*October 7, 2015*

# MTH 9862. Solutions to Quiz 4.

(1) (3 points) Let $(\sigma(t))_{t\ge 0}$, be a stochastic process adapted to the filtration of Brownian motion $(B(t))_{t\ge 0}$ and such that $|\sigma(t)| \le M$ for some $M$ and all $t \ge 0$. Define $S(t)$ by the relations $dS(t) = \sigma(t)S(t)dB(t)$ and $S(0) = 1$. Compute formulas (show all necessary work) for

(1) By Itô-Doeblin formula for Itô processes and the SDE:

$$d\ln S(t) = \frac{1}{S(t)}\,dS(t) - \frac{1}{2S^2(t)}\,d[S]_t = \sigma(t)\,dB(t) - \frac{1}{2}\,\sigma^2(t)\,dt.$$

Conclusion:

$$\ln S(t) - \underbrace{\ln S(0)}_{=0} = \int_0^t \sigma(u)\,dB(u) - \frac{1}{2}\int_0^t \sigma^2(u)\,du.$$

(2) From the last line of (1)

$$S(t) = \exp\left(\int_0^t \sigma(u)\,dB(u) - \frac{1}{2}\int_0^t \sigma^2(u)\,du\right).$$

(3) From the last line of (1), properties of the stochastic integral, and Fubini's theorem,

$$E(\ln S(t)) = -\frac{1}{2}\int_0^t E(\sigma^2(u))\,du.$$

(2) (3 points) Find a closed form formula for

$$\int_0^t e^{\sigma B(s) - \sigma^2 s/2}\,dB(s).$$

Hint: denote the integrand by $\Delta(t)$. Compute $d\Delta(t)$ and derive the answer.

*Solution.* Following the hint we compute using Itô-Doeblin formula for $f(t,x) = e^{\sigma x - \sigma^2 t/2}$

$$d\Delta(t) = -\frac{1}{2}\,\sigma^2\Delta(t)\,dt + \sigma\Delta(t)\,dB(t) + \frac{1}{2}\,\sigma^2\Delta(t)\,dt = \sigma\Delta(t)\,dB(t).$$

Integrating from 0 to $t$ and dividing by $\sigma$ we get:

$$\frac{1}{\sigma}(\Delta(t) - \Delta(0)) = \int_0^t \Delta(u)\,dB(u),$$

which is what we need.

*Answer:* $\frac{1}{\sigma}\left(e^{\sigma B(t) - \sigma^2 t/2} - 1\right)$.

(3) (4 points) Let $X(t)$ be a solution of the SDE

$$dX(t) = -X(t)\,dt + 2dB(t), \quad X(0) = 1.$$

Find a closed form formula for $X(t)$ and find the distribution of $X(t)$. Show all work and explain your reasoning.

*Solution.* As in the HW4 problem, compute using Itô-Doeblin formula for Itô process $X$ and $f(t,x) = e^t x$

$$d(e^t X(t)) = e^t dX(t) + e^t X(t) dt = 2e^t dB(t).$$

Integrating from 0 to $t$ we obtain

$$e^t X(t) - \underbrace{X(0)}_{=1} = 2\int_0^t e^u dB(u) \quad \Rightarrow \quad X(t) = e^{-t} + 2e^{-t}\int_0^t e^u\, dB(u).$$

Therefore $X(t)$ is of the form $c + cY$, where

- $c$ is non-random, $c = e^{-t}$;
- $Y$ is normal as it is a stochastic integral with **deterministic integrand**, $Y = 2\int_0^t e^u\, dB(u)$.

We conclude that $E(Y) = 0$ and the variance of $Y$ is

$$4\int_0^t e^{2u}\, du = 2(e^{2t} - 1),$$

and $X(t)$ is normal with mean $e^{-t}$ and variance $2(1 - e^{-2t})$.