BARUCH, MFE

MTH 9876 Assignment Three

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1 BOOTSTRAPPING OF SURVIVAL CURVES

The enclosed spreadsheet contains a snapshot of actual par spreads (as of October 6, 2015) of several names: General Electric, JPMorgan Chase, Axis Capital, and MBIA. The values of the spreads are expressed in basis points (i.e., the value of 45 means 0.0045). In the calculations below, assume that the recovery rate is 40%, and the riskless rate r is constant and equal 1.5%.

Name	1Y	2Y	3Y	4Y	5Y	7Y	10Y
GE	19.35	25.45	31.85	38	47.9	71	93.6
JPM	38.8	51.5	61.45	72.5	89.6	111.05	131.7
Axis	152.5	200.75	235.75	241.3	264.2	269.8	272.05
MBIA	393.35	463.2	563.25	696.4	727.2	749.4	735.5

(i) Use the valuation formulas and the bootstrap method explained in Lecture 3 to build the survival curves for each of these names.
Solution: According to Lecture 3, the protection leg

$$V_{\text{prot}}(T_i) = -(1-R)\int_0^T P(0,s)\,dS(0,s) \approx \frac{1-R}{2}\sum_{j=1}^{n_i}\left[P(0,s_{j-1})+P(0,s_j)\right]\left[S(0,s_{j-1})-S(0,s_j)\right],$$

where $S(0,t) = e^{-t\bar{\lambda}(t)}$, $P(0,t) = e^{-rt}$, and n_i is the number discretized time steps up to T_i. In practice, it is sufficient to choose the time steps monthly for sufficient accuracy. Assume that we have K CDSs with maturities $T_1 < T_2 < \cdots T_K$. In the simplest approach, we assume that $\bar{\lambda}(t)$ is piecewise constant between the times $T_i, i = 1,2,\cdots,K$. Let $\bar{\lambda}_i$ be the constant forward intensity on the interval $[T_{i-1}, T_i]$, where $T_0 = 0$. On the other hand, according to the approximation given in Lecture 3, the risky annuity of premium leg

$$\mathscr{A}(T_i) = \frac{\delta}{2}\sum_{j=1}^{N_i} P(0,t_j)\left[S(0,t_{j-1})+S(0,t_j)\right],$$

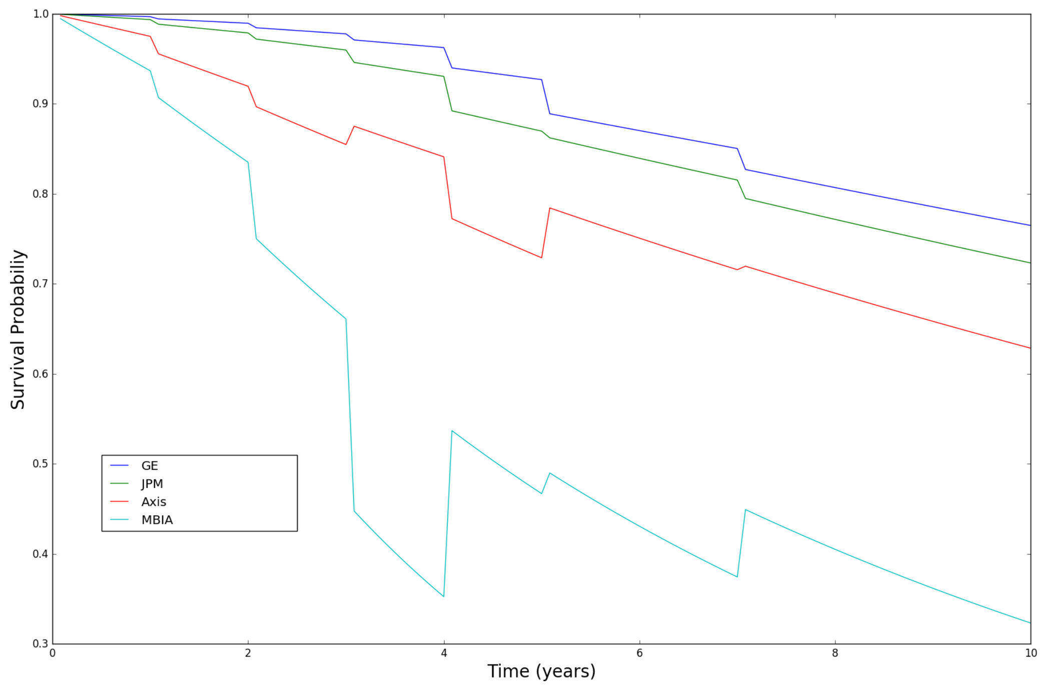
where N_i is the number of quarterly roll dates t_j up to T_i (not including T_0) and $\delta = \frac{1}{4}$. The par credit spread observed from market is

$$C_0(T_i) = \frac{V_{\text{prot}}(T_i)}{\mathscr{A}(T_i)}.$$

More explicitly,

$$C_0(T_i) = \left(\frac{1-R}{\delta}\right)\cdot\frac{\sum_{j=1}^{n_i}\left[e^{-rs_{j-1}}+e^{-rs_j}\right]\left[e^{-\bar{\lambda}(s_{j-1})s_{j-1}}-e^{-\bar{\lambda}(s_j)s_j}\right]}{\sum_{j=1}^{N_i} e^{-rt_j}\left[e^{-\bar{\lambda}(t_{j-1})t_{j-1}}+e^{-\bar{\lambda}(t_j)t_j}\right]},\quad i = 1,2,3,4,5,7,10. \qquad (1.1)$$

In this exercise, we choose $s_0 = t_0 = 0$, $s_j - s_{j-1} = \frac{1}{12}$, $t_j - t_{j-1} = \delta = \frac{1}{4}$, $R = 0.4$, and $r = 1.5\%$. The survival curves are plotted in Fig. 1. For Axis and MBIA, zig-zag patterns are observed in the survival curve which violates the no-arbitrage condition $\frac{dS(0,t)}{dt} \le 0$.



(ii) Based on these curves, calculate the default probabilities of each of these names in $T = 1, 2, \cdots, 10$ years.
Solution: The default probabilities in $T = 1, 2, \cdots, 10$ are tabulated in the following:

Year	GE	JPM	Axis	MBIA
1	0.003214	0.006434	0.025050	0.063342
2	0.010485	0.021250	0.080501	0.164937
3	0.022269	0.040246	0.145195	0.338905
4	0.037525	0.069497	0.158853	0.647462
5	0.073100	0.130316	0.271040	0.532997
6	0.129710	0.160519	0.249224	0.569158
7	0.149630	0.184646	0.284249	0.625570
8	0.193014	0.228376	0.310319	0.594872
9	0.214359	0.252981	0.341616	0.638139
10	0.235140	0.276802	0.371493	0.676786

(iii) For each of these names, compute the par spread for the 2Y into 5Y forward CDS.
Solution: For some conventions in forward starting instruments, refer to MTH9878 *Interest Rate Models* Lecture 3, Page 75, 78. The par spread of a forward CDS is given by a similar formula as Eq.1.1 with the summation restricted to the time window 2Y→7Y, tabulated as follows:

Name	GE	JPM	Axis	MBIA
Par Spread (bps)	63.99	117.70	315.14	1039.01

Python script:

```
import math
import numpy as np
from scipy.optimize import fsolve
import matplotlib.pyplot as plt

f, axarr = plt.subplots(1)

# input parameters
R = 0.4 # recovery rate
r = 0.015 # riskless interest rate
timeStepsPerYear = 12
rollDatesPerYear = 4

# constants
bps = 0.0001

# derived paramters
rollDatesInterval = int(timeStepsPerYear/rollDatesPerYear)
delta = 1.0/rollDatesPerYear # interval between roll dates
dt = 1.0/timeStepsPerYear # time step width for Stieltjes integral
c = (1-R)/delta
numOfMaturities = len(CdsMaturities)
maxMaturity = CdsMaturities[numOfMaturities-1]
maxNumOfTimeSteps = maxMaturity*timeStepsPerYear

def CalcParSpread(x, minUnknownIndex, maxUnknownIndex, knownIntensities):
    numerator = 0.0
    denominator = 0.0
    for i in range(maxUnknownIndex):
        l = 0.0
        if i >= minUnknownIndex: l = x
        else: l = knownIntensities[i]

        P = math.exp(-r*i*dt) + math.exp(-r*(i+1)*dt)
        S = math.exp(-l*i*dt) - math.exp(-l*(i+1)*dt)
        numerator += P*S

    for i in range(0,maxUnknownIndex,rollDatesInterval):
        l = 0.0
        if i >= minUnknownIndex: l = x
        else: l = knownIntensities[i]

        P = math.exp(-r*(i+rollDatesInterval)*dt)
        S = math.exp(-l*i*dt) + math.exp(-l*(i+rollDatesInterval)*dt)
        denominator += P*S

    return c*numerator/denominator
```

```python
def func(x, *data):
    return CalcParSpread(x, data[0], data[1], data[2])-data[3]

def CalcIntensity(parSpreads, knownIntensities):
    minUnknownIndex = None
    maxUnknownIndex = None
    currentParSpread = None
    for i in range(numOfMaturities):
        currentParSpread = parSpreads[i]
        minUnknownIndex = 0
        if i>0: minUnknownIndex = CdsMaturities[i-1]*timeStepsPerYear
        maxUnknownIndex = CdsMaturities[i]*timeStepsPerYear

        lambda0 = currentParSpread/(1-R)
        data = (minUnknownIndex, maxUnknownIndex, knownIntensities,
            currentParSpread)
        y = fsolve(func, lambda0, args=data)
        for j in range(minUnknownIndex,maxUnknownIndex):
            knownIntensities[j] = y[0]

    survivalRates = []
    for i in range(maxNumOfTimeSteps):
        survivalRates.append(math.exp(-(i+1)*dt*knownIntensities[i]))

    return survivalRates

timeAxis = []
for i in range(maxNumOfTimeSteps):
    timeAxis.append((i+1)*dt)

CdsParSpreads = np.array([19.35,25.45,31.85,38.0,47.9,71.0,93.6])*bps
knownIntensities = [None]*maxNumOfTimeSteps
survivalRates = CalcIntensity(CdsParSpreads, knownIntensities)
axarr.plot(timeAxis, survivalRates, label="GE")
```

2 ANDERSEN'S METHOD FOR PRICING A SINGLE PERIOD CMDS

Consider Anderson's method for pricing a single period CMDS as discussed in class. Let α be te participation rate (known). Assume the underlying spread is lognormally distributed,

$$\varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2 T_{i-1}}x} \exp\left(-\frac{\left(\log\left(\frac{x}{C_0}\right) + \frac{1}{2}\sigma^2 T_{i-1}\right)^2}{2\sigma^2 T_{i-1}} \right)$$

The price of a CMDS is often quoted in terms of a *participation rate*. This is the factor which we use to multiply all of the coupons on the CMDS so that their present value equals the present value of the protection leg.

(i) Calculate $V_{\text{cpn}}(0)$ assuming no cap on the spread.

Solution: The example given in Lecture 4 is concerned with the case of

- Settlement date T_0, CMDS coupon payment date T_1;
- $C(T_0) \triangleq C(T_0, T_0, T)$: par spread of the reference swap $T_0 \to T$, known at T_0;
- Under survival measure, $\mathbb{E}^{\mathbb{Q}_{T_0,T}}[C(T_0, T_0, T)] = C(0, T_0, T) \triangleq C_0$, known at time zero;
- Approximation of $f(C) \approx aC + b$, where

$$b = \frac{P_0(T_1)}{A_0(T_0, T)}, \quad a = \frac{1}{C_0}\left(\frac{\mathscr{P}_0(T_1)}{\mathscr{A}_0(T_0, T)} - b\right) = \frac{1}{C_0}\left(\frac{\mathscr{P}_0(T_1)}{\mathscr{A}_0(T_0, T)} - \frac{P_0(T_1)}{A_0(T_0, T)}\right).$$

In this exercise, denote the constant maturity $\Delta T = T - T_0$, the above parameters are adjusted accordingly

- Settlement date T_{i-1}, CMDS coupon payment date T_i;
- $C_0 \triangleq C(0, T_{i-1}, T_{i-1} + \Delta T)$: par spread of the reference swap $T_{i-1} \to T_{i-1} + \Delta T$, valuated at time zero. The survival measure is $\mathbb{Q}_{T_{i-1}, T_{i-1}+\Delta T}$.
- Approximation of $f(C) \approx aC + b$, where

$$\begin{aligned} b &= \frac{P_0(T_i)}{A_0(T_{i-1}, T_{i-1} + \Delta T)}, \\ a &= \frac{1}{C_0}\left(\frac{\mathscr{P}_0(T_i)}{\mathscr{A}_0(T_{i-1}, T_{i-1} + \Delta T)} - b\right) = \frac{1}{C_0}\left(\frac{\mathscr{P}_0(T_i)}{\mathscr{A}_0(T_{i-1}, T_{i-1} + \Delta T)} - \frac{P_0(T_i)}{A_0(T_{i-1}, T_{i-1} + \Delta T)}\right). \end{aligned}$$

Now invoking Anderson's result:

$$\begin{aligned} V_{\text{cpn}}(0) &= \mathscr{A}(0)\delta\mathbb{E}^{\mathbb{Q}_{T_{i-1},T_{i-1}+\Delta T}}\left[f(C)C\right] \\ &\approx \mathscr{A}(0)\delta\mathbb{E}^{\mathbb{Q}_{T_{i-1},T_{i-1}+\Delta T}}\left[aC^2 + bC\right] \\ &= \mathscr{A}(0)\delta\left(a\mathbb{E}^{\mathbb{Q}_{T_{i-1},T_{i-1}+\Delta T}}\left[C^2\right] + b\mathbb{E}^{\mathbb{Q}_{T_{i-1},T_{i-1}+\Delta T}}\left[C\right]\right) \\ &= \mathscr{A}(0)\delta\left(ae^{2\log C_0 - \sigma^2 T_{i-1} + 2\sigma^2 T_{i-1}} + be^{\log C_0 - \frac{1}{2}\sigma^2 T_{i-1} + \frac{1}{2}\sigma^2 T_{i-1}}\right) \\ &= \mathscr{A}(0)\delta\left(aC_0^2 e^{\sigma^2 T_{i-1}} + bC_0\right), \end{aligned}$$

where $\mathscr{A}(0) \triangleq \mathscr{A}(0, T_{i-1}, T_{i-1} + \Delta T)$, and we have used the well-known results on the first two moments of a lognormal variable: $\mathbb{E}[X] = e^{\mu + \frac{1}{2}\sigma^2}$, and $\mathbb{E}[X^2] = e^{2\mu + 2\sigma^2}$.

(ii) Calculate $V_{\text{cpn}}(0)$ assuming a cap of U on the spread.
Solution: Under Anderson's method,

$$\begin{aligned}
V_{\text{cpn}}(0) &= \alpha\mathscr{A}(0)\delta\int_0^\infty (ax+b)\min(x,U)\varphi(x)\,dx \\
&= \alpha\mathscr{A}(0)\delta\left[\int_0^U (ax^2+bx)\varphi(x)\,dx + U\int_U^\infty (ax+b)\varphi(x)\,dx\right].
\end{aligned}$$

Since $\varphi(x)$ is a lognormal distribution, let $y=\log x$ and y is normally distributed

$$\begin{aligned}
\varphi(x)\,dx &= \frac{1}{\sqrt{2\pi\sigma^2 T_{i-1}}x}\exp\left(-\frac{\left(\log\left(\frac{x}{C_0}\right)+\frac{1}{2}\sigma^2 T_{i-1}\right)}{2\sigma^2 T_{i-1}}\right)dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2 T_{i-1}}}\exp\left(-\frac{\left(y-\log C_0+\frac{1}{2}\sigma^2 T_{i-1}\right)^2}{2\sigma^2 T_{i-1}}\right)dy \\
&\triangleq \phi(y)\,dy,
\end{aligned}$$

with mean $\mu=\log C_0-\frac{1}{2}\sigma^2 T_{i-1}$ and variance $s^2=\sigma^2 T_{i-1}$. Thus,

$$V_{\text{cpn}}(0) = \alpha\mathscr{A}(0)\delta\left[\int_{-\infty}^{\log U}(ae^{2y}+be^y)\phi(y)\,dy + U\int_{\log U}^\infty (ae^y+b)\phi(y)\,dy\right].$$

In general, for any $d,u,\beta\in\mathbb{R}$,

$$\begin{aligned}
\int_d^u e^{\beta y}\phi(y)\,dy &= \frac{1}{\sqrt{2\pi}s}\int_d^u e^{-\frac{(y-\mu)^2}{2s^2}+\beta y}\,dy \\
&= \frac{e^{\beta\mu+\frac{1}{2}s^2\beta^2}}{\sqrt{2\pi}s}\int_d^u e^{-\frac{(y-\mu-s^2\beta)^2}{2s^2}}\,dy \\
&= e^{\beta\mu+\frac{1}{2}s^2\beta^2}\int_{\frac{d-\mu-s^2\beta}{s}}^{\frac{u-\mu-s^2\beta}{s}}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy \\
&= e^{\beta\mu+\frac{1}{2}s^2\beta^2}\left[N\left(\frac{u-\mu-s^2\beta}{s}\right)-N\left(\frac{d-\mu-s^2\beta}{s}\right)\right].
\end{aligned}$$

Specifically,

$$\begin{aligned}
a\int_{-\infty}^{\log U} e^{2y}\phi(y)\,dy &= aC_0^2 e^{\sigma^2 T_{i-1}}N\left(\frac{\log\frac{U}{C_0}-\frac{3}{2}\sigma^2 T_{i-1}}{\sigma\sqrt{T_{i-1}}}\right), \\
b\int_{-\infty}^{\log U} e^{y}\phi(y)\,dy &= bC_0 N\left(\frac{\log\frac{U}{C_0}-\frac{1}{2}\sigma^2 T_{i-1}}{\sigma\sqrt{T_{i-1}}}\right), \\
Ua\int_{\log U}^{\infty} e^{y}\phi(y)\,dy &= UaC_0 N\left(-\frac{\log\frac{U}{C_0}-\frac{1}{2}\sigma^2 T_{i-1}}{\sigma\sqrt{T_{i-1}}}\right), \\
Ub\int_{\log U}^{\infty} \phi(y)\,dy &= UbN\left(-\frac{\log\frac{U}{C_0}+\frac{1}{2}\sigma^2 T_{i-1}}{\sigma\sqrt{T_{i-1}}}\right).
\end{aligned}$$

Finally,

$$V_{\text{cpn}}(0) = \alpha\mathscr{A}(0)\delta\left[aC_0^2 e^{\sigma^2 T_{i-1}} N\left(\frac{\log\frac{U}{C_0} - \frac{3}{2}\sigma^2 T_{i-1}}{\sigma\sqrt{T_{i-1}}}\right) + bC_0 N\left(\frac{\log\frac{U}{C_0} - \frac{1}{2}\sigma^2 T_{i-1}}{\sigma\sqrt{T_{i-1}}}\right)\right.$$
$$\left. + UaC_0 N\left(-\frac{\log\frac{U}{C_0} - \frac{1}{2}\sigma^2 T_{i-1}}{\sigma\sqrt{T_{i-1}}}\right) + UbN\left(-\frac{\log\frac{U}{C_0} + \frac{1}{2}\sigma^2 T_{i-1}}{\sigma\sqrt{T_{i-1}}}\right)\right].$$

(iii) What is the value of the cap option?
Solution: Notice that the payoff of a cap option $(x-U)^+ = x - \min(x, U)$, we can write the value of the cap option as $V_{\text{cpn}}^{\textbf{(i)}}(0) - V_{\text{cpn}}^{\textbf{(ii)}}(0)$, in other words, the difference between the premium legs obtained from part **(i)** and **(ii)**.

3 CIR Model of Stochastic Intensity

Consider the CIR model of stochastic intensity

$$d\lambda(t) = \kappa(t)(\theta(t) - \lambda(t))dt + \sigma(t)\sqrt{\lambda(t)}dW(t).$$

Show, by a direct calculation, that the survival probability is of the form

$$S(t,T) = e^{A(t,T) - C(t,T)\lambda(t)},$$

where the coefficients $A(t,T)$ and $C(t,T)$ satisfies Riccati's ordinary differential equations

$$\begin{aligned}
&\frac{dA(t)}{dt} - \kappa(t)\theta(t)C(t) = 0,\\
&\frac{dC(t)}{dt} - \kappa(t)C(t) - \frac{1}{2}\sigma^2(t)C^2(t) + 1 = 0,
\end{aligned}$$

subject to the terminal condition $C(T,T) = A(T,T) = 0$.

Solution: This problem falls into the general category of finding the PDE satisfied by some expression given a stochastic representation of the expression. The key is to find a martingale. Here we have

$$S(t,T) = \mathbb{E}\left[e^{-\int_t^T \lambda(t')dt'}\middle|\mathcal{G}_t\right] = \mathbb{E}\left[e^{\int_0^t \lambda(t')dt' - \int_0^T \lambda(t')dt'}\middle|\mathcal{G}_t\right] = e^{\int_0^t \lambda(t')dt'}\mathbb{E}\left[e^{-\int_0^T \lambda(t')dt'}\middle|\mathcal{G}_t\right].$$

Thus, using the fact that $S(T,T) = 1$,

$$e^{-\int_0^t \lambda(t')dt'}S(t,T) = \mathbb{E}\left[e^{-\int_0^T \lambda(t')dt'}S(T,T)\middle|\mathcal{G}_t\right],$$

we conclude $e^{-\int_0^t \lambda(t')dt'}S(t,T) = e^{-\int_0^t \lambda(t')dt' + A(t,T) - C(t,T)\lambda(t)} \triangleq e^{X(t)}$ is a martingale, where $X(t) = -\int_0^t \lambda(t')dt' + A(t,T) - C(t,T)\lambda(t)$. Apply Itô's lemma,

$$\begin{aligned}
&de^{X(t)}\\
&= e^{X(t)}\left[dX(t) + \frac{1}{2}dX(t)dX(t)\right]\\
&= e^{X(t)}\Big[-\lambda(t)dt + A'(t,T)dt - C'(t,T)\lambda(t)dt\\
&\qquad - C(t,T)\left(\kappa(t)\left(\theta(t) - \lambda(t)\right)dt + \sigma(t)\sqrt{\lambda(t)}dW(t)\right) + \frac{1}{2}C^2(t,T)\sigma^2(t)\lambda(t)dt\Big]\\
&= e^{X(t)}\left[\left(-\lambda(t) + A'(t,T) - C'(t,T)\lambda(t) - C(t,T)\kappa(t)\left(\theta(t) - \lambda(t)\right) + \frac{1}{2}C^2(t,T)\sigma^2(t)\lambda(t)\right)dt + (\cdots)dW(t)\right]\\
&= e^{X(t)}\left[\left(\boxed{A'(t,T) - C(t,T)\kappa(t)\theta(t)} - \lambda(t) \times \boxed{1 + C'(t,T) - C(t,T)\kappa(t) - \frac{1}{2}C^2(t,T)\sigma^2(t)}\right)dt + (\cdots)dW(t)\right].
\end{aligned}$$

Because $e^{X(t)}$ is a martingale, set the drift terms in boxes to zero, we get

$$\begin{aligned}
&\frac{dA(t)}{dt} - C(t,T)\kappa(t)\theta(t) = 0,\\
&\frac{dC(t)}{dt} - \kappa(t)C(t,T) - \frac{1}{2}\sigma^2(t)C^2(t,T) + 1 = 0.
\end{aligned}$$