Basic uses of Itô's formula II

Notation: $(B(t))_{t\geq 0}$ is a standard Brownian motion, $(\mathcal{F}(t))_{t\geq 0}$ is the filtration generated by $(B(t))_{t\geq 0}$.

Tools (continued):

D. (Integration by parts formula, regular case.) The "integration by parts" formula we obtained in Exercise 4 of part 1 for a specific case can be generalized. Here is a version, which is not hard to prove (see p. 46 of B. Øksendal, Stochastic Differential Equations, Sixth Edition). Let $(F(t))_{t\geq 0}$ be a stochastic process with a.s. continuously differentiable trajectories on $[0,t]$ and $(X(t))_{t\geq 0}$ be an Itô process. Then

$$\int_0^t F(u)\,dX(u) = F(u)X(u)\Big|_0^t - \int_0^t X(u)\,dF(u).$$

Derive this useful fact from Itô's formula.

E. Given a "nice" $f(t,x)$ satisfying appropriate integrability conditions, how can we determine whether the process $(f(t,B(t)))_{0\leq t\leq T}$ is an $\mathcal{F}(t)$-martingale?

Solution. Apply Itô's formula to $f(t,B(t))$:

$$df(t,B(t)) = (f_t(t,B(t)) + \frac{1}{2}\,f_{xx}(t,B(t)))\,dt + f_x(t,B(t))\,dB(t).$$

This allows us to say that whenever function f satisfies the partial differential equation (PDE) $f_t(t,x) + \frac{1}{2}\,f_{xx}(t,x) = 0$ for all $(t,x)\in(0,T)\times\mathbb{R}$, then the process $(f(t,B(t)))_{0\leq t\leq T}$ is an $\mathcal{F}(t)$-martingale.

The necessity of this condition is harder to prove. It can be treated as a consequence of the Martingale Representation Theorem (Shreve II, Section 5.3.1) which we shall discuss later. See also B. Øksendal, Stochastic Differential Equations, Sixth Edition, Exercise 4.12 on p. 59 for a direct proof in a slightly more general setting.

Exercises:

(6) Use the method of Exercise 3 to compute the variance of the process $S(t)$, which solves the equation $dS(t) = \sigma S(t)\,dB(t)$, $S(0) = A$. Hint: apply Itô's formula to $S^2(t)$ and then take the expected value. In addition, solve this problem directly by first verifying the fact that $S(t) = Ae^{\sigma B(t)-\sigma^2 t/2}$ and then computing the variance using the density of $B(t)$.

(7) Find the mean and variance of the process $\int_0^t S(u)\,du$, where $dS(t) = \sigma S(t)\,dB(t)$, $S(0) = A$. Give a solution based on integration by parts (similar to the solution of Exercise 4). Use the result of Exercise 6.

(8) As an application of Tool E, determine which of the processes in Exercise 1 are martingales. When it is possible to give an alternative argument based on definition and/or basic properties of martingales or processes in question, provide such an argument as well. For example, the process $t + B(t)$ is not a martingale, since its expectation is not constant.

(9) (Review of properties of conditional expectation.) Using the definition of a martingale (without Itô's formula or any of its consequences) show that the process in Exercise 1(d) is a martingale.

(10) (The Ornstein-Uhlenbeck process.) Let $(X(t))_{t\geq 0}$ satisfy

$$dX(t) = -\beta X(t)\,dt + \sigma dB(t), \quad X(0) = x,$$

where $\beta \in \mathbb{R}$, $\sigma > 0$ are constants. For $\beta > 0$ this is a special case of Vasicek interest rate model (see the next exercise). Apply Itô's formula to $e^{\beta t}X(t)$ and show that $X(t)$ admits a closed form solution

$$X(t) = e^{-\beta t}x + \sigma e^{-\beta t}\int_0^t e^{\beta u}\,dB(u).$$

Use this expression to find the mean and variance of $X(t)$. Then compute the mean and variance not using the solution but applying the same approach as in Exercise 6.

(11) (Vasicek model.) Let $r(t)$ satisfy

$$dr(t) = (\alpha - \beta r(t))\,dt + \sigma\,dB(t).$$

Find a closed form solution of this equation. Compute the mean and variance of $r(t)$.

(12) (Cox-Ingersoll-Ross (CIR) model.) Let $r(t)$ satisfy

$$dr(t) = (\alpha - \beta r(t))\,dt + \sigma\sqrt{r(t)}\,dB(t).$$

(a) Compute the mean and variance of $r(t)$.

(b) Assume that $4\alpha = \sigma^2$. Let $X(t) = \sqrt{r(t)}$. Derive the equation for $X(t)$.

(c) Using part (b) determine the distribution of $r(t)$. Compute its moment generating function.