MTH 9831 Lecture 9

Note Title

① Further examples on connections with PDEs: Pricing of a zero-coupon bond when the interest rate is random.

② Pricing of barrier options:
(a) Probabilistic approach;
(b) PDE approach.

③ Pricing of Asian options. New ideas: augmentation of the state space and *reduction of dimension using the change of numéraire (* next lecture).

① Assume that the interest rate under $\widetilde{\mathbb{P}}$ satisfies

$$(1)\quad dR(t) = \beta(t, R(t))\,dt + \gamma(t, R(t))\,d\widetilde{B}(t)$$

Examples: Vasiček, Hull-White, CIR models

- These are called one factor short rate models.
- For examples of two-factor models and more general HJM framework see Chapter 10 of Shreve II.

The discount process

$$dD(t) = -R(t)D(t)\,dt, \quad D(0)=1$$

$$D(t) = \exp\left(-\int_0^t R(s)\,ds\right).$$

The MMA price

$$M(t) = \frac{1}{D(t)} = \exp\left(\int_0^t R(s)\,ds\right).$$

$$dM(t) = R(t)M(t)\,dt = \frac{R(t)}{D(t)}\,dt, \quad M(0)=1.$$

$B(t,T)$ – the time t price of a unit zero coupon bond maturing at T (pays \$1 at T)

$$B(t,T) = \frac{1}{D(t)}\widetilde{\mathbb{E}}\left(D(T)\cdot 1 \mid \mathcal{F}(t)\right).$$

$$B(t,T) = \widetilde{\mathbb{E}}\left[e^{-\int_t^T R(s)ds} \,\middle|\, \mathcal{F}(t) \right]$$

Yield over the time interval $[t, T]$ is

$$Y(t,T) := -\frac{1}{T-t} \ln B(t,T)$$

$$B(t,T) = e^{-\underbrace{Y(t,T)}(T-t)}$$

constant rate over $[t,T]$

which is consistent with the price $B(t,T)$.

Since $R(t)$ is a solution of an SDE, it is a Markov process we would like to say that

(2) $B(t,T) = f(t, R(t))$ for some function $f(t,y)$.

Rigorously speaking,

$e^{-\int_t^T R(s)\,ds}$ is not of the form $h(R(T))$, so our definition of a Markov process does not allow to conclude (2). But heuristically the only way

$e^{-\int_t^T R(s)\,ds}$ depends of the path $R(s)$ for $0 \le s \le t$ is only through its value at $s = t$.

To find the PDE for f we use the fact that

$D(t)B(t,T) = e^{-\int_0^t R(s)ds} f(t, R(t))$ is a $\widetilde{\mathbb{P}}$-martingale

$$d\left(e^{-\int_0^t R(s)ds} f(t,R(t))\right) = e^{-\int_0^t R(s)ds}\Big(-R(t) f\,dt$$

$$+ f_t\,dt + f_y\big(\beta\,dt + \gamma\,d\widetilde{B}(t)\big) + \tfrac{1}{2} f_{yy}\gamma^2\,dt\Big)$$

$$= e^{-\int_0^t R(s)ds} \Big(\big(-Rf + f_t + \beta f_y + \tfrac{1}{2}\gamma^2 f_{yy} \big) dt + \gamma f_y \, d\tilde{B}(t) \Big) \; ; \; f = f(t, R(t))$$

(3) $f_t(t,y) + \beta(t,y) f_y(t,y) + \frac{1}{2}\gamma^2(t,y) f_{yy}(t,y) = y f(t,y)$

or, in short, $f_t(t,y) + \mathcal{A}_t f(t,y) = y f(t,y)$

$$f(T,y) = 1 \quad \text{for all } y$$

See Example 6.5.1 for a derivation of an explicit formula for the Hull-White model and Example 6.5.2. for CIR model.

- Key idea: these model are affine yield models, which means that

$$f(t,y) = e^{-y C_1(t,T) - C_2(t,T)}$$

for some $C_1(t,T)$ and $C_2(t,T)$ (to be determined). The name comes from the fact that the yield $Y(t,T)$ can be assumed to be of the form $y \dfrac{C_1(t,T)}{T-t} + \dfrac{C_2(t,T)}{T-t}$ – which is an affine function of y

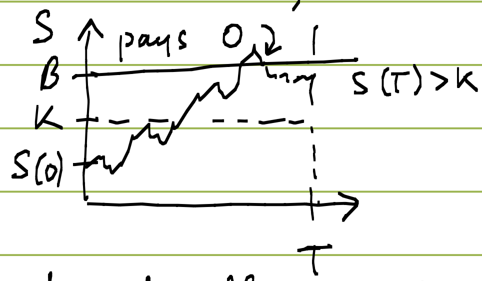
Example 6.5.3 shows that the price $c(t,y)$ of a call option on a bond with expiration $0 \le T_1 \le T$ requires to solve the same PDE (3), i.e.

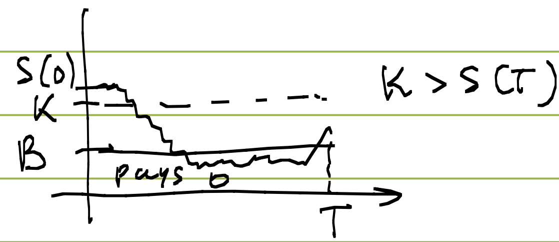
$$c_t + \beta c_y + \tfrac{1}{2}\gamma^2 c_{yy} = yc \quad \text{but with}$$

a different terminal condition

$$c(T_1, y) = \big(f(T_1, y) - K \big)_+ .$$

2. Knock-out and knock-in barrier options:
up-and-out, down-and-out, up-and-in, down-and-in.





Knocked-out call
(up-and-out)

Knocked out put
(down-and-out)

Tools: Joint distribution of BM with a drift and its running maximum (see Lecture 4); stopped martingale; risk-neutral pricing formula; knowing how to determine whether a given function of a diffusion process is a martingale.

Framework: Black-Scholes-Merton. Assume that under the risk-neutral measure the stock price satisfies

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\tilde{B}(t). \quad (4)$$

Option: Up-and-out call with strike K, barrier $B > K$ and expiration T. We assume that $S(0) \le B$.

Goals: (a) use probabilistic approach to set up an integral which gives the price
(b) Write down the PDE and boundary conditions satisfied by the call price.

(a) $S(t) = S(0)\, e^{\sigma \tilde{B}(t) + (r - \frac{\sigma^2}{2})t} = S(0)\, e^{\sigma \hat{B}(t)}$,

where $\hat{B}(t) = \tilde{B}(t) + \alpha t$, and $\alpha = \dfrac{r - \sigma^2/2}{\sigma}$.

Define $\hat{B}^*(t) = \max\limits_{0 \le s \le t} \hat{B}(s)$. Then (since e^x is increasing)

$$\max_{0 \le t \le T} S(t) = S(0)\, e^{\sigma \hat{B}^*(T)}$$

The payoff of the option is

$$V(T) = \left(S(0) e^{\sigma \hat{B}(T)} - K\right)_+ \mathbb{1}_{\{S(0) e^{\sigma \hat{B}^*(T)} < B\}}$$

$$= \left(S(0) e^{\sigma \hat{B}(T)} - K\right) \mathbb{1}_{\{S(0) e^{\sigma \hat{B}(T)} \ge K,\ S(0) e^{\sigma \hat{B}^*(T)} < B\}}$$

$$= \left(S(0) e^{\sigma \hat{B}(T)} - K\right) \mathbb{1}_{\{\hat{B}(T) \ge \underbrace{\frac{1}{\sigma} \log \frac{K}{S(0)}}_{k \in \mathbb{R}};\ \hat{B}^*(T) < \underbrace{\frac{1}{\sigma} \log \frac{B}{S(0)}}_{b > 0}\}}$$

Risk-neutral pricing formula tells us that the value of this option at time $t \in [0, T]$ is equal to

$$V(t) = \tilde{\mathbb{E}}\left(e^{-r(T-t)} V(T) \mid \mathcal{F}(t)\right), \quad 0 \le t \le T.$$

By the tower property, $e^{-rt} V(t)$ is a $\tilde{\mathbb{P}}$-martingale.

To compute $V(0)$ we write

$$V(0) = e^{-rT}\,\widetilde{\mathbb{E}}\left(\left(S(0)e^{\sigma\hat{B}(T)} - K\right)\mathbb{1}_{\{\hat{B}(T)\geq k,\ \hat{B}^*(T)<b\}}\right)$$

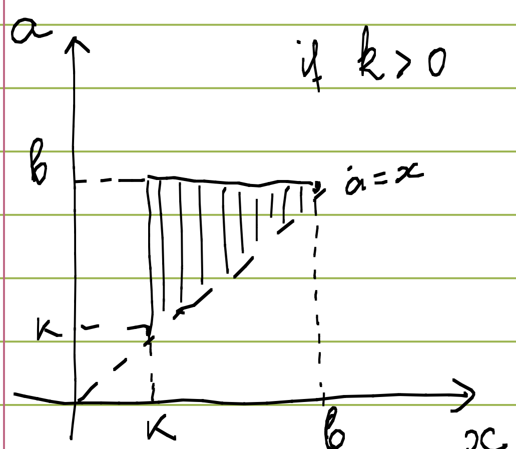
The joint distribution of $(\hat{B}(T), \hat{B}^*(T))$ was computed in Lecture 6. We have (under $\widetilde{\mathbb{P}}$, as it is our reference measure here)

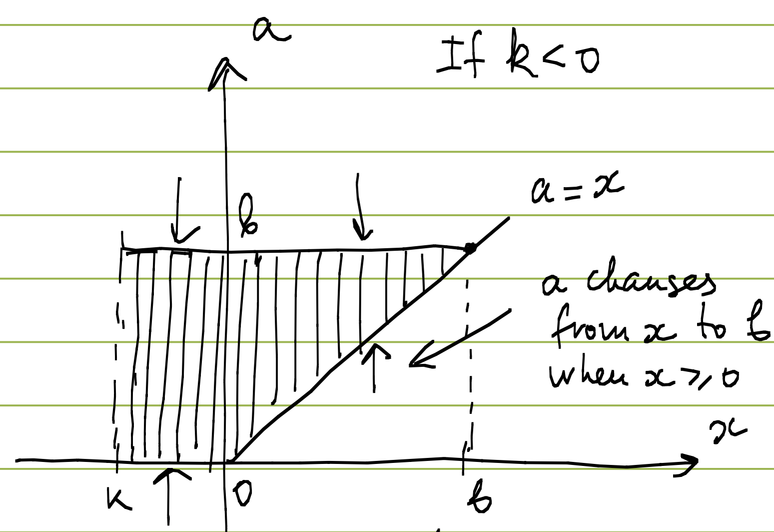
$$\hat{f}(x,a) = \begin{cases} e^{\alpha x - \frac{1}{2}\alpha^2 T}\,\dfrac{2(2a-x)}{T\sqrt{2\pi T}}\,e^{-\frac{1}{2T}(2a-x)^2}; & x \leq a,\ a \geq 0 \\ 0 & \text{otherwise}. \end{cases}$$

(variable x corresponds to $\hat{B}(T)$, and a – to $\hat{B}^*(T)$).

Therefore,

$$V(0) = e^{-rT}\int_k^b\int_{x_+}^b \left(S(0)e^{\sigma x} - K\right)\frac{2(2a-x)}{T\sqrt{2\pi T}}\,e^{\alpha x - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2a-x)^2}\,da\,dx$$





a changes from $0\ (=x_+)$ to b when $x<0$ ($x_+ = 0$ when $x<0$).

This integral can be computed in terms of $N(x)$. This gives a closed form formula. For details see Shreve II (pp. 304–308, 7.3.3).

(b) Let us move to the PDE description.

Theorem 1. Let $v(t,x)$ denote the price at time t of the up-and-out call under the assumption that the call has not knocked out prior to t and $S(t) = x$. Then $v(t,x)$ satisfies the BSM PDE

$$v_t + rx\, v_x(t,x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = rv(t,x)$$

in the rectangle $\{(t,x) : 0 \leq t < T,\ 0 \leq x \leq B\}$ and satisfies boundary conditions

$$v(t,0) = 0, \quad 0 \leq t \leq T \qquad (5)$$

$$v(t,B) = 0, \quad 0 \leq t < T \qquad (6)$$

$$v(T,x) = (x-K)_+, \quad 0 \leq x \leq B \qquad (7)$$

Recall that $V(t) = \widetilde{\mathbb{E}}\left(e^{-r(T-t)} V(T) \mid \mathcal{F}(t)\right)$.
It is clear that $V(t)$ can not be represented as $v(t, S(t))$, since $V(t)$ "remembers" whether it has been knocked out or not, and $v(t, S(t))$ "does not remember" anything that happened prior to t.

Define $\rho = \inf\{t \geq 0 : S(t) = B\}$. Then ρ is the knock-out time (since upon reaching B $S(t)$ will exceed B within any positive additional time increment with probability 1). Since ρ is the first passage time, it is a stopping time (namely $\{\rho \leq t\} \in \mathcal{F}(t)$ for each $t \geq 0$).
We have that $(e^{-rt} V(t))_{0 \leq t \leq T}$ is a $\widetilde{\mathbb{P}}$-martingale.

We know that a martingale, stopped at a stopping time is again a martingale. Thus,

$$\left(e^{-r(t\wedge\rho)} V(t\wedge\rho)\right)_{0\le t\le T} \text{ is a } \tilde{\mathbb{P}}\text{-martingale.}$$

$(t\wedge\rho := \min\{t,\rho\})$.

Lemma 1 For $0\le t\le\rho$ $V(t)$ is representable as $v(t,S(t))$. In particular,

$$\left(e^{-r(t\wedge\rho)} v(t\wedge\rho, S(t\wedge\rho))\right)_{0\le t\le T}$$

is a $\tilde{\mathbb{P}}$-martingale.

Explanation for Lemma 1. On the event $\{\rho>t\}$ the option is alive, so the value of the up-and-out call is the same as for the regular call option.

Recall that for the regular call option we had

$$V(t) = \tilde{\mathbb{E}}\Big(e^{-r(T-t)} \underbrace{h(S(T))}_{(S(T)-K)_+} \,\Big|\, \mathcal{F}(t)\Big) \underset{\text{Markov property}}{=} v(t,S(t)) \text{ for some function } v(t,x).$$

Since $e^{-rt}V(t)$ is a martingale,

$e^{-rt} v(t,S(t))$ is a martingale.

The point here is that the same is true up to the random time ρ, or for all $t\le\rho$. After that we stop our martingale, and the stopped martingale is still a martingale. ☒

Sketch of proof of Theorem 1. By Lemma 1 $V(t) = v(t, S(t))$ for some v and $0 \le t \le \rho$. Moreover $e^{-rt} v(t, S(t))$ is a martingale up to time ρ. Therefore $(0 \le t \le \rho)$

$$d\left(e^{-rt} v(t, S(t))\right) = e^{-rt}\Big(\big(-rv(t, S(t)) + v_t(t, S(t))$$

$$+ v_x(t, S(t))\, S(t)\, r + \tfrac{1}{2} v_{xx}(t, S(t))\, \sigma^2 S^2(t)\big)\, dt$$

$$+ v_x(t, S(t))\, \sigma\, S(t)\, d\tilde{B}(t)\Big), \text{ and}$$

setting the dt term to zero we get $(0 \le t \le \rho)$

$$-rv(t, S(t)) + v_t(t, S(t)) + v_x(t, S(t))\, S(t)\, r$$

$$+ \tfrac{1}{2} v_{xx}(t, S(t))\, \sigma^2 S^2(t) = 0$$

The process $S(t)$, $0 \le t \le \rho$, can reach a neighborhood of any point in $[0, T) \times [0, B]$. with positive probability. This means that the BSM PDE should hold for all $(t, x) \in [0, T) \times [0, B]$:

$$-rv(t, x) + v_t(t, x) + v_x(t, x)\, x r$$

$$+ \tfrac{1}{2} v_{xx}\, \sigma^2 x^2 = 0$$

The boundary conditions simply satisfy the conditions set forth by the option.

Remark. See Shreve II, p. 303-4 for the pitfalls of Δ-hedging strategy for this type of options.

3 Let again $S(t)$ satisfy (4) under $\tilde{\mathbb{P}}$.
The payoff of Asian option with strike K and expiration T is

$$V(T) = \left(\frac{1}{T} \int_0^T S(u)\,du - K \right)_+$$

$$V(t) = e^{-r(T-t)} \tilde{\mathbb{E}}\left(\left(\frac{1}{T} \int_0^T S(u)\,du - K \right)_+ \Big| \mathcal{F}(t) \right)$$

Clearly, the price depends on the whole path $S(u)$, $0 \le u \le t$, so we can not write $V(t)$ as a function of t and $S(t)$.
New idea is needed.

- Augmentation of the state space.

To know $V(t)$ we definitely need to know $\int_0^t S(u)\,du$, since $\int_0^T S(u)\,du = \int_0^t S(u)\,du + \int_t^T S(u)\,du$ and $\int_0^t S(u)\,du$ is $\mathcal{F}(t)$-measurable, so it can be taken out of the conditional expectation.

Introduce an auxilliary process Y: $dY(t) = S(t)\,dt$

– a regular process. Consider a 2-dimentional process $(S(u), Y(u))_{t \le u \le T}$. It is a Markov process

$$dS(u) = rS(u)\,du + \sigma S(u)\,d\tilde{B}(u) \quad ; \quad S(t) = x$$
$$dY(u) = S(u)\,du; \qquad Y(t) = y$$

$$\mathcal{A} v(x,y) = r x v_x + x v_y + \frac{1}{2} \sigma^2 x^2 v_{xx}.$$

$$V(t) = e^{-r(T-t)} \tilde{\mathbb{E}}\left(\left(\tfrac{1}{T} Y(T) - K\right)_+ \mid \mathcal{F}(t)\right)$$

and $V(t)$ can be written as $v(t, S(t), Y(t))$.
for some function $v(t,x,y)$.

<u>Theorem 1.</u> Let $v(t,x,y)$ be the time t value of the Asian call option when $S(t)=x$, $Y(t)=y$.
Then $v(t,x,y)$ satisfies the PDE

$$(8) \quad v_t(t,x,y) + rx\, v_x(t,x,y) + x\, v_y(t,x,y) + \tfrac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y) = r\, v(t,x,y)$$

in $[0,T) \times [0,\infty) \times \mathbb{R}$, and the boundary conditions

$$(9) \quad v(t,0,y) = e^{-r(T-t)}\left(\tfrac{y}{T} - K\right)_+, \quad 0 \le t < T,\ y \in \mathbb{R}.$$

$$(10) \quad \lim_{y \to -\infty} v(t,x,y) = 0, \quad 0 \le t \le T,\ x \geq 0.$$

$$(11) \quad v(T,x,y) = \left(\tfrac{y}{T} - K\right)_+, \quad x \geq 0,\ y \in \mathbb{R}.$$

<u>Remark</u>.
There are several issues here that need to be addressed: (1) why are we looking also at $y<0$?
We know that $Y(t) = \int_0^t S(u)\,du \geq 0$, so how do we end up considering $y<0$?
(2) Usual existence & uniqueness questions for the above problem.
Start with (2): existence is not a problem but for uniqueness in an unbounded domain $(t,x,y) \in [0,T) \times [0,\infty) \times \mathbb{R}$ we need additional conditions at ∞:
What happens if $x \to \infty$ and $y \to \pm\infty$?

We omit this since later we shall be able to reduce the dimension and shall write a simpler PDE with corresponding boundary conditions.

About (1) $Y(t) = \int_0^t S(u)\,du$ is only one specific solution of $dY(u) = S(u)\,du$, $0 \le u \le T$ (the one that satisfies $Y(0) = 0$). More generally, we have to solve this equation for $u \in [t, T]$ ($t \le T$) subject to the condition $Y(t) = y$. So we get

$$Y(u) = y + \int_t^u S(s)\,ds, \quad t \le u \le T.$$

Notice that while ($S(t) = 0 \Rightarrow S(u) = 0$) for all $u \in [t, T]$ and $Y(t) = y$ for all $u \in [t, T]$,

we do not have the same property for the Y process: if $Y(t) = 0$ then $Y(u) = \int_t^u S(s)\,ds$

need not be 0, so we can not determine the value of $v(t, x, 0)$, and we can not provide a boundary condition for $y = 0$. But, at least mathematically, there is no problem considering $y \in \mathbb{R}$, and that is what we do.

"Proof" of Theorem 1. Since Y is a regular process, $d[Y,Y](t)$ and $d[Y,S](t)$ are equal to 0 (a.s.). Recall that $e^{-rt} V(t) = e^{-rt} v(t, S(t), Y(t))$ is a $\tilde{\mathbb{P}}$-martingale. Compute

$$d\left(e^{-rt} v(t, S(t), Y(t))\right) = e^{-rt}\left(-rv + v_t + rSv_x + Sv_y + \tfrac{1}{2}\sigma^2 S^2 v_{xx}\right)dt + e^{-rt}\sigma S v_x \, d\tilde{B}(t).$$

Setting the dt term to zero we get

$$v_t(t, S(t), Y(t)) + rS(t)\, v_x(t, S(t), Y(t)) + S(t)\, v_y(t, S(t), Y(t)) + \frac{1}{2}\sigma^2 S^2(t)\, v_{xx}(t, S(t), Y(t)) = r\, v(t, S(t), Y(t))$$

This equation has to hold at all points (x, y) which can be possibly hit by S and Y processes. Since we did not restrict ourselves to $y > 0$, the process $Y(t)$ can hit any point in $\mathbb{R}$, and $S(t)$ can hit any $x \geq 0$, so we arrive at PDE (8). Turn now to boundary conditions. If $S(t) = 0$ then $S(u) = 0$ for all $u \in [t, T]$, and $Y(t) = y$ for all $u \in [t, T]$. Thus, the value of an Asian call at time t is

$$e^{-r(T-t)}\left(\frac{y}{T} - K\right)_+ .$$

Thus, we got (9).

If $S(t) = x$, $Y(t) = y$ and we let $y \to -\infty$ then it is very unlikely that $\left(\frac{Y(T)}{T} - K\right)_+ > 0$. In other words, $\lim_{y \to -\infty} v(t, x, y) = 0$. This is (10).

(11) is just the payoff of the option.

Remark: we have

$$d\left(e^{-rt} v(t, S(t), Y(t))\right) = \sigma e^{-rt} S(t)\, v_x(t, S(t), Y(t))\, d\tilde{B}(t).$$

On the other hand, the discounted value of the portfolio that has $\Delta(t)$ shares of this stock is given by (see Lecture 5)

$$d\left(e^{-rt} X(t)\right) = e^{-rt} \sigma S(t) \Delta(t)\, d\tilde{B}(t).$$

Thus, to hedge a short position in the Asian call we should have $\Delta(t) = v_x(t, S(t), Y(t))$.