Lecture 3: Option Markets

Modeling and Marketing Making in Foreign Exchange

The FX Options Markets

- The owner of an option has the right on an expiration date to decide whether to enter an FX forward contract
  - Typically a spot contract
  - Mostly physically-settled but some cash-settled options

- Bilateral, over-the-counter transactions
  - Not SEF traded yet because of the lack of clearing

- Traded inter-bank in terms of implied volatility
  - Typically traded delta-neutral – ie with a spot hedge

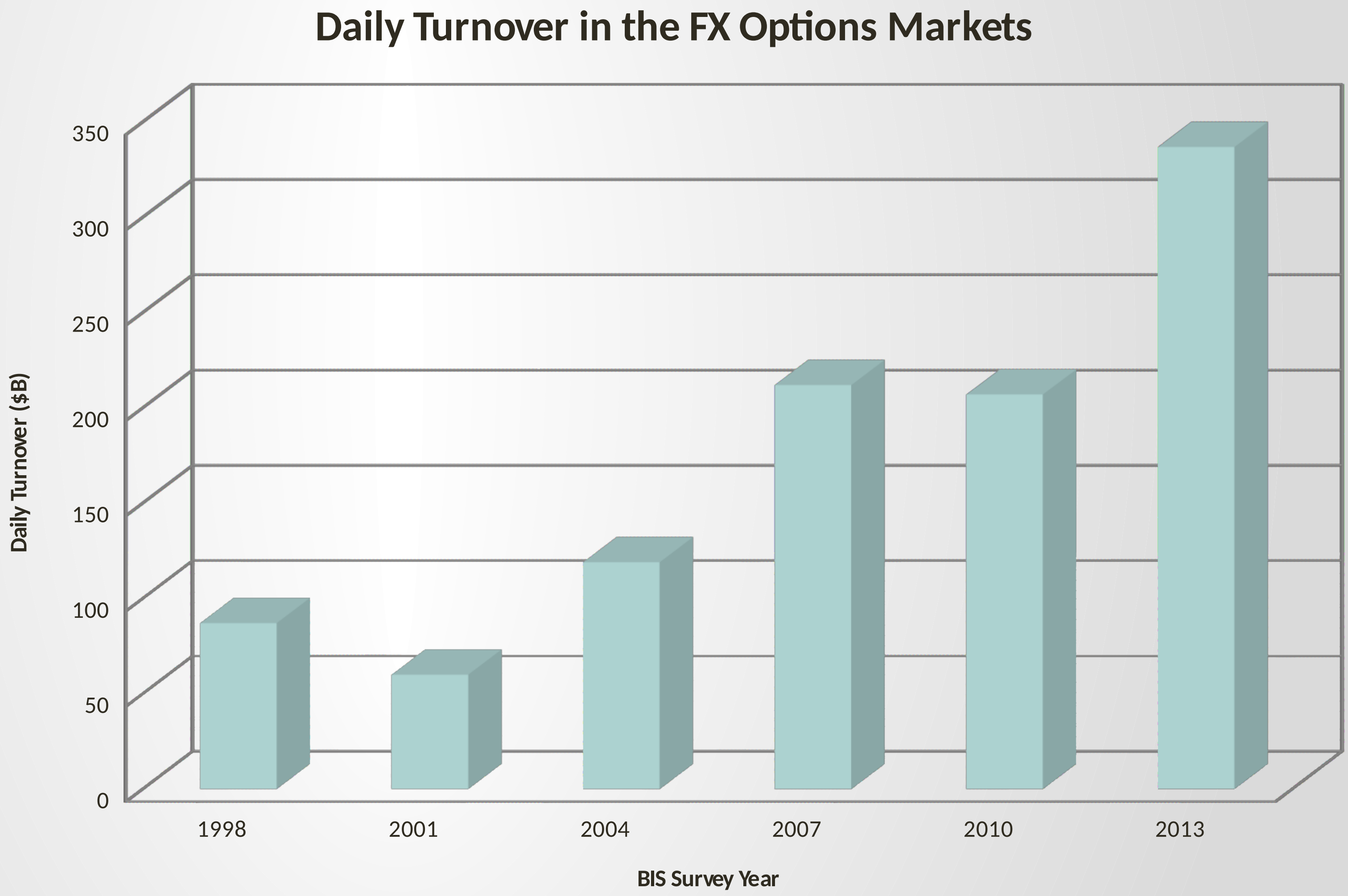
The Futures Options Markets

- Currency futures options trade on the CME
  - Only in EUR, GBP, AUD, CAD, JPY, and a bit in CHF
- Weekly, monthly, and quarterly expirations
  - Maximum expiration is about six months
- Relatively small compared to OTC market
  - $300B/day in OTC options market
  - $6-8B/day on CME futures options
- Futures options are American exercise, vs European exercise for most OTC options

Options Market Statistics

- The FX options market is about the same size as the entire global equities market
  - $300B/day in FX options
  - $300B/day in the global equity markets
- Most trading is inside 1y
  - Most G7 pairs trade out to 2-3y, JPY pairs to 10y
- Smaller fraction of trades executed electronically than spot
  - 37% by volume, vs 65% for spot
  - Electronic market significantly less developed than spot
  - Electronic pricing is common, electronic vega hedging is rare

Option Market Statistics



Options Market Conventions

- Options trade in terms of implied volatility for benchmark tenors and benchmark deltas
  - Not in terms of benchmark strikes!
  - Strikes corresponding to benchmark deltas float as spot moves

- “At-the-money” volatility is the most liquid point
  - ATM means “at-the-money delta neutral”, not spot or forward
  - Delta neutral strike: where a call and a put have equal and opposite delta

- Other liquid points: 25-delta and 10-delta calls and puts
  - Five benchmark points in the strike/delta direction for each expiration date

Options Market Conventions

- What does “delta” mean?
  - Black-Scholes delta using the implied volatility for the strike
  - Usually “spot delta” – delta to spot in the BS delta, including discounting
  - Rarely “forward delta” – delta to forward in the Black delta, not including discounting (notional of forward to use on hedge)
    - Sometimes used for long-dated options

- ATM options use the same convention, which ends up defining the ATM strikes
  - Call delta = -put delta
  - For the same strike and the same implied volatility
  - There is a nice closed-form expression for ATM strike that comes out of this

ATM Strike

$$\Delta_c = -\Delta_p$$

$$\therefore e^{-QT} N(d_1) = -\left(e^{-QT} N(d_1) - e^{-QT} \right)$$

$$\therefore \frac{\ln\left(\frac{S}{K_A}\right) + (R - Q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = 0$$

$$\therefore K_A = Fe^{\frac{\sigma^2 T}{2}}$$

Premium Currency Convention

- The expression above assumes that the premium for the option is paid in the denominated currency

- In some markets, the convention for premium currency is indeed that
  - eg in EURUSD, the convention is to pay premiums in USD

- In some, the convention is for premium payment in the asset currency
  - eg in USDJPY, the convention is to pay premiums in USD
  - In this case, the premium also contributes to the delta and the ATM strike convention must reflect that

Strike For a Delta

- What strike corresponds to eg a 25-delta call option?
  - Need the strike such that the BS call delta formula returns a value of 0.25

$$K_c = F e^{\frac{\sigma_K^2}{2} T - \sigma_K \sqrt{T} N^{-1}(\Delta_c e^{qT})}$$

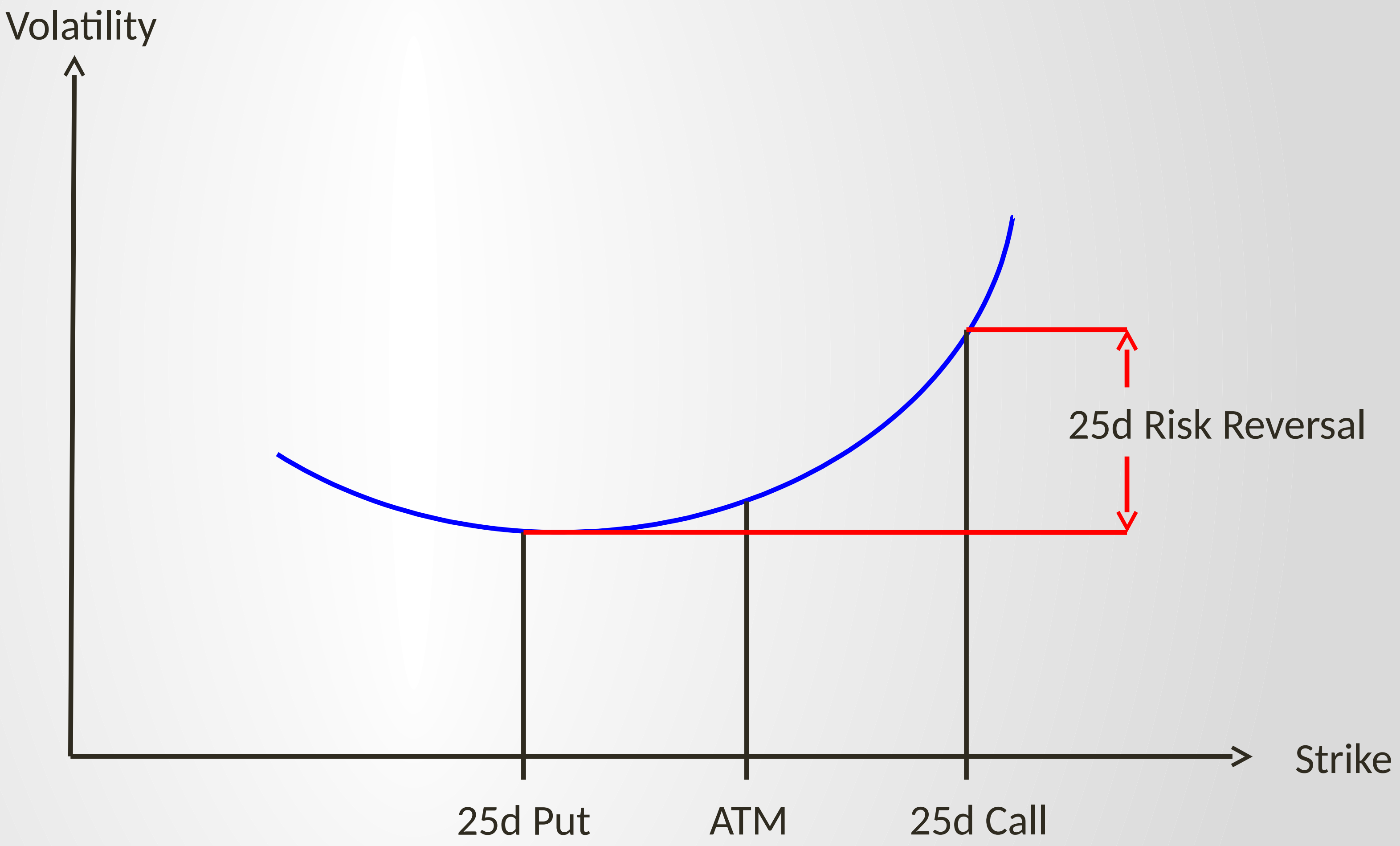
$$K_p = F e^{\frac{\sigma_K^2}{2} T + \sigma_K \sqrt{T} N^{-1}(\Delta_p e^{qT})}$$

- K_c is the strike for a call with delta Δ_c, where $0<\Delta_c<0.5$
- K_p is the strike for a put with delta $-\Delta_p$, where $0<\Delta_p<0.5$
- σ_K is the implied volatility for the given delta (known)

The Risk Reversal

- Implied volatility **skew** is traded as a separate asset in FX options markets, and is called the "risk reversal"
- 25-delta risk reversal means two things:
  - An implied volatility spread: the 25-delta call implied volatility minus the 25-delta put implied volatility
    - A measure of skew
  - An option spread: long a 25-delta call option and short the same amount of a 25-delta put option
    - A position in the option market, not a skew measure
- Similarly for 10-delta risk reversal
  - 10-delta and 25-delta are the two liquid benchmarks

The Risk Reversal



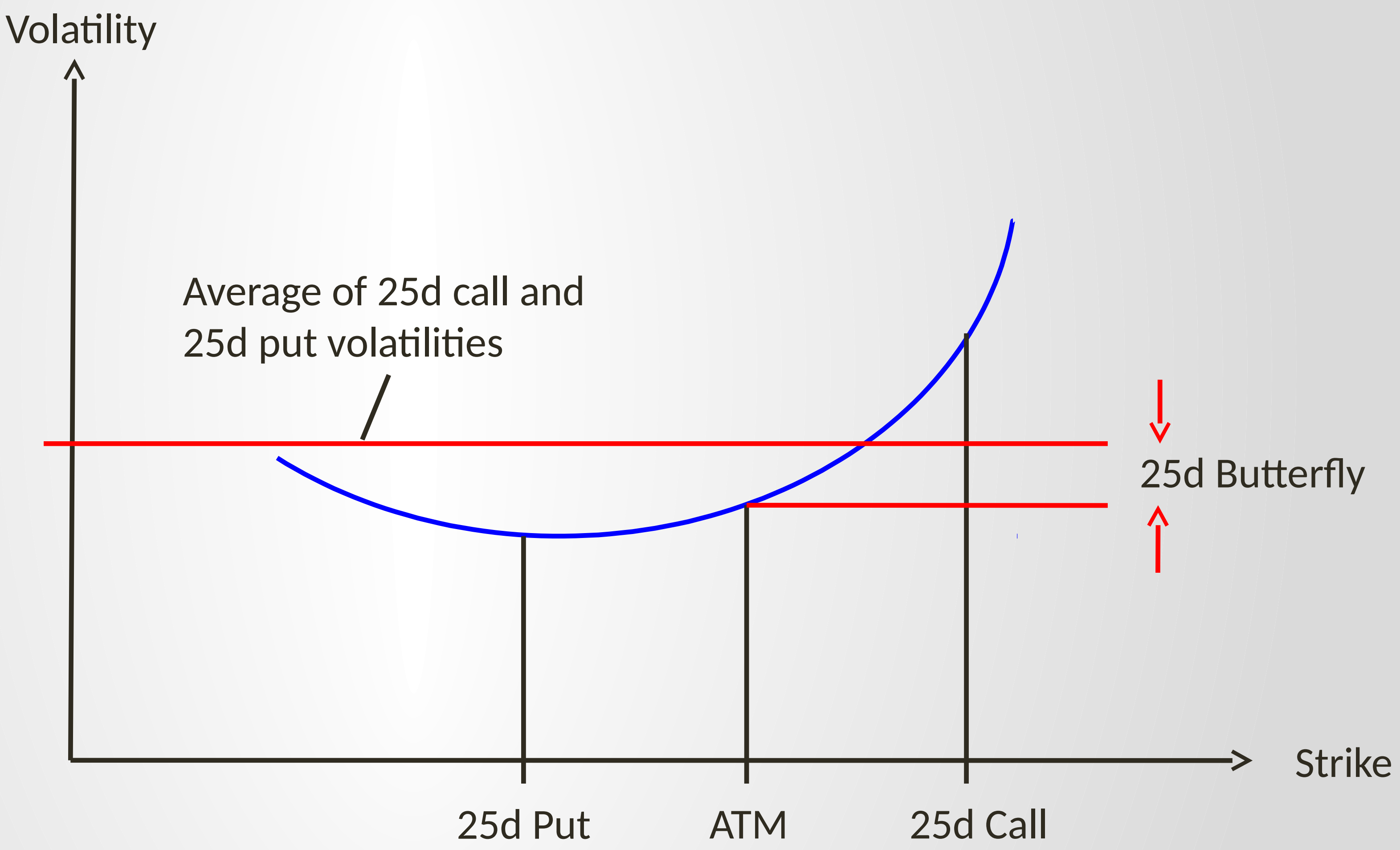
The Butterfly

- Implied volatility **smile** is traded as a separate asset in FX options markets, and is called the “butterfly”
  - Sometimes “smile margin”

- 25-delta butterfly means two things:
  - An implied volatility spread: the average of the 25-delta call and put implied volatilities less the ATM volatility
    - A measure of smile
  - An option spread: long 25-delta call and put options and short some notional (2x, vega-neutral are common variations) of the ATM option
    - A position in the option market, not a smile measure

- Similarly for 10-delta butterfly

The Butterfly

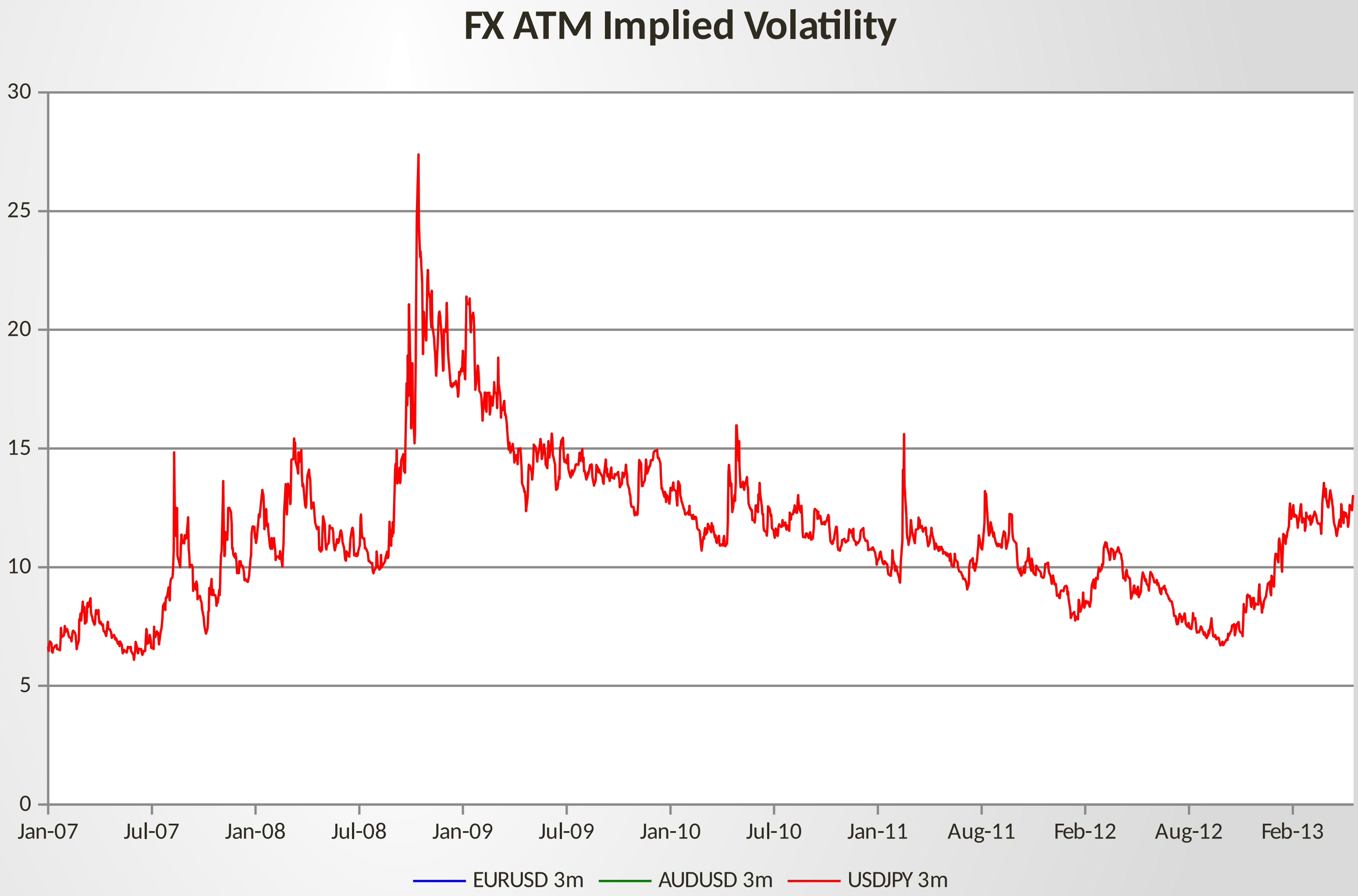


Quoting Convention

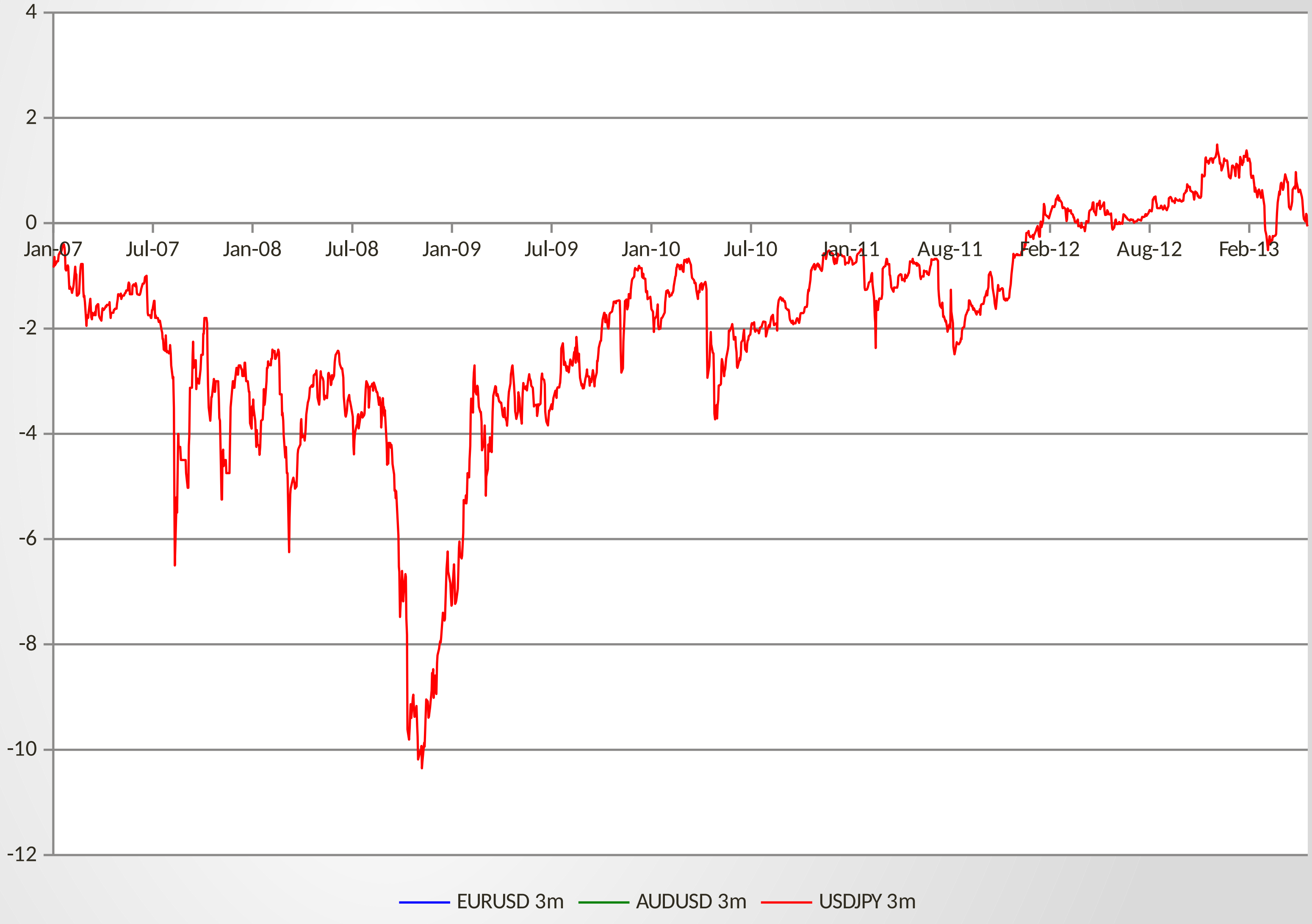
- Options are quoted in the inter-dealer market in terms of implied volatility by delta for a specific asset-currency notional
  - eg for EURUSD ATM 3m, “8.90@8.95 in 30M EUR”

- Those are converted into a specific trade when traders on either side agree on a reference spot, forward points, and discount rate
  - After they agree to buy/sell, both sides agree to the reference levels
  - eg for EURUSD, “spot reference 1.2704, forward points 0.0028, discount rate 0.45%”
  - Pricing is only weakly sensitive to spot because all trades have a delta hedge attached using market-convention delta

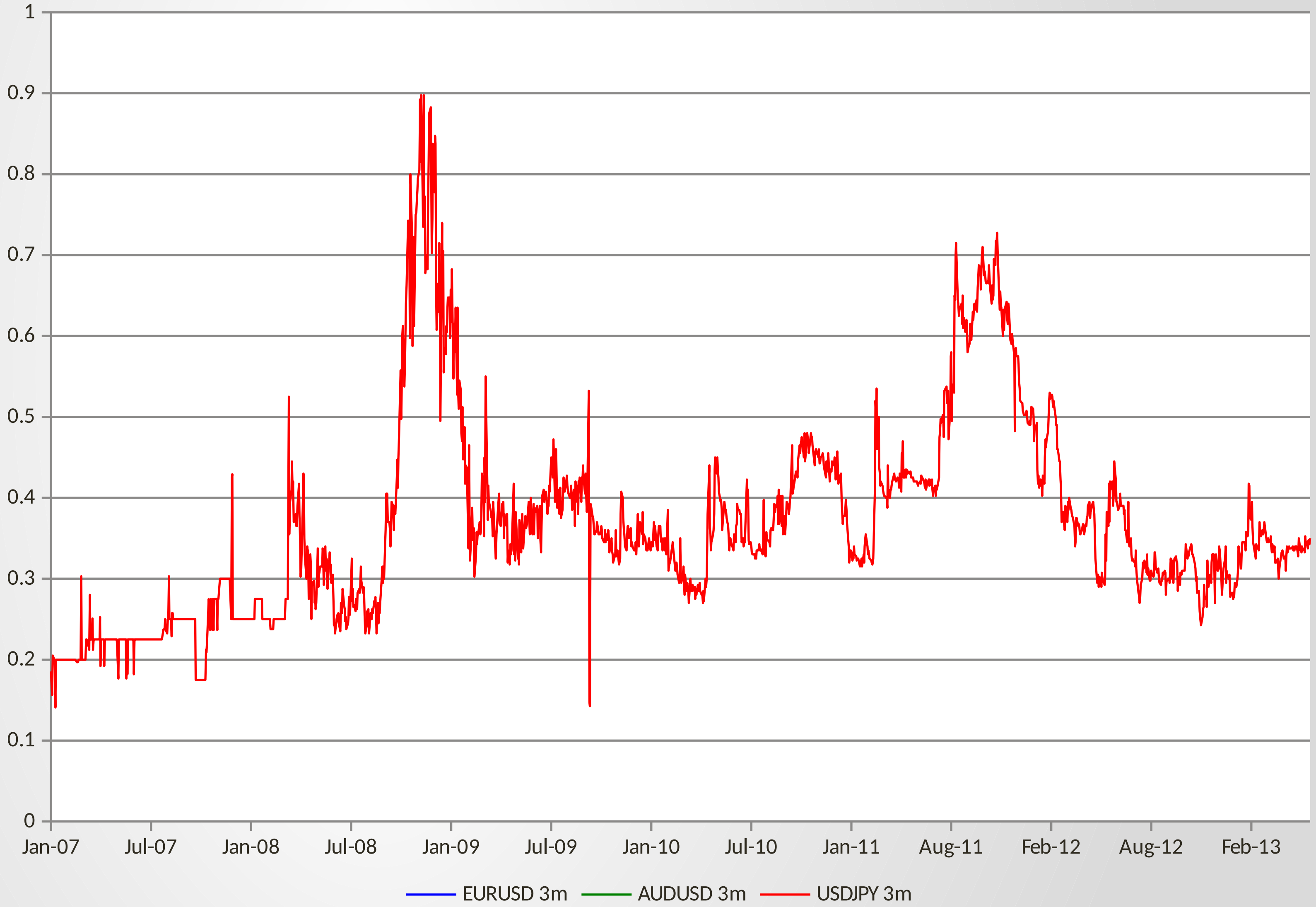
ATM Volatility History



Risk Reversal History



Butterfly History

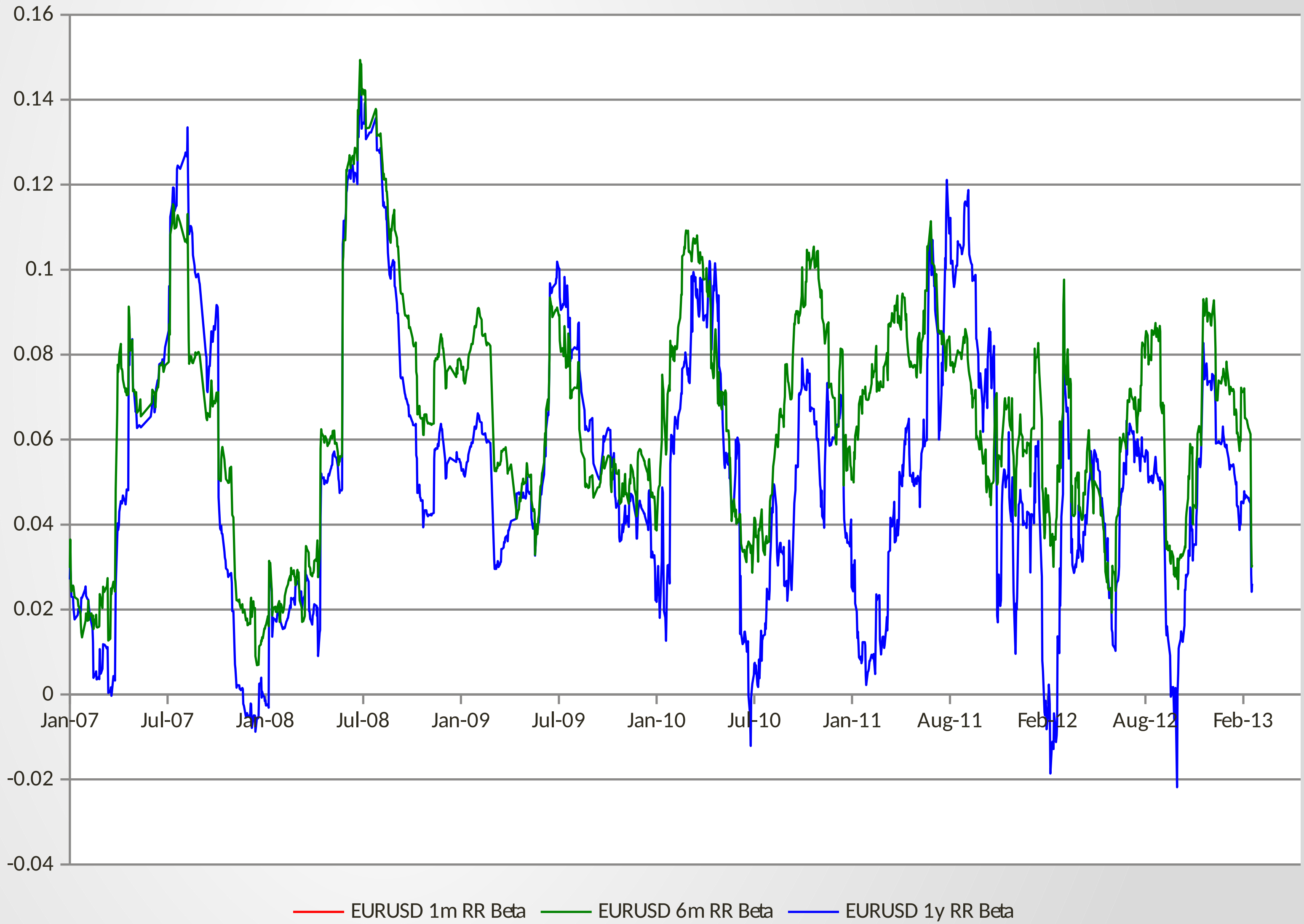


Risk Reversal Dynamics

- Moves in risk reversals have a relatively high correlation with moves in spot
  - This matters a lot for barrier option pricing, as we will see when we talk about exotic derivatives

- This is often quantified as the “risk reversal beta”, or the slope of a linear regression of day-on-day risk reversal change against spot log return
  - A number like 0.2 means “risk reversal gets more positive by 0.2 vols for every 1% move up in spot”

Risk Reversal Beta History



Volatility Interpolation

- The inter-dealer market is the source for implied volatility market data
- It gives you implied volatilities for benchmark expiration tenors
  - 1d, 1w, 2w, 1m, 2m, 3m, 6m, 9m, 1y, 2y, maybe 3y-10y
- And for five benchmark deltas on each expiration
  - 10d put, 25d put, ATM, 25d call, 10d call

- How do you calculate implied volatility for a non-benchmark expiration date or strike?

Volatility Interpolation: Strike

- Let's first think about interpolation in the strike direction
- What are the requirements?
  - dC/dK < 0: call prices decrease as strike increases
  - d^2C/dK^2 > 0: call price curvature wrt strike is positive
- In principle you need just the second derivative arbitrage not to exist, which guarantees that the first will be satisfied
  - In practice you need to check both

Volatility Interpolation: Strike

- $dC/dK>0$ arbitrage
  - Buy call with strike K, sell call with strike $K+dK$
    - Make money putting on the trade due to $dC/dK>0$
  - Payoff is 0 for $spot<K$, $+dK$ for $spot>K+dK$, and positive & linear in between
    - Always flat or make money on payoff

- $d^2C/dK^2<0$ arbitrage
  - Buy call strike $K-dK$, sell 2 units of call strike K, buy call strike $K+dK$
    - Make money putting on the trade due to $d^2C/dK^2<0$
  - Payoff is zero for $spot<K-dK$ or $spot>K+dK$, but triangular in between
    - Always flat or make money on payoff

Volatility Interpolation: Strike

- Since call prices are a function of the implied volatility interpolation, the choice of volatility interpolation scheme determines whether there will be arbitrage

- Two main classes of volatility interpolation:
  - Model-based fits
  - Non-model fits

Model-Based Fits

- Fit some kind of model to implied volatilities on the benchmark strikes for the given expiration date
  - Must be an exact fit, since “interpolation” requires that you hit all the explicitly-marked points
  - Need to have at least as many model parameters as there are marked points
  - The model parameters need to have somewhat independent impact on the explicitly-marked points

- Then you can use the fitted model to calculate implied volatilities for other strikes
  - Guaranteed arb-free since any self-consistent model must be arb-free

Model-Based Fits

- Some examples of models used:
  - Parametric local volatility. Local volatility model where local volatility is parameterized with five points in the spot direction to fit five points in implied volatility vs strike. Slow numerical pricing.
  - SABR. Stochastic volatility model where variance follows a lognormal process. Standard form has only three parameters but the model is often adjusted to include five. Closed-form approximation for pricing usually guarantees arb-free but not always.
  - Heston. Stochastic volatility model where variance follows a mean-reverting square root process. Standard form has five parameters but significant co-dependence between them really means only three are available to fit. Semi-closed form pricing is quite fast to execute.

Non-Model Fits

- SVI ("stochastic volatility inspired")
  - 5-parameter function to fit implied vol vs strike
- Standard cubic spline: fast & easy to implement
  - Need to specify what happens at the boundaries when you describe a cubic spline
  - Can sometimes get "wiggles" in interpolated vols when the shape of implied volatility is unusual
- Also cubic spline variations
  - "Tension" cubic spline: like cubic spline but reduces wiggles

Volatility Extrapolation: Strike

- The market tells you implied vols at the 10d put and call strikes
  - Your interpolation routine extrapolates beyond that, and the details of the interpolation routine determine how that works
  - Smaller-delta options do trade in the inter-dealer market and with clients fairly often
  - But vega of those options is small, so relatively small price impact

- Matter a lot, however, for some exotics

Variance Swap Pricing

$$\sigma^2 = \frac{2}{T}\int_0^\infty \frac{v(K)}{K^2}dK$$

- Formula gives the fair strike for a variance swap with continuous fixings
- v(K) = vanilla price with strike K: calls for K>forward and puts for K<=forward
- Fair strike depends on vanilla prices for all strikes, and has a 1/K^2 dependence for puts
  - Really matters what your vol extrapolation does on the put side

Volatility Interpolation: Time

- Much rarer to see model-based fits in the time direction
  - Some generalizations of SVI do support this
- Generally however the time direction is assumed to be somewhat orthogonal to strike when doing vol interpolation
- One common approach: treat vol-by-time for a fixed delta like an ATM curve in a Black-Scholes world

Volatility Interpolation: Time

- In a (generalized) Black-Scholes world, instantaneous volatility is deterministic
  - Some known function of time

- Volatility interpolation in the time direction assumes some form for the instantaneous volatility and fits it to the market implied volatilities
  - Piecewise-constant instantaneous volatility is a common choice
  - A mean-reverting form is sometimes used

Volatility Interpolation: Time

- Let's stick with piecewise-constant instantaneous vols
  - Simplest and easiest to deal with and explain to people

$$\sigma_I^2(T)T = \int_{t=0}^{T} \sigma^2(t)\,dt = \sum_{i=0}^{N_T} \sigma_i^2\left(t_i - t_{i-1}\right)$$

- σ_I is the implied volatility to time T
- $\sigma(t)$ is the instantaneous volatility (deterministic)
- σ_i is the i^{th} piece of the piecewise-constant instantaneous volatility
- N_T is the number of pieces to expiration T
- t_i is the time to expiration i (t_0=0)

Volatility Interpolation: Time

$$\sigma_N = \sqrt{\frac{\sigma_i^2(T)T - \sum_{i=1}^{N_T-1} \sigma_i^2\left(t_i - t_{i-1}\right)}{t_{N_T} - t_{N_T-1}}}$$

- Note that the argument inside the square root could be negative if the implied volatility to time T is not big enough
  - This is called “negative forward variance”
  - Not allowed – a kind of arbitrage in the time direction, though not a perfect arbitrage because it assumes Black-Scholes dynamics

Trading Time/Calendar Time

- Calendar time is number of calendar days/365
  - Nothing surprising there

- Calendar time is used to quote implied volatilities
  - ie you get the market option price by passing the implied volatility into the Black-Scholes formula along with a time measure in calendar time

- However, variance does not increase smoothly in calendar time
  - Weekends have very little variance; same with holidays
  - Events can cause step moves in variance

Trading Time/Calendar Time

- Best way to think about this is to translate from calendar time to “trading time”
  - Assume variance increases smoothly in trading time
  - Do all time-based vol interpolation in trading time
  - Then convert back to calendar time when displaying implied volatilities, since that is market convention

- Trading time is a monotonically increasing function of calendar time
  - Or could be flat: eg trading time stops over a weekend if we assume weekends have no variance
  - Can have jumps: eg across the Non-Farm Payrolls release we might assume trading time steps up

Weekend Effect

- Many shops assume weekends have no variance
  - Or small variance, but let’s use zero here for simplicity

- This means that Friday->Monday looks like one day in trading time, but three days in calendar time
  - Real variance from Friday->Monday is one “unit” of variance
  - But implied volatility is quoted in terms of calendar days, based on variance = implied volatility squared * calendar time
  - So implied volatility to Monday as seen on Friday is generally much lower (by 1/sqrt(3)) than implied volatility to Friday as seen Thursday
    - 1d volatility tends to bounce around across weekends because of this

Trading Time and Theta

- “Theta” means “how much does the price of my portfolio change when I roll the clock ahead to tomorrow”
  - Should assume that vols move to “forward vols” when you roll the clock ahead
  - “Forward vol” means the implied volatility you expect to see the next day, given that some variance has rolled off as time moves forward

- In a Black-Scholes world you can do this pretty easily
  - Calculate “forward variance” at future time t for an expiration T == implied volatility(T)^2 – implied volatility(t)^2
  - Forward volatility = sqrt(forward variance / (T-t))

Trading Time and Theta

- Unclear what to do when there is implied volatility skew/smile!
  - Mostly people keep RR/BF the same and move ATM to forward vols; or run the forward vol calculation on fixed vol-by-delta for all five benchmark deltas

- Trading time matters a lot to theta over a weekend
  - No trading time, you see three days of theta losses for a long gamma position
  - Trading time, you seen only one day of theta losses
    - Ensures that traders don't start doing dumb trades to hedge a fake theta