November 11, 2015

MTH 9831. Solutions to Quiz 7.

(1) (2 points) (Finding the generator of a diffusion) Let $(X_1(t), X_2(t))$ solve the following system of SDE

$$dX_1(t) = X_1(t)\,dt + \sqrt{X_2(t)}X_1(t)\,dB_1(t)$$
$$dX_2(t) = (1 - X_2(t))\,dt + 2\sqrt{X_2(t)}\,dB_2(t),$$

where $(B_1(t), B_2(t))$ is a 2-dimensional Brownian motion with correlated components, $d[B_1, B_2](t) = dt/2$. Write down the corresponding generator $\mathcal{A}$.

Solution. Since we have a 2-dimensional process, our generator will act on functions of 2 variables, call the variables x_1 and x_2. All we need to do is to compute the coefficients of $\mathcal{A}$. The drifts of X_1 and X_2 determine the coefficients in front of the first derivatives, and from the quadratic and cross variation we can determine the diffusion matrix. We get

$$\mathcal{A} = x_1\frac{\partial}{\partial x_1} + (1 - x_2)\frac{\partial}{\partial x_2} + \frac{x_2x_1^2}{2}\frac{\partial^2}{\partial x_1^2} + x_1x_2\frac{\partial^2}{\partial x_1\partial x_2} + 2x_2\frac{\partial^2}{\partial x_2^2}.$$

(2) (2 points) Suppose that for $0 \le t \le u \le T$

$$dX(u) = (1 - X(u))du + 2\sqrt{X(t)}\,dB(u), \quad X(t) = x.$$

Which PDE and which terminal condition are satisfied by the function $g(t, x) = \int_t^T E^{t,x}(X^2(u))\,du$? (Derivation is not required.)

Answer. (See problem (3) from HW8.)

$$g_t(t, x) + (1 - x)g_x(t, x) + 2xg_{xx}(t, x) + x^2 = 0; \quad g(T, x) = 0.$$

(3) (7=2+3+2 points) (Solving an elementary PDE) Let $u(t, x)$ be a solution to the following terminal value problem:

$$g_t(t, x) - xg_x(t, x) + 2g_{xx}(t, x) = 0, \quad g(T, x) = x^2.$$

(a) Use the Feynman-Kac formula to write a stochastic representation for $g(t, x)$. Do not forget to include the corresponding SDE and the starting point of a stochastic process.

(b) Solve the SDE from part (a) explicitly.

(c) Substitute the solution of the SDE found in (b) into the formula of part (a) and find $g(t, x)$.

Solution. (a) By the Feynman-Kac formula

$$g(t, x) = E^{t,x}((X^2(T)),$$

where $X(u)$, $t \le u \le T$, is a solution to the equation $dX(u) = -X(u)\,dt + 2dB(u)$, $X(t) = x$.

(b) This is an Ornstein-Uhlenbeck process (see (10) of the handout on Itô's formula).

$$d(e^u X(u)) = e^u X(u)\,du + e^u dX(u) = e^u(X(u)\,du - X(u)\,du + 2dB(u)) = 2e^u dB(u).$$

$$e^T X(T) - e^t X(t) = 2\int_t^T e^u\,dB(u)$$

$$X(T) = e^{-(T-t)}x + 2\int_t^T e^{-(T-u)}dB(u).$$

(c) From parts (a), (b), and Itô's identity we have

$$\begin{aligned}
g(t,x) &= E^{t,x}\left(e^{-(T-t)}x + 2\int_t^T e^{-(T-u)}\,dB(u)\right)^2 \\
&= e^{-2(T-t)}x^2 + 0 + 4\int_t^T e^{-2(T-u)}\,dt \\
&= e^{-2(T-t)}x^2 + 2 - 2e^{-2(T-t)} = e^{-2(T-t)}(x^2-2) + 2.
\end{aligned}$$

It is easy to see that $g(T,x) = 0$ and check by substitution into the PDE that $g(t,x)$ is indeed a solution.