2. Yield and Duration

Pricing a set of deterministic cash flows

We saw earlier that an arbitrary risk-free cash flow c occurring at some time t can be priced simply with reference to market risk-free instruments. We defined the discount factor P_t as the present value per unit face amount of the traded zero-coupon bond to that maturity, and in this case the value of the cash flow becomes:

$$V = cP_t$$

The same approach can easily be extended to any arbitrary set of cash flows $c_1, c_2, \ldots, c_n$ whose amounts are known now, occurring at times $t_1, t_2, \ldots, t_n$ also known now, if we disregard the possibility that some or all of these cash flows may not be received as promised. The resulting instrument can be treated as a portfolio made up of n risk-free zero-coupon bonds, and priced correspondingly by:

$$V = \sum_{i=1}^{n} c_i P_{t_i}$$

This general method is useful for pricing the most common form of debt.

Fixed-coupon bonds

When a loan is made with a long maturity, the lender typically requires intermediate payments to mitigate some of the risks associated with the loan. The simplest way to accomplish this is for the bond to make periodic payments of interest up through maturity, at which point the face amount is paid, just as with a zero-coupon bond. In this case, the face amount or notional is typically referred to as the principal of the bond, and this style of repayment of principal is called a bullet repayment.

To specify such an instrument, as before we take a face amount N and a maturity T for the bond. Further, we specify a coupon rate c, which is the annual rate of interest paid on the principal, and a coupon frequency m, indicating how many payments of interest are made per year. Coupons are paid in arrears—that is, the interest for a particular period is paid at the end of the period—and accrue at a rate of simple interest.

Thus, the amount of time in each coupon period is $1 / m$, making each coupon payment the amount Nc / m. These payments are received at times $1 / m$, $2 / m$, and so on up to and including T, at which point the bullet repayment of N also occurs. For the sake of simplicity, we will refer to these times as $t_1, t_2, .., t_n = T$.

If we assume that the payments on the bond are risk-free, we can use the expression we developed above for an arbitrary set of cash flows to price this instrument:

$$V = \sum_{i=1}^{n} N \frac{c}{m} P_{t_i} + NP_T$$

While it is of course useful to decompose the cash flows of this bond into a sequence of zero-coupon bonds, another decomposition may at times also be instructive. We have already seen that the bullet repayment of principal is in effect a zero-coupon bond. It turns out that the strip of coupon payments can also be seen as relating to another financial instrument.

We define an annuity with frequency m and maturity T as an instrument that pays a fixed amount per year spread evenly over m installments each year. If the amount paid per year is N, then the present value of this instrument is:

$$V = \sum_{i=1}^{n} N\frac{1}{m}P_{t_i} = N\sum_{i=1}^{n}\frac{1}{m}P_{t_i}$$

If we express this value per unit notional, then we arrive at a unitless quantity analogous to the discount factor…

$$A_{m,T} = \frac{V}{N} = \sum_{i=1}^{n}\frac{1}{m}P_{t_i}$$

…which can be used to value a coupon strip. Using the annuity, we can express the value of the coupon-bearing bond as…

$$V = NcA_{m,T} + NP_T$$

…indicating that it can also be seen as a portfolio consisting of a zero-coupon bond and an annuity.

The fair coupon of a risk-free bond

In addition to the benefits to the bondholder of periodic interest payments, fixed-coupon bonds offer benefits to the bond's issuer as well. If, for example, the issuer were to choose to take on debt in the form of a long-dated zero-coupon bond, the fact that these bonds trade at a discount would mean that the issuer could raise far less than the face amount of the debt incurred. Typically, however, the issuer prefers to receive an amount of cash similar to the face amount.

At issuance it must be decided what coupon to associate with the bond in order to accomplish this goal. The result is usually inexact; it is seldom the case that the amount raised in a bond issue is precisely the face amount. Nonetheless, it is useful to consider the question of what coupon c^*, at a given moment, causes the bond's price to equal its par amount N.

Our pricing formulas above make this question comparatively easy to answer:

$$N = \sum_{i=1}^{n} N\frac{c^*}{m}P_{t_i} + NP_T$$

$$1 = c^*\sum_{i=1}^{n}\frac{1}{m}P_{t_i} + P_T$$

$$c^* = \frac{1-P_T}{\sum_{i=1}^{n}\frac{1}{m}P_{t_i}} = \frac{1-P_T}{A_{m,T}}$$

The final expression here is particularly intuitive. The numerator is the time value associated with receiving a certain amount today and paying the same amount back at T, per unit notional. The denominator is the value of the coupon strip, also per unit notional; our result can thus be thought of as the size of the strip that must be provided in order to exactly offset the loss of time value on the bond's principal over the term of the loan.

The yield of a set of deterministic cash flows
A party wishing to invest in the fixed-income market is presented with a wide array of choices. Bonds with different coupons, maturities, payment structures, and so on are traded, and even in the straightforward case of debt considered risk-free, as prices move it can be difficult to discern which are cheap or expensive relative to others. If instruments that contain additional risks—such as the appreciable risk of not paying as promised—are added to the collection, understanding bonds in terms of their raw prices can be extremely difficult.

As with many such problems in finance, in this case the common solution is to recast the price information as a more intuitive number. Suppose you have a certain amount V to invest in bonds and wish to analyze a particular bond to determine what return it offers.

To do this, we imagine a bank account that pays some constant rate of interest r, compounded continuously. We can deposit or withdraw at will from this account, and can even carry a negative balance (i.e., borrow from the account) at the same rate. To evaluate the bond, we imagine depositing its price V in the account today and, each time a cash flow is scheduled to occur for the bond, making a withdrawal of precisely that amount. That is, we use the account in an effort to replicate the promised cash flows of the bond.

One possible outcome of this attempt is that the rate r is too large, in which case there is cash remaining in the account once the final withdrawal is made at maturity; alternatively, if r is too small, the balance will turn negative at some point and will remain negative thereafter.

Intuitively, it seems that there should be some rate $r = y$, called the yield of the bond, which causes the balance of the account to be precisely zero after the final withdrawal is made. To determine this rate, we imagine carrying out the process described above and consider the balance B_t at each time. We suppose that the bond we wish to replicate has arbitrary deterministic cash flows $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$.

We make the initial investment of the bond's price V…
$$B_0 = V$$
…and allow interest to accrue at the rate y until the instant before t_1…
$$B_{t_1 -} = Ve^{yt_1}$$
…at which point we withdraw the amount of the first cash flow:
$$B_{t_1} = Ve^{yt_1} - c_1$$

The remaining amount now accumulates interest at y until the moment before t_2…

$$B_{t_2-} = Ve^{yt_1}e^{y(t_2-t_1)} - c_1e^{y(t_2-t_1)} = Ve^{yt_2} - c_1e^{y(t_2-t_1)}$$

…and again we make a withdrawal:

$$B_{t_2} = Ve^{yt_2} - c_1e^{y(t_2-t_1)} - c_2$$

In general, immediately before cash flow *i*, we have…

$$B_{t_i-} = Ve^{yt_i} - \sum_{j=1}^{i-1} c_j e^{y(t_i-t_j)}$$

…and immediately after the withdrawal:

$$B_{t_i} = Ve^{yt_i} - \sum_{j=1}^{i} c_j e^{y(t_i-t_j)}$$

Our goal is for the balance after the n^{th} withdrawal to be zero…

$$B_T = Ve^{yT} - \sum_{j=1}^{n} c_j e^{y(T-t_j)} = 0$$

…which, rearranged, becomes:

$$Ve^{yT} = \sum_{j=1}^{n} c_j e^{y(T-t_j)}$$

$$V = \sum_{j=1}^{n} c_j e^{-yt_j}$$

Thus, the interest rate paid by the bank account that allows us to exactly replicate the bond's cash flows is also the constant rate that, if used to discount the cash flows, returns the bond's price.

Note that this characterization assumes that all cash flows are received; thus, it offers no direct explanation of why, for example, two bonds with the same promised cash flows might trade at different prices. The result in such a case, however, would be to assign a higher yield to the bond trading at the lower price, since a greater rate of return would be required to replicate the same cash flows with a smaller initial investment.

Determining the yield of a fixed-coupon bond

In general, the yield equation above cannot be solved analytically. However, it seems clear that a unique solution y should certainly exist when V and all the cash flows are positive, since as y goes to infinity V goes to zero, as y goes to minus infinity V is arbitrarily large, and…

$$\frac{dV}{dy} = -\sum_{i=1}^{n} t_i c_i e^{-yt_i}$$

…is negative for all y.

Nonlinear equations in a single unknown with unique solutions are actually quite common in finance, and many techniques exist to solve them. Here we will cover three of the most common ones.

The general problem is posed as finding the root of an arbitrary function, that is a value x^* satisfying $f(x^*) = 0$. In this case, we set up the question as…

$$f(y) = \sum_{i=1}^{n} c_i e^{-yt_i} - V = 0$$

To illustrate the method, we will use a fixed-coupon bond with:
$N = 100$
$c = 0.05$
$T = 10$
$m = 1$
$V = 99.5$

The bisection method
As you may recall from your initial calculus class, the intermediate value theorem guarantees us that for any continuous function f and points x_1 and x_2 where $f(x_1)$ has one sign and $f(x_2)$ has the opposite, there exists a point x^* between them such that $f(x^*) = 0$.

The bisection method for solving an iterative equation takes advantage of this fact by requiring two starting points, one of which has $f(x)$ positive, and the other negative. At each step, the algorithm takes the midpoint of the segment and evaluates f there. The midpoint now defines two segments, exactly one of which has one endpoint where f is positive and one where f is negative. The algorithm chooses this segment and iterates again.

Stopping criteria for iterative methods like this generally take one of two forms. One choice is to stop the iteration when the difference between two consecutive iterates is less than a target amount. Given the way the bisection method works, this guarantees that the answer returned lies within a certain x-distance of the root, and moreover we know in advance how many iterations will be required to satisfy this criterion, since the difference between the current value and the next is always half the length of the current segment.

However, if the function in question is steeply sloped near the root, defining a criterion like this may lead to a value of f that is quite far from zero. It is generally more useful to define a stopping criterion based on the absolute value of f, guaranteeing that the final point lies within a certain y-distance of the root.

The below shows the price path taken by the bisection method with our sample bond, taking initial yield points 0.035 and 0.07. We stop the iteration when the absolute value of f is less than 10^{-8}:

point #	yield	f	negEndpt	posEndpt
1	0.035	12.42225149	X	0.035
2	0.07	-15.12713205	0.07	0.035
3	0.0525	-2.457151728	0.0525	0.035
4	0.04375	4.68274455	0.0525	0.04375

5	0.048125	1.040956566	0.0525	0.048125
6	0.0503125	-0.725682971	0.0503125	0.048125
7	0.04921875	0.153194252	0.0503125	0.04921875
8	0.049765625	-0.287349182	0.049765625	0.04921875
9	0.049492188	-0.067354395	0.049492188	0.04921875
10	0.049355469	0.042850605	0.049492188	0.049355469
11	0.049423828	-0.012269214	0.049423828	0.049355469
12	0.049389648	0.015286364	0.049423828	0.049389648
13	0.049406738	0.001507492	0.049423828	0.049406738
14	0.049415283	-0.005381132	0.049415283	0.049406738
15	0.049411011	-0.001936887	0.049411011	0.049406738
16	0.049408875	-0.000214714	0.049408875	0.049406738
17	0.049407806	0.000646385	0.049408875	0.049407806
18	0.04940834	0.000215834	0.049408875	0.04940834
19	0.049408607	5.5968E-07	0.049408875	0.049408607
20	0.049408741	-0.000107077	0.049408741	0.049408607
21	0.049408674	-5.32589E-05	0.049408674	0.049408607
22	0.049408641	-2.63496E-05	0.049408641	0.049408607
23	0.049408624	-1.2895E-05	0.049408624	0.049408607
24	0.049408616	-6.16764E-06	0.049408616	0.049408607
25	0.049408612	-2.80398E-06	0.049408612	0.049408607
26	0.04940861	-1.12215E-06	0.04940861	0.049408607
27	0.049408609	-2.81235E-07	0.049408609	0.049408607
28	0.049408608	1.39223E-07	0.049408609	0.049408608
29	0.049408608	-7.1006E-08	0.049408608	0.049408608
30	0.049408608	3.41084E-08	0.049408608	0.049408608
31	0.049408608	-1.84488E-08	0.049408608	0.049408608
32	0.049408608	7.82981E-09	0.049408608	0.049408608

As you can see here, the method comes fairly close to the eventual value within a few iterations; however, it takes a long time for that estimate to improve, and indeed the absolute value of the function f does not always decrease from step to step. If it happens that, for some iteration, the root lies close to one of the chosen endpoints, improvement in the estimate can be very slow in subsequent steps.

Still, a major strength of this method is that it is guaranteed to converge for continuous functions, even when those functions have discontinuous or zero derivatives. The following methods, while faster, are apt to fail with such functions. In applications where a robust method is required, the bisection method or some faster adaptation of it is a suitable choice.

Newton's method

Suppose we stand at a point x_i on the function, and that additionally we know the values $f(x_i)$ and $f'(x_i)$; we wish to make an educated guess about the next point x_{i+1} at which we should search for a root.

Certainly the signs of the quantities can be helpful. If f is positive but f' is negative, then the curve slopes downward to the right, and thus it seems reasonable to search in this direction. Furthermore, the degree of slope can be helpful in making the determination; if the slope is gentle, then we would be better served choosing a fairly distant point, but if the slope is steep, we ought to search close by.

This is the essential strategy of the Newton-Raphson method (often just called "Newton's method"), which for each point x_i evaluates f and its derivative, then projects along the line passing through that point tangent to the curve to find where it intersects the x-axis. This point becomes x_{i+1}.

The equation of the line defined by these values is…

$$y = f'(x_i)(x - x_i) + f(x_i)$$

…so we can easily solve for x_{i+1}, which is the point on this line where $y = 0$:

$$0 = f'(x_i)(x_{i+1} - x_i) + f(x_i)$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

In cases where the function f is well behaved, this method is extremely fast. For example, here is the iteration path of our sample bond, using an initial yield of 0.035 and the same stopping criterion as above:

point #	yield	f	f'
1	0.035	12.42225149	-920.7369528
2	0.04849164	0.7423707	-813.011663
3	0.049404752	0.003108854	-806.212221
4	0.049408608	5.51206E-08	-806.1836325
5	0.049408608	0	-806.183632

Problems can arise with this method when the function of interest is not monotonic. Suppose, for example, that we stand at a point where f is positive and f' negative as above, but between this point and the root is a local minimum. If the iteration stumbles into the "basin" around this min, then it may become trapped. In the bottom of the basin, the derivative f' is zero at some point, and very small in the neighborhood; since f' appears in the denominator of our iteration expression, a near-zero derivative will throw the iteration very far afield indeed, often resulting in a failure of the method.

Still, in cases where the function is nicely behaved—or if some relatively simple controls are put in place—this is an extremely fast method and is generally the first choice for solving problems of this sort.

The secant method

One problem that limits the usefulness of Newton's method in its pure form is that, particularly in finance, we may be interested in finding roots of a function with no analytic form or derivative. If the function, for example, is based on a numerical pricing algorithm, we cannot directly evaluate its derivative.

One possible solution to this problem is to make a numerical estimate of the derivative by perturbing the current value of *x* by some small value δ and taking advantage of the fact that:

$$f'(x) \approx \frac{f(x+\delta)-f(x)}{\delta}$$

This is the slope of what is called the secant line through two points on the function, and while it is not exactly equal to the derivative, for small δ it can virtually always serve that purpose in Newton's method. Still, it must be noted that, even if an analytic derivative is available, evaluating it or the function *f* itself may be computationally expensive. Since Newton's method requires two such evaluations per iteration (either of the derivative itself, or of *f* to estimate the derivative), it may in the end be more efficient to produce the estimate of the derivative using a less exact but also less expensive method.

The secant method is quite similar to Newton's method, except that it requires two starting points rather than just one. At each iteration, the secant method uses the last two estimates. So, for example, if our current estimate is x_i with function value $f(x_i)$, and we have retained the previous estimate x_{i-1} and its function value $f(x_{i-1})$, then we can use the estimate:

$$f'(x_i) \approx \frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}}$$

This then can be used in our Newton's iteration to determine x_{i+1}. Early on, when our consecutive estimates are apt to be far apart, this quantity will likely be quite different from the actual derivative at this point, and as such this method does not generally converge as quickly as pure Newton's method does. However, it does have the advantage of requiring only one function evaluation per estimate; for some functions, this may make the secant method faster to converge in time terms even though more iterations are required.

In the case of our sample bond, this method follows the path below with initial points 0.035 and 0.07, and the same stopping criterion used previously:

point #	yield	f
1	0.035	12.42225149
2	0.07	-15.12713205
3	0.050781798	-1.100083844
4	0.049274593	0.108107354
5	0.049409456	-0.000683341
6	0.049408609	-4.20997E-07
7	0.049408608	1.62004E-12

In this example, we required seven evaluations of the function *f* in order to arrive at the final answer. Using Newton's method, we required 5 evaluations of *f* and 4 of *f'* (which is essentially the same computational expense as an evaluation of *f*) to arrive at the answer, so in this particular case the secant method can be said to have been more efficient.

Discrete compounding and par yield

As we have seen, the yield serves as a sort of average interest rate appropriate for pricing the bond. We obtained this as the rate of continuously compounded interest paid by the account we used to replicate its promised cash flows.

As with all other interest rates, this one can easily be converted from continuous compounding to any other form of discrete compounding we like. This conversion is time-invariant, meaning that a given continuous yield y corresponds to the same discretely compounded yield y_m with compounding frequency m, irrespective of the bond's maturity.

Thus, for the arbitrary set of cash flows we have previously discussed, the yield using discrete compounding satisfies:

$$V = \sum_{i=1}^{n} N \frac{c}{m} \left(1 + \frac{y_m}{m}\right)^{-mt_i} + N\left(1 + \frac{y_m}{m}\right)^{-mT}$$

An interesting result becomes evident when we consider the case of $c = y_m$. In this case, the above becomes:

$$V = \sum_{i=1}^{n} N \frac{y_m}{m} \left(1 + \frac{y_m}{m}\right)^{-mt_i} + N\left(1 + \frac{y_m}{m}\right)^{-mT}$$

Recalling that $T = t_n$, we can split off the last term of our sum…

$$V = \sum_{i=1}^{n-1} N \frac{y_m}{m} \left(1 + \frac{y_m}{m}\right)^{-mt_i} + N \frac{y_m}{m} \left(1 + \frac{y_m}{m}\right)^{-mt_n} + N\left(1 + \frac{y_m}{m}\right)^{-mt_n}$$

$$V = \sum_{i=1}^{n-1} N \frac{y_m}{m} \left(1 + \frac{y_m}{m}\right)^{-mt_i} + N\left(1 + \frac{y_m}{m}\right)\left(1 + \frac{y_m}{m}\right)^{-mt_n}$$

$$V = \sum_{i=1}^{n-1} N \frac{y_m}{m} \left(1 + \frac{y_m}{m}\right)^{-mt_i} + N\left(1 + \frac{y_m}{m}\right)^{-mt_{n-1}}$$

…and obtain the yield expression for a bond having the same coupon and yield as our original bond, but now maturing one coupon period sooner. If we repeat the process all the way down to the first cash flow, we see that…

$$V = N \frac{y_m}{m} \left(1 + \frac{y_m}{m}\right)^{-1} + N\left(1 + \frac{y_m}{m}\right)^{-1}$$

…since $t_1 = 1 / m$. In other words…

$$V = N\left(1 + \frac{y_m}{m}\right)\left(1 + \frac{y_m}{m}\right)^{-1} = N$$

…and the bond is priced at par.

Earlier, we called the coupon that a bond must pay in order to satisfy this requirement its fair coupon. We see now that when the bond's coupon is equal to this value, then that value is also the bond's yield, as long as the yield is expressed with compounding equal to the payment frequency *m* of the bond.

For this reason, the quantity we earlier solved for using…

$$c^* = y_m^* = \frac{1 - P_T}{A_{m,T}}$$

…is typically referred to as the par yield—the yield that causes the bond to price at par. In the case of a fixed-coupon bond, this yield is equal to the coupon rate.

For this reason, the yield of a bond with coupon frequency *m* is usually reported using the same compounding frequency. If $y_m > c$, then the bond is priced at a discount; if $y_m < c$, it is priced at a premium. And as we will shortly see, the size of the difference can be used in conjunction with another characteristic of the bond to develop a quick approximation of how different the bond's price is from par.

Duration

In our earlier discussion of yield's uniqueness, we saw that the derivative of the price of an arbitrary set of cash flows with respect to its yield is…

$$\frac{dV}{dy} = -\sum_{i=1}^{n} t_i c_i e^{-yt_i}$$

…which in the special case of a fixed-coupon bond becomes:

$$\frac{dV}{dy} = -\sum_{i=1}^{n} t_i N \frac{c}{m} e^{-yt_i} - TNe^{-yT}$$

This is evidently a useful expression for determining the change in a bond's price under a relatively small change in its yield. Yield changes for a particular bond may happen for various reasons, but when risk-free interest rates change, all bonds' yields (and therefore prices) tend to change as well. Moreover, the primary fluctuation that interest rates experience is a parallel shift, where the rates to all maturities change by more or less the same amount. It is thus quite practical—and was particularly so before the advent of computers—to have a single number available that can be used to estimate the effect of such a shift.

The raw derivative, however, is in dollar terms, making it inconvenient to compare bond positions of different sizes to one another. Moreover, from the standpoint of an investor it is primarily the return (or loss) associated with a move that is of interest. Finally, we can see that the derivative is negative in all cases for a fixed-coupon bond. These facts all help to explain why the fundamental bond analytic called duration takes the form:

$$D = -\frac{1}{V}\frac{dV}{dy}$$

This expresses the dollar PL associated with a yield shift as a return on the present value of the investment, and switches the sign so that the reported value will be positive. This is the primary bond characteristic used to describe an instrument's interest rate risk. A bond with long duration has a greater exposure to such shifts than does a bond with short duration.

Interpretations of duration

In the case of a zero-coupon bond with face amount N maturing at T, duration has a clear interpretation:

$$V = Ne^{-yT}$$

$$\frac{dV}{dy} = -TNe^{-yT}$$

$$D = -\frac{1}{V}\frac{dV}{dy} = \frac{TNe^{-yT}}{Ne^{-yT}} = T$$

That is, the duration of such a bond is simply equal to its time to maturity. Note that this result holds for an arbitrary yield y, so it is equally true for both risk-free and risky zero-coupon bonds.

As we have seen, many cash instruments can be treated as portfolios of zero-coupon bonds. Suppose we have such an instrument with deterministic cash flows $c_1, c_2, \ldots, c_n$ as above. The instrument has a present value V that, as we have seen, we can associate with a yield y. For this collection of cash flows, we can thus calculate duration as…

$$V = \sum_{i=1}^{n} c_i e^{-yt_i}$$

$$\frac{dV}{dy} = -\sum_{i=1}^{n} t_i c_i e^{-yt_i}$$

$$D = -\frac{1}{V}\frac{dV}{dy} = \frac{\sum_{i=1}^{n} t_i c_i e^{-yt_i}}{\sum_{i=1}^{n} c_i e^{-yt_i}}$$

The t_i are also the durations of the cash flows considered individually, so the instrument's duration takes the form of a weighted average of these durations, with the weights equal to the present values of the cash flows *discounted using the overall instrument's yield*. This is an important distinction to make, since even in the case of risk-free cash flows, the weight assigned to each cash flow in this calculation is generally not the same as the present value of the cash flow. Still, it is interesting to note that the result of the calculation in this case is indeed a weighted average time, and is most often called the Macaulay duration of the bond.

Modified duration

Earlier, we saw that the yield of a fixed-coupon instrument can be converted easily to any other compounding frequency m and that the resulting value y_m can be compared in appealing ways to the bond's coupon c when the frequency chosen is the "natural" one for the bond.

Given that yields are most often reported per bond in the natural frequency, it also seems worthwhile to consider the sensitivity to yield changes in this case. The resulting analytic is called the modified duration of the bond. By analogy, we define this as:

$$D_m = -\frac{1}{V}\frac{dV}{dy_m}$$

We observe that the continuously compounded yield y can be expressed as a differentiable function of y_m…

$$e^{-yT} = \left(1 + \frac{y_m}{m}\right)^{-mT}$$

$$y = m\ln\left(1 + \frac{y_m}{m}\right)$$

$$\frac{dy}{dy_m} = \frac{1}{m}m\frac{1}{1 + \frac{y_m}{m}} = \left(1 + \frac{y_m}{m}\right)^{-1}$$

…and that if we consider V as a function of y…

$$V(y(y_m)) \Rightarrow \frac{d}{dy_m}V = \frac{dV}{dy}\frac{dy}{dy_m}$$

…giving the result:

$$D_m = -\frac{1}{V}\frac{dV}{dy_m} = -\frac{1}{V}\frac{dV}{dy}\frac{dy}{dy_m} = D\left(1 + \frac{y_m}{m}\right)^{-1}$$

Thus, any duration can be simply converted to a modified duration based on the instrument's yield. Unfortunately, this conversion causes the analytic to lose some of its intuitive appeal, since it means that the duration of a zero-coupon bond is no longer the same as its time to maturity, and that furthermore the degree of the difference depends upon its yield. Still, it is a useful analytic particularly in cases such as US Treasury instruments with maturity greater than a year, where all instruments have the same semiannual coupon frequency, and as a result all yields also are most naturally expressed with semiannual compounding.

Using duration to approximate PL's

If we know that a particular bond we hold is currently trading at V and has duration D, it becomes quite straightforward to calculate a first-order approximation of the PL ΔV caused by a given small yield shift Δy, since…

$$\frac{\Delta V}{\Delta y} \approx \frac{dV}{dy}$$

$$D \approx -\frac{1}{V}\frac{\Delta V}{\Delta y}$$

$$\Delta V \approx -DV\Delta y$$

…or, more simply in return terms:

$$\frac{\Delta V}{V} \approx -D\Delta y$$

The corresponding results hold for modified duration:

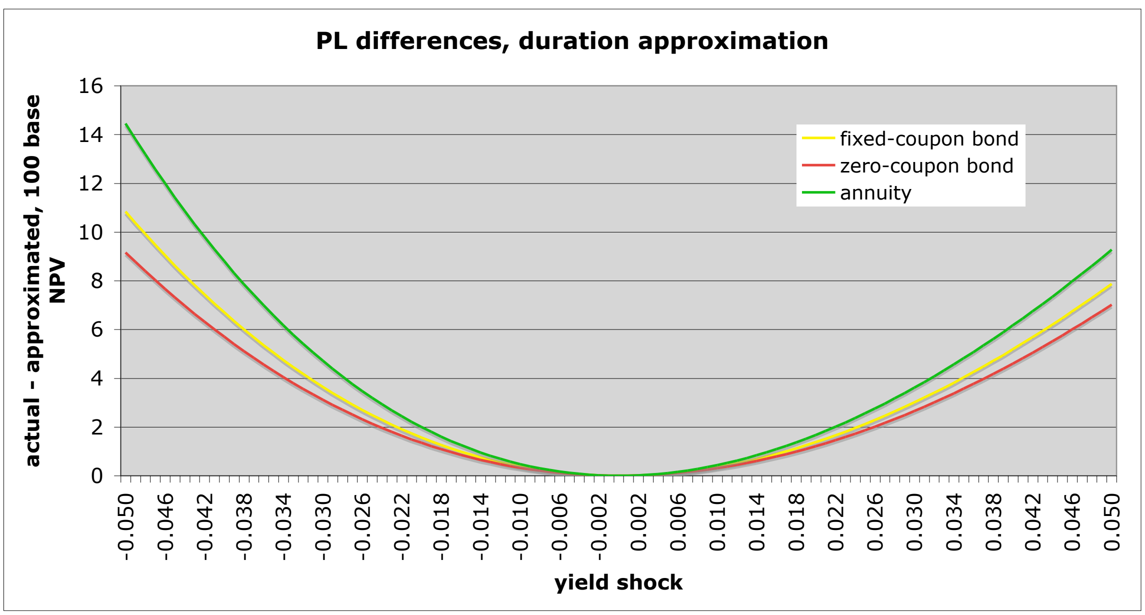
$$\Delta V \approx -D_m V \Delta y_m$$

$$\frac{\Delta V}{V} \approx -D_m \Delta y_m$$

These are most convenient approximations, but how good are they really? To illustrate some characteristics of the approximation, we consider three instruments chosen to have the same present value and duration at the same yield: a fixed-coupon bond, a zero-coupon bond, and an annuity. The characteristics of each are described below:

instrument	T	m	c	N	V	D
bond	10	2	0.05	100	100	7.98944567
zero-coupon bond	7.98944567	n/a	n/a	148.3732057	100	7.98944567
annuity	18.37771106	2	0.082979149	100	100	7.98944567

We can then compare how these instruments respond to changes in yield and evaluate the quality of the duration approximation. The results are summarized below:



Since the instruments have the same starting value and are duration-matched, we would expect their PLs to be essentially identical for small changes in yield. As we can see, though, all three instruments have greater value than predicted by the duration approximation. This seems sensible, since the price functions of bonds are sums of exponential functions in y, which are convex, while the duration approximation is strictly linear.

Furthermore, we see that these instruments do not all exhibit the same amount of excess return over the duration approximation for large shocks. The annuity is most resistant to these shocks, while the zero-coupon bond is least resistant. This suggests that we might achieve a better approximation by including a higher-order effect, and that the strength of this effect provides further useful information about the instrument.

Convexity

If we see the instrument's price V as a function of its yield y, then as we have seen these functions are quite smooth. This means we can express the effect of a yield shock Δy at a time when the bond's price is V_0 and its yield is y_0 using the Taylor polynomial approximation:

$$V\left(y_0 + \Delta y\right) = V_0 + \frac{dV}{dy}\Delta y + \frac{1}{2}\frac{d^2V}{dy^2}\Delta y^2 + \varepsilon$$

$$\Delta V \approx \frac{dV}{dy}\Delta y + \frac{1}{2}\frac{d^2V}{dy^2}\Delta y^2$$

$$\frac{\Delta V}{V} \approx \frac{1}{V}\frac{dV}{dy}\Delta y + \frac{1}{V}\frac{1}{2}\frac{d^2V}{dy^2}\Delta y^2 = -D\Delta y + \frac{1}{2}C\Delta y^2$$

$$\Delta V \approx -DV\Delta y + \frac{1}{2}CV\Delta y^2$$

The second-order term we defined above as…

$$C = \frac{1}{V}\frac{d^2V}{dy^2}$$

…is known as the convexity of the bond. No sign adjustment is needed, since for instruments consisting of deterministic cash flows, this value is always positive. If duration is a measure of an instrument's exposure to changes in interest rates, then convexity is a measure of how resistant the instrument is to these changes. As such, it is a desirable quality to seek among a collection of bonds with similar durations.

Our illustration above of the three duration-matched instruments provides a sense of what characteristics confer convexity. Numerically, the values of these instruments' convexities are:

instrument	C
bond	73.36146312
zero-coupon bond	63.83124214
annuity	91.17921297

The zero-coupon bond, where the only cash flow occurs at its duration, offers the least; the annuity, with its cash flows distributed evenly throughout its term, offers the most. A bond, which is a portfolio of the two instruments, demonstrates, not surprisingly, an intermediate amount of convexity. Generally speaking, an instrument with larger cash flows occurring close to time zero will show greater convexity than an instrument of the same duration with larger cash flows occurring later.

While this is hardly a formal argument, nevertheless it seems intuitive to conclude that the zero-coupon bond in fact offers the least convexity of any instrument with the same duration. Indeed this is true among instruments consisting of deterministic cash flows. The convexity of a zero-coupon bond also has a quite simple form:

$$V = Ne^{-yT}$$

$$\frac{dV}{dy} = -TV$$

$$\frac{d^2V}{dy^2} = T^2V$$

$$C = \frac{1}{V}\frac{d^2V}{dy^2} = T^2$$

While it is less closely monitored than duration, convexity nevertheless plays an important role in the evaluation of fixed income securities and portfolios. Since its size is generally large relative to the size of duration—and also because some market participants prefer to express yield shocks as percentages—convexities quoted by providers of fixed-income analytics are often reported as the value *C* defined above divided by 100.