Assignment 8 – Solutions

Problem (1): (Shreve # 6.4)

(i) Define $\varphi(t) = \exp\left(\frac{1}{2}\sigma^2 \int_t^T C(u,T)\,du\right)$.

Then $\varphi'(t) = -\frac{1}{2}\sigma^2 C(t,T)\varphi(t)$ and

$$\varphi''(t) = -\frac{1}{2}\sigma^2 C'(t,T)\varphi(t) - \frac{1}{2}\sigma^2 C(t,T)\varphi'(t)$$
$$= -\frac{1}{2}\sigma^2 C'(t,T)\varphi(t) + \frac{1}{4}\sigma^4 C^2(t,T)\varphi(t)$$

and hence we get

$$C(t,T) = -\frac{2\varphi'(t)}{\sigma^2\varphi(t)} \qquad (1)$$

$$C'(t,T) = -\frac{2\varphi''(t)}{\sigma^2\varphi(t)} + \frac{1}{2}\sigma^2 C^2(t,T).$$

(ii) Substituting these last two equations into

$$C'(t,T) = b\,C(t,T) + \frac{1}{2}\sigma^2 C^2(t,T) - 1$$

we obtain

$$-\frac{2\varphi''(t)}{\sigma^2\varphi(t)} = -\frac{2b\,\varphi'(t)}{\sigma^2\varphi(t)} - 1$$

which can be rewritten as

$$\varphi''(t) - b\varphi'(t) - \frac{1}{2}\sigma^2\varphi(t) = 0.$$

(iii) From the characteristic equation $\lambda^2 - b\lambda - \frac{1}{2}\sigma^2 = 0$

we have $\lambda_{1/2} = \frac{1}{2}\left(b \pm \sqrt{b^2+2\sigma^2}\right)$.

Using $\gamma = \frac{1}{2}\sqrt{b^2+2\sigma^2}$ we get the general solution

$$\varphi(t) = a_1 \exp\left(\left(\tfrac{1}{2}b+\gamma'\right)t\right) + a_2 \exp\left(\left(\tfrac{1}{2}b-\gamma'\right)t\right).$$

Since a_1, a_2 are arbitrary constants we can rewrite the above equation in a more convenient form:

$$\varphi(t) = \frac{c_1}{\frac{1}{2}b+\gamma'} e^{-\left(\frac{1}{2}b+\gamma'\right)(T-t)} - \frac{c_2}{\frac{1}{2}b-\gamma'} e^{-\left(\frac{1}{2}b-\gamma'\right)(T-t)}, \tag{2}$$

where c_1, c_2 are arbitrary constants. The connection between $a_{1/2}$ and $c_{1/2}$ is as follows

$$a_{1/2} = \frac{c_{1/2}}{\frac{1}{2}b \pm \gamma'} e^{-\left(\frac{1}{2}b\pm\gamma'\right)T}.$$

(iv) Taking the derivative of φ in (2) gives

$$\varphi'(t) = c_1 e^{-\left(\frac{1}{2}b+\gamma'\right)(T-t)} - c_2 e^{-\left(\frac{1}{2}b-\gamma'\right)(T-t)}. \tag{3}$$

The terminal condition $C(T,T)=0$ and (1) gives $\varphi(T) \neq 0$ and $\varphi'(T) = 0$ such that with (3) $\varphi'(T) = c_1 - c_2 = 0$ it follows $c_1 = c_2$.

(v) Now that we know $c_1 = c_2$, we obtain

$$\varphi(t) = c_1 \left(\frac{1}{\frac{1}{2}b+\gamma'} e^{-\left(\frac{1}{2}b+\gamma'\right)(T-t)} - \frac{1}{\left(\frac{1}{2}b-\gamma'\right)} e^{-\left(\frac{1}{2}b-\gamma'\right)(T-t)} \right)$$

$$= c_1 e^{-\frac{1}{2}b(T-t)} \left(\frac{\frac{1}{2}b-\gamma'}{\frac{1}{4}b^2-\gamma^2} e^{-\gamma'(T-t)} - \frac{\frac{1}{2}b+\gamma'}{\frac{1}{4}b^2-\gamma^2} e^{\gamma'(T-t)} \right).$$

Note that $\frac{1}{2}b^2 = \frac{1}{4}b^2 - \gamma^2$ and hence

$$\varphi(t) = -\frac{2c_1}{b^2} e^{-\frac{1}{2}b(T-t)} \left(\tfrac{1}{2}b\left(e^{-\gamma'(T-t)} - e^{\gamma'(T-t)}\right) - \gamma'\left(e^{-\gamma'(T-t)} + e^{\gamma'(T-t)}\right)\right),$$

$$= \frac{2c_1}{b^2} e^{-\frac{1}{2}b(T-t)} \left(b \sinh\left(\gamma'(T-t)\right) + 2\gamma' \cosh\left(\gamma'(T-t)\right)\right)$$

Rewriting (3), we see that

$$\varphi'(t) = -2c_1 e^{-\frac{1}{2}b(T-t)} \sinh(\gamma'(T-t)).$$

Substituting the formulas for φ and φ' in (1) we see that c_1 cancels out and

$$C(t,T) = \frac{\sinh(\gamma'(T-t))}{\gamma' \cosh(\gamma'(T-t)) + \frac{1}{2}b \sinh(\gamma'(T-t))}.$$

(vi) From the ODE for A and (1) we have

$$A'(s,T) = \frac{2a\,\varphi'(s)}{\sigma^2 \varphi(s)}.$$

Integrating from t to T using the fact that

$$\frac{\varphi'(s)}{\varphi(s)} = (\log \varphi(s))'$$

we obtain

$$A(T,T) - A(t,T) = \frac{2a}{\sigma^2} \log(\varphi(T) - \log \varphi(t))$$

$$= \frac{2a}{\sigma^2} \log\left(\frac{\varphi(T)}{\varphi(t)}\right).$$

Since $A(T,T)=0$, $\sinh 0 = 0$, $\cosh = 0$, we get

$$\frac{\varphi(T)}{\varphi(t)} = \frac{\gamma' e^{\frac{1}{2}b(T-t)}}{\frac{1}{2}b \sinh(\gamma'(T-t)) + \gamma' \cosh(\gamma'(T-t))},$$

and conclude that

$$A(t,T) = \frac{2a}{\sigma^2} \log\left(\frac{\gamma' e^{\frac{1}{2}b(T-t)}}{\frac{1}{2}b \sinh(\gamma'(T-t)) + \gamma' \cosh(\gamma'(T-t))}\right).$$

Problem 2 (Shreve #6.8)

This is only a sketch. A complete solution would require to specify conditions on the drift and volatility as well as some a priori estimates of the transition density fct.
The Feynman-Kac Thm. states that for any fct. $h(y)$ the fct. $g(t,x) = E_{t,x}\, h(X(T)) = \int_0^\infty h(y)\,p(t,T,x,y)\,dy$

(note that we ass. $p(t,T,x,y) = 0$ for $y<0$) satisfies the PDE

$$-p_t(t,T,x,y) = \beta(t,x)\,p_x(t,T,x,y) + \frac{1}{2}\gamma^2(t,x)\,p_{xx}(t,T,x,y).$$

Differentiating under the integral sign we get that for all $h(y)$ and all fixed $x>0$, $t \in (0,T)$

$$\int_0^\infty h(y)\,f(y)\,dy = 0$$

where

$$f(y) = p_t(t,T,x,y) + \beta(t,x)p_x(t,T,x,y) + \frac{1}{2}\gamma^2(t,x)p_{xx}(t,T,x,y).$$

Choosing $h(y) = \mathbb{1}_B(y)$ for an arbitrary Borel set $B \subset (0,\infty)$ we see that

$$\int_B f(y)\,dy = 0,$$

which implies $f(y) = 0$ a.e. on $y>0$. Therefore, for a.e. fixed $y>0$ and all fixed $T>0$, the fct. p satisfies the PDE for all $x>0$, $t\in(0,T)$. If we know (which is true under broad cond.) that p together with its appropriate t and x derivatives were continuous in y we would conclude that for every fixed $y, T>0$, the function p satisfies the PDE for all $x>0$, $t \in (0,T)$.

Problem (3) (Pricing down-and-out call options)

(a) In the BS framework we have

$$S(t) = S(0)e^{\sigma \widetilde{W}(t) + (r - \frac{\sigma^2}{2})t} = S(0)e^{\sigma \hat{W}(t)}.$$

Here $\widetilde{W}$ (and hence $-\widetilde{W}$) is a BM under the risk-neutral measure $\widetilde{P}$ and $\hat{W}(t) = \alpha t + \widetilde{W}(t)$ with $\alpha = \frac{r - \sigma^2/2}{\sigma}$.

Define $\hat{m}(t) = \min_{s \in [0,t]} \hat{W}(s)$, then $\min_{t\in[0,T]} S(T) = S(0)e^{\sigma \hat{m}(T)}$.

The payoff of the down-and-out call option is thus given by

$$C(T) = (S(T) - K)^+ \mathbb{1}_{\{S(0)e^{\sigma \hat{m}(T)} > B\}}.$$

Since $\hat{m}(T) = -\max_{t\in[0,T]} (-\hat{W}(t))$, we have

$$S(0)e^{\sigma \hat{m}(T)} > B \iff \hat{m}(T) > \frac{\ln(B/S(0))}{\sigma} =: b$$

$$\iff -\hat{m}(T) < -b \iff \max_{t\in[0,T]} (-\hat{W}(t)) < -b$$

and

$$S(0)e^{\sigma \hat{W}(T)} > K \iff \hat{W}(T) > \frac{\log(K/S(0))}{\sigma} =: k$$

$$\iff -\hat{W}(T) < -k.$$

Note that we have $k > 0$ and $b < k$.
Hence the payoff can be written as

$$C(T) = \left(S(0)e^{-\sigma(-\hat{W}(T))} - K\right) \mathbb{1}_{\{-\hat{W}(T) < -k,\ \max_{t\in[0,T]} -W(t) < -b\}}.$$

The risk-neutral pricing formula tells us that

$$C(0) = \widetilde{E}[e^{-rT} \quad C(T)]$$

and therefore:

$$C(0) = e^{-rT} \int_{-\infty}^{-k} \int_{x^+}^{-b} (S(0)e^{-\sigma x} - K)\, \hat{f}(x,a)\, da\, dx \, .$$

Here $\hat{f}(x,a) = \begin{cases} e^{\alpha x - \frac{1}{2}\alpha^2 T} \cdot \frac{2(2a-x)}{T\sqrt{2\pi T}}\, e^{-(2a-x)^2/(2T)} & (x \le a, a \ge 0) \\ 0 & (\text{otherwise}) \end{cases}$

is the joint density of a BM with drift and its maximum as calculated in Lecture 6.

Note that (x,a) corresponds to $(-\hat{W}(T), \max_{t\in[0,T]} \hat{W}(t))$.

(b) Define $\rho = (\inf_{t\in[0,T]} S(t) = B) \wedge T$, then

$$\left(e^{-rt} v(t,S(t))\right)_{t\in[0,\rho]}$$

is a $\tilde{P}$-martingale, so

$$\begin{aligned} d(e^{-rt} v(t,S(t))) &= -r e^{-rt} v(t,S(t))\, dt + e^{-rt} v_t(t,S(t))\, dt \\ &\quad + e^{-rt} v_x(t,S(t))\, dS(t) + \tfrac{1}{2} e^{-rt} v_{xx}(t,S(t))\, d\langle S,S\rangle_t \\ &= e^{-rt}\left[-rv + v_t + rS(t) v_x + \tfrac{1}{2}\sigma^2 S(t) v_{xx}\right] dt \\ &\quad + e^{-rt} \sigma S(t) v_x\, d\tilde{W}(t). \end{aligned}$$

Obviously, the dt-term is 0 and hence v solves the BSM PDE on $[0,T) \times [B,\infty)$.

The boundary conditions:

(1) If the option is not knocked out, it has the usual call payoff: $v(T,x) = (x-K)^+$ for $x \ge B$.

(2) Whenever $S(t) = B$, the option is knocked out and expires worthless: $v(t,B) = 0$ for $t \in [0,T]$

(3) The call price increases with the stock price at the same rate: $\frac{v(t,x)}{x} \longrightarrow 1$ as $x \to \infty$.

More prec. $\lim_{x\to\infty} \frac{v(t,x)}{x} = \lim_{x\to\infty} \frac{x - Ke^{-r(T-t)}}{x} = 1.$

(c) With the same parameters we have:

$$C_{\text{down-and-in}} = C_{\text{vanilla call}} - C_{\text{down-and-out}}$$

Problem 4: (Shreve 7.2)

(i) We split the interval of integration into two intervals, use a stoch. version of Fubini and the fact that $e^{-ru} S(u)$ is a martingale

$$e^{-r(T-t)} \tilde{E}\left(\frac{1}{T}\int_0^T S(u)\,du \mid \mathcal{F}(t)\right)$$

$$= \frac{1}{T} e^{-r(T-t)} \left(\int_0^t S(u)\,du + \tilde{E}\left(\int_t^T S(u)\,du \mid \mathcal{F}(t)\right)\right)$$

$$= \frac{e^{-r(T-t)}}{T} \left(y + \int_t^T e^{ru}\, \tilde{E}\left[e^{-ru} S(u) \mid \mathcal{F}_t\right] du \right)$$

$$= \frac{e^{-r(T-t)}}{T} \left(y + x \int_t^T e^{r(u-t)}\, du \right)$$

$$= \frac{e^{-r(T-t)}}{T} \left(y + \frac{x}{r}\left(e^{r(T-t)} - 1\right)\right)$$

$$= \frac{1}{T} \left(y e^{-r(T-t)} + \frac{x}{r}\left(1 - e^{-r(T-t)}\right)\right) = v(t,x,y).$$

(ii) I shall skip this part as we have done this multiple times.

(iii) $\Delta(t) = v_x(t,x,y) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right)$

(iv) Let $X(t)$ be our portfolio value at time t. From Itô's formula and (iii) we have

$$d(e^{-rt}X(t)) = \Delta(t)\, d(e^{-rt}S(t))$$
$$= \frac{1}{rT}\left(1-e^{-r(T-t)}\right) d(e^{-rt}S(t))$$
$$= \frac{1}{rT} d(e^{-rt}S(t)) - \frac{1}{rT} e^{-r(T-t)}\left(-re^{-rt}S(t)\,dt + e^{-rt}dS(t)\right)$$
$$= \frac{1}{rT} d(e^{-rt}S(t)) + \frac{1}{T}e^{-rT}S(t)\,dt - \frac{1}{rT}e^{-rT}\,dS(t).$$

Integrating from 0 to T we get:

$$e^{-rT}X(T) - X(0)$$
$$= \frac{1}{rT}\left(e^{-rT}S(T) - S(0)\right) + \frac{1}{T}e^{-rT}\int_0^T S(t)\,dt - \frac{1}{rT}e^{-rT}(S(T)-S(0))$$

Recalling that $X(0) = v(0, S(0), 0) = \frac{1}{rT}S(0)(1-e^{-rT})$, we conclude that

$$X(T) = \frac{1}{T}\int_0^T S(t)\,dt.$$