# MTH 9831 Assignment One

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September 9, 2015

# 1 Problem One

The characteristic functions of $X$ and $Y$ are given by

$$\phi_X(t) = \exp\left(2e^{it} - 2\right), \quad \phi_Y(t) = \left(\frac{3}{4}e^{it} + \frac{1}{4}\right)^{10}.$$

Assume that $X$ and $Y$ are independent.

*Preparation*: In this problem, one needs to find out the underlying probability distributions corresponding to the given characteristic functions to evaluate the required expressions. This is done in a somewhat *ad hoc* way by recognizing the form of characteristic functions of some typical probability distributions.

**(1)** Poisson distribution
For a Poisson distribution with parameter $\lambda$

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}, \quad k \geq 0$$

The characteristic function

$$\begin{aligned}
\phi(t) &= \mathbb{E}\left[e^{itX}\right] \\
&= \sum_{k=0}^{\infty} e^{itk}\frac{\lambda^k}{k!}e^{-\lambda} \\
&= e^{-\lambda}\sum_{k=0}^{\infty}\frac{(e^{it}\lambda)^k}{k!} \\
&= e^{-\lambda}e^{\lambda e^{it}} \\
&= e^{\lambda(e^{it}-1)}.
\end{aligned}$$

Because the characteristic function uniquely determines the probability distribution and vice versa. One recognizes that the random variable $X$ given in the question has a Poisson distribution with parameter $\lambda = 2$. The expectation value of a Poisson random variable

$$\begin{aligned}
\mathbb{E}[X] &= -i\frac{\partial}{\partial t}\mathbb{E}\left[e^{itX}\right]\Big|_{t=0} \\
&= -i\frac{\partial}{\partial t}e^{\lambda(e^{it}-1)}\Big|_{t=0} \\
&= e^{\lambda(e^{it}-1)}\lambda e^{it}\Big|_{t=0} \\
&= \lambda.
\end{aligned}$$

**(2)** Binomial distribution
For a binomial distribution with parameter $(n, p, q = 1 - p)$

$$\mathbb{P}(Y = k) = \binom{n}{k}p^k q^{n-k}, \quad 0 \leq k \leq n.$$

The characteristic function

$$
\begin{aligned}
\phi(t) &= \mathbb{E}\left[e^{itY}\right] \\
&= \sum_{k=0}^{\infty} e^{itk}\binom{n}{k} p^k q^{n-k} \\
&= \sum_{k=0}^{\infty} \binom{n}{k} (pe^{it})^k q^{n-k} \\
&= \left(pe^{it}+q\right)^n,
\end{aligned}
$$

where $p+q=1$. One recognizes that the random variable $Y$ given in the question has a binomial distribution with parameters $n=10$ and $p=3/4$. The expectation value of a binomial random variable

$$
\begin{aligned}
\mathbb{E}[Y] &= -i\frac{\partial}{\partial t}\mathbb{E}\left[e^{itY}\right]\Big|_{t=0} \\
&= -i\frac{\partial}{\partial t}\left(pe^{it}+q\right)^n\Big|_{t=0} \\
&= np\left(pe^{it}+q\right)^{n-1} e^{it}\Big|_{t=0} \\
&= np.
\end{aligned}
$$

Now we are ready to find

**(a)** $\mathbb{E}[XY]$
*Solution*: From independence

$$
\begin{aligned}
\mathbb{E}[XY] &= \mathbb{E}[X]\mathbb{E}[Y] \\
&= \lambda \times (np) \\
&= 2\times 10\times \frac{3}{4} \\
&= 15.
\end{aligned}
$$

**(b)** $\mathbb{P}(XY=0)$
*Solution*: Again, we need independence

$$
\begin{aligned}
\mathbb{P}(XY=0) &= 1-\mathbb{P}(XY\neq 0) \\
&= 1-\mathbb{P}(X\neq 0 \text{ and } Y\neq 0) \\
\text{(independence)} &= 1-\mathbb{P}(X\neq 0)\mathbb{P}(Y\neq 0) \\
&= 1-[1-\mathbb{P}(X=0)][1-\mathbb{P}(Y=0)] \\
&= 1-(1-e^{-\lambda})(1-q^n) \\
&= 1-(1-e^{-2})\left[1-\left(\frac{1}{4}\right)^{10}\right] \\
&\approx 0.135336.
\end{aligned}
$$

**(c)** $\mathbb{P}(X+Y=2)$

*Solution*: Because $X$ can take values in non-negative integers and $Y$ can take values in non-negative integers between 0 and $n=10$,

$$\begin{aligned}
\mathbb{P}(X+Y=2) &= \mathbb{P}(X=0,Y=2)+\mathbb{P}(X=1,Y=1)+\mathbb{P}(X=2,Y=0) \\
\text{(independence)} &= \mathbb{P}(X=0)\mathbb{P}(Y=2)+\mathbb{P}(X=1)\mathbb{P}(Y=1)+\mathbb{P}(X=2)\mathbb{P}(Y=0) \\
&= e^{-\lambda}\binom{n}{2}p^2q^{n-2}+\lambda e^{-\lambda}\binom{n}{1}pq^{n-1}+\frac{\lambda^2}{2}e^{-\lambda}\binom{n}{0}q^n \\
&= e^{-2}\left[\binom{10}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^8+2\binom{10}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^9+\frac{2^2}{2}\left(\frac{1}{4}\right)^{10}\right] \\
&\approx 6\times 10^{-5}.
\end{aligned}$$

## 2 PROBLEM TWO

Show that if $X$ and $Y$ are independent normal random variables with the same variance, then $X+Y$ and $X-Y$ are also independent.
*Solution*: Define

$$U = X + Y, \quad V = X - Y.$$

*One approach* is to start from the characteristic function.

$$\begin{aligned}
\phi_{U,V}(t_1,t_2) &= \mathbb{E}\left[e^{it_1U+it_2V}\right] \\
&= \mathbb{E}\left[e^{it_1(X+Y)+it_2(X-Y)}\right] \\
&= \mathbb{E}\left[e^{i(t_1+t_2)X+i(t_1-t_2)Y}\right] \\
\text{(independence)} &= \mathbb{E}\left[e^{i(t_1+t_2)X}\right]\mathbb{E}\left[e^{i(t_1-t_2)Y}\right] \\
\text{(normality)} &= e^{i(t_1+t_2)\mu_X-\frac{1}{2}\sigma^2(t_1+t_2)^2}e^{i(t_1-t_2)\mu_Y-\frac{1}{2}\sigma^2(t_1-t_2)^2} \\
&= e^{it_1(\mu_X+\mu_Y)+it_2(\mu_X-\mu_Y)-\frac{1}{2}\sigma^2(2t_1^2+2t_2^2)} \\
&= e^{it_1(\mu_X+\mu_Y)-\frac{1}{2}(\sqrt{2}\sigma)^2t_1^2}e^{it_2(\mu_X-\mu_Y)-\frac{1}{2}(\sqrt{2}\sigma)^2t_2^2} \\
&= \phi_U(t_1)\phi_V(t_2),
\end{aligned}$$

where we observed that the joint characteristic function of $U = X+Y, V = X-Y$ factorizes into the product of the characteristic function of $U$ and $V$ individually.

$$\begin{aligned}
\phi_U(t_1) &= e^{it_1(\mu_X+\mu_Y)-\frac{1}{2}(\sqrt{2}\sigma)^2t_1^2} \\
\phi_V(t_2) &= e^{it_2(\mu_X-\mu_Y)-\frac{1}{2}(\sqrt{2}\sigma)^2t_2^2}.
\end{aligned}$$

In other words, $U = X+Y$ and $V = X-Y$ are independent. Furthermore, $U = X+Y$ is a normal random variable with mean $\mu_X+\mu_Y$ and variance $2\sigma^2$, and $V = X-Y$ is a normal random variable with mean $\mu_X-\mu_Y$ and variance $2\sigma^2$, where $\mu_X$ and $\mu_Y$ are the mean of $X$ and $Y$ respectively, and $\sigma^2$ is the common variance of $X$ and $Y$.

*The other approach* is to start from the joint density of $X$ and $Y$,

$$\begin{aligned}
f(x,y) &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu_X)^2}{2\sigma^2}\right)\times\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu_Y)^2}{2\sigma^2}\right) \\
&= \frac{1}{2\pi\sigma^2}\exp\left(-\frac{(x-\mu_X)^2+(y-\mu_Y)^2}{2\sigma^2}\right).
\end{aligned}$$

Notice the following identity, which can be verified by direct computation

$$(x-\mu_X)^2+(y-\mu_Y)^2 = \frac{1}{2}\left[(x-y)-(\mu_X-\mu_Y)\right]^2+\frac{1}{2}\left[(x+y)-(\mu_X+\mu_Y)\right]^2,$$

the probability density function can be re-written in terms of the $(u,v) = (x+y, x-y)$

$$f(u,v) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{\left[u-(\mu_X+\mu_Y)\right]^2+\left[v-(\mu_X-\mu_Y)\right]^2}{4\sigma^2}\right)\times|\det J|$$

where the Jacobian determinant

$$|\det J| = \left|\det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right| = \left|\det \begin{bmatrix} +\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & -\frac{1}{2} \end{bmatrix}\right| = \frac{1}{2}.$$

Thus,

$$\begin{aligned} f(u,v) &= \frac{1}{4\pi\sigma^2} \exp\left(-\frac{\left[u-(\mu_X+\mu_Y)\right]^2 + \left[v-(\mu_X-\mu_Y)\right]^2}{4\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi}(\sqrt{2}\sigma)^2} \exp\left(-\frac{\left[u-(\mu_X+\mu_Y)\right]^2}{2\times 2\sigma^2}\right) \times \frac{1}{\sqrt{2\pi}(\sqrt{2}\sigma)^2} \exp\left(-\frac{\left[v-(\mu_X-\mu_Y)\right]^2}{2\times 2\sigma^2}\right). \end{aligned}$$

From the above expression, one concludes that the joint distribution of $U = X+Y$ and $V = X-Y$ decomposes into a product of the distribution of $U = X+Y$ and $V = X-Y$ individually. In other words, $U = X+Y$ and $V = X-Y$ are independent.

*A third approach.* Since $X$ and $Y$ are both normal, $X+Y$ and $X-Y$ are also normal. The random vector $(X+Y, X-Y)$ is Gaussian because it is a (non-singular) linear transformation of the Gaussian vector $(X,Y)$:

$$\begin{pmatrix} X+Y \\ X-Y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

and any linear combination of $X+Y$ and $X-Y$ with coefficients $a, b \in \mathbb{R}$

$$a(X+Y) + b(X-Y) = (a+b)X + (a-b)Y$$

has a normal distribution because $X$ and $Y$ are normal random variables. Thus, to prove independence, it is sufficient to show that $\text{cov}(X+Y, X-Y) = 0$ (Theorem 1.5, lecture notes 1).

$$\begin{aligned} \text{var}(X) = \text{var}(Y) \quad &\Rightarrow \quad \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\ &\Rightarrow \quad \mathbb{E}[X^2 - Y^2] = \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 \\ &\Rightarrow \quad \mathbb{E}[(X+Y)(X-Y)] = (\mathbb{E}[X] + \mathbb{E}[Y])(\mathbb{E}[X] - \mathbb{E}[Y]) \\ &\Rightarrow \quad \mathbb{E}[(X+Y)(X-Y)] = \mathbb{E}[X+Y]\mathbb{E}[X-Y] \\ &\Rightarrow \quad \text{cov}(X+Y, X-Y) = \mathbb{E}[(X+Y)(X-Y)] - \mathbb{E}[X+Y]\mathbb{E}[X-Y] = 0. \end{aligned}$$

# 3 Problem Three

Let $B(t)_{t\geq 0}$ be a Brownian motion and $\alpha \in (0,1)$. Find the conditional density of $B(\alpha t)$ given that $B(t) = y$.

*Solution*: Because $\alpha \in (0,1)$, $\alpha t \leq t$. The case where $t = 0$ is trivial as $B(0) = 0$ is a constant. We restrict our attention to the case $t > 0$ and $\alpha t < t$. Construct a linear combination of $B(t)$ and $B(\alpha t)$ with $c \in \mathbb{R}$

$$W(t) = B(\alpha t) - cB(t)$$

such that $W$ is independent of $B(t)$. Since $W$ itself is a normal random variable because a Brownian motion is a Gaussian process, it suffices to require $W(t)$ to be uncorrelated with $B(t)$,

$$\operatorname{cov}(W(t), B(t)) = \operatorname{cov}(B(\alpha t), B(t)) - \operatorname{cov}(cB(t), B(t)) = \alpha t - ct = 0 \Rightarrow c = \alpha.$$

In other words, $W(t) = B(\alpha t) - \alpha B(t) \perp B(t)$.

*Note*: the justify the existence of such a random variable $W(t)$, consider the two dimensional vector space spanned by $B(t)$ and $B(\alpha t)$. The Gram-Schmidt orthogonalization scheme transforms the basis $(B(\alpha t), B(t))$ into an orthogonal basis, which implies the existence of $W(t)$.

It's straightforward to verify that $W(t)$ is a Gaussian with the following parameters

$$\begin{aligned}
\mathbb{E}[W(t)] &= \mathbb{E}[B(\alpha t)] - \alpha\mathbb{E}[B(t)] = 0 \\
\operatorname{var}(W(t)) &= \mathbb{E}[W(t)^2] \\
&= \mathbb{E}[B(\alpha t)^2] + \alpha^2\mathbb{E}[B(t)^2] - 2\alpha\mathbb{E}[B(\alpha t)B(t)] \\
&= \alpha t + \alpha^2 t - 2\alpha^2 t \\
&= \alpha(1-\alpha)t.
\end{aligned}$$

*One approach* is to compute the characteristic function of $B(\alpha t)$ conditioned on $B(t)$ (as a function of $\lambda \in \mathbb{R}$)

$$\begin{aligned}
\mathbb{E}[e^{i\lambda B(\alpha t)}|B(t)] &= \mathbb{E}[e^{i\lambda(\alpha B(t)+W(t))}|B(t)] \\
\text{(independence lemma)} &= e^{i\lambda\alpha B(t)}\mathbb{E}[e^{i\lambda W(t)}] \\
&= e^{i\lambda\alpha B(t)} e^{-\frac{1}{2}\alpha(1-\alpha)t\lambda^2} \\
&= e^{i\lambda\alpha B(t) - \frac{1}{2}\alpha(1-\alpha)t\lambda^2}.
\end{aligned} \tag{3.1}$$

One immediately recognizes that the above expression is the characteristic function of a Gaussian random variable with the following parameters

$$\mathbb{E}[B(\alpha t)|B(t)] = \alpha B(t), \quad \operatorname{var}(B(\alpha t)|B(t)) = \alpha(1-\alpha)t.$$

Thus, the corresponding conditional density of $B(\alpha t)$ given that $B(t) = y$ is

$$f(B(\alpha t = x|B(t) = y) = \frac{1}{\sqrt{2\pi\alpha(1-\alpha)t}} \exp\left(-\frac{(x-\alpha y)^2}{2\alpha(1-\alpha)t}\right). \tag{3.2}$$

*Alternatively*, one could compute the conditional cumulative distribution function of $B(\alpha t)|B(t)$:

$$
\begin{aligned}
F_{B(\alpha t)|B(t)}(x|y) &= \mathbb{P}(B(\alpha t) \leq x|B(t) = y) \\
&= \mathbb{P}(\alpha B(t) + W(t) \leq x|B(t) = y) \\
&= \mathbb{P}(W(t) \leq x - \alpha B(t)|B(t) = y) \\
&= \mathbb{P}(W(t) \leq x - \alpha y|B(t) = y) \\
&= \frac{1}{\sqrt{2\pi\alpha(1-\alpha)t}} \int_0^{x-\alpha y} \exp\left(-\frac{z^2}{2\alpha(1-\alpha)t}\right) dz.
\end{aligned}
$$

which again yields the conditional density of $B(\alpha t)|B(t)$ given by Equation. 3.2.

*A third solution*: One can also work out the condition density directly from the definition (see lecture notes 4 from Refresher):

$$
f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \quad \text{if } f_Y(y) \neq 0.
$$

According to one of the equivalent definitions of Brownian motion given in lecture notes, the joint density function of $B(\alpha t)$ and $B(t)$ is given by

$$
\begin{aligned}
f_{B(\alpha t),B(t)}(x,y) &= f_{B(t)|B(\alpha t)}(y|x) f_{B(\alpha t)}(x) \\
\text{(independence)} &= f_{B(t)-B(\alpha t)}(y-x) f_{B(\alpha t)}(x) \\
&= \frac{1}{\sqrt{2\pi(1-\alpha)t}} e^{-\frac{(y-x)^2}{2(1-\alpha)t}} \frac{1}{\sqrt{2\pi(\alpha t)}} e^{-\frac{x^2}{2\alpha t}}.
\end{aligned}
$$

and the marginal density of $B(t)$ is

$$
f_{B(t)}(y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}.
$$

A direct computation yields the expression for

$$
\begin{aligned}
f_{B(\alpha t)|B(t)}(x|y) &= \frac{f_{B(\alpha t),B(t)}(x,y)}{f_{B(t)}(y)} \\
&= \frac{\frac{1}{\sqrt{2\pi(\alpha t)}} e^{-\frac{x^2}{2\alpha t}} \frac{1}{\sqrt{2\pi(1-\alpha)t}} e^{-\frac{(y-x)^2}{2(1-\alpha)t}}}{\frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}} \\
&= \frac{1}{\sqrt{2\pi\alpha(1-\alpha)t}} e^{-\frac{x^2}{2\alpha t} - \frac{(y-x)^2}{2(1-\alpha)t} + \frac{y^2}{2t}} \\
&= \frac{1}{\sqrt{2\pi\alpha(1-\alpha)t}} e^{-\frac{(x-\alpha y)^2}{2\alpha(1-\alpha)t}},
\end{aligned}
$$

which is the same result obtained from the conditional characteristic function in Equation 3.2.

# 4 Problem Four

Let $B(t) = (B_1(t), B_2(t))$, $t \geq 0$ be a two-dimensional Brownian motion (assume that the coordinates are independent). Find the distribution of the distance from $B(t)$ to the origin.
*Solution*: By definition of a two-dimensional Brownian motion, $B_1(t)$ and $B_2(t)$ are independent Brownian motion.The probability density function of the two-dimensional Brownian motion for $t > 0$

$$f(B_1(t) = x, B_2(t) = y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \times \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} = \frac{1}{2\pi t} e^{-\frac{x^2+y^2}{2t}}.$$

(For $t = 0$, this corresponds to a two-dimensional Dirac-$\delta$ measure centered at origin). Thus, the probability of the Brownian motion to fall into an arbitrary two-dimensional region $\Omega$ is

$$\begin{aligned} \mathbb{P}(B(t) \in \Omega) &= \int\int_{\Omega} f(x, y) dx dy \\ &= \int\int_{\Omega} \frac{1}{2\pi t} e^{-\frac{x^2+y^2}{2t}} dx dy \\ \text{(polar coordinates)} &= \int\int_{\Omega} \frac{1}{t} e^{-\frac{r^2}{2t}} r dr \frac{d\phi}{2\pi} \end{aligned} \tag{4.1}$$

where the transformation to polar coordinates is defined by

$$x = r\cos\phi, \quad y = r\sin\phi.$$

From the above expression Equation 4.1, one finds the probability density function of two-dimensional Brownian motion in terms of the radial distance and angular coordinates. The angular distribution is uniform over $[0, 2\pi)$. The cumulative distribution of the distance can be found by integrating the probability density function over a circular region in two dimensions $\Omega = \{(x, y) | r = \sqrt{x^2 + y^2} \leq R\}$

$$F(R) = \mathbb{P}(r \leq R) \quad = \quad \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^R \frac{1}{t} e^{-\frac{r^2}{2t}} r dr = \int_0^R \frac{1}{t} e^{-\frac{r^2}{2t}} r dr$$

Differentiating the cumulative distribution function with respect to the distance to the origin, we get the probability density function

$$f\left(\sqrt{B_1^2(t) + B_2^2(t)} = r\right) = \frac{r}{t} e^{-\frac{r^2}{2t}}, \quad \forall r \geq 0. \tag{4.2}$$

It's easily verified that the radial distance distribution is normalized

$$\int_0^{\infty} f(r) dr = \int_0^{\infty} e^{-\frac{r^2}{2t}} \frac{r dr}{t} = 1.$$

*Note*: since the independent components of Brownian motions $B_1(t)$ and $B_2(t)$ have normal distribution with mean zero and variance $t$, $B_1(t)/\sqrt{t}$ and $B_2(t)/\sqrt{t}$ are independent standard normal random variables. Thus, $X = B_1(t)^2/t + B_2(t)^2/t$ is $\chi^2$-distributed with two degrees of freedom, for which the probability density is well known.

# 5 Problem Five

Let $B(t) = (B_1(t), B_2(t))$, $t \geq 0$, be a two-dimensional Brownian motion (assume that the coordinates are independent), and $\rho \in [-1,1]$. Is the process $X(t) = \rho B_1(t) + \sqrt{1-\rho^2} B_2(t)$ a Brownian motion? What is the correlation between $X(t)$ and $B_1(t)$?

*Solution*: Yes, $X(t) = \rho B_1(t) + \sqrt{1-\rho^2} B_2(t)$ a Brownian motion. To verify by definition

(1) Firstly, the linear combination of continuous functions is still a continuous function. Thus, $X(t)$ is a **continuous** process. To verify that $X(t)$ is a Brownian motion by definition,

(2) $X(0) = \rho B_1(0) + \sqrt{1-\rho^2} B_2(0) = 0$.

(3) Let $\mathbf{X} = (X(t_1), \cdots, X(t_n))$, where $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n$, the incremental random variables

$$\begin{aligned}
& (X(t_1) - X(t_0), X(t_2) - X(t_1), \cdots, X(t_n) - X(t_{n-1})) \\
= \; & \rho(B_1(t_1) - B_1(t_0), B_1(t_2) - B_1(t_1), \cdots, B_1(t_n) - B_1(t_{n-1})) \\
& + \sqrt{1-\rho^2}(B_2(t_1) - B_2(t_0), B_2(t_2) - B_2(t_1), \cdots, B_2(t_n) - B_2(t_{n-1})) \\
= \Big( \; & \rho(B_1(t_1) - B_1(t_0)) + \sqrt{1-\rho^2}(B_2(t_1) - B_2(t_0)), \\
& \rho(B_1(t_2) - B_1(t_1)) + \sqrt{1-\rho^2}(B_2(t_2) - B_2(t_1)), \\
& \cdots, \\
& \rho(B_1(t_n) - B_1(t_{n-1})) + \sqrt{1-\rho^2}(B_2(t_n) - B_2(t_{n-1})) \quad \Big).
\end{aligned}$$

For an arbitrary pair $i \neq j$, $X(t_i) - X(t_{i-1})$ is **independent** of $X(t_j) - X(t_{j-1})$ because

- $B_1(t_i) - B_1(t_{i-1})$ is independent of $B_1(t_j) - B_1(t_{j-1})$ because $B_1(t)$ is a Brownian motion;
- $B_2(t_i) - B_2(t_{i-1})$ is independent of $B_2(t_j) - B_2(t_{j-1})$ because $B_2(t)$ is a Brownian motion;
- $B_1(t_i) - B_1(t_{i-1})$ is independent of $B_2(t_j) - B_2(t_{j-1})$ because $B_1(t)$ and $B_2(t)$ are independent. The same is true that $B_1(t_j) - B_1(t_{j-1})$ is independent of $B_2(t_i) - B_2(t_{i-1})$.

(4) For any $i$, $X(t_i) - X(t_{i-1})$ has a normal distribution because it is a linear combination of two normal random variables

$$X(t_i) - X(t_{i-1}) = \rho(B_1(t_i) - B_1(t_{i-1})) + \sqrt{1-\rho^2}(B_2(t_i) - B_2(t_{i-1}))$$

where $B_1(t_i) - B_1(t_{i-1})$ and $B_2(t_i) - B_2(t_{i-1})$ are both normal random variables with mean zero and variance $t_i - t_{i-1}$ because, again, $B_1(t)$ and $B_2(t)$ are Brownian motions.

- $\mathbb{E}[X(t_i) - X(t_{i-1})] = \rho \mathbb{E}[B_1(t_i) - B_1(t_{i-1})] + \sqrt{1-\rho^2}\mathbb{E}[B_2(t_i) - B_2(t_{i-1})] = 0$
- $\mathrm{var}(X(t_i) - X(t_{i-1})) = \rho^2(t_i - t_{i-1}) + (1-\rho^2)(t_i - t_{i-1}) = (t_i - t_{i-1})$, because $B_1(t)$ and $B_2(t)$ are independent.

Thus, we verified that $X(t_i) - X(t_{i-1})$ has a normal distribution with mean zero and variance $(t_i - t_{i-1})$. In summary, by definition, $X(t)$ is a Brownian motion.

*Note*: We can also carry out the proof by computing the joint characteristic function of $\mathbf{X} = (X(t_1), \cdots, X(t_n))$, where $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n$. Denote accordingly, denote (in **bold** font for vectors)

$$\begin{aligned}
\mathbf{B}_1 &= (B_1(t_1), \cdots, B_1(t_n)), \\
\mathbf{B}_2 &= (B_2(t_1), \cdots, B_2(t_n))
\end{aligned}$$

the joint characteristic function of $\mathbf{X}$

$$\begin{aligned}
\phi(\mathbf{u}) &\triangleq \mathbb{E}\left[e^{i\langle \mathbf{u}, \mathbf{X}\rangle}\right] \\
&= \mathbb{E}\left[e^{i\langle \mathbf{u}, \rho\mathbf{B}_1 + \sqrt{1-\rho^2}\mathbf{B}_2\rangle}\right] \\
&= \mathbb{E}\left[e^{i\langle \rho\mathbf{u}, \mathbf{B}_1\rangle + i\langle \sqrt{1-\rho^2}\mathbf{u}, \mathbf{B}_2\rangle}\right] \\
&= \mathbb{E}\left[e^{i\langle \rho\mathbf{u}, \mathbf{B}_1\rangle} e^{i\langle \sqrt{1-\rho^2}\mathbf{u}, \mathbf{B}_2\rangle}\right] \\
\text{(independence)} &= \mathbb{E}\left[e^{i\langle \rho\mathbf{u}, \mathbf{B}_1\rangle}\right] \mathbb{E}\left[e^{i\langle \sqrt{1-\rho^2}\mathbf{u}, \mathbf{B}_2\rangle}\right] \\
(C_{ij} = t_i \wedge t_j) &= e^{-\frac{1}{2}\langle \rho\mathbf{u}, \mathbb{C}\rho\mathbf{u}\rangle} e^{-\frac{1}{2}\langle \sqrt{1-\rho^2}\mathbf{u}, \mathbb{C}\sqrt{1-\rho^2}\mathbf{u}\rangle} \\
&= e^{-\frac{1}{2}\rho^2\langle \mathbf{u}, \mathbb{C}\mathbf{u}\rangle} e^{-\frac{1}{2}(1-\rho^2)\langle \mathbf{u}, \mathbb{C}\mathbf{u}\rangle} \\
&= e^{-\frac{1}{2}\rho^2\langle \mathbf{u}, \mathbb{C}\mathbf{u}\rangle - \frac{1}{2}(1-\rho^2)\langle \mathbf{u}, \mathbb{C}\mathbf{u}\rangle} \\
&= e^{-\frac{1}{2}\langle \mathbf{u}, \mathbb{C}\mathbf{u}\rangle}
\end{aligned}$$

where $\mathbb{C}$ is the covariance function for Brownian motion. The above expression is the familiar joint characteristic function, which is an equivalent definition for Brownian motion.

To compute the correlation between $X(t)$ and $B_1(t)$, note that $\mathbb{E}[X(t)] = \mathbb{E}[B_1(t)] = \mathbb{E}[B_2(t)] = 0$, the covariance between $X(t)$ and $B_1(t)$

$$\begin{aligned}
\operatorname{cov}(X(t), B_1(t)) &= \mathbb{E}[X(t)B_1(t)] \\
&= \mathbb{E}\left[\left(\rho B_1(t) + \sqrt{1-\rho^2}B_2(t)\right)B_1(t)\right] \\
&= \mathbb{E}\left[\rho B_1(t)B_1(t) + \sqrt{1-\rho^2}B_2(t)B_1(t)\right] \\
&= \rho\mathbb{E}\left[B_1(t)B_1(t)\right] + \sqrt{1-\rho^2}\mathbb{E}\left[B_2(t)B_1(t)\right] \\
\text{(independence)} &= \rho t + \sqrt{1-\rho^2}\mathbb{E}\left[B_2(t)\right]\mathbb{E}\left[B_1(t)\right] \\
&= \rho t
\end{aligned}$$

Since both $X(t)$ and $B_1(t)$ are Brownian motions, $\operatorname{var}(X(t)) = \operatorname{var}(B_1(t)) = t$, the correlation

$$\operatorname{corr}(X(t), B_1(t)) = \frac{\operatorname{cov}(X(t), B_1(t))}{\sqrt{\operatorname{var}(X(t))}\sqrt{\operatorname{var}(B_1(t))}} = \frac{\rho t}{\sqrt{t}\sqrt{t}} = \rho.$$

# 6 Problem Six

Let $B(t)_{t\geq 0}$ be a Brownian motion. Determine, which of the following processes are Brownian motion. Justify your answers.

**(a)** $-B(t)_{t\geq 0}$

*Solution*: This is a Brownian motion. The shortest justification is probably again using the joint characteristic function, but let's instead verify by Definition 2.1 in lecture notes. Denote

$$\bar{B}(t) = -B(t), \quad t \geq 0.$$

(1) $-B(t)$ is a continuous process because $B(t)$ is continuous.
(2) $\bar{B}(0) = -B(0) = 0$.
(3) Let $0 = t_0 < t_1 < \cdots < t_m$, the incremental random variables

$$\begin{aligned}
&\bar{B}(t_1) - \bar{B}(t_0) = B(t_0) - B(t_1) \\
&\bar{B}(t_2) - \bar{B}(t_1) = B(t_1) - B(t_2) \\
&\dots \\
&\bar{B}(t_m) - \bar{B}(t_{m-1}) = B(t_{m-1}) - B(t_m)
\end{aligned}$$

are independent because of the independence of the incremental random variables of the Brownian motion $B(t)$.
(4) For all $s > 0, t > 0$, the increment

$$\bar{B}(t+s) - \bar{B}(t) = -B(t+s) + B(t) = -(B(t+s) - B(t))$$

has a normal distribution with mean zero and variance $s$ because $(B(t+s) - B(t))$ has a normal distribution with mean zero and variance $s$. (The negative of a normal random variable with mean zero is also a normal random variable with mean zero and same variance).

**(b)** $cB(t/c^2)_{t\geq 0}$

*Solution*: This is a Brownian motion. Firstly, $cB(t/c^2)$ is a continuous process because $B(t)$ is continuous and $t/c^2$ is a continuous function of $t$. We justify by computing the joint characteristic function.

$$\begin{aligned}
\phi(\mathbf{u}) &\triangleq \mathbb{E}\left[e^{i\langle \mathbf{u}, c\mathbf{B}(t/c^2)\rangle}\right] \\
&= \mathbb{E}\left[e^{i\langle c\mathbf{u}, \mathbf{B}(t/c^2)\rangle}\right] \\
\left(\mathbb{C}'_{ij} = \frac{1}{c^2} t_i \wedge t_j = \frac{1}{c^2}\mathbb{C}_{ij}\right) \quad &= e^{-\frac{1}{2}\langle c\mathbf{u}, \mathbb{C}' c\mathbf{u}\rangle} \\
&= e^{-\frac{1}{2}c^2\langle \mathbf{u}, \mathbb{C}'\mathbf{u}\rangle} \\
&= e^{-\frac{1}{2}\langle \mathbf{u}, c^2\mathbb{C}'\mathbf{u}\rangle} \\
&= e^{-\frac{1}{2}\langle \mathbf{u}, \mathbb{C}\mathbf{u}\rangle}.
\end{aligned}$$

This is the joint characteristic function of Brownian motion.

**(c)** $\sqrt{t}B(1)_{t\geq 0}$
*Solution*: This is NOT a Brownian motion. To see that, write $B'(t) = \sqrt{t}B(1)$ and compute the covariance

$$\begin{aligned}
\mathbb{E}[B'(t_i)B'(t_j)] &= \mathbb{E}[\sqrt{t_i t_j}B(1)B(1)] \\
&= \sqrt{t_i t_j}\mathbb{E}[B(1)B(1)] \\
&= \sqrt{t_i t_j} \\
&\neq t_i \wedge t_j.
\end{aligned}$$

**(d)** $(B(2t) - B(t))_{t\geq 0}$
*Solution*: This is NOT a Brownian motion. To see that, write $B'(t) = B(2t) - B(t)$ and compute the covariance

$$\begin{aligned}
\mathbb{E}[B'(t)B'(2t)] &= \mathbb{E}[(B(2t) - B(t))(B(4t) - B(2t))] \\
&= \mathbb{E}[B(2t)B(4t) - B(2t)B(2t) - B(t)B(4t) + B(t)B(2t)] \\
&= \mathbb{E}[B(2t)B(4t)] - \mathbb{E}[B(2t)B(2t)] - \mathbb{E}[B(t)B(4t)] + \mathbb{E}[B(t)B(2t)] \\
&= 2t - 2t - t + t \\
&= 0 \\
&\neq t \wedge (2t) = t.
\end{aligned}$$

**(e)** $(B(s) - B(s-t))_{0\leq t\leq s}$, where $s$ is fixed.
*Solution*: This is a Brownian motion. Denote $B'(t) = B(s) - B(s-t)$,
(1) $B'(t) = B(s) - B(s-t)$ is a continuous functions of $t$ because it is a linear combination of two continuous functions.
(2) $B'(0) = B(s) - B(s) = 0$.
(3) Let $0 = t_0 < t_1 < \cdots < t_m < s$, which means that

$$0 < s - t_m < s - t_{m-1} < \cdots < s - t_1 < s - t_0 = s.$$

Thus, the incremental random variables

$$\begin{aligned}
&B'(t_1) - B'(t_0) = B(s - t_0) - B(s - t_1) \\
&B'(t_2) - B'(t_1) = B(s - t_1) - B(s - t_2) \\
&\cdots \\
&B'(t_m) - B'(t_{m-1}) = B(s - t_{m-1}) - B(s - t_m)
\end{aligned}$$

are independent because of the independence of the incremental random variables of Brownian motion $B(t)$.
(4) For all $t' > 0$, $t > 0$ and $t + t' < s$, the increment

$$B'(t + t') - B'(t) = B(s - t - t') - B(s - t) = -(B(s - t - t' + t') - B(s - t - t')).$$

has a normal distribution with mean zero and variance $t'$ because $(B(s-t-t'+t')-B(s-t-t'))$ has a normal distribution with mean zero and variance $t'$. (The negative of a normal random variable with mean zero is also a normal random variable with mean zero and same variance). We thus verified the Definition 2.1 of Brownian motion in lecture notes.

BARUCH, MFE

# MTH 9831 Assignment Two

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September 15, 2015

*Note*: Problem (7) from Homework Assignment One was solved in Theorem 1.7 of Lecture 2. We are copying the solution here for completeness.

Let $B(t)_{t\geq 0}$ be a Brownian motion. Define $B^*(t) := \max_{0\leq s\leq t} B(s)$. For $0 \leq a \leq x$, calculate $\mathbb{P}(B^*(t) \geq a, B(t) \leq x)$.
*Solution*: For all $a > 0$, $x \leq a$, and $t \geq 0$, we have

$$\mathbb{P}(B^*(t) \geq a, B(t) \leq x) = \mathbb{P}(\tau_a < t, B(t) \leq x).$$

The reflection principle says that if we replace each path with the path reflected with respect to the line $x = a$ after time $\tau_a$, then the reflected process will be again Brownian motion. This and the equality $\tau_a = \tilde{\tau}_a$, where $\tilde{\tau}_a$ is the hitting time of $a$ of the reflected process $\tilde{B}$, imply that the last probability above is the same as

$$\mathbb{P}(\tilde{\tau}_a < t, \tilde{B}(t) \geq 2a - x) = \mathbb{P}(\tilde{B}(t) \geq 2a - x) = 1 - N\left(\frac{2a - x}{\sqrt{t}}\right),$$

where $N(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-y^2/2} dy$.

# 1 PROBLEM ONE

Let $B(t)_{t\geq 0}$ be a Brownian motion. Show that

$$\mathbb{P}\left(\max_{0\leq t\leq 1}|B(t)|>a\right)\leq 4\mathbb{P}(B(1)>a).$$

Conclude that all moments of $X:=\max_{0\leq t\leq 1}|B(t)|$ are finite and the moment generating function of $X$ is well-defined on the whole $\mathbb{R}$.
*Proof*: Use **Theorem 1.5** (Reflection principle),

$$\begin{aligned}
\mathbb{P}\left(\max_{0\leq t\leq 1}|B(t)|>a\right) &= \mathbb{P}\left(\max_{0\leq t\leq 1}B(t)>a\ \textbf{or}\ \min_{0\leq t\leq 1}B(t)<-a\right)\\
&\leq \mathbb{P}\left(\max_{0\leq t\leq 1}B(t)>a\right)+\mathbb{P}\left(\min_{0\leq t\leq 1}B(t)<-a\right)\\
(-B(t)\text{ is a Brownian motion}) &= \mathbb{P}\left(\max_{0\leq t\leq 1}B(t)>a\right)+\mathbb{P}\left(\max_{0\leq t\leq 1}-B(t)>a\right)\\
(\text{Reflection principle}) &= 2\mathbb{P}(B(1)>a)+2\mathbb{P}(B(1)>a)\\
&= 4\mathbb{P}(B(1)>a).
\end{aligned}$$

Let $X:=\max_{0\leq t\leq 1}|B(t)|$. For all $p\in\mathbb{N}^+$, [1]

$$\begin{aligned}
\mathbb{E}\left[X^p\right] &= p\int_0^\infty x^{p-1}\mathbb{P}(X>x)dx\\
&\leq 4p\int_0^\infty x^{p-1}\mathbb{P}(B(1)>x)dx\\
(B(1)\text{ is standard normal}) &= 4p\int_0^\infty dx\,x^{p-1}\int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt\\
&= 4p\int_0^\infty dx\,x^{p-1}\int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\mathbb{1}_{\{t\geq x\}}dt\\
(\text{switch order of integration}) &= \frac{4p}{\sqrt{2\pi}}\int_0^\infty dt\,e^{-\frac{1}{2}t^2}\int_0^\infty dx\,x^{p-1}\mathbb{1}_{\{t\geq x\}}\\
&= \frac{4p}{\sqrt{2\pi}}\int_0^\infty dt\,e^{-\frac{1}{2}t^2}\int_0^t dx\,x^{p-1}\\
&= \frac{4}{\sqrt{2\pi}}\int_0^\infty t^p e^{-\frac{1}{2}t^2}dt\\
&< \infty.
\end{aligned}$$

In other words, all moments of $X:=\max_{0\leq t\leq 1}|B(t)|$ are finite.

[1] Here we are using a result introduced in the first TA office hour session. For a non-negative random variable $X\geq 0$, the $p$-moment $\mathbb{E}[X^p]$ and the tail probabilities is related by the following equality

$$\mathbb{E}\left[X^p\right]=p\int_0^\infty x^{p-1}\mathbb{P}(X>x)dx.$$

To prove that the moment generating function of $X$ is well-defined on the whole $\mathbb{R}$, we follow the same derivation that leads us to the result given in footnote 1.

$$
\begin{aligned}
\mathbb{E}\left[e^{uX}\right] &= \mathbb{E}\left[1+u\int_0^X e^{ux}dx\right] \\
&= 1+u\mathbb{E}\left[\int_0^\infty e^{ux}\mathbb{1}_{\{x<X\}}dx\right] \\
&= 1+u\int_\Omega d\mathbb{P}(\omega)\int_0^\infty e^{ux}\mathbb{1}_{\{x<X(\omega)\}}dx \\
\text{(switch order of integration)} &= 1+u\int_0^\infty dx e^{ux}\int_\Omega d\mathbb{P}(\omega)\mathbb{1}_{\{x<X(\omega)\}} \\
&= 1+u\int_0^\infty dx e^{ux}\mathbb{P}(X>x) \\
&\leq 1+4u\int_0^\infty dx e^{ux}\mathbb{P}(B(1)>x) \\
&= 1+4u\int_0^\infty dx e^{ux}\int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt \\
&= 1+4u\int_0^\infty dx e^{ux}\int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}\mathbb{1}_{\{t\geq x\}}dt \\
\text{(switch order of integration)} &= 1+\frac{4u}{\sqrt{2\pi}}\int_0^\infty dt e^{-\frac{1}{2}t^2}\int_0^\infty dx e^{ux}\mathbb{1}_{\{t\geq x\}} \\
&= 1+\frac{4u}{\sqrt{2\pi}}\int_0^\infty dt e^{-\frac{1}{2}t^2}\int_0^t dx e^{ux} \\
&= 1+\frac{4}{\sqrt{2\pi}}\int_0^\infty dt e^{-\frac{1}{2}t^2}\left(e^{ut}-1\right) \\
&= 1+\frac{4}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}t^2+ut}dt-\frac{4}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}t^2}dt \\
\text{(all integrals convergent)} &< \infty.
\end{aligned}
$$

Thus, the moment generating function of $X$ is well-defined.

# 2 Problem Two

Find the joint density of $(B^*(t), B(t))$. Sketch the region where this joint density is non-zero.
*Proof*: Use **Theorem 1.8** (Joint distribution of Brownian motion and its running maximum), for all $y > 0$, $x \leq y$, and $t \geq 0$,

$$\mathbb{P}(B^*(t) \geq y, B(t) \leq x) = 1 - N\left(\frac{2y-x}{\sqrt{t}}\right)$$

where $N(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-z^2/2}dz$.

Notice that the running maximum $B^*(t)$ has a probability density,

$$\begin{aligned}\mathbb{P}(B^*(t) \geq y, B(t) \leq x) + \mathbb{P}(B^*(t) \leq y, B(t) \leq x) &= \mathbb{P}(B(t) \leq x) \\ (Z \text{ is a standard normal}) &= \mathbb{P}\left(Z \leq \frac{x}{\sqrt{t}}\right) \\ &= N\left(\frac{x}{\sqrt{t}}\right).\end{aligned}$$

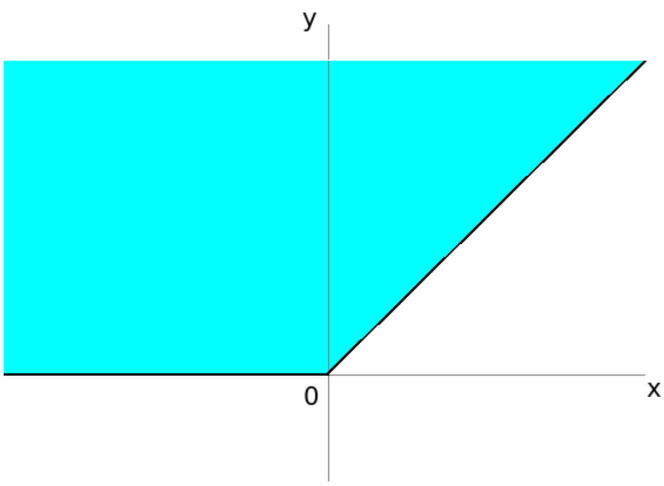
Thus,

$$\mathbb{P}(B^*(t) \leq y, B(t) \leq x) = N\left(\frac{x}{\sqrt{t}}\right) + N\left(\frac{2y-x}{\sqrt{t}}\right) - 1.$$

This is known as the joint (cumulative) distribution function of $B^*(t)$ and $B(t)$, see **Definition 1.13** Refresher Lecture 4. The joint density

$$\begin{aligned}f_{B(t),B^*(t)}(x,y) &= \frac{\partial^2}{\partial x \partial y}\mathbb{P}(B^*(t) \leq y, B(t) \leq x) \\ &= \frac{\partial^2}{\partial x \partial y}\mathbb{P}(B^*(t) \leq y, B(t) \leq x) \\ &= \sqrt{\frac{2}{\pi}}\frac{2y-x}{t^{3/2}}e^{-\frac{(2y-x)^2}{2t}}.\end{aligned}$$

The region where the joint density is non-zero is $x \leq y$ and $y > 0$.



# 3 Problem Three

In the proof of Lemma 2.2, we derived the following statement

> **Lemma (Borel-Cantelli, part 1)**. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $A_n \in \mathscr{F}$, $n \geq 1$. If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}\left(\cap_{N=1}^{\infty} \cup_{n \geq N} A_n\right) = 0$.

The event $\cap_{N=1}^{\infty} \cup_{n \geq N} A_n$ is usually denoted by $\limsup A_n$ or $\{A_n, \text{i.o.}\}$, where i.o. stands for "infinitely often", and consists of those and only those $\omega$ which belong to infinitely many sets $A_n$ of the sequence. Give a detailed proof of Borel-Cantelli lemma and use it to show that if $(X_n)_{n \geq 1}$ is a sequence of identically distributed integrable random variables then

$$\mathbb{P}\left(\lim_{n \to \infty} \frac{X_n}{n} = 0\right) = 1.$$

*Proof*: Notice that

$$\begin{aligned}
\cap_{N=1}^{M} \cup_{n \geq N} A_n \quad &= \quad (A_1 \cup A_2 \cup A_3 \cup A_4 \cup \cdots) \\
&\cap \quad (A_2 \cup A_3 \cup A_4 \cup \cdots) \\
&\cap \quad (A_3 \cup A_4 \cup \cdots) \\
&\cap \quad \cdots \\
&\cap \quad (A_M \cup \cdots) \\
&= \quad (A_M \cup \cdots) \\
&= \quad \cup_{n \geq M} A_n
\end{aligned}$$

the probability

$$\begin{aligned}
\mathbb{P}\left(\cap_{N=1}^{\infty} \cup_{n \geq N} A_n\right) &\triangleq \mathbb{P}\left(\lim_{M \to \infty} \cap_{N=1}^{M} \cup_{n \geq N} A_n\right) \\
&= \mathbb{P}\left(\lim_{M \to \infty} \cup_{n \geq M} A_n\right) \\
&= \lim_{M \to \infty} \mathbb{P}\left(\cup_{n \geq M} A_n\right) \\
&\leq \lim_{M \to \infty} \sum_{n=M}^{\infty} \mathbb{P}(A_n) \\
&= \lim_{M \to \infty} \left(\sum_{n=1}^{\infty} \mathbb{P}(A_n) - \sum_{n=1}^{M-1} \mathbb{P}(A_n)\right) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(A_n) - \lim_{M \to \infty} \left(\sum_{n=1}^{M-1} \mathbb{P}(A_n)\right) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(A_n) - \sum_{n=1}^{\infty} \mathbb{P}(A_n) \\
\left(\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty\right) \quad &= 0.
\end{aligned}$$

Consider the set

$$\left\{\lim_{n\to\infty}\frac{X_n}{n}=0\right\},$$

which is equivalent to saying $\forall m\in\mathbb{N}$, $\exists N$, such that $\forall n\geq N$, $\frac{X_n}{n}<\frac{1}{m}$. The inverse statement

$$\left\{\lim_{n\to\infty}\frac{X_n}{n}\neq 0\right\},$$

is equivalent to saying $\exists m\in\mathbb{N}$, such that $\forall N$, $\exists n\geq N$, $\frac{X_n}{n}\geq\frac{1}{m}$. In other words,

$$\left\{\lim_{n\to\infty}\frac{X_n}{n}\neq 0\right\}=\cup_{m=1}^{\infty}\cap_{N=1}^{\infty}\cup_{n\geq N}\left\{\frac{X_n}{n}\geq\frac{1}{m}\right\}.$$

Then

$$\mathbb{P}\left(\lim_{n\to\infty}\frac{X_n}{n}\neq 0\right)=\mathbb{P}\left(\cup_{m=1}^{\infty}\cap_{N=1}^{\infty}\cup_{n\geq N}\left\{\frac{X_n}{n}\geq\frac{1}{m}\right\}\right)\leq\sum_{m=1}^{\infty}\mathbb{P}\left(\cap_{N=1}^{\infty}\cup_{n\geq N}\left\{\frac{X_n}{n}\geq\frac{1}{m}\right\}\right). \quad (3.1)$$

Define for fixed $m$,

$$A_n^{(m)}\triangleq\left\{\frac{X_n}{n}\geq\frac{1}{m}\right\},$$

then

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbb{P}\left(A_n^{(m)}\right) &= \sum_{n=1}^{\infty}\mathbb{P}\left(\frac{X_n}{n}\geq\frac{1}{m}\right)\\ &= \sum_{n=1}^{\infty}\mathbb{P}\left(mX_n\geq n\right).\end{aligned}$$

Because $(X_n)_{n\geq1}$ is a sequence of identically distributed integrable random variables, $(mX_n)_{n\geq1}$ is also a sequence of identically integrable random variables. According to Refresher Homework 2 Problem (4),

$$\sum_{n=1}^{\infty}\mathbb{P}\left(mX_n\geq n\right)\leq\mathbb{E}\left[mX\right]=m\mathbb{E}\left[X\right]<\infty.$$

Thus, Borel-Cantelli lemma applies,

$$\mathbb{P}\left(\cap_{N=1}^{\infty}\cup_{n\geq N}\left\{\frac{X_n}{n}\geq\frac{1}{m}\right\}\right)=0.$$

From Equation 3.1,

$$\mathbb{P}\left(\lim_{n\to\infty}\frac{X_n}{n}\neq 0\right)=0.$$

In other words,

$$\mathbb{P}\left(\lim_{n\to\infty}\frac{X_n}{n}=0\right)=1.$$

# 4 Problem Four

Use martingales in Theorem 3.2(a,b)

> **Theorem 3.2.** Let $B(t)_{t\geq 0}$ be a Brownian motion and $\mathscr{F}(t)_{t\geq 0}$ be a filtration for this Brownian motion. The following processes are martingales relative to $\mathscr{F}(t)_{t\geq 0}$:
> (a) $B(t)_{t\geq 0}$;
> (b) $(B^2(t)-t)_{t\geq 0}$;
> (c) $\left(e^{\sigma B(t)-\frac{1}{2}\sigma^2 t}\right)_{t\geq 0}$.

to compute **(a)** $\mathbb{P}(\tau_a < \tau_b)$, **(b)** $\mathbb{E}[\tau_a \wedge \tau_b]$, where $a < 0 < b$, $\tau_x := \inf\{t \geq 0 | B(t) = x\}$, and $B(t)_{t\geq 0}$ is a standard Brownian motion.

*Preparation*: Note that $\tau_a$ and $\tau_b$ are stopping times with respect to the filtration $\mathscr{F}(t)_{t\geq 0}$ for the Brownian motion by **Lemma 1.4**, according to **Lemma 2.3** of Refresher lecture 5, $T_{a,b} \triangleq \tau_a \wedge \tau_b$ is a stopping time.

**(a)** $\mathbb{P}(\tau_a < \tau_b)$;
*Solution*: Because $T_{a,b} \triangleq \tau_a \wedge \tau_b$ is a stopping time and $t \wedge T_{a,b}$ is a bounded stopping time, **Theorem 2.5** Optional Sampling Theorem applies for the martingale $B(t \wedge T_{a,b})_{t\geq 0}$:

$$\begin{aligned} 0 &= \mathbb{E}[B(0)] \\ &= \mathbb{E}[B(t \wedge T_{a,b})] \\ &= \mathbb{E}\left[B(t \wedge T_{a,b})\mathbb{1}_{\{T_{a,b}\leq t\}}\right] + \mathbb{E}\left[B(t \wedge T_{a,b})\mathbb{1}_{\{T_{a,b}>t\}}\right] \\ &= \mathbb{E}\left[B(T_{a,b})\mathbb{1}_{\{T_{a,b}\leq t\}}\right] + \mathbb{E}\left[B(t)\mathbb{1}_{\{T_{a,b}>t\}}\right]. \end{aligned} \tag{4.1}$$

The second term

$$\begin{aligned} \mathbb{E}\left[B(t)\mathbb{1}_{\{T_{a,b}>t\}}\right] &\leq \mathbb{E}\left[|B(t)\mathbb{1}_{\{T_{a,b}>t\}}|\right] \\ &\leq \mathbb{E}\left[|a| \vee b\mathbb{1}_{\{T_{a,b}>t\}}\right] \\ &= |a| \vee b \cdot \mathbb{E}\left[\mathbb{1}_{\{T_{a,b}>t\}}\right] \\ \text{(dominated convergence)} &\to 0, \quad \text{as } t \to \infty. \end{aligned}$$

On the other hand, $B(T_{a,b})\mathbb{1}_{\{T_{a,b}\leq t\}} \nearrow B(T_{a,b})$ almost surely as $t \to \infty$. The first term of Equation 4.1, by monotone convergence theorem, reduces to $\mathbb{E}\left[B(T_{a,b})\right]$. Thus

$$\begin{aligned} 0 &= \mathbb{E}\left[B(T_{a,b})\right] \\ &= a\mathbb{P}(B(T_{a,b}) = a) + b\mathbb{P}(B(T_{a,b}) = b) \\ &= a\mathbb{P}(B(T_{a,b}) = a) + b\left(1 - \mathbb{P}(B(T_{a,b}) = a)\right) \\ &= (a-b)\mathbb{P}(B(T_{a,b}) = a) + b. \end{aligned}$$

Thus,

$$\mathbb{P}(\tau_a < \tau_b) = \mathbb{P}(T_{a,b} = \tau_a) = \mathbb{P}(B(T_{a,b}) = a) = \frac{b}{|a| + b}.$$

**(b)** $\mathbb{E}[\tau_a \wedge \tau_b]$
Apply the Optional Stopping Theorem to the martingale $B^2(t\wedge T_{a,b}) - (t\wedge T_{a,b})$

$$\begin{aligned} 0 &= \mathbb{E}\left[B^2(0) - 0\right] \\ &= \mathbb{E}\left[B^2(t\wedge T_{a,b}) - (t\wedge T_{a,b})\right] \\ &= \mathbb{E}\left[B^2(T_{a,b})\mathbb{1}_{\{T_{a,b}\le t\}}\right] + \mathbb{E}\left[B^2(t)\mathbb{1}_{\{T_{a,b}>t\}}\right] - \mathbb{E}\left[t\wedge T_{a,b}\right] \end{aligned} \tag{4.2}$$

The second term

$$\begin{aligned} \mathbb{E}\left[B^2(t)\mathbb{1}_{\{T_{a,b}>t\}}\right] &\le \mathbb{E}\left[a^2\vee b^2\mathbb{1}_{\{T_{a,b}>t\}}\right] \\ &= (a^2\vee b^2)\cdot\mathbb{E}\left[\mathbb{1}_{\{T_{a,b}>t\}}\right] \\ \text{(dominated convergence)} &\to 0, \quad \text{as } t\to\infty. \end{aligned}$$

On the other hand, $B^2(T_{a,b})\mathbb{1}_{\{T_{a,b}\le t\}} \nearrow B^2(T_{a,b})$ and $t\wedge T_{a,b}\nearrow T_{a,b}$ almost surely as $t\to\infty$. By monotone convergence theorem,

$$\mathbb{E}\left[B^2(T_{a,b})\mathbb{1}_{\{T_{a,b}\le t\}}\right]\nearrow\mathbb{E}\left[B^2(T_{a,b})\right], \quad \mathbb{E}\left[t\wedge T_{a,b}\right]\nearrow\mathbb{E}\left[T_{a,b}\right]$$

as $t\to\infty$. Thus, by taking the limit $t\to\infty$,

$$\begin{aligned} \mathbb{E}[T_{a,b}] &= \mathbb{E}\left[B^2(T_{a,b})\right] \\ &= a^2\mathbb{P}(B(T_{a,b})=a) + b^2\mathbb{P}(B(T_{a,b})=b) \\ \text{(results from part } \mathbf{(a)}\text{)} &= a^2\frac{b}{|a|+b} + b^2\frac{|a|}{|a|+b} \\ &= |a|b. \end{aligned}$$

# 5 PROBLEM FIVE

**Theorem 3.3** in Lecture 2. Let $B(t)_{t\geq 0}$ be a Brownian motion and $\tau_a := \inf\{t \geq 0 | B(t) = a\}$, then $\mathbb{E}[e^{-\lambda\tau_a}] = e^{-|a|\sqrt{2\lambda}}, \forall \lambda > 0$. This is the same result given in textbook **Theorem 3.6.2**, which provides the Laplace transform of the density of the first passage time for Brownian motion.

Textbook **Exercise 3.7** and **Theorem 8.3.2** derives the analogous formula for Brownian motions with drift. The same result is given in **Theorem 5.1** in Lecture 2.

Let $W(t)$ be a Brownian motion with filtration $\mathscr{F}(t)$. Fix $m > 0$ and $\mu \in \mathbb{R}$. For $0 \leq t < \infty$, define

$$\begin{aligned} X(t) &= \mu t + W(t), \\ \tau_m &= \min\{t \geq 0 | X(t) = m\}. \end{aligned}$$

As usual, we set $\tau_m = \infty$ if $X(t)$ never reaches the level $m$. Let $\sigma$ be a positive number and set

$$Z(t) = \exp\left\{\sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right) t\right\}.$$

**(i)** Show that $Z(t)_{t\geq 0}$ is a martingale.
*Proof*: This is done in a similar way as the proof of Textbook **Theorem 3.6.1**. Firstly,

$$Z(t) = \exp\left\{\sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right) t\right\} = \exp\left\{\sigma W(t) - \frac{1}{2}\sigma^2 t\right\}$$

is $\mathscr{F}(t)$-measurable because $W(t)$ is $\mathscr{F}(t)$-measurable. From the tail behavior of Brownian motion, we know that all moments of $Z(t)$ are finite. Thus, $\mathbb{E}[Z(t)] < \infty$. For all $0 \leq s < t$, we have

$$\begin{aligned} \mathbb{E}[Z(t)|\mathscr{F}(s)] &= \mathbb{E}\left[\exp\left\{\sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right) t\right\}\Big|\mathscr{F}(s)\right] \\ &= \mathbb{E}\left[\exp\left\{\sigma\left(W(t) + \mu t\right) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right) t\right\}\Big|\mathscr{F}(s)\right] \\ &= \mathbb{E}\left[\exp\left\{\sigma W(t) - \frac{1}{2}\sigma^2 t\right\}\Big|\mathscr{F}(s)\right] \\ &= \mathbb{E}\left[\exp\left\{\sigma\left(W(t) - W(s)\right)\right\}\exp\left\{\sigma W(s) - \frac{1}{2}\sigma^2 t\right\}\Big|\mathscr{F}(s)\right] \\ \text{(taking out what is known)} &= \exp\left\{\sigma W(s) - \frac{1}{2}\sigma^2 t\right\}\mathbb{E}\left[\exp\left\{\sigma\left(W(t) - W(s)\right)\right\}\Big|\mathscr{F}(s)\right] \\ \text{(independence)} &= \exp\left\{\sigma W(s) - \frac{1}{2}\sigma^2 t\right\}\mathbb{E}\left[\exp\left\{\sigma\left(W(t) - W(s)\right)\right\}\right] \\ \text{(normality of } W(t) - W(s)) &= \exp\left\{\sigma W(s) - \frac{1}{2}\sigma^2 t\right\}\exp\left\{\frac{1}{2}\sigma^2(t-s)\right\} \\ &= \exp\left\{\sigma W(s) - \frac{1}{2}\sigma^2 s\right\} \end{aligned}$$

$$
\begin{aligned}
&= \exp\left\{\sigma X(s) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)s\right\} \\
&= Z(s).
\end{aligned}
$$

Thus, we proved that $Z(t)$ is a martingale.

**(ii)** Use **(i)** to conclude that

$$
\mathbb{E}\left[\exp\left\{\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}\right] = 1, \quad t \geq 0.
$$

*Proof*: According to **Theorem 2.5** in Refresher Lecture 5, the martingale $Z(t)$ that is stopped (or "frozen") at a stopping time $Z(t\wedge\tau_m)$ is still a martingale. Furthermore, for any stopping time $\tau$, the time $\tau\wedge t \leq t$ is a bounded stopping time for any $t$, and **Theorem 2.4** (Optional Stopping Theorem) in Refresher Lecture 5 applies

$$
\mathbb{E}\left[Z(t\wedge\tau_m)\right] = \mathbb{E}\left[Z(0\wedge\tau_m)\right] = \mathbb{E}\left[Z(0\right] = 1,
$$

in other words,

$$
\mathbb{E}\left[\exp\left\{\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}\right] = 1, \quad t \geq 0.
$$

**(iii)** Now suppose $\mu \geq 0$. Show that, for $\sigma > 0$,

$$
\mathbb{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathbb{1}_{\{\tau_m<\infty\}}\right] = 1, \quad t \geq 0.
$$

Use this fact to show $\mathbb{P}(\{\tau_m < \infty\}) = 1$ and obtain the Laplace transform

$$
\mathbb{E}\left[e^{-\alpha\tau_m}\right] = e^{m\mu - m\sqrt{2\alpha+\mu^2}}, \quad \forall\alpha > 0.
$$

*Proof*: Use the result obtained in **(ii)**,

$$
\begin{aligned}
1 &= \mathbb{E}\left[\exp\left\{\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}\mathbb{1}_{\{\tau_m<t\}}\right] \\
&\quad + \mathbb{E}\left[\exp\left\{\sigma X(t\wedge\tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t\wedge\tau_m)\right\}\mathbb{1}_{\{\tau_m\geq t\}}\right] \\
&= \mathbb{E}\left[\exp\left\{\sigma \underbrace{X(\tau_m)}_{\triangleq m} - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathbb{1}_{\{\tau_m<t\}}\right] + \mathbb{E}\left[\exp\left\{\sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)t\right\}\mathbb{1}_{\{\tau_m\geq t\}}\right] \\
&= \mathbb{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathbb{1}_{\{\tau_m<t\}}\right] + \mathbb{E}\left[\exp\left\{\sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)t\right\}\mathbb{1}_{\{\tau_m\geq t\}}\right] \quad (5.1)
\end{aligned}
$$

where, because $\mu \geq 0$,

$$\exp\left\{\sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)t\right\}\mathbb{1}_{\{\tau_m \geq t\}} \leq e^{\sigma X(t)}\mathbb{1}_{\{\tau_m \geq t\}} \leq e^{\sigma|m|}\mathbb{1}_{\{\tau_m \geq t\}} \xrightarrow{t\to\infty} 0, \quad \text{almost surely.}$$

By dominated convergence theorem,

$$\mathbb{E}\left[\exp\left\{\sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)t\right\}\mathbb{1}_{\{\tau_m \geq t\}}\right] \xrightarrow{t\to\infty} 0.$$

Thus, by taking the limit $t \to \infty$ in Equation 5.1,

$$\mathbb{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathbb{1}_{\{\tau_m < \infty\}}\right] = 1,$$

or equivalently,

$$\mathbb{E}\left[\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathbb{1}_{\{\tau_m < \infty\}}\right] = e^{-\sigma m}. \tag{5.2}$$

Because $\exp\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\}$ increase monotonically as $\sigma \searrow 0$, by monotone convergence theorem, the above expression yields

$$\mathbb{P}(\{\tau_m < \infty\}) \triangleq \mathbb{E}\left[\mathbb{1}_{\{\tau_m < \infty\}}\right] = 1.$$

Denote $\alpha = \sigma\mu + \frac{1}{2}\sigma^2$, we write $\sigma$ in terms of $\alpha$ and $\mu$

$$\sigma = -\mu + \sqrt{\mu^2 + 2\alpha},$$

we obtain the Laplace transform from Equation 5.2

$$\mathbb{E}\left[e^{-\alpha\tau_m}\mathbb{1}_{\{\tau_m < \infty\}}\right] = e^{m\mu - m\sqrt{\mu^2+2\alpha}}.$$

Because $\tau_m$ is finite with probability one $\mathbb{P}(\{\tau_m < \infty\}) = 1$, we may drop the indicator in the above expression to obtain

$$\mathbb{E}\left[e^{-\alpha\tau_m}\right] = e^{m\mu - m\sqrt{\mu^2+2\alpha}}. \tag{5.3}$$

**(iv)** Show that if $\mu > 0$, then $\mathbb{E}[\tau_m] < \infty$. Obtain a formula for $\mathbb{E}[\tau_m]$.
*Solution*: Differentiate Equation 5.3 with respect to $\alpha$

$$\begin{aligned} \frac{\partial}{\partial\alpha}\mathbb{E}\left[e^{-\alpha\tau_m}\right] &= -\mathbb{E}\left[\tau_m e^{-\alpha\tau_m}\right] \\ &= -\frac{m}{\sqrt{\mu^2+2\alpha}}e^{m\mu - m\sqrt{\mu^2+2\alpha}}. \end{aligned}$$

Taking the limit $\alpha \to 0$, we obtain

$$\mathbb{E}[\tau_m] = \frac{m}{\mu} < \infty.$$

**(v)** Now suppose $\mu < 0$. Show that, for $\sigma > -2\mu$,

$$\mathbb{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathbb{1}_{\{\tau_m<\infty\}}\right] = 1, \quad t \geq 0.$$

Use this fact to show that $\mathbb{P}(\{\tau_m < \infty\}) = e^{-2m|\mu|}$, which is strictly less than one, and to obtain the Laplace transform

$$\mathbb{E}\left[e^{-\alpha\tau_m}\right] = e^{m\mu - m\sqrt{2\alpha+\mu^2}}, \quad \forall\alpha > 0.$$

*Proof*: Again, we follow the same derivation that led us to Equation 5.1 in part **(iii)**, for which we need to show the vanishing of the second term upon $t \to \infty$. Because $\sigma \geq -2\mu$, we have $e^{-\frac{1}{2}\sigma(2\mu+\sigma)t} \leq 1$ and

$$\begin{aligned}
\exp\left\{\sigma X(t) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)t\right\}\mathbb{1}_{\{\tau_m\geq t\}} &= e^{\sigma X(t)}\exp\left\{-\frac{1}{2}\sigma\left(2\mu+\sigma\right)t\right\}\mathbb{1}_{\{\tau_m\geq t\}} \\
&< e^{\sigma X(t)}\mathbb{1}_{\{\tau_m\geq t\}} \\
&\leq e^{\sigma|m|}\mathbb{1}_{\{\tau_m\geq t\}} \\
&\xrightarrow{t\to\infty} 0, \quad \text{almost surely.}
\end{aligned}$$

Thus, the same derivation leading to Equation 5.2 holds valid and we arrived at the same expression

$$\mathbb{E}\left[\exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathbb{1}_{\{\tau_m<\infty\}}\right] = 1, \quad t \geq 0.$$

or equivalently,

$$\mathbb{E}\left[\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\mathbb{1}_{\{\tau_m<\infty\}}\right] = e^{-\sigma m}. \tag{5.4}$$

Now given the constraint $\sigma \geq -2\mu$, we cannot take the limit $\sigma \searrow 0$ to compute $\mathbb{P}(\{\tau_m < \infty\})$ as we did in part **(iii)**. Instead, we let $\sigma = -2\mu = 2|\mu|$ and obtain

$$\mathbb{E}\left[e^{2m|\mu|}\mathbb{1}_{\{\tau_m<\infty\}}\right] = 1 \Rightarrow \mathbb{P}(\{\tau_m < \infty\}) \triangleq \mathbb{E}\left[\mathbb{1}_{\{\tau_m<\infty\}}\right] = e^{-2m|\mu|} < 1.$$

The same derivation leading to Equation 5.3 gives the same formula for Laplace transform the above expression to obtain

$$\mathbb{E}\left[e^{-\alpha\tau_m}\right] = e^{m\mu - m\sqrt{\mu^2+2\alpha}}.$$

# 6 Problem Six

Use the stochastic representation to find the solution of the backward heat equation with the following terminal function:

**(a)** $f(x) = x^2$.
*Solution*:

$$
\begin{aligned}
v(t,x) &= \int_{\mathbb{R}} f(y)p(T-t,x,y)dy \\
&= \int_{-\infty}^{+\infty} y^2 \frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{(y-x)^2}{2(T-t)}\right) dy \\
&= \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} (y+x)^2 \exp\left(-\frac{y^2}{2(T-t)}\right) dy \\
&= \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} y^2 \exp\left(-\frac{y^2}{2(T-t)}\right) dy \\
&\quad + \frac{2x}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} y \exp\left(-\frac{y^2}{2(T-t)}\right) dy \\
&\quad + \frac{x^2}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2(T-t)}\right) dy \\
&= \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} y^2 \exp\left(-\frac{y^2}{2(T-t)}\right) dy + 0 + x^2 \\
&= (T-t) + x^2.
\end{aligned}
$$

Verification, $v_t = -1$ and $v_{xx} = 2$, thus $v_t + \frac{1}{2} v_{xx} = 0$.

**(b)** $f(x) = e^x$.
*Solution*:

$$
\begin{aligned}
v(t,x) &= \int_{\mathbb{R}} f(y)p(T-t,x,y)dy \\
&= \int_{-\infty}^{+\infty} e^y \frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{(y-x)^2}{2(T-t)}\right) dy \\
&= \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(y-x)^2}{2(T-t)} + y\right) dy \\
&= \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} \exp\left(-\frac{[y-(T-t+x)]^2}{2(T-t)} + \frac{T-t+2x}{2}\right) dy \\
&= \frac{\exp\left(\frac{T-t+2x}{2}\right)}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} \exp\left(-\frac{[y-(T-t+x)]^2}{2(T-t)}\right) dy \\
&= \exp\left(\frac{T-t}{2} + x\right).
\end{aligned}
$$

Verification: $v_t = -\frac{1}{2} v$, and $v_{xx} = v$, thus $v_t + \frac{1}{2} v_{xx} = 0$.

**(c)** $f(x) = e^{-x^2}$.
*Solution*:

$$\begin{aligned}
v(t,x) &= \int_{\mathbb{R}} f(y)p(T-t,x,y)dy \\
&= \int_{-\infty}^{+\infty} e^{-y^2} \frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{(y-x)^2}{2(T-t)}\right) dy \\
&= \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(y-x)^2}{2(T-t)} - y^2\right) dy \\
&= \frac{\exp\left(-\frac{1}{1+2(T-t)}x^2\right)}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} \exp\left[-\left(1+\frac{1}{2(T-t)}\right)\left(y-\frac{x}{1+2(T-t)}\right)^2\right] dy \\
&= \frac{\exp\left(-\frac{1}{1+2(T-t)}x^2\right)}{\sqrt{2\pi(T-t)}} \sqrt{\frac{\pi}{1+\frac{1}{2(T-t)}}} \\
&= \frac{1}{\sqrt{1+2(T-t)}} \exp\left(-\frac{1}{1+2(T-t)}x^2\right).
\end{aligned}$$

# 7 PROBLEM SEVEN

Compute the price of the option which pays \$1 at the time the stock price $S(t)$ hits a given level $A < S(0)$, assuming the interest rate $r \in [0, \sigma^2/2]$. Under the risk-neutral measure, the stock price follows the GBM $S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma B(t)}$.

*Solution*: Risk-neutral valuation dictates, using **Theorem 5.1** in Lecture 2,

$$V(t=0) = \tilde{\mathbb{E}}\left[e^{-r\tau_{a,b}}\mathbb{1}_{\tau_{a,b}<\infty}\right] = e^{-a(b+\sqrt{b^2+2r})},$$

where the constants $a$ and $b$ are determined by the following condition

$$\begin{aligned}
& e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma B(t)} = \frac{A}{S(0)} \\
\Rightarrow\quad & e^{-\left(r-\frac{1}{2}\sigma^2\right)t+\sigma(-B(t))} = \frac{S(0)}{A} \\
\Rightarrow\quad & -\left(r-\frac{1}{2}\sigma^2\right)t+\sigma(-B(t)) = \log\left(\frac{S(0)}{A}\right) \\
\Rightarrow\quad & -B(t) = \underbrace{\frac{1}{\sigma}\log\left(\frac{S(0)}{A}\right)}_{\triangleq a} + \underbrace{\frac{1}{\sigma}\left(r-\frac{1}{2}\sigma^2\right)}_{\triangleq b} t,
\end{aligned}$$

where we converted the original scenario of a GBM $S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma B(t)}$ hitting a level $A < S(0)$ to a drifted Brownian motion hitting a positive level $-bt - B(t) = a > 0$,

$$a \triangleq \frac{1}{\sigma}\log\left(\frac{S(0)}{A}\right), \quad b = \frac{r-\frac{1}{2}\sigma^2}{\sigma},$$

for which **Theorem 5.1** applies.

BARUCH, MFE

# MTH 9831 Assignment Three

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September 23, 2015

# 1 PROBLEM ONE: IMITATION OF TECHNIQUE

Let $B(t)_{t\geq 0}$ be a standard Brownian motion. Show that the cross-variation of $B(t)$ and $t$ on $[0,T]$ is equal to zero. Interpret the limit in the $L^2$ sense. This exercise gives a meaning to the informal expression $dB(t)dt = 0$.

*Proof*: The cross-variation of $B(t)$ and $t$ on $[0,T]$ is defined by

$$[B,t]_T \triangleq \lim_{\|\Pi\|\to 0}\sum_{i=1}^{n}\left(B(t_i)-B(t_{i-1})\right)(t_i-t_{i-1}) = \lim_{\|\Pi\|\to 0}\sum_{i=1}^{n}\delta_i,$$

where $\|\Pi\|\triangleq \max_{i\in\{1,\cdots,n\}}(t_i-t_{i-1})$ is the length of the largest interval in a partition of $[0,T]$ : $0=t_0<t_1<\cdots<t_n=T$, and $\delta_i\triangleq\left(B(t_i)-B(t_{i-1})\right)(t_i-t_{i-1})$. To evaluate this cross-variation in the $L^2$ sense, compute

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=1}^{n}\delta_i\right)^2\right] &= \mathbb{E}\left[\sum_{i=1}^{n}\delta_i^2\right]+\mathbb{E}\left[2\sum_{i<j}\delta_i\delta_j\right]\\
&= \sum_{i=1}^{n}\mathbb{E}\left[\delta_i^2\right]+2\sum_{i<j}\mathbb{E}\left[\delta_i\delta_j\right]\\
&= \underbrace{\sum_{i=1}^{n}\mathbb{E}\left[\delta_i^2\right]}_{\text{Term I}}+\underbrace{2\sum_{i<j}\mathbb{E}\left[\delta_i\right]\mathbb{E}\left[\delta_j\right]}_{\text{Term II}} \quad \text{(independent increments).}
\end{aligned}$$

**Term I**:

$$\begin{aligned}
\sum_{i=1}^{n}\mathbb{E}\left[\delta_i^2\right] &= \sum_{i=1}^{n}(t_i-t_{i-1})^2\,\mathbb{E}\left[\left(B(t_i)-B(t_{i-1})\right)^2\right]\\
&= \sum_{i=1}^{n}(t_i-t_{i-1})^2\,\mathrm{var}\left(B(t_i)-B(t_{i-1})\right) \quad \text{(zero mean)}\\
&= \sum_{i=1}^{n}(t_i-t_{i-1})^3 \quad \text{(normality).}\\
&\leq \|\Pi\|^2\sum_{i=1}^{n}(t_i-t_{i-1})\\
&\leq \|\Pi\|^2\,T\\
&\to 0, \quad \text{as } \|\Pi\|\to 0.
\end{aligned}$$

**Term II**: $\mathbb{E}[\delta_i]=(t_i-t_{i-1})\,\mathbb{E}[B(t_i)-B(t_{i-1})]=(t_i-t_{i-1})\left(\mathbb{E}[B(t_i)]-\mathbb{E}[B(t_{i-1})]\right)=0, \forall i$. Thus,

$$\text{Term II} = 2\sum_{i<j}\mathbb{E}[\delta_i]\,\mathbb{E}\left[\delta_j\right]=0.$$

Thus, $\sum_{i=1}^{n}\delta_i$ converges to zero in $L^2$:

$$\lim_{\|\Pi\|\to 0}\mathbb{E}\left[\left(\sum_{i=1}^{n}\delta_i-0\right)^2\right]=\lim_{\|\Pi\|\to 0}\mathbb{E}\left[\left(\sum_{i=1}^{n}\delta_i\right)^2\right]=0.$$

In other words, the cross-variation $[B,t]_T\triangleq\lim_{\|\Pi\|\to 0}\sum_{i=1}^{n}\delta_i$ is zero in the $L^2$ sense. In short hand, we write $dB(t)dt=0$.

## 2 Problem Two: Difference between Ito and Stratonovich

Show that if we let $t_i^* = (t_i + t_{i-1})/2$ in

$$\int_0^T f(t,\omega)dB(t) \overset{L^2}{=} \lim_{\|\Pi\|\to 0} \sum_{i=1}^n f(t_i^*)\left(B(t_i) - B(t_{i-1})\right)$$

and understand the limit in the mean square sense then

$$\int_0^T B(t)dB(t) = \frac{1}{2}B^2(T).$$

The above equality can be written in the differential form as follows:

$$d\left(\frac{1}{2}B^2(t)\right) = B(t)dB(t).$$

*Proof*: We compute the Stratonovich integral

$$\int_0^T B(t)\circ dB(t) \overset{L^2}{=} \lim_{\|\Pi\|\to 0} \sum_{i=1}^n B\left(t_i^*\right)\left(B(t_i) - B(t_{i-1})\right) = \lim_{\|\Pi\|\to 0} \sum_{i=1}^n B\left(\frac{t_i + t_{i-1}}{2}\right)\left(B(t_i) - B(t_{i-1})\right).$$

*Method One*:

$$\begin{aligned}
&\int_0^T B(t)\circ dB(t)\\
\overset{L^2}{=}\;& \lim_{\|\Pi\|\to 0} \sum_{i=1}^n B\left(\frac{t_i + t_{i-1}}{2}\right)\left(B(t_i) - B(t_{i-1})\right)\\
=\;& \lim_{\|\Pi\|\to 0} \sum_{i=1}^n B\left(\frac{t_i + t_{i-1}}{2}\right)\left(B(t_i) - B\left(\frac{t_i + t_{i-1}}{2}\right) + B\left(\frac{t_i + t_{i-1}}{2}\right) - B(t_{i-1})\right)\\
=\;& \lim_{\|\Pi\|\to 0} \sum_{i=1}^n B\left(\frac{t_i + t_{i-1}}{2}\right)\left(B(t_i) - B\left(\frac{t_i + t_{i-1}}{2}\right)\right) + \lim_{\|\Pi\|\to 0} \sum_{i=1}^n B\left(\frac{t_i + t_{i-1}}{2}\right)\left(B\left(\frac{t_i + t_{i-1}}{2}\right) - B(t_{i-1})\right)\\
=\;& \lim_{\|\Pi\|\to 0} \sum_{i=1}^n B\left(\frac{t_i + t_{i-1}}{2}\right)\left(B(t_i) - B\left(\frac{t_i + t_{i-1}}{2}\right)\right) + \lim_{\|\Pi\|\to 0} \sum_{i=1}^n B\left(t_{i-1}\right)\left(B\left(\frac{t_i + t_{i-1}}{2}\right) - B(t_{i-1})\right)\\
&+ \lim_{\|\Pi\|\to 0} \sum_{i=1}^n \left(B\left(\frac{t_i + t_{i-1}}{2}\right) - B(t_{i-1})\right)^2\\
=\;& \underbrace{\lim_{\|\Pi\|\to 0} \sum_{i=1}^n \left[B\left(\frac{t_i + t_{i-1}}{2}\right)\left(B(t_i) - B\left(\frac{t_i + t_{i-1}}{2}\right)\right) + B\left(t_{i-1}\right)\left(B\left(\frac{t_i + t_{i-1}}{2}\right) - B(t_{i-1})\right)\right]}_{\text{Ito integral evaluated on partition } 0=t_0<\frac{1}{2}(t_0+t_1)<t_1<\frac{1}{2}(t_1+t_2)<t_2<\cdots<\frac{1}{2}(t_{n-1}+t_n)<t_n=T}\\
&+ \lim_{\|\Pi\|\to 0} \sum_{i=1}^n \left(B\left(\frac{t_i + t_{i-1}}{2}\right) - B(t_{i-1})\right)^2\\
=\;& \frac{1}{2}\left(B^2(T) - T\right) + \lim_{\|\Pi\|\to 0} \sum_{i=1}^n \left(B\left(\frac{t_i + t_{i-1}}{2}\right) - B(t_{i-1})\right)^2.
\end{aligned} \tag{2.1}$$

To evaluate $\lim_{\|\Pi\|\to 0}\sum_{i=1}^{n}\left(B\left(\frac{t_i+t_{i-1}}{2}\right)-B(t_{i-1})\right)^2$, we follow the same derivation that leads to $[B]_T = T$. Denote

$$\delta_i = \left(B\left(\frac{t_i+t_{i-1}}{2}\right)-B(t_{i-1})\right)^2-\left(\frac{t_i+t_{i-1}}{2}-t_{i-1}\right)=\left(B\left(\frac{t_i+t_{i-1}}{2}\right)-B(t_{i-1})\right)^2-\left(\frac{t_i-t_{i-1}}{2}\right),$$

then

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=1}^{n}\left(B\left(\frac{t_i+t_{i-1}}{2}\right)-B(t_{i-1})\right)^2-\frac{T}{2}\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{n}\left(B\left(\frac{t_i+t_{i-1}}{2}\right)-B(t_{i-1})\right)^2-\sum_{i=1}^{n}\left(\frac{t_i-t_{i-1}}{2}\right)\right)^2\right]\\
&= \mathbb{E}\left[\left(\sum_{i=1}^{n}\delta_i\right)^2\right]\\
&= \mathbb{E}\left[\sum_{i=1}^{n}\delta_i^2\right]+\mathbb{E}\left[2\sum_{i<j}\delta_i\delta_j\right]\\
&= \sum_{i=1}^{n}\mathbb{E}\left[\delta_i^2\right]+2\sum_{i<j}\mathbb{E}\left[\delta_i\delta_j\right]\\
&= \underbrace{\sum_{i=1}^{n}\mathbb{E}\left[\delta_i^2\right]}_{\text{Term I}}+\underbrace{2\sum_{i<j}\mathbb{E}\left[\delta_i\right]\mathbb{E}\left[\delta_j\right]}_{\text{Term II}} \quad \text{(independent increments).}
\end{aligned}$$

**Term I**:

$$\begin{aligned}
\mathbb{E}\left[\delta_i^2\right] &= \mathbb{E}\left[\left(B\left(\frac{t_i+t_{i-1}}{2}\right)-B(t_{i-1})\right)^4\right]-(t_i-t_{i-1})\mathbb{E}\left[\left(B\left(\frac{t_i+t_{i-1}}{2}\right)-B(t_{i-1})\right)^2\right]+\frac{(t_i-t_{i-1})^2}{4}\\
&= 3\times\frac{(t_i-t_{i-1})^2}{4}-\frac{1}{2}(t_i-t_{i-1})^2+\frac{(t_i-t_{i-1})^2}{4}\\
&= \frac{1}{2}(t_i-t_{i-1})^2.
\end{aligned}$$

Thus,

$$\sum_{i=1}^{n}\mathbb{E}\left[\delta_i^2\right]=\sum_{i=1}^{n}\frac{1}{2}(t_i-t_{i-1})^2\le\frac{1}{2}\parallel\Pi\parallel\sum_{i=1}^{n}(t_i-t_{i-1})=\frac{1}{2}\parallel\Pi\parallel T\to 0,\quad \text{as } \parallel\Pi\parallel\to 0.$$

**Term II**:

$$\mathbb{E}\left[\delta_j\right]=\mathbb{E}\left[\left(B\left(\frac{t_i+t_{i-1}}{2}\right)-B(t_{i-1})\right)^2\right]-\left(\frac{t_i-t_{i-1}}{2}\right)=\left(\frac{t_i-t_{i-1}}{2}\right)-\left(\frac{t_i-t_{i-1}}{2}\right)=0.$$

Thus, $2\sum_{i<j}\mathbb{E}[\delta_i]\mathbb{E}\left[\delta_j\right]=0$. In summary, $\sum_{i=1}^{n}\left(B\left(\frac{t_i+t_{i-1}}{2}\right)-B(t_{i-1})\right)^2$ converges to $\frac{T}{2}$ in $L^2$

$$\mathbb{E}\left[\left(\sum_{i=1}^{n}\left(B\left(\frac{t_i+t_{i-1}}{2}\right)-B(t_{i-1})\right)^2-\frac{T}{2}\right)^2\right]=0.$$

In other words,

$$\lim_{\|\Pi\|\to 0}\sum_{i=1}^{n}\left(B\left(\frac{t_i+t_{i-1}}{2}\right)-B(t_{i-1})\right)^2=\frac{T}{2}$$

in the $L^2$ sense. From Equation 2.1,

$$\int_0^T B(t) \circ dB(t) = \frac{1}{2} B^2(T). \tag{2.2}$$

In short hand, we write

$$d\left(\frac{1}{2} B^2(t)\right) = B(t) \circ dB(t).$$

*Method Two*:

$$\sum_{i=1}^{n} B\left(t_i^*\right)(B(t_i) - B(t_{i-1})) = \sum_{i=1}^{n} \left[B\left(t_i^*\right)\left(B(t_i^*) - B(t_{i-1})\right) + B\left(t_i^*\right)\left(B(t_i) - B(t_i^*)\right)\right].$$

Apply the following identities to the above two terms:

$$\begin{aligned} a(a-b) &= \frac{1}{2}\left(a^2 - b^2 + (a-b)^2\right), \\ b(a-b) &= \frac{1}{2}\left(a^2 - b^2 - (a-b)^2\right), \end{aligned}$$

we get

$$\begin{aligned} B\left(t_i^*\right)\left(B(t_i^*) - B(t_{i-1})\right) &= \frac{1}{2}\left[B^2\left(t_i^*\right) - B^2(t_{i-1}) + \left(B(t_i^*) - B(t_{i-1})\right)^2\right], \\ B\left(t_i^*\right)\left(B(t_i) - B(t_i^*)\right) &= \frac{1}{2}\left[B^2(t_i) - B^2(t_i^*) - \left(B(t_i) - B(t_i^*)\right)^2\right]. \end{aligned}$$

Thus,

$$\begin{aligned} B\left(t_i^*\right)(B(t_i) - B(t_{i-1})) &= B\left(t_i^*\right)\left(B(t_i^*) - B(t_{i-1})\right) + B\left(t_i^*\right)\left(B(t_i) - B(t_i^*)\right) \\ &= \frac{1}{2}\left[B^2(t_i) - B^2(t_{i-1}) + \left(B(t_i^*) - B(t_{i-1})\right)^2 - \left(B(t_i) - B(t_i^*)\right)^2\right], \end{aligned}$$

or

$$B\left(t_i^*\right)(B(t_i) - B(t_{i-1})) - \frac{1}{2}\left[B^2(t_i) - B^2(t_{i-1})\right] = \frac{1}{2}\left[\left(B(t_i^*) - B(t_{i-1})\right)^2 - \left(B(t_i) - B(t_i^*)\right)^2\right].$$

Notice that

$$\begin{aligned} \sum_{i=1}^{n} B\left(t_i^*\right)(B(t_i) - B(t_{i-1})) - \frac{1}{2} B^2(T) &= \sum_{i=1}^{n} \left\{B\left(t_i^*\right)(B(t_i) - B(t_{i-1})) - \frac{1}{2}\left[B^2(t_i) - B^2(t_{i-1})\right]\right\} \\ &= \frac{1}{2} \sum_{i=1}^{n} \left[\left(B(t_i^*) - B(t_{i-1})\right)^2 - \left(B(t_i) - B(t_i^*)\right)^2\right] \\ &\triangleq \frac{1}{2} \sum_{i=1}^{n} \left(\alpha_i^2 - \beta_i^2\right) \end{aligned} \tag{2.3}$$

Notice that $\alpha_i \triangleq B(t_i^*) - B(t_{i-1})$ and $\beta_i \triangleq B(t_i) - B(t_i^*)$ are independent as their time intervals do not overlap and both have distribution $N\left(0, \frac{t_i - t_{i-1}}{2}\right)$. Also, $\alpha_i \perp \alpha_j$, $\beta_i \perp \beta_j$ for $i \neq j$; $\alpha_i \perp \beta_j, \forall i, j$. In the $L^2$ sense,

$$
\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=1}^{n}\left(\alpha_i^2 - \beta_i^2\right)\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{n}\left(\alpha_i^2 - \beta_i^2\right)\right)\left(\sum_{j=1}^{n}\left(\alpha_j^2 - \beta_j^2\right)\right)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\alpha_i^2 - \beta_i^2\right)\left(\alpha_j^2 - \beta_j^2\right)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\alpha_i^2\alpha_j^2 - \alpha_i^2\beta_j^2 - \alpha_j^2\beta_i^2 + \beta_i^2\beta_j^2\right)\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\alpha_i^2\alpha_j^2 - \alpha_i^2\beta_j^2 - \alpha_j^2\beta_i^2 + \beta_i^2\beta_j^2\right] \\
&= \sum_{i=1}^{n}\mathbb{E}\left[\alpha_i^4 - \alpha_i^2\beta_i^2 - \alpha_i^2\beta_i^2 + \beta_i^4\right] + \sum_{i\neq j}\underbrace{\mathbb{E}\left[\alpha_i^2\alpha_j^2 - \alpha_i^2\beta_j^2 - \alpha_j^2\beta_i^2 + \beta_i^2\beta_j^2\right]}_{=0,\text{ because of independence.}} \\
&= \sum_{i=1}^{n}\left(\frac{3}{4}(t_i - t_{i-1})^2 - \frac{1}{4}(t_i - t_{i-1})^2 - \frac{1}{4}(t_i - t_{i-1})^2 + \frac{3}{4}(t_i - t_{i-1})^2\right) \\
&= \sum_{i=1}^{n}(t_i - t_{i-1})^2 \\
&\leq \|\Pi\|\sum_{i=1}^{n}(t_i - t_{i-1}) \\
&= \|\Pi\|\, T \\
&\to 0, \quad \text{as } \|\Pi\| \to 0.
\end{aligned}
$$

Combined with Equation 2.3, this leads us to the same conclusion Equation 2.2.

# 3 Problem Three: Stochastic Integral is Martingale

Let $\mathscr{F}(t)_{t\geq 0}$ be a filtration for Brownian motion $B(t)_{t\geq 0}$. Fix a partition of $\Pi$ of $[0,T]$, $0=t_0<t_1<\cdots<t_n=T$, and suppose that

$$\Delta(t,\omega)=\sum_{j=1}^{n}\Delta_{j-1}(\omega)\mathbb{1}_{[t_{j-1},t_j)}(t),$$

where $\Delta_j$ is $\mathscr{F}_j$-measurable square integrable random variable. Define $I(0)\triangleq 0$,,

$$I(t)\triangleq\sum_{j=1}^{k}\Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right)+\Delta_k\left(B(t)-B(t_k)\right),\quad t\in[t_{j-1},t_j).$$

Prove that the process $I(t)$, $0\leq t\leq T$ is an $\mathscr{F}(t)$-martingale. In particular, $\mathbb{E}[I(t)]=\mathbb{E}[I(0)]=0$.
*Proof*: Firstly, for all $t\in[0,T]$,

$$\begin{aligned}
\mathbb{E}[|I(t)|] &= \mathbb{E}\left[\sum_{j=1}^{k}\Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right)+\Delta_k\left(B(t)-B(t_k)\right)\right]\\
&= \sum_{j=1}^{k}\mathbb{E}\left[\Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right)\right]+\mathbb{E}\left[\Delta_k\left(B(t)-B(t_k)\right)\right]\\
&\leq \sum_{j=1}^{k}\mathbb{E}\left[|\Delta_{j-1}|\cdot|\left(B(t_j)-B(t_{j-1})\right)|\right]+\mathbb{E}\left[|\Delta_k|\cdot|\left(B(t)-B(t_k)\right)|\right]\\
&\leq \sum_{j=1}^{k}\sqrt{\mathbb{E}\left[\Delta_{j-1}^2\right]\cdot\mathbb{E}\left[\left(B(t_j)-B(t_{j-1})\right)^2\right]}+\sqrt{\mathbb{E}\left[\Delta_k^2\right]\cdot\mathbb{E}\left[\left(B(t)-B(t_k)\right)^2\right]}\\
&= \sum_{j=1}^{k}\sqrt{(t_j-t_{j-1})\mathbb{E}\left[\Delta_{j-1}^2\right]}+\sqrt{(t-t_k)\mathbb{E}\left[\Delta_k^2\right]}\\
&< \infty,\quad (\Delta_j \text{ is square-integrable}, \forall j).
\end{aligned}$$

Second, note that for $0=t_0<t_1<\cdots<t_k<t$, the Brownian motion at these time points $B(t_0),B(t_1),\cdots,B(t_k),B(t)$ are $\mathscr{F}(t)$-measurable. In addition, $\Delta_j$ is $\mathscr{F}_j$-measurable and $\mathscr{F}_j\subset\mathscr{F}(t),\forall j$ such that $t_j\leq t$. Thus, $\Delta_j$ is $\mathscr{F}(t)$-measurable, $\forall j$ such that $t_j\leq t$. Therefore, $I(t)$ is $\mathscr{F}(t)$-measurable, $t\in[0,T]$.

Next, for all $0\leq t'\leq t\leq T$, consider two cases:

- *Case 1*: there is at least one partition point between $t'$ and $t$;
- *Case 2*: there is no partition point between $t'$ and $t$.

We shall show that in both cases, $\mathbb{E}\left[I(t)|\mathscr{F}(t')\right]=I(t')$.

*Case 1*: Assume $0\leq t_i\leq t'\leq t_{i+1}\leq t_k\leq t\leq t_{k+1}\leq T$,

$$\mathbb{E}\left[I(t)|\mathscr{F}(t')\right] = \mathbb{E}\left[\sum_{j=1}^{k}\Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right)+\Delta_k\left(B(t)-B(t_k)\right)\middle|\mathscr{F}(t')\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\sum_{j=1}^{i} \Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right) + \sum_{j=i+1}^{k} \Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right) + \Delta_k\left(B(t)-B(t_k)\right)\middle|\mathscr{F}(t')\right] \\
&= \mathbb{E}\left[\sum_{j=1}^{i} \Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right)\middle|\mathscr{F}(t')\right] + \mathbb{E}\left[\Delta_i\left(B(t_{i+1})-B(t_i)\right)\middle|\mathscr{F}(t')\right] \\
&\quad + \mathbb{E}\left[\sum_{j=i+2}^{k} \Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right) + \Delta_k\left(B(t)-B(t_k)\right)\middle|\mathscr{F}(t')\right] \\
&= \sum_{j=1}^{i} \Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right) \quad \text{(taking out what is known)} \\
&\quad + \underbrace{\Delta_i\mathbb{E}\left[B(t_{i+1})-B(t_i)\middle|\mathscr{F}(t')\right]}_{=\Delta_i(\mathbb{E}[B(t_{i+1})|\mathscr{F}(t')]-B(t_i))=\Delta_i(B(t')-B(t_i))} \quad \text{(because } B(t) \text{ is martingale)} \\
&\quad + \underbrace{\mathbb{E}\left[\sum_{j=i+2}^{k} \Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right)\middle|\mathscr{F}(t')\right]}_{\sum_{j=i+2}^{k}\mathbb{E}\left[\Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right)\middle|\mathscr{F}(t')\right]} + \mathbb{E}\left[\Delta_k\left(B(t)-B(t_k)\right)\middle|\mathscr{F}(t')\right] \\
&= \sum_{j=1}^{i} \Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right) + \Delta_i\left(B(t')-B(t_i)\right) \\
&\quad + \mathbb{E}\left[\sum_{j=i+2}^{k} \mathbb{E}\left[\Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right)|\mathscr{F}_{j-1}\right]\middle|\mathscr{F}(t')\right] \quad \text{(tower property)} \\
&\quad + \mathbb{E}\left[\mathbb{E}\left[\Delta_k\left(B(t)-B(t_k)\right)|\mathscr{F}_k\right]\middle|\mathscr{F}(t')\right] \quad \text{(tower property)} \\
&= \sum_{j=1}^{i} \Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right) + \Delta_i\left(B(t')-B(t_i)\right) \\
&\quad + \mathbb{E}\left[\sum_{j=i+2}^{k} \Delta_{j-1}\mathbb{E}\left[B(t_j)-B(t_{j-1})|\mathscr{F}_{j-1}\right]\middle|\mathscr{F}(t')\right] \quad \text{(taking out what is known)} \\
&\quad + \mathbb{E}\left[\Delta_k\mathbb{E}\left[B(t)-B(t_k)|\mathscr{F}_k\right]\middle|\mathscr{F}(t')\right] \quad \text{(taking out what is known)} \\
&= \sum_{j=1}^{i} \Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right) + \Delta_i\left(B(t')-B(t_i)\right) \\
&\quad + \mathbb{E}\left[\sum_{j=i+2}^{k} \Delta_{j-1}(\mathbb{E}[B(t_j)|\mathscr{F}_{j-1}]-B(t_{j-1}))\middle|\mathscr{F}(t')\right] \\
&\quad + \mathbb{E}\left[\Delta_k(\mathbb{E}\left[B(t)|\mathscr{F}_k\right]-B(t_k)|\mathscr{F}(t')\right] \\
&= \sum_{j=1}^{i} \Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right) + \Delta_i\left(B(t')-B(t_i)\right) \\
&\quad + \mathbb{E}\left[\sum_{j=i+2}^{k} \Delta_{j-1}\underbrace{(B(t_{j-1})-B(t_{j-1}))}_{=0}\middle|\mathscr{F}(t')\right] + \mathbb{E}\left[\Delta_k\underbrace{(B(t_k)-B(t_k)}_{=0}|\mathscr{F}(t')\right]
\end{aligned}
$$

$$\begin{aligned} &= \sum_{j=1}^{i} \Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right)+\Delta_i\left(B(t')-B(t_i)\right) \\ &= I(t'). \end{aligned}$$

*Case 2*: Assume $0 \le t_i \le t' \le t \le t_{i+1} \le T$,

$$\begin{aligned} \mathbb{E}\left[I(t)|\mathscr{F}(t')\right] &= \mathbb{E}\left[\sum_{j=1}^{i} \Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right)+\Delta_i\left(B(t)-B(t_i)\right)\bigg|\mathscr{F}(t')\right] \\ &= \sum_{j=1}^{i} \Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right)+\Delta_i\left(\mathbb{E}\left[B(t)|\mathscr{F}(t')\right]-B(t_i)\right) \\ &= \sum_{j=1}^{i} \Delta_{j-1}\left(B(t_j)-B(t_{j-1})\right)+\Delta_i\left(B(t')-B(t_i)\right) \\ &= I(t'). \end{aligned}$$

We conclude that $I(t)$ is an $\mathscr{F}(t)$-martingale.

## 4 Problem Four: General Education Problem

Let $B(t)_{t\geq 0}$ be a standard Brownian motion. Show that $\mathbb{E}[B^{2n}(t)] = t^n(2n-1)!!$.
*Proof*: Recall **Theorem 2.1** in Lecture 2, for every non-zero $c \in \mathbb{R}$, the process $cB(c^{-2}t)_{t\geq 0}$ is a Brownian motion. In particular, let $c = \sqrt{\tau}$, $\sqrt{\tau}B(t/\tau)_{t\geq 0}$ is a Brownian motion. Therefore, for a fixed $t = \tau$,

$$\mathbb{E}[B^{2n}(\tau)] = \mathbb{E}\left[\tau^n B^{2n}(1)\right] = \tau^n \mathbb{E}\left[B^{2n}(1)\right],$$

where $B(1)$ is a standard normal random variable, for which the moment generating function is well-known

$$\mathbb{E}\left[e^{uB(1)}\right] = e^{\frac{1}{2}u^2}.$$

Differentiating with respect to $u$ for $2n$ times and set $u = 0$, we get

$$\begin{aligned}
\mathbb{E}\left[B^{2n}(1)\right] &= \mathbb{E}\left[B^{2n}(1)e^{uB(1)}\right]\Big|_{u=0} \\
&= \frac{\partial^{2n}}{\partial u^{2n}}\mathbb{E}\left[e^{uB(1)}\right]\Big|_{u=0} \\
&= \frac{\partial^{2n}}{\partial u^{2n}} e^{\frac{1}{2}u^2}\Big|_{u=0} \\
&= \frac{\partial^{2n}}{\partial u^{2n}} \sum_{k=0}^{\infty} \frac{u^{2k}}{2^k k!}\Big|_{u=0} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^k k!} \frac{\partial^{2n}}{\partial u^{2n}} u^{2k}\Big|_{u=0} \quad \text{(uniform convergence).}
\end{aligned}$$

The only surviving term in the above summation after the differentiation and setting $u = 0$ is the term in the order of $u^{2n}$. Thus,

$$\mathbb{E}\left[B^{2n}(1)\right] = \frac{1}{2^n n!}\frac{\partial^{2n}}{\partial u^{2n}} u^{2k}\Big|_{u=0} = \frac{(2n)!}{2^n n!} = \frac{2^n n!(2n-1)!!}{2^n n!} = (2n-1)!!.$$

In summary, we conclude that

$$\mathbb{E}[B^{2n}(\tau)] = \tau^n(2n-1)!!.$$

Or writing $t$ instead of $\tau$,

$$\mathbb{E}[B^{2n}(t)] = t^n(2n-1)!!.$$

*Alternatively*, a different way to evaluate the moments,

$$\begin{aligned}
\mathbb{E}\left[B^{2n}(t)\right] &= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} x^{2n} e^{-\frac{x^2}{2t}}\,dt \\
\text{(even function)} \quad &= \frac{2}{\sqrt{2\pi t}}\int_{0}^{+\infty} x^{2n} e^{-\frac{x^2}{2t}}\,dt \\
(x = \sqrt{2ts}) \quad &= \frac{2}{\sqrt{2\pi t}}\int_{0}^{+\infty} (2ts)^n e^{-s}\sqrt{\frac{t}{2s}}\,ds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(2t)^n}{\sqrt{\pi}} \int_0^{+\infty} s^{n-\frac{1}{2}} e^{-s} ds \\
&= \frac{(2t)^n}{\sqrt{\pi}} \int_0^{+\infty} s^{n+1-\frac{1}{2}} e^{-s} ds \\
(\Gamma - \text{function}) \quad &= \frac{(2t)^n}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right) \\
&= \frac{(2t)^n}{\sqrt{\pi}} \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \\
&= \frac{(2t)^n}{\sqrt{\pi}} \frac{(2n-1)!!}{2^n} \sqrt{\pi} \\
&= t^n (2n-1)!!.
\end{aligned}
$$

# 5 Problem Five: Textbook Exercise 4.2

Let $W(t)_{0\le t\le T}$ be a Brownian motion, and let $\mathscr{F}(t)_{0\le t\le T}$ be an associated filtration. Let $\Delta(t)_{0\le t\le T}$ be a non-random simple process (i.e., there is a partition $\Pi = \{t_0, t_1, \cdots, t_n\}$ of $[0,T]$ such that for every $j$, $\Delta(t_j)$ is a non-random quantity and $\Delta(t) = \Delta(t_j)$ is constant in $t$ on the subinterval $[t_j, t_{j+1})$). For $t \in [t_k, t_{k+1}]$, define the stochastic integral

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)].$$

**(i)** Show that whenever $0 \le s < t \le T$, the increment $I(t) - I(s)$ is independent of $\mathscr{F}(s)$.
*Proof*: A simplification described in Textbook exercise 4.2: If $s$ is between two partition points, we can always insert $s$ as an extra partition point. Then we can relabel the partition points so that they are still called $t_0, t_1, \cdots, t_n$, but with a larger value of $n$ and now with $s = t_k$ for some value of $k$. Of course, we must set $\Delta(s) = \Delta(t_{k-1})$ so that $\Delta$ takes the same value on the interval $[s, t_{k+1})$ as on the interval $[t_{k-1}, s)$. Similarly, we can insert $t$ as an extra partition point if it is not already one. Consequently, to show that $I(t) - I(s)$ is independent of $\mathscr{F}(s)$ for all $0 \le s < t \le T$, it suffices to show that $I(t_k) - I(t_l)$ is independent of $\mathscr{F}(t_l)$ whenever $t_k$ and $t_l$ are two partition points with $t_l < t_k$.

Now assume $t_k$ and $t_l$ are two partition points with $t_l < t_k$,

$$\begin{aligned} I(t_k) - I(t_l) &= \sum_{j=l}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] \\ &= \Delta(t_l)[W(t_{l+1}) - W(t_l)] + \cdots + \Delta(t_{k-1})[W(t_k) - W(t_{k-1})]. \end{aligned}$$

Because all Brownian motion involved in the above summation are independent of $\mathscr{F}(t_l)$, $I(t_k) - I(t_l)$ is also independent of $\mathscr{F}(t_l)$. According to the simplification scheme described above, we conclude that whenever $0 \le s < t \le T$, the increment $I(t) - I(s)$ is independent of $\mathscr{F}(s)$.

**(ii)** Show that whenever $0 \le s < t \le T$, the increment $I(t) - I(s)$ is a normally distributed random variable with mean zero and variance $\int_s^t \Delta^2(u)du$.
*Proof*: Again, according to the simplification scheme described in the textbook exercise, we work with an arbitrary pair of partition points $t_k < t_l$. The increment

$$I(t_k) - I(t_l) = \sum_{j=l}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)]$$

is a linear combination (with non-random coefficients $\Delta(t_j)$) of independent normal random variables $W(t_{j+1}) - W(t_j)$, for $j \in \{l, \cdots, k-1\}$, which is itself a normal random variable (see **Theorem 1.9** and **Definition 1.1** in Lecture 1). Furthermore, we compute the mean and the variance

$$\mathbb{E}[I(t_k) - I(t_l)] = \sum_{j=l}^{k-1} \Delta(t_j)\mathbb{E}[W(t_{j+1}) - W(t_j)] = 0;$$

$$
\begin{aligned}
\operatorname{var}(I(t_k)-I(t_l)) &= \sum_{j=l}^{k-1} \Delta^2(t_j)\operatorname{var}(W(t_{j+1})-W(t_j)) \\
&= \sum_{j=l}^{k-1} \Delta^2(t_j)(t_{j+1}-t_j) \\
&= \int_{t_l}^{t_k} \Delta^2(u)du, \quad \Delta(t) \text{ is constant in } t \text{ on the subinterval } [t_j, t_{j+1}).
\end{aligned}
$$

Invoking the simplification scheme, we conclude that for $0 \le s < t \le T$, the increment $I(t) - I(s)$ is a normally distributed random variable with mean zero and variance $\int_s^t \Delta^2(u)du$.

**(iii)** Use **(i)** and **(ii)** to show that $I(t)_{0\le t\le T}$, is a martingale.
*Proof*: Firstly, for all $t \in [0, T]$,

$$
\begin{aligned}
\mathbb{E}[|I(t)|] &= \mathbb{E}\left[\left|\sum_{j=0}^{k-1} \Delta(t_j)\left[W(t_{j+1})-W(t_j)\right] + \Delta(t_k)\left[W(t)-W(t_k)\right]\right|\right] \\
&\le \sum_{j=0}^{k-1} \left|\Delta(t_j)\right| \mathbb{E}\left[\left|W(t_{j+1})-W(t_j)\right|\right] + |\Delta(t_k)|\mathbb{E}\left[|W(t)-W(t_k)|\right] \\
&\le \sum_{j=0}^{k-1} \left|\Delta(t_j)\right| \sqrt{\mathbb{E}\left[\left(W(t_{j+1})-W(t_j)\right)^2\right]} + |\Delta(t_k)|\sqrt{\mathbb{E}\left[(W(t)-W(t_k))^2\right]} \\
&\le \sum_{j=0}^{k-1} \left|\Delta(t_j)\right| \sqrt{t_{j+1}-t_j} + |\Delta(t_k)|\sqrt{t-t_k} \\
&< \infty.
\end{aligned}
$$

Second, note that $I(t)$ is $\mathscr{F}(t)$-measurable because it involves Brownian motion up to time $t$.

Finally,

$$
\begin{aligned}
\mathbb{E}\left[I(t)|\mathscr{F}(s)\right] &= \mathbb{E}\left[I(t)-I(s)+I(s)|\mathscr{F}(s)\right] \\
&= \mathbb{E}\left[\underbrace{I(t)-I(s)}_{\text{independent of } \mathscr{F}(s)} |\mathscr{F}(s)\right] + \mathbb{E}\left[\underbrace{I(s)}_{\text{known to } \mathscr{F}(s)} |\mathscr{F}(s)\right] \\
&= \mathbb{E}\left[\underbrace{I(t)-I(s)}_{\text{mean zero}}\right] + I(s) \\
&= I(s).
\end{aligned}
$$

In conclusion, $I(t)_{0\le t\le T}$, is a martingale.

**(iv)** Show that $I^2(t) - \int_0^t \Delta^2(u)du, 0 \le t \le T$, is a martingale.
*Proof*: Again, we invoke the simplification scheme and let $t = t_k$,

$$
\mathbb{E}[I^2(t_k)] = \mathbb{E}\left[\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} \Delta(t_i)\Delta(t_j)[W(t_{i+1})-W(t_i)][W(t_{j+1})-W(t_j)]\right]
$$

$$
\begin{aligned}
&= \sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\Delta(t_i)\Delta(t_j)\mathbb{E}\left[[W(t_{i+1})-W(t_i)][W(t_{j+1})-W(t_j)]\right] \\
&= \sum_{i=1}^{k-1}\Delta^2(t_i)\mathbb{E}\left[[W(t_{i+1})-W(t_i)]^2\right] \\
&= \sum_{i=1}^{k-1}\Delta^2(t_i)(t_{i+1}-t_i) \\
&= \int_0^{t_k}\Delta^2(u)du.
\end{aligned}
$$

Thus,

$$
\mathbb{E}\left[\left|I^2(t_k)-\int_0^{t_k}\Delta^2(u)du\right|\right] \le \mathbb{E}[I^2(t_k)] + \int_0^{t_k}\Delta^2(u)du = 2\int_0^{t_k}\Delta^2(u)du < \infty.
$$

According to the simplification scheme,

$$
\mathbb{E}\left[\left|I^2(t)-\int_0^{t}\Delta^2(u)du\right|\right] < \infty.
$$

Second, it's obvious that $I^2(t)-\int_0^t\Delta^2(u)du$ is $\mathscr{F}(t)$-measurable. Finally, for $0 \le s < t \le T$,

$$
\begin{aligned}
&\mathbb{E}\left[I^2(t)-\int_0^t\Delta^2(u)du\middle|\mathscr{F}(s)\right] \\
=\ &\mathbb{E}\left[(I(t)-I(s)+I(s))^2\middle|\mathscr{F}(s)\right]-\int_0^t\Delta^2(u)du \\
=\ &\mathbb{E}\left[(I(t)-I(s))^2+2I(s)(I(t)-I(s))+I^2(s)\middle|\mathscr{F}(s)\right]-\int_0^t\Delta^2(u)du \\
=\ &\mathbb{E}\left[(I(t)-I(s))^2\middle|\mathscr{F}(s)\right]+2\left[I(s)(I(t)-I(s))\middle|\mathscr{F}(s)\right]+\left[I^2(s)\middle|\mathscr{F}(s)\right]-\int_0^t\Delta^2(u)du \\
=\ &\underbrace{\mathbb{E}\left[(I(t)-I(s))^2\middle|\mathscr{F}(s)\right]}_{=\int_s^t\Delta^2(u)du}+2I(s)\underbrace{\left[I(t)-I(s)\middle|\mathscr{F}(s)\right]}_{=0}+I^2(s)-\int_0^t\Delta^2(u)du \\
=\ &\int_s^t\Delta^2(u)du+I^2(s)-\int_0^t\Delta^2(u)du \\
=\ &I^2(s)-\int_0^s\Delta^2(u)du.
\end{aligned}
$$

We conclude that $I^2(t)-\int_0^t\Delta^2(u)du$ is a martingale.

# 6 PROBLEM SIX: TEXTBOOK EXERCISE 4.3

We now consider a case in which $\Delta(t)$ in Problem Five is simple but random. In particular, let $t_0 = 0$, $t_1 = s$, and $t_2 = t$, and let $\Delta(0)$ be non-random and $\Delta(s) = W(s)$. Which of the following assertions is true? Justify your answers.

**(i)** $I(t) - I(s)$ is independent of $\mathscr{F}(s)$.
*Solution*: No.

$$\begin{aligned} I(t) &= \Delta(t_0)[W(t_1) - W(t_0)] + \Delta(t_1)[W(t_2) - W(t_1)] = \Delta(0)W(s) + \Delta(s)[W(t) - W(s)] \\ I(s) &= \Delta(t_0)[W(t_1) - W(t_0)] = \Delta(0)W(s). \end{aligned}$$

Thus
$$I(t) - I(s) = \Delta(s)[W(t) - W(s)] = W(s)[W(t) - W(s)].$$

From the above expression, $I(t) - I(s)$ is not independent of $\mathscr{F}(s)$ because $W(s)$ is known to $\mathscr{F}(s)$.

**(ii)** $I(t) - I(s)$ is normally distributed.
*Solution*: We compute the kurtosis of $I(t) - I(s)$, which is supposed to be equal to 3 for a normally distributed random variable.

$$\begin{aligned} \mathbb{E}[I(t) - I(s)] &= \mathbb{E}[W(s)[W(t) - W(s)]] = \mathbb{E}[W(s)]\mathbb{E}[W(t) - W(s)] = 0. \\ \mathbb{E}[(I(t) - I(s))^2] &= \mathbb{E}[W^2(s)[W(t) - W(s)]^2] = \mathbb{E}[W^2(s)]\mathbb{E}[(W(t) - W(s))^2] = s(t-s). \\ \mathbb{E}[(I(t) - I(s))^4] &= \mathbb{E}[W^4(s)[W(t) - W(s)]^4] = \mathbb{E}[W^4(s)]\mathbb{E}[(W(t) - W(s))^4] = 3s^2 3(t-s)^2 = 9s^2(t-s)^2. \end{aligned}$$

Thus, the kurtosis
$$\text{Kurt} = \frac{9s^2(t-s)^2}{s^2(t-s)^2} = 9 \neq 3,$$
and $I(t) - I(s)$ is not normally distributed.

**(iii)** $\mathbb{E}[I(t)|\mathscr{F}(s)] = I(s)$.
*Solution*: Yes. By direct computation,

$$\begin{aligned} \mathbb{E}[I(t)|\mathscr{F}(s)] &= \mathbb{E}[\Delta(0)W(s) + \Delta(s)[W(t) - W(s)]|\mathscr{F}(s)] \\ &= \Delta(0)W(s) + W(s)\mathbb{E}[W(t) - W(s)|\mathscr{F}(s)] \quad (\Delta(0) \text{ is non-random}) \\ &= \Delta(0)W(s) + W(s)\mathbb{E}[W(t)|\mathscr{F}(s)] - W(s)\mathbb{E}[W(s)|\mathscr{F}(s)] \\ &= \Delta(0)W(s) + W(s)W(s) - W(s)W(s) \\ &= \Delta(0)W(s) \\ &= I(s). \end{aligned}$$

**(iv)** $\mathbb{E}\left[I^2(t)-\int_0^t \Delta^2(u)du \middle| \mathscr{F}(s)\right] = I^2(s)-\int_0^s \Delta^2(u)du.$
*Solution*: Yes. By direct computation,

$$\begin{aligned}
& \mathbb{E}\left[I^2(t)-\int_0^t \Delta^2(u)du \middle| \mathscr{F}(s)\right] \\
= \; & \mathbb{E}\left[(I(t)-I(s)+I(s))^2-\int_0^s \Delta^2(u)du-\int_s^t \Delta^2(u)du \middle| \mathscr{F}(s)\right] \\
= \; & \mathbb{E}\left[(I(t)-I(s))^2+2I(s)(I(t)-I(s))+I^2(s)-\Delta^2(0)\int_0^s du-W^2(s)\int_s^t du \middle| \mathscr{F}(s)\right] \\
= \; & \mathbb{E}\left[W^2(s)[W(t)-W(s)]^2+2\Delta(0)W^2(s)[W(t)-W(s)]+\Delta^2(0)W^2(s)-\Delta^2(0)\int_0^s du-W^2(s)\int_s^t du \middle| \mathscr{F}(s)\right] \\
= \; & W^2(s)(t-s)+0+\Delta^2(0)W^2(s)-\Delta^2(0)\int_0^s du-W^2(s)\int_s^t du \\
= \; & W^2(s)(t-s)+0+\Delta^2(0)W^2(s)-\Delta^2(0)\int_0^s du-W^2(s)(t-s) \\
= \; & \Delta^2(0)W^2(s)-\Delta^2(0)\int_0^s du \\
= \; & I^2(s)-\int_0^s \Delta^2(u)du.
\end{aligned}$$

## 7 Problem Seven: Brownian Motion Basic Calculus

Use properties of Brownian motion to find the expectation and variance of $\int_0^t B^2(s)ds$.
*Proof*: The expectation

$$
\begin{aligned}
\mathbb{E}\left[\int_0^t B^2(s)ds\right] &= \int_0^t \mathbb{E}\left[B^2(s)\right] ds \\
&= \int_0^t sds \\
&= \frac{1}{2}t^2.
\end{aligned}
$$

The second moment

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^t B^2(s)ds\right)^2\right] &= \mathbb{E}\left[\int_0^t\int_0^t B^2(s)B^2(s')dsds'\right] \\
&= \int_0^t\int_0^t \mathbb{E}\left[B^2(s)B^2(s')\right] dsds' \\
&= 2\int_0^t ds\int_0^s ds'\mathbb{E}\left[B^2(s)B^2(s')\right] \\
&= 2\int_0^t ds\int_0^s ds'\mathbb{E}\left[(B(s)-B(s')+B(s'))^2B^2(s')\right] \\
&= 2\int_0^t ds\int_0^s ds'\mathbb{E}\left[(B(s)-B(s'))^2B^2(s')+2(B(s)-B(s'))B^3(s')+B^4(s')\right] \\
&= 2\int_0^t ds\int_0^s ds'\left((s-s')s'+3s'^2\right) \\
&= 2\int_0^t ds\int_0^s ds'\left(2s'^2+ss'\right) \\
&= \frac{7}{3}\int_0^t dss^3 \\
&= \frac{7}{12}t^4.
\end{aligned}
$$

The variance

$$
\mathrm{var}\left(\int_0^t B^2(s)ds\right) = \frac{7}{12}t^4 - \frac{1}{4}t^4 = \frac{1}{3}t^4.
$$

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# MTH 9831 Assignment Four

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October 7, 2015

# 1 EXERCISE SIX

Use the method of Exercise 3 to compute the variance of the process $S(t)$, which solves the equation $dS(t) = \sigma S(t)dB(t)$, $S(0) = A$. In addition, solve this problem directly by first verifying the fact that $S(t) = Ae^{\sigma B(t)-\sigma^2 t/2}$.
*Solution*: Choose $f(x) = x^2$,

$$f_x(x) = 2x, \quad f_{xx}(x) = 2, \quad f_t(x) = 0.$$

Apply Ito's formula (D):

$$dS^2(t) = 2S(t)dS(t) + dS(t)dS(t) = 2\sigma S^2(t)dB(t) + \sigma^2 S^2(t)dt.$$

Integrate over time (I):

$$S^2(t) - S^2(0) = \int_0^t dS^2(t') = 2\sigma \underbrace{\int_0^t S^2(t')dB(t')}_{\text{martingale}} + \sigma^2 \int_0^t S^2(t')dt'.$$

Take expectation (E):

$$\mathbb{E}\left[S^2(t)\right] - A^2 = 2\sigma \cancelto{0}{\mathbb{E}\left[\int_0^t S^2(t')dB(t')\right]} + \sigma^2 \int_0^t \mathbb{E}\left[S^2(t')\right]dt'.$$

This is equivalent to the initial value problem of the following ODE:

$$\frac{d}{dt}\mathbb{E}\left[S^2(t)\right] = \sigma^2 \mathbb{E}\left[S^2(t)\right], \quad \mathbb{E}\left[S^2(t=0)\right] = A^2,$$

and has the well-known solution

$$\mathbb{E}\left[S^2(t)\right] = A^2 e^{\sigma^2 t}.$$

On the other hand, $dS(t) = \sigma S(t)dB(t)$ means that $S(t)$ is martingale, so that

$$\mathbb{E}\left[S(t)\right] = \mathbb{E}\left[S(0)\right] = A.$$

Thus, the variance

$$\operatorname{var}(S(t)) = \mathbb{E}\left[S^2(t)\right] - \mathbb{E}^2\left[S(t)\right] = A^2\left(e^{\sigma^2 t} - 1\right).$$

*Alternative solution.* Firstly verify that $S(t) = Ae^{\sigma B(t)-\sigma^2 t/2}$ solves the SDE $dS(t) = \sigma S(t)dB(t)$, $S(0) = A$. Let $g(t,x) = Ae^{\sigma x-\sigma^2 t/2}$,

$$g_x(t,x) = \sigma g(t,x), \quad g_{xx}(t,x) = \sigma^2 g(t,x), \quad g_t(t,x) = -\frac{1}{2}\sigma^2 g(t,x).$$

Apply Ito's formula (D) to $S(t) = g(t, B(t))$, which satisfies the initial condition $g(0, B(0)) = A$:

$$dS(t) = \sigma S(t)dB(t) + \frac{1}{2}\sigma^2 S(t)\underbrace{dB(t)dB(t)}_{dt} - \frac{1}{2}\sigma^2 S(t)dt = \sigma S(t)dB(t).$$

Thus, we verified that $S(t) = Ae^{\sigma B(t) - \sigma^2 t/2}$. Compute the moments of $S(t)$ explicitly using the density of $B(t)$,

$$\begin{aligned}
\mathbb{E}\left[S(t)\right] &= \mathbb{E}\left[Ae^{\sigma\sqrt{t}Z - \frac{1}{2}\sigma^2 t}\right] \\
&= A\int_{-\infty}^{+\infty} e^{\sigma\sqrt{t}z - \frac{1}{2}\sigma^2 t}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= A\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2}dz \\
&= A. \\
\mathbb{E}\left[S^2(t)\right] &= \mathbb{E}\left[A^2 e^{2\sigma\sqrt{t}Z - \sigma^2 t}\right] \\
&= A^2\int_{-\infty}^{+\infty} e^{2\sigma\sqrt{t}z - \sigma^2 t}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= A^2 e^{\sigma^2 t}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-2\sigma\sqrt{t})^2}dz \\
&= A^2 e^{\sigma^2 t}.
\end{aligned}$$

Again, the variance

$$\mathrm{var}\left(S(t)\right) = \mathbb{E}\left[S^2(t)\right] - \mathbb{E}^2\left[S(t)\right] = A^2\left(e^{\sigma^2 t} - 1\right).$$

## 2 Exercise Seven

Find the mean and variance of the process $\int_0^t S(u)du$, where $dS(t) = \sigma S(t)dB(t)$, $S(0) = A$. Give a solution based on integration by parts.

*Solution*: Because $S(t)$ is an Itô process and let the deterministic function $F(t) = t$ play the role of the stochastic process with a.s. continuously differentiable trajectories in the formula of *integration by parts*, we have

$$\begin{aligned}\int_0^t u dS(u) &= uS(u)\Big|_0^t - \int_0^t S(u)du \\ &= tS(t) - \int_0^t S(u)du.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}\left[\int_0^t S(u)du\right] &= t\mathbb{E}[S(t)] - \mathbb{E}\left[\int_0^t u dS(u)\right] \\ &= At - \sigma \cancelto{0}{\mathbb{E}\left[\underbrace{\int_0^t uS(u)dB(u)}_{\text{martingale}}\right]} \\ &= At.\end{aligned}$$

*Alternatively*, the mean can be found more straightforwardly

$$\mathbb{E}\left[\int_0^t S(u)du\right] = \int_0^t \mathbb{E}[S(u)]\,du = At.$$

The second moment

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t S(u)du\right)^2\right] &= \mathbb{E}\left[\int_0^t S(u)du \int_0^t S(v)dv\right] \\ &= \int_0^t\int_0^t \mathbb{E}[S(u)S(v)]\,dudv \\ &= 2\int_0^t du \int_0^u dv \mathbb{E}[S(u)S(v)] \\ \text{(tower property)} &= 2\int_0^t du \int_0^u dv \mathbb{E}[\mathbb{E}[S(u)S(v)|\mathscr{F}(v)]] \\ \text{(taking out what is known)} &= 2\int_0^t du \int_0^u dv \mathbb{E}[S(v)\mathbb{E}[S(u)|\mathscr{F}(v)]] \\ (S(t)\text{ is martingale}) &= 2\int_0^t du \int_0^u dv \mathbb{E}[S(v)S(v)] \\ &= 2\int_0^t du \int_0^u dv \mathbb{E}\left[S^2(v)\right]\end{aligned}$$

$$
\begin{aligned}
&= 2A^2 \int_0^t du \int_0^u dv e^{\sigma^2 v} \\
&= \frac{2A^2}{\sigma^2} \int_0^t du \left( e^{\sigma^2 u} - 1 \right) \\
&= \frac{2A^2}{\sigma^2} \left( \frac{e^{\sigma^2 t} - 1}{\sigma^2} - t \right) \\
&= \frac{2A^2}{\sigma^4} \left( e^{\sigma^2 t} - \sigma^2 t - 1 \right).
\end{aligned}
$$

The variance

$$
\mathrm{var}\left( \int_0^t S(u) du \right) = \mathbb{E}\left[ \left( \int_0^t S(u) du \right)^2 \right] - \mathbb{E}^2 \left[ \int_0^t S(u) du \right] = \frac{A^2}{\sigma^4} \left( 2e^{\sigma^2 t} - 2\sigma^2 t - \sigma^4 t^2 - 2 \right).
$$

# 3 Exercise Eight

As an application of Tool E, determine which of the following processes are martingales. When it is possible to given an alternative argument based on definition and/or basic properties of martingales or processes in question, provide such an argument as well.

**(a)** $B^2(t)$
*Answer*: Let $f(t,x) = x^2$, $f_t(t,x) = 0, f_x(t,x) = 2x, f_{xx}(t,x) = 2$, thus

$$f_t(t,x) + \frac{1}{2} f_{xx}(t,x) = 1 \neq 0.$$

The process $B^2(t)$ is not a martingale. Alternatively, from lecture notes 2 Theorem 3.2, we know that $B^2(t) - t$ is a martingale, thus $B^2(t) = (B^2(t) - t) + t$ is a superposition of a martingale and a drift term, which is not a martingale.

**(b)** $tB(t)$
*Answer*: Let $f(t,x) = tx$, $f_t(t,x) = x, f_x(t,x) = t, f_{xx}(t,x) = 0$, thus

$$f_t(t,x) + \frac{1}{2} f_{xx}(t,x) = x \neq 0.$$

The process $tB(t)$ is not a martingale.

**(c)** $(B(t)+t)\exp(-B(t) - t/2)$
*Answer*: Let $f(t,x) = (x+t)\exp(-x - t/2)$,

$$\begin{aligned} f_t(t,x) &= \left(1 - \frac{x+t}{2}\right) \exp\left(-x - \frac{t}{2}\right) \\ f_x(t,x) &= (1 - x - t) \exp\left(-x - \frac{t}{2}\right) \\ f_{xx}(t,x) &= -(2 - x - t) \exp\left(-x - \frac{t}{2}\right) \end{aligned}$$

Thus,

$$f_t(t,x) + \frac{1}{2} f_{xx}(t,x) = \left(1 - \frac{x+t}{2} - \frac{2-x-t}{2}\right) \exp\left(-x - \frac{t}{2}\right) = 0.$$

The process $(B(t)+t)\exp(-B(t) - t/2)$ is a martingale.

**(d)** $t^2 B(t) - 2\int_0^t uB(u)du$
*Answer*: In this case, the process is a functional of $B(t)$. Let $f(t,x) = t^2 x$,

$$f_t(t,x) = 2tx, f_x(t,x) = t^2, f_{xx}(t,x) = 0.$$

Thus,

$$d\left(t^2 B(t) - 2\int_0^t uB(u)du\right) = 2tB(t)dt + t^2 dB(t) - 2tB(t)dt = t^2 dB(t).$$

The process $t^2B(t) - 2\int_0^t uB(u)du$ is a martingale.

**(e)** $\log S(t)$, where $dS(t) = \nu S(t)dt + \sigma S(t)dB(t)$
*Answer*: Let $f(t,x) = \log x$, $f_t(t,x) = 0$, $f_x(t,x) = 1/x$, $f_{xx}(t,x) = -1/x^2$. Thus,

$$\begin{aligned}
d\log S(t) &= \frac{1}{S(t)}dS(t) - \frac{1}{2S^2(t)}dS(t)dS(t) \\
&= \frac{1}{S(t)}\left(\nu S(t)dt + \sigma S(t)dB(t)\right) - \frac{1}{2S^2(t)}\sigma^2 S^2(t)dt \\
&= \nu dt + \sigma dB(t) - \frac{1}{2}\sigma^2 dt \\
&= \left(\nu - \frac{1}{2}\sigma^2\right)dt + \sigma dB(t).
\end{aligned}$$

The process $\log S(t)$ is not a martingale unless $\nu - \frac{1}{2}\sigma^2 = 0$.

*Alternatively*, we have shown that $S(t) = Ae^{\left(\nu - \frac{1}{2}\sigma^2\right)t + \sigma B(t)}$, which gives

$$\log S(t) = \log A + \left(\nu - \frac{1}{2}\sigma^2\right)t + \sigma B(t),$$

the expectation of which is constant if and only if $\nu - \frac{1}{2}\sigma^2 = 0$.

**(f)** $\exp\left(\int_0^t \Delta(u)dB(u) - \frac{1}{2}\int_0^t \Delta^2(u)du\right)$
*Answer*: Let $X(t) = \int_0^t \Delta(u)dB(u) - \frac{1}{2}\int_0^t \Delta^2(u)du$, then

$$dX(t) = \Delta(t)dB(t) - \frac{1}{2}\Delta^2(t)dt.$$

Let $f(t,x) = e^x$, $f_t(t,x) = 0$, $f_x(t,x) = e^x$, $f_{xx}(t,x) = e^x$. Thus,

$$\begin{aligned}
de^{X(t)} &= e^{X(t)}\left(dX(t) + \frac{1}{2}dX(t)dX(t)\right) \\
&= e^{X(t)}\left(\Delta(t)dB(t) - \frac{1}{2}\Delta^2(t)dt + \frac{1}{2}\Delta^2(t)dt\right) \\
&= e^{X(t)}\Delta(t)dB(t).
\end{aligned}$$

The process $e^{X(t)}$ is a martingale.

# 4 Exercise Nine

(Review of properties of conditional expectation) Using the definition of a martingale (without Itô's formula or any of its consequences) show that the process $t^2B(t)-2\int_0^t uB(u)du$ is a martingale.
*Solution*: First, the absolute value of the process $t^2B(t)-2\int_0^t uB(u)du$ is integrable.

$$\mathbb{E}\left[\left|t^2B(t)-2\int_0^t uB(u)du\right|\right] \leq t^2\mathbb{E}\left[|B(t)|\right]+2\int_0^t u\mathbb{E}\left[|B(u)|\right]du<\infty.$$

Also, the process $t^2B(t)-2\int_0^t uB(u)du$ involves the Brownian motion trajectory up to time $t$, hence $\mathscr{F}(t)$-measurable. For every $0\leq s<t$

$$\begin{aligned}\mathbb{E}\left[t^2B(t)-2\int_0^t uB(u)du\middle|\mathscr{F}(s)\right] &= \mathbb{E}\left[t^2B(t)|\mathscr{F}(s)\right]-\mathbb{E}\left[2\int_0^t uB(u)du\middle|\mathscr{F}(s)\right]\\ &= t^2\mathbb{E}\left[B(t)|\mathscr{F}(s)\right]-2\int_0^t u\mathbb{E}\left[B(u)|\mathscr{F}(s)\right]du\\ &= t^2B(s)-2\int_0^t u\mathbb{E}\left[B(u)|\mathscr{F}(s)\right]du.\end{aligned}$$

Invoking the following fact resulting from the martingality of $B(t)$

$$\mathbb{E}\left[B(u)|\mathscr{F}(s)\right]=\begin{cases}B(u), & u\leq s,\\ B(s), & u>s;\end{cases}$$

we have

$$\begin{aligned}\mathbb{E}\left[t^2B(t)-2\int_0^t uB(u)du\middle|\mathscr{F}(s)\right] &= t^2B(s)-2\int_0^s uB(u)du-2B(s)\int_s^t udu\\ &= t^2B(s)-2\int_0^s uB(u)du-B(s)\left(t^2-s^2\right)\\ &= s^2B(s)-2\int_0^s uB(u)du.\end{aligned}$$

We conclude that $t^2B(t)-2\int_0^t uB(u)du$ is a martingale.

# 5 Exercise Ten: Ornstein-Uhlenbeck

(The Ornstein-Uhlenbeck process) Let $X(t)_{t\geq 0}$ satisfy

$$dX(t) = -\beta X(t)dt + \sigma dB(t), \quad X(0) = x,$$

where $\beta \in \mathbb{R}$, $\sigma > 0$ are constants. For $\beta > 0$ this is a special case of Vasicek interest rate model. Apply Ito's formula to $e^{\beta t}X$ and show that $X(t)$ admits a closed form solution

$$X(t) = e^{-\beta t}x + \sigma e^{-\beta t}\int_0^t e^{\beta u}dB(u).$$

use this expression to find the mean and variance of $X(t)$. Then compute the mean and variance not using the solution but applying the same approach as in Exercise Six.
*Solution*: Let $f(t,x) = e^{\beta t}x$, $f_t(t,x) = \beta f(t,x), f_x(t,x) = e^{\beta t}, f_{xx}(t,x) = 0$. Apply Ito's formula to $e^{\beta t}X$,

$$\begin{aligned} d\left(e^{\beta t}X(t)\right) &= \beta e^{\beta t}X(t)dt + e^{\beta t}dX(t) \\ &= \beta e^{\beta t}X(t)dt - \beta e^{\beta t}X(t)dt + \sigma e^{\beta t}dB(t) \\ &= \sigma e^{\beta t}dB(t). \end{aligned}$$

Thus, $e^{\beta t}X(t)$ is a martingale. Integrate over time,

$$e^{\beta t}X(t) - X(0) = \int_0^t d\left(e^{\beta u}X(u)\right) = \sigma\int_0^t e^{\beta u}dB(u);$$

in other words,

$$\boxed{X(t) = e^{-\beta t}x + \sigma e^{-\beta t}\int_0^t e^{\beta u}dB(u).} \tag{5.1}$$

To compute the mean and the variance, we recall Itô's integral for deterministic integrands,

$$X(t) - e^{-\beta t}x \sim N\left(0, \sigma^2 e^{-2\beta t}\int_0^t e^{2\beta u}du\right) \sim N\left(0, \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right)\right).$$

or

$$X(t) \sim N\left(e^{-\beta t}x, \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right)\right). \tag{5.2}$$

The mean

$$\mathbb{E}\left[X(t) - e^{-\beta t}x\right] = 0 \Rightarrow \mathbb{E}\left[X(t)\right] = e^{-\beta t}x.$$

The variance,

$$\mathrm{var}(X(t)) = \mathbb{E}\left[(X(t) - \mathbb{E}[X(t)])^2\right] = \mathbb{E}\left[\left(X(t) - e^{-\beta t}x\right)^2\right] = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right).$$

*Alternatively*, since $e^{\beta t}X(t)$ is a martingale

$$\mathbb{E}\left[e^{\beta t}X(t)\right] = \mathbb{E}[X(0)] = x \Rightarrow \mathbb{E}[X(t)] = e^{-\beta t}x.$$

Furthermore, let $g(t,x) = e^{2\beta t}x^2$, $g_t(x) = 2\beta e^{2\beta t}x^2$, $g_x(x) = 2e^{2\beta t}x$, $g_{xx}(x) = 2e^{2\beta t}$,

$$\begin{aligned}
d\left[e^{2\beta t}X^2(t)\right] &= 2\beta e^{2\beta t}X^2(t)dt + 2e^{2\beta t}X(t)dX(t) + e^{2\beta t}dX(t)dX(t) \\
&= 2\beta e^{2\beta t}X^2(t)dt + 2e^{2\beta t}X(t)\left(-\beta X(t)dt + \sigma dB(t)\right) + \sigma^2 e^{2\beta t}dt \\
&= 2\sigma e^{2\beta t}X(t)dB(t) + \sigma^2 e^{2\beta t}dt.
\end{aligned}$$

Integrate over time,

$$e^{2\beta t}X^2(t) - x^2 = 2\sigma \underbrace{\int_0^t e^{2\beta t'}X(t')dB(t')}_{\text{martingale}} + \sigma^2\int_0^t e^{2\beta t'}dt'.$$

Take expectations,

$$\mathbb{E}\left[e^{2\beta t}X^2(t)\right] - x^2 = \sigma^2\int_0^t e^{2\beta t'}dt' = \frac{\sigma^2}{2\beta}\left(e^{2\beta t}-1\right).$$

Thus, the variance

$$\operatorname{var}(X(t)) = \mathbb{E}\left[X^2(t)\right] - \mathbb{E}^2[X(t)] = \mathbb{E}\left[X^2(t)\right] - \left(e^{-\beta t}x\right)^2 = \frac{\sigma^2}{2\beta}\left(e^{2\beta t}-1\right),$$

which is the same result we obtained from the closed-form solution of $X(t)$.

# 6 Exercise Eleven: Vasicek

Let $r(t)$ satisfy

$$dr(t) = (\alpha - \beta r(t))dt + \sigma dB(t).$$

Find a closed form solution of this equation. Compute the mean and variance of $r(t)$.
*Solution*: We try to solve this SDE by first looking at its deterministic part

$$dR(t) = (\alpha - \beta R(t))dt.$$

This is a well-known ODE solvable by elementary method

$$\frac{dR(t)}{\alpha - \beta R(t)} = dt \Rightarrow R(t) = \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + e^{-\beta t} R(0).$$

Thus, we write $r(t)$ in the following form $r(t) = R(t) + \tilde{r}(t)$ and look for the SDE satisfied by $\tilde{r}(t)$. Straightforwardly,

$$\begin{aligned}
& dr(t) = dR(t) + d\tilde{r}(t) \\
\Rightarrow\quad & (\alpha - \beta r(t))dt + \sigma dB(t) = (\alpha - \beta R(t))dt + d\tilde{r}(t) \\
\Rightarrow\quad & d\tilde{r}(t) = -\beta\left(r(t) - R(t)\right)dt + \sigma dB(t) \\
\Rightarrow\quad & d\tilde{r}(t) = -\beta\tilde{r}(t)dt + \sigma dB(t).
\end{aligned}$$

Notice that the above SDE describes the Ornstein-Uhlenbeck process we solved in the previous exercise, the solution of which is recorded in Equation 5.1. In terms of the notation employed in this exercise, we write

$$\tilde{r}(t) = e^{-\beta t}\tilde{r}(0) + \sigma e^{-\beta t}\int_0^t e^{\beta u} dB(u).$$

Finally,

$$\begin{aligned}
r(t) &= R(t) + \tilde{r}(t) \\
&= e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + e^{-\beta t}\tilde{r}(0) + \sigma e^{-\beta t}\int_0^t e^{\beta u} dB(u) \\
&= e^{-\beta t}r(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t}\int_0^t e^{\beta u} dB(u). \qquad (6.1)
\end{aligned}$$

To compute the mean and the variance, we again recall Itô's integral for deterministic integrands,

$$r(t) - e^{-\beta t}r(0) - \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) \sim N\left(0, \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right)\right).$$

Thus, the mean

$$\mathbb{E}[r(t)] = e^{-\beta t}r(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right);$$

and the variance

$$\operatorname{var}(r(t)) = \frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right).$$

For reference, we provide here an *alternative* way of deriving the explicit solution of $r(t)$ given in Equation 6.1. As usual, let $f(t,x) = e^{\beta t}x$, $f_t(t,x) = \beta e^{\beta t}x$, $f_x(t,x) = e^{\beta t}$, $f_{xx}(t,x) = 0$,

$$\begin{aligned} d\left(e^{\beta t} r(t)\right) &= e^{\beta t} r(t) dt + e^{\beta t} dr(t) \\ &= \beta e^{\beta t} r(t) dt + e^{\beta t}\left((\alpha - \beta r(t))dt + \sigma dB(t)\right) \\ &= \alpha e^{\beta t} dt + \sigma e^{-\beta t} dB(t). \end{aligned}$$

Integrate over time,

$$\begin{aligned} e^{\beta t} r(t) - r(0) &= \int_0^t d\left(e^{\beta t'} r(t')\right) \\ &= \alpha \int_0^t e^{\beta t'} dt' + \sigma \int_0^t e^{-\beta t'} dB(t') \\ &= \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right) + \sigma \int_0^t e^{-\beta t'} dB(t'). \end{aligned}$$

Re-arranging the items, we arrive at the same result obtained in Equation 6.1

$$r(t) = e^{-\beta t} r(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t} \int_0^t e^{-\beta t'} dB(t').$$

The computation of the mean and the variance remains the same.

# 7 Exercise Twelve: Cox-Ingersoll-Ross

Let $r(t)$ satisfy

$$dr(t) = (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}dB(t).$$

**(a)** Compute the mean and variance of $r(t)$.
*Solution*: Use the result obtained in the previous exercise, write

$$r(t) = R(t) + \tilde{r}(t)$$

where $R(t) = \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + e^{-\beta t}R(0)$ solves the deterministic part of the SDE

$$dR(t) = (\alpha - \beta R(t))dt.$$

The remaining part $\tilde{r}(t)$ satisfies the following SDE:

$$\begin{aligned}
d\tilde{r}(t) &= dr(t) - dR(t) \\
&= (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}dB(t) - (\alpha - \beta R(t))dt \\
&= -\beta(r(t) - R(t))dt + \sigma\sqrt{r(t)}dB(t) \\
&= -\beta\tilde{r}(t)dt + \sigma\sqrt{r(t)}dB(t).
\end{aligned}$$

Integrate over time,

$$\begin{aligned}
\tilde{r}(t) - \tilde{r}(0) &= \int_0^t d\tilde{r}(t') \\
&= -\beta\int_0^t \tilde{r}(t')dt' + \sigma\underbrace{\int_0^t \sqrt{r(t')}dB(t')}_{\text{martingale}}.
\end{aligned}$$

Take expectation,

$$\begin{aligned}
\mathbb{E}[\tilde{r}(t)] - \tilde{r}(0) &= -\beta\int_0^t \mathbb{E}[\tilde{r}(t')]dt' + \sigma\cancelto{0}{\mathbb{E}\left[\int_0^t \sqrt{r(t')}dB(t')\right]} \\
&= -\beta\int_0^t \mathbb{E}[\tilde{r}(t')]dt'.
\end{aligned}$$

This is equivalent to the following initial value problem

$$\frac{d}{dt}\mathbb{E}[\tilde{r}(t)] = -\beta\mathbb{E}[\tilde{r}(t)], \quad \mathbb{E}[\tilde{r}(0)] = \tilde{r}(0),$$

which is solved by $\mathbb{E}[\tilde{r}(t)] = \tilde{r}(0)e^{-\beta t}$. Thus,

$$\begin{aligned}
\mathbb{E}[r(t)] &= \mathbb{E}[R(t) + \tilde{r}(t)] = R(t) + \mathbb{E}[\tilde{r}(t)] \\
&= \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + e^{-\beta t}R(0) + e^{-\beta t}\tilde{r}(0) \\
&= \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + e^{-\beta t}r(0).
\end{aligned}$$

To compute the variance of $r(t)$, we try to find an SDE satisfied by $e^{2\beta t} r^2(t)$. Let $f(t,x) = e^{2\beta t}x^2$, similar to what's done in Exercise Ten (Ornstein-Uhlenbeck), $f_t(x) = 2\beta e^{2\beta t}x^2$, $f_x(x) = 2e^{2\beta t}x$, $f_{xx}(x) = 2e^{2\beta t}$,

$$\begin{aligned}
d\left[e^{2\beta t} r^2(t)\right] &= 2\beta e^{2\beta t} r^2(t)dt + 2e^{2\beta t} r(t)dr(t) + e^{2\beta t} dr(t)dr(t) \\
&= 2\beta e^{2\beta t} r^2(t)dt + 2e^{2\beta t} r(t)\left[(\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}dB(t)\right] + \sigma^2 e^{2\beta t} r(t)dt \\
&= 2\alpha e^{2\beta t} r(t)dt + 2\sigma e^{2\beta t} r^{3/2}(t)dB(t) + \sigma^2 e^{2\beta t} r(t)dt \\
&= e^{2\beta t}\left(2\alpha + \sigma^2\right) r(t)dt + 2\sigma e^{2\beta t} r^{3/2}(t)dB(t).
\end{aligned}$$

Integrate over time,

$$e^{2\beta t} r^2(t) - r^2(0) = \int_0^t d\left[e^{2\beta t'} r^2(t')\right] = \left(2\alpha + \sigma^2\right)\int_0^t e^{2\beta t'} r(t')dt' + 2\sigma \underbrace{\int_0^t e^{2\beta t'} r^{3/2}(t')dB(t')}_{\text{martingale}}.$$

Take expectations,

$$\begin{aligned}
e^{2\beta t}\mathbb{E}\left[r^2(t)\right] - r^2(0) &= \left(2\alpha + \sigma^2\right)\int_0^t e^{2\beta t'}\mathbb{E}\left[r(t')\right]dt' + 2\sigma\mathbb{E}\left[\cancelto{0}{\int_0^t e^{2\beta t'} r^{3/2}(t')dB(t')}\right] \\
&= \left(2\alpha + \sigma^2\right)\int_0^t e^{2\beta t'}\left[\frac{\alpha}{\beta}\left(1 - e^{-\beta t'}\right) + e^{-\beta t'} r(0)\right]dt' \\
&= \left(2\alpha + \sigma^2\right)\int_0^t \left[\frac{\alpha}{\beta} e^{2\beta t'} + \left(r(0) - \frac{\alpha}{\beta}\right)e^{\beta t'}\right]dt' \\
&= \frac{2\alpha + \sigma^2}{\beta}\left(r(0) - \frac{\alpha}{\beta}\right)\left(e^{\beta t} - 1\right) + \frac{2\alpha + \sigma^2}{\beta}\cdot\frac{\alpha}{\beta}\left(e^{2\beta t} - 1\right).
\end{aligned}$$

Therefore,

$$\mathbb{E}\left[r^2(t)\right] = e^{-2\beta t} r^2(0) + \frac{2\alpha + \sigma^2}{\beta}\left(r(0) - \frac{\alpha}{\beta}\right)\left(e^{-\beta t} - e^{-2\beta t}\right) + \frac{2\alpha + \sigma^2}{\beta}\cdot\frac{\alpha}{\beta}\left(1 - e^{-2\beta t}\right).$$

The variance,

$$\begin{aligned}
\operatorname{var}(r(t)) &= \mathbb{E}\left[r^2(t)\right] - \mathbb{E}^2[r(t)] \\
&= e^{-2\beta t} r^2(0) + \frac{2\alpha + \sigma^2}{\beta}\left(r(0) - \frac{\alpha}{\beta}\right)\left(e^{-\beta t} - e^{-2\beta t}\right) + \frac{2\alpha + \sigma^2}{\beta}\cdot\frac{\alpha}{\beta}\left(1 - e^{-2\beta t}\right) \\
&\quad - \left[\frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + e^{-\beta t} r(0)\right]^2 \\
&= \frac{\sigma^2}{\beta} r(0)\left(e^{-\beta t} - e^{-2\beta t}\right) + \frac{\alpha\sigma^2}{2\beta^2}\left(1 - e^{-\beta t}\right)^2.
\end{aligned}$$

In the limit $t \to \infty$,

$$\mathbb{E}[r(t)] \to \frac{\alpha}{\beta}, \quad \operatorname{var}\left(r^2(t)\right) \to \frac{\alpha\sigma^2}{2\beta^2}.$$

(Shreve Vol II §4.4.3.)

**(b)** Assume that $4\alpha = \sigma^2$. Let $X(t) = \sqrt{r(t)}$. Derive the equation for $X(t)$.
*Solution*: Let $f(t,x) = \sqrt{x}$, $f_t(t,x) = 0$, $f_x(t,x) = \frac{1}{2\sqrt{x}}$, $f_{xx}(t,x) = -\frac{1}{4\sqrt{x^3}}$,

$$\begin{aligned} d\sqrt{r(t)} &= \frac{1}{2\sqrt{r(t)}}dr(t) - \frac{1}{8\sqrt{r^3(t)}}dr(t)dr(t) \\ &= \frac{1}{2\sqrt{r(t)}}\left[(\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}dB(t)\right] - \frac{1}{8\sqrt{r^3(t)}}\sigma^2 r(t)dt \\ &= \left(\frac{4\alpha - \sigma^2}{8\sqrt{r(t)}} - \frac{\beta}{2}\sqrt{r(t)}\right)dt + \frac{\sigma}{2}dB(t). \end{aligned}$$

Given the assumption $4\alpha = \sigma^2$, we have

$$d\sqrt{r(t)} = -\frac{\beta}{2}\sqrt{r(t)}dt + \frac{\sigma}{2}dB(t),$$

or write in terms of $X(t) = \sqrt{r(t)}$,

$$dX(t) = -\frac{\beta}{2}X(t)dt + \frac{\sigma}{2}dB(t),$$

which is an Ornstein-Uhlenbeck process.

**(c)** Using part **(b)** determine the distribution of $r(t)$. Compute its moment generating function.
*Solution*: Under the assumption given in part (b): $4\alpha = \sigma^2$, we have identified $X(t) = \sqrt{r(t)}$ as an Ornstein-Uhlenbeck process. Invoke the results in Exercise Ten Equation 5.2, we have

$$\sqrt{r(t)} \sim N\left(e^{-\frac{\beta t}{2}}\sqrt{r(0)}, \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)\right).$$

It is known from Statistics Refresher that the square of a normally distributed variable has a non-central $\chi^2$-distribution

$$\frac{r(t)}{\sigma^2} \sim \chi_1^2\left(\frac{\mu^2}{\sigma^2}\right),$$

where $\mu \triangleq e^{-\frac{\beta t}{2}}\sqrt{r(0)}$ and $\sigma^2 \triangleq \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)$. The probability density function of $r(t)$ can be deduced from the normal density. Let

$$p(y) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

be the familiar normal probability density function and $x = y^2$, then

$$p(x)dx = 2p(y)dy,$$

where the factor of 2 comes from the fact that both $\pm y$ contribute to the same value of $x$. Thus,

$$p(x) = 2p(y(x))\frac{dy}{dx}, \quad x \geq 0.$$

In other words, the probability density function of $r(t)$ is given by

$$f(r(t) = x) = \frac{1}{\sigma\sqrt{2\pi x}} e^{-\frac{(\sqrt{x}-\mu)^2}{2\sigma^2}}, \quad x \geq 0.$$

The moment-generating function is given by

$$\begin{aligned}
\mathbb{E}\left[e^{ur(t)}\right] &= \int_0^\infty \frac{1}{\sigma\sqrt{2\pi x}} e^{-\frac{(\sqrt{x}-\mu)^2}{2\sigma^2}} e^{ux} dx \\
&= \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty e^{-\frac{(\sqrt{x}-\mu)^2}{2\sigma^2}+ux} d\sqrt{x} \\
&= \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}+ux^2} dx \\
&= \frac{1}{\sqrt{1-2u\sigma^2}} \exp\left(\frac{\mu^2 u}{1-2u\sigma^2}\right),
\end{aligned}$$

for $2\mu\sigma^2 < 1$. Note: the moment-generating function of the CIR model in the general case can also be solved in a closed form.

BARUCH, MFE

# MTH 9831 Assignment Five

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October 13, 2015

# 1 EXERCISE ONE: LEVY'S CHARACTERIZATION

(Lévy's characterization of BM and Itô's formula) Let $(B_1(t), B_2(t), \cdots, B_n(t))$ be the $n$-dimensional Brownian motion (extend the 2-dimensional definition in an obvious way), $n \geq 2$. Define

$$R(t) = \sum_{i=1}^{n} B_i^2(t); \quad dW(t) = \sum_{i=1}^{n} \frac{B_i(t)}{\sqrt{R(t)}} dB_i(t).$$

The process $R(t), t \geq 0$ is called the squared Bessel process of dimension $n$. What is the distribution of $R(t)$? The process $\sqrt{R(t)}$ (which represents the distance from the origin of the Brownian particle in $n$-dimensions) is called Bessel process of dimension (or order) $n$.

*Answer*: The Brownian motions $B_i(t)$ at any fixed $t$ are normally distributed

$$B_i(t) \sim N(0,t) \Rightarrow \frac{1}{\sqrt{t}} B_i(t) \sim N(0,1) \quad \text{- standard normal}$$

Thus, $R(t)$ can be written as

$$R(t) \triangleq \sum_{i=1}^{n} B_i^2(t) = t \sum_{i=1}^{n} Z_i^2(t),$$

where $Z_i, i = 1, \cdots, n$ are independent standard normal random variables. By definition,

$$\frac{1}{t} R(t) \sim \chi_n^2$$

obeys a $\chi^2$-distribution with $n$ degrees of freedom. Given the probability density function of a $\chi^2$-distribution with $n$ degrees of freedom

$$f(x) = \frac{x^{\frac{1}{2}n-1} e^{-\frac{1}{2}x}}{2^{\frac{1}{2}n} \Gamma\left(\frac{n}{2}\right)}, \quad t \geq 0,$$

the probability density function $R(t)$ is simply given by $t^{-1} f(x/t)$.

**(a)** Show that $W(t), t \geq 0$ is a standard Brownian motion.
*Proof*: Assume $W(0) = 0$, we invoke Lévy's characterization of BM to proceed.

- $W(0) = 0$ by assumption.
- $W(t), t \geq 0$ has continuous paths because

$$W(t) = \int_0^t \sum_{i=1}^{n} \frac{B_i(t')}{\sqrt{R(t')}} dB_i(t') \tag{1.1}$$

  is written as an integral with finite integrand. In other words, since Brownian motions has continuous paths, $dB_i(t) \to 0, \forall i$ as $dt \to 0$, thus

$$dW(t) = \sum_{i=1}^{n} \frac{B_i(t)}{\sqrt{R(t)}} dB_i(t) \to 0, \quad \text{as } dt \to 0.$$

- $W(t), t \geq 0$ is martingale with respect to the natural filtration of the $n$-dimensional Brownian motions because it's sum of martingales expressed in terms of Itô's stochastic integrals, Equation 1.1. A sum of martingales is a martingale because of linearity.
- Quadratic variation

$$\begin{aligned}
[W]_t &= \int_0^t dW(t')dW(t') \\
\text{(Lemma 2.4)} \quad &= \int_0^t \sum_{i=1}^n \sum_{j=1}^n \frac{B_i(t')B_j(t')}{R(t')} \underbrace{dB_i(t')dB_j(t')}_{=\delta_{ij}dt'} \\
&= \int_0^t dt' \frac{\sum_{i=1}^n B_i^2(t')}{R(t')} \\
&= \int_0^t dt' \frac{R(t')}{R(t')} \\
&= \int_0^t dt' \\
&= t
\end{aligned}$$

According to Lévy's characterization of Brownian motion, $W(t), t \geq 0$ is a standard Brownian motion.

**(b)** Show that $R(t), t \geq 0$ satisfies the SDE

$$dR(t) = 2\sqrt{R(t)}dW(t) + n dt.$$

*Proof*: Apply Itô's formula to each individual term in the summation,

$$\begin{aligned}
dR(t) &= \sum_{i=1}^n dB_i^2(t) \\
&= \sum_{i=1}^n \left( 2B_i(t)dB_i(t) + \underbrace{dB_i(t)dB_i(t)}_{dt} \right) \\
&= 2\sum_{i=1}^n B_i(t)dB_i(t) + n dt \\
&= 2\sqrt{R(t)}dW(t) + n dt.
\end{aligned}$$

Compare with Cox-Ingersoll-Ross model

$$dr(t) = (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}dB(t),$$

we identify the following CIR model parameters for the squared Bessel process

$$\alpha = n, \quad \beta = 0, \quad \sigma = 2.$$

**(c)** Show that $X(t) := \sqrt{R(t)}, t \geq 0$ satisfies the SDE

$$dX(t) = dW(t) + \frac{n-1}{2X(t)}dt.$$

*Proof*: Apply Itô's formula to $X(t) = \sqrt{R(t)}$,

$$\begin{aligned} dX(t) &= d\sqrt{R(t)} \\ &= \frac{1}{2\sqrt{R(t)}} dR(t) - \frac{1}{8\sqrt{R^3(t)}} dR(t) dR(t) \\ &= \frac{1}{2\sqrt{R(t)}} \left(2\sqrt{R(t)} dW(t) + n dt\right) - \frac{1}{8\sqrt{R^3(t)}} 4R(t) dt \\ &= dW(t) + \frac{n-1}{2X(t)} dt. \end{aligned}$$

## 2 Exercise Two: Decomposition of Correlated Brownian Motions into Independent Brownian Motions

(Lévy's characterization of BM and computation of cross-variation) Suppose $B_1(t)$ and $B_2(t)$ are Brownian motions and

$$dB_1(t)dB_2(t) = \rho(t)dt,$$

where $\rho(t)$ is a stochastic process taking values strictly between $-1$ and $+1$. Define processes $W_1(t)$ and $W_2(t)$ such that

$$\begin{aligned} B_1(t) &= W_1(t), \\ B_2(t) &= \int_0^t \rho(s)dW_1(s) + \int_0^t \sqrt{1-\rho^2(s)}dW_2(s), \end{aligned} \tag{2.1}$$

and show that $W_1(t)$ and $W_2(t)$ are independent Brownian motions.

*Proof*: In differential form,

$$\begin{aligned} dB_1(t) &= dW_1(t), \\ dB_2(t) &= \rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t), \end{aligned}$$

or in matrix notation

$$\begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho(t) & \sqrt{1-\rho^2(t)} \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}. \tag{2.2}$$

Denote

$$dB(t) \triangleq \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}, \quad dW(t) \triangleq \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}, \quad A(t) \triangleq \begin{pmatrix} 1 & 0 \\ \rho(t) & \sqrt{1-\rho^2(t)} \end{pmatrix}.$$

then we rewrite Equation 2.2 as

$$dB(t) = A(t)dW(t) \Rightarrow dW(t) = A^{-1}(t)dB(t).$$

The $2 \times 2$ matrix $A(t)$ can be straightforwardly inverted

$$A^{-1}(t) = \frac{1}{\sqrt{1-\rho^2(t)}} \begin{pmatrix} \sqrt{1-\rho^2(t)} & 0 \\ -\rho(t) & 1 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix} = \frac{1}{\sqrt{1-\rho^2(t)}} \begin{pmatrix} \sqrt{1-\rho^2(t)} & 0 \\ -\rho(t) & 1 \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix} = \begin{pmatrix} dB_1(t) \\ -\frac{\rho(t)}{\sqrt{1-\rho^2(t)}} dB_1(t) + \frac{1}{\sqrt{1-\rho^2(t)}} dB_2(t) \end{pmatrix}.$$

Check $dW(t)$ is a standard two-dimensional Brownian motion by Lévy's characterization.

- Initial condition:

$$W(t=0) = \begin{pmatrix} B_1(t) \\ -\int_0^t \frac{\rho(t')}{\sqrt{1-\rho^2(t')}} dB_1(t') + \int_0^t \frac{1}{\sqrt{1-\rho^2(t')}} dB_2(t') \end{pmatrix}_{t=0} = 0.$$

  This initial condition is in fact assumed. In the original definition Equation 2.1, we can append an arbitrary constant to $W_2(t)$ without changing the definition.

- $W(t)$ has continuous paths because the Brownian motions $B_1(t)$ and $B_2(t)$ have continuous paths by definition and $W(t)$ is expressed in terms of Itô's stochastic integrals with respect to $B_1(t)$ and $B_2(t)$.
- Both $W_1(t)$ and $W_2(t)$ are Itô's stochastic integral and hence *martingales.*
- Quadratic variation

$$
\begin{aligned}
[W_1]_t &= [B_1]_t = t; \\
[W_2]_t &= \int_0^t dW_2(t')dW_2(t') \\
&= \int_0^t \left(-\frac{\rho(t')}{\sqrt{1-\rho^2(t')}}dB_1(t') + \frac{1}{\sqrt{1-\rho^2(t')}}dB_2(t')\right)^2 \\
&= \int_0^t \left(\frac{\rho^2(t')}{1-\rho^2(t')}dt' - \frac{2\rho^2(t')}{1-\rho^2(t')}dt' + \frac{1}{1-\rho^2(t')}dt'\right) \\
&= \int_0^t \frac{1-\rho^2(t')}{1-\rho^2(t')}dt' \\
&= \int_0^t dt' \\
&= t.
\end{aligned}
$$

- Cross variation

$$
\begin{aligned}
[W_1, W_2]_t &= \int_0^t dW_1(t')dW_2(t') \\
&= \int_0^t dB_1(t')\left(-\frac{\rho(t')}{\sqrt{1-\rho^2(t')}}dB_1(t') + \frac{1}{\sqrt{1-\rho^2(t')}}dB_2(t')\right) \\
&= \int_0^t \left(-\frac{\rho(t')}{\sqrt{1-\rho^2(t')}}dt' + \frac{1}{\sqrt{1-\rho^2(t')}}\rho(t')dt'\right) \\
&= 0.
\end{aligned}
$$

In summary, we have verified that $W(t) = \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}$ is a standard two-dimensional Brownian motion. By the definition of two-dimensional Brownian motion, the processes $W_1(t)$ and $W_2(t)$ are independent Brownian motions.

The approach exemplified in this exercise works for all dimensions. Let $B_1(t), B_2(t), \cdots, B_m(t)$ be $m$ one-dimensional Brownian motions with

$$dB_i(t)dB_j(t) = \rho_{ij}(t)dt, \quad \forall i, j = 1, \cdots, m,$$

where $\rho_{ij}(t)$ are adapted processes taking values in $(-1, +1)$ for $i \neq j$ and $\rho_{ii}(t) = 1, \forall i.$ Assume that the symmetric matrix

$$C(t) = \begin{bmatrix} \rho_{11}(t) & \rho_{12}(t) & \cdots & \rho_{1m}(t) \\ \rho_{21}(t) & \rho_{22}(t) & \cdots & \rho_{2m}(t) \\ \vdots & \vdots & & \vdots \\ \rho_{m1}(t) & \rho_{m2}(t) & \cdots & \rho_{mm}(t) \end{bmatrix}$$

is positive-definite for all $t$ almost surely. Because the matrix $C(t)$ is symmetric and positive definite, it has a matrix square root. In other words, there is a matrix

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1m}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2m}(t) \\ \vdots & \vdots & & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mm}(t) \end{bmatrix}$$

such that $C(t) = A(t)A^T(t)$, which when written componentwise is

$$\rho_{ij}(t) = \sum_{k=1}^{m} a_{ik}(t)a_{jk}(t), \quad \forall i, j = 1, \cdots, m.$$

This matrix can be chosen so that its component $a_{ij}(t)$ are adapted processes. Furthermore, the matrix $A(t)$ has an inverse

$$A^{-1}(t) = \begin{bmatrix} \alpha_{11}(t) & \alpha_{12}(t) & \cdots & \alpha_{1m}(t) \\ \alpha_{21}(t) & \alpha_{22}(t) & \cdots & \alpha_{2m}(t) \\ \vdots & \vdots & & \vdots \\ \alpha_{m1}(t) & \alpha_{m2}(t) & \cdots & \alpha_{mm}(t) \end{bmatrix},$$

which means that

$$\sum_{k=1}^{m} a_{ik}(t)\alpha_{kj} = \sum_{k=1}^{m} \alpha_{ik}(t)a_{kj} = \delta_{ij},$$

where $\delta_{ij}$ is the so-called *Kronecker delta.* We shall show that there exist $m$ independent Brownian motions $W_1(t), \cdots, W_m(t)$ such that

$$B_i(t) = \sum_{j=1}^{m} \int_0^t a_{ij}(t')dW_j(t'), \quad \forall i = 1, \cdots, m.$$

Writing in matrix notation,

$$d\mathbf{B}(t) = \mathbf{A}(t) \cdot d\mathbf{W}(t) \tag{2.3}$$

where we use the bold font to represent matrices and vectors.

- Initial conditions are assumed by construction such that $\mathbf{W}(t = 0) = \mathbf{0}$.
- $\mathbf{W}(t)$ has continuous paths because the vector of Brownian motions $\mathbf{B}(t)$ has continuous paths by definition and $\mathbf{W}(t)$ is expressed in terms of Itô's stochastic integrals with respect to $\mathbf{B}(t)$.
- $\mathbf{W}(t)$ is Itô's stochastic integral and hence *martingales.*
- Quadratic variation and cross variation. Invert the matrix equation 2.3,

  $$d\mathbf{W}(t) = \mathbf{A}^{-1}(t) \cdot d\mathbf{B}(t).$$

  In other words, for any $i = 1, \cdots, m$,

  $$dW_i(t) = \sum_{j=1}^{m} \alpha_{ij}(t)dB_j(t).$$

The quadratic variation ($i = j$) or cross variation ($i \neq j$)

$$
\begin{aligned}
[W_i, W_j]_t &= \int_0^t dW_i(t')dW_j(t') \\
&= \int_0^t \sum_{k=1}^m \sum_{l=1}^m \alpha_{ik}(t')\alpha_{jl}(t') \underbrace{dB_k(t')dB_l(t')}_{\rho_{kl}(t')dt'} \\
&= \int_0^t dt' \sum_{k=1}^m \sum_{l=1}^m \alpha_{ik}(t')\alpha_{jl}(t')\rho_{kl}(t') \\
&= \int_0^t dt' \sum_{k=1}^m \sum_{l=1}^m \alpha_{ik}(t')\alpha_{jl}(t') \sum_{n=1}^m a_{kn}(t')a_{ln}(t') \\
&= \int_0^t dt' \sum_{n=1}^m \left( \sum_{k=1}^m \alpha_{ik}(t')a_{kn}(t') \sum_{l=1}^m \alpha_{jl}(t')a_{ln}(t') \right) \\
&= \int_0^t dt' \sum_{n=1}^m \delta_{in}\delta_{jn} \\
&= \int_0^t dt'\delta_{ij} \\
&= t\delta_{ij}.
\end{aligned}
$$

Thus, we verified both the quadratic variation and the cross variation

$$
\begin{aligned}
[W_i]_t &= t, \quad \forall i; \\
[W_i, W_j]_t &= 0, \quad \forall i \neq j.
\end{aligned}
$$

In summary, $\mathbf{W}(t)$ is an $m$-dimensional Brownian motion where the components $W_1(t), \cdots, W_m(t)$ are $m$ independent Brownian motions.

Note: To find $\mathbf{A}^{-1}$ given $\mathbf{C}$, we first perform the eigenvalue decomposition of $\mathbf{C}$, which exists because $\mathbf{C}$ is positive-definite:

$$\mathbf{C} = \mathbf{X} \cdot \mathbf{D} \cdot \mathbf{X}^{-1},$$

where $\mathbf{X}$ is the matrix composed of eigenvectors of $\mathbf{C}$ and $\mathbf{D}$ is a diagonal matrix with positive eigenvalues of $\mathbf{C}$: $d_i, i = 1, \cdots, m$. The square-root matrix $\mathbf{A}$ of $\mathbf{C}$ is

$$\mathbf{A} = \mathbf{X} \cdot \mathbf{D}^{1/2} \cdot \mathbf{X}^{-1},$$

where $\mathbf{D}^{1/2}$ is a diagonal matrix with positive entries $\sqrt{d_i}, i = 1, \cdots, m$. Similarly, the inverse matrix $\mathbf{A}^{-1}$ of $\mathbf{A}$ can be written as

$$\mathbf{A}^{-1} = \mathbf{X} \cdot \mathbf{D}^{-1/2} \cdot \mathbf{X}^{-1},$$

where $\mathbf{D}^{-1/2}$ is a diagonal matrix with positive entries $1/\sqrt{d_i}, i = 1, \cdots, m$.

# 3 Exercise Three: Creating Correlated Brownian Motions from Independent Brownian Motions

Let $(W_1(t),\cdots,W_d(t))$ be a $d$-dimensional Brownian motion. In particular, these Brownian motions are independent of one another. Let $\sigma_{ij}(t), i=1,\cdots,m; j=1,\cdots,d$ be an $m\times d$ matrix-valued process adapted to the filtration associated with the $d$-dimensional Brownian motion. For $i=1,\cdots,m$, define

$$\sigma_i(t)=\left[\sum_{j=1}^{d}\sigma_{ij}^2(t)\right]^{\frac{1}{2}}$$

and assume this is never zero. Define also

$$B_i(t)=\sum_{j=1}^{d}\int_0^t\frac{\sigma_{ij}(t')}{\sigma_i(t')}dW_j(t').$$

**(i)** Show that, for each $i$, $B_i$ is a Brownian motion.
*Proof*: We use Lévy's characterization to proceed.

- Initial condition:
$$B_i(0)=\sum_{j=1}^{d}\int_0^{t=0}\frac{\sigma_{ij}(t')}{\sigma_i(t')}dW_j(t')=0.$$
- For each $i$, $B_i(t)$ has continuous paths because it's expressed in terms of a sum of Itô's stochastic integrals with respect to Brownian motions $W_j(t), j=1,\cdots,d$, which has continuous paths.
- For each $i$, $B_i(t)$ is martingale because it's a sum of Itô's stochastic integrals with respect to Brownian motions $W_j(t), j=1,\cdots,d$, each term being a martingale.
- In differential form,
$$dB_i(t)=\sum_{j=1}^{d}\frac{\sigma_{ij}(t)}{\sigma_i(t)}dW_j(t).$$

  The quadratic variation

$$\begin{aligned}
[B_i]_t &= \int_0^t dB_i(t')dB_i(t')\\
&= \int_0^t\sum_{j=1}^{d}\sum_{k=1}^{d}\frac{\sigma_{ij}(t')\sigma_{ik}(t')}{\sigma_i^2(t')}\underbrace{dW_j(t')dW_k(t')}_{=\delta_{jk}dt'}\\
&= \int_0^t\sum_{j=1}^{d}\sum_{k=1}^{d}\frac{\sigma_{ij}(t')\sigma_{ik}(t')}{\sigma_i^2(t')}\delta_{jk}dt'\\
&= \int_0^t\frac{\sum_{j=1}^{d}\sigma_{ij}(t')\sigma_{ij}(t')}{\sigma_i^2(t')}dt'\\
&= \int_0^t\frac{\sum_{j=1}^{d}\sigma_{ij}^2(t')}{\sigma_i^2(t')}dt'
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \frac{\sigma_i^2(t')}{\sigma_i^2(t')} dt' \\
&= \int_0^t dt' \\
&= t.
\end{aligned}
$$

According to Lévy's characterization, we verified that for each $i$, $B_i$ is a Brownian motion.

(b) Show that $dB_i(t)dB_j(t) = \rho_{ij}(t)dt$, where

$$\rho_{ij}(t) = \frac{1}{\sigma_i(t)\sigma_j(t)} \sum_{k=1}^{d} \sigma_{ik}(t)\sigma_{jk}(t).$$

*Proof*: By direct computation,

$$
\begin{aligned}
dB_i(t)dB_j(t) &= \sum_{k=1}^{d}\sum_{l=1}^{d} \frac{\sigma_{ik}(t)\sigma_{jl}(t)}{\sigma_i(t)\sigma_j(t)} \underbrace{dW_k(t)dW_l(t)}_{=\delta_{kl}dt} \\
&= \sum_{k=1}^{d}\sum_{l=1}^{d} \frac{\sigma_{ik}(t)\sigma_{jl}(t)}{\sigma_i(t)\sigma_j(t)} \delta_{kl}dt \\
&= \frac{1}{\sigma_i(t)\sigma_j(t)} \sum_{k=1}^{d} \sigma_{ik}(t)\sigma_{jk}(t)dt.
\end{aligned}
$$

# 4 Exercise Four: Self-financing Trading

The fundamental idea behind no-arbitrage pricing is to reproduce the payoff of a derivative security by trading in the underlying asset (which we call a stock) and the money market account. In discrete time, we let $X_k$ denote the value of the hedging portfolio at time $k$ and let $\Delta_k$ denote the number of shares of stock held between times $k$ and $k+1$. Then, at time $k$, after rebalancing (i.e., moving from a position of $\Delta_{k-1}$ to a position $\Delta_k$ in the stock), the amount in the money market account is $X_k - S_k\Delta_k$. The value of the portfolio at time $k+1$ is

$$X_{k+1} = \Delta_k S_{k+1} + (1+r)(X_k - \Delta_k S_k).$$

This formula can be rearranged to become

$$X_{k+1} - X_k = \Delta_k(S_{k+1} - S_k) + r(X_k - \Delta_k S_k),$$

which says that the gain between time $k$ and time $k+1$ is the sum of the capital gain on the stock holdings, $\Delta_k(S_{k+1} - S_k)$, and the interest earnings on the money market account, $r(X_k - \Delta_k S_k)$. The continuous-time analogue is

$$dX(t) = \Delta(t)dS(t) + r\,(X(t) - \Delta(t)S(t))\,dt.$$

Alternatively, one could define the value of a share of the money market account at time $k$ to be

$$M_k = (1+r)^k$$

and formulate the discrete-time model with two processes, $\Delta_k$ as before and $\Gamma_k$ denoting the number of shares of the money market account held at time $k$ after rebalancing. Then

$$X_k = \Delta_k S_k + \Gamma_k M_k,$$

so that

$$\begin{aligned} & X_{k+1} = \Delta_k S_{k+1} + (1+r)\Gamma_k M_k = \Delta_k S_{k+1} + \Gamma_k M_{k+1} \\ \Rightarrow\quad & X_{k+1} - X_k = \Delta_k(S_{k+1} - S_k) + \Gamma_k(M_{k+1} - M_k), \end{aligned}$$

which says that the gain between time $k$ and time $k+1$ is the sum of the capital gain on stock holdings, $\Delta_k(S_{k+1} - S_k)$, and the earnings from the money market investment, $\Gamma_k(M_{k+1} - M_k)$. But $\Delta_k$ and $\Gamma_k$ cannot be chosen arbitrarily. The agent arrives at time $k+1$ with some portfolio of $\Delta_k$ shares of stock and $\Gamma_k$ shares of the money money market and then rebalances. In terms of $\Delta_k$ and $\Gamma_k$, the value of the portfolio upon arrival at time $k+1$, after rebalancing, is

$$X_{k+1} = \Delta_{k+1} S_{k+1} + \Gamma_{k+1} M_{k+1}.$$

Setting these two values equal, we obtain the *discrete-time self-financing* condition

$$S_{k+1}(\Delta_{k+1} - \Delta_k) + M_{k+1}(\Gamma_{k+1} - \Gamma_k) = 0.$$

The first term is the cost of rebalancing in the stock, and the second is the cost of rebalancing in the money market account. If the sum of these two terms is not zero, then money must either be put into the position or can be taken out as a by-product of rebalancing. The point is that when the two processes $\Delta_k$ and $\Gamma_k$ are used to describe the evolution of the portfolio value $X_k$, then two equations

$$\begin{cases} X_{k+1} - X_k = \Delta_k(S_{k+1} - S_k) + \Gamma_k(M_{k+1} - M_k) \\ S_{k+1}(\Delta_{k+1} - \Delta_k) + M_{k+1}(\Gamma_{k+1} - \Gamma_k) = 0. \end{cases}$$

are required rather than the single equation

$$X_{k+1} - X_k = \Delta_k(S_{k+1} - S_k) + r(X_k - \Delta_k S_k)$$

when only the process $\Delta_k$ is used.

Finally, we note that we may rewrite the *discrete-time self-financing* condition as

$$S_k(\Delta_{k+1} - \Delta_k) + (S_{k+1} - S_k)(\Delta_{k+1} - \Delta_k) + M_k(\Gamma_{k+1} - \Gamma_k) + (M_{k+1} - M_k)(\Gamma_{k+1} - \Gamma_k) = 0.$$

This is suggestive of the continuous-time self-financing condition

$$S(t)d\Delta(t) + dS(t)d\Delta(t) + M(t)d\Gamma(t) + dM(t)d\Gamma(t) = 0,$$

which we derive below.

**(i)** In continuous time, let $M(t) = e^{rt}$ be the price of a share of the money market account at time $t$, let $\Delta(t)$ denote the number of shares of stock held at time $t$, and let $\Gamma(t)$ denote the number of shares of the money market account held at time $t$, so that the total portfolio value at time $t$ is

$$X(t) = \Delta(t)S(t) + \Gamma(t)M(t).$$

Derive the continuous-time self-financing condition.
*Solution*: Apply Itô's product rule to $X(t)$,

$$\begin{aligned}
& dX(t) \\
= \; & d(\Delta(t)S(t) + \Gamma(t)M(t)) \\
= \; & \Delta(t)dS(t) + S(t)d\Delta(t) + d\Delta(t)dS(t) + \Gamma(t)dM(t) + M(t)d\Gamma(t) + d\Gamma(t)dM(t) \\
= \; & \left[\Delta(t)dS(t) + \Gamma(t)\underbrace{dM(t)}_{=de^{rt}=rM(t)dt}\right] + [S(t)d\Delta(t) + d\Delta(t)dS(t) + M(t)d\Gamma(t) + d\Gamma(t)dM(t)] \\
= \; & [\Delta(t)dS(t) + r\Gamma(t)M(t)dt] + [S(t)d\Delta(t) + d\Delta(t)dS(t) + M(t)d\Gamma(t) + d\Gamma(t)dM(t)] \\
= \; & \underbrace{[\Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt]}_{\text{portfolio evolution}} + [S(t)d\Delta(t) + d\Delta(t)dS(t) + M(t)d\Gamma(t) + d\Gamma(t)dM(t)].
\end{aligned}$$

Equation the above expression to the portfolio evolution equation where the portfolio's value arises only from the initial investment and from capital gains or losses experienced from the assets held, i.e., injections or withdrawls of funds are not allowed:

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt,$$

we arrive at the continuous-time self-financing condition

$$\boxed{S(t)d\Delta(t) + d\Delta(t)dS(t) + M(t)d\Gamma(t) + d\Gamma(t)dM(t) = 0}.$$

$\square$

A common argument used to derive the Black-Scholes-Merton partial differential equation and delta-hedging formula goes like this. Let $c(t,x)$ be the price of a call at some time $t$ if the stock price at that time is $S(t) = x$. Form a portfolio that is long the call and short $\Delta(t)$ shares of stock, so that the value of the portfolio at time $t$ is $N(t) = c(t, S(t)) - \Delta(t)S(t)$. We want to choose $\Delta(t)$ so this is *instantaneously riskless*, in which case its value would have to grow at the interest rate. Otherwise, according to this argument, we could arbitrage this portfolio against the money market account. This means we should have $dN(t) = rN(t)dt$. We compute the differential of $N(t)$ and get

$$\begin{aligned} dN(t) &= c_t(t,S(t))dt + c_x(t,S(t))dS(t) + \frac{1}{2}c_{xx}(t,S(t))dS(t)dS(t) - \Delta(t)dS(t) \\ &= [c_x(t,S(t)) - \Delta(t)]\, dS(t) + \left[c_t(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right] dt. \end{aligned}$$

In order for this to be instantaneously riskless, we must cancel out the $dS(t)$ term, which contains the risk. This gives us the delta-hedging formula $\Delta(t) = c_x(t, S(t))$. Having chosen $\Delta(t)$ this way, we recall that we expect to have $dN(t) = rN(t)dt$, and this yields

$$rN(t)dt = \left[c_t(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right] dt. \tag{4.1}$$

But

$$N(t) = c(t,S(t)) - \Delta(t)S(t) = c(t,S(t)) - c_x(t,S(t))S(t),$$

and combining the above two equations yields the Black-Scholes-Merton partial differential equation

$$c_t(t,S(t)) + rS(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t)) = rc(t,S(t)).$$

One can question in the first step of this argument, where we failed to use Itô's product rule on the term $\Delta(t)S(t)$ when we differentiated $N(t)$. In discrete time, we hold $\Delta_k$ fixed for a period and let $S$ move, computing the capital gain according to the formula $\Delta_k(S_{k+1} - S_k)$, and we are attempting something analogous to that in continuous time. However, as soon as we set $\Delta(t) = c_x(t,S(t))$, then $\Delta(t)$ moves continuously in time and the differential of $N(t)$ is really

$$\begin{aligned} dN(t) = \;& c_t(t,S(t))dt + c_x(t,S(t))dS(t) + \frac{1}{2}c_{xx}(t,S(t))dS(t)dS(t) \\ & -\Delta(t)dS(t) - S(t)d\Delta(t) - d\Delta(t)dS(t). \end{aligned}$$

This exercise shows that the argument is correct after all. At least, Equation 4.1 is correct, and from that the Black-Scholes-Merton partial differential equation follows.

Recall that if we take $X(0) = c(0, S(0))$ and at each time $t$ hold $\Delta(t) = c_x(t, S(t))$ shares of stock, borrowing or investing in the money market as necessary to finance this, then at each time $t$ we have a portfolio of stock and a money market account valued at $X(t) = c(t, S(t))$. The amount invested in the money market account at each time $t$ is

$$X(t) - \Delta(t)S(t) = c(t, S(t) - \Delta(t)S(t) = N(t),$$

and so the number of money market account shares held at $t$ is

$$\Gamma(t) = \frac{N(t)}{M(t)}.$$

**(ii)** Now replace Equation 4.1 by its corrected version Equation 4.2 and use the continuous-time self-financing condition derived in part (i) to drive Equation 4.1.

*Solution*: Apply Itô's product rule to $N(t) = c(t, S(t)) - \Delta(t)S(t)$ and use the fact that $\Gamma(t)M(t) = X(t) - \Delta(t)S(t) = N(t)$, we have

$$\begin{aligned}
& dN(t) = dc(t, S(t) - d(\Delta(t)S(t)) \\
= \;& c_t(t,S(t))dt + c_x(t,S(t))dS(t) + \frac{1}{2}c_{xx}(t,S(t))dS(t)dS(t) \\
& -\Delta(t)dS(t) - S(t)d\Delta(t) - d\Delta(t)dS(t) \\
= \;& c_t(t,S(t))dt + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))dt \underbrace{-S(t)d\Delta(t) - d\Delta(t)dS(t)}_{=M(t)d\Gamma(t)+dM(t)d\Gamma(t)} \\
& + [c_x(t,S(t)) - \Delta(t)]\, dS(t) \\
\text{(self-financing)} \quad = \;& c_t(t,S(t))dt + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))dt + \underbrace{M(t)d\Gamma(t) + dM(t)d\Gamma(t)}_{=d(M(t)\Gamma(t))-\Gamma(t)dM(t)} \\
& + [c_x(t,S(t)) - \Delta(t)]\, dS(t) \\
\text{(Itô's product)} \quad = \;& c_t(t,S(t))dt + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))dt + d(M(t)\Gamma(t)) - \underbrace{\Gamma(t)dM(t)}_{=\Gamma(t)de^{rt}=r\Gamma(t)M(t)dt} \\
& + [c_x(t,S(t)) - \Delta(t)]\, dS(t) \\
= \;& c_t(t,S(t))dt + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))dt + d(\underbrace{M(t)\Gamma(t)}_{=N(t)}) - r\underbrace{\Gamma(t)M(t)}_{=N(t)}dt \\
& + [c_x(t,S(t)) - \Delta(t)]\, dS(t) \\
\text{(instantaneous riskless)} \quad = \;& c_t(t,S(t))dt + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))dt + dN(t) - rN(t)dt \\
& + [c_x(t,S(t)) - \Delta(t)]\, dS(t).
\end{aligned}$$

Cancelling out the term $dN(t)$ on both sides of the above equation, we have

$$rN(t)dt \quad = \quad \left[c_t(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right] dt + [c_x(t,S(t)) - \Delta(t)]\, dS(t).$$

Eliminate the risky term $dS(t)$ by invoking delta-hedging formula $\Delta(t) = c_x(t, S(t))$, we get

$$rN(t)dt = \left[c_t(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right] dt,$$

which reproduces Equation 4.1 as required. □

# 5 Exercise Five

Let $W(t)$ be a Brownian motion, and define

$$B(t) = \int_0^t \operatorname{sign}(W(s))\,dW(s)$$

where

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

**(i)** Show that $B(t)$ is a Brownian motion.
*Proof*: Use Lévy's characterization.

- Initial condition
$$B(t=0) = \int_0^{t=0} \operatorname{sign}(W(s))\,dW(s) = 0.$$
- $B(t)$ has continuous paths because it's expressed in terms of a sum of Itô's stochastic integrals with respect to a Brownian motion $W(t)$, which has continuous paths and the integrand $\operatorname{sign}(W(s))$ is always finite.
- Martingality. $B(t)$ is martingale because it's a sum of Itô's stochastic integrals with respect to a Brownian motion $W(t)$.
- Quadratic variation

$$\begin{aligned} [B]_t &= \int_0^t dB(t')dB(t') \\ &= \int_0^t \operatorname{sign}(W(t'))^2 dW(t')dW(t') \\ &= \int_0^t dt' \\ &= t. \end{aligned}$$

Thus, $B(t)$ is a Brownian motion.

**(ii)** Use Itô's product rule to compute $d[B(t)W(t)]$. Integrate both sides of the resulting equation and take expectations. Show that $\mathbb{E}[B(t)W(t)] = 0$, i.e. $B(t)$ and $W(t)$ are uncorrelated.
*Proof*: Differentiate:

$$\begin{aligned} d[B(t)W(t)] &= W(t)dB(t) + B(t)dW(t) + dB(t)dW(t) \\ &= \operatorname{sign}(W(t))W(t)dW(t) + B(t)dW(t) + \operatorname{sign}(W(t))dW(t)dW(t) \\ &= \operatorname{sign}(W(t))W(t)dW(t) + B(t)dW(t) + \operatorname{sign}(W(t))dt \end{aligned}$$

Integrate:

$$B(t)W(t) - \underbrace{B(0)W(0)}_{=0} = \underbrace{\int_0^t \operatorname{sign}(W(t'))W(t')dW(t')}_{\text{martingale}} + \underbrace{\int_0^t B(t')dW(t')}_{\text{martingale}} + \int_0^t \operatorname{sign}(W(t'))dt'.$$

Take expectations:

$$\begin{aligned}
\mathbb{E}[B(t)W(t)] &= \int_0^t \mathbb{E}[\text{sign}(W(t'))]dt' \\
&= \int_0^t \mathbb{E}[\mathbb{1}_{W(t')\geq 0} - \mathbb{1}_{W(t')<0}]dt' \\
&= \int_0^t \mathbb{E}[\mathbb{1}_{W(t')\geq 0}]dt' - \int_0^t \mathbb{E}[\mathbb{1}_{W(t')<0}]dt' \\
\text{(normally distributed)} \quad &= \int_0^t \underbrace{\mathbb{P}(W(t')\geq 0)}_{=1/2} dt' - \int_0^t \underbrace{\mathbb{P}(W(t')<0)}_{=1/2} dt' \\
&= \frac{1}{2}t - \frac{1}{2}t \\
&= 0 \quad \text{(uncorrelated)}.
\end{aligned}$$

**(iii)** Verify that

$$dW^2(t) = 2W(t)dW(t) + dt.$$

*Proof*: Because $W(t)$ is a Brownian motion, simply apply Itô's formula

$$dW^2(t) = 2W(t)dW(t) + dW(t)dW(t) = 2W(t)dW(t) + dt.$$

**(iv)** Use Itô's product rule to compute $d[B(t)W^2(t)]$. Integrate both sides of the resulting equation and take expectations to conclude that

$$\mathbb{E}[B(t)W^2(t)] \neq \mathbb{E}B(t)\cdot\mathbb{E}W^2(t).$$

Explain why this shows that, although they are uncorrelated normal random variables, $B(t)$ and $W(t)$ are not independent.
*Proof*: Differentiate:

$$\begin{aligned}
d[B(t)W^2(t)] &= W^2(t)dB(t) + B(t)dW^2(t) + dB(t)dW^2(t) \\
&= W^2(t)\text{sign}(W(t))dW(t) + 2B(t)W(t)dW(t) + B(t)dt \\
&\quad +\text{sign}(W(t))dW(t)(2W(t)dW(t) + dt) \\
&= W^2(t)\text{sign}(W(t))dW(t) + 2B(t)W(t)dW(t) + (B(t) + 2W(t)\text{sign}(W(t)))dt.
\end{aligned}$$

Integrate:

$$\begin{aligned}
B(t)W^2(t) - \underbrace{B(0)W^2(0)}_{=0} &= \underbrace{\int_0^t W^2(t')\text{sign}(W(t'))dW(t')}_{\text{martingale}} + 2\underbrace{\int_0^t B(t')W(t')dW(t')}_{\text{martingale}} \\
&\quad + \int_0^t (B(t') + 2W(t')\text{sign}(W(t')))dt'.
\end{aligned}$$

Take expectations:

$$
\begin{aligned}
\mathbb{E}[B(t)W^2(t)] &= \int_0^t \mathbb{E}[B(t') + 2W(t')\mathrm{sign}(W(t'))]dt' \\
&= \int_0^t \underbrace{\mathbb{E}[B(t')]}_{=0} dt' + 2\int_0^t \mathbb{E}[W(t')\mathrm{sign}(W(t'))]dt' \\
&= 2\int_0^t \mathbb{E}\left[W(t')\left(\mathbb{1}_{W(t')\geq 0} - \mathbb{1}_{W(t')<0}\right)\right] dt' \\
&= 2\left(\int_0^t \mathbb{E}\left[W(t')\mathbb{1}_{W(t')\geq 0}\right] dt' - \int_0^t \mathbb{E}\left[W(t')\mathbb{1}_{W(t')<0}\right] dt'\right) \\
&= 2\left(\int_0^t \mathbb{E}\left[W(t')\mathbb{1}_{W(t')\geq 0}\right] dt' + \int_0^t \mathbb{E}\left[-W(t')\mathbb{1}_{-W(t')>0}\right] dt'\right) \\
(-W(t) \text{ is Brownian motion}) \quad &= 4\int_0^t \mathbb{E}\left[W(t')\mathbb{1}_{W(t')\geq 0}\right] dt' \\
&= 4\int_0^t dt' \int_0^\infty x \frac{1}{\sqrt{2\pi t'}} e^{-\frac{x^2}{2t'}} dx \\
&= \frac{4}{\sqrt{2\pi}} \int_0^t \sqrt{t'} dt' \\
&\neq 0 = \mathbb{E}B(t) \cdot \mathbb{E}W^2(t).
\end{aligned}
$$

This shows that $B(t)$ and $W(t)$ are not independent because otherwise, $B(t)$ and $W^2(t)$ would be independent.

# 6 Exercise Six: Stop-Loss Start-Gain Paradox

Let $S(t)$ be a geometric Brownian motion with mean rate of return zero. In other words,

$$dS(t) = \sigma S(t)dW(t),$$

where the volatility $\sigma$ is constant. We assume the interest rate is $r = 0$.

Suppose we want to hedge a short position in a European call with strike price $K$ and expiration date $T$. We assume that the call is initially out of the money (i.e. $S(0) < K$). Starting with zero capital ($X(0) = 0$), we could try the following portfolio strategy: own one share of the stock whenever its prices strictly exceeds $K$, and own zero shares whenever its price is $K$ or less. In other words, we use the hedging portfolio process

$$\Delta(t) = \mathbb{1}_{(K,\infty)}(S(t)).$$

The value of this hedge follows the stochastic differential equation

$$dX(t) = \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt$$

and since $r = 0$ and $X(0) = 0$, we have

$$X(T) = \sigma \int_0^T \mathbb{1}_{(K,\infty)}(S(t))S(t)dW(t). \tag{6.1}$$

Executing this hedge requires us to borrow from the money market to buy a share of stock whenever the stock price rises across level $K$ and sell the share, repaying the money market debt, when it falls back across level $K$. Recall that we have taken the interest rate to be zero. This situation we are describing can also occur with a non-zero interest rate, but it is more complicated to set up. At expiration, if the stock price $S(T)$ is below $K$, there would appear to have been an even number of crossings of the level $K$, half in the up direction and half in the down direction, so that we would have bought and sold the stock repeatedly, each time at the same price $K$, and at the final time have no stock and zero debt to the money market. In other words, if $S(T) < K$, then $X(T) = 0$. On the other hand, if at the final time $S(T)$ is above $K$, we have bought the stock one more time than we sold it, so that we end with one share of stock and a debt of $K$ to the money market. Hence, if $S(T) > K$, we have $X(T) = S(T) - K$. If at the final time, $S(T) = K$, then we either own a share of stock valued at $K$ and have a money debt $K$ or we have sold the stock and have zero money market debt. In either case, $X(T) = 0$. According to this argument, regardless of the final stock price, we have $X(T) = (S(T) - K)^+$. This kind of hedging is called a *stop-loss start-gain strategy*.

**(i)** Discuss the practical aspects of implementing the stop-loss start-gain strategy described above. Can it be done?

*Answer*: There is an interesting *level-crossing property* of Brownian motion (also mentioned in the TA office hour session on October 3, 2015 Saturday). The level-crossing property of Brownian motion states that for every level $a$, the times $t$ such that $B(t) = a$ form a perfect set (i.e., every point of this set is a limit of points in this set). Thus, when a Brownian motion path reaches a given level at time $t$, it recrosses it *infinitely many time*s in every interval $[t, t+\Delta t]$. This is related to the fact that although continuous, the Brownian motion sample paths are nondifferentiable at any point almost surely.

As a consequence, the transaction cost associated with the strategy is extremely large because every time the asset prices crosses the level $K$, a trade occurs.

Another factor that makes the strategy unrealistic is the existence of bid-ask spread, i.e. one might not be able to buy and sell the share at the same price $K$.

**(ii)** Apart from the practical aspects, does the mathematics of continuous-time stochastic calculus suggest that the stop-loss start-gain strategy can be implemented? In other words, with $X(T)$ defined by Equation 6.1, is it really true that $X(T) = (S(T)-K)^+$?
*Answer*: First notice that Equation 6.1 is a martingale, thus

$$\mathbb{E}[X(T)] = X(0) = 0 \neq \mathbb{E}[(S(T)-K)^+].$$

The statement $X(T) = (S(T)-K)^+$ cannot hold.

The cause of the paradox is that the portfolio strategy

$$\Delta(t) = \mathbb{1}_{(K,\infty)}(S(t)).$$

is not self-financing and the simple form of Itô's lemma Equation 6.1 does not apply. To see the dynamics of portfolio evolution explicitly, one recognizes that according to the strategy $\Delta(t) = \mathbb{1}_{(K,\infty)}(S(t))$, the investor borrows $K$ units of cash from money market, buys one share of stock when $S(t) > K$; otherwise, the investor sells the stock and pays back $K$ units of cash debt in money market account, resulting in zero share of stock and zero cash. In other words, at any time $t$, the portfolio value

$$X(t) = \mathbb{1}_{\{S(t)>K\}}(S(t)-K).$$

Let $f(x) = \mathbb{1}_{\{x>K\}}(x-K)$ and

$$\begin{aligned} f_x(x) &= \underbrace{\delta(x-K)(x-K)}_{=0,\quad x\delta(x)\equiv 0} + \mathbb{1}_{\{x>K\}} = \mathbb{1}_{\{x>K\}} \\ f_{xx}(x) &= \delta(x-K), \end{aligned}$$

where we have used the properties of Dirac-$\delta$ function $\delta(x-K) = \frac{\partial}{\partial x}\mathbb{1}_{\{x>K\}}$. Then apply Itô's formula,

$$\begin{aligned} dX(t) &= \mathbb{1}_{\{S(t)>K\}}dS(t) + \frac{1}{2}\delta(S(t)-K)dS(t)dS(t) \\ &= \sigma\mathbb{1}_{\{S(t)>K\}}S(t)dW(t) + \frac{1}{2}\sigma^2\delta(S(t)-K)S^2(t)dt. \end{aligned}$$

Integrate over time, we get the corrected version of Equation 6.1

$$\boxed{X(T) = \sigma \int_0^T \mathbb{1}_{\{S(t)>K\}} S(t)\,dW(t) + \frac{1}{2}\sigma^2 \int_0^T \delta(S(t)-K) S^2(t)\,dt,} \tag{6.2}$$

where the first term is exactly what we have in the paradoxical Equation 6.1 and the second term missing from Equation 6.1 makes the strategy non-self-financing. Note that the *self-financing* condition in our case of zero interest rate is $dX(t) = \Delta(t)dS(t)$; namely, the portfolio's value arises only from the initial investment and from capital gains or losses experienced from the assets held, and intermediate injections or withdrawls of funds are not allowed.

Note: intuitively, this extra term is nonzero because of the *infinite crossing* property of (geometric) Brownian motion. Once the stock price reaches the exercise price $K$, it returns to this path infinitely, often over any succeeding interval of time, no matter how small. Each time the stock price returns, the investor must make a decision whether to hold a stock position or hold nothing. If the investor holds nothing, as specified by the *stop-loss start-gain* strategy, external financing is required if the stock price subsequently rises. Conversely, if the strategy were altered so that the investor holds a stock position at the money, then external financing is required if the stock price subsequently falls. Thus, irrespective of the decision made, there is some likelihood that the stock price will move in such a way that the strategy will require external financing. For reference, in the general case where interest rate is non-zero, see *The Stop-Loss Start-Gain Paradox and Option Valuation: A New Decomposition into Intrinsic and Time Value*, Peter P. Carr and Robert A. Jarrow.

BARUCH, MFE

# MTH 9831 Assignment Six

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October 21, 2015

## 1 Exercise One: Interpretation of $N(d_+)$

According to the Black-Scholes-Merton formula, the value at time zero of a European call on a stock whose initial price is $S(0) = x$ is given by

$$c(0,x) = xN(d_+(T,x)) - Ke^{-rT}N(d_-(T,x)),$$

where

$$\begin{aligned} d_+(T,x) &= \frac{1}{\sigma\sqrt{T}}\left[\log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)\right], \\ d_-(T,x) &= d_+(T,x) - \sigma\sqrt{T}. \end{aligned}$$

The stock is modelled as a geometric Brownian motion with constant volatility $\sigma > 0$, the interest rate is constant $r$, the call strike is $K$, and the call expiration time is $T$. This formula is obtained by computing the discounted expected payoff of the call under the risk-neutral measure,

$$\begin{aligned} c(0,x) &= \tilde{\mathbb{E}}\left[e^{-rT}(S(T)-K)^+\right] \\ &= \tilde{\mathbb{E}}\left[e^{-rT}\left(x\exp\left\{\sigma\tilde{W}(T) + \left(r - \frac{1}{2}\sigma^2\right)T\right\} - K\right)^+\right], \end{aligned}$$

where $\tilde{W}$ is a Brownian motion under the risk-neutral measure $\tilde{\mathbb{P}}$. The*delta* of this option is computed to be $c_x(0,x) = N(d_+(T,x))$. This problem provides an alternative way to compute $c_x(0,x)$.

**(i)** We begin by with the observation that if $h(s) = (s-K)^+$, then

$$h'(s) = \begin{cases} 0, & \text{if } s < K, \\ 1, & \text{if } s > K. \end{cases}$$

If $s = K$, then $h'(s)$ is undefined, but that will not matter in what follows because $S(T)$ has zero probability of taking the value $K$. Using the formula for $h'(s)$, differentiate inside the expected value of Equation 1.1 to obtain a formula for $c_x(0,x)$.
*Solution*: Compute

$$\begin{aligned} c_x(0,x) &= \frac{\partial}{\partial x}\tilde{\mathbb{E}}\left[e^{-rT}\left(x\exp\left\{\sigma\tilde{W}(T) + \left(r - \frac{1}{2}\sigma^2\right)T\right\} - K\right)^+\right] \\ &= \frac{\partial}{\partial x}\tilde{\mathbb{E}}\left[e^{-rT}h\left(x\exp\left\{\sigma\tilde{W}(T) + \left(r - \frac{1}{2}\sigma^2\right)T\right\}\right)\right] \\ &= \tilde{\mathbb{E}}\left[e^{-rT}\frac{\partial}{\partial x}h\left(x\exp\left\{\sigma\tilde{W}(T) + \left(r - \frac{1}{2}\sigma^2\right)T\right\}\right)\right] \\ &= \tilde{\mathbb{E}}\left[e^{-rT}h'\left(x\exp\left\{\sigma\tilde{W}(T) + \left(r - \frac{1}{2}\sigma^2\right)T\right\}\right) \times \exp\left\{\sigma\tilde{W}(T) + \left(r - \frac{1}{2}\sigma^2\right)T\right\}\right] \\ &= e^{-\frac{1}{2}\sigma^2 T}\tilde{\mathbb{E}}\left[e^{\sigma\tilde{W}(T)}\mathbf{1}_{x\exp\{\sigma\tilde{W}(T) + (r - \frac{1}{2}\sigma^2)T\} > K}\right] \end{aligned}$$

$$
\begin{aligned}
(Z\sim N(0,1)) \quad &= \quad e^{-\frac{1}{2}\sigma^2 T}\tilde{\mathbb{E}}\left[e^{\sigma\sqrt{T}Z}\mathbf{1}_{Z>\frac{1}{\sigma\sqrt{T}}[\log\frac{K}{x}-(r-\frac{1}{2}\sigma^2()T]}\right] \\
&= \quad e^{-\frac{1}{2}\sigma^2 T}\int_{-\infty}^{+\infty} dz\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}e^{\sigma\sqrt{T}z}\mathbf{1}_{z>\frac{1}{\sigma\sqrt{T}}[\log\frac{K}{x}-(r-\frac{1}{2}\sigma^2)T]} \\
&= \quad \int_{-\infty}^{+\infty} dz\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-\sigma\sqrt{T})^2}\mathbf{1}_{z-\sigma T>-\frac{1}{\sigma\sqrt{T}}[\log\frac{x}{K}+(r+\frac{1}{2}\sigma^2)T]} \\
&= \quad \int_{-d_+}^{+\infty} dz'\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z'^2} \\
\text{(symmetry)} \quad &= \quad \int_{-\infty}^{+d_+} dz'\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z'^2} \\
&= \quad N(d+(T,x)).
\end{aligned}
$$

**(ii)** Show that the formula you obtained in **(i)** can be rewritten as

$$c_x(0,x)=\hat{\mathbb{P}}(S(T)>K),$$

where $\hat{\mathbb{P}}$ is a probability measure equivalent to $\tilde{\mathbb{P}}$. Show that

$$\hat{W}(t)=\tilde{W}(t)-\sigma t$$

is a Brownian motion under $\hat{\mathbb{P}}$.
*Proof*: Invoke Girsanov's theorem in one dimension by choosing $\Theta(t)=-\sigma$,

$$
\begin{aligned}
Z(t) \quad &= \quad \exp\left\{-\int_0^t\Theta(u)d\tilde{W}(u)-\frac{1}{2}\int_0^t\Theta^2(u)du\right\}=e^{\sigma\tilde{W}(t)-\frac{1}{2}\sigma^2 t}, \\
\hat{W}(t) \quad &= \quad \tilde{W}(t)+\int_0^t\Theta(u)du=\tilde{W}(t)-\sigma t.
\end{aligned}
$$

The integrability assumption

$$\tilde{\mathbb{E}}\int_0^T\Theta^2(u)Z^2(u)du=\sigma^2\tilde{\mathbb{E}}\int_0^T e^{2\sigma\tilde{W}(u)-\sigma^2 u}du<\infty$$

holds. Let $Z=Z(T)$. Then $\mathbb{E}[\tilde{Z}]=1$ and under the probability measure

$$\hat{\mathbb{P}}(A)=\int_A Z(\omega)d\tilde{\mathbb{P}}(\omega),\quad \forall A\in\mathscr{F},$$

the process $\hat{W}(t)_{0\le t\le T}$ is a Brownian motion according to Girsanov's theorem. Under this probability measure,

$$
\begin{aligned}
c_x(0,x) \quad &= \quad e^{-\frac{1}{2}\sigma^2 T}\tilde{\mathbb{E}}\left[e^{\sigma\tilde{W}(T)}\mathbf{1}_{x\exp\{\sigma\tilde{W}(T)+(r-\frac{1}{2}\sigma^2)T\}>K}\right] \\
&= \quad e^{-\frac{1}{2}\sigma^2 T}\hat{\mathbb{E}}\left[\frac{1}{Z}e^{\sigma\tilde{W}(T)}\mathbf{1}_{x\exp\{\sigma\tilde{W}(T)+(r-\frac{1}{2}\sigma^2)T\}>K}\right] \\
&= \quad e^{-\frac{1}{2}\sigma^2 T}\hat{\mathbb{E}}\left[e^{-\sigma\tilde{W}(t)+\frac{1}{2}\sigma^2 t}e^{\sigma\tilde{W}(T)}\mathbf{1}_{x\exp\{\sigma\tilde{W}(T)+(r-\frac{1}{2}\sigma^2)T\}>K}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \hat{\mathbb{E}}\left[\mathbf{1}_{x\exp\{\sigma\tilde{W}(T)+\left(r-\frac{1}{2}\sigma^2\right)T\}>K}\right] \\
&= \hat{\mathbb{E}}\left[\mathbf{1}_{x\exp\{\sigma\tilde{W}(T)+\left(r-\frac{1}{2}\sigma^2\right)T\}>K}\right] \\
&= \hat{\mathbb{E}}\left[\mathbf{1}_{S(T)>K}\right] \\
&= \hat{\mathbb{P}}(S(T)>K).
\end{aligned}
$$

**(iii)** Rewrite $S(T)$ in terms of $\hat{W}(T)$, and then show that

$$
\hat{\mathbb{P}}(S(T)>K) = \hat{\mathbb{P}}\left\{-\frac{\hat{W}(T)}{\sqrt{T}} < d_+(T,x)\right\} = N(d_+(T,x)).
$$

*Proof*: Rewrite $S(T)$ in terms of $\hat{W}(T)$,

$$
S(T) = x\exp\left\{\sigma\tilde{W}(T) + \left(r - \frac{1}{2}\sigma^2\right)T\right\} = x\exp\left\{\sigma\hat{W}(T) + \left(r + \frac{1}{2}\sigma^2\right)T\right\}.
$$

The probability

$$
\begin{aligned}
\hat{\mathbb{P}}(S(T)>K) &= \hat{\mathbb{P}}\left(x\exp\left\{\sigma\hat{W}(T) + \left(r+\frac{1}{2}\sigma^2\right)T\right\} > K\right) \\
&= \hat{\mathbb{P}}\left(\underbrace{-\frac{\hat{W}(T)}{\sqrt{T}}}_{\sim N(0,1)} < \underbrace{\frac{1}{\sigma\sqrt{T}}\left[\log\frac{x}{K} + \left(r+\frac{1}{2}\sigma^2\right)\right]}_{d_+}\right) \\
&= N\left(d_+(T,x)\right).
\end{aligned}
$$

## 2 EXERCISE TWO: BINOMIAL REPRESENTATION THEOREM

Consider a binomial model with parameters $u, d, r$; $d < 1 + r < u$. Let $\tilde{p}$ be the risk-neutral probability of the stock to go up in any one period (stock movements are assumed to be independent) and $S_n$ be the stock price at time $n$. We know that the discounted stock price process $(\tilde{S}_n)_{n\geq 0}$, wher $\tilde{S}_n = (1+r)^{-n} S_n$, is a martingale (with respect to its natural filtration and the risk-neutral measure). We also know that the discrete stochastic integral of a bounded predictable sequence with respect to $(\tilde{S}_n)_{n\geq 0}$ is again a martingale. The converse of this statement also holds and is known as the Binomial Representation Theorem: let $(\tilde{V}_n)_{n\geq 0}$ be a martingale with respect to the natural filtration $(\tilde{S}_n)_{n\geq 0}$. Then there is a predictable process $(H_n)_{n\geq 1}$ such that

$$\tilde{V}_n = \tilde{V}_0 + \sum_{i=1}^{n} H_i(\tilde{S}_i - \tilde{S}_{i-1}). \tag{2.1}$$

Check that the process defined by

$$H_n(\omega_1 \cdots \omega_{n-1}\omega_n) \triangleq \frac{V_n(\omega_1 \cdots \omega_{n-1} u) - V_n(\omega_1 \cdots \omega_{n-1} d)}{S_n(\omega_1 \cdots \omega_{n-1} u) - S_n(\omega_1 \cdots \omega_{n-1} d)}$$

satisfies the requirements. Here $(\omega_1 \cdots \omega_n)$ represents a path on the $n$-step binomial tree, $\omega_i \in \{u, d\}$.

The significance of this representation is that if we think of $\tilde{V}_n$ as the discounted value of a contingent claim at time $n$ then the process $(H_n)_{n\geq 1}$ is the hedging strategy ($H_n$ is the number of shares of the underlying stock in the replicating portfolio to be held over the $n$-th period). So this proves that every contingent claim can be hedged and tells you exactly how (in this model). Compare with the Martingale Representation Theorem from lecture.

*Proof*: First we note that the process $H_n(\omega_1 \cdots \omega_n)$ defined above is *predictable* because it depends only on the first $(n-1)$ steps $(\omega_1 \cdots \omega_{n-1})$ of the path, which is $\mathscr{F}_{n-1}$-measurable, where $(\mathscr{F}_n)_{n\geq 1}$ is the natural filtration of $(\tilde{S}_n)_{n\geq 0}$. According to **Theorem 1.19** (Discrete stochastic integral with respect to a martingale is a martingale) in Lecture 5 Refresher notes, the discrete stochastic integral Equation 2.1 of a bounded predictable sequence with respect to $(\tilde{S}_n)_{n\geq 0}$ is again a martingale.

Since $\tilde{V}_n$ is a martingale, for any $1 \leq i \leq n$,

$$\begin{aligned} &\tilde{V}_{i-1}(\omega_1 \cdots \omega_{i-1}) = \tilde{\mathbb{E}}\left[\tilde{V}_i(\omega_1 \cdots \omega_i) | \mathscr{F}_{i-1}\right] = \tilde{p}\tilde{V}_i(\omega_1 \cdots \omega_{i-1} u) + \tilde{q}\tilde{V}_i(\omega_1 \cdots \omega_{i-1} d) \\ \Rightarrow\ &\tilde{p}\left[\tilde{V}_{i-1}(\omega_1 \cdots \omega_{i-1}) - \tilde{V}_i(\omega_1 \cdots \omega_{i-1} u)\right] = -\tilde{q}\left[\tilde{V}_{i-1}(\omega_1 \cdots \omega_{i-1}) - \tilde{V}_i(\omega_1 \cdots \omega_{i-1} d)\right] \end{aligned} \tag{2.2}$$

For the same, reason, since $\tilde{S}_n$ is a martingale, for any $1 \leq i \leq n$,

$$\tilde{p}\left[\tilde{S}_{i-1}(\omega_1 \cdots \omega_{i-1}) - \tilde{S}_i(\omega_1 \cdots \omega_{i-1} u)\right] = -\tilde{q}\left[\tilde{S}_{i-1}(\omega_1 \cdots \omega_{i-1}) - \tilde{S}_i(\omega_1 \cdots \omega_{i-1} d)\right]. \tag{2.3}$$

Dividing Equation 2.2 by Equation 2.3, we get

$$\frac{\tilde{V}_i(\omega_1 \cdots \omega_{i-1} u) - \tilde{V}_{i-1}(\omega_1 \cdots \omega_{i-1})}{\tilde{S}_i(\omega_1 \cdots \omega_{i-1} u) - \tilde{S}_{i-1}(\omega_1 \cdots \omega_{i-1})} = \frac{\tilde{V}_i(\omega_1 \cdots \omega_{i-1} d) - \tilde{V}_{i-1}(\omega_1 \cdots \omega_{i-1})}{\tilde{S}_i(\omega_1 \cdots \omega_{i-1} d) - \tilde{S}_{i-1}(\omega_1 \cdots \omega_{i-1})} := \Delta_i(\omega_1 \cdots \omega_{i-1}\omega_i). \tag{2.4}$$

Notice that $\Delta_i(\omega_1\cdots\omega_{i-1}\omega_i)$ defined above is $\mathscr{F}_{i-1}$-measurable and does NOT depend on $\omega_i$. Rewriting the above equation as

$$\begin{aligned}
\left[\tilde{V}_{i-1}(\omega_1\cdots\omega_{i-1}) - \tilde{V}_i(\omega_1\cdots\omega_{i-1}u)\right] &= \Delta_i(\omega_1\cdots\omega_{i-1}\cancel{\omega_i})\left[\tilde{S}_{i-1}(\omega_1\cdots\omega_{i-1}) - \tilde{S}_i(\omega_1\cdots\omega_{i-1}u)\right],\\
\left[\tilde{V}_{i-1}(\omega_1\cdots\omega_{i-1}) - \tilde{V}_i(\omega_1\cdots\omega_{i-1}d)\right] &= \Delta_i(\omega_1\cdots\omega_{i-1}\cancel{\omega_i})\left[\tilde{S}_{i-1}(\omega_1\cdots\omega_{i-1}) - \tilde{S}_i(\omega_1\cdots\omega_{i-1}d)\right],
\end{aligned}$$

subtracting the above two equations from one another, we get

$$\Delta_i(\omega_1\cdots\omega_{i-1}\omega_i) = \frac{\tilde{V}_i(\omega_1\cdots\omega_{i-1}u) - \tilde{V}_i(\omega_1\cdots\omega_{i-1}d)}{\tilde{S}_i(\omega_1\cdots\omega_{i-1}u) - \tilde{S}_i(\omega_1\cdots\omega_{i-1}d)}$$

which is exactly how we defined $H_n(\omega_1\cdots\omega_{n-1}\omega_n)$:

$$H_i(\omega_1\cdots\omega_{i-1}\omega_i) = \Delta_i(\omega_1\cdots\omega_{i-1}\omega_i), \quad \forall 1 \le i \le n.$$

Recalling Equation 7.1, we proved that

$$H_i(\omega_1\cdots\omega_{i-1}\omega_i) \triangleq \frac{\tilde{V}_i(\omega_1\cdots\omega_{i-1}u) - \tilde{V}_i(\omega_1\cdots\omega_{i-1}d)}{\tilde{S}_i(\omega_1\cdots\omega_{i-1}u) - \tilde{S}_i(\omega_1\cdots\omega_{i-1}d)} = \frac{\tilde{V}_i(\omega_1\cdots\omega_{i-1}\omega_i) - \tilde{V}_{i-1}(\omega_1\cdots\omega_{i-1})}{\tilde{S}_i(\omega_1\cdots\omega_{i-1}\omega_i) - \tilde{S}_{i-1}(\omega_1\cdots\omega_{i-1})}.$$

Thus, the remaining task to verify Equation 2.1 is straightforward,

$$\begin{aligned}
\tilde{V}_n &= \tilde{V}_0 + \sum_{i=1}^{n}(\tilde{V}_i - \tilde{V}_{i-1})\\
&= \tilde{V}_0 + \sum_{i=1}^{n}\frac{\tilde{V}_i - \tilde{V}_{i-1}}{\tilde{S}_i - \tilde{S}_{i-1}}(\tilde{S}_i - \tilde{S}_{i-1})\\
&= \tilde{V}_0 + \sum_{i=1}^{n} H_i(\tilde{S}_i - \tilde{S}_{i-1}),
\end{aligned}$$

which is the *Binomial Martingale Representation Theorem*. Compare with the Martingale Representation Theorem from lecture 6

$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u)\,dB(u), \quad 0 \le t \le T,$$

we observe the formal similarity between the two.

## 3 Exercise Three: Martingale Representation Theorem

Prove Corollary 5.3.2 in textbook:

> Let $W(t)_{0\le t\le T}$ be a Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and let $\mathscr{F}(t)_{0\le t\le T}$, be the filtration generated by this Brownian motion. Let $\Theta(t)_{0\le t\le T}$ bean adapted process, define
>
> $$\begin{aligned} Z(t) &= \exp\left\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\right\} \\ \tilde{W}(t) &= W(t) + \int_0^t \Theta(u)du, \end{aligned}$$
>
> and assume that $\mathbb{E}\int_0^T \Theta^2(u)Z^2(u)du < \infty$. Set $Z = Z(T)$. Then $\mathbb{E}[Z] = 1$, and under the probability measure $\tilde{\mathbb{P}}$ given by
>
> $$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega), \quad \forall A \in \mathscr{F},$$
>
> the process $\tilde{W}(t)_{0\le t\le T}$ is a Brownian motion.
>
> Now let $\tilde{M}(t)_{0\le t\le T}$ be a martingale under $\tilde{\mathbb{P}}$. Then there is an adapted process $\tilde{\Gamma}(u)_{0\le u\le T}$, such that
>
> $$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u)d\tilde{W}(u), \quad 0 \le t \le T. \tag{3.1}$$

by the following steps.

**(i)** Compute the differential of $\frac{1}{Z(t)}$.
*Solution*: We do the computation step by step. Let

$$X(t) = \int_0^t \Theta(u)dW(u) + \frac{1}{2}\int_0^t \Theta^2(u)du \Rightarrow dX(t) = \Theta(t)dW(t) + \frac{1}{2}\Theta^2(t)dt.$$

Write $Z(t) = e^{-X(t)}$ and let $f(t,x) = e^{-x}$, then $f_t(t,x) = 0, f_x(t,x) = -e^{-x}, f_{xx}(t,x) = e^{-x}$,

$$dZ(t) = -e^{-X(t)}dX(t) + \frac{1}{2}e^{-X(t)}dX(t)dX(t) = -Z(t)\Theta(t)dW(t).$$

Let $g(t,x) = \frac{1}{x}$, then $g_t(t,x) = 0, g_x(t,x) = -\frac{1}{x^2}, g_{xx}(t,x) = \frac{2}{x^3}$,

$$d\frac{1}{Z(t)} = -\frac{1}{Z^2(t)}dZ(t) + \frac{1}{2}\times\frac{2}{Z^3(t)}dZ(t)dZ(t) = \frac{1}{Z(t)}\left(\Theta(t)dW(t) + \Theta^2(t)dt\right).$$

**(ii)** Let $\tilde{M}(t)_{0\le t\le T}$ be a martingale under $\tilde{\mathbb{P}}$. Show that $M(t) = Z(t)\tilde{M}(t)$ is a martingale under $\mathbb{P}$.

*Proof*: First, $M(t) = Z(t)\tilde{M}(t)$ is integrable $\mathbb{E}[|M(t)|] = \mathbb{E}[|Z(t)| \cdot |\tilde{M}(t)|] < \infty$ under the reasonable integrability condition $\mathbb{E}\int_0^T \Theta^2(u)Z^2(u)du < \infty$. Next, $M(t) = Z(t)\tilde{M}(t)$ is $\mathscr{F}(t)$-measurable because both $Z(t)$ and $\tilde{M}(t)$ are $\mathscr{F}(t)$-measurable. Note that $\tilde{M}(t)$ is an integral of the adapted process $\tilde{\Gamma}(t)$ with respect the Brownian motion up to time $t$.

Finally, invoke **Lemma 5.4** from lecture notes, under probability measure $\mathbb{P}$, $\forall 0 \le s \le t \le T$,

$$\mathbb{E}[M(t)|\mathscr{F}(s)] = \mathbb{E}\left[Z(t)\tilde{M}(t)|\mathscr{F}(s)\right] = Z(s)\tilde{E}\left[\tilde{M}(t)|\mathscr{F}(s)\right] = Z(s)\tilde{M}(s) = M(s).$$

We conclude that $M(t) = Z(t)\tilde{M}(t)$ is a martingale under $\mathbb{P}$.

**(iii)** According to Martingale Representation Theorem in one dimension, there is an adapted process $\Gamma(u)_{0\le u\le T}$ such that

$$M(t) = M(0) + \int_0^t \Gamma(u)dW(u), \quad 0 \le t \le T.$$

Write $\tilde{M}(t) = M(t) \cdot \frac{1}{Z(t)}$ and take its differential using Itô's product rule.
*Solution*: Apply Itô's product rule

$$\begin{aligned}
d\tilde{M}(t) &= d\left[M(t) \cdot \frac{1}{Z(t)}\right] \\
&= M(t)d\frac{1}{Z(t)} + \frac{1}{Z(t)}dM(t) + dM(t)d\frac{1}{Z(t)} \\
&= \frac{M(t)}{Z(t)}\left(\Theta(t)dW(t) + \Theta^2(t)dt\right) + \frac{\Gamma(t)}{Z(t)}dW(t) + \frac{\Gamma(t)\Theta(t)}{Z(t)}dt \\
&= \frac{M(t)\Theta(t) + \Gamma(t)}{Z(t)}\left[dW(t) + \Theta(t)dt\right] \\
&= \frac{M(t)\Theta(t) + \Gamma(t)}{Z(t)}d\tilde{W}(t).
\end{aligned}$$

**(iv)** Show that the differential of $\tilde{M}(t)$ is the sum of an adapted process, which we call $\tilde{\Gamma}(t)$, times $d\tilde{W}(t)$, and zero times $dt$. Integrate to obtain Equation 3.1.
*Proof*: We have already proved in part **(iii)** that

$$d\tilde{M}(t) = \frac{M(t)\Theta(t) + \Gamma(t)}{Z(t)}d\tilde{W}(t).$$

This is the sum of an adapted process, which we call $\tilde{\Gamma}(t)$, times $d\tilde{W}(t)$, and zero times $dt$, where

$$\tilde{\Gamma}(t) \triangleq \frac{M(t)\Theta(t) + \Gamma(t)}{Z(t)}.$$

Thus,

$$d\tilde{M}(t) = \tilde{\Gamma}(t)d\tilde{W}(t).$$

Integrate over time, we obtain Equation 3.1.

# 4 Exercise Four: Every Strictly Positive Price Process is a Generalized GBM

(Every strictly positive asset is a generalized geometric Brownian motion). Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space on which is defined a Brownian motion $W(t)_{0\le t\le T}$. Let $\mathscr{F}(t)_{0\le t\le T}$ be the filtration generated by this Brownian motion. Assume there is a unique risk-neutral measure $\tilde{\mathbb{P}}$, and let $\tilde{W}(t)_{0\le t\le T}$ be the Brownian motion under $\tilde{\mathbb{P}}$ obtained by an application of Girsanov's Theorem. A corollary of the Martingale Representation Theorem asserts that very martingale $\tilde{M}(t)_{0\le t\le T}$ under $\tilde{\mathbb{P}}$ can be written as a stochastic integral with respect to $\tilde{W}(t)_{0\le t\le T}$.

Now let $V(T)$ be an almost surely positive ('almost surely' means with probability one under both $\mathbb{P}$ and $\tilde{\mathbb{P}}$ since these two measures are equivalent), $\mathscr{F}(t)$-measurable random variable. According to the risk-neutral pricing formula, the price at time $t$ of a security paying $V(T)$ at time $T$ is

$$V(t) = \tilde{E}\left[e^{-\int_t^T R(u)du} V(T) \middle| \mathscr{F}(t)\right], \quad 0 \le t \le T. \tag{4.1}$$

**(i)** Show that there exists an adapted process $\tilde{\Gamma}(t)_{0\le t\le T}$, such that

$$dV(t) = R(t)V(t)dt + \frac{\tilde{\Gamma}(t)}{D(t)} d\tilde{W}(t), \quad 0 \le t \le T.$$

*Proof*: Because the discount process

$$D(t) \triangleq e^{-\int_0^t R(s)ds} \quad \Rightarrow \quad dD(t) = -R(t)D(t)dt$$

is $\mathscr{F}(t)$-measurable, it can be moved in and out the expectation conditioned by $\mathscr{F}(t)$. Thus, the discounted process

$$\begin{aligned}
D(t)V(t) &= D(t)\tilde{E}\left[e^{-\int_t^T R(u)du} V(T) \middle| \mathscr{F}(t)\right] \\
&= \tilde{E}\left[D(t)e^{-\int_t^T R(u)du} V(T) \middle| \mathscr{F}(t)\right] \\
&= \tilde{E}\left[e^{-\int_0^T R(u)du} V(T) \middle| \mathscr{F}(t)\right] \\
&= \tilde{E}\left[D(T)V(T)|\mathscr{F}(t)\right], \quad 0 \le t \le T,
\end{aligned}$$

is a martingale under the probability measure $\tilde{\mathbb{P}}$, which can be easily verified by the *tower property* (or *iterated conditioning*). According to Martingale Representation Theorem, there exists an adapted process $\tilde{\Gamma}(t)_{0\le t\le T}$ such that

$$D(t)V(t) = D(0)V(0) + \int_0^t \tilde{\Gamma}(t')d\tilde{W}(t').$$

Differentiate with respect to time, we get

$$V(t)dD(t) + D(t)dV(t) + \underbrace{dD(t)dV(t)}_{=0,\, dD(t) \text{ contains } dt \text{ term only}} = \tilde{\Gamma}(t)d\tilde{W}(t)$$

$$\Rightarrow \quad -V(t)R(t)D(t)dt + D(t)dV(t) = \tilde{\Gamma}(t)d\tilde{W}(t)$$
$$\Rightarrow \quad dV(t) = R(t)V(t)dt + \frac{\tilde{\Gamma}(t)}{D(t)}d\tilde{W}(t), \quad 0 \le t \le T. \tag{4.2}$$

**(ii)** Show that, for each $t \in [0, T]$, the price of the derivative security $V(t)$ at time $t$ is almost surely positive.
*Proof*: Because $V(T)$ is assumed to be almost surely positive under both $\mathbb{P}$ and $\tilde{\mathbb{P}}$ and the discount factor $e^{-\int_t^T R(u)du} > 0$, the price of the derivative security $V(t)$ expressed in terms of the conditional expectation

$$V(t) = \tilde{E}\left[e^{-\int_t^T R(u)du}V(T)\middle|\mathscr{F}(t)\right], \quad 0 \le t \le T.$$

of an almost surely positive quantity is still almost surely positive. Assume that $A = \{V(t) = 0\}$ has a non-zero measure with respect to $\mathbb{P}$ or $\tilde{\mathbb{P}}$, since $A \in \mathscr{F}_t$, we have

$$0 = \tilde{\mathbb{E}}[\mathbf{1}_A V(t)] = \tilde{\mathbb{E}}\left[\mathbf{1}_A e^{-\int_t^T R(u)du}V(T)\right],$$

but this means that $V(T) = 0$ on $A$ which has a non-zero measure, contradicting the assumption that $V(T) > 0$ almost surely.

**(iii)** Conclude from **(i)** and **(iii)** that there exists an adapted process $\sigma(t)_{0\le t\le T}$ such that

$$dV(t) = R(t)V(t)dt + \sigma(t)V(t)d\tilde{W}(t), \quad 0 \le t \le T.$$

In other words, prior to time $T$, the price of every asset with almost surely positive price at time $T$ follows a generalized (because the volatility may be random) geometric Brownian motion.
*Solution*: Because $V(t) > 0$ almost surely, combined with Equation 4.2 obtained in part **(i)**, let

$$\sigma(t) = \frac{\tilde{\Gamma}(t)}{V(t)D(t)},$$

which is accordingly well-defined almost surely, we get

$$dV(t) = R(t)V(t)dt + \sigma(t)V(t)d\tilde{W}(t), \quad 0 \le t \le T.$$

In other words, prior to time $T$, the risk-neutral price valuation process Equation 4.1 with almost surely positive price at time $T$ follows a generalized geometric Brownian motion.

# 5 Exercise Five: Implying the Risk-Neutral Distribution

Let $S(t)$ be the price of an underlying asset, which is not necessarily a geometric Brownian motion (i.e., does not necessarily have constant volatility). With $S(0) = x$, the risk-neutral pricing formula of the price at time zero of a European call on this asset, paying $(S(T) - K)^+$ at time $T$, is

$$c(0,T,x,K) = \tilde{\mathbb{E}}\left[e^{-rT}(S(T)-K)^+\right].$$

(Normally we consider this as a function of the current time 0 and the current stock price x, but in this exercise we shall also treat the expiration time $T$ and the strike price $K$ as variables, and for that reason we include them as arguments of $c$. We denote by $\tilde{p}(0,T,x,y)$ the risk-neutral density in the $y$ variable of the distribution of $S(T)$ when $S(0) = x$. Then we may rewrite the risk-neutral pricing formula as

$$c(0,T,x,K) = e^{-rT}\int_K^\infty (y-K)\tilde{p}(0,T,x,y)dy. \tag{5.1}$$

Suppose we know the market prices for calls of all strikes (i.e., we know $c(0,T,x,K)$ for all $K > 0$. We can then compute $c_K(0,T,x,K)$ and $c_{KK}(0,T,x,K)$, the first and second derivatives of the option price with respect to the strike. Differentiate Equation 5.1 twice with respect to $K$ to obtain the equations

$$\begin{aligned} c_K(0,T,x,K) &= -e^{-rT}\int_K^\infty \tilde{p}(0,T,x,y)dy = -e^{-rT}\tilde{\mathbb{P}}\{S(T) > K\}, \\ c_{KK}(0,T,x,K) &= e^{-rT}\tilde{p}(0,T,x,K). \end{aligned}$$

The second of these equations provides a formula for the risk-neutral distribution of $S(T)$ in terms of call prices:

$$\tilde{p}(0,T,x,K) = e^{rT}c_{KK}(0,T,x,K), \quad \text{for all } K\text{>}0.$$

*Solution*: We verify the formulae for the first and second derivatives.

$$\begin{aligned} c_K(0,T,x,K) &\triangleq \frac{\partial}{\partial K}c(0,T,x,K) \\ &= e^{-rT}\frac{\partial}{\partial K}\int_K^\infty (y-K)\tilde{p}(0,T,x,y)dy \\ &= e^{-rT}\left[-\cancelto{0}{(K-K)\tilde{p}(0,T,x,K)} + \int_K^\infty \frac{\partial}{\partial K}(y-K)\tilde{p}(0,T,x,y)dy\right] \\ &= -e^{-rT}\int_K^\infty \tilde{p}(0,T,x,y)dy \\ &= -e^{-rT}\tilde{\mathbb{P}}\{S(T) > K\}. \end{aligned}$$

$$\begin{aligned} c_{KK}(0,T,x,K) &= \frac{\partial}{\partial K}c_K(0,T,x,K) \\ &= -e^{-rT}\frac{\partial}{\partial K}\int_K^\infty \tilde{p}(0,T,x,y)dy \\ &= e^{-rT}\tilde{p}(0,T,x,K) \quad \Rightarrow \quad \tilde{p}(0,T,x,K) = e^{rT}c_{KK}(0,T,x,K). \end{aligned}$$

# 6 Exercise Six: Chooser option

Consider a model with a unique risk-neutral measure $\tilde{\mathbb{P}}$ and constant interest rate $r$. According to the risk-neutral pricing formula, for $0 \le t \le T$, the price at time $t$ of a European call expiring at time $T$ is

$$C(t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K)^+ \middle| \mathscr{F}(t)\right],$$

where $S(T)$ is the underlying asset price at time $T$ and $K$ is the strike price of the call. Similiarly, the price at time $t$ of a European put expiring at time $T$ is

$$P(t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}(K-S(T))^+ \middle| \mathscr{F}(t)\right].$$

Finally, because $e^{-rt}S(t)$ is a martingale under $\tilde{\mathbb{P}}$, the price at time $t$ of a forward contract for delivery of one share of stock at time $T$ in exchange for a payment for $K$ at time $T$ is

$$\begin{aligned} F(t) &= \tilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K) \middle| \mathscr{F}(t)\right] \\ &= e^{rt}\tilde{\mathbb{E}}\left[e^{-rT}S(T) \middle| \mathscr{F}(t)\right] - e^{-r(T-t)}K \\ &= S(t) - e^{-r(T-t)}K. \end{aligned}$$

Because

$$(S(T)-K)^+ - (K-S(T))^+ = S(T)-K,$$

we have the *put-call parity* relationship

$$\begin{aligned} C(t)-P(t) &= \tilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K)^+ - e^{-r(T-t)}(K-S(T))^+ \middle| \mathscr{F}(t)\right] \\ &= \tilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K) \middle| \mathscr{F}(t)\right] = F(t) \end{aligned}$$

Now consider a date $t_0$ between 0 and $T$, and consider a *chooser option*, which gives the right at time $t_0$ to choose to own either the call or the put.

**(i)** Show that at time $t_0$ the value of the chooser option is

$$C(t_0) + \max\{0, -F(t_0)\} = C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+.$$

*Solution*: Because at time $t_0 \in [0,T]$ one has the right to choose to own either the call or the put, at time $t_0$ the value of the chooser option is the larger of the call and the put:

$$\begin{aligned} \max\{C(t_0), P(t_0)\} &= \max\{C(t_0), C(t_0) - F(t_0)\} \\ &= C(t_0) + \max\{0, -F(t_0)\} \\ &= C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+. \end{aligned}$$

**(ii)** Show that the value of the chooser option at time 0 is the sum of the value of a call expiring at time $T$ with strike price $K$ and the vale of a put expiring at time $t_0$ with strike price $e^{-r(T-t_0)}K$.

*Solution*: Consider a portfolio consisting of

- a call expiring at time $T$ with strike price $K$;
- a put expiring at time $t_0$ with strike price $e^{-r(T-t_0)}K$.

The payoff of this portfolio at time $t_0$

$$\underbrace{C(t_0)}_{\text{value of call at } t_0} + \underbrace{\left(e^{-r(T-t_0)}K - S(t_0)\right)^+}_{\text{payoff of put at expiration } t_0}$$

is exactly the value of the chooser option at time $t_0$ irrespective of the market state at the time. According to the *Law of One Price*: *if two portfolios are guaranteed to have the same value at a future time $t' > t$ regardless of the state of the market at $t'$, then they must have the same value at time $t$*, we conclude that the value of the chooser option at time 0 is the sum of the value of a call expiring at time $T$ with strike price $K$ and the vale of a put expiring at time $t_0$ with strike price $e^{-r(T-t_0)}K$.

# 7 Exercise Seven: Hedging a Cash Flow

Let $W(t)_{0\le t\le T}$, be a Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and let $\mathscr{F}(t)_{0\le t\le T}$ be the filtration generated by this Brownian motion. Let the mean rate of returen $\alpha(t)$, the interest rate $R(t)$, and the volatility $\sigma(t)$ be adapted processes, and assume that $\sigma(t)$ is never zero. Consider a stock price process whose differential is given by:

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \quad 0 \le t \le T.$$

Suppose an agent must pay a cash flow at rate $C(t)$ at each time $t$, where $C(t)_{0\le t\le T}$, is an adapted process. If the agent holds $\Delta(t)$ shares of stock at each time $t$, then the differential of his or her portfolio value will be

$$dX(t) = \Delta(t)dS(t) + R(t)\left(X(t) - \Delta(t)S(t)\right)dt - C(t)dt.$$

Show that there is a nonrandom value of $X(0)$ and a portfolio process $\Delta(t)_{0\le t\le T}$, such that $X(T) = 0$ almost surely.

*Proof*: Define the discount process according to textbook

$$D(t) \triangleq e^{-\int_0^t R(s)ds},$$

which is an adapted process because the $R(t)$ is adapted. Note that $dD(t) = -R(t)D(t)dt$.

Consider the discounted stock price process by Itô's product rule

$$\begin{aligned}
d(D(t)S(t)) &= D(t)dS(t) + S(t)dD(t) + dD(t)dS(t) \\
&= D(t)\left(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)\right) - S(t)R(t)D(t)dt \\
&= D(t)S(t)\left[(\alpha(t) - R(t))dt + \sigma(t)dW(t)\right] \\
&= \sigma(t)D(t)S(t)\left[\frac{\alpha(t) - R(t)}{\sigma(t)}dt + dW(t)\right] \\
&= \sigma(t)D(t)S(t)\left[\Theta(t)dt + dW(t)\right],
\end{aligned}$$

where

$$\Theta(t) \triangleq \frac{\alpha(t) - R(t)}{\sigma(t)}.$$

According to the corollary to Martingale Representation Theorem in one dimension, define

$$\begin{aligned}
\tilde{W}(t) &= W(t) + \int_0^t \Theta(u)du, \\
Z(t) &= \exp\left\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\right\}.
\end{aligned}$$

Set $Z = Z(T)$. Then $\mathbb{E}[Z] = 1$, and under the probability measure $\tilde{\mathbb{P}}$ given by

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega), \quad \forall A \in \mathscr{F},$$

the process $\tilde{W}(t)_{0\le t\le T}$ is a Brownian motion, and the discounted stock prices process can be written as

$$d(D(t)S(t)) = \sigma(t)D(t)S(t)d\tilde{W}(t),$$

which is a martingale under the probability measure $\tilde{\mathbb{P}}$.

Similarly, for the discounted portfolio process

$$\begin{aligned}
d(D(t)X(t)) &= D(t)dX(t) + X(t)dD(t) + dD(t)dX(t) \\
&= D(t)\left[\Delta(t)dS(t) + R(t)\left(X(t) - \Delta(t)S(t)\right)dt - C(t)dt\right] - X(t)R(t)D(t)dt \\
&= \Delta(t)D(t)S(t)\left[(\alpha(t) - R(t))dt + \sigma(t)dW(t)\right] - D(t)C(t)dt \\
&= \Delta(t)d(D(t)S(t)) - D(t)C(t)dt.
\end{aligned}$$

Integrate over time,

$$D(t)X(t) = X(0) + \underbrace{\int_0^t \Delta(t')\sigma(t')D(t')S(t')d\tilde{W}(t')}_{\text{martingale}} - \int_0^t D(t')C(t')dt'.$$

The above equation inspires the introduction of the following martingale process

$$\tilde{M}(t) = \tilde{\mathbb{E}}\left[\int_0^T D(t')C(t')dt' \middle| \mathscr{F}(t)\right], \quad 0 \le t \le T.$$

Note that $\tilde{M}(0) = \tilde{\mathbb{E}}\int_0^T D(t)C(t)dt$ and $\tilde{M}(T) = \int_0^T D(t)C(t)dt$. According to Martingale Representation Theorem, there exists an adapted process $\tilde{\Gamma}(t)$ such that

$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(t')d\tilde{W}(t').$$

Thus, if we choose the strategy process

$$\Delta(t) = \frac{\tilde{\Gamma}(t)}{\sigma(t)D(t)S(t)}, \tag{7.1}$$

the discounted portfolio can be rewritten as

$$D(t)X(t) = X(0) + \int_0^t \tilde{\Gamma}(t')d\tilde{W}(t') - \int_0^t D(t')C(t')dt',$$

and at time $T$,

$$\begin{aligned}
D(T)X(T) &= X(0) + \int_0^T \tilde{\Gamma}(t')d\tilde{W}(t') - \int_0^T D(t')C(t')dt' \\
&= X(0) + \tilde{M}(T) - \tilde{M}(0) - \int_0^T D(t')C(t')dt' \\
&= X(0) - \tilde{M}(0) + \cancelto{0}{\tilde{M}(T) - \int_0^T D(t')C(t')dt'}
\end{aligned}$$

$$= \quad X(0) - \tilde{M}(0).$$

To achieve the terminal condition $X(T) = 0$ almost surely $(D(T) \neq 0)$, we can choose a non-random $X(0)$ such that

$$X(0) = \tilde{M}(0) = \tilde{\mathbb{E}} \int_0^T D(t)C(t)dt. \tag{7.2}$$

In summary, the combination of the portfolio strategy process Equation 7.1 and the non-random initial capital given in Equation 7.2 ensures that $X(T) = 0$ almost surely.

BARUCH, MFE

# MTH 9831 Assignment Seven

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November 14, 2015

# 1 PROBLEM ONE

Use the Feynman-Kac formula to solve explicitly ($\sigma$ and $r$ are given non-negative constants)

$$g_t(t,x) + \frac{1}{2}\sigma^2 g_{xx}(t,x) = r g(t,x), \quad g(T,x) = x^4.$$

Check that your solution is correct by substituting it in the PDE.
*Solution*: Comparing with **Theorem 2** (Feynman-Kac formula) in lecture notes, we identify

$$b(t,x) \equiv 0, \quad \sigma^2(t,x) \equiv \sigma, \quad r(t,x) \equiv r, \quad h(x) = x^4.$$

Let $X(t')_{t \le t' \le T}$ solve the SDE

$$dX(t') = \sigma dB(t')$$

with initial condition $X(t) = x$, we get

$$X(t') = x + \sigma\left(B(t') - B(t)\right) \Rightarrow X(T) = x + \sigma\left(B(T) - B(t)\right) = x + \sigma\sqrt{T-t}Z,$$

where $Z \sim N(0,1)$. The Feynman-Kac formula gives

$$\begin{aligned}
g(t,x) &= \mathbb{E}^{t,x}\left[e^{-\int_t^T r(t',X(t'))dt'} h(X(T))\right] \\
&= e^{-r(T-t)}\mathbb{E}^{t,x}\left[X^4(T)\right] \\
&= e^{-r(T-t)}\mathbb{E}\left[X^4(T)|X(t)=x\right] \\
&= e^{-r(T-t)}\mathbb{E}\left[\left(x+\sigma\sqrt{T-t}Z\right)^4\right] \\
&= e^{-r(T-t)}\mathbb{E}\left[x^4 + 4\sigma\sqrt{T-t}x^3 Z + 6\sigma^2(T-t)x^2Z^2 + 4\sigma^3(T-t)^{3/2}Z^3 + \sigma^4(T-t)^2Z^4\right] \\
&= e^{-r(T-t)}\left[x^4 + 6\sigma^2(T-t)x^2 + 3\sigma^4(T-t)^2\right],
\end{aligned}$$

where we have used $\mathbb{E}[Z]=0, \mathbb{E}[Z^2]=1, \mathbb{E}[Z^3]=0, \mathbb{E}[Z^4]=3$.

Verification by substitution,

$$\begin{aligned}
g_t(t,x) &= r g(t,x) - e^{-r(T-t)}\left[6\sigma^2x^2 + 6\sigma^4(T-t)\right], \\
g_{xx}(t,x) &= e^{-r(T-t)}\left[12x^2 + 12\sigma^2(T-t)\right] \Rightarrow \frac{1}{2}\sigma^2 g_{xx}(t,x) = e^{-r(T-t)}\left[6\sigma^2x^2 + 6\sigma^4(T-t)\right],
\end{aligned}$$

Thus, the PDE is satisfied

$$g_t(t,x) + \frac{1}{2}\sigma^2 g_{xx}(t,x) = r g(t,x),$$

and the terminal condition

$$g(T,x) = e^{-r(T-T)}\left[x^4 + 6\sigma^2(T-T)x^2 + 3\sigma^4(T-T)^2\right] = x^4.$$

## 2 Problem Two: Solving a general linear SDE

**(a)** Solve $dZ(u) = b(u)Z(u)du + \sigma(u)Z(u)dW(u)$ for $u \ge t$ with the initial condition $Z(t) = 1$.
*Solution*: Apply Itô's formula to $\ln Z$,

$$\begin{aligned}
d\ln Z &= \frac{1}{Z}dZ - \frac{1}{2Z^2}dZdZ \\
&= \frac{1}{Z(u)}\left(b(u)Z(u)du + \sigma(u)Z(u)dW(u)\right) - \frac{1}{2Z^2(u)}\sigma^2(u)Z^2(u)du \\
&= b(u)du + \sigma(u)dW(u) - \frac{1}{2}\sigma^2(u)du \\
&= \left(b(u) - \frac{1}{2}\sigma^2(u)\right)du + \sigma(u)dW(u).
\end{aligned}$$

Integrate over time,

$$\ln Z(u) - \underbrace{\ln Z(t)}_{=\ln(1)=0} = \int_t^u \left(b(t') - \frac{1}{2}\sigma^2(t')\right)dt' + \int_t^u \sigma(t')dW(t').$$

Thus,

$$Z(u) = \exp\left\{\int_t^u \left(b(t') - \frac{1}{2}\sigma^2(t')\right)dt' + \int_t^u \sigma(t')dW(t')\right\}.$$

**(b)** Solve $dY(u) = k(u)du + l(u)dW(u)$ for $u \ge t$ with the initial condition $Y(t) = x$.
*Solution*: Integrate over time,

$$Y(u) - Y(t) = \int_t^u k(t')dt' + l(t')dW(t') \Rightarrow Y(u) = x + \int_t^u \left[k(t')dt' + l(t')dW(t')\right].$$

**(c)** Set $X(u) = Y(u)Z(u)$. Find $k(u)$ and $l(u)$ such that $X(u)$ solves the original SDE. Express the solution $X(u)$ in terms of the original parameters.
*Solution*: Apply Itô's product rule to $X(u) = Y(u)Z(u)$,

$$\begin{aligned}
dX(u) &= d\left(Y(u)Z(u)\right) \\
&= Z(u)dY(u) + Y(u)dZ(u) + dZ(u)dY(u) \\
&= Z(u)\left(k(u)du + l(u)dW(u)\right) + Y(u)\left(b(u)Z(u)du + \sigma(u)Z(u)dW(u)\right) + \sigma(u)Z(u)l(u)du \\
&= \left[Z(u)k(u) + Z(u)Y(u)b(u) + Z(u)\sigma(u)l(u)\right]du + \left[Z(u)l(u) + Z(u)Y(u)\sigma(u)\right]dW(u) \\
&= \left[Z(u)k(u) + Z(u)\sigma(u)l(u) + b(u)X(u)\right]du + \left[Z(u)l(u) + \sigma(u)X(u)\right]dW(u).
\end{aligned}$$

Compare with the original SDE

$$dX(u) = \left[a(u) + b(u)X(u)\right]du + \left[\gamma(u) + \sigma(u)X(u)\right]dW(u),$$

we make the following identification

$$
\begin{aligned}
a(u) &= Z(u)\left[k(u)+\sigma(u)l(u)\right], \\
\gamma(u) &= Z(u)l(u).
\end{aligned}
$$

Solve for $k(u)$ and $l(u)$, we get

$$
\begin{aligned}
k(u) &= \frac{a(u)-\sigma(u)\gamma(u)}{Z(u)}, \\
l(u) &= \frac{\gamma(u)}{Z(u)}.
\end{aligned}
$$

Thus, express the solution $X(u)$ in terms of the original parameters,

$$
\begin{aligned}
X(u) &= Y(u)Z(u) \\
&= \left\{x+\int_t^u \left[k(t')dt'+l(t')dW(t')\right]\right\}\times Z(u) \\
&= \left[x+\int_t^u \frac{a(t')-\sigma(t')\gamma(t')}{Z(u)}dt'+\int_t^u \frac{\gamma(t')}{Z(t')}dW(t')\right]\times Z(u),
\end{aligned}
$$

where

$$
Z(u)=\exp\left\{\int_t^u \left(b(t')-\frac{1}{2}\sigma^2(t')\right)dt'+\int_t^u \sigma(t')dW(t')\right\}.
$$

# 3 Problem Three

Suppose that for $0 \le t \le u \le T$,

$$dX(u) = b\,(u, X(u))\,du + \sigma\,(u, X(u))\,dB(u), \quad X(t) = x.$$

Let $f(x)$ and $h(x)$ be given deterministic functions. Find the PDE satisfied by

$$g(t,x) = \mathbb{E}^{t,x}\left[h(X(T))\right] + \int_t^T \mathbb{E}^{t,x}\left[f(X(u))\right] du.$$

*Solution*: We first write down the terminal condition satisfied by $g(t,x)$ at $t = T$,

$$g(T,x) = \mathbb{E}^{T,x}\left[h(X(T))\right] = \mathbb{E}\left[h(X(T))|X(T) = x\right] = \mathbb{E}\left[h(x)\right] = h(x).$$

According to this observation, we can re-write $g(t,x)$ in the following form

$$g(t,x) = \mathbb{E}^{t,x}\left[g(T,X(T))\right] + \int_t^T \mathbb{E}^{t,x}\left[f(X(u))\right] du = \mathbb{E}^{t,x}\left[g(T,X(T)) + \int_t^T f(X(u))du\right].$$

Similar to the derivation of Feynman-Kac formula given in the lecture notes, we allow the terminal time $T$ of

$$g(T,X(T)) + \int_t^T f(X(u))du$$

to vary and differentiate with respect to the this time

$$\begin{aligned}
&d\left[g(u,X(u)) + \int_t^u f(X(s))ds\right] \\
&= g_t(u,X(u))du + g_x(u,X(u))dX(u) + \frac{1}{2}g_{xx}(u,X(u))dX(u)dX(u) + f(X(u))du \\
&= g_t(u,X(u))du + g_x(u,X(u))\left(b\,(u,X(u)) + \sigma\,(u,X(u))\,dB(u)\right) + \frac{1}{2}\sigma^2(u,X(u))g_{xx}(u,X(u))du \\
&\quad + f(X(u))du \\
&= \left(g_t(u,X(u)) + b\,(u,X(u))\,g_x(u,X(u)) + \frac{1}{2}\sigma^2(u,X(u))g_{xx}(u,X(u)) + f(X(u))\right)du \\
&\quad + \sigma\,(u,X(u))\,g_x(u,X(u))dB(u).
\end{aligned}$$

Integrate over time from $t$ to $T$,

$$\begin{aligned}
&g(T,X(T)) - g(t,X(t)) + \int_t^T f(X(u))du \\
&= \int_t^T \left(g_t(u,X(u)) + b\,(u,X(u))\,g_x(u,X(u)) + \frac{1}{2}\sigma^2(u,X(u))g_{xx}(u,X(u)) + f(X(u))\right)du \\
&\quad + \int_t^T \sigma\,(u,X(u))\,g_x(u,X(u))dB(u).
\end{aligned}$$

Take expectations, we arrive at the following equation

$$
\begin{aligned}
0 &= \mathbb{E}^{t,x}\left[g(T,X(T)) + \int_t^T f(X(u))\,du\right] - g(t,x) \\
&= \mathbb{E}^{t,x}\int_t^T \left(g_t(u,X(u)) + b(u,X(u))\,g_x(u,X(u)) + \frac{1}{2}\sigma^2(u,X(u))g_{xx}(u,X(u)) + f(X(u))\right)du \\
&\quad + \cancelto{0}{\mathbb{E}^{t,x}\int_t^T \sigma(u,X(u))\,g_x(u,X(u))\,dB(u)},
\end{aligned}
$$

which holds for an arbitrary $t$. Thus, the PDE satisfied by $g(t,x)$ is

$$\boxed{g_t(t,x) + b(t,x)g_x(t,x) + \frac{1}{2}\sigma^2(t,x)g_{xx}(t,x) + f(x) = 0.}$$

# 4 Problem Four: From a PDE to an SDE

Find a diffusion process (i.e., an SDE) whose generator $\mathscr{A}$ has the following form.

**(a)** (d=1) $\mathscr{A} = (2-x)\frac{\partial}{\partial x} + 2x^2\frac{\partial^2}{\partial x^2}$. Can you solve the obtained SDE?
*Solution*: Compare with the standard form of generator, we make the following identification:

$$
\begin{aligned}
a(t,x) &= a(x) = 4x^2 = \sigma^2(x) \Rightarrow \sigma(x) = \pm 2x;\\
b(t,x) &= b(x) = 2-x.
\end{aligned}
$$

The associated one-dimensional diffusion process is

$$dX(t) = (2 - X(t))dt \pm 2X(t)dB(t).$$

This SDE can be solved in the general formalism presented in Problem Two, with the following identification: $a = 2, b = -1, \gamma = 0, \sigma = \pm 2$, then

$$
\begin{aligned}
Z(t) &= \exp\left\{\int_0^t \left(-1 - \frac{1}{2}(\pm 2)^2\right) dt' \pm 2\int_0^t dB(t')\right\} = \exp\{-3t \pm 2B(t)\}\\
X(t) &= \left[X(t=0) + \int_0^t \frac{a(t') - \sigma(t')\gamma(t')}{Z(t')} dt' + \int_0^t \frac{\gamma(t')}{Z(t')} dB(t')\right] \times Z(t)\\
&= \left[X(t=0) + 2\int_0^t \exp\left\{3t' \mp 2B(t')\right\} dt'\right] \times \exp\{-3t \pm 2B(t)\},
\end{aligned}
$$

under the initial condition given at $t = 0$.

**(b)** (d=2) $\mathscr{A} = \frac{1+x^2}{2}\frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial x \partial y} + \frac{1}{2}\frac{\partial^2}{\partial y^2} + 2y\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y}$.
*Solution*: Compare with the standard form of generator, we make the following identification:

$$
\begin{aligned}
\mathbf{b}(t,x,y) &= \mathbf{b}(x,y) = \begin{pmatrix} 2y \\ 2x \end{pmatrix},\\
\mathbf{a}(t,x,y) &= \mathbf{a}(x,y) = \begin{pmatrix} 1+x^2 & x \\ x & 1 \end{pmatrix}.
\end{aligned}
$$

One possible solution to the matrix equation $\sigma\sigma^T = \mathbf{a}$ is

$$\sigma = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \Rightarrow \sigma\sigma^T = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1+x^2 & x \\ x & 1 \end{pmatrix} = \mathbf{a}(x,y).$$

The associated two-dimensional diffusion process is

$$
\begin{aligned}
dX(t) &= 2Y(t)dt + dB_1(t) + X(t)dB_2(t),\\
dY(t) &= 2X(t)dt + dB_2(t),
\end{aligned}
$$

where $(B_1(t), B_2(t))$ is a standard two-dimensional Brownian motion.

Multiply $\sigma$ from the right by an arbitrary orthogonal matrix yields a new matrix satisfying the same requirement. For example, in two dimensions, an orthogonal matrix representing a rotation in two dimensions can be written as, for $\theta \in \mathbb{R}$,

$$O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Thus the matrix $\sigma O$ satisfies the equation

$$(\sigma O)(\sigma O)^T = \sigma \underbrace{OO^T}_{=\mathbb{1}} \sigma^T = \sigma\sigma^T = \mathbf{a}(x, y).$$

For example, let $\theta = \frac{\pi}{2}$,

$$O = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \sigma' \triangleq \sigma O = \begin{pmatrix} -x & 1 \\ -1 & 0 \end{pmatrix}$$

is another choice for $\sigma'\sigma'^T = \mathbf{a}(x, y)$. The associated two-dimensional diffusion process is

$$\begin{aligned} dX(t) &= 2Y(t)dt - X(t)dB_1(t) + dB_2(t), \\ dY(t) &= 2X(t)dt - dB_1(t). \end{aligned}$$

# 5 Problem Five: Feyman-Kac + Girsanov

Consider the following terminal value problem:

$$g_t + \frac{1}{2} g_{xx} + \theta(x) g_x = 0, \quad (x,t) \in \mathbb{R} \times [0,T), \quad g(T,x) = h(x).$$

Show that the solution of this problem can be written as

$$g(t,x) = \mathbb{E}^{t,x}\left[e^{\int_t^T \theta(B(s))dB(s) - \frac{1}{2}\int_t^T \theta^2(B(s))ds} h(B(T))\right].$$

You may assume that $h$ and $\theta$ are such that the problem has a unique solution and both theorems are applicable.
*Solution*: Comparing with the conditions assumed by Feynman-Kac theorem, we make the following identifications:

$$b(t,x) = \theta(x), \quad \sigma(t,x) = \pm 1, \quad r(t,x) \equiv 0,$$

thus,

$$g(t,x) = \mathbb{E}^{t,x}\left[h(X(T))\right],$$

where $X(u)_{0\le u \le T}$, obeys the following SDE by letting $\sigma = +1$,

$$dX(u) = \theta(X(u))du + dB(u) \triangleq d\tilde{B}(u), \quad X(t) = x$$

where we have given $X(u)$ a new name $\tilde{B}(u)$. Now invoking Girsanov's theorem, since $\bar{\theta}(u) \triangleq \theta(X(u)) = \theta(\tilde{B}(u))$ is an adapted process with respect to the filtration generated by the process $X(t)$, let

$$\begin{aligned}
Z(u) &= \exp\left\{-\int_t^u \theta(\tilde{B}(s))dB(s) - \frac{1}{2}\int_t^u \theta^2(\tilde{B}(s))ds\right\} \\
&= \exp\left\{-\int_t^u \theta(\tilde{B}(s))\left(d\tilde{B}(s) - \theta(\tilde{B}(s)ds)\right) - \frac{1}{2}\int_t^u \theta^2(\tilde{B}(s))ds\right\} \\
&= \exp\left\{-\int_t^u \theta(\tilde{B}(s))d\tilde{B}(s) + \frac{1}{2}\int_t^u \theta^2(\tilde{B}(s))ds\right\}
\end{aligned}$$

Then under the measure defined by $\tilde{\mathbb{P}}(A) = \int_A Z(T)d\mathbb{P}(\omega), \forall A \in \Omega$, the process $X(t) = \tilde{B}(t)$ is a Brownian motion with the initial condition $\tilde{B}(t) = x$. Thus,

$$\begin{aligned}
g(t,x) &= \mathbb{E}^{t,x}\left[h(X(T))\right] \\
&= \mathbb{E}^{t,x}\left[Z(T)Z^{-1}(T)h(X(T))\right] \\
&= \tilde{\mathbb{E}}^{t,x}\left[Z^{-1}(T)h(\tilde{B}(T))\right] \\
&= \tilde{\mathbb{E}}^{t,x}\left[\exp\left\{\int_t^u \theta(\tilde{B}(s))d\tilde{B}(s) - \frac{1}{2}\int_t^u \theta^2(\tilde{B}(s))ds\right\} h(\tilde{B}(T))\right].
\end{aligned}$$

*The solution given by TA*:

At this point, think about the structure of this SDE while considering Girsanov's theorem: the function $\theta(X(u))$ is an adapted function of $u$. Therefore, we can shift $X(u)$ in space and time in the following way, via Girsanov, and end up with a new Brownian motion on a new measure. Define

$$\tilde{B}(u) := X(u+t) - x, \quad 0 \le u \le T-t.$$

In other words, $X(u) = x + \tilde{B}(u-t), t \le u \le T$. Then, integrating the $dX(u)$ SDE, we get the translation

$$\begin{aligned}
&dX(u) = \theta(X(u))du + dB(u), \quad X(t) = x, t \le u \le T \\
&\Rightarrow X(u) - x = \int_t^u \theta(X(v))\, dv + B(u) - B(t) \\
&\Rightarrow \tilde{B}(u-t) = \int_t^u \theta\left(x + \tilde{B}(v-t)\right) dv + B(u) - B(t) \\
&\Rightarrow \tilde{B}(u-t) = \int_0^{u-t} \theta\left(x + \tilde{B}(v)\right) dv + \underbrace{B(u) - B(t)}_{\overset{(d)}{=} B(u-t)}
\end{aligned}$$

$$(u-t \to u) \Rightarrow \tilde{B}(u) \overset{(d)}{=} \int_0^u \theta\left(x + \tilde{B}(v)\right) dv + B(u), \quad 0 \le u \le T-t,$$

where $\tilde{B}(u)$ is a standard Brownian motion on the measure $\tilde{P}$. We say *equal in distribution* in the last equality because we've swapped out $B(u+t)$ for an increment. (Since we are only taking an expectation value, it's OK to do this). How do we shift between measures? We need the Radon-Nicodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = Z(T-t)$$

from the process

$$\begin{aligned}
Z(u) &= \exp\left\{-\int_0^u \theta\left(\tilde{B}(v) + x\right) dB(v) - \frac{1}{2}\int_0^u \theta^2\left(\tilde{B}(v) + x\right) dv\right\}, \quad 0 \le u \le T-t \\
&= \exp\left\{-\int_0^u \theta\left(\tilde{B}(v) + x\right) d\tilde{B}(v) + \frac{1}{2}\int_0^u \theta^2\left(\tilde{B}(v) + x\right) dv\right\}
\end{aligned}$$

since $-dB(v) = -d\tilde{B}(v) + \theta(x + \tilde{B}(v))dv$. Finally, this yields the F-K solution for $g(t,x)$, with a change of measure from $\mathbb{P}$ with $B(t) = x$ to $\tilde{\mathbb{P}}$ with $\tilde{B}(0) = 0$, and at the end a change back to $\mathbb{P}$ since we are talking about a standard Brownian motion inside an expectation, so the notation no longer matters:

$$\begin{aligned}
g(t,x) &= \mathbb{E}^{t,x}\left[h(X(T))\right] = \tilde{\mathbb{E}}^{0,0}\left[Z^{-1}(T-t)h\left(x + \tilde{B}(T-t)\right)\right] \\
&= \tilde{\mathbb{E}}^{0,0}\left[\exp\left\{\int_0^{T-t} \theta\left(\tilde{B}(v) + x\right) d\tilde{B}(v) - \frac{1}{2}\int_0^{T-t} \theta^2\left(\tilde{B}(v) + x\right) dv\right\} h\left(x + \tilde{B}(T-t)\right)\right] \\
&= \tilde{\mathbb{E}}^{t,x}\left[\exp\left\{\int_t^T \theta(B(v))\, dB(v) - \frac{1}{2}\int_t^T \theta^2(B(v))\, dv\right\} h(B(T))\right].
\end{aligned}$$

## 6 Problem Six: Implying the Volatility Surface

The stock price evolves according to the SDE

$$dS(u) = \underbrace{rS(u)}_{\beta(u,x)=rx}\,du + \underbrace{\sigma(u,S(u))S(u)}_{\gamma(u,x)=\sigma(u,x)x}\,d\tilde{W}(u).$$

The transition density $\tilde{p}(t,T,x,y)$ satisfies the Kolmogorov forward equation

$$\begin{aligned}\frac{\partial}{\partial T}\tilde{p}\big(t,T,x,y\big) &= -\frac{\partial}{\partial y}\big(\beta(t,y)\tilde{p}(t,T,x,y)\big) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\gamma^2(t,y)\tilde{p}(t,T,x,y)\big)\\ &= -\frac{\partial}{\partial y}\big(ry\tilde{p}(t,T,x,y)\big) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\sigma^2(T,y)y^2\tilde{p}(t,T,x,y)\big).\end{aligned}$$

Let

$$c(0,T,x,K) = e^{-rT}\int_K^\infty (y-K)\tilde{p}(0,T,x,y)\,dy$$

denote the time-zero price of a call expiring at time $T$, struck at $K$, when the initial stock price is $S(0)=x$. Note that

$$c_T(0,T,x,K) = -rc(0,T,x,K) + e^{-rT}\int_K^\infty (y-K)\tilde{p}_T(0,T,x,y)\,dy.$$

**(i)** Integrate once by parts to show that

$$-\int_K^\infty (y-K)\frac{\partial}{\partial y}\big(ry\tilde{p}(0,T,x,y)\big)\,dy = \int_K^\infty ry\tilde{p}(0,T,x,y)\,dy.$$

You may assume that

$$\lim_{y\to\infty}(y-K)ry\tilde{p}(0,T,x,y) = 0.$$

*Proof*:

$$\begin{aligned}&-\int_K^\infty (y-K)\frac{\partial}{\partial y}\big(ry\tilde{p}(0,T,x,y)\big)\,dy\\ &= \cancelto{0}{-(y-K)ry\tilde{p}(0,T,x,y)\Big|_{y=K}^{y=\infty}} + \int_K^\infty ry\tilde{p}(0,T,x,y)\,dy\\ &= \int_K^\infty ry\tilde{p}(0,T,x,y)\,dy.\end{aligned}$$

**(ii)** Integrate by parts and then integrate again to show that

$$\frac{1}{2}\int_K^\infty (y-K)\frac{\partial^2}{\partial y^2}\big(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\big)\,dy = \frac{1}{2}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K).$$

You may assume that

$$\lim_{y\to\infty}(y-K)\frac{\partial}{\partial y}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)=0.$$
$$\lim_{y\to\infty}\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)=0.$$

*Proof*:

$$\begin{aligned}
&\frac{1}{2}\int_K^\infty (y-K)\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)dy\\
&=\frac{1}{2}(y-K)\frac{\partial}{\partial y}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)\Big|_{y=K}^{y=\infty}-\frac{1}{2}\int_K^\infty\frac{\partial}{\partial y}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)dy\\
&=-\frac{1}{2}\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\Big|_{y=K}^{y=\infty}\\
&=\frac{1}{2}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K).
\end{aligned}$$

**(iii)**

$$\begin{aligned}
c_T(0,T,x,K)&=-rc(0,T,x,K)+e^{-rT}\int_K^\infty(y-K)\tilde{p}_T(0,T,x,y)dy\\
&=-re^{-rT}\int_K^\infty(y-K)\tilde{p}(0,T,x,y)dy\\
&\quad+e^{-rT}\int_K^\infty(y-K)\left[-\frac{\partial}{\partial y}\left(ry\tilde{p}(0,T,x,y)\right)+\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)\right]dy\\
&=-re^{-rT}\int_K^\infty(y-K)\tilde{p}(0,T,x,y)dy\\
&\quad+e^{-rT}\int_K^\infty ry\tilde{p}(0,T,x,y)dy+e^{-rT}\frac{1}{2}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K)\\
&=rKe^{-rT}\int_K^\infty\tilde{p}(0,T,x,y)dy+\frac{1}{2}e^{-rT}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K).
\end{aligned}$$

From Homework Assignment Six Exercise Five, we know that

$$\tilde{p}(0,T,x,K)=e^{rT}c_{KK}(0,T,x,K).$$

Thus,

$$\begin{aligned}
c_T(0,T,x,K)&=rK\int_K^\infty c_{KK}(0,T,x,y)dy+\frac{1}{2}\sigma^2(T,K)K^2c_{KK}(0,T,x,K)\\
&=rKc_K(0,T,x,y)\Big|_{y=K}^{y=\infty}+\frac{1}{2}\sigma^2(T,K)K^2c_{KK}(0,T,x,K)\\
&=-rKc_K(0,T,x,K)+\frac{1}{2}\sigma^2(T,K)K^2c_{KK}(0,T,x,K),
\end{aligned}$$

under the assumption that $\lim_{y\to\infty}c_K(0,T,x,y)=0$. This is the end of the problem. Note that under the assumption that $c_{KK}(0,T,x,K)\neq 0$, the above equation can be solved for the volatility term $\sigma^2(T,K)$ in terms of the quantities $c_T(0,T,x,K)$, $c_K(0,T,x,K)$, and $c_{KK}(0,T,x,K)$, which can be inferred from market prices.

BARUCH, MFE

# MTH 9831 Assignment Eight

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November 16, 2015

# 1 PROBLEM ONE: BOND PRICE FOR CIR MODEL

This exercise solves the ordinary differential equations

$$
\begin{aligned}
C'(t,T) &= bC(t,T) + \frac{1}{2}\sigma^2 C^2(t,T) - 1 && (1.1)\\
A'(t,T) &= -aC(t,T) && (1.2)
\end{aligned}
$$

to produce the solutions $C(t,T)$ and $A(t,T)$.

**(i)** Define the function

$$\phi(t) = \exp\left\{\frac{1}{2}\sigma^2 \int_t^T C(u,T)\,du\right\}.$$

Show that

$$
\begin{aligned}
C(t,T) &= -\frac{2\phi'(t)}{\sigma^2\phi(t)},\\
C'(t,T) &= -\frac{2\phi''(t)}{\sigma^2\phi(t)} + \frac{1}{2}\sigma^2 C^2(t,T).
\end{aligned}
$$

*Proof*: Differentiate $\phi(t)$ with respect to $t$,

$$
\begin{aligned}
\phi'(t) &= \exp\left\{\frac{1}{2}\sigma^2 \int_t^T C(u,T)\,du\right\}\left(-\frac{1}{2}\sigma^2 C(t,T)\right) = -\frac{1}{2}\sigma^2 C(t,T)\phi(t),\\
\phi''(t) &= -\frac{1}{2}\sigma^2 C'(t,T)\phi(t) - \frac{1}{2}\sigma^2 C(t,T)\phi'(t) = -\frac{1}{2}\sigma^2 C'(t,T)\phi(t) + \frac{1}{4}\sigma^4 C^2(t,T)\phi(t).
\end{aligned}
$$

Solving the above two equations for $C(t,T)$ and $C'(t,T)$, we get

$$
\begin{aligned}
C(t,T) &= -\frac{2\phi'(t)}{\sigma^2\phi(t)},\\
C'(t,T) &= -\frac{2\phi''(t)}{\sigma^2\phi(t)} + \frac{1}{2}\sigma^2 C^2(t,T).
\end{aligned}
$$

**(ii)** Use Equation 1.1 to show that

$$\phi''(t) - b\phi'(t) - \frac{1}{2}\sigma^2\phi(t) = 0.$$

*Proof*: Substitute the results obtained in part **(i)** into Equation 1.1, we get

$$
\begin{aligned}
&-\frac{2\phi''(t)}{\sigma^2\phi(t)} + \frac{1}{2}\sigma^2 C^2(t,T) = -b\frac{2\phi'(t)}{\sigma^2\phi(t)} + \frac{1}{2}\sigma^2 C^2(t,T) - 1\\
&\Rightarrow \phi''(t) - b\phi'(t) - \frac{1}{2}\sigma^2\phi(t) = 0.
\end{aligned}
$$

This is a constant-coefficient linear ordinary differential equation. All solutions are of the form

$$\phi(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t},$$

where $\lambda_1$ and $\lambda_2$ are solutions of the so-called characteristic equation $\lambda^2 - b\lambda - \frac{1}{2}\sigma^2 = 0$, and $a_1$ and $a_2$ are constants.

**(iii)** Show that $\phi(t)$ must be of the form

$$\phi(t) = \frac{c_1}{\frac{1}{2}b+\gamma} e^{-\left(\frac{1}{2}b+\gamma\right)(T-t)} - \frac{c_2}{\frac{1}{2}b-\gamma} e^{-\left(\frac{1}{2}b-\gamma\right)(T-t)}$$

for some constants $c_1$ and $c_2$, where $\gamma = \frac{1}{2}\sqrt{b^2+2\sigma^2}$.
*Proof*: Solve the quadratic equation $\lambda^2 - b\lambda - \frac{1}{2}\sigma^2 = 0$, wet get two roots

$$\lambda = \frac{b \pm \sqrt{b^2+2\sigma^2}}{2} = \frac{b}{2} \pm \gamma.$$

Thus, for some constants $a_1$ and $a_2$,

$$\phi(t) = a_1 e^{\left(\frac{1}{2}b+\gamma\right)t} + a_2 e^{\left(\frac{1}{2}b-\gamma\right)t}.$$

Let

$$\begin{aligned} c_1 &= a_1\left(\frac{1}{2}b+\gamma\right)e^{\left(\frac{1}{2}b+\gamma\right)T}, \\ c_2 &= -a_2\left(\frac{1}{2}b-\gamma\right)e^{\left(\frac{1}{2}b-\gamma\right)T}, \end{aligned}$$

we get

$$\phi(t) = \frac{c_1}{\frac{1}{2}b+\gamma} e^{-\left(\frac{1}{2}b+\gamma\right)(T-t)} - \frac{c_2}{\frac{1}{2}b-\gamma} e^{-\left(\frac{1}{2}b-\gamma\right)(T-t)}.$$

The terminal conditions are $\phi(T) = 1, \phi'(T) = -\frac{1}{2}\sigma^2 C(T,T)\phi(T) = 0$.

**(iv)** Show that

$$\phi'(t) = c_1 e^{-\left(\frac{1}{2}b+\gamma\right)(T-t)} - c_2 e^{-\left(\frac{1}{2}b-\gamma\right)(T-t)}.$$

*Proof*: Straightforwardly,

$$\begin{aligned} \phi'(t) &= \frac{c_1}{\frac{1}{2}b+\gamma}\left(\frac{1}{2}b+\gamma\right) e^{-\left(\frac{1}{2}b+\gamma\right)(T-t)} - \frac{c_2}{\frac{1}{2}b-\gamma}\left(\frac{1}{2}b-\gamma\right) e^{-\left(\frac{1}{2}b-\gamma\right)(T-t)} \\ &= c_1 e^{-\left(\frac{1}{2}b+\gamma\right)(T-t)} - c_2 e^{-\left(\frac{1}{2}b-\gamma\right)(T-t)}. \end{aligned}$$

Use the terminal condition $\phi'(T) = -\frac{1}{2}\sigma^2 C(T,T)\phi(T) = 0$,

$$0 = \phi'(T) = c_1 - c_2 \Rightarrow c_1 = c_2.$$

**(v)** Show that

$$\begin{aligned}\phi(t) &= c_1 e^{-\frac{1}{2}b(T-t)}\left[\frac{\frac{1}{2}b-\gamma}{\frac{1}{4}b^2-\gamma^2}e^{-\gamma(T-t)} - \frac{\frac{1}{2}b+\gamma}{\frac{1}{4}b^2-\gamma^2}e^{\gamma(T-t)}\right] \\ &= \frac{2c_1}{\sigma^2}e^{-\frac{1}{2}b(T-t)}\left[b\sinh\gamma(T-t)+2\gamma\cosh\gamma(T-t)\right], \\ \phi'(t) &= -2c_1 e^{-\frac{1}{2}b(T-t)}\sinh\gamma(T-t).\end{aligned}$$

*Proof*: Because $c_1 = c_2$ as shown in part **(iv)**,

$$\begin{aligned}\phi(t) &= \frac{c_1}{\frac{1}{2}b+\gamma}e^{-\left(\frac{1}{2}b+\gamma\right)(T-t)} - \frac{c_2}{\frac{1}{2}b-\gamma}e^{-\left(\frac{1}{2}b-\gamma\right)(T-t)} \\ &= c_1 e^{-\frac{1}{2}b(T-t)}\left[\frac{1}{\frac{1}{2}b+\gamma}e^{-\gamma(T-t)} - \frac{1}{\frac{1}{2}b-\gamma}e^{\gamma(T-t)}\right] \\ &= c_1 e^{-\frac{1}{2}b(T-t)}\left[\frac{\frac{1}{2}b-\gamma}{\underbrace{\frac{1}{4}b^2-\gamma^2}_{-\frac{1}{2}\sigma^2}}e^{-\gamma(T-t)} - \frac{\frac{1}{2}b+\gamma}{\underbrace{\frac{1}{4}b^2-\gamma^2}_{-\frac{1}{2}\sigma^2}}e^{\gamma(T-t)}\right] \\ &= \frac{2c_1}{\sigma^2}e^{-\frac{1}{2}b(T-t)}\left[-\left(\frac{1}{2}b-\gamma\right)\left(\cosh\gamma(T-t)-\sinh\gamma(T-t)\right)+\left(\frac{1}{2}b+\gamma\right)\left(\cosh\gamma(T-t)+\sinh\gamma(T-t)\right)\right] \\ &= \frac{2c_1}{\sigma^2}e^{-\frac{1}{2}b(T-t)}\left[b\sinh\gamma(T-t)+2\gamma\cosh\gamma(T-t)\right]. \\ \phi'(t) &= c_1 e^{-\left(\frac{1}{2}b+\gamma\right)(T-t)} - c_2 e^{-\left(\frac{1}{2}b-\gamma\right)(T-t)} \\ &= c_1\left[e^{-\left(\frac{1}{2}b+\gamma\right)(T-t)} - e^{-\left(\frac{1}{2}b-\gamma\right)(T-t)}\right] \\ &= c_1 e^{-\frac{1}{2}b(T-t)}\left[e^{-\gamma(T-t)} - e^{\gamma(T-t)}\right] \\ &= -2c_1 e^{-\frac{1}{2}b(T-t)}\sinh\gamma(T-t).\end{aligned}$$

We conclude that

$$C(t,T) = -\frac{2\phi'(t)}{\sigma^2\phi(t)} = \frac{\sinh\gamma(T-t)}{\gamma\cosh\gamma(T-t)+\frac{b}{2}\sinh\gamma(T-t)}.$$

**(vi)** From Equation 1.2, we have

$$A'(t,T) = -aC(t,T) = \frac{2a\phi'(t)}{\sigma^2\phi(t)}.$$

Integrate over time from $t$ to $T$ and show that

$$A(t,T) = -\frac{2a}{\sigma^2}\log\left[\frac{\gamma e^{\frac{1}{2}b(T-t)}}{\gamma\cosh\gamma(T-t)+\frac{b}{2}\sinh\gamma(T-t)}\right].$$

*Proof*: Integrate over time,

$$
\begin{aligned}
\underbrace{A(T,T)}_{=0} - A(t,T) &= \int_t^T A'(s,T)\,ds \\
&= \frac{2a}{\sigma^2}\int_t^T \frac{\phi'(s)}{\phi(s)}\,ds \\
&= \frac{2a}{\sigma^2}\int_t^T \frac{d\phi(s)}{\phi(s)} \\
&= \frac{2a}{\sigma^2}\log\phi(s)\Big|_t^T \\
&= \frac{2a}{\sigma^2}\log\frac{\phi(T)}{\phi(t)} \\
&= \frac{2a}{\sigma^2}\log\frac{e^{-\frac{1}{2}b(T-T)}\left[b\sinh\gamma(T-T)+2\gamma\cosh\gamma(T-T)\right]}{e^{-\frac{1}{2}b(T-t)}\left[b\sinh\gamma(T-t)+2\gamma\cosh\gamma(T-t)\right]} \\
&= \frac{2a}{\sigma^2}\log\frac{2\gamma e^{\frac{1}{2}b(T-t)}}{b\sinh\gamma(T-t)+2\gamma\cosh\gamma(T-t)} \\
&= \frac{2a}{\sigma^2}\log\frac{\gamma e^{\frac{1}{2}b(T-t)}}{\frac{b}{2}\sinh\gamma(T-t)+\gamma\cosh\gamma(T-t)}
\end{aligned}
$$

Thus,

$$
A(t,T) = -\frac{2a}{\sigma^2}\log\frac{\gamma e^{\frac{1}{2}b(T-t)}}{\gamma\cosh\gamma(T-t)+\frac{b}{2}\sinh\gamma(T-t)}.
$$

## 2 PROBLEM TWO: KOLMOGOROV BACKWARD EQUATION

Consider the stochastic differential equation

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u).$$

We assume that, just as with a geometric Brownian motion, if we begina process at an arbitrary initial positive value $X(t) = x$ at an arbitrary initial time $t$ and evolve it forward using this equation, its value at each time $T > t$ could be any positive number but cannot be less than or equal to zero. For $0 \le t \le T$, let $p(t, T, x, y)$ be the transition density for the solution to this equation (i.e., if we solve the equation with the initial condition $X(t) = x$, then the random variable $X(T)$ has density $p(t, T, x, y)$ in the $y$-variable). We are assuming that $p(t, T, x, y) = 0$ for $0 \le t < T$ and $y \le 0$.

Show that $p(t, T, x, y)$ satisfies the Kolmogorov backward equation

$$-p_t(t, T, x, y) = \beta(t, x)p_x(t, T, x, y) + \frac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y).$$

*Proof*: We know from Feynman-Kac Theorem that, for any function $h(y)$, the function

$$g(t, x) = \mathbb{E}^{t,x}[h(X(T))] = \int_0^\infty h(y)p(t, T, x, y)dy$$

satisfies the partial differential equation

$$\begin{aligned}
&g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma^2(t, x)g_{xx}(t, x) = 0 \\
&\Rightarrow \int_0^\infty h(y)p_t(t, T, x, y)dy + \beta(t, x)\int_0^\infty h(y)p_x(t, T, x, y)dy + \frac{1}{2}\gamma^2(t, x)\int_0^\infty h(y)p_{xx}(t, T, x, y)dy = 0 \\
&\Rightarrow \int_0^\infty h(y)\left(p_t(t, T, x, y) + \beta(t, x)p_x(t, T, x, y) + \frac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y)\right)dy = 0.
\end{aligned}$$

Note that the above equation holds for an arbitrary Borel-measurable function $h(x)$. In particular, one can choose a series of functions $h_n(y)$, for example, Gaussians, that converges to the Dirac-Delta function $\delta(y - y_0)$ for an arbitrarily given $y_0 \ge 0$. Substitute $h(y) = \delta(y - y_0)$ to the above equation, we get

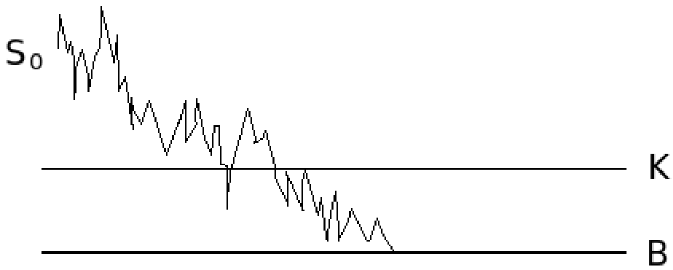
$$\begin{aligned}
0 &= \int_0^\infty h(y)\left(p_t(t, T, x, y) + \beta(t, x)p_x(t, T, x, y) + \frac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y)\right)dy \\
&= \int_0^\infty \delta(y - y_0)\left(p_t(t, T, x, y) + \beta(t, x)p_x(t, T, x, y) + \frac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y)\right)dy \\
&= p_t(t, T, x, y_0) + \beta(t, x)p_x(t, T, x, y_0) + \frac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y_0), \quad \forall y_0 \ge 0.
\end{aligned}$$

In other words,

$$p_t(t, T, x, y) + \beta(t, x)p_x(t, T, x, y) + \frac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y) = 0, \quad 0 \le t < T, \quad y \ge 0.$$

## 3 PROBLEM THREE: PRICING DOWN-AND-OUT CALL OPTIONS

**(a)** Use a probabilistic technique to find the time zero price of a down-and-out call option with strike $K$, barrier $B < K$, and expiration $T$. Assume the Black-Scholes framework with a constant interest rate $r$. You may leave the answers an integral, but make sure that limits of integration are spelled out.




*Solution*: In the Black-Scholes framework with constant interest rate,

$$S(t) = S(0)e^{\sigma\tilde{B}(t)+\left(r-\frac{1}{2}\sigma^2\right)t} \triangleq S(0)e^{-\sigma\hat{B}(t)},$$

where $\hat{B}(t) = -\tilde{B}(t) - \alpha t$, and $\alpha = \frac{r-\sigma^2/2}{\sigma}$. Note that $-\tilde{B}(t)$ is a standard Brownian motion under probability measure $\tilde{\mathbb{P}}$ if $\tilde{B}(t)$ is a standard Brownian motion.

Define $\hat{B}^*(t) = \max_{0\le s\le t}\hat{B}(s)$. Then

$$\min_{0\le t\le T} S(t) = S(0)e^{-\hat{B}^*(T)}.$$

The payoff of the down-and-out option is

$$\begin{aligned}
V(T) &= \left(S(0)e^{-\sigma\hat{B}(T)} - K\right)^+ \mathbb{1}_{\{S(0)e^{-\sigma\hat{B}^*(T)}>B\}} \\
&= \left(S(0)e^{-\sigma\hat{B}(T)} - K\right) \mathbb{1}_{\{S(0)e^{-\sigma\hat{B}(T)}\ge K;S(0)e^{-\sigma\hat{B}^*(T)}>B\}} \\
&= \left(S(0)e^{-\sigma\hat{B}(T)} - K\right) \mathbb{1}_{\{\hat{B}(T)\le\frac{1}{\sigma}\log\frac{S(0)}{K};\hat{B}^*(T)<\frac{1}{\sigma}\log\frac{S(0)}{B}\}} \\
&= \left(S(0)e^{-\sigma\hat{B}(T)} - K\right) \mathbb{1}_{\{\hat{B}(T)\le k;\hat{B}^*(T)<b\}},
\end{aligned}$$

where we have defined

$$k \triangleq \frac{1}{\sigma}\log\frac{S(0)}{K}, \quad b \triangleq \frac{1}{\sigma}\log\frac{S(0)}{B}.$$

such that $k < b$. According to risk-neutral pricing formula, the time zero price of the down-and-out call option considered here is given by

$$V(0) = e^{-rT}\tilde{\mathbb{E}}\left[\left(S(0)e^{-\sigma\hat{B}(T)} - K\right)\mathbb{1}_{\{\hat{B}(T)\le k;\hat{B}^*(T)<b\}}\right].$$

The joint distribution of $(\hat{B}, \hat{B}^*(T))$ under probability measure $\tilde{P}$ was given in lecture notes

$$\hat{f}(x,a) = \begin{cases} \frac{2(2a-x)}{T\sqrt{2\pi T}} e^{-\frac{(2a-x)^2}{2T} - \alpha x - \frac{1}{2}\alpha^2 T}, & x \le a, \quad a \ge 0 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$V(0) = e^{-rT} \int_{-\infty}^{k} dx \int_{x^+}^{b} da \big(S(0)e^{-\sigma x} - K\big) \frac{2(2a-x)}{T\sqrt{2\pi T}} e^{-\frac{(2a-x)^2}{2T} - \alpha x - \frac{1}{2}\alpha^2 T},$$

where, again, $\alpha = \frac{r - \sigma^2/2}{\sigma}$.

**(b)** Let $v(t,x)$ be the price of the above option at time $t$ if $S(t) = x$ assuming that the call has not been knocked out prior to $t$. Show that $v(t,x)$ solves the BSM PDE in $[0,T) \times [B,\infty)$ and argue that it has to satisfy the following boundary conditions

$$\begin{aligned} &v(T,x) = (x-K)^+, \quad x \ge B; \\ &v(t,B) = 0, \quad t \in [0,T]; \\ &\frac{v(t,x)}{x} \to 1, \quad \text{as } x \to \infty. \end{aligned}$$

*Solution*: As usual, by tower property, $e^{-rt}V(t)$ is a $\tilde{\mathbb{P}}$-martingale. Define

$$\rho = \inf\{t \ge 0 : S(t) = B\}.$$

Then $\rho$ is the knock-out time. Since $\rho$ is the first passage time, it is a stopping time. A martingale stopped at a stopping time is again a martingale. Thus, $e^{-r(t\wedge\rho)}V(t\wedge\rho)_{0\le t\le T}$ is a $\tilde{\mathbb{P}}$-martingale.

By Lemma 1 in Lecture 9, $V(t) = v(t, S(t))$ for some $v$ and $0 \le t \le \rho$. Moreover, $e^{-rt}v(t,S(t))$ is a martingale up to time $\rho$. Therefore, for $0 \le t \le \rho$,

$$\begin{aligned} d\left(e^{-rt}v(t,S(t))\right) = e^{-rt}&\left[-rv(t,S(t)) + v_t(t,S(t)) + v_x(t,S(t))rS(t) + \frac{1}{2}v_{xx}(t,S(t))\sigma^2 S^2(t)\right]dt \\ &+ e^{-rt}v_x(t,S(t))\sigma S(t)d\tilde{B}(t). \end{aligned}$$

Setting the $dt$ term to zero, we get for $0 \le t \le \rho$,

$$-rv(t,S(t)) + v_t(t,S(t)) + v_x(t,S(t))rS(t) + \frac{1}{2}v_{xx}(t,S(t))\sigma^2 S^2(t) = 0.$$

The process $S(t)_{0\le t\le\rho}$ can reach a neighbourhood of any point in $[0,T) \times [B,\infty)$ with positive probability. This means that the BSM PDE should hold for all $(t,x) \in [0,T) \times [B,\infty)$:

$$-rv(t,x) + v_t(t,x) + v_x(t,x)rx + \frac{1}{2}v_{xx}(t,x)\sigma^2 x^2 = 0.$$

Boundary conditions:

- At time $T$, if $x \geq B$, the down-and-out option is never knocked-out, and it will have the same payoff at maturity as a plain vanilla option. Thus, $v(T, x) = (x - K)^+, x \geq B$.

- For $S(t) = B$ at any time $t$, the down-and-out option is knocked out ad expires worthless. Thus, $v(t, B) = 0, t \in [0, T]$.

- For $S(t) \to \infty$, it is highly unlikely the stock price will hit the barrier and the down-and-out option gets knocked-out. In other words, the down-and-out option will very likely become a plain vanilla option *deep in the money* and end up with being in the money. In this case, the price of the down-and-out option and also the call is almost as much as the price of a forward contract:

$$\lim_{x\to\infty} \frac{v(t,x)}{x} = \lim_{x\to\infty} \frac{x - e^{-r(T-t)}K}{x} = 1.$$

**(c)** What is the relationship between the price of a down-and-out call option and the price of a down-and-in call option with the same parameters? Using this relationship write down the price of a down-and-in call option with the same parameters.
*Solution*: Long positions in one down-and-out option and one down-and-in option with all other parameters, i.e., barrier, maturity, and strike of the underlying option, being the same is equivalent to a long position in the corresponding plain vanilla option. To see, this, for every possible path for the price of the underlying asset, the final payoff of the plain vanilla option and of a portfolio made of the down-and-out and the down-and-in options are the same.

- If the barrier is hit before the expiry of the options, then the down-and-out option expires worthless, while the down-and-in option becomes a plain vanilla option. Thus, the payoff of the portfolio at maturity will be the same as the payoff of a plain vanilla option.
- If the price of the underlying asset never reaches the barrier, then the down-and-in option is never knocked in, and will expire worthless, while the down-and-out option is never knocked out, and it will have the same payoff at maturity as a plain vanilla option. The payoff at maturity of the portfolio made the barrier options will be the same as the payoff of a plain vanilla option.

Thus, the following relationship holds

$$\text{Price of down-and-out call } + \text{ Price of down-and-in call } = \text{Price of plain vanilla call}\,.$$

Using this relationship,

$$\begin{aligned} V^{\text{down-and-in}}(0) &= V^{\text{plain vanilla}}(0) - V^{\text{down-and-out}}(0) \\ &= c - e^{-rT} \int_{-\infty}^{k} dx \int_{x^+}^{b} da \big(S(0)e^{-\sigma x} - K\big) \frac{2(2a-x)}{T\sqrt{2\pi T}} e^{-\frac{(2a-x)^2}{2T} - \alpha x - \frac{1}{2}\alpha^2 T}, \end{aligned}$$

where $V^{\text{plain vanilla}}(0) = c$ is the usual Black-Scholes formula for a plain vanilla call option with the same parameters.

## 4 Problem Four: Zero strike Asian call option

Consider a zero-strike Asian call whose payoff at time $T$ is

$$V(T) = \frac{1}{T}\int_0^T S(u)du.$$

**(i)** Suppose at time $t$ we have $S(t) = x \geq 0$ and $\int_0^t S(u)du = y \geq 0$. Use the fact that $e^{-ru}S(u)$ is a martingale under $\widetilde{\mathbb{P}}$ to compute

$$e^{-r(T-t)}\widetilde{\mathbb{E}}\left[\frac{1}{T}\int_0^T S(u)du \middle| \mathscr{F}(t)\right].$$

Call your answer $v(t,x,y)$.
*Solution*: Invoking stochastic Fubini's theorem,

$$\begin{aligned}
e^{-r(T-t)}\widetilde{\mathbb{E}}\left[\frac{1}{T}\int_0^T S(u)du \middle| \mathscr{F}(t)\right] &= e^{-r(T-t)}\frac{1}{T}\int_0^T \widetilde{\mathbb{E}}\left[S(u)|\mathscr{F}(t)\right]du \\
&= e^{-r(T-t)}\frac{1}{T}\int_0^T e^{ru}\widetilde{\mathbb{E}}\left[\underbrace{e^{-ru}S(u)}_{\text{martingale}}|\mathscr{F}(t)\right]du \\
&= e^{-r(T-t)}\frac{1}{T}\int_0^T e^{ru}e^{-rt\wedge u}S(t\wedge u)du \\
&= e^{-r(T-t)}\frac{1}{T}\left[\int_0^t e^{ru}e^{-ru}S(u)du + \int_t^T e^{ru}e^{-rt}S(t)du\right] \\
&= e^{-r(T-t)}\frac{1}{T}\left[\int_0^t S(u)du + e^{-rt}S(t)\int_t^T e^{ru}du\right] \\
&= e^{-r(T-t)}\frac{1}{T}\left[\int_0^t S(u)du + \frac{1}{r}\left(e^{r(T-t)}-1\right)S(t)\right] \\
&= e^{-r(T-t)}\frac{1}{T}\underbrace{\int_0^t S(u)du}_{=Y(t)} + \frac{1}{rT}\left(1-e^{-r(T-t)}\right)S(t) \\
&\triangleq v(t,S(t),Y(t)).
\end{aligned}$$

Thus,

$$\boxed{v(t,x,y) = \frac{e^{-r(T-t)}}{T}y + \frac{1}{rT}\left(1-e^{-r(T-t)}\right)x.}$$

**(ii)** Verify that the function $v(t,x,y)$ you obtained in **(i)** satisfies the Black-Scholes-Merton equation

$$v_t(t,x,y) + rxv_x(t,x,y) + xv_y(t,x,y) + \frac{1}{2}\sigma^2x^2v_{xx}(t,x,y) = rv(t,x,y), \quad 0 \leq t < T, x \geq 0, y \in \mathbb{R}$$

and the boundary conditions

$$v(t,0,y) = e^{-r(T-t)}\left(\frac{y}{T}-K\right)^+, \quad 0 \le t < T, y \in \mathbb{R},$$
$$v(T,x,y) = \left(\frac{y}{T}-K\right)^+, \quad x \ge 0, y \in \mathbb{R}.$$

We do not try to verify

$$\lim_{y \downarrow -\infty} v(t,x,y) = 0, \quad 0 \le t < T, x \ge 0.$$

because the computation of $v(t,x,y)$ outlined here works only for $y \ge 0$.
*Solution*: Direct computation,

$$\begin{aligned}
v_t(t,x,y) &= \frac{re^{-r(T-t)}}{T}y - \frac{e^{-r(T-t)}}{T}x,\\
v_x(t,x,y) &= \frac{1}{rT}\left(1-e^{-r(T-t)}\right) \Rightarrow rxv_x(t,x,y) = \frac{x}{T}\left(1-e^{-r(T-t)}\right),\\
v_y(t,x,y) &= \frac{e^{-r(T-t)}}{T} \Rightarrow xv_y(t,x,y) = \frac{x}{T}e^{-r(T-t)},\\
v_{xx}(t,x,y) &= 0.
\end{aligned}$$

Putting all the terms together, we verify that

$$v_t(t,x,y) + rxv_x(t,x,y) + xv_y(t,x,y) + \frac{1}{2}\sigma^2x^2v_{xx}(t,x,y) = rv(t,x,y), \quad 0 \le t < T, x \ge 0, y \in \mathbb{R}.$$

Boundary conditions:

- If at some time $0 \le t < T$, the stock price becomes zero and will remain zero for the period of time $[t,T]$, and as a result, $Y(u)$ is constant $Y(u) = Y(t) = y$, $t \le u \le T$. The value of the Asian call at time $T$ is $(y/T-K)^+$, discounted back from $T$ back to $t$, we get

$$v(t,0,y) = e^{-r(T-t)}\left(\frac{y}{T}-K\right)^+, \quad 0 \le t < T, y \in \mathbb{R}.$$

- At maturity $t = T$, the value of the option reduces to the payoff function, namely,

$$v(T,x,y) = \left(\frac{y}{T}-K\right)^+, \quad x \ge 0, y \in \mathbb{R}.$$

**(iii)** Determine the process $\Delta(t) = v_x(t,S(t),Y(t))$, and observe that it is not random.
*Solution*: Differentiate $v(t,x,y)$ with respect to $x$,

$$v_x(t,x,y) = \frac{1}{rT}\left(1-e^{-r(T-t)}\right).$$

Thus,

$$\Delta(t) = v_x(t,S(t),Y(t)) = \frac{1}{rT}\left(1-e^{-r(T-t)}\right)$$

is not random as it does not depend on the $S(t)$.

**(iv)** Use the Itô-Doeblin formula to show that if you begin with initial capital $X(0) = v(0, S(0), 0)$ and at each time you hold $\Delta(t)$ shares of the underlying asset, investing or borrowing at the interest rate $r$ in order to do this, then at time $T$ the value of your portfolio will be

$$X(T) = \frac{1}{T}\int_0^T S(u)du.$$

*Proof*: On the one hand, the differential of the discounted portfolio value is given

$$d\left(e^{-rt}X(t)\right) = \sigma\Delta(t)\left(e^{-rt}S(t)\right)d\tilde{B}(t) = \sigma\frac{e^{-rt}-e^{-rT}}{rT}S(t)d\tilde{B}(t). \tag{4.1}$$

On the other hand,

$$\begin{aligned}
d\left(e^{-rt}v(t,S(t),Y(t))\right) &= -re^{-rt}v(t,S(t),Y(t))\,dt + e^{-rt}dv(t,S(t),Y(t)) \\
&= -re^{-rt}v(t,S(t),Y(t))\,dt \\
(v_{xx}=0) \qquad &\quad + e^{-rt}\left[v_t(t,S(t),Y(t))dt + v_x(t,S(t),Y(t))dS(t) + v_y(t,S(t),Y(t))dY(t)\right] \\
&= -re^{-rt}v(t,S(t),Y(t))\,dt \\
&\quad + e^{-rt}\left[\frac{rY(t)-S(t)}{T}e^{-r(T-t)}dt + \frac{1-e^{-r(T-t)}}{rT}dS(t) + \frac{S(t)}{T}e^{-r(T-t)}dt\right] \\
&= -re^{-rt}v(t,S(t),Y(t))\,dt + e^{-rt}\left[\frac{rY(t)}{T}e^{-r(T-t)}dt + \frac{1-e^{-r(T-t)}}{rT}dS(t)\right] \\
&= -re^{-rt}v(t,S(t),Y(t))\,dt \\
&\quad + e^{-rt}\left[\frac{rY(t)}{T}e^{-r(T-t)}dt + \frac{1-e^{-r(T-t)}}{rT}\left(rS(t)dt + \sigma S(t)d\tilde{B}(t)\right)\right] \\
&= -re^{-rt}v(t,S(t),Y(t))\,dt \\
&\quad + e^{-rt}\left[r\underbrace{\left(\frac{Y(t)}{T}e^{-r(T-t)} + \frac{1-e^{-r(T-t)}}{rT}S(t)\right)}_{=v(t,S(t),Y(t))}dt + \frac{1-e^{-r(T-t)}}{rT}\sigma S(t)d\tilde{B}(t)\right] \\
&= -re^{-rt}v(t,S(t),Y(t))\,dt + e^{-rt}rv(t,S(t),Y(t))\,dt + \frac{e^{-rt}-e^{-rT}}{rT}\sigma S(t)d\tilde{B}(t) \\
&= \frac{e^{-rt}-e^{-rT}}{rT}\sigma S(t)d\tilde{B}(t).
\end{aligned}$$

Comparing with Equation 4.1, we have

$$d\left(e^{-rt}X(t)\right) = d\left(e^{-rt}v(t,S(t),Y(t))\right), \quad \forall t \in [0,T).$$

Integrate over time, we get, note that $X(0) = v(0, S(0), 0)$,

$$\begin{aligned}
&e^{-rT}X(T) - X(0) = e^{-rT}v(T,S(T),Y(T)) - v(0,S(0),Y(0)) \\
&\Rightarrow X(T) = v(T,S(T),Y(T)) = \frac{1}{T}\int_0^T S(u)du.
\end{aligned}$$

BARUCH, MFE

# MTH 9831 Assignment Nine

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November 24, 2015

# 1 Problem One: Asian Option, zero interest rate

Consider the continuously sampled Asian option, but assume now that that interest rate is $r = 0$. Find an initial capital $X(0)$ and a nonrandom function $\gamma(t)$ to replace

$$\gamma(t) = \begin{cases} \frac{1-e^{-rc}}{rc}, & 0 \le t \le T-c, \\ \frac{1-e^{-r(T-t)}}{rc}, & T-c \le t \le T. \end{cases} \tag{1.1}$$

and

$$X(0) = \frac{1-e^{-rc}}{rc} S(0) - e^{-rT} K \tag{1.2}$$

so that

$$X(T) = \frac{1}{c} \int_{T-c}^{T} S(u) du - K$$

still holds. Give the formula for the resulting process $X(t)_{0 \le t \le T}$.

*Solution*: Taking the limit $r \to 0$ in Equation 1.1,

$$\gamma(t) = \begin{cases} 1, & 0 \le t \le T-c, \\ \frac{T-t}{c}, & T-c \le t \le T. \end{cases}$$

and in Equation 1.2

$$X(0) = S(0) - K.$$

We verify that, with the above choice $\gamma(t)$ and $X(0)$ for zero interest rate, the terminal condition still holds.

In the time interval $[0, T-c]$, the process $\gamma(t)$ mandates a buy-and-hold strategy. At time zero, we buy one share of the risky asset, which costs $S(0)$. Our initial capital is insufficient to do this, and we must borrow $K$ from the money market account. For $0 \le t \le T-c$, the value of our holdings in the risky asset is $S(t)$ and we owe $K$ to the money market account. Therefore,

$$X(t) = S(t) - K, \quad 0 \le t \le T-c.$$

In particular,

$$X(T-c) = S(T-c) - K.$$

For $T-c \le t \le T$, we have $d\gamma(t) = -dt/c$ and we compute $X(t)$ by integrating

$$d\left(e^{r(T-t)} X(t)\right) = d\left(e^{r(T-t)} \gamma(t) S(t)\right) - e^{r(T-t)} S(t) d\gamma(t)$$

from $T-c$ to $t$, with $r = 0$,

$$\begin{aligned} X(t) - \underbrace{X(T-c)}_{=S(T-c)-K} &= \gamma(t')S(t')\Big|_{T-c}^{t} - \int_{T-c}^{t} S(t') d\gamma(t') \\ &= \gamma(t)S(t) - \underbrace{\gamma(T-c)}_{=1} S(T-c) + \frac{1}{c}\int_{T-c}^{t} S(t')dt' \\ &= \frac{T-t}{c} S(t) - S(T-c) + \frac{1}{c}\int_{T-c}^{t} S(t')dt'. \end{aligned}$$

Thus,

$$X(t) = \frac{T-t}{c} S(t) + \frac{1}{c}\int_{T-c}^{t} S(t')dt' - K, \quad T-c \le t \le T.$$

In summary,

$$X(t) = \begin{cases} S(t) - K, & 0 \le t \le T-c, \\ \frac{T-t}{c} S(t) + \frac{1}{c}\int_{T-c}^{t} S(t')dt' - K, & T-c \le t \le T. \end{cases}$$

It's straightforward to verify that

$$X(T) = \frac{1}{c}\int_{T-c}^{T} S(t')dt' - K$$

still holds.

## 2 Problem Two: Floating strike Asian option

Assume the BSM model ($r \neq 0$). Consider the floating strike Asian call option with payoff at time $T$ given by

$$\left(\frac{1}{c}\int_{T-c}^{T} S(t)dt - S(T)\right)^{+}. \tag{2.1}$$

Derive an analog of Večeř's Theorem (Theorem 7.5.3 in Shreve II) for this option.
*Solution*: To price this call, we want to create a process whose value at time $T$ is

$$X'(T) = \frac{1}{c}\int_{T-c}^{T} S(t)dt - S(T).$$

Recall the portfolio process $X(t)$ used to price the fixed strike Asian call option with the same maturity satisfies

$$X(T) = \frac{1}{c}\int_{T-c}^{T} S(t)dt - K,$$

the difference between the two payoffs at maturity

$$X(T) - X'(T) = S(T) - K \triangleq F(T)$$

is simply the payoff of a forward contract expiring at $T$. Thus, we can form the portfolio process $X'(t)_{0\le t\le T}$ by a combination of the portfolio $X(t)$ and a forward contract $F(t) = S(t) - Ke^{-r(T-t)}$:

$$\begin{aligned} X'(t) &= X(t) - F(t) = X(t) - S(t) + Ke^{-r(T-t)} \\ &= \begin{cases} \frac{1-rc-e^{-rc}}{rc} S(t), & 0 \le t \le T-c \\ \frac{1-rc-e^{-r(T-t)}}{rc} S(t) + \frac{e^{-r(T-t)}}{c}\int_{T-c}^{t} S(t')dt', & T-c \le t \le T. \end{cases} \end{aligned} \tag{2.2}$$

Differentiate with respect to time $t$ and use the portfolio process for $X(t)$ given in lecture notes

$$\begin{aligned} dX'(t) &= dX(t) - dS(t) + rKe^{-r(T-t)} \\ &= \gamma(t)dS(t) + r(X(t) - \gamma(t)S(t))dt - dS(t) + rKe^{-r(T-t)} \\ &= (\gamma(t)-1)dS(t) + r(X'(t) + S(t) - Ke^{-r(T-t)} - \gamma(t)S(t))dt + rKe^{-r(T-t)} \\ &= (\gamma(t)-1)dS(t) + r(X'(t) - (\gamma(t)-1)S(t))dt \\ &\triangleq \gamma'(t)dS(t) + r(X'(t) - \gamma'(t)S(t))dt, \end{aligned}$$

where $\gamma(t)$, given by Equation 1.1, is the number of shares of the risky asset held by the $X(t)$ portfolio and $\gamma'(t)$ is the number of shares of the risky asset held by the $X'(t)$ portfolio. From the above equation, we can read off $\gamma'(t)$:

$$\boxed{\gamma'(t) = \gamma(t) - 1 = \begin{cases} \frac{1-rc-e^{-rc}}{rc}, & 0 \le t \le T-c, \\ \frac{1-rc-e^{-r(T-t)}}{rc}, & T-c \le t \le T, \end{cases}} \tag{2.3}$$

and the initial capital expressed in terms of $X(0)$ given by Equation 1.2:

$$\boxed{X'(0) = X(0) - S(0) + Ke^{-rT} = \frac{1 - rc - e^{-rc}}{rc} S(0).}$$

Thus, the payoff of the floating strike Asian call option is

$$V(T) = X'^{+}(T).$$

The risk-neutral price of the option at time $t$ prior to expiration is

$$V(t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)} X'^{+}(T) | \mathscr{F}(t)\right] = S(t)\tilde{\mathbb{E}}^{S}\left[Y'^{+}(T) | \mathscr{F}(t)\right] = S(t) \times g(t, Y'(t)),$$

where $Y'(t) = X'(t)/S(t)$. The derivation is a repetition of one done in the lecture, with a replacement

$$X(t) \to X'(t),$$
$$Y(t) \to Y'(t),$$
$$\gamma(t) \to \gamma'(t).$$

**Analog of Večeř's Theorem (Theorem 7.5.3 in Shreve II)**. For $0 \le t \le T$, the price $V(t)$ at time $t$ of the continuously averaged Asian call with payoff Equation 2.1 at time $T$ is

$$V(t) = S(t) g\left(t, \frac{X'(t)}{S(t)}\right),$$

where $g(t, y)$ satisfies

$$g_t(t, y) + \frac{1}{2}\sigma^2 \left(\gamma'(t) - y\right)^2 g_{yy}(t, y) = 0, \quad 0 \le t < T, \quad y \in \mathbb{R}.$$

and $\gamma'(t)$ is given by Equation 2.3 and $X'(t)$ is given by Equation 2.2. Boundary conditions for the function $g(t, y)$:

- $g(T, y) = y^{+}, \quad y \in \mathbb{R};$
- $\lim_{y \to -\infty} g(t, y) = 0, \quad 0 \le t \le T;$
- $\lim_{y \to +\infty} \left[g(t, y) - y\right] = 0, \quad 0 \le t \le T.$

# 3 Problem Four: Solving the linear complementarity condition

Suppose $v(x)$ is bounded conditions function having a continuous derivative and satisfying the linear complementary conditions. This exercise shows that $v(x)$ must be the function $v_{L*}(x)$ given by

$$v_{L^*}(x) = \begin{cases} K - x, & 0 \le x \le L_*, \\ (K - L_*)\left(\frac{x}{L_*}\right)^{-\frac{2r}{\sigma^2}}, & x \ge L_*, \end{cases} \tag{3.1}$$

with $L_*$ given by

$$L_* = \frac{2r}{2r + \sigma^2} K. \tag{3.2}$$

We assume that $K$ is strictly positive.

**(i)** First consider an interval of $x$-values in which $v(x)$ satisfies

$$r v(x) - r x v'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) = 0. \tag{3.3}$$

Equation 3.3 is a linear, second-order differential equation, and it has two solutions of the form $x^p$, the solutions differing because of different values of $p$. Substitute $x^p$ into Equation 3.3 and show that the only values of $p$ that cause $x^p$ to satisfy the equation are $p = -2r/\sigma^2$ and $p = 1$.

*Solution*: Substitute $x^p$ into Equation 3.3,

$$0 = r x^p - r p x^p - \frac{1}{2}\sigma^2 p(p-1) x^p = \left(r - rp - \frac{1}{2}\sigma^2 p(p-1)\right) x^p, \quad \forall x$$
$$\Rightarrow r - rp - \frac{1}{2}\sigma^2 p(p-1) = 0$$
$$\Rightarrow \sigma^2 p^2 + (2r - \sigma^2) p - 2r = 0$$
$$\Rightarrow p = 1 \text{ or } -\frac{2r}{\sigma^2}.$$

**(ii)** The functions $x^{-\frac{2r}{\sigma^2}}$ and $x$ are said to be linearly independent solutions of Equation 3.3, and every function that satisfies Equation 3.3 on an interval must be of the form

$$f(x) = A x^{-\frac{2r}{\sigma^2}} + Bx$$

for some constants $A$ and $B$. Use this fact and the fact that both $v(x)$ and $v'(x)$ are continuous to show that there cannot be an interval $[x_1, x_2]$ where $0 < x_1 < x_2 < \infty$, such that $v(x)$ satisfies Equation 3.3 on $[x_1, x_2]$ and satisfies

$$v(x) = (K - x)^+, \quad \forall x \ge 0$$

for $x$ at and immediately to the left of $x_1$ and for $x$ at and immediately to the right of $x_2$ unless $v(x)$ is identically zero on $[x_1, x_2]$.

*Solution*: To examine the conditions for

$$f(x) = Ax^{-\frac{2r}{\sigma^2}} + Bx,$$
$$f'(x) = -\frac{2Ar}{\sigma^2}x^{-\frac{2r}{\sigma^2}-1} + B,$$

we consider the following cases

- $0 < x_1 < K < x_2$, then in the left neighbourhood of $x' \in (x_2 - \epsilon, x_2]$ for an arbitrarily small $\epsilon > 0$, $f(x') = (K - x')^+ = 0$ and the right derivative $f'(x') = \frac{\partial}{\partial x'}(K - x')^+ = 0$. Thus there exists $x' \in [x_1, x_2]$ such that $f(x') = f'(x') = 0$:

$$Ax'^{-\frac{2r}{\sigma^2}} + Bx' = 0,$$
$$-\frac{2Ar}{\sigma^2}x'^{-\frac{2r}{\sigma^2}-1} + B = 0$$

  This set of linear equations of $A$ and $B$ is non-singular, thus, $A = B = 0$.

- $0 < x_1 < x_2 < K$,

$$Ax_1^{-\frac{2r}{\sigma^2}} + Bx_1 = K - x_1$$
$$Ax_2^{-\frac{2r}{\sigma^2}} + Bx_2 = K - x_2$$
$$-\frac{2Ar}{\sigma^2}x_1^{-\frac{2r}{\sigma^2}-1} + B = -1$$
$$-\frac{2Ar}{\sigma^2}x_2^{-\frac{2r}{\sigma^2}-1} + B = -1$$

  Because $x_1 \neq x_2$, the last two equations imply $A = 0$ and $B = -1$. Substituting $A = 0$ into the first two equations, we get

$$\frac{K - x_1}{x_1} = \frac{K - x_2}{x_2} = -1 \Rightarrow K = 0 \text{ or } x_1 = x_2.$$

  Both contradicts the assumptions.

In summary, the only solution $f(x)$ that satisfies both the PDE and the boundary conditions is the identically zero solution.

**(iii)** Use the fact that $v(0)$ must equal $K$ to show that there cannot be a number $x_2 > 0$ such that $v(x)$ satisfies Equation 3.3 on $[0, x_2]$.
*Solution*: If $v(x)$ satisfies Equation 3.3 on $[0, x_2]$, then

$$v(x) = Ax^{-\frac{2r}{\sigma^2}} + Bx$$

for some constants $A$ and $B$ and let $x \to 0^+$,

$$v(0) \triangleq \lim_{x \to 0^+} v(x) \sim Ax^{-\frac{2r}{\sigma^2}} \to \infty,$$

violating the requirement $v(0) = K$.

**(iv)** Explain why $v(x)$ cannot satisfy Equation 3.3 for all $x \geq 0$.
*Answer*: Equation 3.3 cannot hold in the vicinity of $x = 0^+$. This is shown in part **(iii)**.

**(v)** Explain why $v(x)$ cannot satisfy

$$v(x) = (K - x)^+, \quad \forall x \geq 0.$$

*Answer*: $v(x) = (K-x)^+, \forall x \geq 0$ does not have continuous derivative around $x = K$, violating the assumption of this problem.

**(vi)** From part **(iv)** and **(v)** and the so-called *linear complementarity condition* that for each $x \geq 0$, either PDE 3.3 or $v(x) = (K-x)^+$ holds, we see that $v(x)$ sometimes satisfies $v(x) = (K-x)^+$ and sometimes does not, in which case it must satisfy Equation 3.3. From part **(ii)** and **(iii)** we see that the region in which $v(x)$ does not satisfy $v(x) = (K-x)^+$ and satisfies Equation 3.3 is not an interval $[x_1, x_2]$, where $0 < x_1 < x_2 < \infty$, nor can this region be a union of disjoint intervals of this form. Therefore, it must be a half-line $[x_1, \infty)$, where $x_1 > 0$. In the region $[0, x_1]$, $v(x)$ satisfies $v(x) = (K-x)^+$. Show that $x_1$ must equal $L_*$ given by Equation 3.2 and $v(x)$ must be $v_{L*}(x)$ given by Equation 3.1.
*Solution*: According to part (i) and the description given in this problem, we have

$$v(x) = \begin{cases} (K-x)^+, & x \in [0, x_1] \\ Ax^{-\frac{2r}{\sigma^2}} + Bx, & x \in [x_1, \infty). \end{cases}$$

The continuity condition on the function $v(x)$ and its derivative requires that $x_1 < K$ because $(K-x)^+$ is not differentiable at $K$ and the following matching conditions

$$\begin{cases} K - x_1 = Ax_1^{-\frac{2r}{\sigma^2}} + Bx_1, \\ -1 = -\frac{2Ar}{\sigma^2} x_1^{-\frac{2r}{\sigma^2}-1} + B. \end{cases}$$

Because $v(x)$ is a bounded function by assumption, we must have $B = 0$; otherwise, $\lim_{x\to\infty} v(x) \sim Bx \to \infty$. Thus,

$$\begin{cases} K - x_1 = Ax_1^{-\frac{2r}{\sigma^2}}, \\ -1 = -\frac{2Ar}{\sigma^2} x_1^{-\frac{2r}{\sigma^2}-1}. \end{cases}$$

Solving the above equations, we get

$$x_1 = \frac{2r}{2r+\sigma^2} K \triangleq L_*, \quad Ax_1^{-\frac{2r}{\sigma^2}} = \frac{\sigma^2}{2r} x_1 = \frac{\sigma^2}{2r} L_* = K - L_*.$$

Thus,

$$v(x) = \begin{cases} K - x, & x \in [0, L_*] \\ (K - L_*) \left(\frac{x}{L_*}\right)^{-\frac{2r}{\sigma^2}}, & x \in [L_*, \infty). \end{cases}$$

# 4 PROBLEM FIVE: PERPETUAL AMERICAN PUT PAYING DIVIDENDS

Consider a perpetual American put on a geometric Brownian motion asset price paying dividends at a constant rate $a > 0$. The differential of this asset is

$$dS(t) = (r-a)S(t)dt + \sigma S(t)d\tilde{W}(t),$$

where $\tilde{W}(t)$ is a Brownian motion under a risk-neutral measure $\tilde{\mathbb{P}}$.

**(i)** Suppose we adopt the strategy of exercising the put the first time the asset price is at or below $L$. What is the risk-neutral expected discounted payoff of this strategy? Write this as a function $v_L(x)$ of the initial asset price $x$.

*Solution*: As hinted in the problem (by an application of Itô's Lemma), the SDE

$$dS(t) = (r-a)S(t)dt + \sigma S(t)d\tilde{W}(t)$$

is solved by

$$S(t) = S(0)\exp\left\{\sigma\tilde{W}(t) + \left(r-a-\frac{1}{2}\sigma^2\right)t\right\}.$$

Assume $S(0) = x$, then the asset process $S(t)$ hits the level $L$ when

$$S(t) = L \Rightarrow x\exp\left\{\sigma\tilde{W}(t) + \left(r-a-\frac{1}{2}\sigma^2\right)t\right\} = L \Rightarrow \tilde{W}(t) + \frac{1}{\sigma}\left(r-a-\frac{1}{2}\sigma^2\right)t = -\frac{1}{\sigma}\log\frac{x}{L} < 0.$$

This condition can be re-written as

$$-\tilde{W}(t) \underbrace{-\frac{1}{\sigma}\left(r-a-\frac{1}{2}\sigma^2\right)}_{\triangleq\mu} t = \frac{1}{\sigma}\log\frac{x}{L} > 0,$$

where $-\tilde{W}(t)$ is also a standard Brownian motion. Since the level hit is now positive, we can apply Theorem 8.3.2 in textbook or Problem Five in Homework Assignment 2:

$$\tilde{\mathbb{E}}\left[e^{-r\tau_L}\right] = \exp\left\{-\frac{1}{\sigma}\log\frac{x}{L}\left[\frac{1}{\sigma}\left(r-a-\frac{1}{2}\sigma^2\right) + \sqrt{\frac{1}{\sigma^2}\left(r-a-\frac{1}{2}\sigma^2\right)^2 + 2r}\right]\right\} \triangleq \exp\left\{-\gamma\log\frac{x}{L}\right\} = \left(\frac{x}{L}\right)^{-\gamma},$$

where

$$\gamma \triangleq \frac{1}{\sigma^2}\left(r-a-\frac{1}{2}\sigma^2\right) + \frac{1}{\sigma}\sqrt{\frac{1}{\sigma^2}\left(r-a-\frac{1}{2}\sigma^2\right)^2 + 2r}.$$

Thus the risk-neutral expected discounted payoff of this strategy is

$$v_L(x) = \begin{cases} K-x, & 0 \le x \le L, \\ (K-L)\left(\frac{x}{L}\right)^{-\gamma}, & x > L. \end{cases} \tag{4.1}$$

**(ii)** Determine $L_*$, the value of $L$ that maximizes the risk-neutral expected discounted payoff computed in **(i)**.

*Solution*: Differentiate $v_L(x)$ with respect to $L$ gives the extremization condition of the risk-neutral expected discounted payoff:

$$\frac{\partial}{\partial L} v_L(x) = -\left(\frac{x}{L}\right)^{-\gamma} + \gamma(K-L)\left(\frac{x}{L}\right)^{-\gamma-1}\frac{x}{L^2} = -\left(\frac{x}{L}\right)^{-\gamma}\left(1-\gamma\frac{K-L}{L}\right) = 0 \Rightarrow L_* = \frac{\gamma K}{1+\gamma}.$$

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# MTH 9831 Assignment Ten

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December 2, 2015

# 1 Problem One

A function $f(x)$ defined for $x > 0$ is said to be *convex* if, every $0 \leq x_1 \leq x_2$ and every $0 \leq \lambda \leq 1$, the in equality

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)f(x_1) + \lambda f(x_2)$$

holds. Suppose $f(x)$ and $g(x)$ are convex functions defined for $x \geq 0$. Show that

$$h(x) = \max\{f(x), g(x)\}$$

is also convex.
*Proof*: For every $0 \leq x_1 \leq x_2$ and every $0 \leq \lambda \leq 1$,

$$\begin{aligned} h((1-\lambda)x_1 + \lambda x_2) &\triangleq \max\{f((1-\lambda)x_1 + \lambda x_2), g((1-\lambda)x_1 + \lambda x_2)\} \\ \text{(convexity)} \quad &\leq \max\{(1-\lambda)f(x_1) + \lambda f(x_2), (1-\lambda)g(x_1) + \lambda g(x_2)\} \\ \text{(because } \lambda \geq 0, 1-\lambda \geq 0) \quad &\leq (1-\lambda)\max\{f(x_1), g(x_1)\} + \lambda \max\{f(x_2), g(x_2)\} \\ &= (1-\lambda)h(x_1) + \lambda h(x_2). \end{aligned}$$

Thus, $h(x)$ is also convex.

# 2 Problem Two

Let $M(t)$ be the compensated Poisson process.
**(i)** Show that $M^2(t)$ is a submartingale.
*Proof*: We first verify that $M^2(t)$ is integrable.

$$\begin{aligned}\mathbb{E}\left[|M^2(t)|\right] &= \mathbb{E}\left[N^2(t) - 2\lambda t N(t) + \lambda^2 t^2\right] \\ &= \mathbb{E}\left[N^2(t)\right] - 2\lambda t \mathbb{E}\left[N(t)\right] + \lambda^2 t^2 \\ &= \lambda^2 t^2 + \lambda t - 2\lambda^2 t^2 + \lambda^2 t^2 \\ &= \lambda t < \infty.\end{aligned}$$

Second, $M^2(t)$ is $\mathcal{F}(t)$-measurable because $N(t)$, depending on the jumps occurred before time $t$, is is $\mathcal{F}(t)$-measurable. Finally, we verify the sub-martingality, for $0 \leq s < t$,

$$\begin{aligned}\mathbb{E}\left[M^2(t)|\mathcal{F}(s)\right] &= \mathbb{E}\left[(M(t) - M(s) + M(s))^2|\mathcal{F}(s)\right] \\ &= \mathbb{E}\left[(M(t) - M(s))^2 - 2M(s)(M(t) - M(s)) + M^2(s)|\mathcal{F}(s)\right] \\ &= \mathbb{E}\left[(M(t) - M(s))^2\right] - 2M(s)\mathbb{E}\left[M(t) - M(s)\right] + M^2(s) \\ &= \lambda(t - s) + M^2(s) \\ &> M^2(s).\end{aligned}$$

**(ii)** Show that $M^2(t) - \lambda t$ is a martingale.
*Proof*: The process $M^2(t) - \lambda t$ is integrable and $\mathcal{F}(t)$-measurable for exactly the same reason outlined in the previous part for $M^2(t)$, the derivation of which we will not duplicate here. We verify the martingality, for $0 \leq s < t$,

$$\begin{aligned}\mathbb{E}\left[M^2(t) - \lambda t|\mathcal{F}(s)\right] &= \mathbb{E}\left[(M(t) - M(s) + M(s))^2 - \lambda t|\mathcal{F}(s)\right] \\ &= \mathbb{E}\left[(M(t) - M(s))^2 - 2M(s)(M(t) - M(s)) + M^2(s) - \lambda t|\mathcal{F}(s)\right] \\ &= \mathbb{E}\left[(M(t) - M(s))^2\right] - 2M(s)\mathbb{E}\left[M(t) - M(s)\right] + M^2(s) - \lambda t \\ &= \lambda(t - s) + M^2(s) - \lambda t \\ &= M^2(s) - \lambda s.\end{aligned}$$

# 3 Problem Three

Suppose $N_1(t)$ and $N_2(t)$ are Poisson processes with intensities $\lambda_1$ and $\lambda_2$, respectively, both defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and relative to the same filtration $\mathcal{F}(t)_{t\geq 0}$. Assume that almost surely $N_1(t)$ and $N_2(t)$ have no simultaneous jump. Show that, for each fixed $t$, the random variable $N_1(t)$ and $N_2(t)$ are independent. (In fact, the whole path of $N_1$ is independent of the whole path of $N_2$, although you are not being asked to prove this stronger statement.)

*Proof*: Let $u_1$ and $u_2$ be fixed real numbers and define

$$Y(t) = \exp\left\{u_1 N_1(t) + u_2 N_2(t) - \lambda_1(e^{u_1} - 1)t - \lambda_2(e^{u_2} - 1)t\right\}.$$

We use Itô formula to show that $Y$ is a martingale. To do this, we define

$$X(s) = u_1 N_1(s) + u_2 N_2(s) - \lambda_1(e^{u_1} - 1)s - \lambda_2(e^{u_2} - 1)s$$

and $f(x) = e^x$, so that $Y(s) = f(X(s))$. The process $X(s)$ has Riemann integral part

$$R(s) = -\lambda_1(e^{u_1} - 1)s - \lambda_2(e^{u_2} - 1)s$$

such that $dX^c(s) = -\lambda_1(e^{u_1} - 1)ds - \lambda_2(e^{u_2} - 1)ds$, and pure jump part

$$J(s) = u_1 N_1(s) + u_2 N_2(s).$$

We next observe that if $Y$ has a jump at time $s$, then either $N_1$ or $N_2$ has a jump at time $s$, but not both, then

$$\begin{aligned} Y(s) &= \begin{cases} \exp\left\{u_1(N_1(s-)+1) + u_2 N_2(s) - \lambda_1(e^{u_1} - 1)s - \lambda_2(e^{u_2} - 1)s\right\}, & N_1 \text{ jumps} \\ \exp\left\{u_1 N_1(s) + u_2(N_2(s-)+1) - \lambda_1(e^{u_1} - 1)s - \lambda_2(e^{u_2} - 1)s\right\}, & N_2 \text{ jumps} \end{cases} \\ &= \begin{cases} e^{u_1} Y(s-), & N_1 \text{ jumps} \\ e^{u_2} Y(s-), & N_2 \text{ jumps}. \end{cases} \end{aligned}$$

Therefore,

$$Y(s) - Y(s-) = \begin{cases} (e^{u_1} - 1) Y(s-) \Delta N_1(s), & N_1 \text{ jumps} \\ (e^{u_2} - 1) Y(s-) \Delta N_2(s), & N_2 \text{ jumps}. \end{cases}$$

Because almost surely $N_1(t)$ and $N_2(t)$ have no simultaneous jump, we can rewrite the above expression as

$$Y(s) - Y(s-) = (e^{u_1} - 1) Y(s-) \Delta N_1(s) + (e^{u_2} - 1) Y(s-) \Delta N_2(s).$$

According to Itô formula for jump processes,

$$\begin{aligned} Y(t) &= f(X(t)) \\ &= f(X(0)) + \int_0^t f'(X(s)) dX^c(s) + \cancelto{0}{\frac{1}{2}\int_0^t f''(X(s)) dX^c(s) dX^c(s)} + \sum_{0<s\leq t} \left[f(X(s)) - f(X(s-))\right] \end{aligned}$$

$$
\begin{aligned}
&= 1 - \lambda_1(e^{u_1} - 1)\int_0^t Y(s)ds - \lambda_2(e^{u_2} - 1)\int_0^t Y(s)ds + \sum_{0<s\le t}[Y(s) - Y(s-)] \\
&= 1 - \lambda_1(e^{u_1} - 1)\int_0^t Y(s)ds - \lambda_2(e^{u_2} - 1)\int_0^t Y(s)ds \\
&\quad + (e^{u_1} - 1)\int_0^t Y(s-)dN_1(s) + (e^{u_2} - 1)\int_0^t Y(s-)dN_2(s) \\
&= 1 - (e^{u_1} - 1)\int_0^t Y(s-)dM_1(s) - (e^{u_2} - 1)\int_0^t Y(s-)dM_2(s),
\end{aligned}
$$

where $M_i(s) = N_i(s) - \lambda_i s$, $i = 1, 2$, are compensated Poisson processes. Here we have used the fact that because both $N_1$ and $N_2$ have only finitely many jumps, $Y$ also has only finitely many jumps, such that

$$
\int_0^t Y(s)ds = \int_0^t Y(s-)ds.
$$

We have expressed $Y(t)$ as the sum of two integrals of left-continuous process $Y(s-)$ with respect to the martingales $M_1(s)$ and $M_2(s)$. Therefore, $Y$ is a martingale. (The sum of martingales with relative to the same filtration is still a martingale).

Because $Y(0) = 1$ and $Y$ is a martingale, we have $\mathbb{E}[Y(t)] = 1$ for all $t$. In other words,

$$
\mathbb{E}\exp\{u_1N_1(t) + u_2N_2(t) - \lambda_1(e^{u_1} - 1)t - \lambda_2(e^{u_2} - 1)t\} = 1, \quad \forall t \ge 0.
$$

We have obtained the joint moment-generating function formula

$$
\mathbb{E}\exp\{u_1N_1(t) + u_2N_2(t)\} = \exp\{\lambda_1(e^{u_1} - 1)t + \lambda_2(e^{u_2} - 1)t\} = \exp\{\lambda_1(e^{u_1} - 1)t\}\exp\{\lambda_2(e^{u_2} - 1)t\},
$$

which factors into the product of the moment-generating functions of $N_1$ and $N_2$. Therefore, $N_1(t)$ and $N_2(t)$ are independent.

# 4 Problem Four

Evaluate the following stochastic integrals:
**(a)** $\int_0^t N(s)dN(s)$;
*Solution*: By definition,

$$\int_0^t N(s)dN(s) = \sum_{0<s\le t} N(s)\Delta N(s) = \sum_{\text{all jumps } 0<s'\le t} N(s') = \sum_{n=1}^{N(t)} n = \frac{N(t)(N(t)+1)}{2}.$$

**(b)** $\int_0^t N(s-)dN(s)$;
*Solution*: By definition,

$$\int_0^t N(s-)dN(s) = \sum_{0<s\le t} N(s-)\Delta N(s) = \sum_{\text{all jumps } 0<s'\le t} N(s'-) = \sum_{n=1}^{N(t)} (n-1) = \frac{N(t)(N(t)-1)}{2}.$$

**(c)** $\int_0^t M(s-)dM(s)$;
*Solution*: Recall that $M(t) = N(t) - \lambda t$, so

$$dM(t) = dN(t) - \lambda dt.$$

If $M$ has a jump at $s$,

$$M^2(s) = (M(s-)+1)^2 = M^2(s-)+2M(s-)+1 \Rightarrow M^2(s)-M^2(s-) = 2M(s-)+1 = [2M(s-)+1]\,\Delta N(s)$$

Let $f(x) = x^2$, apply Itô's formula,

$$\begin{aligned}
M^2(t) &= f(M(t)) \\
&= \cancelto{0}{f(M(0))} - 2\lambda \int_0^t M(s)ds + \sum_{0<s\le t} \left[M^2(s) - M^2(s-)\right] \\
&= -2\lambda \int_0^t M(s)ds + \sum_{0<s\le t} [2M(s-)+1]\,\Delta N(s) \qquad (4.1) \\
&= -2\lambda \int_0^t M(s)ds + 2\int_0^t M(s-)\underbrace{dN(s)}_{=dM(s)+\lambda ds} + N(t) \\
&= 2\int_0^t M(s-)dM(s) + N(t).
\end{aligned}$$

Rearranging the terms, we get

$$\int_0^t M(s-)dM(s) = \frac{M^2(t) - N(t)}{2}.$$

**(d)** $\int_0^t M(s)dM(s)$;
*Solution*: Follow essentially the same procedure in the previous part, starting from the third equality (4.1),

$$\begin{aligned}
M^2(t) &= -2\lambda \int_0^t M(s)ds + \sum_{0<s\le t} [2M(s-)+1]\,\Delta N(s) \\
&= -2\lambda \int_0^t M(s)ds + \sum_{0<s\le t} [2M(s)-1]\,\Delta N(s) \\
&= -2\lambda \int_0^t M(s)ds + 2\int_0^t M(s) \underbrace{dN(s)}_{=dM(s)+\lambda ds} - N(t) \\
&= 2\int_0^t M(s)dM(s) - N(t).
\end{aligned}$$

Rearranging the terms, we get

$$\int_0^t M(s)dM(s) = \frac{M^2(t)+N(t)}{2}.$$

## 5 Problem Five

Suppose that $\sigma > -1$ and a jump process $S(t)_{t\geq 0}$ satisfies

$$S(t) = S(0) + \sigma \int_0^t S(u-)dM(u).$$

Show that $S(t) = S(0)e^{-\lambda\sigma t}(1+\sigma)^{N(t)}$.
*Proof*: Write the jump process in differential form,

$$dS(t) = \sigma S(t-) \underbrace{dM(t)}_{=-\lambda dt + dN(t)} = \underbrace{-\lambda\sigma S(t-)dt}_{=dS^c(t)} + \sigma S(t-)dN(t).$$

In integral form,

$$S(t) = S(0) + \int_0^t dS^c(t') + \sigma \int_0^t S(t'-)dN(t') = S(0) + \int_0^t dS^c(t') + \sigma \sum_{0<t'\leq t} S(t'-)\Delta N(t').$$

Thus, if $S$ has a jump at time $t$, $S(t) = S(t-) + \sigma S(t-)\Delta N(t) = (1+\sigma)S(t-)\Delta N(t)$.

Let $f(x) = \log x$. Apply Itô's formula to $\log S(t)$:

$$\begin{aligned}
\log S(t) &= \log S(0) + \int_0^t \frac{1}{S(t')} dS^c(t') - \frac{1}{2}\int_0^t \frac{1}{S^2(t')} dS^c(t')dS^c(t') + \sum_{0<t'\leq t} \left[\log S(t') - \log S(t'-)\right] \\
&= \log S(0) - \lambda\sigma \int_0^t \frac{1}{S(t')} S(t'-)dt' + \sum_{0<t'\leq t} \log \frac{S(t')}{S(t'-)} \\
&= \log S(0) - \lambda\sigma \int_0^t \underbrace{\frac{1}{S(t')} S(t'-)}_{\text{almost surely one}} dt' + \sum_{0<t'\leq t} \log(1+\sigma)\Delta N(t') \\
&= \log S(0) - \lambda\sigma t + (1+\sigma)^{N(t)}.
\end{aligned}$$

Exponentiate both sides of the above equation, we get

$$S(t) = S(0)e^{-\lambda\sigma t}(1+\sigma)^{N(t)}.$$

# 6 Problem Six

Let $\lambda, \tilde{\lambda} \in (0, \infty)$. Apply Itô's formula to show that

$$Z(t) \triangleq Z(0) e^{(\lambda - \tilde{\lambda})t} \left( \frac{\tilde{\lambda}}{\lambda} \right)^{N(t)}, \quad t \geq 0$$

satisfies the equation

$$dZ(t) = \frac{\tilde{\lambda} - \lambda}{\lambda} Z(t-) dM(t).$$

Conclude that $Z(t)_{t \geq 0}$ is a martingale.

*Proof*: Write

$$Z(t) = Z(0) e^{(\lambda - \tilde{\lambda})t + N(t) \log \frac{\tilde{\lambda}}{\lambda}}.$$

Let $X(t) = (\lambda - \tilde{\lambda})t + N(t) \log \frac{\tilde{\lambda}}{\lambda} = X^c(t) + J(t)$, where $X^c(t) = (\lambda - \tilde{\lambda})t$ is the continuous part and $J(t) = N(t) \log \frac{\tilde{\lambda}}{\lambda}$ is a pure jump process.

Let $f(x) = e^x$ and apply Itô's formula to $\frac{Z(t)}{Z(0)} = f(X(t))$:

$$\begin{aligned}
\frac{Z(t)}{Z(0)} &= 1 + \int_0^t f'(X(s)) dX^c(s) + \cancelto{0}{\frac{1}{2} \int_0^t f''(X(s)) dX^c(s) dX^c(s)} + \sum_{0 < s \leq t} \left[ f(X(s)) - f(X(s-)) \right] \\
&= 1 + (\lambda - \tilde{\lambda}) \int_0^t e^{X(s)} ds + \sum_{0 < s \leq t} \left[ \underbrace{f(X(s)) - f(X(s-))}_{= \left( \frac{\tilde{\lambda}}{\lambda} - 1 \right) f(X(s-)) \Delta N(s)} \right] \\
&= 1 + (\lambda - \tilde{\lambda}) \int_0^t \underbrace{e^{X(s)}}_{= e^{X(s-)} \text{ almost surely}} ds + \frac{\tilde{\lambda} - \lambda}{\lambda} \int_0^t e^{X(s-)} dN(s) \\
&= 1 + \frac{\tilde{\lambda} - \lambda}{\lambda} \int_0^t e^{X(s-)} \left[ \lambda ds + dN(s) \right] \\
&= 1 + \frac{\tilde{\lambda} - \lambda}{\lambda} \int_0^t e^{X(s-)} dM(s) \\
&= 1 + \frac{\tilde{\lambda} - \lambda}{\lambda} \int_0^t f(X(s-)) dM(s) \\
&= 1 + \frac{\tilde{\lambda} - \lambda}{\lambda} \int_0^t \frac{Z(s-)}{Z(0)} dM(s).
\end{aligned}$$

In differential notations,

$$dZ(t) = \frac{\tilde{\lambda} - \lambda}{\lambda} Z(t-) dM(t).$$

We conclude that $Z(t)_{t \geq 0}$ is a martingale since Z(t-) is left continuous.

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# MTH 9831 Assignment Eleven

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December 9, 2015

Let $N(t)_{t\geq 0}$ be Poisson process with intensity $\lambda$, $M(t) = N(t) - \lambda t, t \geq 0$, be a compensated Poisson process, $Q(t)_{t\geq 0}$ be a compound Poisson process with jump distribution $\mathbb{P}(Y_1 = y_m) = p_m, m \in \{1, 2, \cdots, M\}$, and $M_Q(t) = Q(t) - \beta\lambda t, t \geq 0$ where $\beta = \mathbb{E}Y_1$ be a compensated Poisson process.

**(1)** Find the process $[Q, Q](t)_{t\geq 0}$, and compute its expectation. What is $[M_Q, M_Q](t)_{t\geq 0}$?
*Solution*: Because $Q(t)$ is a right-continuous pure jump process,

$$\begin{aligned}
[Q, Q](t) &= \sum_{0<s\leq t} \Delta Q^2(s) \\
&= \sum_{i=1}^{N(t)} Y_i^2.
\end{aligned}$$

To compute its expectation,

$$\begin{aligned}
\mathbb{E}[Q, Q](t) &= \mathbb{E} \sum_{i=1}^{N(t)} Y_i^2 \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left[\sum_{i=1}^{k} Y_i^2 \middle| N(t) = k\right] \mathbb{P}(N(t) = k) \\
&= \mathbb{E}\left[Y_1^2\right] \sum_{k=0}^{\infty} k\mathbb{P}(N(t) = k) \\
&= \mathbb{E}\left[Y_1^2\right] \lambda t \\
&= \lambda t \sum_{m=1}^{M} p_m y_m^2.
\end{aligned}$$

The continuous part of $M_Q(t)$ does not contain Itô integral, thus

$$[M_Q, M_Q](t) = [Q, Q](t), \quad t \geq 0.$$

**(2)** Let $\varphi(t)_{t\geq 0}$, be a left-continuous square-integrable process adapted to the filtration of $M(t)_{t\geq 0}$. Show that

$$\mathbb{E}\left(\int_0^t \varphi(s) dM(s)\right)^2 = \lambda \mathbb{E} \int_0^t \varphi^2(s) ds.$$

Find

$$\mathrm{Var}\left(\int_0^t 2^{M(s-)} dM(s)\right).$$

*Solution*: Let

$$X(t) \triangleq \int_0^t \varphi(s) dM(s) \Rightarrow dX(t) = \varphi(t) dM(t), X(0) = 0.$$

Apply the alternative form of Itô formula introduced in Lecture 12,

$$X^2(t) = \int_0^t 2X(s-) dX(s) + \sum_{0<s\leq t} \left[(X(s-) + \Delta X(s))^2 - X^2(s-) - 2X(s-)\Delta X(s)\right]$$

$$
\begin{aligned}
&= 2\int_0^t X(s-)dX(s) + \sum_{0<s\le t} \underbrace{\Delta X(s)^2}_{\varphi^2(s)\Delta N(s)} \\
&= \underbrace{2\int_0^t X(s-)dX(s)}_{\text{martingale}} + \int_0^t \varphi^2(s)dN(s).
\end{aligned}
$$

Taking the expectation,

$$
\begin{aligned}
\mathbb{E}\left(\int_0^t \varphi(s)dM(s)\right)^2 &= \mathbb{E}X^2(t) \\
&= \cancelto{0}{2\mathbb{E}\int_0^t X(s-)dX(s)} + \mathbb{E}\int_0^t \varphi^2(s)dN(s) \\
\text{(left-continuous)} \quad &= \mathbb{E}\int_0^t \underbrace{\varphi^2(s)}_{=\varphi^2(s-)} dM(s) + \lambda\mathbb{E}\int_0^t \varphi^2(s)ds \\
&= \cancelto{0}{\underbrace{\mathbb{E}\int_0^t \varphi^2(s-)dM(s)}_{\text{martingale}}} + \lambda\mathbb{E}\int_0^t \varphi^2(s)ds \\
&= \lambda\mathbb{E}\int_0^t \varphi^2(s)ds.
\end{aligned}
$$

Because $\int_0^t 2^{M(s-)}dM(s)$ is a martingale,

$$
\begin{aligned}
\operatorname{Var}\int_0^t 2^{M(s-)}dM(s) &= \mathbb{E}\left(\int_0^t 2^{M(s-)}dM(s)\right)^2 \\
&= \lambda\mathbb{E}\int_0^t 2^{2M(s-)}ds \\
&= \lambda\mathbb{E}\int_0^t 2^{2N(s-)-2\lambda s}ds \\
&= \lambda\mathbb{E}\int_0^t 2^{2N(s)-2-2\lambda s}ds \\
&= \lambda\int_0^t 2^{-2-2\lambda s}\mathbb{E}e^{2\log 2\times N(s)}ds \\
\text{(M.G.F. of Poisson)} &= \lambda\int_0^t 2^{-2-2\lambda s}\exp\left\{\lambda s\left(e^{2\log 2}-1\right)\right\}ds \\
&= \lambda\int_0^t 2^{-2-2\lambda s}e^{3\lambda s}ds \\
&= \frac{\lambda}{4}\int_0^t e^{\lambda(3-2\log 2)s}ds \\
&= \frac{e^{\lambda(3-2\log 2)t}-1}{4(3-2\log 2)}.
\end{aligned}
$$

**(3)** Let $\varphi(t)_{t\geq 0}$, be a left-continuous square-integrable process adapted to the filtration of $M_Q(t)_{t\geq 0}$. Show that

$$\mathbb{E}\left(\int_0^t \varphi(s)dM_Q(s)\right)^2 = \lambda\mathbb{E}\left[Y_1^2\right]\mathbb{E}\int_0^t \varphi^2(s)ds.$$

*Solution*: Let

$$X(t) \triangleq \int_0^t \varphi(s)dM_Q(s) \Rightarrow dX(t) = \varphi(t)dM_Q(t), X(0) = 0.$$

Apply the alternative form of Itô formula introduced in Lecture 12,

$$\begin{aligned} X^2(t) &= 2\int_0^t X(s-)dX(s) + \sum_{0<s\leq t} \underbrace{\Delta X(s)^2}_{\varphi^2(s)\Delta Q(s)^2=\varphi^2(s)Y_1^2\Delta N(s)} \\ &= 2\underbrace{\int_0^t X(s-)dX(s)}_{\text{martingale}} + Y_1^2\int_0^t \varphi^2(s)dN(s). \end{aligned}$$

Taking the expectation,

$$\begin{aligned} \mathbb{E}\left(\int_0^t \varphi(s)dM_Q(s)\right)^2 &= \mathbb{E}X^2(t) \\ &= \cancel{2\mathbb{E}\int_0^t X(s-)dX(s)} \to 0 + \mathbb{E}\left[Y_1^2\int_0^t \varphi^2(s)dN(s)\right] \\ \text{(independence)} \quad &= \mathbb{E}\left[Y_1^2\right]\mathbb{E}\left[\int_0^t \varphi^2(s)dN(s)\right] \\ \text{(left-continuous)} \quad &= \mathbb{E}\left[Y_1^2\right]\left[\mathbb{E}\int_0^t \underbrace{\varphi^2(s)}_{=\varphi^2(s-)}\, dM(s) + \lambda\mathbb{E}\int_0^t \varphi^2(s)ds\right] \\ &= \mathbb{E}\left[Y_1^2\right]\left[\cancel{\underbrace{\mathbb{E}\int_0^t \varphi^2(s-)dM(s)}_{\text{martingale}}} \to 0 + \lambda\mathbb{E}\int_0^t \varphi^2(s)ds\right] \\ &= \lambda\mathbb{E}\left[Y_1^2\right]\mathbb{E}\int_0^t \varphi^2(s)ds. \end{aligned}$$

**(4)** Let $W(t)$ be a Brownian motion and let $Q(t)$ be a compound Poisson process, both defined the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and relative to the same filtration $\mathscr{F}(t)_{t\geq 0}$. Show that, for each $t$, the random variables $W(t)$ and $Q(t)$ are independent. (In fact, the whole path of $W$ is independent of the whole path of $Q$, although you are not being asked to prove this stronger statement.)

*Proof*: Let

$$X(t) = u_1 W(t) - \frac{1}{2}u_1^2 t + u_2 Q(t) - \lambda t(\varphi(u_2) - 1),$$

where $\varphi$ is the moment-generating function of the jump size $Y$. Apply Itô's formula,

$$\begin{aligned}
e^{X(t)} - 1 &= \int_0^t e^{X(s)}\left(u_1 dW(s) - \frac{1}{2}u_1^2 ds - \lambda(\varphi(u_2)-1)ds\right) + \frac{1}{2}\int_0^t e^{X(s)}u_1^2 ds + \sum_{0<s\le t}\left(e^{X(s)} - e^{X(s-)}\right) \\
&= \int_0^t e^{X(s)}\left(u_1 dW(s) - \lambda(\varphi(u_2)-1)ds\right) + \sum_{0<s\le t}\left(e^{X(s)} - e^{X(s-)}\right).
\end{aligned}$$

Note $\Delta X(t) = u_2 \Delta Q(t) = u_2 Y \Delta N(t)$, where $N(t)$ is the Poisson process associated with $Q(t)$. When $X$ has a jump at $s$,

$$e^{X(s)} - e^{X(s-)} = e^{X(s-)}\left(e^{\Delta X(s)} - 1\right) = e^{X(s-)}\left(e^{u_2 Y} - 1\right)\Delta N(s).$$

Thus,

$$\begin{aligned}
e^{X(t)} - 1 &= \int_0^t e^{X(s)}\left(u_1 dW(s) - \lambda(\varphi(u_2)-1)ds\right) + \int_0^t e^{X(s-)}\left(e^{u_2 Y} - 1\right) dN(s) \\
&= u_1 \underbrace{\int_0^t e^{X(s)} dW(s)}_{\text{martingale}} + \int_0^t e^{X(s-)} d\left[\left(e^{u_2 Y} - 1\right)N(s) - (\varphi(u_2)-1)\lambda s\right].
\end{aligned}$$

Now we prove that $H(t) \triangleq \left(e^{u_2 Y} - 1\right)N(t) - (\varphi(u_2)-1)\lambda t$ is a martingale. First, we note that $H(t)$ is $\mathscr{F}(t)$-measurable. For $0 \le s < t$,

$$\begin{aligned}
\mathbb{E}[H(t)|\mathscr{F}(s)] &= \mathbb{E}[H(t) - H(s)|\mathscr{F}(s)] + \mathbb{E}[H(s)|\mathscr{F}(s)] \\
&= \mathbb{E}\left[\left(e^{u_2 Y} - 1\right)(N(t) - N(s)) - (\varphi(u_2)-1)(t-s)|\mathscr{F}(s)\right] + H(s) \\
&= \mathbb{E}\left[\left(e^{u_2 Y} - 1\right)(N(t) - N(s))|\mathscr{F}(s)\right] - (\varphi(u_2)-1)(t-s) + H(s) \\
\text{(independence)} \quad &= \mathbb{E}\left[e^{u_2 Y} - 1\right]\mathbb{E}[N(t) - N(s)|\mathscr{F}(s)] - (\varphi(u_2)-1)(t-s) + H(s) \\
&= \mathbb{E}\left[e^{u_2 Y} - 1\right]\mathbb{E}[N(t) - N(s)] - (\varphi(u_2)-1)(t-s) + H(s) \\
&= \underbrace{(\varphi(u_2)-1)(t-s) - (\varphi(u_2)-1)(t-s)}_{=0} + H(s) \\
&= H(s).
\end{aligned}$$

Therefore,

$$e^{X(t)} - 1 = u_1 \underbrace{\int_0^t e^{X(s)} dW(s)}_{\text{martingale}} + \underbrace{\int_0^t e^{X(s-)} dH(s)}_{\text{martingale}}$$

is a martingale. Taking expectations, we get

$$\mathbb{E}\left[e^{X(t)}\right] \equiv 1 \Rightarrow \mathbb{E}\left[e^{u_1 W(t) + u_2 Q(t)}\right] = e^{\frac{1}{2}u_1^2 t} e^{\lambda t(\varphi(u_2)-1)} = \mathbb{E}\left[e^{u_1 W(t)}\right]\mathbb{E}\left[e^{u_2 Q(t)}\right].$$

The moment-generating function factorizes. In other words, $W(t)$ and $Q(t)$ are independent.

**(5)** Let under $\mathbb{P}$ the process $S(t)_{t\geq 0}$ be a solution to

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) + S(t-)dQ(t).$$

Find a change of measure, such that under the new measure the process $e^{-rt}S(t)_{t\geq 0}$ is a martingale.

*Solution*: First, we apply Itô's formula to $e^{-rt}S(t)$:

$$d\left[e^{-rt}S(t)\right] = -re^{-rt}S(t)dt + e^{-rt}dS(t) = e^{-rt}\left[(\mu - r)S(t)dt + \sigma S(t)dB(t) + S(t-)dQ(t)\right].$$

We want to look for constants $a$ and $b$: $a + b = \mu - r$,

$$d\left[e^{-rt}S(t)\right] = e^{-rt}\left[\underbrace{\sigma S(t)\left(dB(t) + \frac{a}{\sigma}dt\right)}_{\text{term I}} + \underbrace{S(t-)(dQ(t) + bdt)}_{\text{term II}}\right].$$

such that under the new measure, both term I and II form martingales. For term II to be a martingale, we let $b = -\beta\lambda$, such that

$$e^{-rt}S(t-)(dQ(t) + bdt) = e^{-rt}S(t-)dM_Q(t),$$

where $M_Q(t)$ is the compensated Poisson process. Consequently, $a = \mu - r + \beta\lambda$. We define

$$\theta \triangleq \frac{a}{\sigma} = \frac{\mu - r + \beta\lambda}{\sigma}.$$

Define the change of measure,

$$Z(t) = \exp\left\{-\theta B(t) - \frac{1}{2}\theta^2 t\right\}.$$

Under the probability measure

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)d\mathbb{P}, \quad \forall A \in \mathscr{F},$$

the process $\tilde{B}(t) = B(t) + \theta t$ is a Brownian motion, and

$$d\left[e^{-rt}S(t)\right] = e^{-rt}\left[\sigma S(t)d\tilde{B}(t) + S(t-)dM_Q(t)\right],$$

where both terms on the right hand side form a martingale under $\tilde{\mathbb{P}}$, so does $e^{-rt}S(t)$.

**(6)** Suppose that under a risk-neutral measure the stock price can be represented as follows:

$$S(t) = S^*(t)e^{Q(t)},$$

where $S^*(t) = e^{\sigma B(t)+\mu t}$, $B(t)$ is a standard Brownian motion, $Q(t) = \sum_{i=1}^{N(t)} Y_i$ is a compound Poisson process with intensity $\lambda$, random variables $Y_i, i \geq 1$, are normal with mean $\mu_0$ and variance $\sigma_0^2$, and $\mu = r - \sigma^2/2 - \lambda(\mathbb{E}e^{Y_1} - 1)$, where $r$ is the annual nominal interest rate. Find the time zero cost of a European call option with strike price $K$ and expiration $t$.
*Solution*: Compare with **Theorem 11.7.3** in textbook (or **Theorem 6** in lecture notes 12), the process

$$S(t) = S(0)\exp\left\{\sigma B(t) + \left(\alpha - \beta\lambda - \frac{1}{2}\sigma^2\right)t\right\}\prod_{i=1}^{N(t)}(\bar{Y}_i + 1)$$

solves the SDE

$$dS(t) = \alpha S(t)dt + \sigma S(t)dB(t) + S(t-)d(Q(t) - \beta\lambda t),$$

where $\beta = \mathbb{E}\bar{Y}_i, \forall i$. Translating the symbols to the ones given in this problem,

$$S(t) = \exp\{\sigma B(t) + \mu t\}\prod_{i=1}^{N(t)} e^{Y_i} = \exp\left\{\sigma B(t) + \left(r - \frac{1}{2}\sigma^2 - \lambda(\mathbb{E}e^{Y_1} - 1)\right)t\right\}\prod_{i=1}^{N(t)} e^{Y_i},$$

we recognize the following identifications:

$$\begin{aligned} S(0) &= 1, \\ \alpha &= r, \\ \bar{Y}_i &\triangleq e^{Y_i} - 1 \Rightarrow \beta = \mathbb{E}\bar{Y}_i = \mathbb{E}e^{Y_i} - 1. \end{aligned}$$

Therefore, $\mu = r - \sigma^2/2 - \lambda(\mathbb{E}e^{Y_1} - 1) = r - \sigma^2/2 - \lambda\beta$. The asset price process given in this question

$$S(t) = \exp\{\sigma B(t) + \mu t\}\prod_{i=1}^{N(t)} e^{Y_i} = \exp\left\{\sigma B(t) + \left(r - \frac{1}{2}\sigma^2 - \lambda\beta\right)t\right\}\prod_{i=1}^{N(t)}(\bar{Y}_i + 1)$$

satisfies the expressions given in **Theorem 11.7.3** in textbook (or **Theorem 6** in lecture notes 12). Furthermore, because $r$ is the annual nominal interest rate, $S(t)$ evolves in risk-neutral measure. To compute the risk-neutral time zero cost of a European call option with strike price $K$ and expiration $t$, we can perform the calculation directly in the original probability measure, which is already the risk-neutral measure.

According to **Theorem 11.7.5** in textbook,

$$V(0) = \mathbb{E}\left[e^{-rt}(S(t) - K)^+ | \mathscr{F}(0)\right] = c(0, S(0) = 1),$$

where

$$c(0, x) = e^{-\lambda t}\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}\mathbb{E}\kappa\left(t, xe^{\beta\lambda t + \sum_{i=1}^{j} Y_i}\right), \tag{0.1}$$

and $\kappa(\tau, x) = xN(d_+(\tau, x)) - Ke^{-r\tau}N(d_-(\tau, x))$, where

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right]$$

and

$$N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}z^2}\,dz.$$

Because $Y_i, i\geq 1$ are independent normal random variables with mean $\mu_0$ and variance $\sigma_0^2$, the partial sum $\sum_{i=1}^{j} Y_i$ is a normal random variable with mean $j\mu_0$ and variance $j\sigma_0^2$. We can write the expression Equation (0.1) for the time zero cost of the option more explicitly,

$$\begin{aligned}
V(0) &= e^{-\lambda t}\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}\mathbb{E}\kappa\left(t, e^{\beta\lambda t+\sum_{i=1}^{j}Y_i}\right)\\
&= e^{-\lambda t}\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}\int_{-\infty}^{+\infty}\kappa\left(t, e^{\beta\lambda t+z}\right)\frac{1}{\sqrt{2\pi j\sigma_0^2}}e^{-\frac{(z-j\mu_0)^2}{2j\sigma_0^2}}\,dz.
\end{aligned}$$