FOURIER ANALYSIS NOTES

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1   INTRODUCTION

The general Fourier transform in one dimension is written as

$$\tilde{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt, \tag{1}$$

$$f(t) = \int_{-\infty}^{+\infty} \tilde{f}(\omega) e^{i\omega t} \frac{d\omega}{2\pi}. \tag{2}$$

Here the notation is motivated by a transform between *time* domain and *frequency* domain. In the language of engineers, Equation 1 is called *analysis* and Equation 2 is called *synthesis*. Another common pair of domains for Fourier transform is space and wave vector, also known as the *reciprocal* space, typically in higher dimensions,

$$\tilde{f}(\mathbf{k}) = \int f(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^D\mathbf{r}, \tag{3}$$

$$f(\mathbf{r}) = \int \tilde{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d^D\mathbf{k}}{(2\pi)^D}, \tag{4}$$

where D is the number of spatial dimensions. A change in sign convention $\omega \longleftrightarrow -\omega$ will be made when Fourier transform is performed in space and time together, such as in relativistic theories.

2   PERIODICITY IN TIME

Suppose the function $f(t)$ is periodic in time domain

$$f(t + nT) = f(t), \quad \forall n \in \mathbb{N}, \tag{5}$$

where T is the time period. According to the theory of Fourier series, the spectral decomposition of periodic function reduces to a Fourier series of a discrete set of frequencies. Here is a proof:

$$\begin{aligned}
\tilde{f}(\omega) &= \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \\
&= \int_{-\infty}^{+\infty} f(t+T) e^{-i\omega t} dt \\
&= \int_{-\infty}^{+\infty} f(t) e^{-i\omega(t-T)} dt \\
&= e^{i\omega T} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \\
&= e^{i\omega T} \tilde{f}(\omega).
\end{aligned}$$

The above condition can be satisfied only when

$$\omega T = 2\pi \times \text{Integer} \quad \text{or} \quad f(\omega) = 0. \tag{6}$$

Thus, we proved that $\tilde{f}(\omega) \neq 0$ only when the ω takes on a discrete set frequencies

$$\omega_m = \frac{2\pi m}{T}, \quad m \in \mathbb{N}. \tag{7}$$

The above results can be generalized to higher dimensions in reciprocal space.

Based on the above results, we can write the synthesis integral reduces to a series

$$\begin{aligned}
f(t) &= \sum_m \int_{\omega_m - \epsilon}^{\omega_m + \epsilon} \tilde{f}(\omega) e^{i\omega t} \frac{d\omega}{2\pi} \\
&= \sum_m e^{i\omega_m t} \int_{\omega_m - \epsilon}^{\omega_m + \epsilon} \tilde{f}(\omega) \frac{d\omega}{2\pi} \\
&\triangleq \sum_m \tilde{f}_m e^{i\omega_m t}
\end{aligned} \tag{8}$$

where $(\omega_m - \epsilon, \omega_m + -\epsilon)$ denotes an infinitesimal neighborhood of ω_m in frequency domain. The integral

$$\tilde{f}_m \triangleq \int_{\omega_m - \epsilon}^{\omega_m + \epsilon} \tilde{f}(\omega) \frac{d\omega}{2\pi} \tag{9}$$

should be finite and independent of ϵ based on intuition, such that $\tilde{f}(\omega)$ has to be a series of Dirac Delta located at the discrete set of frequencies. This can be verified by the corresponding analysis equation

$$\begin{aligned}
\tilde{f}(\omega) &= \int f(t) e^{-i\omega t} dt \\
&= \sum_m \tilde{f}_m \int e^{i(\omega_m - \omega)t} dt \\
&= 2\pi \sum_m \tilde{f}_m \delta(\omega - \omega_m).
\end{aligned}$$

It is easy to check that the Fourier series Equation 8 does possess the assumed periodicity. The Fourier coefficients can be determined by the analysis integral over one period in time

$$\begin{aligned}
\int_T f(t) e^{-i\omega_m t} dt &= \sum_{m'} \tilde{f}_{m'} \int_T e^{i(\omega'_m - \omega_m)t} dt \\
&= \sum_{m'} \tilde{f}_{m'} T \delta_{mm'} \\
&= T \tilde{f}_m.
\end{aligned}$$

Thus, the Fourier transform of a periodic function takes the following form

$$\tilde{f}_m = \frac{1}{T} \int_T f(t) e^{-i\omega_m t} dt, \tag{10}$$

$$f(t) = \sum_m \tilde{f}_m e^{i\omega_m t}. \tag{11}$$

In summary

Periodicity in time with period $T \Leftrightarrow$ Discretiztion in frequency at $\omega_m = \frac{2\pi m}{T}$

3   REPLICATION WITH DIFFERENT PERIODS

Suppose $f(t)$ represents a signal localized in time such that it's vanishingly small beyond a certain extent, then we are allowed to replicate $f(t)$ along the time axis with different lengths of time period without losing much information as long as the replication period is larger than the length of the localization window. How would the resulting spectra in frequency domain compare?

4   SAMPLING IN TIME

Sampling in continuous time produces an *impulse train*

$$f(t) = \sum_{n=1}^{N} f_n \delta(t - t_n). \tag{12}$$

Note that the Dirac Delta is necessary here in order for function $f(t)$ to have a non-zero integral over time, or *gain* in the language of engineers. The Fourier transform to frequency domain is

$$\tilde{f}(\omega) = \sum_n f_n \int \delta(t - t_n) e^{-i\omega t} dt = \sum_n f_n e^{-i\omega t_n}. \tag{13}$$

If we further assume that the function in time domain is uniformly sampled, i.e. $t_n = n\Delta t$, $n \in \mathbb{N}$, the spectrum in frequency domain

$$\tilde{f}(\omega) = \sum_n f_n e^{-i\omega n \Delta t}. \tag{14}$$

is replicated with period $2\pi/\Delta t$

$$\tilde{f}\left(\omega + \frac{2\pi m}{\Delta t}\right) = \tilde{f}(\omega), \quad m \in \mathbb{N}. \tag{15}$$

Equation 14 can be inverted by the synthesis integral over one replication period in frequency

$$
\begin{aligned}
\int_{-\pi/\Delta t}^{+\pi/\Delta t} \tilde{f}(\omega) e^{i\omega n \Delta t} \frac{d\omega}{2\pi} &= \int_{-\pi/\Delta t}^{+\pi/\Delta t} \sum_{n'} f_{n'} e^{i\omega(n-n')\Delta t} \frac{d\omega}{2\pi} \\
&= \sum_{n'} f_{n'} \int_{-\pi/\Delta t}^{+\pi/\Delta t} e^{i\omega(n-n')\Delta t} \frac{d\omega}{2\pi} \\
&= \frac{1}{\Delta t} \sum_{n'} f_{n'} \int_{-\pi}^{+\pi} e^{i\phi(n-n')} \frac{d\phi}{2\pi} \\
&= \frac{1}{\Delta t} \sum_{n'} f_{n'} \delta_{nn'} \\
&= \frac{1}{\Delta t} f_n.
\end{aligned} \tag{16}
$$

Thus, the Fourier transform of a uniformly sampled function in time takes the form

$$\tilde{f}(\omega) = \sum_n f_n e^{-i\omega n \Delta t}, \tag{17}$$

$$f_n = \Delta t \int_{-\pi/\Delta t}^{+\pi/\Delta t} \tilde{f}(\omega) e^{i\omega n \Delta t} \frac{d\omega}{2\pi}. \tag{18}$$

In summary

$$\text{Sampling in time at } t = n\Delta t \Leftrightarrow \text{Replication in frequency with period } \frac{2\pi}{\Delta t}$$

5 SAMPLING AT DIFFERENT INTERVALS

Discretization differs from sampling in that the time domain is a discrete. This is achieved by incorporating the time pixel Δt into the angular frequency ω in Equation 17 to form the phase change per time pixel

$$\phi \triangleq \omega \Delta t, \tag{19}$$

leaving the index n alone to represent time, and

$$\tilde{f}(\phi) \triangleq \tilde{f}\left(\omega = \frac{\phi}{\Delta t}\right), \tag{20}$$

where we are overloading the symbol $\tilde{f}$ in frequency domain and phase-change domain. The Fourier transform takes the following form

$$\tilde{f}(\phi) = \sum_n f_n e^{-i\phi n}, \tag{21}$$

$$f_n = \int_{-\pi}^{+\pi} \tilde{f}(\phi) e^{i\phi n} \frac{d\phi}{2\pi}. \tag{22}$$

The question then arises: how to compare the discrete Fourier transform sampled at different intervals, say Δt and $\Delta t'$. Suppose that the total time window is T, sampling at two different intervals Δt and $\Delta t'$ produces two series with different lengths

Sampling at $\Delta t : f_1, f_2, f_3, f_4, \cdots, f_N, \quad N = T/\Delta t,$
Sampling at $\Delta t' : f_{1'}, f_{2'}, f_{3'}, f_{4'}, \cdots, f_{N'}, \quad N = T/\Delta t'.$

Notices that in the above two series, for a given index n, f_n and $f_{n'}$ are not comparable because they represent different time points $n\Delta t$ and $n\Delta t'$. Similarly, in the domain of phase changes, the Fourier transforms are not directly comparable corresponding to different sampling intervals

$$\sum_{n=1}^{N} f_n e^{-i\phi n} \not\sim \sum_{n=1}^{N'} f_n e^{-i\phi n}. \tag{23}$$

In the simplest case where $\phi = 0$, the two summations involve different number terms, $\sum_{n=1}^{N} f_n \not\sim \sum_{n=1}^{N'} f_n$. This discrepancy for comparison is not surprising at all because the two impulse trains we start with

$$\sum_{n=1}^{N} f_n \delta(t - t_n) \not\sim \sum_{n=1}^{N'} f_n \delta(t - t_n) \tag{24}$$

are totally different functions, and of course, their spectra in frequency domain are different. Because the two impulse trains are defined to approximate some smooth function to different levels of accuracies, each sampling

point represent a neighbourhood with length Δt, the comparable quantities are impulse trains weighted by the length of time interval

$$\sum_{n=1}^{N} f_n \delta(t - t_n)\Delta t \sim \sum_{n=1}^{N'} f_n \delta(t - t_n)\Delta t' \tag{25}$$

such that the integrals of the above two expressions over any finite time intervals are comparable approximations to the integral of the same smooth function. Similarly, in the frequency domain, the fair comparison is

$$\Delta t \sum_{n=1}^{N} f_n e^{-i\phi n} \sim \Delta t' \sum_{n=1}^{N'} f_n e^{-i\phi n}. \tag{26}$$