Stochastic calculus for jump processes
Stochastic intensity based models
Stochastic recovery models

Credit Risk Models

5. Advanced credit models

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Stochastic calculus for jump processes
Stochastic intensity based models
Stochastic recovery models

Outline

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3 Stochastic recovery models

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Jump processes

- Stochastic calculus, used in the study of diffusion processes driven by Brownian motion, can be extended to stochastic processes driven by jump processes such as counting processes (Poisson, compound Poisson, Cox).

- We shall take a few slides to review basic facts about jump processes and their stochastic calculus.

- The processes we are considering in this course are known in mathematics as CADLAG (which is the acronym of French for "right-continuous with left limits").

- We will be assuming that a process X can jump only at finitely many times. All the jumps are assumed to be of finite magnitude.

- The jump amount of X at time t is denoted by $\Delta X(t)$.

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Jump processes

● For a jump process X, we denote:

$$X(t_-) = \lim_{s \nearrow t} X(s).$$

● We also introduce the following notation:

(i) The jump size at time t is thus:

$$\Delta X(t) = X(t) - X(t_-).$$

(ii) The discontinuous part of the process:

$$X^{\mathrm{d}}(t) = \sum_{s \leq t} \Delta X(s).$$

(iii) The continuous part of the process:

$$X^{\mathrm{c}}(t) = X(t) - X^{\mathrm{d}}(t).$$

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Ito's lemma for jump processes

- Ito's lemma is a powerful tool allowing us to easily manipulate complex stochastic processes.
- Let $X = (X_1, \ldots, X_n)$ be an n-dimensional process with a finite number of jumps. For a smooth function $f(t, X)$ we have

$$
f(t, X(t)) = f(0, X_0) + \int_0^t \frac{\partial f(s, X(s_-))}{\partial s} \, ds + \sum_{1 \leq i \leq n} \int_0^t \frac{\partial f(s, X(s_-))}{\partial X_i} \, dX_i^c(s)
$$
$$
+ \frac{1}{2} \sum_{1 \leq i,j \leq n} \int_0^t \frac{\partial^2 f(s, X(s_-))}{\partial X_i \partial X_j} \, dX_i^c(s) \, dX_j^c(s) + \sum_{s_k \leq t} \Delta f(s_k, X(s_k)),
$$

where the last (finite) sum extends over all jump times s_k of the process.

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Ito's lemma for jump processes

- Alternatively, we can formulate Ito's lemma in the differential form:

$$df(t, X(t)) = \frac{\partial f(t, X(t_-))}{\partial t} dt + \sum_{1 \leq i \leq n} \frac{\partial f(t, X(t_-))}{\partial X_i} dX_i^c(t)$$
$$+ \frac{1}{2} \sum_{1 \leq i,j \leq n} \frac{\partial^2 f(t, X(t_-))}{\partial X_i \partial X_j} dX_i^c(t) dX_j^c(t) + \Delta f(t, X(t)) dN(t),$$

where $N(t)$ is the process counting the number of jumps up to time t:

$$N(t) = \sum_{s_k \leq t} 1_{s_k \leq t}.$$

- The last term in the formula above can explicitly be written as

$$\sum_{s_k \leq t} \Delta f(s_k, X(s_k)) \delta(t - s_k) dt,$$

where $\delta(t)$ denotes Dirac's delta.

- We will see applications of this formula later in this lecture and in the following lectures.

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Stochastic intensity based models

- The methodology used to model CMDS contracts can easily be extended to any European style payoff function $u(x)$ of the par spread at expiration: we just set $g(x) = f(x) u(x)$ in the formulas derived in Lecture Notes #4.
- Clearly, the swaption replication method continues to hold, and we can price an option whose payoff is any function of the par CDS spread.
- This does not include exotic options whose payoffs depend on the term structure of spreads. Such options include barrier options and Bermudan options.
- Modeling such options requires dynamic intensity models, very much like exotic interest rate options require term structure models.
- Let us consider a Cox process intensity with a single factor risk neutral diffusion dynamics given by:

$$d\lambda(t) = \mu(t, \lambda(t))dt + \sigma(t, \lambda(t))dW(t).$$

This is a generalization of the Gaussian models considered in Lecture Notes #2.

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Stochastic intensity based models

- For simplicity, we assume that interest rates are deterministic, and consider a credit risky derivative $V(t, \lambda(t))$.
- By $R(t) = R(t, \lambda(t))$ we denote the recovery amount associated with the claim $V(t)$.
- Applying Ito's lemma with jumps yields

$$dV = \Big(\frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial \lambda} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial \lambda^2}\Big)dt + \sigma \frac{\partial V}{\partial \lambda}\,dW + (R - V)dN(t)\,,$$

  where $N(t)$ is the underlying counting process.
- Notice the presence of the jump term $(R - V)dN(t)$ in the equation above: when N jumps, V becomes the recovery value R.

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Stochastic intensity based models

- Now, we take an expectation of this equation:

$$\mathsf{E}[dV(t)] = \Big(\frac{\partial V}{\partial t} + \mu\,\frac{\partial V}{\partial \lambda} + \frac{1}{2}\,\sigma^2\,\frac{\partial^2 V}{\partial \lambda^2}\Big)dt + \lambda(R - V)dt,$$

  where we have used the fact that, for a counting process, $\mathsf{E}[dN(t)] = \lambda(t)\,dt$.

- In the risk neutral measure, this expectation is equal to $r(t)\,V(t)\,dt$.

- As a consequence, we have obtain the following PDE for $V = V(t, \lambda(t))$:

$$\frac{\partial V}{\partial t} + \mu\,\frac{\partial V}{\partial \lambda} + \frac{1}{2}\,\sigma^2\,\frac{\partial^2 V}{\partial \lambda^2} + \lambda R = (r + \lambda)V.$$

- Note the following differences with the standard backward Kolmogorov equations of option pricing:

    (i) The right hand side contains the factor $r + \lambda$ rather than r.
    (ii) The left hand side contains the term $\lambda R(t, \lambda)$ without partial derivatives.

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Stochastic intensity based models

- This equation is tractable by means of the usual numerical methods such as Crank-Nicolson, fully implicit, etc.
- It is subject to a terminal boundary condition at option maturity, and, possibly, to intermediate conditions if the option has barriers, early exercise, or other exotic features.
- For example:
    - (i) For a zero-recovery, risky zero coupon bond maturing at time T, $R = 0$, and the terminal condition is $V(T, \lambda) = 1$.
    - (ii) For a risk-free zero-coupon bond, $R(t, \lambda(t)) = P(t, T)$, and $V(T, \lambda(T)) = 1$.
    - (iii) For a contract paying \$1 at the time of default, provided that the default occurs before time T, $R(t, \lambda(t)) = 1$ and $V(T, \lambda(T)) = 0$.

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Affine intensity models

- Terminal boundary conditions in the PDE can be specified as function $q(\lambda)$, i.e. $V(T, \lambda) = q(\lambda)$.

- In practice, the boundary conditions arising in pricing applications are written in terms of the survival probabilities $S(t, s)$ depending on a range of s.

- This raises the question, under what conditions can $S(t, s)$ (where $t \leq s$) be expressed as a function of $\lambda(t)$.

- We say that the model admits a *reconstitution formula*, if the entire survival curve $S(t, s)$ depends on the intensity process only through $\lambda(t)$.

- An example of such a model is the Gaussian Hull-White model for $\lambda(t)$ which we discussed in Lecture Notes #2, namely

$$S(t, s) = e^{A(s, t) - h_\lambda(s - t)\lambda(t)},$$

  with deterministic $A(s, t)$.

- The Hull-White model has some limitations. This is especially true for low credit spreads, for which the probabilities of generating negative hazard rates may be significant.

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Affine intensity models

- An alternative choice is a CIR process:

$$d\lambda(t) = \kappa(t)(\theta(t) - \lambda(t))dt + \sigma(t)\sqrt{\lambda(t)}dW(t).$$

- This is an *affine model* (as is the Hull-White model) in the sense that the survival probability is of the form

$$S(t, T) = e^{A(t,T) - C(t,T)\lambda(t)}.$$

- The coefficients $A(t, T)$ and $C(t, T)$ satisfy Riccati's differential equations:

$$\frac{dA(t)}{dt} - \kappa(t)\theta(t)C(t) = 0,$$
$$\frac{dC(t)}{dt} - \kappa(t)C(t) - \frac{1}{2}\sigma(t)^2 C(t)^2 + 1 = 0,$$

subject to the terminal condition $C(T, T) = A(T, T) = 0$.

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Affine intensity models

- These ODEs can be solved for all t and T using standard numerical algorithms, such as the Runge-Kutta method.

- As a result, if we need to impose a terminal condition in terms of $S(T, s)$, we set

$$V(T, \lambda) = e^{A(T,s) - C(T,s)\lambda}.$$

- The same is true for various intermediate boundary conditions required for exotic options.

- Very importantly, the existence of the reconstitution formula allows us, for given $\sigma(t)$ and $\kappa(t)$, to calibrate the function $\theta(t)$ to match exactly the initial risky zero coupon curve $\mathcal{P}(0, T)$, for all T.

- The same is, of course, true for the Hull-White model.

- $\sigma(t)$ is calibrated to match the market prices of CDS swaptions.

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Recovery modeling

- So far, we have treated the recovery rate on a CDS as a distraction. The recovery rate is assumed to be a constant determined by the market convention (say, 40%).

- In fact, modeling recovery rates is an inexact science, as available market information is insufficient.

- There is a (rather illiquid) market for *recovery rate locks*, which in principle provides information about the implied recovery rates.

- A recovery rate lock is an instrument structured similarly to a forward rate agreement, on which the counterparties exchange, in case of a default of the reference name, the net of an actual recovery rated and a fixed coupon.

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Recovery modeling

- We will now show how to extend the reduced form modeling framework to include the stochastic nature of the recovery rate.
- We thus assume that the *recovery process* $R(t)$ is adapted with respect to $\mathcal{G}_t$.
- The time 0 price $\mathcal{P}_0^{\text{rec}}(T)$ of a nonzero recovery risky zero coupon bond is given by

$$
\begin{aligned}
\mathcal{P}_0^{\text{rec}}(T) &= \mathsf{E}^{\mathsf{Q}}\Big[e^{-\int_0^T r(s)ds}1_{\tau \geq T} + e^{-\int_0^\tau r(s)ds}1_{\tau \leq T}R(\tau)\Big] \\
&= \mathsf{E}^{\mathsf{Q}}\Big[e^{-\int_0^T (r(s)+\lambda(s))ds}\Big] + \int_0^T \mathsf{E}^{\mathsf{Q}}\Big[e^{-\int_0^t (r(s)+\lambda(s))ds}R(t)\lambda(t)\Big]dt \\
&= \mathcal{P}_0(T) + \int_0^T \mathsf{E}^{\mathsf{Q}}\Big[e^{-\int_0^t (r(s)+\lambda(s))ds}R(t)\lambda(t)\Big]dt.
\end{aligned}
$$

- It straightforward to extend this formula to the case of a coupon bearing bond.

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Recovery modeling

- Various approaches to modeling the recovery process have been proposed. We focus on the following three approaches:
    (i) recovery of treasury,
    (ii) recovery of market value,
    (iii) recovery of par.
- These models do not attempt to model the outcomes of actual restructuring and bankruptcy proceedings. Their purpose is to model the value of the settlement, and they may be thought of as reflecting various different types of rules used to distribute the recovery amount among the claimants.

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Recovery of treasury

- In the *recovery of treasury* (RT) model, the recovery amount is expressed in terms of the credit riskless zero coupon bond, namely

$$
R(t) = x P_t(T) \\
= x e^{-\int_t^T r(s) ds},
$$

where x is a constant factor (we shall relax this assumption shortly).

- Consequently,

$$
\begin{aligned}
\mathcal{P}_0^{\text{rec}}(T) &= \mathcal{P}_0(T) + x \int_0^T \mathsf{E}^{\mathsf{Q}} \Big[e^{-\int_0^t (r(s) + \lambda(s)) ds} e^{-\int_t^T r(s) ds} \lambda(t) \Big] dt \\
&= \mathcal{P}_0(T) + x \mathsf{E}^{\mathsf{Q}} \Big[e^{-\int_0^T r(s) ds} \int_0^T e^{-\int_0^t \lambda(s) ds} \lambda(t) \, dt \Big] \\
&= \mathcal{P}_0(T) + x \mathsf{E}^{\mathsf{Q}} \Big[e^{-\int_0^T r(s) ds} \Big(1 - e^{-\int_0^T \lambda(s) ds} \Big) \Big].
\end{aligned}
$$

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Recovery of treasury

- Rearranging the terms we see that, under RT,

$$\mathcal{P}_0^{\text{rec}}(T) = xP_0(T) + (1-x)\mathcal{P}_0(T).$$

- In other words, the price of an RT zero coupon bond is the sum of x units of a riskless zero coupon bond and $1 - x$ units of a zero recovery zero coupon bond.

- The same result holds for a coupon bearing bond $\mathcal{B}$:

$$\mathcal{B}_0^{\text{rec}} = xB_0 + (1-x)\mathcal{B}_0,$$

where B is the price of the equivalent default free bond.

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Recovery of treasury

- The RT model is viewed as weak.
- Under RT coupon bonds can recover more than their par value. This occurs when the risky bond has a high default risk, long maturity, and trades close to par. Under these circumstances, the equivalent riskless bond will trade well above par. A large recovery rate may results in a value larger than the par value of the bond.
- Under RT with constant fraction x, bonds of different maturities and different coupons have different loss severities.

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Recovery of market value

- Under the recovery of market value (RMV) model, the defaulted zero coupon bond pays a fraction of its market value preceding the default:

$$R(t) = x\mathcal{P}_{t_-}^{\text{rec}}(T),$$

where $\mathcal{P}_{t_-}^{\text{rec}}(T)$ denotes the price of the bond just prior to the default.

- We shall now show that, explicitly,

$$\mathcal{P}_0^{\text{rec}}(T) = \mathsf{E}^{\mathsf{Q}}\left[e^{-\int_0^T (r(s)+(1-x)\lambda(s))ds}\right].$$

- To this end, we notice first that

$$\mathsf{E}^{\mathsf{Q}}\left[1_{\tau \geq T} \ldots | \mathscr{F}_t\right] = 1_{\tau \geq t}\mathsf{E}^{\mathsf{Q}}\left[1_{\tau \geq T} \ldots | \mathscr{F}_t\right].$$

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Recovery of market value

- Therefore, denoting $V(t) = \mathcal{P}_t^{\text{rec}}(T)$, we find that

$$
\begin{aligned}
1_{t \leq \tau} V(t) &= \mathsf{E}^{\mathsf{Q}} \Big[1_{\tau \geq T} e^{-\int_t^T r(s) ds} + \big(1_{t \leq \tau} - 1_{\tau \geq T} \big) x e^{-\int_t^\tau r(s) ds} V(\tau) \,|\, \mathscr{F}_t \Big] \\
&= 1_{t \leq \tau} \mathsf{E}^{\mathsf{Q}} \Big[e^{-\int_t^T (r(s) + \lambda(s)) ds} + x \int_t^T e^{-\int_t^u (r(s) + \lambda(s)) ds} V(u) \lambda(u) \, du \,|\, \mathscr{F}_t \Big].
\end{aligned}
$$

- Now, let us introduce the martingale

$$
M(t) = \mathsf{E}^{\mathsf{Q}} \Big[e^{-\int_0^T (r(s) + \lambda(s)) ds} + x \int_0^T e^{-\int_0^u (r(s) + \lambda(s)) ds} V(u) \lambda(u) \, du \,|\, \mathscr{F}_t \Big].
$$

- Then, we can express $V(t)$ in terms of $M(t)$ as follows:

$$
V(t) = e^{\int_0^t (r(s) + \lambda(s)) ds} \Big(M(t) - x \int_0^t e^{-\int_0^u (r(s) + \lambda(s)) ds} V(u) \lambda(u) \, du \Big).
$$

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Recovery of market value

- Ito's lemma implies that

$$d\left(e^{-\int_0^t (r(s)+(1-x)\lambda(s))ds} V(t)\right) = e^{-x\int_0^t \lambda(s)ds} dM(t).$$

- Taking the expectation value yields

$$E^Q\left[d\left(e^{-\int_0^t (r(s)+(1-x)\lambda(s))ds} V(t)\right)\right] = 0,$$

since M is a martingale and so $E^Q[dM(t)] = 0$.

- Integrating the equation above from 0 to T, yields

$$E^Q\left[e^{-\int_0^T (r(s)+(1-x)\lambda(s))ds} V(T)\right] - V(0) = 0.$$

Keeping in mind that $V(T) = 1$, we get the desired result.

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Recovery of par

- Under the recovery of par (RP) model, the defaulted bond pays a fraction of its par value, $R(t) = x$, for all $t \geq 0$.
- The idea of the RP model is a liquidation under a bankruptcy court supervision. Typically, the court distributes the proceeds in proportion to the claims filed.
- For the zero coupon case, the RP model yields

$$\mathcal{P}_0^{\text{rec}}(T) = \mathcal{P}_0(T) + x \int_0^T \mathsf{E}^{\mathsf{Q}} \Big[e^{-\int_0^t (r(s)+\lambda(s))ds} \lambda(t) \Big] dt.$$

- In general, the value of a claim under RP is the sum of the claim under zero recovery and $x \int_0^T \mathsf{E}^{\mathsf{Q}} \big[e^{-\int_0^t (r(s)+\lambda(s))ds} V(t) \lambda(t) \big] dt$, where $V(t)$ is the value of the claim at time t.

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Stochastic recovery rates

- We have been assuming so far that the value of the recovery fraction x is a constant. This simplifies the calculations but ignores the *recovery risk*.
- Recovery risk may have a significant impact on the prices of credit sensitive instruments.
- In order to model this risk we assume that x is drawn from a probability distribution, $x \sim F$.
- The prices of bonds under the recovery models discussed above are thus given in terms of expected values under the joint distribution Q and F.

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Stochastic recovery rates

- Assuming that Q and F are independent, we thus obtain:

  (i) Under RT:

  $$\mathcal{P}_0^{\text{rec}}(T) = \mathsf{E}^F[x]P_0(T) + (1 - \mathsf{E}^F[x])\mathcal{P}_0(T).$$

  (ii) Under RMV:

  $$\mathcal{P}_0^{\text{rec}}(T) = \mathsf{E}^F\left[\mathsf{E}^Q\left[e^{-\int_0^T(r(s)+(1-x)\lambda(s))ds}\right]\right].$$

  (iii) Under RP:

  $$\mathcal{P}_0^{\text{rec}}(T) = \mathcal{P}_0(T) + \mathsf{E}^F[x]\int_0^T \mathsf{E}^Q\left[e^{-\int_0^t(r(s)+\lambda(s))ds}\lambda(t)\right]dt.$$

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Stochastic recovery rates

- A possible choice of a parametric distribution for recovery fraction modeling is the *beta distribution* $B(p, q)$, where $p, q > 0$.
- Its PDF $f(x)$ is given by

$$f(x) = \frac{1}{B(p, q)} \, x^{p-1}(1 - x)^{q-1},$$

for $x \in [0, 1]$.
- The beta function $B(p, q)$ is defined by

$$B(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1} dt.$$

- Its mean and variance are given by

$$E[X] = \frac{p}{p + q},$$
$$Var[X] = \frac{pq}{(p + q)^2(p + q + 1)},$$

respectively.

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Stochastic recovery rates

- An alternative choice is given by a transformed Gaussian distribution: its domain $\mathbb{R}$ should be mapped in a one-to-one manner on the interval $[0, 1]$.
- The logit transform:

$$l(x) = \frac{e^x}{1 + e^x},$$

and its inverse

$$x = \log \frac{l}{1 - l}$$

are a convenient choice.

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