10. Properties of Option Combinations

Option combinations

As we have seen, options provide exposure to a variety of characteristics of the underlying. While options are primarily volatility instruments, by combining them a market participant can isolate particular characteristics and outcomes of interest and tailor an instrument that expresses a precise view of the underlying. Furthermore, option combinations can offer reduced collateral requirements that might otherwise make a strategy too capital-intensive to justify its gains.

The simplest position is, of course, a long position in puts or calls on an equity. Contracts are typically sets of 100 options, making this the smallest increment in which the underlying can be traded via options. The advantage of such an exposure is leverage: a relatively small amount of capital can be used to control a large number of shares, and this can be done more or less irrespective of the spot price of the underlying. Thus, a market participant can gain access to the appreciation in price of 100 shares of an underlying trading at hundreds or thousands of dollars per share with less outlay than the cost of a single share would require.

Long positions are also the safest of all "naked" options exposures. For a long call or put—and, more generally, for any payoff whose value is always strictly nonnegative—loss is limited to the initial premium. As a result, there is no need to post additional collateral, and in limited cases it may be possible to purchase long-dated options on margin, as one can purchase ordinary stocks. Of course, the idea of the "safety" of long options positions is relative: In pure cash trading of shares or bonds, a total loss of the initial investment is extremely rare. In options trading, it is fairly common.

Naked short option positions are, by contrast, extremely risky. In theory, the possible loss on a short call position is limitless; for a short put position, it is limited to the strike times the number of underlying shares. As a result, options exchanges require that short positions be heavily collateralized. The particular rules may be slightly involved, but generally speaking a party short an option must maintain at least the market value of the option plus some percentage of the shares to which the contract refers. This collateral requirement may be satisfied by bonds of sufficient quality instead of cash.

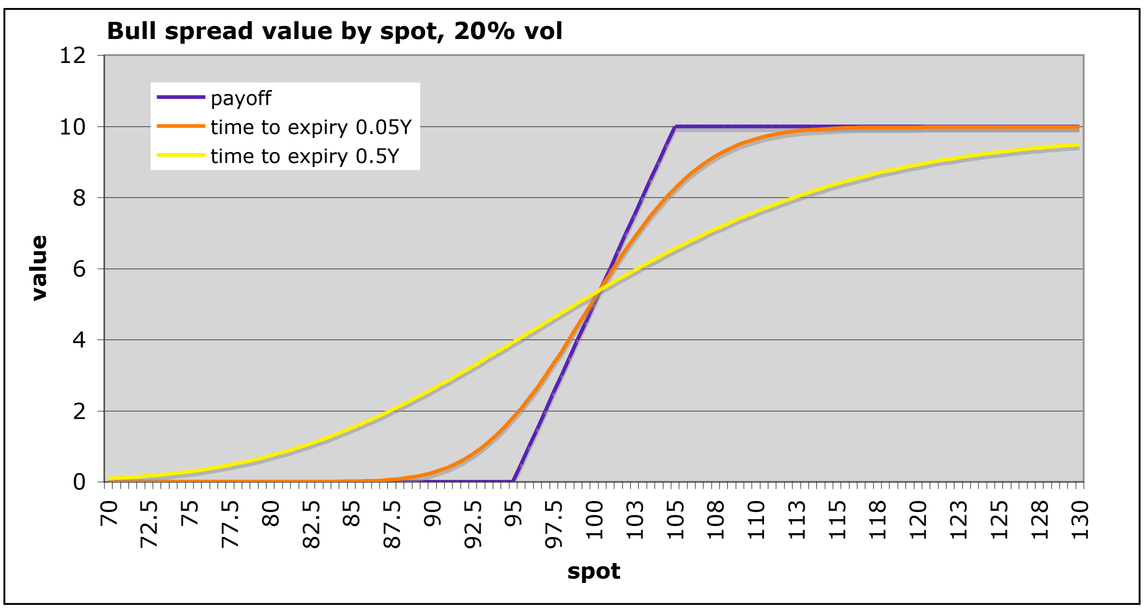
In the particular case of a short call, the collateral requirement can be satisfied by posting the underlying shares themselves. Since the call is a contract to deliver shares at a particular price, there is no risk at all in this method of collateralization. As a result, for retail investors the strategy of being long the underlying and short calls on it—called a "covered call" strategy—is appealing as a source of extra income on positions they intend to hold anyway. For short puts, however, the corresponding solution—going short the equity—does not work as collateral since a short position is not truly an asset.

A solution that works for both puts and calls is to reduce the total possible loss by pairing a short option position with a long option position on the same underlying—a combination called a "spread." Exchanges typically list and trade such combinations as units, rather than requiring participants to construct them out of individual legs themselves. Moreover, the collateral requirements on such combinations are significantly less as long as the spread position is not broken up by, for example, selling the long position but retaining the short. It may be impossible even to break up the spread in this way, depending upon what assets you have available to provide as collateral.

The below covers a few of the most common combinations traded in options markets. Many go by several interchangeable names. We will assume that all the options involved are European, and under this assumption there are usually at least two ways any of these combinations can be constructed. Many of these combinations lead to fundamental results in options theory that are hold irrespective of the model chosen.

Bull spread

As mentioned above, perhaps the simplest and among the most useful combinations is to be short an option at one strike and long an option at another with the same expiry, creating a payoff with limited gain and loss. Generally, these types of positions are referred to as “vertical spreads.” Specifically, if the combination consists of a long call position at strike K and a short call position at strike $K + \mathrm{d}K$, then the position is referred to as a bull spread or, more specifically, a bull spread of calls.



The identical payoff can be constructed by a long put position struck at K and a short put position struck at $K + \mathrm{d}K$ with the addition of a bond paying $\mathrm{d}K$ at expiry, as can be easily be shown using put-call parity:

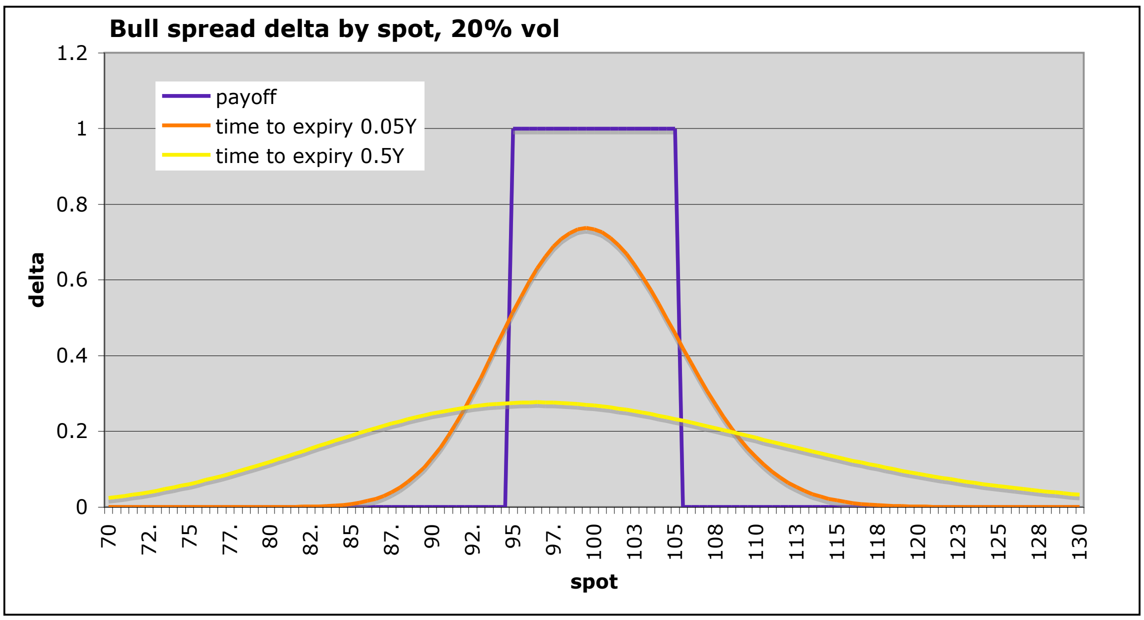
$$C_K - C_{K+\delta K} = P_K + F_K - \left(P_{K+\delta K} - F_{K+\delta K}\right) =$$

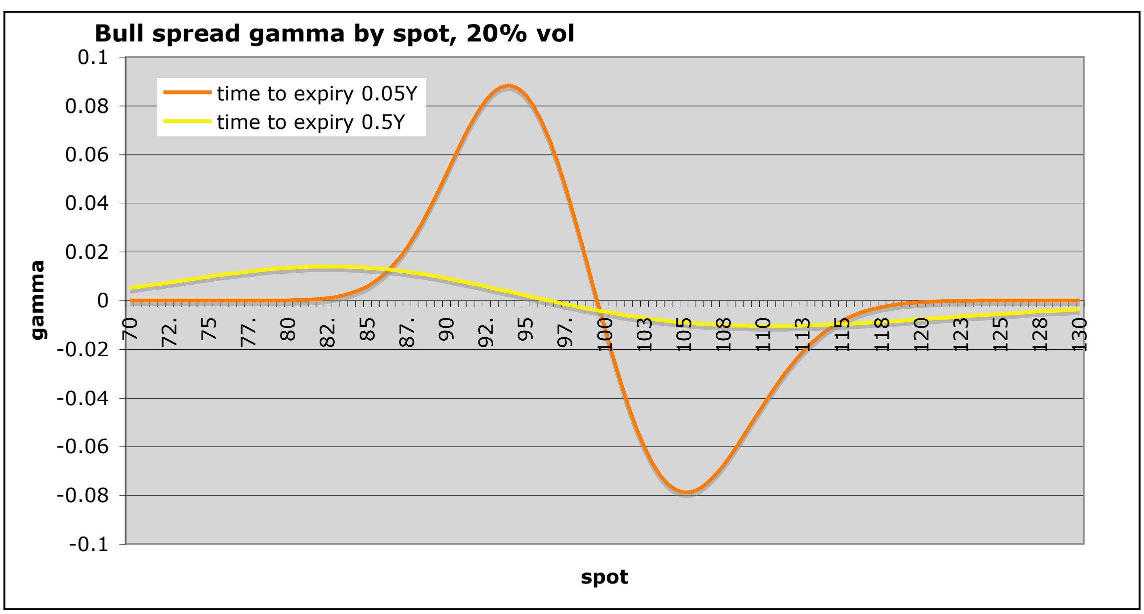
$$P_K - P_{K+\delta K} + S - Ke^{-rT} - S + \left(K + \delta K\right)e^{-rT} =$$

$$P_K - P_{K+\delta K} + \delta K e^{-rT}$$

If constructed with calls, then this vertical combination is what is known as a *debit spread*: The call purchased is worth more than the call sold, requiring an initial outlay of cash. If the bull spread is instead constructed with put options, then the payoff is always nonpositive, since the put with the higher strike is sold and a put with a lower strike is bought. Such a combination, called a *credit spread*, results in an initial cash flow to the party setting it up—which must, of course, be posted initially as collateral.

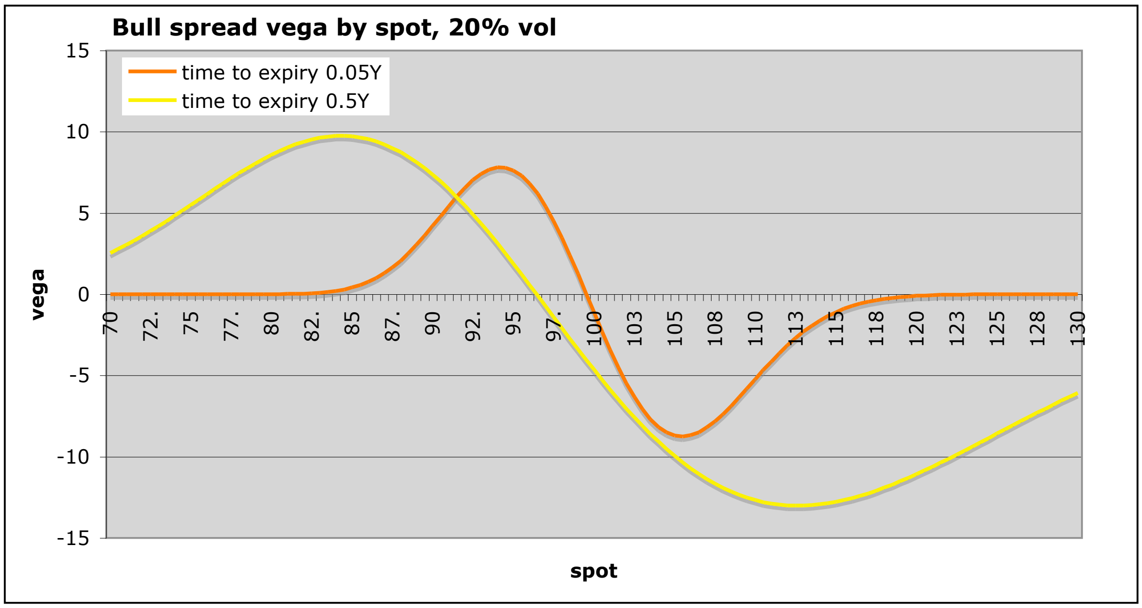
Whichever construction method is chosen, the maximum gain / loss of the position is dK, and when the spot is far from K these exposures require hardly any delta-hedging at all. However, near the region of interest to the combination, their delta-hedging profiles can be quite extreme:





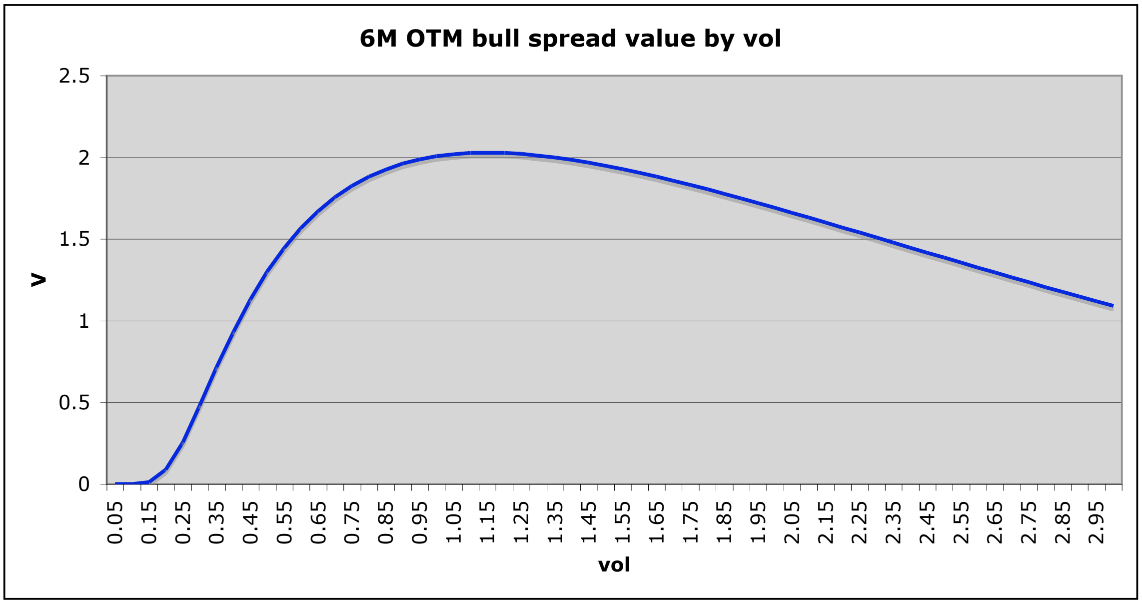
In this case, our spacing between strikes is relatively large, but even so with a short time to expiry our delta hedge can require us to adjust from 0.7 shares down to 0.3 for moves of a few dollars in either direction. Particularly in a position where the gain or loss is capped, attempting to delta-hedge such an exposure with short time to expiry can incur significant cost.

The position's exposure to volatility is also quite interesting:



As expected, the longer-dated position has a greater overall vol sensitivity, but here we encounter a payoff whose vega may be positive or negative depending upon the situation. The reason for this is clear with a little thought: If the forward price of the underlying is greater than $K + \mathrm{d}K$—the higher strike—then we would be assured of receiving the maximum payoff of the position if the volatility were to go to zero. So an increase in volatility reduces the option's value. By contrast, if the option is out of the money forward, then it seems sensible that increasing the volatility would increase the probability of receiving the maximum payoff.

Interestingly, even this relationship is not as straightforward as it might seem. Taking the option with 6 months to expiry at a spot of 70, the value of the option depends upon volatility in the following way:

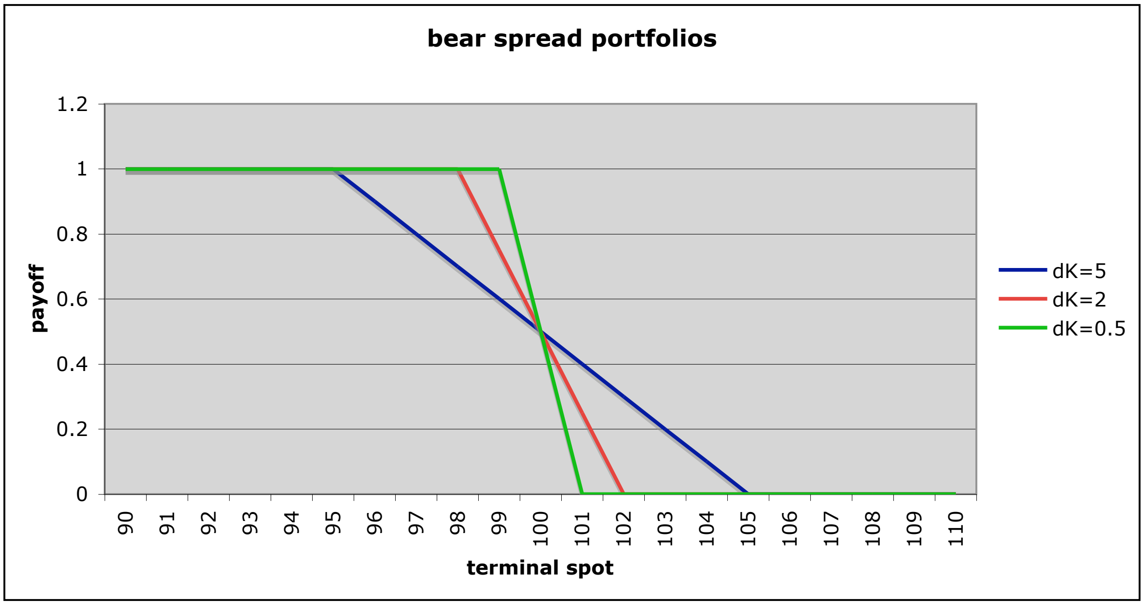


To understand this, consider the behavior of the distribution of the terminal spot price as the volatility is increased. The forward price of the asset remains the same, but as volatility increases we increase both the number of extremely high realizations and the number of extremely low realizations in the process; in particular, the median value of the distribution decreases as the volatility is increased. For a vanilla call option, the extremely high realizations overwhelm the increased probability of the option finishing OTM, and the value of the call increases. In the case of a bear spread, however, since the payoff is capped, there is no equivalent compensation for the increasing likelihood of very low realizations, and eventually the combination's value begins to decrease with increasing vol, even when the lower option is OTM.

Bear spreads and European binary options

As with the above, it is possible to construct a payoff that is fixed positive if the terminal spot price is below a certain value by going long a put option at one strike and short a put option on the same underlying with the same expiry at a lower strike. This is another instance of a vertical spread—called a bear spread of puts—and has many properties similar to the above. Equivalently, this payoff can be constructed with call options and a zero-coupon bond.

It is instructive to consider the family of bear spreads with the following properties: Choose a value K and a positive dK. We construct a bear spread of 1 / 2dK long puts struck at K + dK and the same number of short puts struck at K – dK. The resulting payoff is shown below for a few choices of dK.



The limiting payoff of this process as dK approaches zero is clearly $I(S_T < K)$—an indicator function for the relationship of the terminal spot price and the central strike K. This European payoff goes by various names, among them a binary put, a digital put, or a cash-or-nothing put.

If we assume that the terminal spot price is not correlated with the risk-free interest rate, then our familiar risk-neutral pricing formula shows that…

$$V_{P,binary}(K) = e^{-rT} E^*\left[I_{S_T<K}\right] = e^{-rT} P^*(S_T < K)$$

$$P^*(S_T < K) = e^{rT} V_{P,binary}(K)$$

…which, aside from our interest rate stipulation, is a model-free result. Thus, with market prices of this instrument available, we can define the risk-neutral cumulative distribution function:

$$F^*_{S_T}(s) = P(S_T < s) = e^{rT} V_{P,binary}(s)$$

Furthermore, if we assume that European puts are traded at a continuum of strikes, then we can extend this result using the relationship of the payoffs…

$$I_{S_T<K} = \lim_{\delta K \to 0} \frac{V_{P,T}(K+\delta K) - V_{P,T}(K-\delta K)}{2\delta K} = \frac{d}{dK} V_{P,T}(K)$$

…where the values in the right-hand equations are the payoffs of put options at expiry. Assuming that the derivative on the far right exists, we observe that taking a risk-neutral expectation should not interfere with our taking of the limit, since expectation is a linear operator. Thus…

$$E^*\left[I_{S_T<K}\right] = \frac{d}{dK} E^*\left[V_{P,T}(K)\right] = e^{rT} \frac{d}{dK} V_P(K)$$

…where V_P signifies the present value of the put. This leads us at last to the result…

$$F^*_{S_T}(s) = e^{rT} \frac{d}{dK} V_P(s)$$

…where the quantity $V_P(s)$ is the price of the put option with strike s.

While it is true that options do not trade with a continuum of strikes, if market values for the derivative of a European put's price with respect to its strike can be obtained or approximated, then the result is a risk-neutral cumulative distribution function of the terminal spot price motivated only by the actual traded prices of the options, not by any model.

In the case of our vanilla Black-Scholes model, it is hardly necessary to take this derivative with respect to K, since we have already observed that in our familiar formula…

$$V_P = Ke^{-rT}N(-d_2) - S_0e^{-qT}N(-d_1)$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T\right)$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

…the present value of the strike K is weighted by the probability of its being received under the risk-neutral distribution of the terminal spot that arises from geometric Brownian motion. Thus, the value of the European binary put is in the Black-Scholes world:

$$V_{P,binary} = e^{-rT}N(-d_2)$$

And we can see almost immediately that binary options must also have their own version of put-call parity…

$$1 = I_{S_T<K} + I_{S_T\geq K}$$

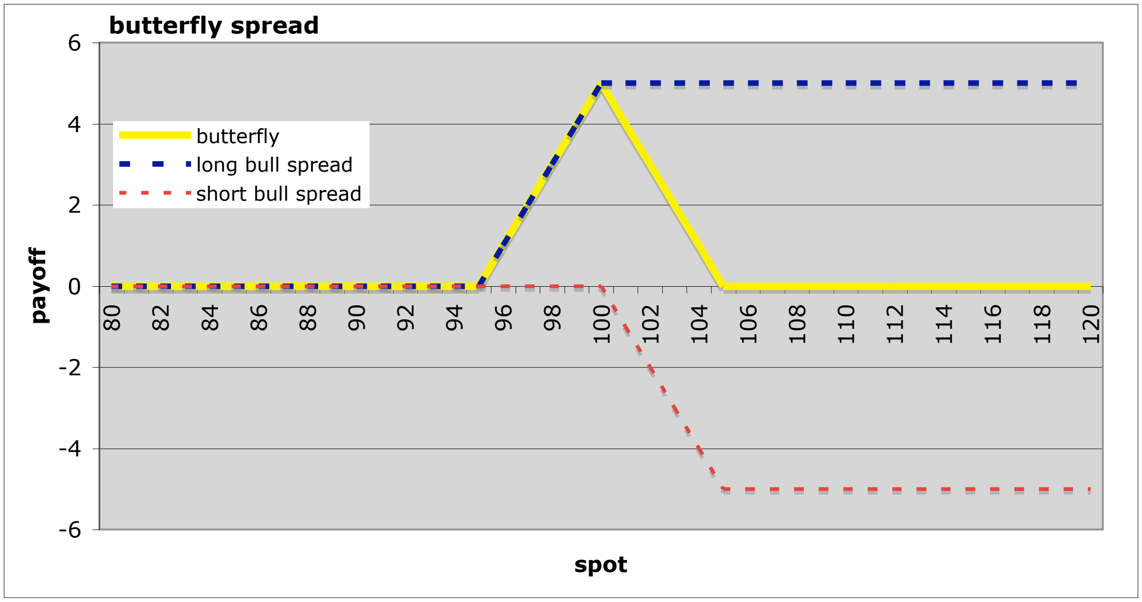
$$e^{-rT} = V_{P,binary} + V_{C,binary}$$

…leading once again to a result familiar from our initial consideration of the Black-Scholes value of a call option:

$$V_{C,binary} = e^{-rT}N(d_2)$$

Butterfly spreads and the Breeden-Litzenberger result

A combination closely related to the vertical spread is called a butterfly spread. Just as a vertical spread is constructed by buying an option at one strike and selling an option of the same type at another strike with the same expiry, a butterfly spread is formed by buying a vertical spread at one pair of strikes and selling a vertical spread with the same spacing at a pair of strikes that overlaps with the first. The resulting payoff gives a positive result only when the spot finishes in a narrow range. We illustrate this below with bull spreads, although bear spreads can be used just as easily:



Butterfly spreads are standard options combinations that can usually be traded as a unit on options exchanges. As one would expect, the narrow range of positive payoffs makes these options extremely affordable, and short butterflies have small collateral requirements compared to virtually any other short option position.

It may be that some purchase butterfly spreads based on a firm belief about the range in which an underlying will trade, but its main use can best be illustrated by considering another limiting case of these options. Taking a strike K and a positive value dK, a long butterfly spread of European puts consists of:

A long put at strike K + dK
Two short puts at strike K
A long put at strike K – dK

If we hold 1 / dK^2 of this portfolio, then at payoff this value is:

$$V_{\delta K,T} = \frac{P_{K-\delta K,T} - 2P_{K,T} + P_{K+\delta K,T}}{\delta K^2} = \frac{1}{\delta K}\frac{\left(P_{K+\delta K,T} - P_{K,T}\right) - \left(P_{K,T} - P_{K-\delta K,T}\right)}{\delta K}$$

These expressions should look familiar. Taking limits, we see that…

$$V_{\delta,T} = \lim_{\delta K \to 0} \frac{P_{K-\delta K,T} - 2P_{K,T} + P_{K+\delta K,T}}{\delta K^2} = \frac{d^2}{dK^2} P_{K,T} = \frac{d}{dK} I_{S_T < K}$$

…since above we saw that the indicator function is simply the derivative of a put payoff with respect to K. We also used this to characterize a risk-neutral distribution of the terminal spot price…

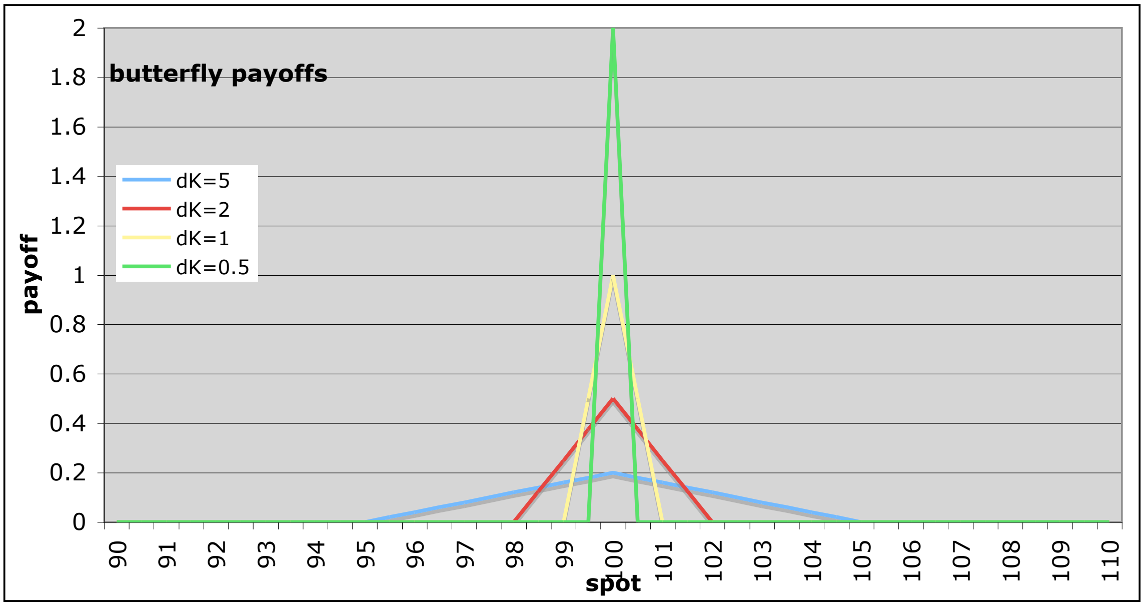
$$F_{S_T}^{*}(s) = E^{*}\left[I_{S_T < s}\right] = e^{rT}\frac{d}{dK}V_P(s)$$

…meaning that we can obtain the risk-neutral density by taking another derivative with respect to K in the put prices…

$$f_{S_T}^*(s) = e^{rT} \frac{d^2}{dK^2} V_P(s)$$

…where as above the notation $V_P(s)$ indicates the price of the European put with strike s. This model-free result was obtained by Breeden and Litzenberger in 1978 and is not only elegant in itself, but also is made use of extensively in more complex models that extend the Black-Scholes results to include more sophisticated volatility behaviors and / or jumps in the price of the underlying.

A look at the limiting payoff may make it more intuitive why this is so:



At the limit, this payoff is zero everywhere except at K, but each payoff in the sequence has integral 1, so we infer that the limiting payoff does as well. This is the Dirac delta function, which has the well-known property that in expectation:

$$E[\delta_s(X)] = P(X = s) = f_X(s)$$

That is, in expectation this function "picks out" the value of the density function at the chosen value. Since the Dirac delta function is the second derivative of a put payoff with respect to strike, this is precisely what our limiting butterfly does.

Since the same function is also the second derivative of a call payoff, one would additionally expect that…

$$f_{S_T}^*(s) = e^{rT} \frac{d^2}{dK^2} V_C(s)$$

…which can be easily seen, once again, by put-call parity:

$$Se^{-qT} - Ke^{-rT} = V_C - V_P$$

$$-e^{-rT} = \frac{d}{dK}V_C - \frac{d}{dK}V_P$$

$$0 = \frac{d^2}{dK^2}V_C - \frac{d^2}{dK^2}V_P$$

$$\frac{d^2}{dK^2}V_C = \frac{d^2}{dK^2}V_P$$

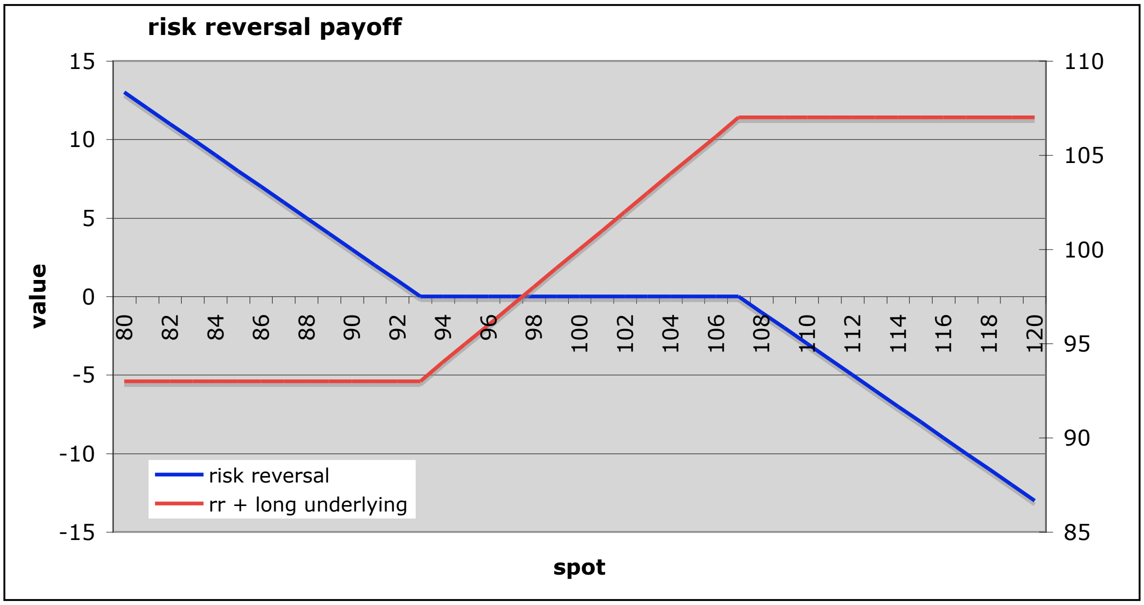
The butterfly spread thus is a precise instrument for gaining access to particular areas of the risk-neutral distribution of the terminal spot price. The fact that the combination consists of options at three strikes means that the bid-ask spread, particularly at remote strikes, can make the cost of these spreads prohibitive, but in favorable circumstances they are an excellent way to pick out particular regions that are underbought or overbought and gain exposure only to those outcomes.

As suggested by the Breeden-Litzenberger result, butterflies enjoy a special status in options theory, Since they are quite directly related to the probability distribution of the terminal spot, it can be shown that the collection of prices is arbitrage-free if and only if all of the butterflies at that expiry have positive value.

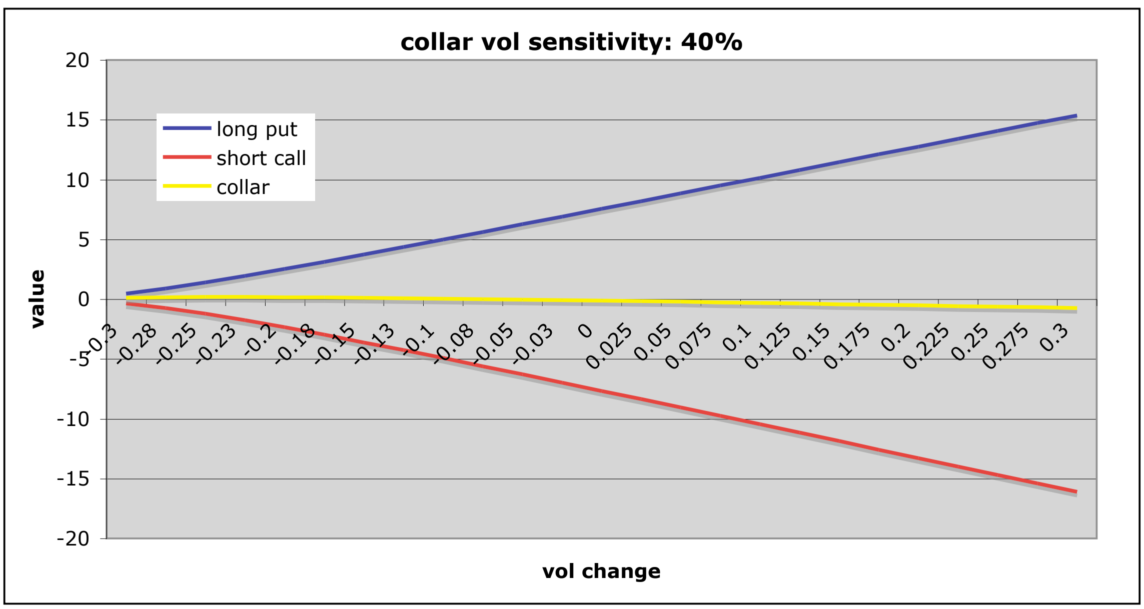
Risk reversals

A risk reversal is a long put position at one strike (usually OTM) funded by a short call position at a different strike (usually also OTM). This combination goes by several other names in other circumstances: With interest rate or inflation options, it is often called a collar; when the party is short this combination rather than long it, it is sometimes called a range forward contract.

This combination is more economical to hold than a simple naked long put position, as the proceeds of the call can be used to cover some or all of the cost of the put. Its primary name stems from the payoff obtained by going long this combination and additionally holding a long position in the underlying:

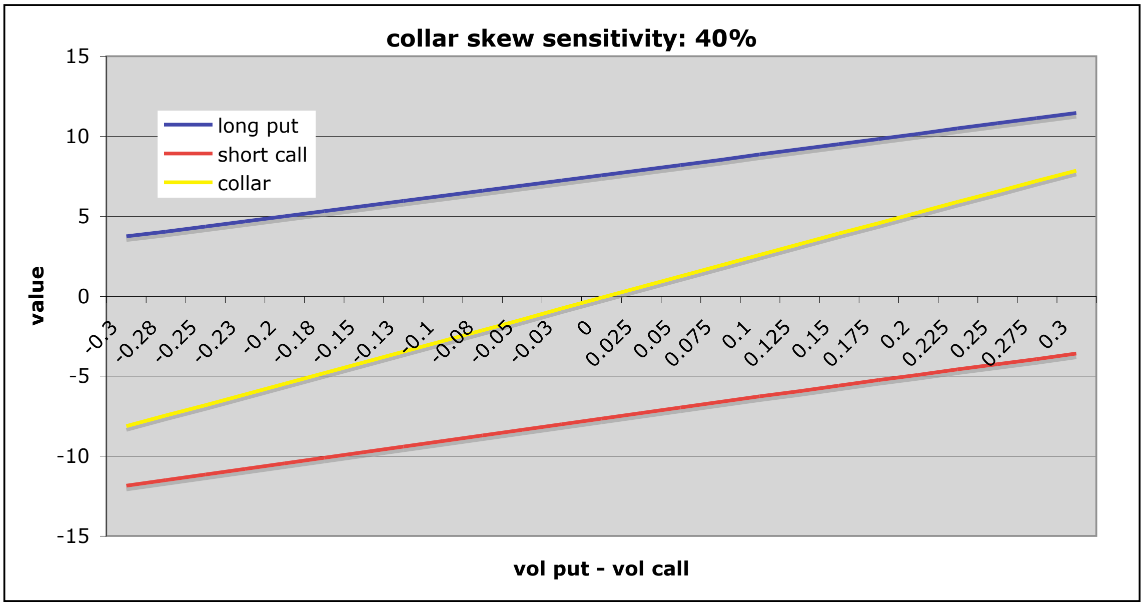


The addition of a risk reversal to the underlying confines exposure to changes in its price to a narrow range. In conventional parlance, one might say then that this hedged position is protected against excessive “volatility,” but in technical terms this is not precisely correct. In fact, one of the characteristics of a risk reversal in the conventional Black-Scholes world is that its volatility exposure in between the two strikes around which it is constructed is quite low, since the long put and short call positions’ exposures nearly offset in this region:



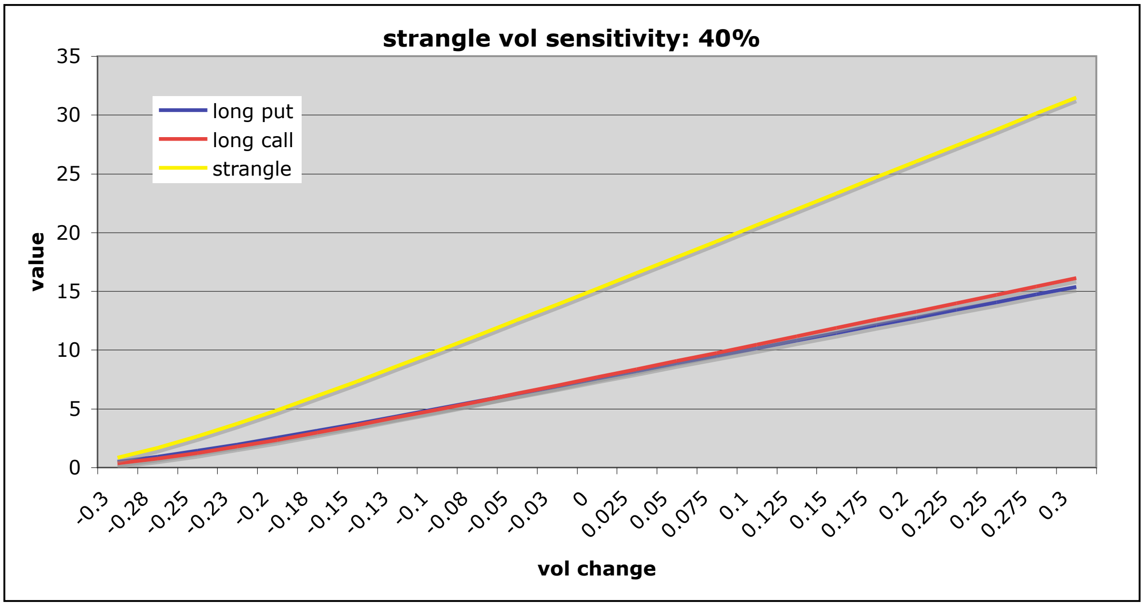
As a practical matter—particularly in equity markets—the risk reversal has an even more interesting feature. While in the Black-Scholes world the volatility for options at all strikes should be the same, in reality the market tends to assign higher volatilities to OTM puts than it does to OTM calls. This feature is called the volatility skew, and it becomes particularly pronounced in times of market stress.

As we have already seen, the risk reversal is relatively insensitive to changes in the level of the vol. In the below, however, we show the result of changes to the vol skew. The x-axis below shows the difference put vol – call vol and the impact on the combination's value:

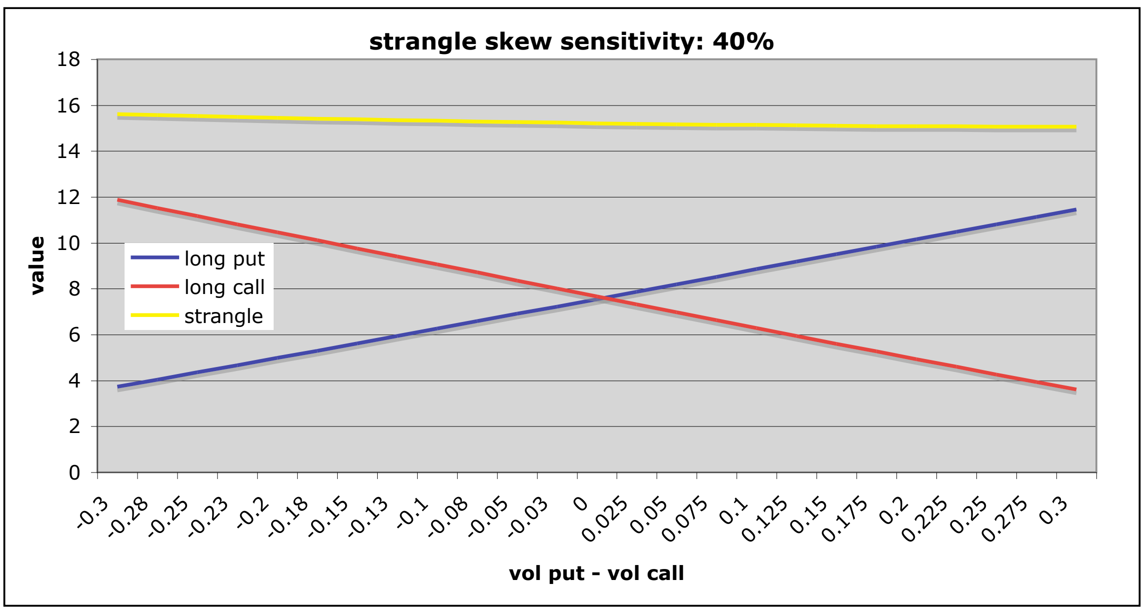


Strangles and straddles

A strangle is a position where, instead of selling the OTM call as we did with a risk reversal, we buy the OTM call. This produces a position whose payoff is positive as long as the spot finishes far enough from the strike. Moreover, it creates precisely switched vol sensitivities from the ones described above for the risk reversal. The strangle is extremely sensitive to the level of volatility…



…but relatively insensitive to changes in skew:



When the strikes of the long call and long put positions are the same, the position is referred to as a straddle. Clearly the straddle cannot have skew sensitivity, since the call and put strike are the same. The vega of this position is also quite simple to calculate…

$$vega_{straddle,K} = vega_{C,K} + vega_{P,K} = 2vega_{C,K}$$

…since vega of a call and put at the same strike are equal. Straddles tend to be quite expensive positions, and thus their status in the common understanding of options is likely misunderstood. It is commonly thought that these are good positions to take in times when large moves in the spot price are expected, but their direction is uncertain. Given the expense of setting up the position, however, it is usually the case that the underlying must move far indeed in order for it to be economical to buy and hold such a position.

Calendar spreads

Calendar spreads are combinations that involve options at two different expiries. These can be so-called “horizontal spreads,” which involve options at different expiries but the same strike, or “diagonal spreads,” which involve options at different expiries and different strikes.

Like butterflies, as a practical matter these can be arranged to gain exposure to quite specific characteristics of traded options, and also like butterflies they have a special status in the world of options theory. Consider two call options at expiries $t < T$ and an underlying whose spot price is S. We assume for the sake of simplicity that the underlying pays no dividends and that interest rates are constant. The strike of the option expiring at t is K_t, and the strike of the option expiring at T is $K_T = K_t\ e^{r(T-t)}$.

Construct a portfolio long the call expiring at T and short the call expiring at t. Our goal is to determine whether we can say anything with certainty about this portfolio without requiring any particular model of the spot price’s evolution.

Suppose that, when t arrives, either $S_t = K_t$ or $S_t < K_t$. Then the short call expires worthless, and the remainder of the portfolio is a long call with expiry $T - t$. This outcome clearly has positive value.

Alternatively, suppose that $S_t > K_t$. Then the call is exercised, and the portfolio’s holder receives K_t and must deliver the share. The holder takes a short position in the underlying and invests the cash amount K_t at the risk-free rate. At T, the holder is still short the share and receives $K_t\ e^{r(T-t)}$ in cash. If the long call position is at the money or in the money at that time, then the holder may pay the cash (the strike of the option) and receive a share, with which the holder can close the short. The value of this outcome is exactly zero.

If the call expires worthless, then $S_T < K_t\ e^{r(T-t)}$. Thus, the holder must pay no more than the strike at T to close the short. The holder is sure to have sufficient cash to do this, so the value of this outcome is strictly positive.

As a result, we see that this portfolio must have, to say the least, nonnegative value since the payoff is nonnegative under all possible realizations.

This result describes a no-arbitrage requirement for call prices at different expiries along the series of strikes with constant present value. This calendar spread requirement can be used to determine a relationship that the Black-Scholes implied volatilities of these options must obey. Consider the Black-Scholes valuation formula for the call option with the longer expiry…

$$V_{C,T} = SN(d_{1,T}) - K_T e^{-rT} N(d_{2,T}) =$$

$$N(d_{1,T}) - K_t e^{r(T-t)} e^{-rT} N(d_{2,T}) =$$

$$SN(d_{1,T}) - K_t e^{-rt} N(d_{2,T})$$

…with…

$$d_{1,T} = \frac{1}{\sigma_T\sqrt{T}}\left[\ln\left(\frac{S}{K_t e^{r(T-t)}}\right) + rT + \frac{\sigma_T^2 T}{2}\right] =$$

$$\frac{1}{\sigma_T\sqrt{T}}\left[\ln\left(\frac{S}{K_t}\right) - r(T-t) + rT + \frac{\sigma_T^2 T}{2}\right] =$$

$$\frac{1}{\sigma_T\sqrt{T}}\left[\ln\left(\frac{S}{K_t}\right) + rt + \frac{\sigma_T^2 T}{2}\right]$$

…where σ_T indicates the Black-Scholes implied volatility of the longer-dated option. Note that this is the same as the pricing formula for the call expiring at t…

$$V_{C,t} = SN(d_{1,t}) - K_t e^{-rt} N(d_{2,t})$$

$$d_{1,t} = \frac{1}{\sigma_t\sqrt{t}}\left[\ln\left(\frac{S}{K_t}\right) + rt + \frac{\sigma_t^2 t}{2}\right]$$

…with the exception of the terms involving volatility. We note that…

$$\sigma_T^2 T = \frac{\sigma_T^2 T}{\sigma_t^2 t}\sigma_t^2 t = k\sigma_t^2 t$$

…and that our no-arbitrage requirement then becomes:

$$V_{C,T} = V_{C,t}\left(vol = \sigma_t\sqrt{k}\right) \geq V_{C,t}\left(vol = \sigma_t\right)$$

We have already seen that the price of a call option is monotonically increasing in the volatility. Thus, in order for our no-arbitrage requirement to be satisfied, we must have:

$$k \geq 1$$

$$\frac{\sigma_T^2 T}{\sigma_t^2 t} \geq 1$$

$$\sigma_T^2 T \geq \sigma_t^2 t$$

…meaning that the total variance of the return, accumulated over the term of the longer-dated option, must be no less than the total variance of the return accumulated over the term of the short-dated option. This is a quite sensible result to expect in the context of the Black-Scholes framework.

This, then, becomes the second no-arbitrage requirement for the Black-Scholes implied volatilities of a collection of options on the same underlying. Combined with the butterfly requirement detailed above, the two requirements can be shown to amount to a necessary and sufficient condition for a surface of Black-Scholes implied volatilities across all strikes and expiries to be arbitrage-free.