MTH 9862 Solutions to selected problems of Assignment 8 (04/05/2013 - 04/11/2013).

(1) Use the Feynman-Kac formula to solve explicitly (σ and r are given non-negative constants)

$$g_t(t,x) + \frac{1}{2}\sigma^2 g_{xx}(t,x) = rg(t,x), \quad g(T,x) = x^4.$$

Check that your solution is correct by substituting it in the PDE.

(2) (Solving a general linear SDE) Solve Exercise 6.1 following these steps:

(a) Solve $dZ(u) = b(u)Z(u)du + \sigma(u)Z(u)dW(u)$ for $u \geq t$ with the initial condition $Z(t) = 1$.

(b) Solve $dY(u) = k(u)du + \ell(u)dW(u)$ for $u \geq t$, with the initial condition $Y(t) = x$.

(c) Set $X(u) = Y(u)Z(u)$. Find $k(u)$ and $\ell(u)$ such that $X(u)$ solves the original SDE. Express the solution $X(u)$ in terms of the original parameters.

Which of the processes and models that you already know satisfy a linear SDE?

(3) Suppose that for $0 \leq t \leq u \leq T$

$$dX(u) = b(u, X(u))\, du + \sigma(u, X(u))\, dB(u), \quad X(t) = x.$$

Let $f(x)$ and $h(x)$ be given deterministic functions. Find the PDE satisfied by

$$g(t,x) = E^{t,x}[h(X(T)] + \int_t^T E^{t,x}[f(X(u))]\, du.$$

Hint: the game here is to find a relevant martingale, after that the PDE is obtained by applying Ito's formula and setting the drift term to 0. If I give you the martingale, the game is over. Instead, I shall tell you how to see it, since this point of view is applicable to many situations that involve finding a PDE. Start by looking at the following sequence of equalities, think why they are true (all but one are just rewriting of the same thing), and find a martingale:

$$g(t,x) = E^{t,x}[h(X(T)] + \int_t^T E^{t,x}[f(X(u))]\, du;$$

$$g(t,x) = E\Big[h(X(T) + \int_t^T f(X(u))\, du \,\Big|\, X(t) = x\Big];$$

$$g(t,X(t)) = E\Big[h(X(T) + \int_t^T f(X(u))\, du \,\Big|\, X(t)\Big];$$

$$g(t,X(t)) = E\Big[h(X(T) + \int_t^T f(X(u))\, du \,\Big|\, \mathcal{F}(t)\Big];$$

$$g(t,X(t)) = E\Big[h(X(T) + \int_0^T f(X(u))\, du \,\Big|\, \mathcal{F}(t)\Big] - \int_0^t f(X(u))\, du;$$

$$g(t,X(t)) + \int_0^t f(X(u))\, du = E\Big[g(T,X(T)) + \int_0^T f(X(u))\, du \,\Big|\, \mathcal{F}(t)\Big].$$

Which transition from line to line is non-trivial? Justify this transition.

Solution. The transition from line 3 to line 4 of the above chain of formulas requires justification. We have

$$
\begin{aligned}
g(t, X(t)) &= E\Big[h(X(T) + \int_t^T f(X(u))\,du \,\Big|\, X(t)\Big] \\
&= E[h(X(T) \,|\, X(t)] + E\Big[\int_t^T f(X(u))\,du \,\Big|\, X(t)\Big] \\
&= E[h(X(T) \,|\, X(t)] + \int_t^T E[f(X(u))|\, X(t)]\,du \\
&= E[h(X(T) \,|\, \mathcal{F}(t)] + \int_t^T E[f(X(u)) \,|\, \mathcal{F}(t)]\,du \\
&= E\Big[h(X(T) + \int_t^T f(X(u))\,du \,\Big|\, \mathcal{F}(t)\Big].
\end{aligned}
$$

We interchanged the expectation and integration using Fubini's theorem and we replaced the conditioning on $X(t)$ (i.e. on $\sigma(X(t))$) with the conditioning on $\mathcal{F}(t)$ using the Markov property of the process $X(t)$, $0 \le t \le T$.

From the last line of the chain of formulas given in the hint it follows that the process

$$
g(t, X(t)) + \int_0^t f(X(u))\,du, \quad 0 \le t \le T,
$$

is a martingale. An application of Itô formula gives

$$
\begin{aligned}
&d\left(g(t, X(t)) + \int_0^t f(X(s))\,ds\right) \\
&\quad = g_t(t, X(t))\,dt + g_x(t, X(t))\,dX(t) + \frac{1}{2}\,g_{xx}(t, X(t))\,d[X, X](t) + f(X(t))\,dt \\
&\quad = (g_t(t, X(t)) + g_x(t, X(t))b(t, X(t)) + \frac{1}{2}\,g_{xx}(t, X(t))\sigma^2(t, X(t)) + f(X(t)))dt \\
&\qquad\qquad + g_x(t, X(t))\sigma(t, X(t))\,dB(t).
\end{aligned}
$$

Therefore, for all $0 \le t \le T$

$$
g_t(t, X(t)) + g_x(t, X(t))b(t, X(t)) + \frac{1}{2}\,g_{xx}(t, X(t))\sigma^2(t, X(t)) + f(X(t)) = 0.
$$

This implies that for all (t, x) which can be "reached by the process" $X(t)$, $0 \le t < T$, the following PDE should hold

$$
g_t(t, x) + g_x(t, x)b(t, x) + \frac{1}{2}\,g_{xx}(t, x)\sigma^2(t, x) + f(x) = 0.
$$

(4) (From a PDE to an SDE) Find a diffusion process (i.e. an SDE) whose generator $\mathcal{A}$ has the following form:

(a) $(d=1)$ $\mathcal{A} = (2-x)\dfrac{\partial}{\partial x} + 2x^2\dfrac{\partial^2}{\partial x^2}$. Can you solve the obtained SDE?

(b) $(d=2)$

$$\mathcal{A} = \frac{1+x^2}{2}\frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial x \partial y} + \frac{1}{2}\frac{\partial^2}{\partial y^2} + 2y\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y}.$$

Give at least two different answers.

Solution. (a) $dX(t) = (2 - X(t))dt + 2X(t)\,dB(t)$. This is a linear SDE, which you solved in problem (2). The solution to this SDE is also given in (11) of the handout on Itô formula.

(b) We seek a 2-dimensional SDE is of the form

$$d\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = b(X(t),Y(t))dt + \sigma(X(t),Y(t))\,dB(t),$$

where $b(x,y)$ is a 2-dimensional vector, $\sigma(x,y)$ is a 2×2 matrix, and $B(t)$ is the standard 2-dimensional Brownian motion. From the generator we have

$$b(x,y) = \begin{pmatrix} 2y \\ 2x \end{pmatrix}, \quad a(x,y) = \frac{1}{2}(\sigma\sigma^T)(x,y) = \frac{1}{2}\begin{pmatrix} 1+x^2 & x \\ x & 1 \end{pmatrix}.$$

The drift $b(x,y)$ is determined uniquely. We need to find $\sigma(x,y)$. One choice comes, say, from the Cholesky decomposition

$$\begin{pmatrix} 1+x^2 & x \\ x & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{1+x^2} & 0 \\ \frac{x}{\sqrt{1+x^2}} & \frac{1}{\sqrt{1+x^2}} \end{pmatrix}\begin{pmatrix} \sqrt{1+x^2} & \frac{x}{\sqrt{1+x^2}} \\ 0 & \frac{1}{\sqrt{1+x^2}} \end{pmatrix}.$$

There are infinitely many choices. For every orthogonal 2×2 matrix $O(x,y)$ (i.e. $OO^T(x,y) = I_2$) and every $\sigma(x,y)$

$$\sigma\sigma^T(x,y) = (\sigma(OO^T)\sigma^T)(x,y) = (\sigma O)(\sigma O)^T(x,y).$$

So if we find one matrix $\sigma(x,y)$ such that $a = \frac{1}{2}\sigma\sigma^T$ then σO will also work. Which one is better? You want σ such that the SDE is solvable and probably such that it is reasonably simple. For example, another choice of $\sigma(x,y)$ is

$$\begin{pmatrix} 1+x^2 & x \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

In short, two possible answers are

$$\begin{aligned} dX(t) &= 2Y(t)dt + \sqrt{1+X^2(t)}\,dB_1(t) && (1)\\ dY(t) &= 2X(t)dt + \frac{X(t)}{\sqrt{1+X^2}}\,dB_1(t) + \frac{1}{\sqrt{1+X^2}}\,dB_2(t); \quad \text{or} \\ dX(t) &= 2Y(t)dt + dB_1(t) + X(t)dB_2(t) && (2)\\ dY(t) &= 2X(t)dt + dB_2(t). \end{aligned}$$

(5) (Feynman-Kac+Girsanov) Consider the following terminal value problem:

$$g_t + \frac{1}{2} g_{xx} + \theta(x) g_x = 0, \quad (x,t) \in \mathbb{R} \times [0,T), \quad g(T,x) = h(x).$$

Show that the solution of this problem can be written as

$$g(t,x) = E^{t,x} \left(e^{\int_t^T \theta(B(s)) dB(s) - \frac{1}{2} \int_t^T \theta^2(B(s)) ds} h(B(T)) \right).$$

Hint: use Feynman-Kac and Girsanov theorems. You may assume that h and θ are such that the problem has a unique solution and both theorems are applicable.

Solution. By the Feynman-Kac formula,

$$g(t,x) = E^{t,x}(h(X(T)),$$

where

$$dX(u) = \theta(X(u))du + dB(u), \quad t \le u \le T, \ X(t) = x.$$

The next step is not necessary (the problem can be done directly) but it will bring us to the standard set up for Girsanov's theorem and all processes will start at 0 at time 0. Rewrite $g(t,x)$ as follows: i.e.

$$g(t,x) = E^{0,0}(h(x + \tilde{B}(T-t))),$$

where

$$d\tilde{B}(u) = \theta(x + \tilde{B}(u))du + dB(u), \quad 0 \le u \le T-t, \ \tilde{B}(0) = 0.$$

(The relationship between X and $\tilde{B}$ is simply $x + \tilde{B}(u) \stackrel{d}{=} X(u+t)$, $0 \le u \le T-t$, so $x + \tilde{B}(T-t) \stackrel{d}{=} X(T)$.) In the integral form,

$$\tilde{B}(u) = \int_0^u \theta(x + \tilde{B}(s))ds + B(u), \ 0 \le u \le T-t.$$

Let $\tilde{\mathbb{P}}$ be the measure under which the process $\tilde{B}(u)$ is a standard Brownian motion on $[0, T-t]$. Then by Girsanov's theorem with

$$\begin{aligned} Z := Z(T-t) &= \exp\left(- \int_0^{T-t} \theta(x + \tilde{B}(u))\, dB(u) - \frac{1}{2} \int_0^{T-t} \theta^2(x + \tilde{B}(u))\, du \right) \\ &= \exp\left(- \int_0^{T-t} \theta(x + \tilde{B}(u))\, d\tilde{B}(u) + \frac{1}{2} \int_0^{T-t} \theta^2(x + \tilde{B}(u))\, du \right); \text{ we have} \\ g(t,x) &= E^{0,0}(h(x + \tilde{B}(T-t))) = \tilde{E}^{0,0}(Z^{-1} h(x + \tilde{B}(T-t))) \\ &= \tilde{E}^{0,0} \left(e^{\int_0^{T-t} \theta(x + \tilde{B}(u))\, d\tilde{B}(u) - \frac{1}{2} \int_0^{T-t} \theta^2(x + \tilde{B}(u))\, du} h(x + \tilde{B}(T-t)) \right). \end{aligned}$$

Moving the starting point back to (t,x) and removing tildes (since we know that $\tilde{B}$ is a standard BM under $\tilde{\mathbb{P}}$, we can drop the tilde simultaneously from the process and the measure), we get

$$g(t,x) = E^{t,x} \left(e^{\int_t^T \theta(B(u))\, dB(u) - \frac{1}{2} \int_t^T \theta^2(B(u))\, du} h(B(T)) \right),$$

where $B(u)$, $t \leq u \leq T$, is a Brownian motion, which starts at x at time t (i.e. $B(t+u) \stackrel{d}{=} x + \tilde{B}(u)$, $0 \leq u \leq T-t$).

(6) (Implying the volatility surface) Solve Exercise 6.10.