MTH9821. Implicit Finite Difference Method for American Options Based on Projected LU-Decomposition

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Abstract. This note outlines an implicit finite difference method based on LU-decomposition and projected backward substitution for American options. Unlike the projected SOR method which iteratively improves the solution to a given tolerance, this algorithm takes a finite number of operations to complete and yields full arithmetic precision. Numerical tests on an example of American option in Homework Assignment #9 are given at the end.

1 Matrix Inequalities for Backward Euler Method

The backward Euler method for American options involves the solution of the following matrix inequalities at each time step:

$$\mathbf{A} \cdot \mathbf{u} \geq \mathbf{b}, \tag{1.1}$$

$$\mathbf{u} \geq \mathbf{g}, \tag{1.2}$$

while entry-wise, one of two inequalities is an equality. Here,

$$\mathbf{A} = \begin{pmatrix} 1+2\alpha & -\alpha & 0 & \cdots & 0 \\ -\alpha & 1+2\alpha & -\alpha & & 0 \\ 0 & -\alpha & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1+2\alpha & -\alpha \\ 0 & \cdots & 0 & -\alpha & 1+2\alpha \end{pmatrix},$$

the vector $\mathbf{b}$ is given by a combination of the state vector at the previous time step and the boundary conditions, and the vector $\mathbf{g}$ is the early exercise premium.

The LU-decomposition of the matrix $\mathbf{A}$ can be written as

$$\begin{pmatrix} 1+2\alpha & -\alpha & 0 & \cdots & 0 \\ -\alpha & 1+2\alpha & -\alpha & & 0 \\ 0 & -\alpha & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1+2\alpha & -\alpha \\ 0 & \cdots & 0 & -\alpha & 1+2\alpha \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\frac{\alpha}{y_0} & 1 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\frac{\alpha}{y_{N-1}} & 1 \end{pmatrix}}_{\triangleq \mathbf{L}} \cdot \underbrace{\begin{pmatrix} y_0 & -\alpha & 0 & \cdots & 0 \\ 0 & y_1 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -\alpha \\ 0 & \cdots & 0 & 0 & y_N \end{pmatrix}}_{\triangleq \mathbf{U}},$$

where

$$y_0 = 1 + 2\alpha, \tag{1.3}$$

$$y_n = (1+2\alpha) - \frac{\alpha^2}{y_{n-1}}, \quad n = 1, \cdots, N; \tag{1.4}$$

2 Transformation to Triangular Matrix Inequalities

Lemma 2.1. *The matrix inequality 1.1 is equivalent to the upper-triangular matrix inequalities:*

$$\mathbf{v} \triangleq \mathbf{U} \cdot \mathbf{u} \geq \mathbf{L}^{-1} \cdot \mathbf{b} \tag{2.1}$$

if α is sufficiently small.

Proof. The above statement amounts to saying that $y_n > 0$, $n = 0, \cdots, N$, so that the following sequence of inequalities hold true:

$$\begin{aligned} & v_0 \geq b_0, \\ & v_1 \geq b_1 + \frac{\alpha}{y_0} v_0 \geq b_1 + \frac{\alpha}{y_0} b_0, \\ & \quad \cdots \\ & v_N \geq b_N + \frac{\alpha}{y_{N-1}} v_{N-1} \geq \cdots \geq b_N + \frac{\alpha}{y_{N-1}} \Big(b_{N-1} + \frac{\alpha}{y_{N-2}} \Big(b_{N-2} + \frac{\alpha}{y_{N-3}} \Big(b_{N-3} + \cdots + \frac{\alpha}{y_0} b_0 \Big)\Big)\Big). \end{aligned}$$

Define $Y_n = \Pi_{i=0}^n y_n$, we get a 2^{nd} order linear recurrence relation $Y_n - (1+2\alpha)Y_{n-1} + \alpha^2 Y_{n-2} = 0$. The fact that $y_n > 0$, $n = 0, \cdots, N$ for sufficiently small α can be proved by solving the above recurrence relation explicitly and let $y_n = \frac{Y_n}{Y_{n-1}}$. Detail is omitted. □

Lemma 2.2. *If the matrix inequiality 1.1 is an equality for an entry n, i.e. $(\mathbf{A}\cdot\mathbf{u}-\mathbf{b})_n = 0$, then the triangular matrix inequality 2.1 is also an equality for the same entry, i.e. $(\mathbf{U}\cdot\mathbf{u}-\mathbf{L}^{-1}\cdot\mathbf{b})_n = 0$.*

Proof. The proof is based on the existence of an early exercise boundary entry $\hat{n}$ such that

$$\begin{aligned}
&(\mathbf{A}\cdot\mathbf{u}-\mathbf{b})_n = 0, \quad (\mathbf{u}-\mathbf{g})_n > 0, \quad \text{for } 0 \le n < \hat{n};\\
&(\mathbf{A}\cdot\mathbf{u}-\mathbf{b})_n > 0, \quad (\mathbf{u}-\mathbf{g})_n = 0, \quad \text{for } \hat{n} \le n \le N,
\end{aligned}$$

which is heuristically evident but by itself requires a proof. Detail is omitted. Note that we adopted an indexing convention that in the finite difference scheme for an American option, the right boundary of the computational domain is labeled by $n = 0$ and the left boundary by $n = N$. If the indexing convention is the other way around, we need a UL-decomposition instead.

Now, for a given n, if $(\mathbf{A}\cdot\mathbf{u}-\mathbf{b})_n = 0$, then

$$(\mathbf{A}\cdot\mathbf{u}-\mathbf{b})_{n'} = 0, \quad \forall n' < n.$$

thus, invoking the lower-triangularity of $\mathbf{L}^{-1}$, we have

$$\begin{aligned}
(\mathbf{U}\cdot\mathbf{u}-\mathbf{L}^{-1}\cdot\mathbf{b})_n &= \left[\mathbf{L}^{-1}\cdot(\mathbf{A}\cdot\mathbf{u}-\mathbf{b})\right]_n\\
&= \sum_{n'\le n} \mathbf{L}^{-1}_{nn'}\cdot(\mathbf{A}\cdot\mathbf{u}-\mathbf{b})_{n'}\\
&= 0.
\end{aligned}$$

□

The above result establishes the equivalence between the matrix inequalities 1.1 - 1.2 and

$$\begin{aligned}
\mathbf{U}\cdot\mathbf{u} &\ge \mathbf{L}^{-1}\cdot\mathbf{b},\\
\mathbf{u} &\ge \mathbf{g},
\end{aligned}$$

while entry-wise, one of two inequalities is an equality.

Lemma 2.3. *Denote $\mathbf{c} \triangleq \mathbf{L}^{-1}\cdot\mathbf{b}$, the triangular matrix inequalities*

$$\mathbf{U}\cdot\mathbf{u} \ge \mathbf{c}, \tag{2.2}$$

$$\mathbf{u} \ge \mathbf{g}, \tag{2.3}$$

while entry-wise, one of two inequalities is an equality, can be solved by the following projected backward substitution procedure:

$$\begin{aligned}
u_N &= \max\left\{\frac{c_N}{y_N}, g_N\right\},\\
u_n &= \max\left\{\frac{c_n + \alpha u_{n+1}}{y_n}, g_n\right\}, \quad n = N-1, \cdots, 0.
\end{aligned}$$

Proof. The proof is essentially an induction based on the triangularity of the system and the positiveness of y_n, $0 \le n \le N$. Detail is omitted. □

3 Numerical Tests

Numerical tests are performed on the example of American option valuation in Homework #9, focusing on the comparison between backward Euler method by projected LU and by projected SOR. Similar tests are done for Crank-Nicolson.

Method	Error I	Error II	Δ	Γ	Θ	Error III
Backward Euler by LU	0.003959021	0.004001460	-0.379403240	0.030962375	-2.767599900	0.001595845
Backward Euler by SOR	0.003959004	0.004001443	-0.379403240	0.030962375	-2.767599886	0.001595643
Crank-Nicolson by LU	0.001037275	0.001079720	-0.379476930	0.030920454	-2.765847325	0.000887689
Crank-Nicolson by SOR	0.001037275	0.001079720	-0.379476930	0.030920453	-2.765847326	0.000887694

Note: The domain in discretized by $M = 256$ and $\alpha_{\text{temp}} = 0.45$. In SOR method, $\omega = 1.2$ and $tol = 10^{-6}$.