Interest Rate Models

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Outline

1. Dynamics of the LIBOR market model

2. Calibration of the LMM

3. The SABR/LMM model

LIBOR market model

- Various term structure models differ in the choice of state variables that parametrize the rate curve. The models differ in the number of factors and, consequently, the property of the volatility.
- Over the past few years the industry standard is the LIBOR Market Model that we will abbreviate as LMM.
- We have covered the short rate models in which the underlying state variable is the instantaneous rate.
- The instantaneous rate is not an observed quantity.
- The state variables in LMM will be LIBOR forwards. They are observable from the market and the obtained model captures the dynamics of the entire curve of interest rates.
- The time evolution of the forwards will be required to satisfy the SDEs that would prevent arbitrage from occurring.

LIBOR market model

- The model is multi-factor and will be able to capture various aspects of the curve dynamics: parallel shifts, steepenings / flattenings, butterflies, etc.
- We will cover two LMM methods:
  (i) The classic LMM with local volatility specification;
  (ii) The stochastic volatility (SABR) extension.
- Older (pre-LMM) models produce unrealistic volatility structure. They create "hump" in the short end of the volatility curve. This results in overvaluation of instruments depending on forward volatility.
- LMM allows for approximately stationary volatility and correlation structure of LIBOR forwards.
- LMM is not as tractable as Hull-White and is not Markovian. All valuations based on LMM must be done using Monte Carlo simulations.

LIBOR market model

- The sequence of approximately equally spaced times $0 = T_0 < T_1 < \cdots < T_N$ will form the *standard tenors*.
- L_j denotes the standard LIBOR rate associated with a FRA with start T_j and maturity T_{j+1}.
- Usually, $N = 120$ and L_j are 3-month LIBORs.
- These dates refer to the actual start and end dates rather than the LIBOR fixing dates.
- We will simplify the notation by disregarding the difference between the fixing date and the start date.
- Each LIBOR rate is modeled as a continuos time stochastic process $L_j(t)$ that gets killed at T_j.
- The dynamics is driven by an N-dimensional Wiener processes $W_1, \ldots, W_N$ with correlations $\rho_{jk}dt = \mathbb{E}\left[dW_j\, dW_k\right]$.

No arbitrage condition

- Each of the rates is stochastic and assumed to follow

$$dL_j(t) = \Delta_j(t)\,dt + C_j(t)\,dW_j(t),$$

  where $\Delta_j(t)$ and $C_j(t)$ may depend on $L_j(t)$ as well but we suppress that from notation to make it pretty.
- Under the T_{k+1}-forward measure $\mathbb{Q}_{T_{k+1}}$, the rate L_k is a martingale and its dynamics is

$$dL_k(t) = C_k(t)\,dW_k(t).$$

- For $j \neq k$ we have

$$dL_j(t) = \Delta_j(t)\,dt + C_j(t)\,dW_j(t).$$

- The rate $L_j(t)$ is killed at $t = T_j$.

No arbitrage condition

- Define $\gamma : [0, T_N] \to \mathbb{N}$ as $\gamma(t) = T_{m+1}$ for $t \in [T_m, T_{m+1})$.
- Then the discount factor $P(t, T_q)$ is given by

$$P(t, T_q) = P(t, \gamma(t)) \cdot \prod_{i=\gamma(t)}^{q} \frac{1}{1 + \delta_i F_i(t)},$$

  where $F_i(t)$ is the OIS forward for the period $[T_i, T_{i+1})$.
- If $(\mathbb{P}, \mathcal{N})$ and $(\mathbb{Q}, \mathcal{M})$ are two associated measure-numeraire pairs and if the asset X satisfies

$$\begin{aligned} dX(t) &= \Delta^{\mathbb{P}}(t)\, dt + C(t) dW^{\mathbb{P}}(t) \\ dX(t) &= \Delta^{\mathbb{Q}}(t)\, dt + C(t) dW^{\mathbb{Q}}(t), \end{aligned}$$

  then

$$\Delta^{\mathbb{Q}}(t) - \Delta^{\mathbb{P}}(t) = \frac{d}{dt} \left[X, \log \frac{\mathcal{M}}{\mathcal{N}} \right] (t).$$

No arbitrage condition

- We now have

$$\begin{aligned} dL_j(t) &= C_j(t)\, dW^{\mathbb{Q}_{T_{j+1}}}(t) \\ dL_j(t) &= \Delta_j^{\mathbb{Q}_{t_{k+1}}}(t) + C_j(t)\, dW^{\mathbb{Q}_{T_{k+1}}}(t). \end{aligned}$$

The numeraire for $\mathbb{Q}_{T_{j+1}}$ is $P(t, T_{j+1})$ and the numeraire for $\mathbb{Q}_{T_{k+1}}$ is $P(t, T_{k+1})$ hence

$$\Delta_j^{\mathbb{Q}_{t_{k+1}}}(t) = \frac{d}{dt}\left[L_j, \log\frac{P(t, T_{k+1})}{P(t, T_{j+1})}\right](t).$$

- We now use that LIBOR/OIS spread is deterministing. Assuming that $j < k$, we get

$$\begin{aligned} \Delta_j^{\mathbb{Q}_{t_{k+1}}}(t) &= -\frac{d}{dt}\left[L_j, \log\prod_{i=j+1}^{k}(1+\delta_i F_i)\right](t) = -\sum_{i=j+1}^{k}\frac{d}{dt}\frac{dL_j \cdot \delta_i dF_i}{1+\delta_i F_i} \\ &= -C_j(t)\sum_{i=j+1}^{k}\frac{\rho_{ij}\delta_i C_i(t)}{1+\delta_i F_i(t)}. \end{aligned}$$

No arbitrage condition

- Similarly, for $j > k$ we have

$$\Delta_j^{\mathbb{Q}_{t_{k+1}}}(t) \quad = \quad C_j(t) \sum_{i=k+1}^{j} \frac{\rho_{ij}\delta_i C_i(t)}{1+\delta_i F_i(t)}.$$

- Therefore for every j and k and every $t < \min\{T_j, T_k\}$ we must have

$$\begin{aligned} dL_j(t) \quad &= \quad C_j(t)\, dW_j(t) \\ &\quad +C_j(t) \cdot \begin{cases} 0, & \text{if } j = k \\ -\sum_{i=j+1}^{k} \frac{\rho_{ij}\delta_i C_i(t)}{1+\delta_i F_i(t)}\, dt, & \text{if } j < k \\ \sum_{i=k+1}^{j} \frac{\rho_{ij}\delta_i C_i(t)}{1+\delta_i F_i(t)}\, dt, & \text{if } j > k. \end{cases} \end{aligned}$$

- We also have initial conditions $L_j(0) = L_{j0}$ for every j, where L_{j0} is the current value of the forward.

No arbitrage condition

- It is often convenient to use the spot measure. Its numeraire represents the value of the money market account at time t if the deposit of \$1 was made at time 0. The numeraire $B(t)$ can be expressed in terms of the discount factors as

$$B(t) = \frac{P(t, T_{\gamma(t)})}{\prod_{i=1}^{\gamma(t)} P(T_{i-1}, T_i)}.$$

- Under the spot measure the LMM dynamics becomes

$$dL_j(t) = C_j(t) \left(\sum_{i=\gamma(t)}^{j} \frac{\rho_{ij}\delta_i C_i(t)}{1 + \delta_i F_i(t)} \, dt + dW_j(t) \right).$$

Structure of the instantaneous volatility

- The most common choices for C_j are:

$$C_j(t) = \begin{cases} \sigma_j(t) & \text{normal model} \\ \sigma_j(t)L_j(t) & \text{lognormal model} \\ \sigma_j(t)L_j^{\beta_j}(t) & \text{CEV model} \\ \theta_j(t) + \sigma_j(t)L_j(t) & \text{shifted lognormal model} \end{cases}$$

- We will study the CEV model. The dynamics of the LIBOR forwards under $\mathbb{Q}_{T_{k+1}}$ are given by

$$\begin{aligned} dL_j(t) &= \sigma_j(t)L_j^{\beta_j}(t)\, dW_j(t) \\ &\quad + \sigma_j(t)L_j^{\beta}(t) \cdot \begin{cases} 0, & \text{if } j = k \\ -\sum_{i=j+1}^{k} \frac{\rho_{ij}\delta_i\sigma_i(t)L_i^{\beta_i}(t)}{1+\delta_i F_i(t)}\, dt, & \text{if } j < k \\ \sum_{i=k+1}^{j} \frac{\rho_{ij}\delta_i\sigma_i(t)L_i^{\beta_i}(t)}{1+\delta_i F_i(t)}\, dt, & \text{if } j > k. \end{cases} \end{aligned}$$

Structure of the instantaneous volatility

- Under the spot measure the dynamics is

$$dL_j(t) = \sigma_j(t) L_j^{\beta_j}(t)\, dW_j(t)) \left(\sum_{i=\gamma(t)}^{j} \frac{\rho_{ij}\delta_i\sigma_i(t) L_i^{\beta_i}(t)}{1+\delta_i F_i(t)}\, dt + dW_j(t) \right).$$

- We will assume the Dirichlet boundary condition at 0.
- This means that if the realization of L_j hits 0, then it gets killed and stays zero forever.

Factor reduction

- If the forward curve spans 30 years, then there are 120 quarterly LIBOR forwards.
- If the model had 120 stochastic factors, then we would see very big performance hit.
- Also, the parameters of the model become under-determined. This results in an unstable calibration.
- We will have only a small number of independent Brownian motions to drive the process. Typically this number is 1, 2, 3, or 4.
- We will denote by d the number of Brownian motions.
- For two integers $a, b \in \{1, 2, \ldots, d\}$ we will assume

$$\mathbb{E}\left[dZ_a\, dZ_b\right] = \delta_{ab}\, dt.$$

Factor reduction

- We define
$$dW_j(t) = \sum_{i=1}^{d} U_{ji} dZ_i(t).$$
- Here U is an $n \times d$ matrix and UU^T is close to the correlation matrix.
- The dynamics of the model can now be written in terms of independent Brownian motions Z.

$$\begin{aligned} dL_j(t) &= \Delta_j(t)\,dt + \sum_{j=1}^{d} B_{ja}(t)\,dZ_a(t), \text{where} \\ B_{ja}(t) &= U_{ja}C_j(t). \end{aligned}$$

- This is called the *factor reduced LMM dynamics*.

Calibration of the LMM model

- Calibration is the choice of model parameters that result in the rates that replicate correctly the prices of certain benchmark securities.
- The choice of the calibrating instruments has to be tailored towards the characteristics of the portfolio that is to be modeled.
- LMM model gives pricing formulas for caps and floors that are consistent with the practice of quoting these prices in terms of the Black's model.
- The calibration with caps and floors is easy.
- Swaptions are exotic instruments from the point of view of LMM model. Their pricing is a serious challenge.
- Fast and accurate swaption valuations are the key tasks with LMM model.

Approximate valuation of swaptions

- The SDE or the swap rate implied by the LMM model cannot be solved in a closed form. Recall that the swap rate is a non-linear function of the underlying LIBOR forward rate.
- Pricing swaptions requires Monte Carlo simulations.
- Monte Carlo simulations are time consuming.
- Consider the forward starting swap that starts at T_m and ends at T_n. The level function is

$$A_{mn}(t) = \sum_{j=m}^{n-1} \alpha_j P(t, T_{j+1}).$$

- In general α_j and T_j could be different from the LIBOR day count fractions and LIBOR payment dates.
- Denote by $S_{mn}(t)$ the corresponding forward swap rate. We will suppress the subscripts.

Approximate valuation of swaptions

- Since S depends on F_m, F_{m+1}, $\ldots$, F_{n-1} we can use Ito's formula to obtain the following representation

$$
\begin{aligned}
dS(t) &= \Omega(t,F)\,dt + \sum_{j=m}^{n-1} \Lambda_j(t,F)\,dW_j(t), \quad \text{where} \\
\Omega(t,F) &= \sum_{j=m}^{n-1} \frac{\partial S}{\partial F_j}\Delta_j + \frac{1}{2}\sum_{i,j=m}^{n-1} \rho_{ij}\frac{\partial^2 S}{\partial F_i \partial F_j} C_i C_j \quad \text{and} \\
\Lambda_j(t,F) &= \frac{\partial S}{\partial F_j} \cdot C_j.
\end{aligned}
$$

- The forward swap rate is not a martingale with respect to the spot measure. It is a martingale with respect to the swap measure.

Approximate valuation of swaptions

- Let
$$\nu^2(t) = \sum_{i,j=m}^{n-1} \rho_{ij} \Lambda_i(t,F) \lambda_j(t,F).$$
- Then we have
$$\begin{aligned} dS(t) &= \Omega(t,F)\,dt + \sum_{j=m}^{n-1} \Lambda_j(t,F)\,dW_j(t) \\ &= \nu(t) \cdot \frac{\Omega(t,F)\,dt + \sum_{j=m}^{n-1} \Lambda_j(t,F)\,dW_j(t)}{\nu(t)} \\ &= \nu(t)\,dW(t) \quad \text{where} \\ dW(t) &= \frac{\Omega(t,F)\,dt + \sum_{j=m}^{n-1} \Lambda_j(t,F)\,dW_j(t)}{\nu(t)}. \end{aligned}$$

Approximate valuation of swaptions

- We assume that L_j are frozen to their initial states $L_{j,0}(T_j, T_{j+1})$. Then F_j are also frozen and placing those values into formulas $\Lambda_j = \frac{\partial S}{\partial F_j} C_j$ we obtain $\Lambda_j(t, F_0)$ that are only functions of t.
- In the same way we obtain $\Omega(t, F_0)$ to replace $\Omega(t, F)$.
- Let us denote by $\nu_0(t)$ the $\nu(t)$ obtained when $\Omega(t, F_0)$ and $\Lambda(t, F_0)$ are placed instead of $\Omega(t, F)$ and $\Lambda(t, F)$ in the formula for ν.
- Then S is following the normal model

$$S(t) = S_0(t) + \int_0^t \nu_0(s)\, dW(s).$$

- Its implied normal volatility satisfies

$$\zeta_{mn}^2(t) \quad \approx \quad \frac{1}{T_m} \int_0^t \nu_0^2(s)\, ds = \frac{1}{T_m} \int_0^t \sum_{i,j=m}^{n-1} \rho_{ij} \Lambda_i(s, F_0) \Lambda_j(s, F_0)\, ds.$$

Parametrization of the volatility surface

- We will do the calibration under the assumption that the deterministic CEV volatilities $\sigma_j(t)$ featured in the equations for C_j are constant on each of the intervals $[T_i, T_{i+1})$.

	$[T_0, T_1)$	$[T_1, T_2)$	$\cdots$	$[T_{N-1}, T_N)$
$\sigma_0(t)$	0	0	$\cdots$	0
$\sigma_1(t)$	$\sigma_{1,0}$	0	$\cdots$	0
$\sigma_2(t)$	$\sigma_{2,0}$	$\sigma_{2,1}$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\sigma_{N-1}(t)$	$\sigma_{N-1,0}$	$\sigma_{N-1,1}$	$\cdots$	0

- The problem is overparametrized and we make additional assumption that the instantaneous volatility is stationary, i.e. $\sigma_{i,j} = \sigma_{i-j,0}$.

Parametrization of the volatility surface

- With the assumption $\sigma_{i,j} = \sigma_{i-j,0}$, the table becomes:

	$[T_0, T_1)$	$[T_1, T_2)$	$\cdots$	$[T_{N-1}, T_N)$
$\sigma_0(t)$	0	0	$\cdots$	0
$\sigma_1(t)$	σ_1	0	$\cdots$	0
$\sigma_2(t)$	σ_2	σ_1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\sigma_{N-1}(t)$	σ_{N-1}	σ_{N-2}	$\cdots$	0

- Such assumption is justified because it will make the cap volatility to behave the same way in the future as it behaves at present.

Parametrization of the volatility surface

- The number of parameters can be reduced even further. We look for a function h to generate volatilities with $\sigma_i = h(T_i)$.
- The hump function is a popular choice for h. It is defined as

$$h(t) = (at + b)e^{-\lambda t} + \mu.$$

- The stationarity assumption makes it impossible to calibrate the model. The reason is the mean reversion of long term rates.
- It is not possible to introduce a mean reversion through some Vasicek type drift (or Ornstein-Uhlenbeck process). Doing so would violate the arbitrage free requirement.

Parametrization of the volatility surface

- An alternative is another choice for volatility parameters.
- The long term volatility will be affected by coefficients chosen as $\bar{\sigma}_j = \bar{h}(T_i)$, for $i = 1, 2, \ldots, N-1$, where $\bar{h}$ is anoether hump shaped function.
- Then we define

$$\sigma_{j,i} = p_i \sigma_{j-i} + q_i \bar{\sigma}_{j-i}.$$

- This way σ are mixture of short-term σ and long-term $\bar{\sigma}$. We require $p_i + q_i = 1$, $p_i, q_i \geq 0$, and $p_i \to 0$ as $i \to \infty$.
- As we move along the time axis, the volatility structure approaches its long term limit.

Parametrization of the volatility surface

- The table with volatilities becomes

	$[T_0, T_1)$	$[T_1, T_2)$	$\cdots$	$[T_{N-1}, T_N)$
$\sigma_0(t)$	0	0	$\cdots$	0
$\sigma_1(t)$	$p_0\sigma_1 + q_0\bar{\sigma}_1$	0	$\cdots$	0
$\sigma_2(t)$	$p_0\sigma_2 + q_0\bar{\sigma}_2$	$p_1\sigma_1 + p_1\bar{\sigma}_1$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\sigma_{N-1}(t)$	$p_0\sigma_{N-1} + q_0\bar{\sigma}_{N-1}$	$p_1\sigma_{N-2} + p_1\bar{\sigma}_{N-2}$	$\cdots$	0

- The above lower triangular matrix is called the LMM volatility surface.
- The calibration is a bit tedious and we leave out the details.
- The parametrization reduces to a relatively small number of parameters $(\theta_1, \ldots, \theta_d)$ (such as parameters of the hump functions and the weights p_i) and $\sigma_{j,i} = \sigma_{j,i}(\theta)$ can be calibrated.

Parametrization of the correlation matrix

- The model has to be calibrated to the cap-floor market and to the swaption market at the same time.
- This can be achieved by calibration of the correlation matrix $\rho = \{\rho_{jk}\}_{0 \le j,k \le N-1}$.
- The number of entries in the matrix is $\frac{N(N+1)}{2}$ which is too hight to achieve the stable calibration.
- We will make ρ_{ij} to depend on fewer parameters.
- The correlations of late LIBORs that are far apart will converge to a desired value ρ (which is one of parameters to be calibrated). We will have another parameter α to control the speed of convergence.

Parametrization of the correlation matrix

- The parametrization is:

$$\begin{aligned}\rho_{ij} &= \bar{\rho}_{\min(i,j)} + \left(1 - \bar{\rho}_{\min(i,j)}\right) e^{-\beta_{\min(i,j)}|T_i - T_j|}, \\ &\text{where } \ \bar{\rho}_k = \rho \tanh\left(\alpha T_k\right) \\ &\text{and } \ \beta_k = \beta T_k^{-\kappa}.\end{aligned}$$

- Parameters β_k are decay rate of correlations, and κ is an asymmetry parameter.
- The parameters can be calibrated using historical data.
- One has to be careful because the above parametrization produces a matrix that may not be positive definite, although it is approximately positive definite.

Optimization

- We will calibrate σ_i to fit the at the money caplet and swaption volatilities. Let us denote by ζ_m and ζ_{mn} the implied volatilities of the caplet expiring at T_m and of the ATM swaption, respectively.
- Since the stochastic term in $dL_j(t)$ is $\sigma_j(t)L_j^{\beta_m}(t)\,dW_j(t)$ and since $\sigma_j(t)$ is piecewise constant, if we assume the frozen curve approximation for $L_j(t)$ we get that

$$
\begin{aligned}
\zeta_m^2(\theta) &= \frac{1}{T_m} L_{m0}^{2\beta_m} \sum_{i=1}^{m-1} \sigma_{m,i}^2(\theta)\delta_i, \\
\zeta_{mn}^2(\theta) &= \frac{1}{T_m} \sum_{i=0}^{m-1} \sum_{j,k=m}^{n-1} \rho_{jk}\Lambda_{j,i}(\theta)\Lambda_{k,i}(\theta)\delta_i,
\end{aligned}
$$

where $\delta_i = T_{i+1} - T_i$.

Optimization

- $\Lambda_{j,i}(s, F_0)$ are calculated using the frozen curve F_0.
- The objective function for minimization is

$$\mathcal{L}(\theta) = \sum_m w_m \left(\zeta_m(\theta) - \bar{\zeta}_m\right)^2 + \sum_{m,n} w_{mn} \left(\zeta_{mn}(\theta) - \bar{\zeta}_{mn}\right)^2 .$$

- The weights w_m and w_{mn} can be chosen by the user to select the calibration instrument to give the desired importance to the volatility matchings that are relevant to the portfolio.
- A Tikhonov regularizer is often added to the objective function to penalize the high mean curvature. The form of the regularizer is

$$\lambda \int_{\text{LMM vol surface}} R(u,v)^2 \, du dv.$$

LMM and smile dynamics

- There is a drawback of the classic LMM model.
- If it is calibrated to price the ATM caps and swaptions, then it will missprice the out of the money instruments.
- The solution is to have stochastic volatility models such as SABR.
- We now assume that the instantaneous volatilities of the forward rates are

$$C_j(t) = \sigma_j(t) L_j(t)^{\beta_j},$$

where volatility parameters $\sigma_j(t)$ are stochastic.

LMM and smile dynamics

- Under the T_k-forward measure $\mathbb{Q}_{T_k}$ the model is assumed to satisfy the system:

$$\begin{aligned} dL_k(t) &= C_k(t)\, dW_k(t), \\ d\sigma_k(t) &= D_k(t)\, dZ_k(t), \end{aligned}$$

  where the diffusion coefficient is of the form $D_k(t) = \alpha_k(t)\sigma_k(t)$ and $\alpha_k(t)$ is assumed to be deterministic.
- We assume the following correlations

$$\mathbb{E}\left[dW_j(t) dZ_k(t)\right] = \rho_{jk}\, dt \quad \text{and} \quad \mathbb{E}\left[dZ_j(t) dZ_k(t)\right] = \eta_{jk}\, dt.$$

- The block matrix $\Pi = \begin{bmatrix} \rho & r \\ r' & \eta \end{bmatrix}$ is positive definite.

LMM and smile dynamics

- In an analogous way as in the case of deterministic volatility we obtain for every j and k and every $t < \min\{T_j, T_k\}$ that

$$
\begin{aligned}
dL_j(t) &= C_j(t)\, dW_j^{\mathbb{Q}_k}(t) \\
&\quad + C_j(t) \cdot \begin{cases} 0, & \text{if } j = k \\ -\sum_{i=j+1}^{k} \frac{\rho_{ij}\delta_i C_i(t)}{1+\delta_i F_i(t)}\, dt, & \text{if } j < k \\ \sum_{i=k+1}^{j} \frac{\rho_{ij}\delta_i C_i(t)}{1+\delta_i F_i(t)}\, dt, & \text{if } j > k. \end{cases}
\end{aligned}
$$

- Under the spot measure the LMM dynamics becomes

$$
dL_j(t) = C_j(t) \left(\sum_{i=\gamma(t)}^{j} \frac{\rho_{ij}\delta_i C_i(t)}{1 + \delta_i F_i(t)}\, dt + dW_j(t) \right).
$$

LMM and smile dynamics

- We have that σ_j is a martingale under $\mathbb{Q}_{T_j}$. However, under $\mathbb{Q}_{T_k}$ it has a drift term. If we call that term Γ_j we obtain that for $j < k$

$$
\begin{aligned}
\Gamma_j(t) &= \frac{d}{dt}\left[\sigma_j, \log\frac{P(t,T_{k+1})}{P(t,T_{j+1})}\right] \\
&= -\frac{d}{dt}\left[\sigma_j, \log\prod_{i=j+1}^{k}(1+\delta_i F_i)\right](t) = -\sum_{i=j+1}^{k}\frac{d}{dt}\frac{d\sigma_j\cdot\delta_i dF_i}{1+\delta_i F_i} \\
&= -D_j(t)\sum_{i=j+1}^{k}\frac{r_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}.
\end{aligned}
$$

- If $j > k$ we have

$$
\Gamma_j(t) = D_j(t)\sum_{i=k+1}^{j}\frac{r_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}.
$$

LMM and smile dynamics

- We can summarize the previous conclusions into the following dynamics of σ_j under the measure $\mathbb{Q}_{T_k}$:

$$
\begin{aligned}
d\sigma_j(t) &= D_j(t)\, dZ_j^{\mathbb{Q}_k}(t) \\
&\quad + C_j(t) \cdot \begin{cases} 0, & \text{if } j = k \\ -\sum_{i=j+1}^{k} \frac{r_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}\, dt, & \text{if } j < k \\ \sum_{i=k+1}^{j} \frac{r_{ji}\delta_i C_i(t)}{1+\delta_i F_i(t)}\, dt, & \text{if } j > k. \end{cases}
\end{aligned}
$$

- Under the spot measure we have

$$
d\sigma_j(t) = D_j(t) \left(\sum_{i=\gamma(t)}^{j} \frac{r_{ji}\delta_i C_i(t)}{1 + \delta_i F_i(t)}\, dt + dZ_j(t) \right).
$$

LMM and smile dynamics

- With the choices $C_j(t) = \sigma_j(t) L_j^{\beta_j}(t)$ and $D_j(t) = \alpha_j(t)\sigma_j(t)$ we obtain

$$
\begin{aligned}
dL_j(t) &= \sigma_j(t) L_j^{\beta_j}(t)\, dW_j^{\mathbb{Q}_k}(t) \\
&\quad + \sigma_j(t) L_j^{\beta_j}(t) \cdot \begin{cases} 0, & \text{if } j = k \\ -\sum_{i=j+1}^{k} \frac{\rho_{ij}\delta_i\sigma_i(t) L_i^{\beta_i}(t)}{1+\delta_i F_i(t)}\, dt, & \text{if } j < k \\ \sum_{i=k+1}^{j} \frac{\rho_{ij}\delta_i\sigma_i(t) L_i^{\beta_i}(t)}{1+\delta_i F_i(t)}\, dt, & \text{if } j > k, \end{cases} \\
d\sigma_j(t) &= \alpha_j(t)\sigma_j(t)\, dZ_j^{\mathbb{Q}_k}(t) \\
&\quad + C_j(t) \cdot \begin{cases} 0, & \text{if } j = k \\ -\sum_{i=j+1}^{k} \frac{r_{ji}\delta_i\sigma_i(t) L_i^{\beta_i}(t)}{1+\delta_i F_i(t)}\, dt, & \text{if } j < k \\ \sum_{i=k+1}^{j} \frac{r_{ji}\delta_i\sigma_i(t) L_i^{\beta_i}(t)}{1+\delta_i F_i(t)}\, dt, & \text{if } j > k, \end{cases} \\
&\quad L_j(0) = L_{j0}, \quad \text{and} \quad \sigma_j(0) = \sigma_{j0}.
\end{aligned}
$$

SABR/LMM implementation

- We will not go into details of how SABR/LMM is executed in practice. The main idea is to follow the steps of classical LMM implementation.
- One first does the factor reduction, and appropriate parametrizations are necessary for volatilities and correlations.
- Since SABR does not have closed form solutions, it is not possible to have closed form valuations for caps and floors in SABR/LMM. Recall that such valuations were possible in classical LMM.
- It is necessary to perform Monte Carlo simulations.
- The prices of caps and floors is consistent with the market prices.

SABR/LMM implementation

- SABR/LMM has rich correlation structure. We can specify the correlations the forwards, the correlations between their volatilities, and the correlations between the forwards and the volatilities. This gives us a big control over the shape of volatility smile.
- The SABR specifies β_j for each of the forwards, but it does not specify the exponents for forward swap rates S_{mn}. These forward swap rates also follow SABR model and their β_{mn} are implied by the model but are difficult to determine.

SABR/LMM implementation

- The following formulas are used in practice to approximate these exponents:

$$
\begin{aligned}
\beta_{mn} &= \sum_{j=m}^{n-1} a_{mn,j}\beta_j + b_{mn}, \quad \text{where} \\
a_{mn,k} &= \frac{2\log L_{k0}}{(n-m)^2} \sum_{j=m}^{n-1} \frac{1}{\log L_{j0} + \log L_{k0}}, \\
b_{mn} &= \frac{1}{(n-m)^2} \sum_{j,k=m}^{n-1} \frac{\log \rho_{jk}}{\log L_{j0} + \log L_{k0}}.
\end{aligned}
$$

- Notice that the numbers $a_{mn,k}$ satisfy

$$
\sum_{k=m}^{n-1} a_{mn,k} = 1.
$$

SABR/LMM implementation

- The CEV exponent of the swaption is a weighted average of the CEV exponents of the forwards plus a convexity correction.
- If were equal, i.e. $a_{mn,j} = \frac{1}{m-n}$ for all j then the convexity correction would be very small.
- Typically the convexity corrections is of order of magnitude 10^{-3} and for all practical purposes it is assumed to be 0.
- Term structure model should be calibrated to a suitable set of swaptions because the swaptions are the most liquid volatility instruments.
- Calibration requires understanding the relationship between swaption SABR parameters (discussed earlier) and the caplet parameters discussed in the SABR/LMM model.

SABR/LMM implementation

- We have described the relationship between the CEV exponents.
- Other relations can be established between caplet and swaption SABR parameters can also be established. For example, one may study the relation between vol of vols, or the relations between correlation coefficients in the SABR model and the correlation coefficients in the SABR/LMM model.
- These relations are easier to derive if “frozen curve” assumption is introduced but the programming is still a tedious task.

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