Interest Rate Models

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Outline

Fixed income markets

- *Fixed income markets* consists of debt related products.
- Two kinds of instruments are traded: *cash instruments (bonds)* and *derived instruments (interest rate swaps, bond futures, credit default swaps)*
- The purpose of bonds is to raise funds by various entities.
- Fixed income instruments have complex risk profiles. They depend on the term structure of interest rates, credit characteristics, foreign exchange levels, ets.
- We will deal only with interest rate risks.

Fixed income markets

- The main source of fluctuations in the market are the current perceptions of future interest rates.
- *Rally* refers to falling interest rates; *Sell off* refers to rising interest rates.
  Reason for this is: falling interest rates are equivalent to increase in bond prices.
- Each fixed income instrument is a stream of known or contingent cash flows. Cash flows are calculated as a fixed or floating coupons applied to a notional principal.
- Conventions to apply coupon in US dollar market are *money market convention* act/360 (actual number of days/360), and the *bond market convention* 30/360 (based on a 30 day month over a 360 day year).

Zero coupon bonds

- A *zero coupon bond* (or *discount bond*) pays predefined principal amount (default value for this course: \$1) at time T. The settlement date is S is a time that satisfies $S < T$ and at which the payment to the issuer is made.
- We are interested in $P(t, S, T)$ of the forward zero coupon bond for valuation date $t \leq S$.
- $P(t, S, T)$ is also called the *forward discount factor*.
- We will assume that for $t \leq S < T$:

$$\begin{aligned} P(t,S,T) &< 1, & (1) \\ \frac{\partial}{\partial T} P(t,S,T) &< 0. & (2) \end{aligned}$$

Zero coupon bonds

Theorem (Homework 1)

If $P(t, S, T)$, $P(t, t, T)$, and $P(t, t, S)$ are known at time t then

$$P(t, S, T) = \frac{P(t, t, T)}{P(t, t, S)}. \tag{3}$$

- If the valuation date is now, then $t = 0$ and we denote $P_0(S, T) = P(0, S, T)$.
- The equality (3) becomes

$$P_0(S, T) = \frac{P_0(0, T)}{P_0(0, S)}.$$

Forward rates

- For given t we define *instantaneous forward rate* as a function $f(t,s)$ that satisfies

$$P(t,S,T) = e^{-\int_S^T f(t,s)\,ds}. \tag{4}$$

- The equivalent form of (4) is

$$f(t,s) = -\frac{\partial}{\partial s} \log P(t,S,s).$$

- When $t=0$ we will write $f_0(s)$ instead of $f(0,s)$.

LIBOR rates

- London Interbank Offered Rate (*LIBOR*) is the interest rate at which banks from British Banking Association (BBA) offer unsecured deposits to each other. They are short term rates (overnight to 1 year).
- Daily *fixings* are published by Thompson Reuters on each London business day at 11:30 am. See www.bbalibor.com.
- USD, GBP, EUR, JPY, CHF, CAD, AUD, DKK, SEK, NZD
- For each currency up to 15 quotes: 1D, 1W, 2W, 1M, 2M, . . . , 12M
- For USD only 7 quotes: 1D, 1W, 1M, 2M, 3M, 6M, 12M
- MTH9878: LIBOR = 3M USD LIBOR

LIBOR rates

- For USD, LIBOR spot date is two business days from the current date and maturity is on *anniversary date* of that settlement date. The following two rules are used to determine the anniversary date:
  (i) *Modified following business day convention:* If the anniversary date is not a business day, move forward to the next business day, except if this takes you over a calendar month end, in which case you move back to the last business day.
  (ii) If the settlement date is the last business day of a calendar month, all anniversary dates are last business days of their calendar months.
- In addition to spot transactions, the following vanilla instruments are based on LIBOR: LIBOR futures, forward rate agreements, interest rate swaps.

Forward rate agreements

- Abbreviation: FRAs.
- They are over the counter transactions.
- The counterparties A and B first define the notional amount (e.g. 1 million dollars). A agrees to pay B the LIBOR settling t years from now applied to the notional amount. In exchange the counterparty B agrees to pay the pre-agreed interest rate r applied to the same notional.
- The contract matures on an anniversary T of the settlement date, and interest is compounded on an act/360 day count basis. Anniversary dates follow the modified following business day convention.
- FRAs are quoted in terms of the annualized forward interest rate applied to the accrual period of the transaction.

Forward rate agreements

- Referenced by three dates: *fixing*, *settle*, and *maturity* dates.
- Notation is "$A * B$ over C."
  1. Start with the current LIBOR *spot* date.
  2. Advance A months forward from 1.
  3. The C^{th} date of the month in 2 is the settle date.
  4. Advance B months forward from 1.
  5. The C^{th} date of the month in 4 is the maturity date.
  6. $B - A$ month LIBOR is the reference rate.
  7. Two London business days prior to 3 is the fixing date.
- FOMC FRA fixes the day after a Federal Open Markets Committee meeting.
- FOMC switch involves taking opposite positions before and after FOMC meeting.
- *Turn switch* : before and after the end of a month/year (like Aug 31. vs Sep 1, Dec 31 vs Jan 1).

Eurodollar futures

- Also known as LIBOR futures.
- Future contracts traded on the 3 month LIBOR rate.
- Traded at Chicago Mercantile Exchange (a.k.a. "the Merc")
- The parties involved maintain margin account and the Merc makes daily "mark to market."
- The terms are standardized. Notional is $\$1,000,000$. Interest is computed on act/360 day count basis assuming 90 day accrual period.

Eurodollar futures

- For 1 basis point (1% of a 1%) movement in the underlying LIBOR forward rate, the mark to market adjustment is

$$0.0001 \cdot \frac{90}{360} \cdot \$1,000,000 = \$25.$$

- The value of the margin goes up if the rate drops, and goes down if the LIBOR goes up.
- Eurodollar quotes are made to look like bond prices. The forward rate R underlying the contract is quoted in terms of "price" defined as $100(1-R)$.

Eurodollar futures

- 44 Eurodollar contracts are listed on Merc. The quarterly 40 contracts expire on the third Wednesday of the months of March, June, September, and December for the next 10 years (called IMM dates and are never holidays in New York or London). IMM=International Monetary Market.
- The Monday before IMM is the last date to trade the contract.
- Of the 40 quarterly contracts, the last 20 are rarely traded.
- Grouped in *packs* of four (correspond to a year). The first 4 are called White (or Front). The others are Red, Green, Blue, and Gold.
- 2 *year bundle* = White + Red.

Eurodollar futures

- The remaining 4 contracts are called *serial contracts* and are defined to be the ones corresponding to the first 4 months not covered by the above 40 contracts. Their maturity is also on the third Wednesday of the given month.
- For example, on February 1, 2016 the White pack consists of March, June, September, and December. The serial contracts are: February, April, May, and July. Typically only the first two are liquid.
- The codes for months are F, G, H, J, K, M, N, Q, U, V, X, Z. The ticker EDU16 means the contract expiring in September 2016.

Eurodollar futures

ticker	price
*EDG*16	99.3675
*EDH*16	99.335
*EDJ*16	99.305
*EDK*16	–
*EDM*16	99.275
*EDN*16	–
*EDU*16	99.205
*EDZ*16	99.115
*EDH*17	99.030
*EDM*17	98.925
*EDU*17	98.825
*EDZ*17	98.720

ticker	price
*EDH*18	98.630
*EDM*18	98.545
*EDU*18	98.460
*EDZ*18	98.375
*EDH*19	98.300
*EDM*19	98.235
*EDU*19	98.165
*EDZ*19	98.090
*EDH*20	98.020
*EDM*20	97.965
*EDU*20	97.895
*EDZ*20	97.835

Swaps

- Full name: A fixed for floating interest rate swap.
- Short name: Swap.
- P (fixed payer) vs R (fixed receiver).
- They agree on *notional* amount.
- P pays a fixed coupon C (say 0.62%) on the notional, and in return receives from R the LIBOR applied to the same notional.
- *Spot starting swaps* start 2 business days from the current day and mature and pay interest on anniversary dates that use the same modified following business day convention as the LIBOR index.
- Interest is computed on an act/360 day basis on the floating side of the swap and on 30/360 day basis on the fixed side.
- Fixed payment dates ("coupon dates") are semiannual. Floating payment dates are quarterly.

Swaps

- In *forward starting swaps* the counterparties agree on some other first accrual period as long as it is after the spot.
- Swaps are quoted in terms of the fixed coupon.
- Examples of swap rates on January 26, 2016:

Tenor	Rate
$2Y$	0.87%
$5Y$	1.31%
$10Y$	1.81%

Overnight Indexed Swap (OIS)

- LIBOR is obtained from polls and is not a result of trading.
- Each bank in US is assigned a minimum level of reserves that it must maintain in the end of each business day. If the reserves would go below that level, the bank must borrow from Fed at a rate that is called *discount rate*. Usually other banks with excess reserves are willing to negotiate and eventually lend money at a better rate than the discount rate.
- The weighted average of these rates (excluding the discount rate) is calculated each day and called *Fed funds effective rate*.
- OIS is a fixed for floating interest rate swap where the floating rate is the fed funds effective rate. The rate is based on a short term (overnight), daily compounded (instead of quarterly as in the case of LIBOR swap) interest rate.

OIS

- The fixing is published the following day between 7 and 8 am.
- Both the fixed and floating legs accrue interest on act/360 basis.
- Short maturity: few days to five years.
- The Federal Reserve Bank influences the fed funds effective rate. In the periods of *monetary* easing, the target fed funds rate is lowered, while in the periods of *tightening*, it is raised. It enforces its targets through open market operations at the Domestic Trading Desc at the Federal Bank of New York.
- The rates are usually announced at FOMC meetings.

OIS

- OIS rates in other currencies are referenced by the following overnight rates:
- EONIA – euro overnight index average. The banks contributing to EONIA are the same as the banks contributing to EURIBOR.
- SONIA – sterling overnight index average.
- SARON – Swiss average rate overnight.

LIBOR/OIS basis swap

- The LIBOR/OIS spread is defined as the difference between 3 month LIBOR and 3 month OIS.
- A wider spread means that major banks are less willing to lend to each other.
- LIBOR/OIS spread is around 10 basis points. The all time high was 364 basis points in October 2008. It dropped below 100 bp in January 2009, and by September 2009 it went to 10-15 bp. In December 2011 it widened to the levels of 40-60bp.

Fed Fund futures

- Similar to Eurodollar futures, except that the underlying rate is the fed funds effective rate rather than LIBOR.
- 36 monthly futures contracts are traded. The notional principal is $\$5,000,000$. Because it was (falsely) believed that daily fixings of fed funds effective rate are more volatile than LIBOR rates, the settlement value of each contract was defined as the monthy arithmetic average of the daily fixings.
- The first 12 of the contracts are the most liquid.

Fed fund futures

ticker	price
$FFG16$	99.635
$FFH16$	99.605
$FFJ16$	99.570
$FFK16$	99.550
$FFM16$	99.530
$FFN16$	99.500
$FFQ16$	99.490
$FFU16$	99.485
$FFV16$	99.455
$FFX16$	99.435
$FFZ16$	99.400
$FFF17$	99.380

Multicurve paradigm

- The evaluation of future cash flows reduces to determining the following two quantities: *Discount factors*, and *Forward rates*.
- Until 2008, LIBOR rate was used for both purposes.
- Since then the industry has adopted *multicurve paradigm* for swap valuation.
- OIS is a better indicator of the cost of funding and as such is used for discounting. LIBOR remained suitable as the index rate.

Forward rates

- Recall that *instantaneous forward rate* is a function $f(t, s)$ that
$$P(t, S, T) = e^{-\int_S^T f(t,s)\,ds}.$$
- The OIS forward rate $F(t, S, T)$ for start S and maturity T, as observed at time t is defined in the following way:
$$F(t, S, T) = \frac{1}{\delta}\left(\frac{1}{P(t, S, T)} - 1\right),$$
where δ is the day count fraction.
- If we denote by $f(t, s)$ the instantaneous forward rate for *OIS* then
$$F(t, S, T) = \frac{1}{\delta}\left(e^{\int_S^T f(t,s)\,ds} - 1\right).$$

Forward rates

- Similarly, the LIBOR forward rate $L(t, S, T)$ with start S, maturity T, and tenor $T - S$ can be expressed in terms of instantaneous forward rate $l(t, s)$ as

$$L(t, S, T) = \frac{1}{\delta}\left(e^{\int_S^T l(t,s)\,ds} - 1\right).$$

- The LIBOR/OIS spread $B(t, S, T)$ is given by

$$B(t, S, T) = L(t, S, T) - F(t, S, T).$$

Valuation of swaps

- Consider a swap with spot date T_0 that matures at T. Assume that the notional principal is \$1.
- Denote by $T_1^c < \cdots < T_{n_c}^c = T$ the coupon dates of the swap and denote by $t \leq T_0$ the valuation date. The present value on the fixed leg is

$$P_0^{\text{fix}}(t) \;=\; \sum_{j=1}^{n_c} \alpha_j C \cdot P_0\left(t, T_j^c\right). \tag{5}$$

  C is the coupon rate and α_j are the day count fractions.
- The quantity $A_0(t) = \sum_{j=1}^{n_c} \alpha_j P_0\left(t, T_j^c\right)$ is called *annuity function* or *level function* of the swap. It lets us re-write (6) as $P_0^{\text{fix}} = CA_0(t)$.

Valuation of swaps

- Denote by $T_1^f < \cdots < T_{n_f}^f = T$ the LIBOR payment date. The present value on the floating leg is

$$P_0^{\text{float}}(t) \;=\; \sum_{j=1}^{n_f} \delta_j L_j \cdot P_0\left(t, T_j^f\right). \tag{6}$$

  $L_j = L_0(T_{j-1}^f, T_j^f)$ is the coupon rate and δ_j are the day count fractions.

- For the standard USD swaps the conventions are that α_j correspond to modified following 30/360 business day convention. The coupons for fixed leg are semi-annual. The LIBOR payments are quarterly, and daycount franctions follow modified following act/360 business day convention.

Valuation of swaps

- The present value of the swap is

$$P_0(t) = P_0^{\text{fix}}(t) - P_0^{\text{float}}(t).$$

- A *break-even* (or *mid-market*) swap has zero PV, or equivalently, $P_0^{\text{fix}}(t) = P_0^{\text{float}}(t)$.
- The *break-even swap rate* is the value of the coupon that makes the swap break-even:

$$S_0(T_0, T) = \frac{P^{\text{float}}(t)}{A_0(t)}.$$

Valuation of LIBOR/OIS basis swaps

- LIBOR/OIS basis swap has maturity T and payment dates $T_1,\ T_2,\ \ldots,\ T_n = T$.
- The present value on LIBOR and OIS legs are:

$$
\begin{aligned}
P_0^{\text{LIBOR}}(t) &= \sum_{i=1}^{n} \delta_i L_i P_0(t, T_i), \\
P_0^{\text{OIS}}(t) &= \sum_{i=1}^{n} \delta_i \left(F_i + B\right) P_0(t, T_i),
\end{aligned}
$$

  where B denotes the fixed basis spread for the maturity T.
- The break-even basis spread is

$$
B_0(T_0, T) = \frac{\sum_{i=1}^{n} \delta_i (L_i - F_i) P_0(t, T_i)}{\sum_{i=1}^{n} \delta_i P_0(t, T_i)}.
$$

Curve stripping

- $P_0(S, T)$ and $L_0(S, T)$ are not known for all $S < T$. The markets are not liquid enough to provide all these quotes.
- *Curve stripping* is the process of curve construction out of liquid market inputs.
- We will be simultaneously stripping two interest rate curves: OIS and LIBOR. There are different curves that we may want to find:
- $S \mapsto L_0(S, T)$ with fixed tenor $T - S$ is called the *LIBOR forward curve*.
- $T \mapsto P_0(0, T)$ is called *discount curve* or *zero coupon curve*.
- A collection of spot starting swaps rates for all tenors is called *par swap curve*.

Polynomial interpolation

- If x_0, x_1, ..., x_n are distinct real numbers and y_0, y_1, ..., y_n any real numbers, there is a unique polynomial $p(x)$ of degree n such that $p(x_i) = y_i$ for all $i \in \{1, 2, \ldots, n\}$.
- This polynomial is called *interpolation polynomial*.
- There are many ways to write this very same polynomial.
- Lagrange's way is $p(x) = b_0 + b_1 x + \cdots + b_n x^n$. The coefficients b_i are easy to guess.
- Newton's way is:

$$\begin{aligned} p(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \\ &\quad + \cdots + a_{n-1}(x - x_0)(x - x_1)\cdots(x - x_{n-1}). \end{aligned}$$

Polynomial interpolation

- Define the *finite differences* recursively as

$$f[x_0, x_1, \ldots, x_{j+1}] = \frac{f[x_1, x_2, \ldots, x_{j+1}] - f[x_0, x_1, \ldots, x_j]}{x_{j+1} - x_0},$$

- Newton's theorem: $a_j = f[x_0, x_1, \ldots, x_{j+1}]$.
- Other equalities hold for finite differences:

$$\begin{aligned} f[x_0, x_1, \ldots, x_j] &= \sum_{i=0}^{j} \frac{f(x_i)}{\prod_{k \neq i}(x_i - x_k)} \\ f[x_0, x_1, \ldots, x_j] &= \int \cdots \int_{T_k} f^{(j)}(t_0 x_0 + \cdots + t_j x_j)\, dt_1 \cdots dt_j \end{aligned}$$

Piecewise polynomial interpolation

- $s(x)$ is a *spline of order* m if $s(x) \in \mathcal{C}^{m-2}(\mathbb{R})$ and if it is a polynomial of degree $< m$ on each subinterval $[x_{i-1}, x_i]$.
- Splines of order 4 are called *cubic splines*.
- There are many cubic splines that satisfy $s(x_i) = y_i$ for every $i \in \{0, 1, 2, \ldots, n\}$. More precisely, once the system of equations is written, we get two degrees of freedom.
- If we set two additional constraints $s'(x_0) = y'_0$ and $s'(x_n) = y'_n$, then there is a unique s.
- We can get this s by solving a linear system.

B-splines

- Alternative way is to express $s(x)$ as a linear combination of *basic splines*.
- B-spline of degree d with respect to points $\{x_0, \ldots, x_{d+1}\}$ is defined as

$$B^{(d)}(x) = (x_{d+1} - x_0) \cdot f_x[x_0, x_1, \ldots, x_{d+1}],$$

  where $f_x(t) = ((t - x)_+)^d$.
- $B^{(d)}(x) \in [0, 1]$ and $B^{(d)}(x) = 0$ outside of $\mathcal{H}(x_0, \ldots, x_{d+1})$.

B-splines

- Given set of knot points $\cdots < t_{-2} < t_{-1} < t_0 < t_1 < \ldots$, define $B_k^{(d)}(x)$ as the B-spline corresponding to $\{t_k, t_{k+1}, \ldots, t_{k+d+1}\}$.
- For $d = 0$ we have $B_k^{(0)}(t) = 1_{[t_k, t_{k+1}]}(t)$.
- The following recursion holds:

$$\begin{aligned} B_k^{(d)}(t) &= \frac{t - t_k}{t_{k+d} - t_k} B_k^{(d-1)}(t) \\ &\quad + \frac{t_{k+d+1} - t}{t_{k+d+1} - t_{k+1}} B_{k+1}^{(d-1)}(t). \end{aligned} \tag{7}$$

- B-splines form a partition of unity:

$$\sum_k B_k^{(d)}(t) = 1.$$

B-splines

- Their derivatives satisfy

$$\frac{d}{dt}B_k^{(d)}(t) = \frac{d}{t_{k+d}-t_k}B_k^{(d-1)}(t) - \frac{d}{t_{k+d+1}-t_{k+1}}B_{k+1}^{(d-1)}(t). \tag{8}$$

- Their integrals satisfy:

$$\int_{-\infty}^{t} B_k^{(d)}(s)\,ds = \sum_{i=k}^{\infty} \frac{t_{k+d+1}-t_k}{d+1}B_i^{(d+1)}(t). \tag{9}$$

B-splines

- In the case of cubic splines the integrals $\int_a^b B_k''(t) B_l''(t)\, dt$ can be easily calculated.
- If $w_1, \ldots, w_m$ are the knot points between a and b then

$$\begin{aligned}\int_a^b B_k''(t) B_l''(t)\, dt &= B_k'(b) B_l''(b) - B_k'(a) B_l''(a) \\ &\quad - \sum_{j=1}^{m+1} B_l'''(w_{j-1}) \left(B_k(w_j) - B_k(w_{j-1}) \right).\end{aligned} \tag{10}$$

- B_l''' are constant.

B-splines

- Our goal is to construct the instantaneous OIS forward rate $f_0(t)$ and the instantaneous 3 month LIBOR forward rate $l_0(t)$ on $[T_0, T_{\max}]$.
- We will use cubic B-splines. Choose $T_0 = 0$ and choose K knot points t_{-3}, t_{-2}, t_{N+4} with $t_0 = 0$ and $T_{\max} \in (t_{N-1}, t_N)$.
- Denote $B_k(t) = B_k^{(3)}(t)$.
- $$f_0(t) = \sum_{k=-3}^{N} f_k B_k(t), \quad l_0(t) = \sum_{k=-3}^{N} l_k B_k(t)$$

B-splines

- Assume that we know the sequences (f_k) and (l_k). Later we will talk on how to get them.
- The discount factors $P_0(S, T)$ satisfy

$$\begin{aligned} P_0(S,T) &= e^{-\int_S^T f_0(s)\,ds} = e^{-\sum_{k=-3}^{N}\int_S^T f_k B_k(s)\,ds} \\ &= e^{-\sum_{k=-3}^{N} f_k \gamma_k(S,T)}, \end{aligned}$$

where
$$\gamma_k(S,T) = \int_S^T B_k(s)\,ds.$$

- The coefficients $\gamma_k(S, T)$ are computed using formula (9). Thus knowing (f_k) allows us to calculate all discount factors.
- In order to get analogous equations for $L_0(S, T)$ recall that $L(t, S, T) = \frac{1}{\delta}\left(e^{\int_S^T l(t,s)\,ds} - 1\right)$.

B-splines

- Thus
$$L(t,S,T) \;=\; \frac{1}{\delta}\left(e^{\sum_{k=-3}^{N} l_k \gamma_k(S,T)} - 1\right).$$
- We now get all LIBOR/OIS basis spreads and all swap rates in terms of (l_k) and (f_k).
- The sequences (f_k) and (l_k) are chosen to fit market data. If we are given some benchmark rates $\bar{R}_j$, for $j = 1, 2, \ldots, m$, (fed funds effective rate, LIBOR, Eurodollar implied LIBOR forward rates, swap rates, LIBOR/OIS bases), we can evaluate those rates for every given sequence of (f_k) and (l_k).
- Denote by R_j these values.
- Then we define
$$Q(f,l) = \frac{1}{2}\sum_{j=1}^{m}\left(R_j - \bar{R}_j\right)^2 + \frac{1}{2}\lambda \int_{T_0}^{T_{\max}} \left(f''(t)^2 + l''(t)^2\right)\, dt.$$

B-splines

- $Q(f,l) = \frac{1}{2}\sum_{j=1}^{m}\left(R_j - \bar{R}_j\right)^2 + \frac{1}{2}\lambda\int_{T_0}^{T_{\max}}\left(f''(t)^2 + l''(t)^2\right)\,dt.$
- The quantity $Q(f,l)$ depends on the constant λ and sequences (f_k) and (l_k). We minimize the functional Q. The integral on the right-hand side is called a *Tikhonov regularizer* and it can be calculated using the formula (10).
- Experience has shown that it is better to use the Tikhonov regularizer of the form $\int_{T_0}^{T_{\max}}\lambda(t)\left(f''(t)^2 + l''(t)^2\right)\,dt$, where λ is smaller in the front of the curve and larger in the back end of the curve.
- The minimum can be foiund using standard Newton-type optimization algorithms such as Levenberg-Marquardt algorithm.

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