MTH 9831 Lecture 12

Note Title

1. A useful form of Itô-Doeblin formula ($d=1$). Itô's isometry for jump processes.
2. Itô-Doeblin formula for $d=2$. "Product rule" for jump processes.
3. Change of measure for
   (a) Poisson process;
   (b) Compound Poisson process when the jump distribution is supported on a finite set;
   (c) Compound Poisson process when the jump distribution has a density;
   (d) Compound Poisson process and Brownian motion.
4. Pricing a European call in a jump model.

(1) Let X be a jump process, that is

(JP) • $X(t) = \underbrace{X(0) + I(t) + R(t)}_{X^c(t)} + J(t)$, where

$$I(t) = \int_0^t \Gamma(s)\,dB(s)\ ;\quad R(t) = \int_0^t \Theta(s)\,ds.$$

Recall that:

$$\Delta X(t) = X(t) - X(t-) = J(t) - J(t-) = \Delta J(t)$$

$$[X,X](t) = [X^c,X^c](t) + [J,J](t) = \int_0^t \Gamma^2(s)\,ds + \sum_{0<s\le t} (\Delta J(s))^2.$$

Itô formula states that for a function $f \in C^2(\mathbb{R})$

$$(1)\quad f(X(t)) = f(X(0)) + \int_0^t f'(X(s-))\,dX^c(s) + \frac{1}{2}\int_0^t f''(s)\,d[X^c,X^c](s) + \sum_{0<s\le t}\big(f(X(s)) - f(X(s-))\big)$$

Add and subtract $\int_0^t f'(X(s-))\,dJ(s) = \sum_{0<s\le t} f'(X(s-))\,\Delta X(s)$

then (1) can be rewritten as

$$(2)\quad f(X(t)) = f(X(0)) + \int_0^t f'(X(s-))\,dX(s) + \frac{1}{2}\int_0^t f''(s)\,d[X^c, X^c](s) + \sum_{0<s\le t}\left(f(X(s)) - f(X(s-)) - f'(X(s-))\,\Delta X(s)\right)$$

Example 1. $M(t) = N(t) - \lambda t$; $f(x) = x^2$

Find $\operatorname{Var}\left(\int_0^t M(s-)\,dM(s)\right)$.

We know that $\int_0^t M(s-)\,dM(s)$ is a martingale. Thus $E\int_0^t M(s-)\,dM(s) \equiv 0$ and

$$(3)\quad \operatorname{Var}\left(\int_0^t M(s-)\,dM(s)\right) = E\left[\left(\int_0^t M(s-)\,dM(s)\right)^2\right].$$

At home you computed that

$$(4)\quad \int_0^t M(s-)\,dM(s) = \frac{1}{2}\left(M^2(t) - N(t)\right).$$

Therefore, we can compute the variance (3) from this identity. It is not difficult but is not very short. Let us compute (3) without using (4).

Set $X(t) = \int_0^t M(s-)\,dM(s)$

$$= \int_0^t M(s-)\,d(-\lambda s) + \sum_{0<s\le t} M(s-)\,\Delta N(s)$$

$$= X^c(t) + J(t).$$

Note that $[X^c, X^c](t) \equiv 0$. Now apply (2).

(5) $$X^2(t) = \int_0^t 2X(t-)\,dX(t) + \sum_{0<s\le t}\left((X(s-)+\Delta X(s))^2 - X^2(s-) - 2X(t-)\Delta X(t)\right)$$

$$= 2\int_0^t X(t-)\,dX(t) + \sum_{0<s\le t}(\Delta X(s))^2$$

$\uparrow$ martingale $\qquad$ $\uparrow$ $[J,J](t) = \sum_{0<s\le t} M^2(s-)\underbrace{(\Delta N(s))^2}_{=\Delta N(s)}$

$$= 2\int_0^t X(t-)\,dX(t) + \int_0^t M^2(s-)\,dN(s)$$

$$E(X^2(t)) = E\left(\int_0^t M^2(s-)\,dN(s)\right)$$

$$= E\left(\underbrace{\int_0^t M^2(s-)\,dM(s)}_{\text{martingale}} + \lambda\int_0^t M^2(s-)\,ds\right)$$

(6) $$= \lambda\int_0^t E M^2(s)\,ds = \lambda^2\int_0^t s\,ds = \frac{1}{2}\lambda^2 t^2$$

There is a general fact, which is a version of Itô's isometry.

Proposition 1. Let $X(t)$, $t\ge 0$, be a jump process of the form (JP) (see p. 1). Assume that $X(t)$, $t\ge 0$, is a martingale. Then

$$E(X^2(t)) - X^2(0) = E([X,X](t)) = E([X^c,X^c](t)) + E\sum_{0<s\le t}(\Delta J(s))^2$$

The proof is left as an exercise : apply Itô's formula to $x^2(t)$ and take expectations of both sides.

The same method which we used when we derived (6) can be used to show that for a left-continuous adapted square-integrable process Φ

$$E\left[\left(\int_0^t \Phi(s)\, dM(s)\right)^2\right] = \lambda \int_0^t E\Phi^2(s)\, ds .$$

More generally, if $M_Q(t) := Q(t) - \beta\lambda t$ where Q is a compound Poisson process, $\beta = EY_1$, then for a left-continuous adapted square-integrable process Φ

$$E\left[\left(\int_0^t \Phi(s)\, dM_Q(s)\right)^2\right] = \lambda\, E(Y_1^2) \int_0^t E\Phi^2(s)\, ds .$$

Proofs are left as exercises.

Now we turn to the case of more than 1 dimension

2 Theorem 1. Let X_1, X_2 be jump processes and $f(t, x_1, x_2)$ be a $C^{1,2}$ function.

Then

$$f(t,X_1(t),X_2(t)) = f(0,X_1(0),X_2(0)) + \int_0^t f_t(s,X_1(s),X_2(s))\,ds + \int_0^t f_{x_1}(s,X_1(s),X_2(s))\,dX_1^c(s)$$
$$+ \int_0^t f_{x_2}(s,X_1(s),X_2(s))\,dX_2^c(s) + \frac{1}{2}\int_0^t f_{x_1x_1}(s,X_1(s),X_2(s))\,d[X_1^c,X_1^c](s) + \int_0^t f_{x_1x_2}(s,X_1(s),X_2(s))\,d[X_1^c,X_2^c](s)$$
$$+ \frac{1}{2}\int_0^t f_{x_2x_2}(s,X_1(s),X_2(s))\,d[X_2^c,X_2^c](s)$$
$$+ \sum_{0<s\le t}\Big(f(s,X_1(s),X_2(s)) - f(s,X_1(s-),X_2(s-))\Big).$$

<u>Corollary</u> 1 (Itô's product formula for jump processes). Let X_1, X_2 be jump processes. Then

$$X_1(t)X_2(t) = X_1(0)X_2(0) + \int_0^t X_2(s)\,dX_1^c(s) + \int_0^t X_1(s)\,dX_2^c(s) + [X_1^c,X_2^c](t) + \sum_{0<s\le t}\big(X_1(s)X_2(s) - X_1(s-)X_2(s-)\big)$$
$$\overset{(*)}{=} X_1(0)X_2(0) + \int_0^t X_2(s-)\,dX_1(s) + \int_0^t X_1(s-)\,dX_2(s) + [X_1,X_2](t).$$

Proofs are omitted.

To see (*): $\sum_{0<s\le t}(X_1(s)X_2(s) - X_1(s-)X_2(s-)) = \sum_{0<s\le t}\big[(X_1(s-)+J_1(s))(X_2(s-)+J_2(s)) - X_1(s-)X_2(s-)\big] = \sum_{0<s\le t}\big(X_2(s-)J_1(s) + X_1(s-)J_2(s) + J_1(s)J_2(s)\big)$
$= \int_0^t X_2(s-)\,dJ_1(s) + \int_0^t X_1(s-)\,dJ_2(s) + [J_1,J_2](t).$

Example 2. $tN(t) = \int_0^t N(s)\,ds + \int_0^t s\,dN(s)$.
(cross variation is 0)

3 (a) Let $\tilde{\lambda} > 0$. At home, you have shown that

$$Z(t) := e^{(\lambda - \tilde{\lambda})t} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(t)}$$ satisfies

$$dZ(t) = \frac{\tilde{\lambda} - \lambda}{\lambda} Z(t-)\, dM(t)$$ and, in particular, is a martingale.

Fix $T > 0$ and define

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)\, d\mathbb{P} \quad \text{for all } A \in \mathcal{F}(T)$$

<u>Theorem 2</u>. Under $\tilde{\mathbb{P}}$ the process $N(t)$, $0 \le t \le T$, is Poisson with intensity $\tilde{\lambda}$.

Not a proof but an idea as to why the result holds:

$$\tilde{\mathbb{E}}(e^{uN(t)}) \underset{\substack{\uparrow \\ \text{Lemma 5.4 from Lecture 5}}}{=} \mathbb{E}(e^{uN(t)} Z(t)) =$$

$$= \mathbb{E}\left(e^{uN(t)} e^{(\lambda - \tilde{\lambda})t} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(t)}\right)$$

$$= e^{(\lambda - \tilde{\lambda})t}\, \mathbb{E}\, e^{(u + \ln(\tilde{\lambda}/\lambda))N(t)}$$

$$= e^{(\lambda - \tilde{\lambda})t} e^{\lambda t(e^{u + \ln(\tilde{\lambda}/\lambda)} - 1)}$$

$$= e^{(\lambda - \tilde{\lambda})t} e^{\lambda t(\frac{\tilde{\lambda}}{\lambda} e^u - 1)} = e^{\tilde{\lambda} t(e^u - 1)}$$

$\Rightarrow N(t) \sim \text{Poisson}(\tilde{\lambda} t)$.

Remark.

We checked out the distribution of $N(t)$ under $\tilde{\mathbb{P}}$, and saw that $N(t) \sim \text{Poisson}(\tilde{\lambda} t)$. But how does one figure out what should be $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$ if I have $N(t) \sim \text{Poisson}(\lambda t)$ under $\mathbb{P}$ and I want $N(t) \sim \text{Poisson}(\tilde{\lambda} t)$ under $\tilde{\mathbb{P}}$.

We did this in the refresher in general for finite state spaces (see pp. 5-6 of Lecture 3 (refresher)).

Countably infinite state spaces are treated in the same way.

Let $\Omega = \{0, 1, 2, \dots\}$, $\mathcal{F}$ = all subsets of Ω

$$\mathbb{P}(\omega) = p_\omega \quad ; \quad p_\omega > 0 \, ; \quad \sum_{\omega \in \Omega} p_\omega = 1$$

$$\mathbb{P}(A) = \sum_{\omega \in A} p_\omega \quad ; \quad \mathbb{E} X(\omega) = \sum_{\omega \in \Omega} X(\omega) p_\omega .$$

Want $\tilde{\mathbb{P}}(\omega) = \tilde{p}_\omega$, $\tilde{p}_\omega > 0$, $\sum_{\omega \in \Omega} \tilde{p}_\omega = 1$.

$$\tilde{\mathbb{P}}(A) = \sum_{\omega \in A} \tilde{p}_\omega = \sum_{\omega \in A} \frac{\tilde{p}_\omega}{p_\omega} p_\omega = \sum_{\omega \in A} Z(\omega) p_\omega$$

$$\tilde{\mathbb{E}} X(\omega) = \sum_{\omega \in \Omega} X(\omega) \tilde{p}_\omega = \sum_{\omega \in \Omega} X(\omega) \frac{\tilde{p}_\omega}{p_\omega} p_\omega$$

$$= \mathbb{E}\left(X(\omega) Z(\omega) \right), \quad Z(\omega) = \frac{\tilde{p}_\omega}{p_\omega}, \quad \omega \in \Omega$$

In our case $\omega = i$, $i = 0, 1, 2, \dots$

$$\frac{\tilde{p}_i}{p_i} = \frac{\frac{(\tilde{\lambda} t)^i}{i!} e^{-\tilde{\lambda} t}}{\frac{(\lambda t)^i}{i!} e^{-\lambda t}} = \left(\frac{\tilde{\lambda}}{\lambda} \right)^i e^{(\lambda - \tilde{\lambda}) t}$$

The claim of Theorem 2 is much more profound than this, since the change of measure is done not for a single fixed t, but for a whole process, on the space of paths of Poisson process. But the above tells you how to guess and quickly recover the answer.

(b) Let $Q(t) = \sum_{i=1}^{N(t)} Y_i$ and $\mathbb{P}(Y_1 = y_m) = p_m$,

$m = 1, 2, \dots, M$; $p_m \in (0,1)$ and $\sum_{m=1}^{M} p_m = 1$.

Recall that $N(t) = \sum_{m=1}^{M} N_m(t)$; $Q(t) = \sum_{m=1}^{M} y_m N_m(t)$

(see Lecture 11), where $N_1, \dots, N_M$ are independent Poisson processes with intensities $p_1\lambda, \dots, p_M\lambda$.

Set $\lambda_m := \lambda p_m$

<u>Remark</u>. Which parameters can we hope to change? We can change λ to $\tilde{\lambda}$ and p_m to $\tilde{p}_m$ $(m = 1, \dots, M)$ but we can not change y_m's, since then $\tilde{\mathbb{P}}$ will never be absolutely continuous w.r.t. $\mathbb{P}$. Say,

if $\mathbb{P}(Y_1 = 3) > 0$ and $\tilde{\mathbb{P}} \sim \mathbb{P}$ then
$\tilde{\mathbb{P}}(Y_1 = 3) > 0$ (can not be 0)

If $\mathbb{P}(Y_1 = 2) = 0$ and $\tilde{\mathbb{P}} \sim \mathbb{P}$ then
$\tilde{\mathbb{P}}(Y_1 = 2) = 0$ (can not be > 0)

Thus they have to "charge" the same collection of y_i's.

Let $\tilde{\lambda}$ and $\tilde{p}_1, \dots, \tilde{p}_M$ be given ($\tilde{p}_m > 0$ and $\sum_{m=1}^{M} \tilde{p} = 1$). Set $\tilde{\lambda}_m = \tilde{\lambda} \tilde{p}_m$

and define $Z_m(t) := e^{(\tilde{\lambda}_m - \lambda_m)t} \left(\dfrac{\tilde{\lambda}_m}{\lambda_m} \right)^{N_m(t)}$

$Z(t) := \prod_{m=1}^{M} Z_m(t)$. Fix $T > 0$.

Let $\tilde{\mathbb{P}}(A) = \int_A Z(T)\, d\mathbb{P} \quad \forall A \in \mathcal{F}(T)$

Theorem 3. Under $\tilde{\mathbb{P}}$ the process $Q(t)$, $0 \le t \le T$, is a compound Poisson process with intensity $\tilde{\lambda}$ and jump distribution $\tilde{\mathbb{P}}(Y_1 = y_m) = \tilde{p}_m$.

(c) Assume now that Y_1 has density $f(y)$. Let us guess how $Z(t)$ should look like in this case.

In (b) we had

$$Z(t) = \prod_{m=1}^{M} Z_m(t) = e^{\sum_{m=1}^{M} (\lambda_m - \tilde{\lambda}_m)t} \cdot \prod_{m=1}^{M} \left(\frac{\tilde{\lambda} \tilde{p}_m}{\lambda p_m} \right)^{N_m(t)}$$

$$= e^{(\tilde{\lambda} - \lambda)t} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda} \tilde{p}(Y_i)}{\lambda p(Y_i)},$$

where $\tilde{p}(Y_i) := \tilde{p}_m$, $p(Y_i) := p_m$ when the i-th jump is of size y_m;

Our guess then is that

$$Z(t) = e^{(\tilde{\lambda} - \lambda)t} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda} \tilde{f}(Y_i)}{\lambda f(Y_i)}$$

We again fix $T > 0$ and define

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)\, d\mathbb{P}, \quad \forall A \in \mathcal{F}(T)$$

Theorem 4. Under $\tilde{\mathbb{P}}$ the process $Q(t)$, $0 \le t \le T$, is a compound Poisson process with intensity

$\tilde{\lambda}$ and jump distribution $\tilde{\mathbb{P}}(Y_1 \le y) = \int_{-\infty}^{y} \tilde{f}(x)\,dx$.

(d) Suppose that on the same $(\Omega, \mathcal{F}, \mathbb{P})$ we have Brownian motion $B(t)$, $t \geq 0$, and a compound Poisson process $Q(t)$, $t \geq 0$. We assume that they are adapted to the same filtration $\mathcal{F}(t)$, $t \geq 0$. By Exercise 11.6 (see HW 11, (4)) we know that B and Q are independent.

We would like to change the measure in such a way that under $\tilde{\mathbb{P}}$

$$(7) \quad \tilde{B}(t) := B(t) + \int_0^t \theta(s)\,ds, \text{ where}$$

$\theta(s)$ is an adapted process, is a standard Brownian motion and Q is a compound Poisson process with intensity $\tilde{\lambda}$ and modified jump distribution. Moreover, we would like $\tilde{B}$ and Q be independent under $\tilde{\mathbb{P}}$.

Define $Z_1(t) = e^{-\int_0^t \theta(u)\,dB(u) - \frac{1}{2}\int_0^t \theta^2(u)\,du}$

$$Z_2(t) = \begin{cases} e^{(\lambda-\tilde{\lambda})t} \prod_{i=1}^{N(t)} \dfrac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)} & \text{if jumps have density } f \text{ and we want it to be } \tilde{f}. \\[2em] e^{(\lambda-\tilde{\lambda})t} \prod_{i=1}^{N(t)} \dfrac{\tilde{\lambda}\tilde{p}(Y_i)}{\lambda p(Y_i)} & \text{if } Y_1 \text{ has only finitely many jump sizes.} \end{cases}$$

$Z(t) = Z_1(t) Z_2(t)$. Fix $T > 0$

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)\, d\mathbb{P} \quad \forall A \in \mathcal{F}(T).$$

<u>Theorem 5</u>. Under $\tilde{\mathbb{P}}$ the process (7), $0 \le t \le T$ is a standard BM and $Q(t)$, $0 \le t \le T$, is a compound Poisson process with intensity $\tilde{\lambda}$ and jump distribution given by density $\tilde{f}$ or probability mass function $\tilde{p}_m$, $1 \le m \le M$. Moreover $\mathbb{P}$ and Q are independent.

(5) Let $dS(t) = \alpha S(t)\, dt + \sigma S(t)\, dB(t)$

(8) $\qquad + S(t-)\, d(Q(t) - \beta\lambda t)$ under $\mathbb{P}$.

The expected rate of return of this stock is α. We assume that possible jumps of Q are $-1 < y_1 < y_2 < \dots < y_M$, none of which is 0. $P(Y_1 = y_m) = p_m$, $m = 1, 2, \dots, M$.

<u>Theorem 6</u>. The solution to (8) is

$$S(t) = S(0) \underbrace{e^{\sigma B(t) + (\alpha - \beta\lambda - \frac{1}{2}\sigma^2)t}}_{X(t)} \underbrace{\prod_{i=1}^{N(t)} (Y_i + 1)}_{J(t)}$$

Proof. $S(t) = X(t) J(t)$

$$dX(t) = (\alpha - \beta\lambda) X(t)\, dt + \sigma(t) X(t)\, dB(t)$$

$$\Delta J(t) = J(t) - J(t-) = J(t-)\left(\frac{J(t)}{J(t-)} - 1\right)$$

$$= J(t-) \underbrace{Y_{N(t)} \Delta N(t)}_{\Delta Q(t)}$$

$$S(t) = X(t)J(t) = S(0) + \int_0^t X(s-)\,dJ(s) + \int_0^t J(s)\,dX(s)$$

$+ [X,J](t)$. Since J is a pure jump process and X is continuous, $[X,J](t) \equiv 0$.

$$S(t) = S(0) + \int_0^t X(s-)J(s-)\,dQ(s) + (\alpha - \beta\lambda)\int_0^t J(s)X(s)\,ds$$

$$+ \sigma \int_0^t J(s)X(s)\,dB(s)$$

In differential form,

(9) $$dS(t) = S(t-)\,dQ(t) + (\alpha-\beta\lambda)S(t)\,dt + \sigma S(t)\,dB(t)$$

which is exactly (8).

We want to construct $\tilde{\mathbb{P}}$ such that under $\tilde{\mathbb{P}}$

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\tilde{B}(t) + S(t-)\,d(Q(s) - \tilde{\beta}\tilde{\lambda}s), \text{ where } d\tilde{B}(t) = dB(t) + \theta\,dt$$

$$dS(t) = (r + \theta\sigma - \tilde{\beta}\tilde{\lambda})S(t)\,dt + \sigma S(t)\,dB(t) + S(t-)\,dQ(t)$$

Comparing with (9) we see that

$$r + \theta\sigma - \tilde{\beta}\tilde{\lambda} = \alpha - \beta\lambda$$

$$\theta\sigma + \beta\lambda - \tilde{\beta}\tilde{\lambda} = \alpha - r$$

$$\theta\sigma + \sum_{m=1}^{M} y_m p_m \lambda + \sum_{m=1}^{M} y_m \tilde{p}_m \tilde{\lambda} = \alpha - r$$

(10) $$\theta\sigma + \sum_{m=1}^{M} y_m(\lambda_m - \tilde{\lambda}_m) = \alpha - r$$

We have 1 equation and $M+1$ unknowns $\theta, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_M$ $\left(\tilde{\lambda} = \tilde{\lambda}_1 + \cdots + \tilde{\lambda}_M ;\ \tilde{p}_m = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}, m = 1, \ldots, M \right)$.

$\Rightarrow$ the model is incomplete.

These extra parameters can be used to calibrate the model to market prices.

Pick any one solution of (10). Under $\tilde{\mathbb{P}}$

$$S(t) = S(0)\, e^{\sigma \tilde{B}(t) + (r - \tilde{\beta}\tilde{\lambda} - \frac{1}{2}\sigma^2)t} \prod_{i=1}^{N(t)} (Y_i + 1)$$

where N has intensity $\tilde{\lambda}$ under $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{P}}(Y_1 = y_m) = \tilde{p}_m = \tilde{\lambda}_m / \tilde{\lambda}$.

The call price at time t is now

$$e^{-r(T-t)} \tilde{\mathbb{E}}\left((S(T) - K)_+ \mid \mathcal{F}(t) \right).$$

There is an explicit expression (see Shreve II, Theorem 11.7.5). But I shall not state or derive it.

<u>Theorem 6</u>. Let $c(t,x)$ be the price of a call at time t when $S(t) = x$. Then $c(t,x)$ satisfies

$$-rc(t,x) + c_t(t,x) + (r - \tilde{\beta}\tilde{\lambda})\, x\, c_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) + \tilde{\lambda} \sum_{m=1}^{M} \left[\tilde{p}_m\, c(t, (y_m + 1)x) - c(t,x) \right] = 0$$

$0 \le t < T$, $x > 0$, and the terminal condition

$$c(T,x) = (x - K)_+, \quad x \geq 0.$$

Proof is left as an exercise. The process $e^{-rt} c(t, S(t))$

should be a martingale. Apply Itô-Doeblin formula and set the "dt" term to zero. The details are given in Shreve II, proof of Theorem 11.7.7.