MTH 9893 HW6

Problem 1 (a) $y_n = 0.0047 + 0.35\, y_{n-1} + 0.18\, y_{n-2} - 0.14\, y_{n-3} + \epsilon_n$

z-transforms. Write as $Q(z)Y(z) = \text{const} \times U(z) + \epsilon(z)$

where $Q(z) = (1-p_1 z^{-1})(1-p_2 z^{-1})(1-p_3 z^{-1}) = 1 - 0.35 z^{-1} - 0.18 z^{-2} + 0.14 z^{-3}$

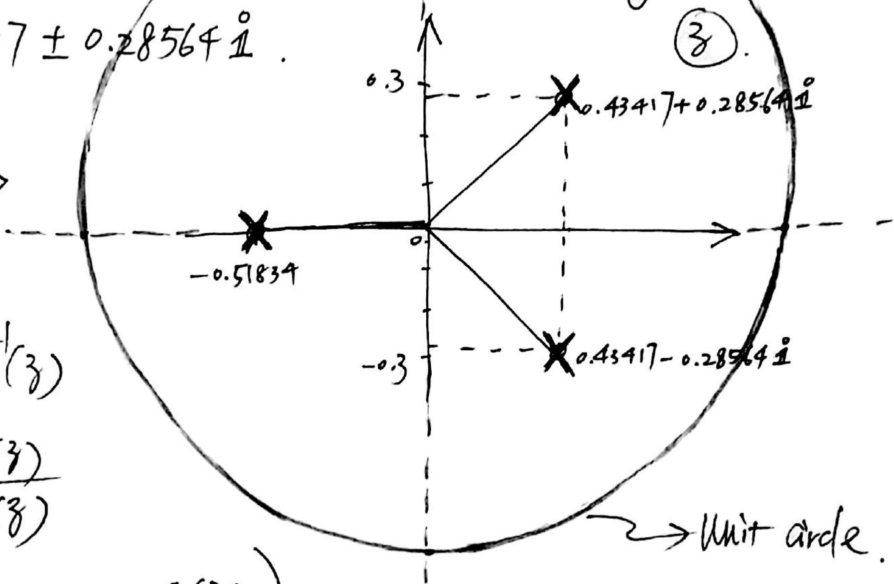
and p_1, p_2, and p_3 are roots of $Q(z)$

$\therefore\ \underbrace{0.14 - 0.18 z - 0.35 z^2 + z^3}_{\text{a polynomial}} = (z-p_1)(z-p_2)(z-p_3)$

(b) Run numerical root-finder of Wolfram-Alpha, we get

$z \cong -0.51834,\ 0.43417 \pm 0.28564\, i$.

(c) pole-zero diagram $\Rightarrow$ of $\dfrac{1}{Q(z)}$



(d) Multiply through by $Q^{-1}(z)$

$$Y(z) = \text{const} \times \frac{U(z)}{Q(z)} + \frac{\epsilon(z)}{Q(z)}$$

$$= H(z) \cdot \left(\text{const} \times U(z) + \epsilon(z)\right)$$

where

$$H(z) = \frac{1}{Q(z)} = \frac{1}{\prod_{k=1}^{3}(1-p_k z^{-1})} = \sum_{k=1}^{3} \frac{A_k}{1-p_k z^{-1}}$$

Perform a partial fraction expansion, we get residues: $A_1 = -0.14083$

$A_{2,3} = 0.24542 \pm 0.02870\, i$.

See Python script: partial_fraction_expansion.py

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(e) The inverse z-transform

$$h[n] = \sum_{k=1}^{3} A_k p_k^n u[n]$$

↳ unit-step series

(f) $p = \frac{N_{eff}}{1+N_{eff}} \Rightarrow \frac{1}{p} = 1 + \frac{1}{N_{eff}} \Rightarrow N_{eff} = \frac{p}{1-p}$

$\therefore$ $p_1 = -0.51834 \Rightarrow N_{eff} = -0.34138$

$p_{2,3} = 0.43417 \pm 0.28564\, i \Rightarrow |p_{2,3}| = 0.51970 \Rightarrow N_{eff} \cong 1.082$.

(g) Usually, the AR(3) model coefficients are determined by fitting the model with historical data.

The intent is to get low error in out-of-sample updates.

Since $N_{eff} \approx 1$, the smoothing capability of this impulse response is NOT very good. Some simpler alteranative models include low-order MA model or poly-ema model.

Problem 2 Autocorrelation and Polynomial Emas.

(a) Stability (BIBO : bounded-input bounded output)

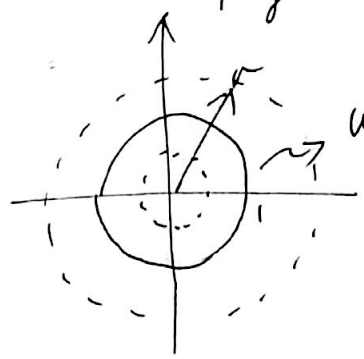
$$H_{m,cand}(z)=\frac{1}{\left(1-\frac{p}{m}z^{-1}\right)^{m}}$$ with degenerate pole at $z=\frac{p}{m}$ if $m>1$

$\Rightarrow$ impulse response $=\frac{1}{(m-1)!}\left(\frac{p}{m}\right)^{n}\prod_{s=1}^{m-1}(n+s)$

$\Rightarrow$ Convergence series $\sum_{n=0}^{\infty}|x[n]|\,r^{-n}\propto\sum_{n=0}^{\infty}\left(\frac{p}{rm}\right)^{n}\underbrace{\prod_{s=1}^{m-1}(n+s)}_{\text{a polynomial of } n.}$

This series converges when

$$\frac{p}{rm}<1\Rightarrow r>\frac{p}{m}$$



BIBO stability requires

$$\frac{p}{m}<1$$

(b) Gain Adjustment.

$$\text{Gain}=M_0=\sum_{n=-\infty}^{+\infty}h[n]=H(z=1)=\frac{1}{\left(1-\frac{p}{m}\right)^{m}}$$

$$\therefore H_m(z)=g^{-1}H_{m,cand}(z)=\frac{\left(1-\frac{p}{m}\right)^{m}}{\left(1-\frac{p}{m}z^{-1}\right)^{m}}=\left[\frac{1-\frac{p}{m}}{1-\frac{p}{m}z^{-1}}\right]^{m}.$$

(c) First Moment ($\zeta=z^{-1}$)

$$M_1^{(H)}=\frac{\partial}{\partial\zeta}H(\zeta)\Big|_{\zeta=1}=\frac{\partial}{\partial\zeta}\cdot\frac{1}{\left(1-\frac{p}{m}\zeta\right)^{m}}\Big|_{\zeta=1}$$

$$=\frac{\cancel{m}\times\frac{p}{\cancel{m}}}{\left(1-\frac{p}{m}\zeta\right)^{m+1}}\Big|_{\zeta=1}$$

$$=\frac{p}{\left(1-\frac{p}{m}\right)^{m+1}}\ \checkmark\ \Rightarrow M_1^{(H_m)}=\left(1-\frac{p}{m}\right)^{m}\times\frac{p}{\left(1-\frac{p}{m}\right)^{m+1}}=\frac{p}{1-\frac{p}{m}}$$

(d) Relative half width (RHW) — note: use $H(\zeta)$ to do the calculation, the result is the same by using $H_m(\zeta)$ because RHW is dimensionless

$$M_2 - M_1 = \frac{\partial^2}{\partial \zeta^2} H(\zeta)\Big|_{\zeta=1} = \frac{\partial}{\partial \zeta} \frac{p}{\left(1-\frac{p}{m}\zeta\right)^{m+1}}\Big|_{\zeta=1}$$

$$= \frac{p(m+1)\frac{p}{m}}{\left(1-\frac{p}{m}\zeta\right)^{m+2}}\Big|_{\zeta=1}$$

$$= \frac{m+1}{m}\cdot\frac{p^2}{\left(1-\frac{p}{m}\right)^{m+2}}$$

$$\therefore M_2 = \frac{m+1}{m}\cdot\frac{p^2}{\left(1-\frac{p}{m}\right)^{m+2}} + \frac{p}{\left(1-\frac{p}{m}\right)^{m+1}}$$

$$= \frac{\frac{m+1}{m}p^2 + \left(1-\frac{p}{m}\right)p}{\left(1-\frac{p}{m}\right)^{m+2}}$$

$$= \frac{\left(1+\frac{1}{m}\right)p^2 + p - \frac{p^2}{m}}{\left(1-\frac{p}{m}\right)^{m+2}}$$

$$= \frac{p(p+1)}{\left(1-\frac{p}{m}\right)^{m+2}}$$

$$\therefore M_2 - M_1^2 = \frac{p(p+1)}{\left(1-\frac{p}{m}\right)^{m+2}} - \frac{p^2}{\left(1-\frac{p}{m}\right)^{2m+2}} = \frac{p(p+1)\left(1-\frac{p}{m}\right)^m - p^2}{\left(1-\frac{p}{m}\right)^{2m+2}}$$

$$\therefore \frac{M_2 - M_1^2}{M_1^2} = \frac{p(p+1)\left(1-\frac{p}{m}\right)^m - p^2}{p^2}$$

$$= \left(1+\frac{1}{p}\right)\left(1-\frac{p}{m}\right)^m - 1$$

$$\therefore RHW = \sqrt{\left(1+\frac{1}{p}\right)\left(1-\frac{p}{m}\right)^m - 1}$$

(2) Here is a derivation of impulse responses, which I think is more straightforward than the one given in textbook.

$$h[n] = \frac{1}{2\pi i}\oint H(z)\, z^{n-1}\, dz$$

$$= \frac{1}{2\pi i}\oint \frac{z^{n-1}\, dz}{\left(1-\frac{p}{m}z^{-1}\right)^{m}}$$

$$= \frac{1}{2\pi i}\oint \frac{z^{m+n-1}\, dz}{\left(z-\frac{p}{m}\right)^{m}}$$

$$= \frac{1}{2\pi i}\oint \frac{\left(z-\frac{p}{m}+\frac{p}{m}\right)^{m+n-1}}{\left(z-\frac{p}{m}\right)^{m}}\, dz$$

$$= \frac{\left(p/m\right)^{m+n-1}}{2\pi i}\oint \frac{\left(1+\frac{z-p/m}{p/m}\right)^{m+n-1}}{\left(z-\frac{p}{m}\right)^{m}}\, dz$$

$$= \frac{\left(\frac{p}{m}\right)^{m+n-1}}{2\pi i}\oint \sum_{k=0}^{m+n-1} \frac{\binom{m+n-1}{k}\left(\frac{z-p/m}{p/m}\right)^{k}}{\left(z-\frac{p}{m}\right)^{m}}\, dz$$

$$= \frac{\left(\frac{p}{m}\right)^{m+n-1}}{2\pi i}\sum_{k=0}^{m+n-1}\binom{m+n-1}{k}\frac{1}{\left(\frac{p}{m}\right)^{k}}\cdot \underbrace{\oint \frac{1}{\left(z-\frac{p}{m}\right)^{m-k}}\, dz}_{\delta_{k,m-1}\cdot 2\pi i}.$$

$$= \frac{\left(\frac{p}{m}\right)^{m+n-1}}{\cancel{2\pi i}}\sum_{k=0}^{m+n-1}\binom{m+n-1}{k}\frac{1}{\left(\frac{p}{m}\right)^{k}}\delta_{k,m-1}$$

$$= \left(\frac{p}{m}\right)^{m+n-1}\binom{m+n-1}{m-1}\frac{1}{\left(\frac{p}{m}\right)^{m-1}}$$

$$= \binom{m+n-1}{m-1}\left(\frac{p}{m}\right)^{n}$$

$$= \underline{\frac{1}{(m-1)!}\prod_{s=1}^{m-1}(n+s)\cdot\left(\frac{p}{m}\right)^{n}}.$$

$m=1$, $h_1[n] = p^n \times (1-p)$

$m=2$, $h_2[n] = (n+1)\left(\frac{p}{2}\right)^n \times \left(1-\frac{p}{2}\right)^2$

$m=3$, $h_3[n] = \frac{1}{2}(n+1)(n+2)\left(\frac{p}{3}\right)^n \times \left(1-\frac{p}{3}\right)^3$

$m=4$, $h_4[n] = \frac{1}{6}(n+1)(n+2)(n+3)\left(\frac{p}{4}\right)^n \times \left(1-\frac{p}{4}\right)^4$

$m=5$, $h_5[n] = \frac{1}{24}(n+1)(n+2)(n+3)(n+4)\left(\frac{p}{5}\right)^n \times \left(1-\frac{p}{5}\right)^5$.

*plot see enclose picture files

p is determined by $M_1 = \dfrac{p}{1-\frac{p}{m}} = 50 \Rightarrow$

(f)

m	M_0	M_1	RHW.
1	1.0	50.0	1.00995
2	1.0	50.0	0.72111
3	1.0	50.0	0.594418
4	1.0	50.0	0.519615
5	1.0	50.0	0.469042.

(g) $H_1 = \dfrac{1-p}{1-pz^{-1}} \Rightarrow y_n = p y_{n-1} + (1-p)x_n$.

$H_2 = \dfrac{\left(1-\frac{p}{2}\right)^2}{\left(1-\frac{p}{2}z^{-1}\right)^2} \Rightarrow y_n = p y_{n-1} - \frac{p^2}{4} y_{n-2} + \left(1-\frac{p}{2}\right)^2 x_n$

$H_3 = \dfrac{\left(1-\frac{p}{3}\right)^3}{\left(1-\frac{p}{3}z^{-1}\right)^3} \Rightarrow y_n = p y_{n-1} - 3\times\left(\frac{p}{3}\right)^2 y_{n-2} + \left(\frac{p}{3}\right)^3 y_{n-3} + \left(1-\frac{p}{3}\right)^3 x_n$

$H_4 = \dfrac{\left(1-\frac{p}{4}\right)^4}{\left(1-\frac{p}{4}z^{-1}\right)^4} \Rightarrow y_n = p y_{n-1} - 6\times\left(\frac{p}{4}\right)^2 y_{n-2} + 4\left(\frac{p}{4}\right)^3 y_{n-3} - \left(\frac{p}{4}\right)^4 y_{n-4} + \left(1-\frac{p}{4}\right)^4 x_n$.

$H_5 = \dfrac{\left(1-\frac{p}{5}\right)^5}{\left(1-\frac{p}{5}z^{-1}\right)^5} \Rightarrow y_n = p y_{n-1} - 10\times\left(\frac{p}{5}\right)^2 y_{n-2} + 10\left(\frac{p}{5}\right)^3 y_{n-3} - 5\times\left(\frac{p}{5}\right)^4 y_{n-4} + \left(\frac{p}{5}\right)^5 y_{n-5} + \left(1-\frac{p}{5}\right)^5 x_n$

(h) See enclosed code and graph: plot_poly_ema_impulse_responses.png

(i). See enclosed code and graph: plot_poly_ema_system.png

(j)

m	l^* such that $K_h(l^*) \simeq 5\%$
1	152
2	121
3	102
4	90
5	81

See enclosed code for numerical calculation.

(k). $l^*(m=5)$ is the smallest, thus $m=5$ poly ema is the most favorable filter with respect to auto-correlation length.
$l^*(m=1)$ is the least favorable.