September 30, 2015

MTH 9862. Solutions to Quiz 3.

Let $B(t)$, $t \geq 0$, be a standard Brownian motion.

(1) (2 points) Let $S(t) = S(0)e^{\mu t + \sigma B(t)}$, $\mu \in \mathbb{R}$, $\sigma > 0$, $S(0) > 0$. Find $d(\ln S(t)) - \dfrac{dS(t)}{S(t)}$.

Solution. Let $f(t,x) = S(0)e^{\mu t + \sigma x}$. Then $S(t) = f(t, B(t))$, $f_t(t,x) = \mu f(t,x)$, $f_x(t,x) = \sigma f(t,x)$, $f_{xx}(t,x) = \sigma^2 f(t,x)$, and

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) + \frac{1}{2}\,\sigma^2 S(t)\,dt;$$
$$\frac{dS(t)}{S(t)} = \left(\mu + \frac{\sigma^2}{2}\right) dt + \sigma\, dB(t).$$

On the other hand, $\log S(t) = \log S(0) + \mu t + \sigma B(t)$, so $d \log S(t) = \mu dt + \sigma dB(t)$. Thus,

$$d(\ln S(t)) - \frac{dS(t)}{S(t)} = -\frac{\sigma^2}{2}\, dt.$$

(2) (3 points) Let $X(t) = \exp\left(6 \int_0^t u\, dB(u) - 6t^3\right)$. Compute $dX(t)$ and find $E(X(t))$.

Solution. Let $f(t,x) = e^{6x - 6t^3}$ and $I(t) = \int_0^t u\, dB(u)$. Then $I(t)$, $t \geq 0$, is an Itô process with $\Delta(t) = t$ and $\Theta(t) \equiv 0$, and $X(t) = f(t, I(t))$. Therefore,

$$dX(t) = f_t(t, I(t))dt + f_x(t, I(t))\, dI(t) + \frac{1}{2}\, f_{xx}(t, I(t))\, d[I]_t.$$

Substituting $f_t(t,x) = -18t^2 f(t,x)$, $f_x(t,x) = 6f(t,x)$, $f_{xx}(t,x) = 36f(t,x)$, $dI(t) = t\, dB(t)$ and $d[I]_t = t^2\, dt$ we get after cancellations

$$dX(t) = 6X(t)t\, dB(t) = 6te^{6\int_0^t u\, dB(u) - 6t^3}\, dB(t).$$

We conclude that $X(t)$, $t \geq 0$, is a martingale. Therefore $E(X(t)) = E(X(0)) = 1$.

(3) (5=2+3 points) Let

$$X(t) = \int_0^t s\, dB(s) \quad \text{and} \quad Y(t) = \int_0^t B(s)\, ds.$$

Find the distribution of (a) $X(t)$; (b) $Y(t)$.

Solution. (a) The integrand is a deterministic function. Therefore, by Tool C $X(t)$ has a normal distribution with mean 0 and variance $\int_0^t u^2\,du = t^3/3$.

(b) Since $d(tB(t)) = t\,dB(t) + B(t)\,dt$, integrating and rearranging the terms we get

$$\int_0^t B(s)\,ds = tB(t) - \int_0^t s\,dB(s) = \int_0^t t\,dB(s) - \int_0^t s\,dB(s) = \int_0^t (t-s)\,dB(s).$$

By the same Tool C the last integral has a normal distribution with mean 0 and variance

$$\int_0^t (t-s)^2\,ds \overset{u=t-s}{=} \int_0^t u^2\,du = \frac{t^3}{3}.$$