^2(t)+1)dt - 2\rho(t)d[B_1,B_2](t)}{1-\rho^2(t)} = dt$$

because $\rho(t)dt = d[B_1, B_2](t)$. Therefore, $W_2(t)$ is a Brownian motion.

Finally,

$$\begin{aligned} d[W_1, W_2](t) &= \frac{-\rho(t)}{\sqrt{1-\rho^2(t)}} dB_1(t)dB_1(t) + \frac{1}{\sqrt{1-\rho^2(t)}} dB_1(t)dB_2(t) \\ &= \frac{-\rho(t)}{\sqrt{1-\rho^2(t)}} dt + \frac{\rho(t)}{\sqrt{1-\rho^2(t)}} dt = 0, \end{aligned}$$

so W_1 and W_2 are independent.

For the n-dimensional case, Exercise 4.16 describes the situation: we have an $n \times n$ correlation matrix $C(t) = [\rho_{ik}(t)]$, where $|\rho_{ik}(t)| < 1$ if $i \neq k$ and $\rho_{ii} = 1$ for each $i = 1, 2, ..., n$. Again, $C(t) = A(t)A^T(t)$ since this correlation matrix Shreve gives us is assumed to be symmetric positive definite [usually we can only expect semidefinite], and so we have an inverse matrix $A^{-1}(t)$ and all the algebra above follows to give that $W(t) = (W_1(t), W_2(t), ..., W_n(t))^T$ is an n-dimensional BM. In short, for $W(t) = A^{-1}(t)dB(t)$, we have

$$\begin{aligned} dW(t)dW(t)^T &= A^{-1}(t)dB(t)(A^{-1}(t)dB(t))^T = A^{-1}(t)dB(t)dB(t)^T A^{-1}(t)^T \\ &= A^{-1}(t)C(t)A^{-1}(t)^T dt = A^{-1}(t)A(t)A(t)^T A^{-1}(t)^T dt = I\,dt. \end{aligned}$$

Problem 3 (Shreve 4.15 - Lévy characterization and cross-variation) Now we flip the situation from the previous problem. We'll start with $W(t) = (W_1(t), W_2(t), ..., W_n(t))^T$ as an n-dimensional BM, and define, for $i = 1, 2, ..., m$,

$$\sigma_i(t) = \left[\sum_{j=1}^{d} \sigma_{ij}^2(t)\right]^{1/2}, \quad B_i(t) = \sum_{j=1}^{d} \int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)} dW_j(u),$$

assuming $\sigma_i(t) \neq 0$. (The σ_i are clearly to be interpreted as squares of dot products of rows and columns in a covariance matrix Σ of the n-dimensional BM B.) We need to show that

(i) B_i is a (one-dimensional) BM for each i: by the same reasoning as the previous problem, as a sum of Itô integrals, $B_i(0) = 0$, B_i has continuous paths a.s., and B_i is a martingale. All we need to show is that the quadratic variation process $[B_i, B_i](t) = t$, which we'll do via the differential, recalling that $d[W_j, W_k](t) = \delta_{jk}$, the Kronecker delta, which $= 1$ if $j = k$ and 0 if $j \neq k$:

$$d[B_i, B_i](t) = \sum_{j=1}^{d}\sum_{k=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)} \frac{\sigma_{ik}(t)}{\sigma_i(t)} d[W_j, W_k](t) = \sum_{j=1}^{d} \frac{\sigma_{ij}^2(t)}{\sigma_i^2(t)} dt = dt.$$

(ii) Now for correlations: defining

$$\rho_{ik}(t) = \frac{1}{\sigma_i(t)\sigma_k(t)} \sum_{j=1}^{d} \sigma_{ij}(t)\sigma_{kj}(t),$$

we show that $d[B_i, B_k](t) = \rho_{ik}(t)dt$, i.e. the ρ_{ik} are the correlations of the coordinates of B. This is just an easy computation, changing some indices in (i):

$$d[B_i, B_k](t) = \sum_{j=1}^{d}\sum_{l=1}^{d} \frac{\sigma_{ij}(t)}{\sigma_i(t)} \frac{\sigma_{kl}(t)}{\sigma_k(t)} d[W_j, W_l](t) = \sum_{j=1}^{d} \frac{\sigma_{ij}(t)\sigma_{kj}(t)}{\sigma_i(t)\sigma_k(t)} dt = \rho_{ik}(t)dt.$$

Problem 4 (Shreve 4.10 - self-financing property) Recalling Itô's product rule:

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + d[X, Y](t),$$

we proceed to compute.

(i) We are given

$$\begin{aligned} dM(t) &= rM(t)dt \\ (4.10.16)\ \ X(t) &= \Delta(t)S(t) + \Gamma(t)M(t) \\ (4.10.9)\ \ dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt \end{aligned}$$

and told to derive the continuous self-financing condition

$$(4.10.15)\ \ 0 = S(t)d\Delta(t) + d[S, \Delta](t) + M(t)d\Gamma(t) + d[\Gamma, M](t).$$

This is simply a matter of computation:

$$\begin{aligned} dX(t) &= S(t)d\Delta(t) + \Delta(t)dS(t) + dS(t)d\Delta(t) + \Gamma(t)dM(t) + M(t)d\Gamma(t) + d\Gamma(t)dM(t) \\ &= S(t)d\Delta(t) + \Delta(t)dS(t) + dS(t)d\Delta(t) + r\Gamma(t)M(t)dt + M(t)d\Gamma(t) + d\Gamma(t)dM(t) \\ \implies \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt &= S(t)d\Delta(t) + \Delta(t)dS(t) + dS(t)d\Delta(t) \\ &\quad + r\Gamma(t)M(t)dt + M(t)d\Gamma(t) + d\Gamma(t)dM(t) \\ \implies r(X(t) - \Delta(t)S(t))dt &= S(t)d\Delta(t) + dS(t)d\Delta(t) \\ &\quad + r(X(t) - \Delta(t)S(t))dt + M(t)d\Gamma(t) + d\Gamma(t)dM(t) \\ \implies 0 &= S(t)d\Delta(t) + dS(t)d\Delta(t) + M(t)d\Gamma(t) + d\Gamma(t)dM(t). \end{aligned}$$

(ii) Armed with

$$\begin{aligned} X(t) - \Delta(t)S(t) &= c(t, S(t)) - \Delta(t)S(t) = N(t), \\ \Gamma(t) &= \frac{N(t)}{M(t)} \implies d(M(t)\Gamma(t)) = \Gamma(t)dM(t) + M(t)d\Gamma(t) + d[M, \Gamma](t) \\ &= r\Gamma(t)M(t)dt + M(t)d\Gamma(t) + d[M, \Gamma](t) \\ &= rN(t)dt + M(t)d\Gamma(t) + d[M, \Gamma](t), \\ \Delta(t) &= c_x(t, S(t)), \end{aligned}$$

we find that

$$\begin{aligned}
(4.10.21)\ \ dN(t) &= c_t(t,S(t))dt + c_x(t,S(t))dS(t) + \frac{1}{2}c_{xx}(t,S(t))d[S,S](t) \\
&\quad - \Delta(t)dS(t) - S(t)d\Delta(t) - d[\Delta,S](t) \\
&= \left[c_t(t,S(t)) + \frac{\sigma^2}{2}c_{xx}(t,S(t))S^2(t)\right]dt + M(t)d\Gamma(t) + d[M,\Gamma](t) \\
&= \left[c_t(t,S(t)) + \frac{\sigma^2}{2}c_{xx}(t,S(t))S^2(t)\right]dt + dN(t) - rN(t)dt \\
\implies rN(t)dt &= \left[c_t(t,S(t)) + \frac{\sigma^2}{2}c_{xx}(t,S(t))S^2(t)\right]dt.
\end{aligned}$$

Problem 5 (Shreve 4.19 - sign process) For $W(t)$ a Brownian motion, and $B(t) = \int_0^t sign(W(s))dW(s)$,

(i) Lévy again: $B(0) = 0$, stochastic integrals are continuous, and

$$d[B,B](t) = sign^2(W(t))dt = dt,$$

so $B(t)$ is a Brownian motion.

(ii) By Itô's product rule and the fact that $x\, sign(x) = |x|$, $dB(t) = sign(W(t))dW(t)$, $sign(x)$ is an odd function, and stochastic Fubini's theorem,

$$\begin{aligned}
d(B(t)W(t)) &= B(t)dW(t) + W(t)dB(t) + d[W,B](t) \\
&= \left(\int_0^t sign(W(s))dW(s)\right)dW(t) + W(t)sign(W(t))dW(t) + sign(W(t))dW(t)dW(t) \\
&= \left(\int_0^t sign(W(s))dW(s)\right)dW(t) + |W(t)|dW(t) + sign(W(t))dt \\
E(B(t)W(t)) &= E\left[\int_0^t \left(\int_0^u sign(W(s))dW(s) + |W(u)|\right)dW(u)\right] + \int_0^t E(sign(W(s)))ds \\
&= 0 \text{ (martingale) } + 0 = 0.
\end{aligned}$$

(iii) Easy:

$$d(W^2(t)) = 2W(t)dW(t) + \frac{1}{2}(2)d[W,W](t) = 2W(t)dW(t) + dt.$$

(iv) The usual drill: differentiation, integration, expectation... what's left?

$$\begin{aligned}
d(B(t)W^2(t)) &= B(t)d(W^2(t)) + W^2(t)dB(t) + d[W^2,B](t) \\
&= B(t)(2W(t)dW(t) + dt) + W^2(t)sign(W(t))dW(t) \\
&\quad + (2W(t)dW(t) + dt)(sign(W(t))dW(t)) \\
&= [2B(t)W(t) + W^2(t)sign(W(t))]dW(t) \\
&\quad + [B(t) + 2W(t)sign(W(t))]dt \\
\implies E[B(t)W^2(t)] &= \int_0^t [E(B(s)) + 2E|W(s)|]ds = 2\int_0^t E|W(s)|ds > 0,
\end{aligned}$$

but $E[B(t)] \cdot E[W^2(t)] = 0 \cdot t = 0.$

Hence, $B(t)$ and $W(t)$ are not independent, since there exists a pair of functions of $B(t)$ and $W(t)$, respectively, that do not successfully factor the expectation.