MTH 9831 Fall 2015 HW #6 Solutions

Please note that these are *possible* solutions, and that there are often many ways to approach and solve a problem or prove a theorem.

Problem 1 (Girsanov Theorem, Shreve #5.3)

(i) We differentiate with respect to x and use a standard normal distributed random variable Z to obtain

$$\begin{aligned}
c_x(0,x) &= \widetilde{E}\left[e^{-rT}\partial_x\left(xe^{\sigma\widetilde{W}_T+(r-\frac{1}{2}\sigma^2)T}-K\right)^+\right] \\
&= e^{-\sigma^2T/2}\widetilde{E}\left[e^{\sigma\widetilde{W}_T}1_{xe^{\sigma\widetilde{W}_T+(r-\sigma^2/2)T}>K}\right] \\
&= e^{-\sigma^2T/2}\widetilde{E}\left[e^{\sigma\sqrt{T}Z}1_{Z-\sigma\sqrt{T}>-d_+(T,x)}\right] \\
&= e^{-\sigma^2T/2}\int_{-\infty}^{\infty}e^{\sigma\sqrt{T}z}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}1_{z-\sigma\sqrt{T}>-d_+(T,x)}dz \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(z-\sigma\sqrt{T})^2/2}1_{z-\sigma\sqrt{T}>-d_+(T,x)}dz \\
&= \int_{-d_+(T,x)}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-u^2/2}du \\
&= 1-N(-d_+(T,x)) = N(d_+(T,x))
\end{aligned}$$

(ii) We set $\widehat{Z}(T)=e^{\sigma\widetilde{W}_T-\sigma^2T/2}$ and $Z(t)=\widetilde{E}[\widehat{Z}(T)|\mathcal{F}_t]$, then $\widehat{Z}$ is a $\widetilde{P}$-martingale, $\widehat{Z}>0$ and $\widetilde{E}Z(T)=1$. We can define a new measure $d\widehat{P}=\widehat{Z}(T)d\widetilde{P}$ which is equivalent to $\widetilde{P}$, furthermore we have from (i)

$$c_x(0,x)=\widetilde{E}[\widehat{Z}(T)1_{S(T)>K}]=\widehat{E}[1_{S(T)>K}]=\widehat{P}(S(T)>K).$$

By Girsanov $\widetilde{W}(t)+\int_0^t(-\sigma)ds=\widetilde{W}(t)-\sigma t=\widehat{W}(t)$ is a $\widehat{P}$-Brownian motion.

(iii) We rewrite $S(T)$ in terms of $\widehat{W}(T)$ $S(T)=S(0)e^{\sigma\widehat{W}(T)+(r+\sigma^2/2)T}$ and thus

$$\begin{aligned}
\widehat{P}(S(T)>K) &= \widehat{P}(S(0)e^{\sigma\widehat{W}(T)+(r+\sigma^2/2)T}>K)=\widehat{P}(\frac{\hat{W}(T)}{\sqrt{T}}>-d_+(T,x)) \\
&= \widehat{P}(Z>-d_+(T,x))=N(d_+(T,x)).
\end{aligned}$$

Problem 2 (Binomial Representation Theorem)
Note that

$$\tilde{V}_{n+1}=\tilde{V}_0+\sum_{i=1}^{n+1}H_{i+1}(\tilde{S}_{i+1}-\tilde{S}_i)=\tilde{V}_n+H_{n+1}(\tilde{S}_{n+1}-\tilde{S}_n).$$

Since $\tilde{S}_n=(1+r)^{-n}S_n$ is a risk-neutral martingale itself, and H_{n+1} as stated is $\mathcal{F}_n$-measurable, then

$$\tilde{E}(\tilde{V}_{n+1}\,|\,\mathcal{F}_n)=\tilde{V}_n+H_{n+1}(\tilde{E}(\tilde{S}_{n+1}\,|\,\mathcal{F}_n)-\tilde{S}_n)=\tilde{V}_n,$$

as expected. Note that, since H_n is predictable, its value only relies on time $n-1$ and before, so, writing $H_n(\omega_n)$ to show emphasis of only the last coin flip, $H_n(u)=H_n(d)$. Note that

$\tilde{V}_n(u) \neq \tilde{V}_n(d)$ and $\tilde{S}_n(u) \neq \tilde{S}_n(d)$.

$$\begin{aligned}
&\tilde{V}_n = \tilde{V}_{n-1} + H_n(\tilde{S}_n - \tilde{S}_{n-1}) \\
\implies &\tilde{V}_n(u) = \tilde{V}_{n-1} + H_n(\tilde{S}_n(u) - \tilde{S}_{n-1}), \tilde{V}_n(d) = \tilde{V}_{n-1} + H_n(\tilde{S}_n(d) - \tilde{S}_{n-1}) \\
\implies &H_n = \frac{\tilde{V}_n(u) - \tilde{V}_n(d)}{\tilde{S}_n(u) - \tilde{S}_n(d)} = \frac{V_n(u) - V_n(d)}{S_n(u) - S_n(d)}.
\end{aligned}$$

Problem 3 (Martingale Representation Theorem, Shreve #5.5)
(i) Compute $d(1/Z_t)$:

$$\begin{aligned}
&\frac{1}{Z_t} = f(X_t); f(x) = e^x, X_t = \int_0^t \Theta_u dW_u + \frac{1}{2}\int_0^t \Theta_u^2 du \\
\implies &df(X_t) = f_x(X_t)dX_t + \frac{1}{2} f_{xx}(X_t)dX_t dX_t = f(X_t)dX_t + \frac{1}{2} f(X_t)dX_t dX_t \\
&= \frac{\Theta_t dW_t + \frac{1}{2}\Theta_t^2 dt + \frac{1}{2}\Theta_t^2 dt}{Z_t} = \frac{\Theta_t dW_t + \Theta_t^2 dt}{Z_t}.
\end{aligned}$$

(ii) $\tilde{M}_t$ is a $\tilde{P}$-martingale. Therefore, by Lemma 5.2.2, p. 212:

$$\tilde{E}(\tilde{M}_t|\mathcal{F}_s) = \tilde{M}_s = \frac{1}{Z_s} E(\tilde{M}_t Z_t|\mathcal{F}_s) \implies M_t = \tilde{M}_t Z_t \text{ is a } P-\text{martingale.}$$

(iii) Now, by Thm. 5.3.1, p. 221, $\exists \Gamma_u$ an adapted process such that $M_t = M_0 + \int_0^t \Gamma_u dW_u$. Writing $\tilde{M}_t = M_t/Z_t$, we have

$$\begin{aligned}
d\tilde{M}_t &= dM_t(1/Z_t) + M_t d(1/Z_t) + dM_t d(1/Z_t) \\
&= \Gamma_t dW_t/Z_t + M_t \tfrac{\Theta_t dW_t + \Theta_t^2 dt}{Z_t} + \Gamma_t dW_t \tfrac{\Theta_t dW_t + \Theta_t^2 dt}{Z_t} \\
&= \tfrac{1}{Z_t}\Bigg[\Gamma_t dW_t + M_t(\Theta_t dW_t + \Theta_t^2 dt) + \Gamma_t \Theta_t dt\Bigg] \\
&= \tfrac{1}{Z_t}\Bigg[(\Gamma_t + M_t\Theta_t)dW_t + \Theta_t(\Gamma_t + M_t\Theta_t)dt\Bigg] \\
&= \tfrac{\Gamma_t + M_t\Theta_t}{Z_t}\Bigg[dW_t + \Theta_t dt\Bigg].
\end{aligned}$$

iv) Note that, since $d\tilde{W}_t = dW_t + \Theta_t dt$, the last result is really $d\tilde{M}_t = \frac{\Gamma_t + M_t\Theta_t}{Z_t} d\tilde{W}_t$, which means, setting $\tilde{\Gamma}_t = \frac{\Gamma_t + M_t\Theta_t}{Z_t}$, we have (5.3.2): $\tilde{M}_t = \tilde{M}_0 + \int_0^t \tilde{\Gamma}_u d\tilde{W}_u$.

Problem 4 (Every strictly positive price process is a generalized GBM, Shreve #5.8)
We know a risk-neutral martingale $\tilde{M}(t)$ under $\tilde{\mathbb{P}}$ (with $\tilde{W}(t)$ its BM) has representation

$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u) d\tilde{W}(u), \ 0 \le t \le T.$$

(i) We are given

$$V(t) = \tilde{E}\left[e^{-\int_t^T R(u)du} V(T) \,\middle|\, \mathcal{F}(t)\right], \ 0 \le t \le T.$$

Then the discount process $D(t) = e^{-\int_0^t R(u)du}$ (5.2.17) shows that $D(t)V(t)$ is a $\tilde{P}$-martingale, i.e. $D(t)V(t) = \tilde{E}\left[D(T)V(T) \,|\, \mathcal{F}(t)\right]$.

Thus, by the MRT, $\exists\tilde{\Gamma}(t)$ such that $D(t)V(t) = D(0)V(0) + \int_0^t \tilde{\Gamma}(u)d\tilde{W}(u)$. But, in differential form, this is, with $dD(t) = -R(t)D(t)dt$,

$$\begin{aligned}
d(D(t)V(t)) &= \tilde{\Gamma}(t)d\tilde{W}(t) \\
&= D(t)dV(t) + V(t)dD(t) + dD(t)dV(t) = D(t)[dV(t) - V(t)R(t)dt] \\
\implies dV(t) &= \frac{\tilde{\Gamma}(t)}{D(t)}d\tilde{W}(t) + R(t)V(t)dt.
\end{aligned}$$

(ii) Given $V(T) > 0$ a.s. as a hypothesis, $V(t)$ is the discounted expectation of the a.s. positive RV $e^{-\int_t^T R(u)du}V(T)$, so $V(t) > 0$ a.s.
(iii) Hence, since $D(t)V(t) > 0$ a.s. as well, we set $\sigma(t) := \frac{\tilde{\Gamma}(t)}{D(t)V(t)}$ and rewrite (i) as a generalized GBM:

$$dV(t) = \frac{\tilde{\Gamma}(t)}{D(t)}d\tilde{W}(t) + R(t)V(t)dt = \sigma(t)V(t)d\tilde{W}(t) + R(t)V(t)dt.$$

Problem 5 (Finding the risk-neutral distribution from call prices across all strikes, Shreve #5.9)
We are given

$$\begin{aligned}
c(0,T,x,K) &= \tilde{\mathbb{E}}\left[e^{-rT}(S(T)-K)_+\right] \\
&= e^{-rT}\textstyle\int_K^\infty (y-K)\tilde{p}(0,T,x,y)dy.
\end{aligned}$$

for the price of a call where $S(0) = x$, expiry time is T, and strike price is K. All we need to do is differentiate this twice with respect to K.

$$\begin{aligned}
c(0,T,x,K) &= e^{-rT}\int_K^\infty (y-K)\tilde{p}(0,T,x,y)dy \\
&= e^{-rT}\left[\int_K^\infty y\tilde{p}(0,T,x,y)dy - K\int_K^\infty \tilde{p}(0,T,x,y)dy\right] \\
\implies c_K(0,T,x,K) &= e^{-rT}\left[(y\tilde{p}(0,T,x,y)|_{y=K}^\infty - \int_K^\infty \tilde{p}(0,T,x,y)dy - K\tilde{p}(0,T,x,y)|_{y=K}^\infty\right] \\
&= e^{-rT}\left[[(y-K)\tilde{p}(0,T,x,y)|_{y=K}^\infty - \int_K^\infty \tilde{p}(0,T,x,y)dy\right] \\
&= -e^{-rT}\int_K^\infty \tilde{p}(0,T,x,y)dy = -e^{-rT}\tilde{P}(S(T) > K); \\
c_{KK}(0,T,x,K) &= -e^{-rT}\tilde{p}(0,T,x,y)|_{y=K}^{y=\infty} = e^{-rT}\tilde{p}(0,T,x,K),
\end{aligned}$$

which gives a formula for the risk-neutral distribution of $S(T)$ in terms of call prices:

$$\tilde{p}(0,T,x,K) = e^{rT}c_{KK}(0,T,x,K), \; K > 0.$$

Problem 6 (Chooser option, Shreve #5.10)
Put-call parity sets the time-t value of a long call and short put to the value of a forward:

$$\begin{aligned}
C(t) - P(t) &= \tilde{E}\left[e^{-r(T-t)}[(S(T)-K)^+ - (K-S(T))^+] \,|\, \mathcal{F}(t)\right] \\
&= \tilde{E}\left[e^{-r(T-t)}[S(T)-K] \,|\, \mathcal{F}(t)\right] = F(t).
\end{aligned}$$

(i) Set t_0 to be the chooser option time, $0 < t_0 < T$, and let its value at time t be $A(t)$. There are two cases, both of which set the value of the chooser to

$$A(t_0) = \max\{C(t_0), P(t_0)\} = C(t_0) + \max\{0, -F(t_0)\} = C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+ :$$

- $C(t_0) \geq P(t_0)$: here you would choose the call over the put, so the value of the chooser is $C(t_0)$. Note that $F(t_0) = C(t_0) - P(t_0) \geq 0$ here, so $C(t_0) + \max\{0, -F(t_0)\} = C(t_0)$.
- $C(t_0) < P(t_0)$: here you would choose the put over the call, so the value of the chooser is $P(t_0) = C(t_0) - F(t_0)$ by put-call parity.

(ii) $D(t)A(t)$ is a $\tilde{P}$-martingale, just like any other asset. Thus, we have the conditional expectations

$$A(0) = D(0)A(0) = \tilde{E}(D(T)V(T)) = e^{-rT}\tilde{E}(A(T))$$
$$D(t_0)A(t_0) = \tilde{E}(D(T)V(T) \,|\, \mathcal{F}(t_0)) = e^{-rt_0}\left[C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right].$$

Combining the two and using the tower property at t_0,

$$\begin{aligned} A(0) &= \tilde{E}[\tilde{E}(D(T)V(T) \,|\, \mathcal{F}(t_0))] \\ &= \tilde{E}\left(e^{-rt_0}\left[C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right]\right) \\ &= \tilde{E}\left[e^{-rt_0}C(t_0)\right] + \tilde{E}\left[e^{-rt_0}\left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right] = C(0) + P^*(0), \end{aligned}$$

where $C(0)$ is the time-0 price of a call with expiration T and strike K, and $P^*(0)$ is the time-0 price of a put with expiration t_0 and strike $e^{-r(T-t_0)}K$.

Problem 7 (Hedging a cash flow, Shreve #5.11)
We are given the setup

$$\begin{aligned} dS(t) &= \alpha(t)S(t)dt + \sigma(t)S(t)dW(t),\ 0 \leq t \leq T \\ dX(t) &= \Delta(t)dS(t) + R(t)[X(t) - \Delta(t)S(t)]dt - C(t)dt \end{aligned}$$

where $C(t)$ is the cash flow being continuously paid out of the portfolio. With the discount process $D(t)$ given above, we need to show that there is a nonzero $X(0)$ and a portfolio process $\Delta(t)$ that hedges this portfolio, leaving the investor with $X(T) = 0$ a.s.
First, we need the risk-neutral measure $\tilde{P}$ that makes our discounted assets martingales: this comes from applying Girsanov's Theorem with the process $\Theta(t)$ (what is it??): the $\tilde{P}$-BM will be

$$\tilde{W}(t) = \int_0^t \Theta(u)du + W(t);\ d\tilde{W}(t) = \Theta(t)dt + dW(t)$$

and we get

$$\begin{aligned} d(D(t)S(t)) &= D(t)dS(t) + S(t)dD(t) + dD(t)dS(t) \\ &= D(t)S(t)\left[(\alpha(t) - R(t))dt + \sigma(t)dW(t)\right] \\ &= D(t)S(t)\left[(\alpha(t) - R(t))dt + \sigma(t)(d\tilde{W}(t) - \Theta(t)dt)\right] \\ &= D(t)S(t)\left[(\alpha(t) - R(t) - \sigma(t)\Theta(t))dt + \sigma(t)d\tilde{W}(t)\right]. \end{aligned}$$

Thus, $\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}$, the *market price of risk* process (5.2.21), p. 216, is the process we need in Girsanov's Theorem here to kill the dt term in our discounted assets. Applying this to the discounted portfolio process $D(t)X(t)$, we have

$$\begin{aligned}
d(D(t)X(t)) &= D(t)dX(t) + X(t)dD(t) + dD(t)dX(t) \\
&= D(t)\left[dX(t) - R(t)X(t)dt\right] \\
&= D(t)\left[\Delta(t)dS(t) - R(t)\Delta(t)S(t)dt - C(t)dt\right] \\
&= D(t)\left[\Delta(t)[\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)] - R(t)\Delta(t)S(t)dt - C(t)dt\right] \\
&= D(t)\Delta(t)S(t)\sigma(t)d\tilde{W}(t) - D(t)C(t)dt.
\end{aligned}$$

Now we integrate and take expectations, more precisely we set

$$\tilde{M}(t) := \tilde{E}\left[\int_0^T D(s)C(s)ds \,\middle|\, \mathcal{F}(t)\right].$$

By the MRT, $\tilde{M}$ has a representation $\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(s)d\tilde{W}(s)$. Now from above we have

$$D(T)X(T) = D(0)X(0) + \int_0^T D(t)\Delta(t)S(t)\sigma(t)d\tilde{W}(t) - \int_0^T D(t)C(t)dt,$$

which, choosing $\Delta(t) = \frac{\tilde{\Gamma}(t)}{D(t)S(t)\sigma(t)}$, can be written as

$$\begin{aligned}
D(T)X(T) &= X(0) + \int_0^T \tilde{\Gamma}(t)d\tilde{W}(t) - \int_0^T D(t)C(t)dt \\
&= X(0) + (\tilde{M}(T) - \tilde{M}(0)) - \int_0^T D(t)C(t)dt
\end{aligned}$$

Note that $\tilde{M}(0) = \tilde{E}\left[\int_0^T D(t)C(t)dt\right]$, $\tilde{M}(T) = \int_0^T D(t)C(t)dt$ and hence

$$\begin{aligned}
D(T)X(T) &= X(0) + \int_0^T \tilde{\Gamma}(t)d\tilde{W}(t) - \int_0^T D(t)C(t)dt \\
&= X(0) - \tilde{M}(0)
\end{aligned}$$

Finally if we choose $X(0) = \tilde{M}(0) = \tilde{E}\left[\int_0^T D(t)C(t)dt\right]$ as our initial portfolio value, we get $X(T) = 0$ a.s.