MTH 9831 Fall 2015 HW #2 Solutions

Please note that these are *possible* solutions, and that there are often many ways to approach and solve a problem or prove a theorem.

Problem 1 Let $(B(t))_{t\geq 0}$ be a Brownian motion. To show that

$$P\left(\max_{0\leq t\leq 1}|B(t)|>a\right)\leq 4P(B(1)>a),$$

we first recognize that $-B(t)$ is also a Brownian motion, and setting the events

$$A=\left\{\max_{0\leq t\leq 1}B(t)>a\right\},\quad C=\left\{\max_{0\leq t\leq 1}(-B(t))>a\right\},$$

we have, via the reflection principle on A and C, and the inclusion-exclusion principle,

$$\begin{aligned}P\left(\max_{0\leq t\leq 1}|B(t)|>a\right)&=P\left(\left\{\max_{0\leq t\leq 1}B(t)>a\right\}\cup\left\{\min_{0\leq t\leq 1}B(t)<-a\right\}\right)\\&=P\left(A\cup C\right)=P(A)+P(C)-P(A\cap C)\\&\leq P(A)+P(C)\leq 4P(B(1)>a).\end{aligned}$$

From this we conclude that the random variable $X:=\max_{0\leq t\leq 1}|B(t)|$ has all moments finite: we can bound the kth moment of X by using the identity, for random variables $Y\geq 0$ a.s.,

$$E(Y)=\int_0^\infty P(Y>y)dy.$$

We know that $|B(1)|\geq 0$ a.s. and that $E(|B(1)|^k)<\infty$ for all k (all standard normal absolute moments are finite). Thus, using $P(|B(1)|>a)=2P(B(1)>a)$ for any $a\geq 0$,

$$\begin{aligned}\implies E(X^k)&=\int_0^\infty P(X^k>y)dy=\int_0^\infty P(X>y^{1/k})dy\\&\leq 4\int_0^\infty P(B(1)>y^{1/k})dy=2\int_0^\infty P(|B(1)|>y^{1/k})dy\\&=2\int_0^\infty P(|B(1)|^k>y)dy=2E(|B(1)|^k)<\infty.\end{aligned}$$

Problem 2 We start with Theorem 1.8 from Lecture 2: for $B(t)$ a Brownian motion, $B^*(t)=\max_{0\leq s\leq t}B(s)$, $a>0$, and $x\leq a$, we have

$$P(B^*(t)\geq a,\,B(t)\leq x)=1-N\left(\frac{2a-x}{\sqrt{t}}\right).$$

Using this, we get the joint distribution with $\phi(x)=\frac{d}{dx}N(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ the density of a standard normal, the joint density of $(B^*(t),\,B(t))$ is the mixed partial derivative

$$f_{B^*(t),B(t)}(a,x)=-\frac{\partial^2}{\partial a\partial x}P(B^*(t)\geq a,\,B(t)\leq x)=\frac{\partial}{\partial a}\left[\frac{1}{\sqrt{t}}\phi\left(\frac{2a-x}{\sqrt{t}}\right)\right]=\frac{2(2a-x)}{t\sqrt{2\pi t}}e^{-\frac{(2a-x)^2}{2t}}.$$

Problem 3 First, we prove Borel-Cantelli Lemma #1 (this proof is due to Durrett):

Let $N := \sum_{n=1}^{\infty} 1_{A_n}$ be the number of events A_n that occur. Then

$$E[N] = E\left[\sum_{n=1}^{\infty} 1_{A_n}\right] = \sum_{n=1}^{\infty} E[1_{A_n}] = \sum_{n=1}^{\infty} P(A_n) < \infty$$

implies that $N < \infty$ a.s. (i.e. with probability 1). Hence, $P(N = \infty) = P(A_n \text{ i.o.}) = 0$.

Now, let $(X_n)_{n\geq 1}$ be a sequence of IID integrable random variables, and set $B_n = \{X_n > n\varepsilon\}$ for some $\varepsilon > 0$. Since the X_n are IID and integrable, i.e. $E[X_n] = \mu < \infty$ for some μ, for all n, then by the SLLN,

$$P\left(\left|\sum_{n=1}^{\infty} \frac{X_n}{n} - \mu\right| > \varepsilon\right) = 0.$$

Thus, if $M := \sum_{n=1}^{\infty} 1_{B_n}$, we have $E[M] < \infty$ since only finitely many X_n can exceed $n\varepsilon$ a.s. if the SLLN is to hold. Therefore, $P(B_n \text{ i.o.}) = 0$, i.e. $P(X_n > n\varepsilon \text{ for infinitely many } n) = 0$, which implies that

$$P(X_n > n\varepsilon \text{ only finitely many times}) = P(\exists R = R(\varepsilon) : X_n \leq n\varepsilon \ \forall n > R) = 1.$$

Since this was done with an arbitrary $\varepsilon > 0$, we use the dominated convergence theorem to send $\varepsilon \to 0$ and get $P\left(\lim_{n\to\infty} \frac{X_n}{n} = 0\right) = 1$.

Problem 4 We assume $a < 0 < b$. First, for the martingale $B(t)$, note that by Theorem 2.5, Refresher Lecture 5, $B(\tau_a \wedge \tau_b \wedge t)$ is also a martingale, and so by the optional stopping theorem, since $\tau_a \wedge \tau_b \wedge t \leq t$ for a fixed t,

$$0 = E(B(0)) = E(B(\tau_a \wedge \tau_b \wedge t)) = aP(\tau_a < \tau_b \wedge t) + bP(\tau_b < \tau_a \wedge t) + E(B(t);\, t < \tau_a \wedge \tau_b).$$

(a) Sending $t \to \infty$ in the above equation, we get

$$0 = E(B(\tau_a \wedge \tau_b)) = aP(\tau_a < \tau_b) + bP(\tau_b < \tau_a).$$

The limit passes through the expectations and probabilities by the dominated convergence theorem, since, on $\{t < \tau_a \wedge \tau_b\}$, $a < B(t) < b$, and

$$\lim_{t\to\infty} P(t < \tau_a \wedge \tau_b) \leq P(B^*(t) < |a| \wedge b) \to 0.$$

(Compare this to Refresher Lecture 5, Exercise 9, the gambler's ruin problem.) Thus, $P(\tau_b < \tau_a) = 1 - P(\tau_a < \tau_b)$, reducing the above to

$$0 = aP(\tau_a < \tau_b) + b(1 - P(\tau_a < \tau_b)) \implies P(\tau_a < \tau_b) = \frac{b}{b-a}.$$

(b) This is now easy via optional stopping with the martingale $B(t)^2 - t$:

$$\begin{aligned}
0 = E(B(0)^2 - 0) = E(B(\tau_a \wedge \tau_b)^2 - (\tau_a \wedge \tau_b)) &= E(B(\tau_a \wedge \tau_b)^2) - E(\tau_a \wedge \tau_b) \\
\implies E(\tau_a \wedge \tau_b) &= E(B(\tau_a \wedge \tau_b)^2) \\
&= a^2 P(\tau_a < \tau_b) + b^2(1 - P(\tau_a < \tau_b)) \\
&= \frac{a^2 b + ab^2}{b-a} = (-ab) \cdot \frac{-a+b}{b-a} = -ab.
\end{aligned}$$

Problem 5 This problem has 5 parts. We are given that $X(t) = \mu t + W(t)$ is a Brownian motion with drift, $\tau_m = \inf\{t \geq 0 :\, X(t) = m\}$ is the first hitting time of $X(t)$ at the level m (with $\tau_m = \infty$ if X never reaches m), and $Z(t) = \exp\{\sigma X(t) - (\sigma\mu + \frac{1}{2}\sigma^2)t\}$.

(i) To show that $Z(t)$ is a martingale[1], we first notice that $E(Z(t)) = 1 < \infty$ (due to the usual Gaussian integral with square completion), $Z(t)$ is adapted to $\mathcal{F}(t)$, the natural filtration of $W(t)$, and $Z(t)$ satisfies the martingale property: for $0 \le s \le t$, since $W(t) - W(s)$ and $W(s)$ are independent, we have that $\frac{Z(t)}{Z(s)}$ and $Z(s)$ are independent, and so

$$E[Z(t) \,|\, \mathcal{F}(s)] = E\left[\frac{Z(t)}{Z(s)} Z(s) \,|\, \mathcal{F}(s)\right] = Z(s) E\left[\frac{Z(t)}{Z(s)}\right] = Z(s) E\left[Z(t-s)\right] = Z(s).$$

(ii) Since $Z(t)$ is a martingale, and $t \wedge \tau_m \le t$ is a bounded stopping time, then by optional stopping,

$$E\left[\exp\{\sigma X(t \wedge \tau_m) - (\sigma\mu + \frac{1}{2}\sigma^2)(t \wedge \tau_m)\}\right] = E[Z(t \wedge \tau_m)] = E[Z(0)] = 1.$$

(iii) Suppose $\mu \ge 0$, and let $\sigma > 0$. Then $Z(t \wedge \tau_m) \le e^{\sigma m} < \infty$ since, on $\{\tau_m < \infty\}$, $Z(\tau_m) = \exp\{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\} \le e^{\sigma m}$, and on $\{\tau_m = \infty\}$, $Z(t) \le e^{\sigma m - \frac{1}{2}\sigma^2 t} \to 0$ as $t \to \infty$. Therefore, by the bounded convergence theorem and (ii),

$$\begin{aligned}
E[Z(\tau_m)] &= \lim_{t\to\infty} E[Z(t \wedge \tau_m)] = 1 \text{ and} \\
E[Z(\tau_m)] &= \lim_{t\to\infty} E[1_{\{\tau_m < \infty\}} Z(t \wedge \tau_m)] + \lim_{t\to\infty} E[1_{\{\tau_m = \infty\}} Z(t \wedge \tau_m)] \\
&= E[1_{\{\tau_m < \infty\}} \lim_{t\to\infty} Z(t \wedge \tau_m)] + E[1_{\{\tau_m = \infty\}} \lim_{t\to\infty} Z(t \wedge \tau_m)] \\
&= E[1_{\{\tau_m < \infty\}} \lim_{t\to\infty} Z(t \wedge \tau_m)] + E[1_{\{\tau_m = \infty\}} \lim_{t\to\infty} Z(t \wedge \tau_m)] \\
&= E[1_{\{\tau_m < \infty\}} Z(\tau_m)] = E[1_{\{\tau_m < \infty\}} \exp\{\sigma m - (\sigma\mu + \frac{1}{2}\sigma^2)\tau_m\}].
\end{aligned}$$

Therefore, the Laplace transform of τ_m is, via the quadratic formula (with positive root only) and (ii),

$$\begin{aligned}
0 < \alpha := \sigma\mu + \frac{1}{2}\sigma^2 &\implies 0 < \sigma = -\mu + \sqrt{\mu^2 + 2\alpha} \\
E[e^{-\alpha\tau_m}] = e^{-\sigma m} &= e^{m\left(\mu - \sqrt{\mu^2 + 2\alpha}\right)}.
\end{aligned}$$

(iv) If $\mu > 0$, then by (iii) and monotone convergence,

$$E[\tau_m] = -\lim_{\alpha \downarrow 0} \frac{\partial}{\partial \alpha} E[e^{-\alpha\tau_m}] = \lim_{\alpha\downarrow 0} e^{m\left(\mu - \sqrt{\mu^2+2\alpha}\right)} \cdot \frac{m}{\sqrt{\mu^2 + 2\alpha}} = \frac{m}{\mu} < \infty.$$

(v) Finally, if $\mu < 0$ and $\sigma > -2\mu > 0$, then $\sigma\mu + \frac{\sigma^2}{2} > 0$ (with equality at $\sigma = -2\mu$) and we have the same bounds on $Z(t \wedge \tau_m)$ that we have in (iii). Hence, the result of (iii) holds. In addition, we get by monotone convergence

$$P(\tau_m < \infty) = \lim_{\sigma \downarrow -2\mu} E[1_{\{\tau_m < \infty\}} Z(t \wedge \tau_m)] = e^{\sigma m} = e^{2\mu m}.$$

Problem 6 First, the stochastic representation (4.2) of Lecture 2 for the solution of the backwards heat equation $v_t + \frac{1}{2}v_{xx} = 0$ is

$$v(t, x) = u(T - t, x) = E[f(B(T - t)) \,|\, B(0) = x] = \int_{\mathbb{R}} f(y) p(T - t, x, y) dy$$

[1] This is called the *exponential martingale*.

with terminal condition $v(T,x) = f(x)$. Our plan of attack is to convert this integral into expectations of standard normals whenever possible, since we know the following: if $Z \sim N(0,1)$, then

$$E[Z] = 0,\ E[Z^2] = 1,\ E[e^{sZ}] = e^{s^2/2}.$$

To do this conversion, we'll use the substitution

$$u = \frac{y-x}{\sqrt{T-t}},\ du = \frac{dy}{\sqrt{T-t}},\ y = \sqrt{T-t}\,u + x,$$

to turn $p(T-t,x,y)\,dy$ into $p(1,0,u)\,du$. Why? For fixed t and T, with $T \geq t \geq 0$,

$$''B(T-t)\,|\,B(0)=x'' \equiv'' [B(T-t)-B(0)] + B(0)\,|\,B(0)=x'' \equiv'' \sqrt{T-t}\,Z + x''.$$

Thus,

$$v(t,x) = E[f(B(T-t))\,|\,B(0)=x] = E[f(\sqrt{T-t}\,Z+x)] = \int_{\mathbb{R}} f(\sqrt{T-t}\,u+x)p(1,0,u)du.$$

(You should check to see that each of these solutions satisfies its terminal condition and PDE by evaluating at $t=T$ and checking $v_t + \frac{1}{2}v_{xx} = 0$.)

(a) $f(x) = x^2$, combined with (4.2), yields the backwards heat equation solution

$$\begin{aligned}
v(t,x) = \int_{\mathbb{R}} f(y)p(T-t,x,y)dy &= \frac{1}{\sqrt{2\pi(T-t)}}\int_{-\infty}^{\infty} y^2 e^{-\frac{(y-x)^2}{2(T-t)}}\,dy \\
\text{(substitution)} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\sqrt{T-t}\,u+x\right)^2 e^{-\frac{u^2}{2}}\,du \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left((T-t)u^2 + 2x\sqrt{T-t}\,u + x^2\right) e^{-\frac{u^2}{2}}\,du \\
&= (T-t)E[Z^2] + 2x\sqrt{T-t}E[Z] + x^2 = T-t+x^2.
\end{aligned}$$

(b) $f(x) = e^x$ yields

$$\begin{aligned}
v(t,x) &= \frac{1}{\sqrt{2\pi(T-t)}}\int_{-\infty}^{\infty} e^y e^{-\frac{(y-x)^2}{2(T-t)}}\,dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\sqrt{T-t}\,u+x} e^{-\frac{u^2}{2}}\,du = e^x E[e^{\sqrt{T-t}Z}] = e^{x+\frac{T-t}{2}}.
\end{aligned}$$

(c) $f(x) = e^{-x^2}$ yields (here we go straight with the substitution)

$$\begin{aligned}
v(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(\sqrt{T-t}\,u+x)^2} e^{-\frac{u^2}{2}}\,du \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(2(T-t)+1)u^2 + 4\sqrt{T-t}\,xu + 2x^2}{2}}\,du.
\end{aligned}$$

Here we need another square-completion substitution: let $v = au + bx$, $dv = a\,du$, for the functions a, b, c of $T-t$ such that the exponent on the integrand is

$$-\frac{(2(T-t)+1)u^2 + 4\sqrt{T-t}\,xu + 2x^2}{2} = -\frac{(au+bx)^2}{2} + cx^2.$$

Then, this integral reduces to

$$v(t,x) = \frac{e^{cx^2}}{a\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{v^2}{2}}\,dv = \frac{e^{cx^2}}{a}.$$

We have $a = \sqrt{2(T-t)+1}$, $b = \frac{2\sqrt{T-t}}{a}$, and $c = \frac{1}{2}b^2 - 1$, simplifying to

$$v(t,x) = \frac{e^{cx^2}}{a} = \frac{e^{-\frac{x^2}{2(T-t)+1}}}{\sqrt{2(T-t)+1}}.$$

Problem 7 The option in question is a perpetual option which pays \$1 at

$$\tau_A = \inf\{t \geq 0 : S(t) = A\} \text{ for a fixed } A < S(0),$$

assuming the interest rate is $r \in [0, \sigma^2/2]$ and that the price follows the GBM $S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma B(t)}$, $t \geq 0$, under the risk-neutral measure.

Let C_t be the value of the option at time $t \geq 0$. The time-zero price, as usual, is the discounted payoff under the risk-neutral measure; that is, since the payoff is $C_{\tau_A} = 1_{\{\tau_A < \infty\}}$,

$$C_0 = \tilde{E}[e^{-r\tau_A} C_{\tau_A}] = \tilde{E}[e^{-r\tau_A} 1_{\{\tau_A < \infty\}}].$$

Theorem 5.1 in Lecture 2 states that, for $\lambda > 0$, $a > 0$, and any $b \in \mathbb{R}$, the first hitting time by a Brownian motion of the line $a + bt$ has Laplace transform

$$E(e^{-\lambda \tau_{a,b}}) = e^{-a(b+\sqrt{b^2+2\lambda})}.$$

Note that in this problem,

$$\begin{aligned}
\tau_A = t &\iff S(t) = A \\
&\iff S(0)e^{(r-\sigma^2/2)t+\sigma B(t)} = A \\
&\iff (r-\sigma^2/2)t + \sigma B(t) = \ln(A/S(0)) < 0 \\
&\iff B(t) = \frac{-(r-\sigma^2/2)t + \ln(A/S(0))}{\sigma}.
\end{aligned}$$

We know that $\tilde{B}(t) := -B(t)$ is also a Brownian motion, so, by setting $a := -\ln(A/S(0))/\sigma = \ln(S(0)/A)/\sigma$ and $b := (r-\sigma^2/2)/\sigma$, we satisfy the theorem and can conclude that

$$\tilde{E}[e^{-r\tau_A} 1_{\{\tau_A < \infty\}}] = e^{-a(b+\sqrt{b^2+2r})} = \left(\frac{S(0)}{A}\right)^{-\frac{b+\sqrt{b^2+2r}}{\sigma}} = \left(\frac{S(0)}{A}\right)^{-\frac{2r}{\sigma^2}}.$$