MTH 9821 Numerical Methods for Finance

Fall 2015

Homework 4

Assigned: September 15; Due: September 29

This homework is to be done as a group. Each team will hand in one homework solution, and each member of the team should write at least one problem. On the cover page of the homework, please indicate the members of the team and who wrote each problem.

General Questions

(1) Show that for a one period binomial model for an asset paying dividends continuously at rate q to be arbitrage-free, the following condition must be satisfied:

$$d \;<\; e^{(r-q)\delta t} \;<\; u.$$

Hint: Recall that having a long position between time 0 and time δt in one unit of an asset paying dividends continuously at rate q is equivalent to having a long position between time 0 and time δt in $e^{-q\,\delta t}$ units of a non–dividend–paying asset with the same drift and volatility.

(2) The classical way of drawing a tree with equal jumps up and down is, in fact, not correct. On a two dimensional plot having the time t on the horizontal axis and the price of the stock $S(t)$ on the vertical axis, draw a binomial tree with $N = 24$ time periods for $T = 6$ months, for a stock with spot price $S(0) = 20$, assuming that in one time interval $\delta t = 1/48$ the price of the stock can go either up by a factor of 1.04 or down by a factor of 1/1.04.

(3) Recall that the risk–neutral probability of an up move in a binomial tree with time step δt is

$$p \;=\; \frac{e^{(r-q)\delta t} - d}{u - d},$$

where $u = e^{\sigma\sqrt{\delta t}}$ and $d = e^{-\sigma\sqrt{\delta t}}$. Use Taylor expansions to show that

$$\frac{e^{(r-q)\delta t} - d}{u - d} \;=\; \frac{1}{2} \;+\; \frac{1}{2}\left(\frac{r-q}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\delta t} \;+\; O\left(\delta t\right),$$

as $\delta t \to 0$.

(4) If a stock follows a binomial tree with parameters

$$u \;=\; e^{\sigma\sqrt{\delta t}}; \quad d \;=\; e^{-\sigma\sqrt{\delta t}}; \quad p \;=\; \frac{1}{2} \;+\; \frac{1}{2}\left(\frac{r-q}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\delta t},$$

where p is the probability of an up move over the time δt, then $\ln S_N(T)$ has a binomial distribution, where $N = \frac{T}{\delta t}$. Show that, as $\delta t \to 0$, S_N converges to a lognormal variable, i.e.,

$$\begin{aligned}\lim_{N\to\infty} \ln\left(\frac{S_N}{S(0)}\right) \;&\sim\; \mathcal{N}\left(\left(r - q - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)\\ &=\; \left(r - q - \frac{\sigma^2}{2}\right)T \;+\; \sigma\sqrt{T}Z,\end{aligned}$$

where Z is the standard normal distribution.

Binomial Tree Methods for Derivatives Valuation and Hedging Parameters Computation

Throughout this homework, the following parameterizations will be used for binomial tree methods:

Binomial Trees:

$$u = e^{\sigma\sqrt{\delta t}}; \quad d = e^{-\sigma\sqrt{\delta t}}; \quad p_{RN} = \frac{e^{(r-q)\delta t} - d}{u - d}. \tag{1}$$

Consider a one year European put and a one year American put, both with strike \$40, on an asset with spot price \$41 paying dividends continuously at rate 1%, and following a lognomal process with volatility 30%. Assume the risk free interest rates are constant at 3%.

Price both the European and the American put option using trees with $N = 10 : 100$ time intervals, and using the following tree methods: binomial trees, average binomial trees, BBS, BBSR. Do not report the numbers; plot the approximate values of all methods on the same graph for the European put, and plot the approximate values of all methods on another graph for the American put. Comment on the results.

Binomial Tree Methods for European Options

Compute the Black–Scholes option value V_{BS}, and the following Greeks: Δ_{BS}, Γ_{BS}, and Θ_{BS}.

Price the European put option using the following tree methods:

- Binomial Tree with $N \in \{10, 20, 40, \ldots, 1280\}$ time steps;
- Average binomial Tree with N=10,11; with N=20,21; with N=40,41; ... with N=1280,1281;
- Binomial Black–Scholes with $N \in \{10, 20, 40, \ldots, 1280\}$ time steps;
- Binomial Black–Scholes with Richardson Extrapolation, with $N \in \{20, 40, \ldots, 1280\}$ time steps. (Recall that $V_{BBSR}(N) = 2V_{BBS}(N) - V_{BBS}(N/2)$.)

For each method, record the first six decimals of the following values in the solution template file hw_sol_template-BINOMIAL-European.xls:

- $V(N)$, the value given by the tree method with N time steps;
- $|V(N) - V_{BS}|$, the approximation error of the tree method;
- $N\ |V(N) - V_{BS}|$ and $N^2\ |V(N) - V_{BS}|$, terms that indicate whether the convergence of the tree method is linear or quadratic;
- the following approximations for the Delta, Gamma, and Theta of the option:

$$\Delta_1 = \frac{V_{1,0} - V_{1,1}}{S_{1,0} - S_{1,1}};$$

$$\Gamma_1 = \frac{\frac{V_{2,0}-V_{2,1}}{S_{2,0}-S_{2,1}} - \frac{V_{2,1}-V_{2,2}}{S_{2,1}-S_{2,2}}}{(S_{2,0} - S_{2,2})/2};$$

$$\Theta_1 = \frac{V_{2,1} - V_{0,0}}{2\delta t},$$

  and the approximation errors $|\Delta_1 - \Delta_{BS}|$, $|\Gamma_1 - \Gamma_{BS}|$, and $|\Theta_1 - \Theta_{BS}|$.

Rank the methods in terms of convergence speed and comment on the order of the convergence.

Binomial Tree Methods for American Options

Compute the value of an American Put with the same parameters by using an average binomial tree with $10,000$ and $10,001$ time steps and denote it by V_{exact}.

Price the American put option using the following tree methods:

- Binomial Tree with $N \in \{10, 20, 40, \ldots, 1280\}$ time steps;
- Average binomial Tree with N=10,11; with N=20,21; with N=40,41; ... with N=1280,1281;
- Binomial Black–Scholes with $N \in \{10, 20, 40, \ldots, 1280\}$ time steps;
- Binomial Black–Scholes with Richardson Extrapolation, with $N \in \{20, 40, \ldots, 1280\}$ time steps.

For each method, record the first six decimals of the following values in the solution template file hw_sol_template-BINOMIAL-American.xls:

- $V(N)$, the value given by the tree method with N time steps;
- $|V(N) - V_{exact}|$, the approximation error of the tree method;
- $N\,|V(N) - V_{exact}|$ and $N^2\,|V(N) - V_{exact}|$, terms that indicate whether the convergence of the tree method is linear or quadratic;
- the following approximations for the Delta, Gamma, and Theta of the option:

$$\begin{aligned} \Delta_1 &= \frac{V_{1,0} - V_{1,1}}{S_{1,0} - S_{1,1}}; \\ \Gamma_1 &= \frac{\frac{V_{2,0}-V_{2,1}}{S_{2,0}-S_{2,1}} - \frac{V_{2,1}-V_{2,2}}{S_{2,1}-S_{2,2}}}{(S_{2,0} - S_{2,2})/2}; \\ \Theta_1 &= \frac{V_{2,1} - V_{0,0}}{2\delta t}, \end{aligned}$$

  and the approximation errors $|\Delta_1 - \Delta_{exact}|$, $|\Gamma_1 - \Gamma_{exact}|$, and $|\Theta_1 - \Theta_{exact}|$.

Repeat the process by using Variance Reduction for each method.

Rank the methods in terms of convergence speed and comment on the order of the convergence.