Credit Risk Models

12. Monte Carlo methods for counterparty credit risk

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Outline

Motivating example

- This is a mathematical interlude on backward stochastic differential equations (BSDE), which provide a natural framework for counterparty credit risk modeling.
- To motivate, consider an ODE

$$\frac{d}{dt} Y(t) = f(t, Y(t)), \text{ where } 0 \leq t \leq T,$$

with terminal value condition

$$Y(T) = \xi,$$

instead of an initial value condition.

- This problem is not more difficult to solve, than an initial value problem.
- We set $Y^*(t) = Y(T - t)$, and note that then

$$\begin{aligned} -\frac{d}{dt} Y^*(t) &= f(T - t, Y^*(t)), \\ Y^*(0) &= \xi, \end{aligned}$$

which is an initial value problem.

Motivating example

- Consider now a simple SDE with a terminal value condition:

$$\begin{aligned} dY(t) &= 0, \\ Y(T) &= \xi, \end{aligned}$$

where ξ is a (constant, measurable with respect to $\mathscr{F}_T$) random variable.

- We require additionally that the solution be adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$.
- This problem has no solution unless ξ is a deterministic constant. Indeed, the naive solution reads $Y(t) = \xi$, which is not adapted for a random ξ.
- Consider the process

$$Y(t) \triangleq \mathrm{E}_t[\xi].$$

It is adapted and satisfies $Y(T) = \xi$, but $dY(t) \neq 0$.

- On the other hand, from the martingale representation theorem, since $\mathrm{E}_t[\xi] - \mathrm{E}[\xi]$ is a martingale, there exists an adapted process $Z(t)$ such that

$$\mathrm{E}_t[\xi] - \mathrm{E}[\xi] = \int_0^t Z(s)\, dW(s)\,.$$

Motivating example

- Since

$$\xi = \mathsf{E}[\xi] + \int_0^T Z(s)\, dW(s)\,,$$

the expression for $\mathsf{E}_t[\xi]$ can be written as

$$Y_t = \xi - \int_t^T Z(s)\, dW(s)\,.$$

- This provides an adapted solution to the following terminal value problem:

$$\begin{aligned} dY(t) &= Z(t)\, dW(t)\,, \\ Y(T) &= \xi, \end{aligned}$$

rather than the original problem.

BSDEs

- We are thus led to the idea of the solution of a BSDE as a pair $(Y(t), Z(t))$ of adapted processes such that

$$\begin{aligned} -dY(t) &= f(t, Y(t), Z(t))dt - Z(t)\, dW(t), \\ Y(T) &= \xi, \end{aligned}$$

  where the function $f(t, y, z)$ is called the *generator* of the BSDE.
- Surprisingly, the pair $(Y(t), Z(t))$ is unique (if exists). From Ito's lemma,

$$\begin{aligned} d(Y(t))^2 &= 2Y(t)\, dY(t) + (dY(t))^2 \\ &= 2Y(t)\, Z(t)\, dW(t) + Z(t)^2\, dt. \end{aligned}$$

- Integrating from t to T and taking expected value yields

$$\int_t^T d\mathrm{E}[Y(s)^2] = \int_t^T \mathrm{E}[Z(s)^2] ds.$$

BSDEs

- This can be written as

$$\mathrm{E}[\xi^2] = \mathrm{E}[Y(t)^2] + \int_t^T \mathrm{E}[Z(s)^2]ds.$$

- Therefore, if the BSDE has two solutions $(Y_1(t), Z_1(t))$ and $(Y_2(t), Z_2(t))$, then

$$0 = \mathrm{E}[(Y_2(t) - Y_1(t))^2] + \int_t^T \mathrm{E}[(Z_2(s) - Z_1(s))^2]ds.$$

- Each term on the right hand side is nonnegative, and so $Y_2(t) = Y_1(t)$ and $Z_2(t) = Z_1(t)$ (almost everywhere).
- The counterintuitive part of this theorem is that the process $Z(t)$ is uniquely determined.

Example: linear BSDE

- A BSDE with a linear generator $f(t, y, z) = a(t) + b(t)\, y + c(t)\, z$ is called *linear*.
- Linear BSDEs are among the very few BSDEs that can be solved in closed form.
- Explicitly a linear BSDE has the form:

$$
\begin{aligned}
-dY(t) &= \big(a(t)\, Y(t) + b(t)\, Z(t) + c(t)\big) dt - Z(t)\, dW(t)\,, \\
Y(T) &= \xi,
\end{aligned}
$$

- To solve it, we proceed as follows.
- *Step 1*. Consider the SDE

$$
\begin{aligned}
d\gamma(t) &= \gamma(t)\, \big(a(t)\, dt + b(t)\, dW(t)\big), \\
\gamma(0) &= 1.
\end{aligned}
$$

Its solution reads

$$
\gamma(t) = \exp\Big(\int_0^t b(s)\, dW(s) + \int_0^t \big(a(s) - \tfrac{1}{2} b(s)^2\big) ds\Big).
$$

Example: linear BSDE

- *Step 2*. From Ito's lemma,

$$\begin{aligned} d\big(\gamma(t)\,Y(t)\big) &= d\gamma(t)\,Y(t) + \gamma(t)\,dY(t) + d\gamma(t)\,dY(t) \\ &= \gamma(t)\,\big(b(t)\,Y(t)\,dW(t) - c(t)\,dt + Z(t)\,dW(t)\big), \end{aligned}$$

  or

$$d\Big(\gamma(t)\,Y(t) + \int_0^t \gamma(s)\,c(s)\,ds\Big) = \gamma(t)\,\big(b(t)\,Y(t) + Z(t)\big)\,dW(t)\,.$$

- *Step 3*. As a consequence, $M(t) \triangleq \gamma(t)\,Y(t) + \int_0^t \gamma(s)\,c(s)\,ds$ is a martingale, and so $M(t) = \mathrm{E}_t[M(T)]$. This yields

$$\gamma(t)\,Y(t) + \int_0^t \gamma(s)\,c(s)\,ds = \mathrm{E}_t\Big[\gamma(T)\,\xi + \int_0^T \gamma(s)\,c(s)\,ds\Big],$$

  and so the solution $Y(t)$ reads:

$$Y(t) = \gamma(t)^{-1}\,\mathrm{E}_t\Big[\gamma(T)\,\xi + \int_t^T \gamma(s)\,c(s)\,ds\Big].$$

FBSDEs

- In financial applications, a BSDE usually comes up as part of a forward backward stochastic differential equation (FBSDE). This consists of a forward part:

$$
\begin{aligned}
dX(t) &= \mu(t, X(t))dt + \sigma(t, X(t))dW(t), \\
X(0) &= X_0,
\end{aligned}
$$

  and a backward part:

$$
\begin{aligned}
-dY(t) &= f(t, X(t), Y(t), Z(t))dt - Z(t)\, dW(t), \\
Y(T) &= g(X(T)),
\end{aligned}
$$

  where $g(x)$ is a deterministic function.
- The forward process $X(t)$ is usually the state variable describing e.g. the underlying asset value, $Y(t)$ is the value of a derivative contract, and $Z(t)$ is the hedging strategy.

FBSDEs

- As an example, let us consider a strike K European call option on a lognormal stock:

$$
\begin{aligned}
dX(t) &= rX(t)\,dt + \sigma X(t)\,dW(t)\,,\\
X(0) &= X_0
\end{aligned}
$$

(Black-Scholes model).

- Let $Y(t)$ denote the time t value of the option. Then

$$
Y(T) = (X(T) - K)^+.
$$

- Under risk neutral valuation, $dY(t) = rY(t)\,dt +$ martingale, i.e.

$$
dY(t) = rY(t)\,dt + Z(t)\,dW(t)\,.
$$

FBSDEs and PDEs

- Consider again a general FBSDE. We seek its solution (Y, Z) by means of the following ansatz:

$$Y_t = \Phi(t, X_t),$$

  where $\Phi(t, x)$ is a smooth function.
- From Ito's lemma

$$d\Phi = \Big(\frac{\partial \Phi}{\partial t} + \mathcal{A}_t \Phi\Big) dt + \sigma \, \frac{\partial \Phi}{\partial x} \, dW_t,$$

  where

$$\mathcal{A}_t = \frac{1}{2}\, \sigma^2 \, \frac{\partial^2}{\partial x^2} + \mu \, \frac{\partial}{\partial x}$$

  is the generator.

FBSDEs and PDEs

- This yields the conditions:

$$\begin{aligned} \dot{\Phi} + \mathcal{A}_t \Phi + f(t, X, \Phi, Z) &= 0, \\ \Phi(T) &= g(X(T)), \end{aligned}$$

  and

$$Z(t) = \sigma \frac{\partial}{\partial x} \Phi(t, X(t)).$$

- Consequently, we find that Φ satisfies the nonlinear PDE:

$$\begin{aligned} \dot{\Phi} + \mathcal{A}_t \Phi + f\big(t, x, \Phi, \sigma \frac{\partial \Phi}{\partial x}\big) &= 0, \\ \Phi(T) &= g(X(T)). \end{aligned}$$

- In summary, the solution to the FBSDE is given by the pair of processes:

$$\begin{aligned} Y(t) &= \Phi(t, X(t)), \\ Z(t) &= \sigma \frac{\partial \Phi}{\partial x}(t, X(t)). \end{aligned}$$

FBSDEs and PDEs

- In the case of the Black-Scholes model discussed above, this yields the familiar PDE:

$$
\begin{aligned}
\dot{\Phi} + \frac{1}{2}\,\sigma^2 x^2\,\frac{\partial^2\Phi}{\partial x^2} + rx\,\frac{\partial\Phi}{\partial x} &= r\Phi,\\
\Phi(T) &= (X(T) - K)^+,
\end{aligned}
$$

  whose solution is the standard call option valuation formula.

- The process

$$
\begin{aligned}
Z(t) &= \sigma\,\frac{\partial\Phi}{\partial x}(t, X(t))\\
&= \sigma\Delta(t, X(t))
\end{aligned}
$$

  is (up to the factor σ) the option delta, its hedging strategy.

Numerical solution to FBSDEs

- As is the case with all interesting equations, explicit solutions to BSDEs are rarely available and we have to resort to numerical techniques.
- We present here a numerical scheme for solving FBSDEs. For simplicity, we assume a one factor model.
- We choose a discrete time grid $0 = T_0 < T_1 < \ldots < T_m = T$, and consider the discrete system. For $i = 1, 2, \ldots, m$,

$$X_i = X_{i-1} + \mu(T_{i-1}, X_{i-1})\Delta T_i + \sigma(T_{i-1}, X_{i-1})\Delta W_i,$$

  where $\Delta T_i = T_i - T_{i-1}$ and $\Delta W_i = W(T_i) - W(T_{i-1})$. This is simply Euler's scheme applied to the forward SDE.
- For the BSDE, naive Euler's scheme reads

$$\begin{aligned} Y_m &= g(X_m), \\ Y_{i-1} &= Y_i + f(T_{i-1}, X_{i-1}, Y_{i-1}, Z_{i-1})\Delta T_i - Z_i \Delta W_i, \end{aligned}$$

  for $i = m, m-1, \ldots, 0$.

Numerical solution to FBSDEs

- Since Y_i should be adapted, the right hand side should be conditioned on T_{i-1}:

$$\begin{aligned} Y_m &= g(X_m), \\ Y_{i-1} &= \mathrm{E}_{i-1}[Y_i] + f(T_{i-1}, X_{i-1}, Y_{i-1}, Z_{i-1})\Delta T_i, \\ \mathrm{E}_{i-1}[Y_i \Delta W_i] &= Z_i \Delta T_i. \end{aligned}$$

- Notice that the scheme above is implicit (Y_{i-1}) shows up on both sides of the equation). We replace it with the following explicit scheme:

$$\begin{aligned} Y_m &= g(X_m), \\ Y_{i-1} &= \mathrm{E}_{i-1}\big[Y_i + f(T_{i-1}, X_{i-1}, Y_i, Z_{i-1})\Delta T_i\big], \\ Z_i &= \frac{1}{\Delta T_i}\,\mathrm{E}_{i-1}[Y_i \Delta W_i]. \end{aligned}$$

- Notice that the term including the generator is now inside the expected value.

Numerical solution to FBSDEs

- The main computational issue is to find an efficient method for calculating the conditional expected values $\mathrm{E}_{i-1}[\,\cdot\,]$.
- Notice the similarity between this problem and the problem of valuing American options by means of Monte Carlo simulations that was discussed in the Interest Rate Modeling class.
- The least square regression method (using Hermite polynomials) discussed there is directly applicable in the current context. This choice is natural as expressions involving conditional expectations of Hermite polynomials of Gaussian random variables lead to convenient closed form expressions.

Numerical solution to FBSDEs

- Let $\mathrm{He}_k(x)$, $k = 0, 1, \ldots$, denote the k-th normalized Hermite polynomial corresponding to the standard Gaussian measure $d\mu(x) = (2\pi)^{-1/2}\, e^{-x^2/2} dx$. These functions form an orthonormal basis for the space $L^2(\mathbb{R}, \mu)$.
- The key property of $\mathrm{He}_k(x)$ is the following addition formula for $\chi \in [0, 1]$ and $w, x \in \mathbb{R}^n$:

$$\mathrm{He}_k(\sqrt{\chi}\, w + \sqrt{1-\chi}\, x) = \sum_{0 \le j \le k} \binom{k}{j}^{1/2} \chi^{j/2}(1-\chi)^{(k-j)/2} \mathrm{He}_j(w) \mathrm{He}_{k-j}(x).$$

- Consequently, integrating over x with respect to μ yields the following conditioning rules:

$$\begin{aligned} \mathsf{E}\left[\mathrm{He}_k(\sqrt{\chi}\, w + \sqrt{1-\chi}\, x) \mid w\right] &= \chi^{k/2} \mathrm{He}_k(w), \\ \mathsf{E}\left[\mathrm{He}_k(\sqrt{\chi}\, w + \sqrt{1-\chi}\, x)\, x \mid w\right] &= \chi^{(k-1)/2}(1-\chi)^{1/2}\, \mathrm{He}'_k(w). \end{aligned}$$

Numerical solution to FBSDEs

- Here, w, x are independent standard normal random variables. The latter rule is found using the addition formula and orthonormality of Hermite polynomials with respect to the standard Gaussian measure.
- We shall use these rules in order to estimate the conditional expected values above.
- We set $W_{t_i} = \sqrt{t_i}\, w_i$, for $i = 1, \ldots, m$, where w_i is a standard normal random variable. We notice that

$$w_{i+1} = \sqrt{\chi_i}\, w_i + \sqrt{1 - \chi_i}\, X_i,$$

  where $\chi_i = t_i / t_{i+1}$, and where X_i is standard normal and independent of w_i.
- In the following, we shall use this decomposition in conjunction with the conditioning rule above.

Numerical solution to FBSDEs

- Now, we assume the following linear architecture:

$$Y_{i+1} = \sum_{k:\, k \leq K} g_{k,i+1}\, \mathrm{He}_k(w_{i+1}),$$

  where K is the cutoff value of the order of the Hermite polynomials.
- This is simply a truncated expansion of the random variable Y_{i+1} in terms of the orthonormal basis $\mathrm{He}_k(w_{i+1})$. The values of the Fourier coefficients are estimated by means of ordinary least square regression.
- Then, as a consequence of the first conditioning rule,

$$\mathsf{E}_i[Y_{i+1}] = \sum_{k:\, k \leq K} g_{k,i+1}\, \chi_i^{k/2}\, \mathrm{He}_k(w_i).$$

- In other words, conditioning Y_{i+1} on w_i is equivalent to multiplying its Fourier coefficients g_k by the factor $\chi_i^{k/2}$.

Numerical solution to FBSDEs

- In practice, the formula for Z_i given in the recursion is hard to use. Instead, we find an explicit expression using the above Hermite architecture.
- The following identity holds:

$$
\begin{aligned}
Z_i &= \frac{\partial}{\partial W_i}\,\mathsf{E}_i[Y_{i+1}] \\
&= \frac{1}{\sqrt{t_i}} \sum_{k\leq K} g_{k,i+1}\chi_i^{k/2}\, k\mathrm{He}_{k-1}(w_i).
\end{aligned}
$$

- To establish this identity, we use the two conditioning rules stated above to find that

$$
\begin{aligned}
\mathsf{E}_i[\mathrm{He}_k(w_{i+1})\Delta W_i] &= \sqrt{\Delta_i}\mathsf{E}_i[\mathrm{He}_k(\sqrt{\chi_i}\, w_i + \sqrt{1-\chi_i}\, X_i)X_i] \\
&= \frac{\Delta_i}{\sqrt{t_i}}\,\chi_i^{k/2}\,\mathrm{He}_k'(w_i), ,
\end{aligned}
$$

Numerical solution to FBSDEs

- Consequently, from the recursion we find that

$$\begin{aligned} Z_i &= \frac{1}{\Delta_i} E_i\Big[\sum_{k\leq K} g_{k,i+1}\mathrm{He}_k(w_{i+1})\Delta W_i\Big] \\ &= \sum_{k\leq K} g_{k,i+1}\chi_i^{k/2}\,\mathrm{He}'_k(w_i). \end{aligned}$$

  Since $\mathrm{He}'_k(w) = k\mathrm{He}_k(w)$, we see that the claim holds.
- Now that we have found a practical representation for Z_i, we proceed calculating Y_i in the discretizing recursion. To this end, we repeat the calculations for Y_{i+1} with Y_{i+1} replaced by $Y_{i+1} + f(X_i, Y_i, Z_i)\Delta_i$.

The fundamental BSDE

- We shall now apply the framework of FBSDEs to counterparty credit risk modeling. Its major elements parallel the PDE approach of Lecture Notes #11. To streamline the notation we again focus on a single asset.
- Let S denote the price process, namely a Markov process with infinitesimal generator $\mathcal{A}_t$. The dynamics of the asset under the measure P are

$$dS(t) = \mu(t, S(t))dt + \sigma(t, S(t))dW(t).$$

- We let τ_B and τ_C be the random default times of the bank and the counterparty, respectively, and denote their indicator processes by counting processes

$$J_B(t) = 1_{\tau_B \leq t},$$

and

$$J_C(t) = 1_{\tau_C \leq t}.$$

- The natural filtration generated by J^B and J^C constitutes the default event filtration $\mathscr{G}$, i.e. $\mathscr{G}_t = \sigma(J_B(s), J_C(s) : s \leq t)$.

The fundamental BSDE

- Both counting processes J_B, J_C are assumed to be Cox processes, i.e. they have stochastic, time-dependent intensities λ_B, λ_C,

$$\lambda_B(t)\,dt = \mathrm{E}[dJ_B(t)\,|\,\mathscr{G}_{t-}],$$

and

$$\lambda_C(t)\,dt = \mathrm{E}[dJ_C(t)\,|\,\mathscr{G}_{t-}].$$

- Although not absolutely necessary, it is convenient to assume that $\lambda_B(t)$ and $\lambda_B(t)$ are Ito diffusions,

$$\begin{aligned} d\lambda_B(t) &= \vartheta_B(t,\lambda_B(t))dt + \eta_B(t,\lambda_B(t))dW_B(t)\,, \\ d\lambda_C(t) &= \vartheta_C(t,\lambda_C(t))dt + \eta_C(t,\lambda_C(t))dW_C(t)\,, \end{aligned}$$

with correlated Brownian motions $W_B(t)$ and $W_C(t)$,

$$dW_B(t)\,dW_C(t) = \rho_{BC}dt.$$

The fundamental BSDE

- This specification lends itself to including the wrong way risk into consideration. We assume that

$$\begin{aligned} dW_B(t)\, dW(t) &= \rho_{BS} dt, \\ dW_C(t)\, dW(t) &= \rho_{CS} dt. \end{aligned}$$

- We denote the bank's and counter-party's default risky, zero-recovery ZCBs P_B and P_C with maturity T. They follow the dynamics

$$\begin{aligned} \frac{dP_B(t)}{P_B(t-)} &= r_B(t)\, dt - dJ_B(t), \\ \frac{dP_C(t)}{P_C(t-)} &= r_C(t)\, dt - dJ_C(t). \end{aligned}$$

Here the stochastic processes r_B, r_C are the yields on P_B and P_C, respectively.

The fundamental BSDE

- Now we would like to find the credit-risky value $\hat{V}(t) = \hat{V}(t, T)$ of the netted set of derivatives positions on S with the maximum maturity T.
- Assume that the net cash (contingent) flows $C(T_1, S), \ldots, C(T_N, S)$ on the dates $T_1 < \ldots < T_N = T$. We let $V(t) = V(t, T)$ be the value of the same set of netted positions between the two parties without any default risk, i.e.

$$V(t) = \mathsf{E}_t^{\mathsf{Q}}\Big[\sum_{i:T_i \geq t} e^{-\int_t^{T_i} r(s)ds} C(T_i, S)\Big].$$

- The bank can hedge the portfolio $\hat{V}(t)$ with a self-financing portfolio $\Pi(t)$ that consists of δ units of S, α_B units of P_B, α_C units of P_C, and β units of cash.
- Repeating verbatim the arguments and notation of Lecture Notes #11, we find that

$$d\beta = \delta(\gamma_S - q_S)Sdt + \big(-r(\widehat{V} + \alpha_B P_B) + s_F(-\widehat{V} - \alpha_B P_B)^-\big)dt - \alpha_C r P_C dt.$$

The fundamental BSDE

- Putting everything together we obtain

$$\begin{aligned} d\Pi &= \delta(\mu dt + \sigma dW) + \alpha_B(r_B P_B dt - P_B dJ_B) + \alpha_C(r_C P_C dt - P_C dJ_C) \\ &\quad + \delta(\gamma_S - q_S)Sdt + \big(- r(\widehat{V} + \alpha_B P_B) + s_F(-\widehat{V} - \alpha_B P_B)^- \big)dt - \alpha_C r P_C dt \\ &= \delta\sigma dW - \alpha_B P^B dJ_B - \alpha_C P_C dJ_C + \delta\mu dt + \alpha_B r_B P_B dt + \alpha_C r_C P_C dt \\ &\quad + \delta(\gamma_S - q_S)Sdt - r\alpha_C P_C dt + (r(-\hat{V} - \alpha_B P_B) + s_F(-\hat{V} - \alpha_B P_B)^-)dt, \end{aligned}$$

- This simplifies to

$$\begin{aligned} d\Pi = \big(&\delta\mu + \alpha_B(r_B - r)P_B + \alpha_C(r_C - r)P_C \\ &+ \delta(\gamma_S - q_S)S - r\hat{V} + s_F(-\hat{V} - \alpha_B P_B)^- \big)dt \\ &+ \delta\sigma dW - \alpha_B P_B dJ_B - \alpha_C P_C dJ_C. \end{aligned}$$

The fundamental BSDE

- Now we set

$$
\begin{aligned}
Z\,(t) &= -\delta\,(t)\,\sigma(t, S\,(t)),\\
U_B\,(t) &= \alpha_B\,(t)\,P_B(t-),\\
U_B\,(t) &= \alpha_C\,(t)\,P_C(t-).
\end{aligned}
$$

- With these definitions, we can write

$$
\begin{aligned}
-d\hat{V}_t = \big(&-Z\sigma^{-1}\mu + (r_B - r)U_B + (r_C - r)U_C\\
&- (\gamma_S - q_S)(\sigma^{-1}Z)S - r\hat{V} + s_F(-\hat{V} - U_B)^-\big)\,dt\\
&- ZdW - U_B dJ_B - U_C dJ_C.
\end{aligned}
$$

- We see that this is a jump BSDE with the driver

$$
\begin{aligned}
f(t, x, \hat{v}, z, u_B, u_C) = &-z\sigma(t,x)^{-1}\mu(t,x) + (r_B\,(t) - r\,(t))u_B + (r_C\,(t) - r\,(t))u_C\\
&- (\gamma_S\,(t) - q_S\,(t))\sigma(t,x)^{-1}zx - r\,(t)\,\hat{v} + s_F\,(t)\,(-\hat{v} - u_B)^-.
\end{aligned}
$$

The fundamental BSDE

- Let us denote the first default time by $\tau = \tau_B \wedge \tau_C$. Then $(\hat{V}, Z, U_B, U_C)$ is a solution to the following jump BSDE

$$
\begin{aligned}
-d\hat{V}(t) = & \, f(t, S(t), \hat{V}(t), Z(t), U_B(t), U_C(t))dt \\
& - Z(t)\, dW(t) - U_B(t-)dJ_B(t) - U_C(t-)dJ_C(t), \text{ for } t \in [0, \tau \wedge T], \\
\hat{V}(\tau \wedge T) = & \, 1_{\tau > T} C(T, S) + 1_{\tau \leq T} \theta(\tau).
\end{aligned}
$$

- The terminal condition consists of the last cash flow $C(T, S)$ if there is no default before the final maturity T. In case either the bank or the counterparty defaults before T, the cash settlement at default $\theta(\tau)$ is

$$
\theta(\tau) = 1_{\tau_C < \tau_B}(R_C M^+(\tau) + M^-(\tau)) + 1_{\tau_B < \tau_C}(M^+(\tau) + R_B M^-(\tau)),
$$

  where, as before, M denotes the portfolio's mark to market, and R_B, R_C denote the recovery rates of the bank and the counter-party, respectively.
- This is the fundamental BSDE of counterparty credit risk modeling. Along with the (forward) SDE for S, it forms the fundamental FBSDE of counterparty credit risk modeling.

The fundamental BSDE

- Having the jump process as part of the BSDE is a little bothersome. We shall now show how it can be eliminated by modifying the process dynamics.
- We consider the jump BSDE

$$\begin{aligned} -dY(t) &= f(t, Y(t), Z(t), U(t))dt \\ &\quad - Z(t)\, dW(t) - U_1(t)\, dJ_1(t) - U_2(t)\, dJ_2(t), \text{ for } t \in [0, \tau \wedge T], \\ Y(\tau \wedge T) &= 1_{\tau > T}\widetilde{\theta} + 1_{\tau \leq T}\big(\theta_1(\tau)1_{\tau=\tau_1} + \theta_2(\tau)1_{\tau=\tau_2}\big), \end{aligned}$$

  with counting process $J(t) = (J_1(t), J_2(t))$, and first default time $\tau = \tau_1 \wedge \tau_2$. The terminal condition is given by an $\mathscr{F}_T$- measurable random variable $\widetilde{\theta}$, and a stochastic process $\theta(t) = (\theta_1(t), \theta_2(t))$.
- We solve the above BSDE by transforming it in the following way into a BSDE without jumps and a fixed terminal time:

$$\begin{aligned} -d\widetilde{Y}(t) &= f(t, \widetilde{Y}(t), \widetilde{Z}(t), \theta(t) - \widetilde{Y}(t))dt - \widetilde{Z}(t)\, dW(t), \text{ for } t \in [0, T], \\ \widetilde{Y}(T) &= \widetilde{\theta}. \end{aligned}$$

The fundamental BSDE

- Namely, we claim that if $(\widetilde{Y}(t), \widetilde{Z}(t))$ is a solution to the transformed BSDE, then the solution (Y, Z, U) to the original jump BSDE is given by

$$
\begin{aligned}
Y(t) &= \widetilde{Y}(t)\,1_{t<\tau} + \big(\theta_1(\tau)1_{\tau=\tau_1} + \theta_2(\tau)1_{\tau=\tau_2}\big)1_{t\geq\tau},\\
Z(t) &= \widetilde{Z}(t)\,1_{t\leq\tau},\\
U(t) &= (\theta(t) - \widetilde{Y}(t))1_{t\leq\tau}.
\end{aligned}
$$

- In order to these this equivalence, we consider three cases.
- *Case 1.* No default occurs before the terminal time, i.e. $\tau > T$. Then

$$
\begin{aligned}
Y(t) &= \widetilde{Y}(t),\\
Z(t) &= \widetilde{Z}(t),\\
U(t) &= \theta(t) - \widetilde{Y}(t),
\end{aligned}
$$

for all $t \in [0, T]$.

The fundamental BSDE

- Since $(\widetilde{Y}(t), \widetilde{Z}(t))$ solves the transformed equation, we have

$$\begin{aligned} -dY(t) &= f(t, Y(t), Z(t), U(t))dt - Z(t)\, dW(t), \text{ for } t \in [0, T], \\ Y(T) &= \widetilde{\theta} \\ &= 1_{\tau > T}\widetilde{\theta} + 1_{\tau \leq T}\big(\theta_1(\tau)1_{\tau=\tau_1} + \theta_2(\tau)1_{\tau=\tau_2}\big). \end{aligned}$$

  Additionally, we know that

$$U_1(s)\, dJ_1(s) + U_2(s)\, dJ_2(s) = 0,$$

  if $\tau > T$, and hence we arrive at the jump BSDE.
- *Case 2*. Assume that a default occurs between now and T, i.e. $t < \tau \leq T$. Then, for all $s < \tau$, we have $Y(s) = \widetilde{Y}(s)$, $Z(s) = \widetilde{Z}(s)$, and $U(s) = \theta(s) - Y(s)$.

The fundamental BSDE

- Using the fact that $(\widetilde{Y}, \widetilde{Z})$ solves the transformed BSDE, we obtain

$$
\begin{aligned}
Y(t) &= \widetilde{Y}(\tau) + \int_t^\tau f(s, Y(s), Z(s), U(s))ds - Z(s)\, dW(s) \\
&= \theta_1(\tau) 1_{\tau=\tau_1} + \theta_2(\tau) 1_{\tau=\tau_2} + \int_t^\tau f(s, Y(s), Z(s), U(s))ds \\
&\quad - \int_t^\tau Z(s)\, dW(s) - (\theta_1(\tau) 1_{\tau=\tau_1} + \theta_2(\tau) 1_{\tau=\tau_2} - \widetilde{Y}(\tau)),
\end{aligned}
$$

  for $t \in [0, \tau]$.
- Since $U = \theta - \widetilde{Y}$,

$$
\begin{aligned}
&\int_t^\tau U_1(s) dJ_1(s) + U_2(s)\, dJ_2(s) \\
&\qquad = U_1(\tau)(J_1(\tau) - J_1(\tau^-)) + U_2(\tau)(J_2(\tau) - J_2(\tau^-)) \\
&\qquad = (\theta_1(\tau) - \widetilde{Y}(\tau))(J_1(\tau) - J_1(\tau-)) + (\theta_2(\tau) - \widetilde{Y}(\tau))(J_2(\tau) - J_2(\tau-)) \\
&\qquad = \theta_1(\tau) 1_{\tau=\tau_1} + \theta_2(\tau) 1_{\tau=\tau_2} - \widetilde{Y}(\tau),
\end{aligned}
$$

  which again yields the jump BSDE.

The fundamental BSDE

- *Case 3*. The last case considers the situation when the default happens before or at time t, i.e. $\tau \leq t$. We have $Y(t) = \theta_1(\tau) 1_{\tau=\tau_1} + \theta_2(\tau) 1_{\tau=\tau_2}$.
- As a consequence, for $\tau \leq t$ we get

$$
\begin{aligned}
Y(t) &= \theta_1(\tau) 1_{\tau=\tau_1} + \theta_2(\tau) 1_{\tau=\tau_2} \\
&= 1_{\tau > T} \widetilde{\theta} + 1_{\tau \leq T} \big(\theta_1(\tau) 1_{\tau=\tau_1} + \theta_2(\tau) 1_{\tau=\tau_2} \big) \\
&\quad + \int_{t\wedge\tau}^{T\wedge\tau} f(s, Y(s), Z(s), U(s)) ds \\
&\quad - \int_{t\wedge\tau}^{T\wedge\tau} Z(s)\, dW(s) + U_1(s)\, dJ_1(s) + U_2(s)\, dJ_2(s),
\end{aligned}
$$

which is the jump BSDE written in the integrated form.

The fundamental BSDE

- The advantage of the transformed equation is that the jump process does not explicitly enter the dynamics.
- As a consequence, the numerical algorithm for solving FBSDEs discussed earlier can be used to this equation.
- Using the Feynman-Kac formula we can show that, in case of deterministic rates and default intensities, the BSDE approach is equivalent to the PDE approach discussed in Lecture Notes #11.

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