# Interest Rate Models

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# Outline

1. Convexity in LIBOR
2. CMS rates and instruments
3. The uses of Girsanov's theorem

# Convexity

- In finance context, *convexity* refers to nonlinear behavior of the price of an instrument. Convexity may appear because of some optionality embedded in the instrument.
- We often have to include *convexity corrections* to various popular interest rates. Such necessities are often a consequence of pricing under a wrong martingale measure.
- The notation for prices of zero-coupon bonds is used both for deterministic quantities and a stochastic processes.
- The present value of the forward discount factor, $P_0(t, T)$, is deterministic. We use subscript 0 to emphasize that. On the other hand, the quantity $P(t, t, T)$ denotes the time $t$ value of the stochastic process describing the price of the zero coupon bond with maturity $T$.

# LIBOR in arrears

- The first example on which we will show the convexity correction is *LIBOR in arrears swap*.
- Unlike regular LIBOR swap in which each payment is made at the end of accrual period $T$, *LIBOR in arrears swap* requires the payment to be made at the beginning $S$ of the accrual period.
- Let us denote by $L_0^A(S,T)$ the LIBOR in arrears swap rate.
- The present value of the payment is $P = P_0(0,S) \cdot L_0^A(S,T)$.
- Assume you have entered as a fixed receiver in this LIBOR in arrears FRA.
- The way to replicate your position is to enter into a proper LIBOR swap for the time interval $[S_1, S]$ where $S_1$ is a number $S_1 < S$ for which the FRAs are quoted. Typically, $S_1$ is 3 months before $S$.

# LIBOR in arrears

- The value $L(S, T)$ will be a random variable at time $S_1$ when the rate $L(S_1, S)$ settles. The martingale measure for $L(S_1, S)$ is $\mathbb{Q}_S$, and therefore

$$L_0^A(S, T) = \mathbb{E}^{\mathbb{Q}_S}\left[L(S, T)\right].$$

- Therefore the present value of the payment is

$$P = P_0(0, S)\mathbb{E}^{\mathbb{Q}_S}[L(S, T)].$$

- Here $\mathbb{Q}_S$ denotes the $S$-forward measure.
- $T$-forward measure $\mathbb{Q}_T$ is a nice measure. The corresponding numeraire is $P_0(0, T)$ and $L(S, T)$ is a martingale.

# LIBOR in arrears

- Therefore we obtain

$$
\begin{aligned}
P &= P_0(0,S)\mathbb{E}^{\mathbb{Q}_S}[L(S,T)] \\
&= P_0(0,S)\mathbb{E}^{\mathbb{Q}_T}\left[L(S,T)\cdot\frac{d\mathbb{Q}_S}{d\mathbb{Q}_T}\right] \\
&= P_0(0,S)\mathbb{E}^{\mathbb{Q}_T}\left[L(S,T)\cdot\frac{\mathcal{N}_T(0)\tilde{\mathcal{N}}_S(S)}{\tilde{\mathcal{N}}_S(0)\mathcal{N}_T(S)}\right] \\
&= P_0(0,S)\mathbb{E}^{\mathbb{Q}_T}\left[L(S,T)\cdot\frac{P_0(0,T)P(S,S)}{P_0(0,S)P(S,T)}\right] \\
&= P_0(0,T)\mathbb{E}^{\mathbb{Q}_T}\left[\frac{L(S,T)}{P(S,T)}\right].
\end{aligned}
$$

# LIBOR in arrears

- Therefore we conclude

$$P = P_0(0, T)\mathbb{E}^{\mathbb{Q}_T}\left[\frac{L(S, T)}{P(S, T)}\right].$$

- Notice that above formula is obvious if we started with $\mathbb{Q}_T$.
- We can now write

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}_S}[L(S, T)] &= \mathbb{E}^{\mathbb{Q}_T}[L(S, T)] \\ &\quad + \mathbb{E}^{\mathbb{Q}_T}\left[L(S, T)\left(\frac{P_0(S, T)}{P(S, T)} - 1\right)\right].\end{aligned}$$

- Recall that $\mathbb{E}^{\mathbb{Q}_T}[L(S, T)] = L_0(S, T)$.

# LIBOR in arrears

- Let us define $\Delta(S,T)$ by

$$\mathbb{E}^{\mathbb{Q}_S}[L(S,T)] = L_0(S,T) + \Delta(S,T). \tag{1}$$

- $\Delta(S,T)$ is called *in arrears convexity correction* and
- We now use that $P(S,T) = \frac{1}{1+\delta F(S,T)}$ and our standing assumption that LIBOR/OIS spread, $B(S,T)$ is deterministic.
- Assume that $L$ follows the Black's model, i.e.

$$L(S,T) = L_0(S,T)e^{\sigma W(S) - \frac{1}{2}\sigma^2 S}.$$

# LIBOR in arrears

- Homework: Prove that

$$\Delta(S,T) = \left(e^{\sigma^2 S} - 1\right) \frac{\delta L_0^2(S,T)}{1 + \delta F_0(S,T)}. \qquad (2)$$

- Using the Taylor's formula for the expansion of the exponent implies that

$$\Delta(S,T) \approx \frac{\delta L_0^2(S,T)}{1 + \delta F_0(S,T)} \cdot \sigma^2 S.$$

- In real life, since $B(S,T)$ is stochastic there is additional small convexity correction.

# Eurodollar futures / FRAs convexity corrections

- Another example of convexity correction is the one between a Eurodollar future and a FRA.
- Eurodollar future is cash settled at maturity rather than at the end of the accrual period. Therefore, LIBOR from the price of Eurodollar futures contract is higher than the corresponding LIBOR forward.
- This could be also seen by analyzing an optimal behavior of the investor with a long position in a Eurodollar contract. The investor's balance in the margin account is negatively correlated with the dynamics of the interest rates: If the interest rate goes up, the price of the contract goes down and the investor has to add to the margin account. This is an opportunity loss for the investor, because the money that is being added to the margin account could have been invested at the interest rate that is increasing.

# Eurodollar futures / FRAs convexity corrections

- Because of the daily variation margin adjustment, the appropriate measure for Eurodollar future is the spot measure $\mathbb{Q}_0$. Therefore the implied LIBOR rate $L_0^{\text{fut}}(S, T)$ from the futures is
$$L_0^{\text{fut}}(S, T) = \mathbb{E}^{\mathbb{Q}_0}\left[L(S, T)\right].$$
- The Eurodollar/Fra convexity correction is given by
$$\begin{aligned}\Delta_{\text{ED/FRA}}(S, T) &= L_0^{\text{fut}}(S, T) - L_0(S, T) \\ &= \mathbb{E}^{\mathsf{Q}_0}\left[L(S, T)\right] - \mathbb{E}^{\mathsf{Q}_T}\left[L(S, T)\right].\end{aligned}$$
- We will see later that it is best to use short rate term structure model for numerical calculations of $\Delta_{\text{ED/FRA}}(S, T)$.

# Constant Maturity Swap (CMS)

- CMS represents a future fixing of the swap rate.
- For example it may refer to the 10 year swap rate which will set 2 years from now.
- We will see that CMS are different from the forward swap rates.
- CMS rates are convenient alternatives for LIBOR as a floating index rate. One way to understand CMS is rates is as the opinion of the market participants about the long term rates.
- A number of CMS based instruments are traded. They include CMS swaps, and CMS caps and floors. Other commonly traded instruments are CMS spread options.

# CMS swaps

- A fixed for floating CMS swap is a periodic exchange of interest applied on a fixed notional principal. The floating rate is indexed by a reference swap rate (e.g. the 10 year swap rate).
- The fixed leg pays a fixed coupon, quarterly, on the act/360 basis.
- The floating leg pays the 10 year swap rate which fixes two business days before the start of each accrual period. The payments are quarterly on the act/360 basis and are made at the end of each accrual period.

# CMS swaps

- LIBOR for CMS swap also exists. It differs in that instead of fixed leg we have LIBOR leg that pays the floating coupon equal to the LIBOR rate plus a fixed spread applied to the same notional, on quarterly act/360 basis.
- The floating leg pays the 10 year swap rate which fixes two business days before the start of each accrual period. The payments are quarterly on the act/360 basis and are made at the end of each accrual period.
- CMS swaps are used by corporations and investors seeking to maintain constant asset or liability duration of their portfolios.

# CMS caps and floors

- CMS cap is a basket of calls on a swap rate with fixed tenor (say 10 years) structured in analogy to a LIBOR cap or floor.
- A 5 year cap on 10 years CMS struck at $K$ is a basket of CMS caplets each of which pays

$$\max\left\{\text{10 year CMS rate} - K, 0\right\}.$$

- The CMS fixes two days before the start of each accrual period.
- The payments are quarterly on act/360 basis, and are made at the end of each accrual period.
- The CMS floor is defined in an analogous way.

# CMS caps and floors

- CMS caps and floors are useful to those exposed to risk of long term interest rates.
- For example, mortgage servicing companies are short the mortgage rate. The reason for this is that rallying (i.e. falling) interest rates cause higher prepayments and decline in the business. Since the mortgage is believed to be strongly correlated with the 10 year swap rate, the mortgage companies are buying low strike CMS floors.
- A CMS spread option is a European call or put on the difference of two CMS rates.

# CMS caps and floors

- 1 year 2Y/10Y CMS spread call with strike $K$ has the payoff

$$\max\{(10 \text{ year CMS rate}) - (2 \text{ year CMS rate}) - K, 0\}$$

  applied to a notional principal where the two CMS rates fix two business days before the option expiration date.
- The CMS spread options provide a way to hedge the future shape of the swap curve.
- Having swap rate as the floating rate makes the instruments more difficult to price than u usual LIBOR based swap.

# Valuation of CMS swaps, caps, and floors

- Consider a single period $[T_0, T_p]$ CMS swaplet whose fixed leg pays coupon $C$.
- The present value of the fixed leg is

$$P^{\text{fix}} = C\delta P_0(0, T_p).$$

  Here $\delta$ is the day count fraction for the time $[T_0, T_p]$.
- The present value of the floating leg is

$$P^{\text{float}} = \delta P_0(0, T_p)\mathbb{E}^{\mathbb{Q}_{T_p}}\left[S(T_0, T)\right].$$

- $T$ denotes the maturity of the reference swap starting on $T_0$ and not the end of the accrual period.

# Valuation of CMS swaps, caps, and floors

- Consequently,

$$\begin{aligned} P^{\text{CMS swaplet}} &= P^{\text{fix}} - P^{\text{float}} \\ &= \delta P_0(0, T_p)\mathbb{E}^{\mathbb{Q}_{T_p}}\left[C - S(T_0, T)\right]. \end{aligned}$$

- The present value of the CMS swap is obtained by summing up all the contribution from corresponding swaplets.
- Analogous results hold for CMS caplets and floorlets:

$$\begin{aligned} P^{\text{CMS caplet}} &= \delta P_0(0, T_p)\mathbb{E}^{\mathbb{Q}_{T_p}}\left[(S(T_0, T) - K)^+\right]; \\ P^{\text{CMS floorlet}} &= \delta P_0(0, T_p)\mathbb{E}^{\mathbb{Q}_{T_p}}\left[(K - S(T_0, T))^+\right]. \end{aligned}$$

- This implies the put-call parity: The present value of the CMS floorlet - PV of the CMS caplet with the same strike is equal to the present value of the CMS swaplet paying $K$.

# Valuation of CMS swaps, caps, and floors

- Denote by $C(T_0, T|T_p)$ the break-even CMS rate, given by

$$C(T_0, T|T_p) = \mathbb{E}^{\mathbb{Q}_{T_p}}\left[S(T_0, T)\right].$$

- Let us specify the notation once again:
  (i) $T_0$ denotes the start rate of the reference swap. This is also the start of the accrual period of the swaplet.
  (ii) $T$ denotes the maturity date of the reference swap.
  (iii) $T_p$ denotes the payment date of the swaplet. This is also the end of the accrual period of the swaplet.
- One more date plays the role: the date on which the swap is fixed. It is usually two days before $T_0$ and we will neglect its impact.
- Example: $T_0$ is 1 year from now, $T$ can be chosen to be 10 years from $T_0$, and $T_p$ to be 3 months from $T_0$.

# Valuation of CMS swaps, caps, and floors

- Recall that $A(t, T_{\mathrm{val}}, T_0, T)$ denotes the forward annuity function. It is defined as the present value of the cash flow that pays \$1 on each coupon day of the swap accrued according to the swap's day count conventions.

$$A(t, T_{\mathrm{val}}, T_0, T) = \sum_{j=1}^{n_c} \alpha_j P\left(t, T_{\mathrm{val}}, T_j^c\right).$$

- The annuity numeraire is defined as $N_{T_0,T}(t) = A(t, t, T_0, T)$ and the corresponding EMM is the swap measure $\mathbb{Q}$. Recall that $S(t, T_0, T)$ is a martingale with respect to $\mathbb{Q}$.

# CMS rate

- In order to simplify the notation let us introduce the following:

$$
\begin{aligned}
A_0(T_0, T) &= A(0, 0, T_0, T), \\
A_0(T_{\text{val}}, T_0, T) &= A(0, T_{\text{val}}, T_0, T), \\
A(T_0, T) &= A(t, t, T_0, T).
\end{aligned}
$$

- The first two quantities are deterministic, while the third is a stochastic process.
- We can now write

$$
\begin{aligned}
C(T_0, T|T_p) &= \mathbb{E}^{\mathbb{Q}_{T_p}}\left[S(T_0, T)\right] = \mathbb{E}^{\mathbb{Q}}\left[S(T_0, T) \cdot \frac{d\mathbb{Q}_{T_p}}{d\mathbb{Q}}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[S(T_0, T) \cdot \frac{\mathcal{N}_{T_p}(T_0) \cdot \mathcal{N}_{T_0,T}(0)}{\mathcal{N}_{T_p}(0) \cdot \mathcal{N}_{T_0,T}(T_0)}\right]
\end{aligned}
$$

# CMS rate

$$
\begin{aligned}
C(T_0, T|T_p) &= \mathbb{E}^{\mathbb{Q}}\left[S(T_0, T)\cdot\frac{\mathcal{N}_{T_p}(T_0)\cdot\mathcal{N}_{T_0,T}(0)}{\mathcal{N}_{T_p}(0)\cdot\mathcal{N}_{T_0,T}(T_0)}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[S(T_0, T)\cdot\frac{P(T_0, T_p)\cdot A_0(T_0, T)}{P(0, T_p)\cdot A(T_0, T)}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[S(T_0, T)\right] + \Delta(T_0, T|T_p), \text{ where}
\end{aligned}
$$

$$
\Delta(T_0, T|T_p) = \mathbb{E}^{\mathbb{Q}}\left[S(T_0, T)\left(\frac{P(T_0, T_p)\cdot A_0(T_0, T)}{P(0, T_p)\cdot A(T_0, T)} - 1\right)\right].
$$

- We now use that $\mathbb{E}^{\mathbb{Q}}\left[S(T_0, T)\right] = S_0(T_0, T)$ and obtain

$$
C(T_0, T|T_p) = S_0(T_0, T) + \Delta(T_0, T|T_p).
$$

# CMS rate

- $\Delta(T_0, T|T_p)$ is the convexity correction and is influenced by two factors:
  (i) The substantial difference between the two EMMs that are caused by the substantial difference between the numeraires. One of them is the single discount factor and the other is a more comprehensive swap rate. We can almost think of this as a correlation effect between LIBOR and swap rate. This technically misleading, but serves as a good start for building the intuition.
  (ii) Payment day delay.
- The convexity correction can be decomposed into two parts that are related to the above two factors.

$$\begin{aligned}\frac{P(T_0, T_p)\cdot A_0(T_0, T)}{P(0, T_p)\cdot A(T_0, T)} - 1 \quad &= \quad \left(\frac{A_0(T_0, T)}{A(T_0, T)} - 1\right) \\ &\quad + \frac{A_0(T_0, T)}{A(T_0, T)}\left(\frac{P(T_0, T_p)}{P(0, T_p)} - 1\right)\end{aligned}$$

# CMS rate

- Let us introduce the notation:

$$\begin{aligned}\Delta_{\text{corr}}(T_0, T) &= \mathbb{E}^{\mathbb{Q}}\left[S(T_0, T)\left(\frac{A_0(T_0, T)}{A(T_0, T)} - 1\right)\right]\\ \Delta_{\text{delay}}(T_0, T|T_p) &= \mathbb{E}^{\mathbb{Q}}\left[S(T_0, T)\frac{A_0(T_0, T)}{A(T_0, T)}\left(\frac{P(T_0, T_p)}{P(0, T_p)} - 1\right)\right].\end{aligned}$$

- Then we have

$$\Delta(T_0, T|T_p) = \Delta_{\text{corr}}(T_0, T) + \Delta_{\text{delay}}(T_0, T|T_p).$$

- Notice that if the CMS rate is paid at the beginning of the accrual period, then $\Delta_{\text{delay}}(T_0, T|T_p) = 0$.

# Calculating the CMS convexity correction

- So far the formulas that we have derived do not depend on the model for the dynamics of the forward swap rate.
- The challenge lies in the fact that the martingale measure $\mathbb{Q}$ is difficult to calculate.
- These are some common approaches to the problem.
    (i) Black model style assumptions. In this approach we assume that the forward swap rate follows a lognormal process.
    (ii) Replication method. We attempt to replicate the CMS payoff using European swaptions of different strikes. This way we are not making any assumptions on the underlying process. This allows us to capture the effects of volatility smile (say by using the SABR model).
    (iii) Monte Carlo simulation together with a term structure model. This is slow and success depends on the accuracy of the term structure model that we will study later.

# Calculating the CMS convexity correction

- We will now make few assumptions. First, the that day count fractions will be assumed to be $\frac{1}{f}$, where $f$ is the frequency of payments. Usually $f = 2$.
- All the discounting will be in terms of a single swap rate, $S(t, T_0, T)$.
- We further simplify the notation of the stochastic process to have $S(t) = S(t, T_0, T)$ and $A(t) = A(t, t, T_0, T)$
- With these approximations, the level function becomes

$$A(t) = \frac{1}{f} \sum_{j=1}^{n} \frac{1}{\left(1 + \frac{S(t)}{f}\right)^{j}} = \frac{1}{S(t)} \left(1 - \frac{1}{\left(1 + \frac{S(t)}{f}\right)^{n}}\right).$$

# Calculating the CMS convexity correction

- In order to calculate the discount factor from the start date to the payment date we first introduce the symbol $f_{\text{CMS}}$ to denote the frequency of the payments on the CMS swap (typically it is equal to 4). Then we have

$$P(t) = \frac{1}{1+\frac{S(t)}{f_{\text{CMS}}}} \approx \frac{1}{\left(1+\frac{S(t)}{f}\right)^{\frac{f}{f_{\text{CMS}}}}}.$$

# Black's model

- Since the formulas for $\Delta_{\text{corr}}$ and $\Delta_{\text{delay}}$ involve $\frac{1}{A(t)}$ we will first use the Taylor's expansion of first order to replace this quantity by polynomial.
- Notice that

$$\frac{1}{A(t)} = \varphi(S(t)), \quad \text{where} \quad \varphi(x) = \frac{x\left(1+\frac{x}{f}\right)^n}{\left(1+\frac{x}{f}\right)^n - 1}.$$

- Using the Taylor expansion of first degree of the function $\varphi(x)$ around $x_0 = S_0$ and substituting $x = S(t)$ we can prove that

$$\frac{1}{A(t)} \quad = \quad \frac{1}{A_0}\left(1+\frac{1}{S_0}\left(1-\frac{1}{1+\frac{S_0}{f}}\cdot\frac{\frac{nS_0}{f}}{\left(1+\frac{S_0}{f}\right)^n-1}\right)(S(t)-S_0)\right) \quad (3)$$

# Black's model

- If we denote $\theta_c = 1 - \frac{1}{1+\frac{S_0}{f}} \cdot \frac{\frac{nS_0}{f}}{\left(1+\frac{S_0}{f}\right)^n - 1}$ then (3) becomes

$$\frac{1}{A(t)} = \frac{1}{A_0}\left(1 + \theta_c \frac{S(t) - S_0}{S_0}\right).$$

- Since $\theta_c$ is deterministic, the previous approximation leaves the formula for $\Delta_{\mathrm{corr}}$ with only one stochastic process $S(t)$.

$$S(t) = S_0 e^{\sigma W(t) - \frac{\sigma^2}{2} t}.$$

- Integration with respect to the normal distribution for $t = T_0$ yields

$$\Delta_{\mathrm{corr}}(T_0, T) \approx S_0(T_0, T)\theta_c \left(e^{\sigma^2 T_0} - 1\right).$$

# Black's model

- In an analogous way we obtain

$$\Delta_{\text{delay}}(T_0, T|T_p) \approx -S_0(T_0, T)\theta_d \left(e^{\sigma^2 T_0} - 1\right),$$

$$\text{where} \quad \theta_d = \frac{\frac{S_0}{f_{\text{CMS}}}}{1 + \frac{S_0}{f}}.$$

- Once the two convexity corrections are combined we obtain

$$\Delta(T_0, T|T_p) \approx S_0(T_0, T)\,(\theta_c - \theta_d)\left(e^{\sigma^2 T_0} - 1\right), \text{ where}$$

$$\begin{aligned}
\theta_c - \theta_d &= 1 - \frac{1}{1 + \frac{S_0}{f}} \cdot \frac{\frac{nS_0}{f}}{\left(1 + \frac{S_0}{f}\right)^n - 1} - \frac{\frac{S_0}{f_{\text{CMS}}}}{1 + \frac{S_0}{f}} \\
&= 1 - \frac{\frac{S_0}{f}}{1 + \frac{S_0}{f}} \cdot \left(\frac{f}{f_{\text{CMS}}} + \frac{n}{\left(1 + \frac{S_0}{f}\right)^n - 1}\right).
\end{aligned}$$

# Replication method

- We will first prove a special case of Tanaka's formula. We start with Taylor's formula of the first degree with integral remainder:

$$\varphi(x) \;=\; \varphi(x_0) + \varphi'(x_0)(x - x_0) + \int_{x_0}^{x} \varphi''(u)\,(x-u)\;du.$$

- Notice that the remainder term can be expressed in the following way

$$\begin{aligned}\int_{x_0}^{x} \varphi''(u)\,(x-u)\;du \;&=\; \int_{-\infty}^{x_0} \varphi''(u)\,(u-x)^{+}\;du \\ &\quad + \int_{x_0}^{+\infty} \varphi''(u)\,(x-u)^{+}\;du.\end{aligned}$$

- The remainder term now looks like a mixture of calls and puts.

# Replication method

- Assume that we are given a diffusion process $X(t)$ and use the above formula for $x = X(T)$.

$$\begin{aligned}\varphi(X(T)) &= \varphi(X_0) + \varphi'(X_0)(X(T) - X_0) \\ &\quad + \int_0^{X_0} \varphi''(K)\,(K - X(T))^+ \; dK \\ &\quad + \int_{X_0}^{+\infty} \varphi''(K)\,(X(T) - K)^+ \; dK.\end{aligned}$$

- Let $\mathbb{Q}$ be a martingale measure for $X$. When we take the expectations of both sides with respect to $\mathbb{Q}$ we get

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\varphi(X(T))\right] &= \varphi(X_0) + \int_0^{X_0} \varphi''(K) B^{\text{put}}(T, K, X_0)\, dK \\ &\quad + \int_{X_0}^{+\infty} \varphi''(K) B^{\text{call}}(T, K, X_0)\, dK.\end{aligned}$$

# Replication method

$$\mathbb{E}^{\mathbb{Q}}\left[\varphi(X(T))\right] = \varphi(X_0) + \int_0^{X_0} \varphi''(K) B^{\mathrm{put}}(T, K, X_0)\, dK + \int_{X_0}^{+\infty} \varphi''(K) B^{\mathrm{call}}(T, K, X_0)\, dK. \quad (4)$$

- $B^{\mathrm{call/put}}(T, K, X_0) = \mathbb{E}^{\mathbb{Q}}\left[(X(T) - K)^{\pm}\right]$ in (4).
- The interpretation of the above formula is as follows: If $\varphi(x)$ is a payoff of an instrument, then its expected value at time $T$ is the sum of the following components:
  - (i) Today's value;
  - (ii) The basket of out the money puts and calls integrated against the second derivative of the payoff.

# Replication method

- An alternative way of writing (4) by using the integration by part twice.
- First we recall that

$$\begin{aligned} B^{\mathrm{put}}(T, X_0, X_0) - B^{\mathrm{call}}(T, X_0, X_0) &= 0, \\ \frac{\partial}{\partial K} B^{\mathrm{put}}(T, X_0, X_0) - \frac{\partial}{\partial K} B^{\mathrm{call}}(T, X_0, X_0) &= 1. \end{aligned}$$

- Consequently,

$$\mathbb{E}^{\mathbb{Q}}\left[\varphi(X(T))\right] = \varphi(X_0) + \int_0^{+\infty} \varphi(K) G(T, K, X_0)\, dK, \text{ where}$$

$$\begin{aligned} G(T, K, X_0) &= \frac{\partial^2}{\partial K^2} B^{\mathrm{call}}(T, K, X_0) = \frac{\partial^2}{\partial K^2} B^{\mathrm{put}}(T, K, X_0) \\ &= \mathbb{E}^{\mathbb{Q}}\left[\delta\left(X(T) - K\right)\right]. \end{aligned}$$

# Replication method applied to swaptions

- We made an approximation to $S(t)$ and made it appear to depend only on $t$.
- Then we approximated $A(t) = \ell(S(t))$ and $P(t) = p(S(t))$ where

$$\ell(S) = \frac{1}{S}\left(1 - \frac{1}{\left(1+\frac{S}{f}\right)^n}\right) \text{ and } p(S) = \frac{1}{\left(1+\frac{S}{f}\right)^{\frac{f}{f_{\text{CMS}}}}}.$$

- The Radon-Nikodym derivative is now $R(S) = \frac{\ell(S_0)p(S)}{\ell(S)p(S_0)}$. Now the break even CMS rate satisfies

$$C(T_0, T|T_p) \approx \mathbb{E}^{\mathbb{Q}}\left[S(T_0, T)R(S(T_0, T))\right].$$

- We will now apply Tanaka's formula for $\varphi(S) = SR(S)$. Observe that $\varphi(S_0) = S_0$.

# Replication method applied to swaptions

$$
\begin{aligned}
C(T_0, T|T_p) \quad &= \quad S_0(T_0, T) + \int_0^{S_0} \varphi''(K) B^{\text{put}}(T, K, X_0)\, dK \\
&\quad + \int_{S_0}^{+\infty} \varphi''(K) B^{\text{call}}(T, K, X_0)\, dK.
\end{aligned}
$$

- The quantities $B^{\text{call}}$ and $B^{\text{put}}$ are related to the prices of payer and receiver swaptions. We then obtain

$$
\begin{aligned}
\Delta(T_0, T|T_p) \quad &\approx \quad \int_0^{S_0} \frac{\varphi''(K)}{\ell(K)} P^{\text{rec}}(T, K, X_0)\, dK \\
&\quad + \int_{S_0}^{+\infty} \frac{\varphi''(K)}{\ell(K)} P^{\text{pay}}(T, K, X_0)\, dK.
\end{aligned}
$$

- The above formula links the CMS convexity correction with the swaption market. In practice, the integrals are discretized and each of the swaption prices is calculated based on the calibrated SABR model.

# Replication method applied to caplets

- A CMS caplet that matures $T_0$ years from now has the price

$$P^{\text{CMS caplet}} = P(0, T_p)\mathbb{E}^{\mathbb{Q}_{T_p}}\left[(S(T_0, T_0, T) - K)^+\right] \cdot \delta \cdot \text{notional}$$

- The dynamics of the forward swap rate with respect to the $T_p$ forward measure cannot be explicitly determined. We need to use approximations to calculate the expected value from above.
- Let $g_{T_0}(F, F_0)$ denote the terminal probability density of the forward swap rate at expiry. Then we need to calculate

$$P = P_0(0, T_p) \cdot \int (F - K)^+ g_{T_0}(F, F_0)\, dF. \qquad (5)$$

- $g_{T_0}(F, F_0)$ is unkown.

# Replication method applied to caplets

- Under the Black model approximation $g_{T_0}(F, F_0)$ is a lognormal distribution applied to the process

$$\begin{aligned} dS(t) &= \sigma_{\mathrm{CMS}} S(t)\, dW(t), \\ S(0) &= C(T_0, T | T_p). \end{aligned}$$

- The CMS rate $C(T_0, T | T_p)$ is calculated as before and $\sigma_{\mathrm{CMS}}$ is the approximate implied volatility.
- Replication method gives different prices. We construct terminal probability distribution from the swaption market prices. Since we are taking a second derivative we have to make sure that the interpolation is twice differentiable. We can use interpolation based by the SABR model. Then option prices are obtained by numerical integration of (5).

# Triangle arbitrage

- CMS rate options and CMS spread options are traded on the market. Sometimes the valuation model for spread options is inconsistent with the model used to price CMS options. However, there is an obvious inequality between payoffs that must translates into an inequality between the prices.
- Let us denote by $S_1$ and $S_2$ two different CMS rates (say 2 year and 10 years). Denote by $S_{12}$ the spread between them, i.e. $S_{12} = S_2 - S_1$. Let $K_1$ and $K_2$ be the strikes for which the calls are quoted on $S_1$ and $S_2$. Let $K_{12} = K_2 - K_1$.
- The triangle inequality $(x+y)^+ \leq x^+ + y^+$ implies the following

$$(S_2 - K_2)^+ - (S_1 - K_1)^+ \leq (S_{12} - K_{12})^+ \leq (S_2 - K_2)^+ + (K_1 - S_1)^+ .$$

# Triangle arbitrage

- Hence

$$\begin{aligned}\mathbb{E}\left[(S_2 - K_2)^+\right] - \mathbb{E}\left[(S_1 - K_1)^+\right] &\leq \mathbb{E}\left[(S_{12} - K_{12})^+\right] \\ &\leq \mathbb{E}\left[(S_2 - K_2)^+\right] + \mathbb{E}\left[(K_1 - S_1)^+\right].\end{aligned}$$

- After multiplying by appropriate discount factors we obtain the inequality between prices of options.
- The arbitrage can be avoided using the models developed based on copulas methodology.

# References


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