1. Discount Factors and Interest Rates

Notes

Interest rates are expressed with continuous compounding unless otherwise specified.

We will not concern ourselves with industry daycount or business-day conventions. To translate maturities into times in years…

For maturities expressed in days, assume a 365-day year (i.e., a 1D maturity corresponds to $T = 1 / 365$).

For maturities expressed in weeks, assume 52 weeks per year.

For maturities expressed in months, assume 12 months per year.

Exercises

1. At a given moment, you observe the following prices for various maturities of risk-free zero-coupon bonds:

maturity	bond price (per 100 face)
1D	99.9995
1W	99.994
1M	99.971
3M	99.9
6M	99.65
1Y	99.05

(a) What is the term structure of risk-free zero rates implied by these bond prices, expressed with continuous compounding?
(b) What is the term structure of single-period forward rates (i.e., the forwards between adjacent maturities), also expressed with continuous compounding?
(c) Verify that the 1Y zero rate is the time-weighted average of the single-period forward rates you calculated.

2. The USD LIBOR fixings for a particular day are as follows:

USD	Rate (percent)
o/n	0.12800
1w	0.16270
2w	0.17170
1m	0.19043
2m	0.22413
3m	0.25288
4m	0.29463
5m	0.35013
6m	0.40313
7m	0.45913
8m	0.51288

9m	0.56463
10m	0.61913
11m	0.67125
12m	0.72950

For all parts of this problem, recall that LIBOR is quoted as a rate of simple interest, and make the simplifying assumptions detailed above concerning daycount and business-day conventions. Treat the o/n (overnight) rate as a one-day rate.

(a) What is the term structure of discount factors implied by these fixings?
(b) What are the one-month forward rates, expressed as rates of simple interest, for months 1-11 implied by these fixings?
(c) What is the term structure of interest rates implied by these fixings, with all rates expressed with monthly compounding?

3. Using market data, you extract the following term structure of discount factors:

maturity	discount factor
0.00274	0.99999
0.01923	0.99989
0.08333	0.9995
0.25	0.99875
0.5	0.99576
1	0.98807

(a) Using linear interpolation in the continuously compounded spot rates, determine the discount factor at 4 months.
(b) Using the assumption of piecewise constant forward rates, determine the discount factor at 10 months.
(c) Under each of the interpolation methods used above, what is the six-month forward rate in four months, expressed with continuous compounding?

4. The risk-free zero rate to the maturity 1 year is 1.25%. The risk-free zero rate to the maturity 3 years is 1.6%. Under each of the following interpolation methods, determine the instantaneous forward rate at 1 year (with the limit taken from the right) and 3 years (with the limit taken from the left).

(a) Constant forward rate
(b) Linear zero rates
(c) Cubic spline in the zero rates, where the maturities at 1 and 3 years are connected by a segment with the equation:

$$r(t) = -0.001138889t^3 + 0.001638889t^2 + 0.01t + 0.002$$

Here r is the rate expressed simply as a number—i.e., 1% is 0.01.

Applications
You are free to use the provided Sandbox utilities for this problem. If you do, please make your solution sheet reference Sandbox, but *do not alter* it. Submit your sheet, and the TA will use the main sheet in checking it.

Nelson-Siegel IR fitting
While the exact term structure of interest rates is necessary for full pricing, it has limited analytical or intuitive meaning. Mechanically, the various points on the interest rate curve may be shifted independently; practically, interest rate movements at various maturities are related, and for analytical purposes, it is useful to parameterize the curve in some way that is intuitive and reproduces trading behavior at the various maturities.

The conventional view of yield curves sees three primary sources of variation: level, steepness, and curvature. The level of the curve is associated with parallel shifts in interest rates. Steepness determines the difference between short-term rates and long-term rates. Curvature mainly pertains to medium-term rates (the "belly" of the curve) and dictates the degree to which the curve is humped.

The original Nelson-Siegel parameterization of the zero rate curve has the functional form:

$$r_{NS}(t) = \beta_0 + \beta_1 \frac{\lambda}{t}\left(1 - e^{-\frac{t}{\lambda}}\right) + \beta_2 \left[\frac{\lambda}{t}\left(1 - e^{-\frac{t}{\lambda}}\right) - e^{-\frac{t}{\lambda}}\right]$$

Where...
t is maturity
$r_{NS}(t)$ is the model-derived zero rate to this maturity
$\beta_0, \beta_1, \beta_2$ are the key parameters
λ is a decay parameter

A later adaptation of the model due to Svensson adds a fourth term with a second decay parameter that can produce an additional "hump" in the curve to improve fit; we will work with the basic NS model in this exercise.

Because this functional form cannot exactly match all zero rates, deviations are typically handled as a residual to the model rate:

$$r(t) = r_{NS}(t) + r_{residual}(t)$$

Once the NS model is calibrated to observed rates via a least squares fit, sensitivity of instruments to the various parameters of the model can be determined by shocking the parameter, interpolating in the residual, and repricing off the modified curve. In this exercise, we will first calibrate an NS model to a set of rates and will then determine the sensitivity of various cash flows to the model parameters.

(i) Derive the limiting values of the NS parameterization for $t = 0$ and as $t \to +\infty$. What interpretation of the model parameters does this suggest?

(ii) The term structure of zero rates in Euro observed recently was:

timeName	t	r(t) (continuous)
3M	0.25	-0.002486014
6M	0.5	-0.002612929
1Y	1	-0.002692741
2Y	2	-0.002295355
3Y	3	-0.001372911
5Y	5	0.001295265
7Y	7	0.004275038
10Y	10	0.008286853
20Y	20	0.014627831
30Y	30	0.014882457

Create a sheet in which you use Excel's built-in solver to produce the best-fit NS parameters for this term structure of interest rates and calculate the residual rate curve values at these maturities.

(iii) Create a user-defined function that takes the following inputs as Excel ranges...

A vector of maturities
The NS model parameters
The term structure of the residual curve

...and produces the vector of interpolated zero rates. Use cubic spline interpolation in the residual interest rates, with the residual equal to 0 at maturity = 0 and at maturity = 50.

Demonstrate that your function works by creating a sheet showing the NS rates, both with and without residual, at maturities from 0 years to 50 years inclusive, spaced quarterly. Graph both results.

(iv) Consider a hypothetical portfolio consisting of a cash flow of 0.5 every 6 months, with the first cash flow at 6 months and the final cash flow at 30 years. Use your interpolation function, with cubic spline interpolation in the residuals you calculated, to determine the present value of each cash flow and the present value of the overall portfolio using your Nelson-Siegel fit.

(v) Show the sensitivity of each cash flow in the portfolio above in part iv to the three key NS parameters. Specifically:

Shock β_0 downward proportionally by 1%, and show the *return* on each cash flow; also calculate the portfolio return.

Shock β_1 upward proportionally by 1%, and do the same. (Note: For a negative parameter, consider an "upward" shock to mean increasing the absolute value of the parameter.)

Shock β_2 upward proportionally by 1% as above, and show the corresponding results.

Include a graph that visually summarizes your results.

(vi) Discuss the meaning of your result in part v above, particularly with regard to what it shows about the relationship between the NS parameters and the conventional components of yield curve risk (parallel shift, steepen / flatten, belly move).

(vii) Use the functional form of the zero rates to derive the functional form of the instantaneous forward rates in the Nelson-Siegel parameterization.