9. The Black-Scholes PDE and Greeks of Vanilla Options

A deterministic price process

Recall that the lognormal model of an asset price led us to the following expression of the geometric Brownian motion governing its dynamics:

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t$$

We are often interested in determining the dynamics of a function of this process—in particular, we may be interested in the process followed by a derivative of this underlying security whose pricing formula is V.

To begin this discussion, consider a function $V(t)$ alone. Assume for the moment that the function is twice continuously differentiable in some neighborhood of t_0. Then we may express the value of V…

$$V(t_0+\Delta t) = V(t_0) + V'(t_0)\Delta t + \frac{1}{2}V''(t_0)\Delta t\Delta t + \varepsilon$$

…making the discrete increment in V over the period Dt…

$$\Delta V = V_t \Delta t + \frac{1}{2}V_{tt}\Delta t\Delta t + \varepsilon$$

…where V_t and V_{tt} represent the first and second derivatives with respect to t. To reach the result we are familiar with in this case, we allow the length of the increment to approach zero. In this case, the fact that V is twice continuously differentiable assures us that, at the limit, ε approaches zero. So we disregard this term at the limit and, for the moment, use the somewhat odd notation:

$$dV = V_t dt + \frac{1}{2}V_{tt} dtdt$$

It is a commonplace of calculus that, as the finite increment approaches zero, its square approaches zero "faster"—so much so that in the application of the result, we may disregard whether our function V has a second derivative at all at the point t_0. Thus, in terms of the notation above, we effectively conclude:

$$dtdt = 0$$

$$dV = V_t dt$$

This is simply the chain rule.

Suppose for a moment that we set the volatility to zero in the process of the underlying asset, thus making its path nonrandom, and consider the differential of a pricing function $V(t, S_t)$ in a similar manner:

$$dS_t = (r-q)S_t dt$$

$$V(t_0+\Delta t, S_0+\Delta S) = V(t_0,S_0) + V_t(t_0,S_0)\Delta t + V_S(t_0,S_0)\Delta S +$$

$$\frac{1}{2}V_{tt}(t_0,S_0)\Delta t\Delta t + V_{tS}(t_0,S_0)\Delta t\Delta S + \frac{1}{2}V_{SS}(t_0,S_0)\Delta S\Delta S + \varepsilon$$

$$dV = V_t dt + V_S dS + \frac{1}{2}V_{tt}dtdt + V_{tS}dtdS + \frac{1}{2}V_{SS}dSdS$$

Again, we make use of the notational convenience we have established, and can for reasons as outlined above make the simplification:

$$dtdt = dtdS = dSdS = 0$$

$$dV = V_t dt + V_S dS$$

…in the case when S is nonrandom. This is, once again, the chain rule—now in two variables.

As an illustration of the result above, take the case where V is the value of a forward contract. From our discussions before, it is clear that this formula still holds in the case when the evolution of S is nonrandom. So…

$$V = S_t e^{-q(T-t)} - Ke^{-r(T-t)}$$

$$V_t = qS_t e^{-q(T-t)} - rKe^{-r(T-t)}$$

$$V_S = e^{-q(T-t)}$$

$$dV = V_t dt + V_S dS = \left[qS_t e^{-q(T-t)} - rKe^{-r(T-t)}\right]dt + e^{-q(T-t)}\left[(r-q)S_t dt\right]$$

$$dV = \left[qS_t e^{-q(T-t)} - rKe^{-r(T-t)} + rS_t e^{-q(T-t)} - qS_t e^{-q(T-t)}\right]dt$$

$$dV = r\left[S_t e^{-q(T-t)} - Ke^{-r(T-t)}\right]dt = rVdt$$

…meaning that, in this case, the value of the forward contract grows constantly at the risk-free rate. Certainly this is what we have come to expect from our earlier discussions of risk-neutral pricing.

Ito's Lemma

In order to extend the result above to the case of geometric Brownian motion, we must consider the behavior of the increment dW_t in this situation. Recall that we introduced this as an infinitesimal "normal" increment independent of all other increments with the property that…

$$\int_t^{t+\Delta t} dW_\tau = W_{t+\Delta t} - W_t = \Delta W = Z\sqrt{\Delta t}$$

…is normally distributed with mean zero and variance Δt. In essence, this becomes an extension of the result for a sum of a finite number of independent normal random variables to the continuous case. We wish to know how to work with some function f of this Brownian increment:

$$f(W_t + \Delta W) = f(W_t) + f'(W_t)\Delta W + \frac{1}{2}f''(W_t)\Delta W\Delta W + \varepsilon$$

$$\Delta f = f_W \Delta W + \frac{1}{2}f_{WW}\Delta W\Delta W + \varepsilon$$

This is a rather different proposition from the case above. We are interested in the value of this increment at the infinitesimal limit of a change in W, which is, however, random. A formal proof of the outcome is beyond the scope of these notes, as is the reassurance that the error term ε can be disregarded at the infinitesimal limit. Nevertheless, we can gain an intuition of the result by considering the expected value of the finite increment above, after dropping the error term…

$$E[\Delta f] = f_W E[\Delta W] + \frac{1}{2} f_{WW} E\left[(\Delta W)^2\right]$$

$$E[\Delta f] = \frac{1}{2} f_{WW} \Delta t$$

At the limit, this makes it seem reasonable that in fact the proper expression of the increment is:

$$df = f_W dW + \frac{1}{2} f_{WW} dt$$

This can be seen as an extension of the chain rule for the special case of a Wiener process. The first term of this sum is the only one needed for nonrandom W: The change in the function is simply the product of the underlying's instantaneous change and f's first-order sensitivity to that change.

The second term is a deterministic drift that arises from the random character of the increment. Evidently, this term serves as a correction in the mean of the random variable df to account for the curvature of the function f. In the case of a nonrandom increment dx, no such correction is necessary since:

$$E[f(x+dx)] = f(x+dx) = f(E[x+dx])$$

However, in the case of a random increment dX when the function f is convex…

$$E[f(x+dX)] > f(E[x+dX])$$

…by Jensen's inequality, and by extension the reverse is true when f is concave. Thus, the drift adjustment indicated above adjusts the mean upward when the function is locally convex—that is, when its second derivative is positive—and does the reverse when the function is concave.

For a function of both t and W_t, we obtain the expression…

$$df = f_t dt + f_W dW + \frac{1}{2} f_{tt} dt dt + f_{tW} dt dW + \frac{1}{2} f_{WW} dW dW$$

…at second order. We have already seen in the deterministic case that $dtdt$ is zero. What should we do with the $dtdW$ term? Since the term dt is nonrandom, at the discrete approximation we see that…

$$E[f_{tW} \Delta t \Delta W] = f_{tW} \Delta t E[\Delta W] = 0$$

…and conclude that this term does not demand any adjustment to the function's drift, at least. Moreover, as in the case of $\Delta t \Delta t$ above, we would expect to observe that the product of the random increment with the nonrandom increment shrinks substantially faster than ΔW or Δt do by themselves. Indeed, this intuition is correct, and we will treat the product of increments $dtdW$ as zero as well, making the final statement of the differential:

$$df = f_t dt + f_W dW + \frac{1}{2} f_{WW} dt$$

We now have seen all the cases necessary to understand how to extend the above to a general stochastic process. For a function V of both t and S, we will state the result in differential form as:

$$dV = V_t dt + V_S dS + \frac{1}{2} V_{SS} dSdS$$

This is known as Ito's Lemma, and is a key result for analysis of the continuous-time dynamics of derivative prices and other functions of an asset price. For example, we can use it to determine the instantaneous dynamics of the log of S. First, we determine the needed derivatives:

$$\frac{\partial}{\partial t} \ln S = 0$$

$$\frac{\partial}{\partial S} \ln S = \frac{1}{S}$$

$$\frac{\partial^2}{\partial S^2} \ln S = -\frac{1}{S^2}$$

So…

$$d \ln S = 0dt + \frac{1}{S} dS - \frac{1}{2} \frac{1}{S^2} dSdS$$

From our risk-neutral lognormal evolution of S, we see that…

$$dS = (r-q)Sdt + \sigma SdW$$

$$dSdS = (r-q)^2 S^2 dtdt + 2(r-q)\sigma S^2 dtdW + \sigma^2 S^2 dWdW$$

$$dSdS = \sigma^2 S^2 dt$$

…since the product of the dW terms is the only one that survives in the expression of $dSdS$. Then our differential becomes…

$$d \ln S = \frac{1}{S}\left[(r-q)Sdt + \sigma SdW\right] - \frac{1}{2}\frac{1}{S^2}\sigma^2 S^2 dt$$

$$d \ln S = \left(r - q - \frac{1}{2}\sigma^2\right)dt + \sigma dW$$

…which is exactly the expression we arrived at from our lognormal model.

Self-financing replication revisited

Earlier, we saw in the context of our binomial model that options could be replicated using a portfolio of stock and cash rebalanced at each time step. The above allows us to consider whether the same is true in continuous time, and the answer to this question yields at least one interesting result.

For the sake of this discussion, we will dispense with the risk-neutral requirement for the underlying asset S. We suppose that it evolves according to geometric Brownian motion with some arbitrary constant drift rate μ and constant volatility σ. We also suppose that the asset pays continuous dividends at the rate q. Thus, its price follows the process:

$$dS = \mu Sdt + \sigma SdW$$

We wish to replicate a derivative dependent upon S whose value is given by the function $V(t, S)$. By Ito's lemma as described above, the increment of the derivative's value is:

$$dV = V_t dt + V_S dS + \frac{1}{2} V_{SS} dSdS$$

$$dV = V_t dt + V_S \left(\mu S dt + \sigma S dW \right) + \frac{1}{2} V_{SS} \sigma^2 S^2 dt$$

$$dV = \left(V_t + \mu S V_S + \frac{1}{2} \sigma^2 S^2 V_{SS} \right) dt + \sigma S V_S dW$$

To replicate this increment, we begin with a portfolio with equal value $\Pi = V$ consisting of Δ shares of the equity, borrowing from or depositing excess cash into the money market account as needed. The increment in value thus arises from three sources: the random change in the value of the equity position, the growth of the money market position at the risk-free rate r, and the dividends received or owed through the equity position:

$$d\Pi = \Delta dS + r\left(\Pi - \Delta S\right) dt + q \Delta S dt$$

$$d\Pi = \Delta\left(\mu S dt + \sigma S dW \right) + \left(r\Pi - \left(r - q\right) \Delta S \right) dt$$

$$d\Pi = \left(r\Pi + \mu \Delta S - \left(r - q\right) \Delta S \right) dt + \Delta \sigma S dW$$

To perform the replication, we as mentioned above require that the initial value of the portfolio and the initial value of the option be the same. Then, we must choose the replicating portfolio so that the increments are equal as well. To do this, we treat the deterministic and random components of the process separately. First, if we equate the random portions, we see that:

$$\sigma S V_S = \Delta \sigma S$$

$$\Delta = V_S$$

So, in order to offset the random variation in S, our shares position must be the derivative of the option's value with respect to S, which seems sensible.

With this information in hand, we can then equate the deterministic portions:

$$V_t + \mu S V_S + \frac{1}{2} \sigma^2 S^2 V_{SS} = rV + \mu S V_S - \left(r - q\right) S V_S$$

$$V_t + \left(r - q\right) S V_S + \frac{1}{2} \sigma^2 S^2 V_{SS} - rV = 0$$

This result is better known as the Black-Scholes PDE.

As is evident from our derivation, this PDE is the one that an option valuation formula must obey in order for its value to be arbitrage-free under the dynamics we have posited for S. As we have already seen above, clearly our valuation formula for a forward contract satisfies this equation:

$$V = Se^{-q(T-t)} - Ke^{-r(T-t)}$$
$$V_t = qSe^{-q(T-t)} - rKe^{-r(T-t)}$$
$$V_S = e^{-q(T-t)}$$
$$V_{SS} = 0$$
$$V_t + (r-q)SV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV =$$
$$qSe^{-q(T-t)} - rKe^{-r(T-t)} + (r-q)Se^{-q(T-t)} - r\left(Se^{-q(T-t)} - Ke^{-r(T-t)}\right) =$$
$$r\left(Se^{-q(T-t)} - Ke^{-r(T-t)}\right) - r\left(Se^{-q(T-t)} - Ke^{-r(T-t)}\right) = 0$$

This is interesting because our formula for a forward contract was obtained by taking the discounted *risk-neutral* expectation of the asset price at expiry, which as we have seen before posits that the growth rate of the underlying equity is r, the risk-free rate. Yet, approaching the problem using this method, we made no such assumption about its growth rate: the result evidently holds even when μ is some other rate.

This derivation, then, provides another insight into the rationale behind risk-neutral pricing. Earlier, we saw that the risk-free rate is the rate we must assume for the growth of the asset in order for the forward contract's value to be arbitrage-free; we now see further that this idea applies for an arbitrary derivative of the asset. The risk-neutral measure is then an alternate view of the asset's dynamics that we must employ in order to replicate the option's payoff under all circumstances. It makes no statement about the real-world growth of the asset's price, however we understand that concept.

Anatomy of the Black-Scholes PDE

In effect, the Black-Scholes PDE is itself a statement of a no-arbitrage requirement. As we have seen before, if we have some cash amount V, in our risk-neutral framework it should not matter in which traded asset we invest this amount; while their payoffs may vary, all ought to grow in expectation at the same risk-free rate r.

Thus, if we rewrite the PDE this way…

$$rV = V_t + (r-q)SV_s + \frac{1}{2}\sigma^2 S^2 V_{SS}$$

…then we see that the left-hand side of this equation is the growth expected of the cash amount V per unit time at the risk-free rate. The right-hand side can then be understood to pertain to the derivative itself.

The first term is the change in the value of the derivative through time; for many options, this value is negative, and this term is sometimes referred to as the "time decay" of the instrument. More commonly, the value of this quantity goes by the name theta.

The second term is the expected growth in our effective position in the underlying asset through the derivative. As we saw above, the first derivative of the instrument's value with respect to S is also the quantity of the underlying we would hold in a replicating portfolio. This, like any cash asset, should grow at the risk-free rate r, less any dividend rate q that the underlying may pay. This degree of exposure to the underlying is known as delta.

The final term is the only one involving volatility. Again thinking in terms of what we have seen so far, another aspect of replicating this derivative is the continuous rebalancing of the replicating portfolio. Seen this way, one of the purposes of buying a derivative instrument is to provide the value accompanying continuous rebalancing without requiring it to be actually done, which in practical fact is physically impossible and, even if it were possible, would be prohibitively expensive. The derivative V_{SS} is known in options theory as gamma, and it can be thought of as relating to the PL associated with the dynamic replication process. Note that for a forward contract, which is linear, no rebalancing is required—it can be replicated using a static hedge—and its gamma is, not coincidentally, zero.

Greeks of option positions

The price of an option, as we have repeatedly seen, depends upon the values of a variety of market variables. As a result, the price itself, while important, tends to be an unwieldy quantity for a market participant to use in determining the nature of the exposure entailed by an option or a collection of options. Moreover, since most market participants who trade options do so in an active way, there is a need to aggregate exposure to the various factors of importance to an options book.

The method most commonly used to determine these exposures is a collection of values known as option "Greeks." We have already seen three of them above: theta, delta, and gamma. The linearity of derivatives makes such values particularly useful for aggregating exposures, since the net delta of an options book with respect to a particular underlying asset is simply the sum of the individual positions' deltas, and the same applies to other Greeks as well. Thus, they provide an excellent means for market participants to hedge unusual derivatives through trading in more vanilla instruments.

Delta

As we have seen above, the delta of an option is its first derivative with respect to the price of the underlying asset. A delta hedge is thus a means of immunizing a portfolio against relatively small changes in the price of the underlying asset.

For a vanilla European call, the Black-Scholes valuation formula is…

$$V_C = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2)$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T\right)$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

…making an initial expression of its delta:

$$\Delta_C = e^{-qT}N(d_1) + Se^{-qT}N'(d_1)\frac{d}{dS}d_1 - Ke^{-rT}N'(d_2)\frac{d}{dS}d_2$$

We observe that…
$$\frac{d}{dS}d_2 = \frac{d}{dS}\left(d_1 - \sigma\sqrt{T}\right) = \frac{d}{dS}d_1$$
…and further consider the expression…
$$Ke^{-rT}N'(d_2) = Ke^{-rT}N'\left(d_1 - \sigma\sqrt{T}\right) = Ke^{-rT}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(d_1^2 - 2\sigma\sqrt{T}d_1 + \sigma^2 T\right)} =$$
$$Ke^{-rT}e^{\sigma\sqrt{T}d_1 - \frac{\sigma^2}{2}T}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2} = Ke^{-rT}e^{\ln\left(\frac{S}{K}\right)}e^{(r-q)T}N'(d_1) =$$
$$Se^{-qT}N'(d_1)$$
…thus making our delta:
$$\Delta_C = e^{-qT}N(d_1) + Se^{-qT}N'(d_1)\frac{d}{dS}d_1 - Se^{-qT}N'(d_1)\frac{d}{dS}d_1$$
$$\Delta_C = e^{-qT}N(d_1)$$

For a European put, it is often most convenient to make use of put-call parity:
$$F_K = Se^{-qT} - Ke^{-rT} = V_C - V_P$$
$$e^{-qT} = \Delta_C - \Delta_P$$
$$\Delta_P = e^{-qT}N(d_1) - e^{-qT}$$
$$\Delta_P = -e^{-qT}\left(1 - N(d_1)\right) = -e^{-qT}N(-d_1)$$

Gamma

As mentioned above, a delta hedge protects against relatively small changes in the price of the underlying. However, "relatively small" can mean different things for different positions. If the delta of the option is insensitive to changes in the underlying, then the delta hedge need not be rebalanced often, and a delta hedge is both reliable and economical. For some positions, however, the delta may change radically even for small changes in the spot price; delta hedges of such options need to be adjusted more often. In some cases, the variation may be so extreme that delta hedging is almost totally impractical.

As we have seen above, this concept is covered by gamma, the second derivative of the option's value with respect to the spot price. Gamma can be thought of as an indication of the cost or gain associated with rebalancing a delta hedge.

For a vanilla call option, we see that…
$$\Gamma_C = \frac{d^2}{dS^2}V_C = \frac{d}{dS}\Delta_C = \frac{d}{dS}e^{-qT}N(d_1)$$
$$\Gamma_C = e^{-qT}N'(d_1)\frac{d}{dS}d_1 = e^{-qT}N'(d_1)\frac{d}{dS}\left[\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T\right)\right]$$
$$\Gamma_C = \frac{1}{S\sigma\sqrt{T}}e^{-qT}N'(d_1)$$

…and as before, we can use put-call parity to determine the same for puts…
$\Gamma_P = \Gamma_C - \Gamma_{F_K} = \Gamma_C$
…since, as discussed above, the forward contract is a linear payoff and thus has zero gamma.

Vega
In a sense, this extremely important Greek is a contradiction in terms. Our specified dynamics of the underlying relied explicitly on the idea that volatility was a constant. This assumption can be easily extended to the case when volatility is solely a nonrandom function of time. However, it is clear that in practice, the instantaneous volatility of an underlying is not simple to measure and is most likely random. Thus, the valuation formulas we have used for European puts and calls do not reflect the true dynamics of the underlying asset.

This fact is actually not such a tragedy once it is acknowledged and accepted. While the Black-Scholes volatility may not be altogether correct, nevertheless it offers a convenient and intuitive statistic to use in comparing options on the same underlying to one another. In essence, the Black-Scholes implied volatility—like bond yield in the fixed income space—is a convenient way of representing the instrument's price. Still, since the relationship between the implied volatility and price is complex, it is necessary to know the price impact of a change in volatility.

For a European call, we have:
$$vega_C = \frac{d}{d\sigma}V_C = Se^{-qT}N'(d_1)\frac{d}{d\sigma}d_1 - Ke^{-rT}N'(d_2)\frac{d}{d\sigma}d_2$$

From the above discussion of delta, we saw that…
$$Ke^{-rT}N'(d_2) = Se^{-qT}N'(d_1)$$
…and we furthermore observe that…
$$\frac{d}{d\sigma}d_2 = \frac{d}{d\sigma}\left(d_1 - \sigma\sqrt{T}\right) = \frac{d}{d\sigma}d_1 - \sqrt{T}$$
…which means that…
$$vega_C = Se^{-qT}N'(d_1)\left(\frac{d}{d\sigma}d_1 - \frac{d}{d\sigma}d_1 + \sqrt{T}\right) = S\sqrt{T}e^{-qT}N'(d_1)$$

And as above with gamma, since the forward contract has no sensitivity to volatility, we see that…
$$F_K = V_C - V_P$$
$$0 = vega_C - vega_P$$
$$vega_P = vega_C$$

From the standpoint of vol sensitivity, then, calls and puts at the same strike are essentially the same instrument.

Theta

As we saw above in the Black-Scholes PDE, the time decay of an option is one of its fundamental characteristics. For vanilla calls and puts, since the payoffs are nonnegative and convex, the time value of these options is always positive and in ordinary cases increases with increasing time to expiry. Thus, if we consider the case of a vanilla call option:

$$\frac{d}{dT}V_C = \frac{d}{dT}\left(Se^{-qT}N(d_1) - Ke^{-rT}N(d_2)\right) =$$

$$-qSe^{-qT}N(d_1) + Se^{-qT}N'(d_1)\frac{d}{dT}d_1 + rKe^{-rT}N(d_2) - Ke^{-rT}N'(d_2)\frac{d}{dT}d_1$$

We use once again the relationship found in our derivation of delta…

$$Ke^{-rT}N'(d_2) = Se^{-qT}N'(d_1)$$

…and the fact that…

$$\frac{d}{dT}d_2 = \frac{d}{dT}\left(d_1 - \sigma\sqrt{T}\right) = \frac{d}{dT}d_1 - \frac{\sigma}{2\sqrt{T}}$$

…giving…

$$\frac{d}{dT}V_C = -r\left(Se^{-qT}N(d_1) - Ke^{-rT}N(d_2)\right) + (r-q)Se^{-qT}N(d_1) + Se^{-qT}N'(d_1)\left(\frac{d}{dT}d_1 - \frac{d}{dT}d_1 + \frac{\sigma}{2\sqrt{T}}\right)$$

$$\frac{d}{dT}V_C = -r\left(Se^{-qT}N(d_1) - Ke^{-rT}N(d_2)\right) + (r-q)Se^{-qT}N(d_1) + \frac{\sigma S}{2\sqrt{T}}e^{-qT}N'(d_1)$$

$$\frac{d}{dT}V_C = -rV_C + (r-q)S\Delta_C + \frac{\sigma S}{2\sqrt{T}}e^{-qT}N'(d_1)$$

$$\frac{d}{dT}V_C = -rV_C + (r-q)S\Delta_C + \frac{1}{2}\sigma^2 S^2 \frac{1}{S\sigma\sqrt{T}}e^{-qT}N'(d_1)$$

$$\frac{d}{dT}V_C = -rV_C + (r-q)S\Delta_C + \frac{1}{2}\sigma^2 S^2\Gamma_C$$

Clearly, this is not exactly theta of the call option, since our derivative here is with respect to the time to expiry *T*. As time passes, this value decreases rather than increasing, so we must reverse the sign of this result:

$$\Theta_C = -\frac{d}{dT}V_C = rV_C - (r-q)S\Delta_C - \frac{1}{2}\sigma^2 S^2\Gamma_C$$

This may not be the most computationally efficient expression for theta of a call option, but it does provide verification of an important fact: The pricing formula for a call option satisfies the Black-Scholes PDE. Clearly, then, the equivalent formula holds for a European put.

Rho

While usually given less consideration than the other risk factors seen so far, the risk-free interest rate also has an influence on the value of the option. Rho is the Greek used to indicate this sensitivity. For a European call, it is…

$$\mathrm{P}_C = \frac{d}{dr}V_C = Se^{-qT}N'(d_1)\frac{d}{dr}d_1 - Ke^{-rT}N'(d_2)\frac{d}{dr}d_2 + TKe^{-rT}N(d_2)$$

…and since…

$$\frac{d}{dr}d_2 = \frac{d}{dr}\left(d_1 - \sigma\sqrt{T}\right) = \frac{d}{dr}d_1$$

…as with our delta derivation, the first two terms cancel, leaving…

$$\mathrm{P}_C = TKe^{-rT}N\left(d_2\right)$$

This seems a sensible result: Rho is the time to expiry multiplied by the present value of the strike weighted by the risk-neutral probability of exercise. Since in a call option, the holder pays the strike at exercise, an increase in *r* results in a gain for the holder of a call. By analogy, one would expect rho of a put option to be…

$$\mathrm{P}_P = -TKe^{-rT}N\left(-d_2\right)$$

…which can easily be verified through put-call parity.

Other Greeks

The sensitivities described here only reflect the major ones that are most often reported. There is a whole assortment of other sensitivities that may be useful to certain practitioners depending upon their purposes in accessing the options market. Determining which sensitivities are of importance in a particular context is part of the day-to-day—or minute-to-minute—practice of risk management in an options trading operation.