Interest Rate Models

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Outline

Frictionless markets

- *A frictionless market* consist of N risky securities that satisfy
  (i) Each of the assets is liquid; at each time any bid or ask order can be executed.
  (ii) There are no transaction costs. Each bid and ask order at a given time can be executed at the same price level.
- The model of the process is the vector $S(t) = (S_1(t), \ldots, S_N(t))$, where $S_i(t)$ is the price of the i-th asset. For a given d dimensional Wiener process $(W_1, \ldots, W_d)$ on $(\Omega, \mathcal{F}_t)$ each S_i satisfies

$$dS_i(t) = \Delta_i(t, S(t))\, dt + \sum_k C_{ik}(t, S(t))\, dW_k(t).$$

- The drift coefficients Δ_i and the diffusion coefficients C_{ik} are adapted.

Black-Scholes model

- This is the simplest model. We assume that there are only two securities, and that the first one is not random.

$$\begin{aligned} dB(t) &= rB(t)\,dt \\ dS(t) &= \mu S(t)\,dt + \sigma S(t)\,dW(t). \end{aligned}$$

- r is *riskless rate*, and μ is the rate of return on the risky asset.
- In reality the riskless rate does not exist, and OIS is the closest approximation to it.
- We will build the models in which we will not assume the existence of the riskless rate.

Self-financing portfolios and arbitrage

- The portfolio is a sequence of weights $w(t) = (w_1(t), \ldots, w_N(t))$ that are adapted with respect to $\mathcal{F}_t$ and add up to one. The value of the portfolio is the process defined as

$$V(t) = \sum w_i(t) S_i(t).$$

- The portfolio is self financing if

$$dV(t) = \sum w_i(t)\, dS_i(t).$$

- The arbitrage opportunity occurs if there is a self-financing portfolio such that: $V(0) = 0$; $\mathbb{P}\left(V(T) \geq 0\right) = 1$ and $\mathbb{P}\left(V(T) > 0\right) > 0$.

Complete markets

- A contingent claim expiring at T is a random variable $X \in \mathcal{F}_T$ that is square integrable.
- Main example of a contingent claim is a payoff of a derived security $X = g(S(T))$.
- A call option is a contingent claim with payoff $(S(T) - K)^+$.
- A financial market is *complete* if each contingent claim can be obtained as the terminal value of a self-financing portfolio.

Complete markets

- In the case of Black-Scholes model if φ is the solution to the terminal value problem

$$\begin{aligned} \frac{\partial \varphi}{\partial t} + rx\frac{\partial \varphi}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \varphi}{\partial x^2} &= r\varphi, \\ \varphi(T, x) &= g(x), \end{aligned}$$

then
$$\begin{aligned} w_1(t) &= \frac{\varphi(t, S(t)) - S(t)\frac{\partial}{\partial x}\varphi(t, S(t))}{B(t)} \\ w_2(t) &= \frac{\partial}{\partial x}\varphi(t, S(t)) \end{aligned}$$
is the replicating portfolio for g.

- The market is complete when $N = d$, $C(t, S(t))$ is invertible for all $t \leq T$, and some further integrability conditions hold.

Mumeraires and EMMs

- A *numeraire* is any tradeable asset with price process $\mathcal{N}(t)$ that is positive for all time.
- The *relative price* process of asset i is its price expressed in the units of numeraire. It is formally defined as

$$S_i^{\mathcal{N}}(t) = \frac{S_i(t)}{\mathcal{N}(t)}.$$

- A probability measure $\mathbb{Q}$ is called an *equivalent martingale measure* (EMM) if
  (i) It is equivalent to $\mathbb{P}$, i.e. $\frac{d\mathbb{P}}{d\mathbb{Q}}(\omega) = D_{PQ}(\omega)$, and $\frac{d\mathbb{Q}}{d\mathbb{P}} = D_{QP}$ for some $D_{PQ}(\omega) > 0$ and $D_{QP}(\omega) > 0$;
  (ii) The relative price processes $S_i^{\mathcal{N}}$ are martingales with respect to $\mathbb{Q}$, i.e. $S_i^{\mathcal{N}}(s) = \mathbb{E}^{\mathbb{Q}}\left[S_i^{\mathcal{N}}(T) \middle| \mathcal{F}_s \right]$.

Girsanov's theorem

- Assume that $W(t)$ is a Brownian motion with respect to $(\Omega, \mathcal{F}, \mathbb{P})$.
- Assume that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ and assume that D is the Radon-Nikodym derivative $D = \frac{d\,\mathbb{Q}}{d\,\mathbb{P}}$. In other words for every $A \in \mathcal{F}$ we have

$$\mathbb{Q}(A) = \int_A D(\omega)\,\mathbb{P}(\omega).$$

- We also require that D is adapted to $\mathcal{F}_t$.
- The measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$ and $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$.

Girsanov's theorem

- Consider the diffusion process

$$dX(t) = \Delta(t, X(t))\,dt + C(t, X(t))\,dW(t). \qquad (1)$$

- With a suitable function D, the diffusion process X can be changed into a diffusion with a different drift, $\tilde{\Delta}$

$$dX(t) = \tilde{\Delta}(t, X(t))\,dt + C(t, X(t))\,d\tilde{W}(t). \qquad (2)$$

Girsanov's theorem

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$$
\begin{aligned}
dX(t) &= \Delta(t)\,dt + C(t)\,dW(t) \\
&= \tilde{\Delta}(t)\,dt + C(t)\cdot\left(\frac{\Delta(t)-\tilde{\Delta}(t)}{C(t)}\,dt + dW(t)\right) \\
&= \tilde{\Delta}(t) + C(t)\,d\tilde{W}(t),
\end{aligned}
$$

where $d\tilde{W}(t) = \frac{\Delta-\tilde{\Delta}}{C}\,dt + dW(t)$, or

$$
\tilde{W}(t) = W(t) - \int_0^t \theta(s)\,ds, \quad \text{for} \quad \theta(s) = \frac{\tilde{\Delta}(s)-\Delta(s)}{C(s)}.
$$

Girsanov's theorem

- $$\tilde{W}(t) = W(t) - \int_0^t \theta(s)\, ds, \quad \text{for} \quad \theta(s) = \frac{\tilde{\Delta}(s) - \Delta(s)}{C(s)}.$$
- The process $\tilde{W}$ is a Brownian motion under the measure $Q = D\, d\mathbb{P}$, where
$$D(t) = e^{\int_0^t \theta(s)\, dW(s) - \frac{1}{2}\int_0^t \theta^2(s)\, ds}.$$
- The above formula (a.k.a. Girsanov's theorem) holds under the Novikov's condition
$$\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^t \theta^2(s)\, ds}\right] < +\infty.$$
- The process $D(t)$ is a martingale under $\mathbb{P}$ and satisfies
$$dD(t) = \theta(t) D(t)\, dW(t).$$

Multi-dimensional Girsanov's theorem

- Assume that $W(t) = (W_1(t), \ldots, W_n(t))$ is a standard Brownian motion (the components are uncorrelated). Let $X(t) = (X_1(t), \ldots, X_d(t))$ be a d-dimensional diffusion process:

$$dX(t) = \Delta(t, X(t))\, dt + C(t, X(t))\, dW(t).$$

- Define

$$\begin{aligned} D(t) &= e^{\int_0^t \theta^T(s) \cdot dW(s) - \frac{1}{2}\int_0^t \theta^T(s)\theta(s)\, ds}, \quad \text{where} \\ \theta(t) &= C(t, X(t))^{-1} \left(\tilde{\Delta}(t, X(t)) - \Delta(t, X(t)) \right). \end{aligned}$$

Multi-dimensional Girsanov's theorem

- Under the Novikov's condition

$$\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^t \theta^T(s)\theta(s)\,ds}\right] < +\infty$$

the process $D(t)$ is a matringale that satisfies

$$dD(t) = D(t)\theta^T(t)\,dW(t)$$

and $\tilde{W}$ is a Wiener process under Q.

The First Fundamental Theorem

- *The First Fundamental Theorem of arbitrage free pricing*. A market is arbitrage free if and only if for each numeraire there exists an equivalent martingale measure $\mathbb{Q}$.
- In other words, when the prices of all assets are represented in the units of numeraire, these prices are martingales with respect to $\mathbb{Q}$.
- We will not prove this theorem. We will see that there is no arbitrage if the EMM exists. Assume that $\mathcal{N}$ is a numeraire and $\mathbb{Q}$ the EMM under which all $S_i^{\mathcal{N}}$ are martingales.

The First Fundamental Theorem

- Assume that there is an arbitrage and that there is a self-financing portfolio with weights $(w_1(t), \ldots, w_N(t))$. Then the portfolio with the same weights expressed in terms of $\left(S_1^{\mathcal{N}}, \ldots, S_N^{\mathcal{N}}\right)$ is also self-financing. (HW2)
- $dV^{\mathcal{N}}(t) = \sum w_i(t)\, dS_i^{\mathcal{N}}(t)$.
- $S_i^{\mathcal{N}}$ are martingales, hence $dS_i^{\mathcal{N}}(t) = \sum A_{ij}(t)\, dW_j(t)$. Thus $V^{\mathcal{N}}$ is martingale.
- $V^{\mathcal{N}}(0) = \mathbb{E}^{\mathbb{Q}}\left[V^{\mathcal{N}}(T) \middle| \mathcal{F}_0 \right]$.
- Since $\mathbb{Q}\left(V^{\mathcal{N}}(T) > 0\right) > 0$ and $V^{\mathcal{N}}(T) \geq 0$ almost surely (because $\mathbb{Q}$ and $\mathbb{P}$ are equivalent), we have that the last expectation is positive. This contradicts the assumption that $V^{\mathcal{N}}(0) = 0$.

The First Fundamental Theorem

- In Black-Scholes model if we consider the numeraire $\mathcal{N}(t) = B(t) = e^{rt}$.
- With this numeraire we have

$$dS^B(t) = (\mu - r)S^B(t)\,dt + \sigma S^B(t)\,dW(t),$$
$$\text{with} \quad S^B(t) = \frac{S(t)}{B(t)} = e^{-rt}S(t).$$

- We want to change the measure so that $dS^B(t) = \sigma S^B(t)\,dW(t)$.
- Then

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(t) = e^{-\lambda W(t) - \frac{1}{2}\lambda^2 t}, \quad \text{where } \lambda = \frac{\mu - r}{\sigma}.$$

The Second Fundamental Theorem

- *The Second Fundamental Theorem of arbitrage free pricing.* An arbitrage free market is complete if and only if, for each numeraire $\mathcal{N}(t)$, the equivalent martingale measure is unique.
- Important conequence of the Second Fundamental Theorem is the arbitrage pricing law:

$$\frac{V(s)}{\mathcal{N}(s)} = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{V(t)}{\mathcal{N}(t)}\right|\mathcal{F}_s\right] \qquad \text{for all } s < t.$$

- If we used a different numeraire we would obtain

$$\frac{V(s)}{\mathcal{N}'(s)} = \mathbb{E}^{\mathbb{Q}'}\left[\left.\frac{V(t)}{\mathcal{N}'(t)}\right|\mathcal{F}_s\right]$$

Change of numeraire

- $$\frac{V(s)}{\mathcal{N}(s)} = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{V(t)}{\mathcal{N}(t)}\right| \mathcal{F}_s\right] \quad \text{and} \quad \frac{V(s)}{\mathcal{N}'(s)} = \mathbb{E}^{\mathbb{Q}'}\left[\left.\frac{V(t)}{\mathcal{N}'(t)}\right| \mathcal{F}_s\right].$$
- Therefore
$$\frac{d\mathbb{Q}'}{d\mathbb{Q}} = \frac{\mathcal{N}(0)\mathcal{N}'(t)}{\mathcal{N}(t)\mathcal{N}'(0)}.$$

Examples of EMMs

- The *banking account numeraire* is \$1 deposited in a bank and accruing the instantaneous rate. It is given by

  $$\mathcal{N}(t) = e^{\int_0^t r(s)\, ds},$$

  where $r(s) = f(s, s)$ is the instantaneous forward rate.
- The corresponding EMM is called the *spot measure*.
- Another numeraire is the zero coupon bond with maturity T. It is given by

  $$\mathcal{N}_T(t) = P(t, t, T).$$
- The corresponding EMM is called the *T-forward measure*.
- It arises naturally in pricing instruments based on forwards with maturity T. Forward rates for maturity T are martingales under this EMM.

Examples of EMMs

- Consider a forward starting swap that settles in T_0 and matures T years from now. Consider the annuity (or level function) of this swap as a numeraire. It is defined as

$$A(t, T_{\mathrm{val}}, T_0, T) = \sum_{j=1}^{n_c} \alpha_j P\left(t, T_{\mathrm{val}}, T_j^c\right)$$

  and corresponds to a present value of a stream that pays \$1 per annum on each coupon day of the swap, accrued according to the swap's day count conventions. Here t is the observation time, T_{val} the valuation time, T_0 the settlement time, and T_j^c the coupon times.
- We define the annuity numeraire as $\mathcal{N}_{T_0,T}(t) = A(t, t, T_0, T)$.
- The corresponding EMM is called the *swap measure*.

Mechanics of changing of numeraire

- Assume that the financial asset X satisfies

$$dX(t) = \Delta^{\mathbb{P}}(t)\,dt + C(t)\,dW^{\mathbb{P}}(t).$$

- Our goal is to relate this dynamics to the one of the same asset under an equivalent measure $\mathbb{Q}$:

$$dX(t) = \Delta^{\mathbb{Q}}(t)\,dt + C(t)\,dW^{\mathbb{Q}}(t).$$

- Assume that $\mathbb{P}$ is associated with the numeraire $\mathcal{N}(t)$ whose dynamics is given by

$$d\mathcal{N}(t) = A_{\mathcal{N}}(t)\,dt + B_{\mathcal{N}}(t)dW^{\mathbb{P}}(t),$$

  and $\mathbb{Q}$ is associated with the numeraire $\mathcal{M}$ whose dynamics is

$$d\mathcal{M}(t) = A_{\mathcal{M}}(t)\,dt + B_{\mathcal{M}}(t)dW^{\mathbb{P}}(t).$$

Mechanics of changing of numeraire

- Let $D(t) = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_t$. Then D is a martingale under $\mathbb{P}$ and satisfies

$$dD(t) = \theta(t)D(t)\,dW^{\mathbb{P}}(t), \quad \text{where} \quad \theta(t) = \frac{\Delta^{\mathbb{Q}}(t) - \Delta^{\mathbb{P}}(t)}{C(t)}.$$

- We can write D explicitly as

$$D(t) = e^{\int_0^t \theta(s)\,dW^{\mathbb{P}}(s) - \frac{1}{2}\int_0^t \theta^2(s)\,ds}.$$

- From the second fundamental theorem of asset pricing we have

$$D(t) = \frac{\mathcal{N}(0)}{\mathcal{M}(0)} \cdot \frac{\mathcal{M}(t)}{\mathcal{N}(t)}.$$

Mechanics of changing of numeraire

- Since $D(t)$ is a martingale with respect to $\mathbb{P}$, then so is $\frac{\mathcal{M}(t)}{\mathcal{N}(t)}$ hence

$$d\left(\frac{\mathcal{M}(t)}{\mathcal{N}(t)}\right) = \frac{\mathcal{M}(t)}{\mathcal{N}(t)}\left(\frac{B_{\mathcal{M}}(t)}{\mathcal{M}(t)} - \frac{B_{\mathcal{N}}(t)}{\mathcal{N}(t)}\right) dW^{\mathbb{P}}(t). \qquad (3)$$

- $dD(t) = \theta(t)D(t)\,dW^{\mathbb{P}}(t)$ implies $\theta(t) = \frac{B_{\mathcal{M}}(t)}{\mathcal{M}(t)} - \frac{B_{\mathcal{N}}(t)}{\mathcal{N}(t)}$.

$$\begin{aligned}
\text{Therefore } \Delta^{\mathbb{Q}}(t) - \Delta^{\mathbb{P}}(t) &= C(t)\left(\frac{B_{\mathcal{M}}(t)}{\mathcal{M}(t)} - \frac{B_{\mathcal{N}}(t)}{\mathcal{N}(t)}\right) \\
&= \frac{d}{dt}\int_0^t dX(s)d\left(\log\frac{\mathcal{M}(s)}{\mathcal{N}(s)}\right).
\end{aligned}$$

- Equivalent form $\Delta^{\mathbb{Q}}(t) = \Delta^{\mathbb{P}}(t) + \frac{d}{dt}\left[X, \log\frac{\mathcal{M}}{\mathcal{N}}\right](t)$.

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