Solutions to MTH 9831 Spring 2015 HW #4 (Itô handouts)

Please note that these are *possible* solutions, and that there are often many ways to approach and solve a problem or prove a theorem.

Problem 1 is done here for use in Problem 8.

(a)
$$d(B^2(t)) = 2B(t)dB(t) + \frac{1}{2}(2dt) = 2B(t)dB(t) + dt.$$

(b)
$$d(tB(t)) = tdB(t) + B(t)dt.$$

(c) Let $f(t,x) = (x+t)e^{-x-t/2}$. Then

$$\begin{aligned}
f_t(t,x) &= -\frac{1}{2}f(t,x) + e^{-x-t/2} \\
f_x(t,x) &= -f(t,x) + e^{-x-t/2} \\
f_{xx}(t,x) &= -f_x(t,x) - e^{-x-t/2} = f(t,x) - 2e^{-x-t/2} \\
\implies df(t,B(t)) &= \left(f_t + \frac{1}{2}f_{xx}\right)dt + f_x dB(t) \\
&= e^{-B(t)-t/2}\left\{\left[\left(-\frac{1}{2}(B(t)+t)\right) + 1 + \frac{1}{2}((B(t)+t) - 2)\right]dt + (1 - B(t) - t)dB(t)\right\} \\
&= (1 - (B(t)+t))e^{-B(t)-t/2}dB(t).
\end{aligned}$$

(d)
$$\begin{aligned}
d\left[t^2B(t) - 2\int_0^t uB(u)du\right] &= d[t^2B(t)] - 2d\left[\int_0^t uB(u)du\right] \\
&= 2tB(t)dt + t^2dB(t) - 2tB(t)dt = t^2dB(t).
\end{aligned}$$

(e) For $dS(t) = \nu S(t)dt + \sigma S(t)dB(t)$ a GBM,

$$\begin{aligned}
d(\log(S(t))) &= \frac{dS(t)}{S(t)} - \frac{1}{2}\frac{dS(t)dS(t)}{S(t)^2} \\
&= \nu dt + \sigma dB(t) - \frac{1}{2}\sigma^2 dt = \left(\nu - \frac{1}{2}\sigma^2\right)dt + \sigma dB(t).
\end{aligned}$$

(f) The differential of the exponential martingale $X(t) = \exp\left(\int_0^t \Delta(u)dB(u) - \frac{1}{2}\int_0^t \Delta^2(u)du\right)$ is

$$dX(t) = X(t)\left[-\frac{1}{2}\Delta^2(t) + \Delta(t)dB(t) - \frac{1}{2}\Delta^2(t)dt\right] = \Delta(t)X(t)dB(t).$$

Problem 6 Given $dS(t) = \sigma S(t)dB(t)$ (which we should know as a GBM known as an exponential martingale), $S(0) = A$, we use two approaches to find $S(t)$ (even though we see the answer in #1(f)):

(a) Apply Ito's formula to $S^2(t)$ and take the expected value: letting $m_k(t) := E[S^k(t)]$,

$$\begin{aligned}
d(S^2(t)) &= 2S(t)dS(t) + dS(t)dS(t) = 2\sigma S^2(t)dB(t) + \sigma^2S^2(t)dt \\
\implies E(S^2(t) - S^2(0)) &= 2\sigma E\left[\int_0^t S^2(u)dB(u)\right] + \sigma^2 E\left[\int_0^t S^2(u)du\right] \\
\implies m_2(t) &= \sigma^2A^2 + \sigma^2\int_0^t m_2(u)du \\
\implies m_2'(t) &= \sigma^2 m_2(t),\ m_2(0) = A^2.
\end{aligned}$$

The ODE results in $m_2(t) = A^2 e^{\sigma^2 t}$, and so, since

$$E[S(t) - S(0)] = E\left[\int_0^t dS(u)\right] = E\left[\int_0^t \sigma S(u) dB(u)\right] = 0,$$

we have $E[S(t)] = A$ and therefore $\mathrm{Var}(S(t)) = A^2(e^{\sigma^2 t} - 1)$.

(b) Starting with $S(t) = Ae^{\sigma B(t) - \sigma^2 t/2}$, we can easily find, via $S^2(t) = A^2 e^{2\sigma B(t) - \sigma^2 t}$ and some calculus, the same result with a square completion. Using the density of $B(t) \sim N(0,t)$,

$$\begin{aligned} E(S^2(t)) &= \int_{-\infty}^{\infty} A^2 e^{2\sigma x - \sigma^2 t} f_{B(t)}(x) dx = \frac{A^2}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{2\sigma x - \sigma^2 t - (x^2/2t)} dx \\ &= \frac{A^2 e^{\sigma^2 t}}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{(x - 2\sigma t)^2/2t} dx = A^2 e^{\sigma^2 t}. \end{aligned}$$

Either way, $\mathrm{Var}(S(t)) = E(S^2(t)) - E(S^2(t))^2 = A^2(e^{\sigma^2 t} - 1)$.

Problem 7 Now for the mean and variance of $\int_0^t S(u)du$: by the IBP trick, being careful with the fact that $S(0) = A$, we see that

$$\begin{aligned} tS(t) &= t[S(t) - S(0) + S(0)] = t\left(\int_0^t dS(u) + A\right) \\ \Longrightarrow\ E[tS(t)] &= E\left[t\left(\int_0^t dS(u) + A\right)\right] = \sigma t E\left[\int_0^t S(u) dB(u)\right] + At = At. \end{aligned}$$

Thus, differentiating, integrating, and taking expectations,

$$\begin{aligned} &d(tS(t)) = t\, dS(t) + S(t)\, dt \\ &\quad \Longrightarrow tS(t) - 0S(0) = tS(t) = \int_0^t u dS(u) + \int_0^t S(u) du \\ &\quad \Longrightarrow \int_0^t S(u) du = tS(t) - \int_0^t u dS(u) = tS(t) - \sigma \int_0^t u S(u) dB(u) \\ &\quad \Longrightarrow E\left[\int_0^t S(u) du\right] = E[tS(t)] = At. \end{aligned}$$

The variance requires the second moment: by the result above, the Itô isometry (Tool C), and Problem 6,

$$\begin{aligned} E\left[\left(\int_0^t S(u)du\right)^2\right] &= E\left[\left(tS(t) - \sigma \int_0^t uS(u) dB(u)\right)^2\right] \\ &= t^2 E[S^2(t)] - 2\sigma t E\left[S(t) \int_0^t uS(u)dB(u)\right] + \sigma^2 \int_0^t u^2 E\left[S^2(u)\right] du \\ &= (At)^2 e^{\sigma^2 t} - 2\sigma t E\left[S(t) \int_0^t uS(u) dB(u)\right] + (A\sigma)^2 \int_0^t u^2 e^{\sigma^2 u} du. \end{aligned}$$

We rewrite $S(t)$ as the integral $\int_0^t dS(v) + A$ again, noting that in the double integral form (similar to the Gaussian integral trick), the differential product $dB(u)dB(v) = dt\, 1_{\{u=v\}}$. This integral only has nonzero value (due to the independence of Brownian increments) on the diagonal of the square

$(u, v) \in [0, t]^2$.

$$
\begin{aligned}
&= (At)^2 e^{\sigma^2 t} - 2\sigma t E\left[\left(\sigma \int_0^t S(u)dB(v) + A\right)\left(\int_0^t uS(u)dB(u)\right)\right] + (A\sigma)^2 \int_0^t u^2 e^{\sigma^2 u} du \\
&= (At)^2 e^{\sigma^2 t} - 2\sigma t E\left[\sigma \int_0^t uS^2(u)du + A\int_0^t uS(u)dB(u)\right] + (A\sigma)^2 \int_0^t u^2 e^{\sigma^2 u} du \\
&= (At)^2 e^{\sigma^2 t} - 2\sigma^2 t \int_0^t uE[S^2(u)]du + (A\sigma)^2 \int_0^t u^2 e^{\sigma^2 u} du \\
&= (At)^2 e^{\sigma^2 t} - 2A^2\sigma^2 t \int_0^t ue^{\sigma^2 u} du + (A\sigma)^2 \int_0^t u^2 e^{\sigma^2 u} du \\
&= A^2\sigma^2 \left[\int_0^t e^{\sigma^2 u}\left(t^2 - 2tu + u^2\right) du + \frac{t^2}{\sigma^2}\right] \\
&= A^2\sigma^2 \left[\int_0^t (t-u)^2 e^{\sigma^2 u} du + \frac{t^2}{\sigma^2}\right] = A^2\sigma^2 \int_0^t (t-u)^2 e^{\sigma^2 u} du + (At)^2
\end{aligned}
$$

which yields the variance (with substitution $v = t - u$)

$$
\begin{aligned}
\implies \operatorname{Var}\left(\int_0^t S(u)du\right) &= A^2\sigma^2 \int_0^t (t-u)^2 e^{\sigma^2 u} du = A^2\sigma^2 e^{\sigma^2 t} \int_0^t v^2 e^{-\sigma^2 v} dv \\
&= \frac{A^2}{\sigma^2}\left[\frac{2}{\sigma^4}\left(e^{\sigma^2 t} - 1\right) - t^2 - \frac{2t}{\sigma^2}\right].
\end{aligned}
$$

Problem 8 From Problem 1, (c), (d), and (f) are martingales (their dt terms are zero), and by the same reasoning, (e) is a martingale $\iff \nu = \frac{\sigma^2}{2}$.

Problem 9 The main observation here is to note that

$$
E(B(u) \,|\, \mathcal{F}(s)) = \begin{cases} B(u) & u \le s \\ B(s) & u > s. \end{cases}
$$

Thus, setting $s < t$, we need to break the integral up into two pieces: $[0, s]$ and $(s, t]$, yielding

$$
\begin{aligned}
E\left(t^2 B(t) - 2\int_0^t uB(u)du \,|\, \mathcal{F}(s)\right) &= t^2 B(s) - 2\int_0^s uB(u)du - 2B(s)\int_s^t u\,du \\
&= t^2 B(s) - (t^2 - s^2)B(s) - 2\int_0^s uB(u)du \\
&= s^2 B(s) - 2\int_0^s uB(u)du.
\end{aligned}
$$

We know that

$$
E\left|t^2 B(t) - 2\int_0^t uB(u)du\right| \le t^2 E|B(t)| + 2\int_0^t uE|B(u)|du < \infty,
$$

so we're done.

Problem 10 (Ornstein-Uhlenbeck) Our SDE is

$$
dX(t) = -\beta X(t)dt + \sigma dB(t),\; X(0) = x.
$$

(a) Let $g(t, x) = e^{\beta t} x$, thinking about IBP to try to kill a dt term. The partials are

$$
g_t(t, x) = \beta e^{\beta t} x;\; g_x(t, x) = e^{\beta t};\; g_{xx}(t, x) = 0
$$

and the Itô differential is

$$\begin{aligned}dg(t, X(t)) &= \beta e^{\beta t} X(t)dt + e^{\beta t} dX(t) \\ &= \beta e^{\beta t} X(t)dt + e^{\beta t}[-\beta X(t)dt + \sigma dB(t)] = e^{\beta t} \sigma dB(t)\end{aligned}$$

which integrates to

$$\begin{aligned}g(t, X(t)) = e^{\beta t} X(t) &= X(0) + \sigma \int_0^t e^{\beta u} dB(u) = x + \sigma \int_0^t e^{\beta u} dB(u) \\ \Rightarrow X(t) &= e^{-\beta t} \left(x + \sigma \int_0^t e^{\beta u} dB(u) \right).\end{aligned}$$

(b) $E(X(t)) = e^{-\beta t} x$ is obvious from (a) since $X(t)$ is a martingale, and using the Itô isometry,

$$\begin{aligned}Var(X(t)) &= E(X^2(t)) - E(X(t))^2 \\ &= e^{-2\beta t} x^2 + 2e^{-\beta t} x\sigma E\left[\int_0^t e^{\beta u} dB(u)\right] + e^{-2\beta t}\sigma^2 E\left[\left(\int_0^t e^{\beta u} dB(u)\right)^2\right] - e^{-2\beta t} x^2 \\ &= \sigma^2 e^{-2\beta t} \int_0^t e^{2\beta u} du = \frac{\sigma^2 e^{-2\beta t}}{2\beta}(e^{2\beta t} - 1) = \frac{\sigma^2(1 - e^{-2\beta t})}{2\beta}.\end{aligned}$$

(c) Using the approach of Problem 6: in expectation, the SDE

$$dX(t) = -\beta X(t)dt + \sigma dB(t),\ X(0) = x$$

becomes the ODE

$$\begin{aligned}\phi(t) = EX(t) &= x - \beta \int_0^t EX(u)du = x - \beta \int_0^t \phi(u)du \\ \Longrightarrow \phi'(t) &= -\beta\phi(t),\ \phi(0) = x \Longrightarrow \phi(t) = e^{-\beta t} x.\end{aligned}$$

The variance comes from the second moment, as usual:

$$\begin{aligned}&d(X^2(t)) = 2X(t)dX(t) + d[X, X](t) = (\sigma^2 - 2\beta X^2(t))dt + \sigma dB(t) \\ &\Longrightarrow E(X^2(t)) = \sigma^2 t - 2\beta \int_0^t E(X^2(w))dw \\ &\Longrightarrow \psi(t) = \sigma^2 t - 2\beta \int_0^t \psi(w)dw \Longrightarrow \psi'(t) = \sigma^2 - 2\beta\psi(t),\ \psi(0) = x^2.\end{aligned}$$

IBP integrating factor $u = e^{2\beta t}, v = \psi(t)$

$$\begin{aligned}&\Longrightarrow u\,dv = e^{2\beta t}\psi'(t) = \sigma^2 e^{2\beta t} - 2\beta e^{2\beta t}\psi(t) = d(uv) - v\,du \\ &\Longrightarrow uv = e^{2\beta t}\psi(t) = \psi(0) + \int_0^t d(uv) = x^2 + \int_0^t \sigma^2 e^{2\beta w} dw = x^2 + \frac{\sigma^2(e^{2\beta t} - 1)}{2\beta} \\ &\Longrightarrow \psi(t) = e^{-2\beta t} x^2 + \frac{\sigma^2(1 - e^{-2\beta t})}{2\beta} \\ &\Longrightarrow Var(X(t)) = \frac{\sigma^2(1 - e^{-2\beta t})}{2\beta}.\end{aligned}$$

Problem 11 generalizes Problem 10 with α extra drift:

$$dr(t) = (\alpha - \beta r(t))dt + \sigma dB(t).$$

Using $g(t,x) = e^{\beta t}x$ again, and assuming $r(0)$ constant,

$$
\begin{aligned}
dg(t, r(t)) &= \beta e^{\beta t} r(t)dt + e^{\beta t} dr(t) \\
&= \beta e^{\beta t} r(t)dt + e^{\beta t}[(\alpha - \beta r(t))dt + \sigma dB(t)] = \alpha e^{\beta t}dt + e^{\beta t}\sigma dB(t) \\
\implies g(t, r(t)) = e^{\beta t} r(t) &= r(0) + \frac{\alpha}{\beta}(e^{\beta t} - 1) + \sigma \int_0^t e^{\beta u} dB(u) \\
\implies r(t) &= e^{-\beta t} r(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma \int_0^t e^{-\beta(t-u)} dB(u) \\
\implies E(r(t)) &= e^{-\beta t} r(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) \\
r^2(t) &= E(r(t))^2 + 2E(r(t))\sigma \int_0^t e^{-\beta(t-u)} dB(u) + \sigma^2 \left(\int_0^t e^{-\beta(t-u)} dB(u) \right) \\
\implies Var(r(t)) = Var(X(t)) &= \frac{\sigma^2(1 - e^{-2\beta t})}{2\beta}.
\end{aligned}
$$

Note that α, only attached to the dt term in the SDE and being deterministic, contributes only to the drift (and so, to the mean), and not to the variance.

Problem 12 (CIR) $r(t)$ satisfies

$$dr(t) = (\alpha - \beta r(t))dt + \sigma \sqrt{r(t)} dB(t).$$

(a) The mean and variance come easily: using the same ODE trickery that would have worked for Problem 11,

$$
\begin{aligned}
Er(t) &= r(0) + \alpha t - \beta \int_0^t Er(u)du \implies \phi'(t) = \alpha - \beta\phi(t),\ \phi(0) = r(0) \\
\implies Er(t) = \phi(t) &= r(0)e^{-\beta t} + \frac{\alpha(1 - e^{-\beta t})}{\beta} = \left(r(0) - \frac{\alpha}{\beta} \right) e^{-\beta t} + \frac{\alpha}{\beta}.
\end{aligned}
$$

$$
\begin{aligned}
d(r^2(t)) &= 2r(t)dr(t) + d[r, r](t) \\
&= 2r(t)(\alpha - \beta r(t))dt + 2\sigma r(t)^{3/2} dB(t) + \sigma^2 r(t)dt \\
&= (2\alpha + \sigma^2 - \beta r(t))r(t)dt + 2\sigma r(t)^{3/2} dB(t) \\
\implies r^2(t) &= r^2(0) + \int_0^t (2\alpha + \sigma^2 - \beta r(u))r(u)du + 2\sigma \int_0^t r(u)^{3/2} dB(u) \\
\implies E(r^2(t)) = \psi(t) &= \psi(0) + E\left[\int_0^t \left((2\alpha + \sigma^2)r(u) - 2\beta r^2(u) \right) du \right] \\
&= (2\alpha + \sigma^2) \int_0^t \phi(u)du - 2\beta \int_0^t \psi(u)du \\
\implies \psi'(t) &= -2\beta\psi(t) + (2\alpha + \sigma^2)\phi(t),\ \ \psi(0) = r(0)^2.
\end{aligned}
$$

Using an IF of $e^{2\beta t}$, and integrating,

$$
\begin{aligned}
[e^{2\beta t}\psi(t)]' &= (2\alpha + \sigma^2)e^{2\beta t}\phi(t) \\
\implies e^{2\beta t}\psi(t) &= \psi(0) + (2\alpha + \sigma^2) \int_0^t e^{2\beta u}\phi(u)du \\
\implies \psi(t) &= e^{-2\beta t} r(0)^2 + (2\alpha + \sigma^2) \int_0^t e^{-2\beta(t-u)}\phi(u)du
\end{aligned}
$$

$$\begin{aligned}
\implies Var(r(t)) &= \psi(t) - \phi(t)^2 \\
&= e^{-2\beta t} r(0)^2 + (2\alpha + \sigma^2) \int_0^t e^{-2\beta(t-u)} \phi(u) du - \left[\left(r(0) - \frac{\alpha}{\beta} \right) e^{-\beta t} + \frac{\alpha}{\beta} \right]^2 \\
&= \dots
\end{aligned}$$

(b) Assuming $4\alpha = \sigma^2$, we reduce $dX(t) = d(\sqrt{r(t)})$ to the O-U process, with parameters half of what they are in Problem 10:

$$\begin{aligned}
dX(t) &= \frac{1}{2} r^{-1/2}(t) dr(t) - \frac{1}{8} r^{-3/2}(t) d[r, r](t) \\
&= \frac{1}{2} r^{-1/2}(t) \left[\left(\frac{\sigma^2}{4} - \beta r(t) \right) dt + \sigma r^{1/2}(t) dB(t) \right] - \frac{\sigma^2}{8} r^{-1/2}(t) dt \\
&= -\frac{\beta}{2} r^{1/2}(t) dt + \frac{\sigma}{2} dB(t) = -\frac{\beta}{2} X(t) dt + \frac{\sigma}{2} dB(t).
\end{aligned}$$

Thus, by Problem 10, and assuming $X(0) = x$,

$$X(t) = e^{-\beta t/2} \left(x + \frac{\sigma}{2} \int_0^t e^{\beta u/2} dB(u) \right).$$

(c) $X(t) = \sqrt{r(t)}$ means $r(t) = X^2(t)$, so, since $X(t) \sim N\left(e^{-\beta t/2} x, \frac{\sigma^2(1-e^{-\beta t})}{4\beta}\right)$, the distribution of $r(t)$ is noncentral χ^2; that is, $r(t) = e^{-\beta t} x^2 + 2e^{-\beta t/2} xY + Y^2$, where $Y \sim N\left(0, \frac{\sigma^2(1-e^{-\beta t})}{4\beta}\right)$.