Basic uses of Itô's formula I.

Notation: $(B(t))_{t\geq 0}$ is a standard Brownian motion, $(\mathcal{F}(t))_{t\geq 0}$ is the natural filtration of $(B(t))_{t\geq 0}$.

Tools:

A. (Itô's formula for Brownian motion.) Let $f(t,x)$ be a twice continuously differentiable function on $[0,\infty)\times\mathbb{R}$. Then with probability 1 for all $t>0$

$$f(t,B(t))-f(0,B(0)) = \int_0^t f_x(u,B(u))\,dB(u) + \int_0^t f_t(u,B(u))\,du + \frac{1}{2}\int_0^t f_{xx}(u,B(u))\,du.$$

To proceed to the next step it is convenient to think about du in the last integral as $d[B,B](u)$ (since $[B,B](t)\equiv t$ a.s.). Then the next formula will not look surprising.

B. (Itô's formula for Itô processes.) Let

$$X(t) = X(0) + \int_0^t \Delta(u)\,dB(u) + \int_0^t \Theta(u)\,du$$

be an Itô process and $f(t,x)$ be a twice continuously differentiable function on $[0,\infty)\times\mathbb{R}$. Then with probability 1 for all $t>0$

$$f(t,X(t))-f(0,X(0)) = \int_0^t f_x(u,X(u))\,dX(u)+\int_0^t f_t(u,X(u))\,dt+\frac{1}{2}\int_0^t f_{xx}(u,X(u))\,d[X,X](u).$$

I prefer not to substitute $dX(u)=\Delta(u)\,dB(u)+\Theta(u)\,du$ and $d[X,X](u)=\Delta^2(u)\,du$ in the right hand side of the formula, since this substitution hides the natural structure of the formula.

C. (Itô's integral for deterministic integrands.) Let $(\Delta(t))_{t\geq 0}$ be a non-random square integrable function on $[0,t]$. Then

$$I(t)=\int_0^t \Delta(u)\,dB(u) \;\sim\; N\left(0,\int_0^t \Delta^2(u)\,du\right).$$

Exercises:

(1) Apply Itô's formula to the following processes:

 (a) $B^2(t)$;

 (b) $tB(t)$;

(c) $(B(t)+t)\exp(-B(t)-t/2)$;

(d) $t^2 B(t) - 2\int_0^t uB(u)\,du$;

(e) $\log S(t)$, where $dS(t) = \nu S(t)\,dt + \sigma S(t)\,dB(t)$;

(f) $\exp\left(\int_0^t \Delta(u)\,dB(u) - \frac{1}{2}\int_0^t \Delta^2(u)\,du\right)$.

(2) Use Itô's formula to compute

$$\int_0^t B(u)\,dB(u).$$

(3) Use Itô's formula to compute the moment generating function of $B(t)$.

(4) Compute the distribution of the signed area under the graph of Brownian motion on the interval $[0,t]$,

$$\int_0^t B(u)\,du.$$

(5) (From Black-Karasinski to Vasicek model.) Let α, β, σ be positive constants. A (special case of) Black-Karasinski interest rate model states that the interest rate process satisfies

$$dR(t) = \left(\alpha + \frac{1}{2}\sigma^2 - \beta \log R(t)\right) R(t)\,dt + \sigma R(t)\,dB(t).$$

Set $r(t) = \log R(t)$ and find the equation on $r(t)$.