MTH 9831, FALL 2015. LECTURE 2

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Abstract. In this lecture we continue to study Brownian motion.
1. Strong Markov property, reflection principle. Application: joint distribution of Brownian motion and its running maximum.
2. Scaling properties of Brownian motion.
3. Continuous time martingales. Martingales associated to the Brownian motion. Lévy's characterization of Brownian motion.
4. Stochastic representation of solutions of the heat equation.
5. Geometric Brownian motion. Applications.

1. Strong Markov property, reflection principle. Application: joint distribution of Brownian motion and its running maximum.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. At first we recall definitions of a filtration and a stopping time (now in a continuous setting).

Definition 1.1. *A family of σ-algebras $(\mathcal{F}(t))_{t\geq 0}$ is called a **filtration** if $\mathcal{F}(s) \subset \mathcal{F}(t)$ for all $0 \leq s \leq t$.*

Definition 1.2. *A random variable $\tau : \Omega \to [0, \infty]$ is said to be a stopping time relative to the filtration $(\mathcal{F}(t))_{t\geq 0}$ if $\{\tau \leq t\} \in \mathcal{F}(t)$ for every $t \geq 0$.*

Strong Markov property is the extension of Markov property to stopping times. Let τ be a stopping time relative to filtration $(\mathcal{F}(t))_{t\geq 0}$.[1] Define

$$\mathcal{F}_\tau = \{A \in \mathcal{F} \,:\, A \cap \{\tau \leq t\} \subset \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Here is a version of the strong Markov property for Brownian motion.

Theorem 1.3. *Let $(B(t))_{t\geq 0}$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}(t))_{t\geq 0}$ be its filtration (in the sense of Definition 5.5 of Lecture 1), and τ be a stopping time relative to $(\mathcal{F}(t))_{t\geq 0}$. Then for all $A \in \mathcal{B}$ and $t \geq 0$*

$$P(B(t+\tau) \in A \,|\, \mathcal{F}_\tau) = p(t, B(\tau), A), \text{ where}$$
$$p(t, x, A) := P(B(t) \in A \,|\, B(0) = x) = \frac{1}{\sqrt{2\pi t}} \int_A e^{-(y-x)^2/(2t)}\, dy.$$

Informally speaking, the strong Markov property says that the process $(B(t+\tau) - B(\tau))_{t\geq 0}$ is again a Brownian motion and is independent from $\mathcal{F}_\tau$ or, in other words, "Brownian motion starts afresh at every stopping time". The proof is omitted. An important consequence of the strong Markov property is the reflection principle. We shall need the following fact.

Lemma 1.4. *Let $(B(t))_{t\geq 0}$ be a Brownian motion and $\tau_a := \inf\{t \geq 0 \,|\, B(t) = a\}$. Then τ_a is a stopping time relative to the natural filtration of $(B(t))_{t\geq 0}$.*

The proof can be found in the textbook on pp. 341-342.

[1] i.e. $\{\tau \leq t\} \subset \mathcal{F}(t)$ for all $t \geq 0$.

Theorem 1.5 (Reflection principle)**.** *Let $(B(t))_{t\geq 0}$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$. Then for every $a \geq 0$ and $t > 0$*

$$P(\max_{0\leq s\leq t} B(s) > a) = 2P(B(t) > a).$$

Sketch of a proof. Let $\tau_a = \inf\{t > 0 : B(t) = a\}$. Then τ_a is a stopping time and $B(\tau_a) = a$. By the strong Markov property, $B(t + \tau_a) - B(\tau_a)$ is a Brownian motion which is independent of $\mathcal{F}(\tau_a)$. Notice that $P(\max_{0\leq s\leq t} B(s) > a) = P(\tau_a < t)$. We have

$$\begin{aligned} P(\tau_a < t) &= P(\tau_a < t, B(t) > a) + P(\tau_a < t, B(t) \leq a) \\ &= P(B(t) > a) + P(B(t) \leq B(\tau_a) \,|\, \tau_a < t)P(\tau_a < t) \\ &= P(B(t) > a) + P(B(t) - B(\tau_a) \leq 0 \,|\, \tau_a < t)P(\tau_a < t) = P(B(t) > a) + \frac{1}{2}P(\tau_a < t). \end{aligned}$$

This gives $P(\tau_a < t) = 2P(B(t) > a)$ as claimed. □

Corollary 1.6. *Let $(B(t))_{t\geq 0}$ be a standard Brownian motion and $\tau_a := \inf\{t \geq 0 : B(t) = a\}$. For $a \neq 0$ the density of τ_a is given by*

$$f_{\tau_a}(t) = \frac{|a|}{t\sqrt{2\pi t}} e^{-a^2/2t} \mathbb{1}_{(0,\infty)}(t).$$

Proof. Since $-B(t)$ is also a standard Brownian motion, the distribution of τ_a is the same as that of τ_{-a}. Therefore, we can assume that $a > 0$. It is clear that $P(\tau_a < t) = 0$ for all $t \leq 0$. From the reflection principle we have for all $t > 0$

$$P(\tau_a < t) = P(\max_{0\leq s\leq t} B(s) > a) = 2P(B(t) > a) = 2P\left(Z > \frac{a}{\sqrt{t}}\right) = 2 - 2N\left(\frac{a}{\sqrt{t}}\right).$$

Differentiating with respect to t we get for all $t > 0$

$$f_{\tau_a}(t) = -2N'\left(\frac{a}{\sqrt{t}}\right)\left(-\frac{a}{2t^{3/2}}\right) = \frac{a}{t\sqrt{2\pi t}} e^{-a^2/2t}.$$

□

Exercise 1. Let $(B(t))_{t\geq 0}$ be a Brownian motion. Show that

$$P(\max_{0\leq t\leq 1} |B(t)| > a) \leq 4P(B(1) > a).$$

Conclude that all moments of $X := \max_{0\leq t\leq 1} |B(t)|$ are finite and the moment generating function of X is well-defined on the whole $\mathbb{R}$.

The next statement (given without a proof) is another version of the reflection principle which is often used in applications.

Theorem 1.7 (Reflection principle)**.** *Let $(B(t))_{t\geq 0}$ be a Brownian motion and τ be a stopping time with respect to the filtration of this Brownian motion. Then the process*

$$\widetilde{B}(t) := \begin{cases} B(t), & \text{if } t \leq \tau; \\ 2B(\tau) - B(t), & \text{if } t > \tau \end{cases}$$

is also a Brownian motion.

Theorem 1.8 (Joint distribution of Brownian motion and its running maximum)**.** *Let $(B(t))_{t\geq 0}$ be a Brownian motion and $B^*(t) := \max_{0\leq s\leq t} B(s)$. For all $a > 0$, $x \leq a$, and all $t \geq 0$*

$$P(B^*(t) \geq a, B(t) \leq x) = 1 - N\left(\frac{2a - x}{\sqrt{t}}\right),$$

where $N(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-y^2/2}\, dy$.

Proof. For all $a > 0$, $x \le a$, and $t \ge 0$ we have

$$P(B^*(t) \ge a, B(t) \le x) = P(\tau_a \le t, B(t) \le x).$$

The reflection principle says that if we replace each path with the path reflected with respect to the line $x = a$ after time τ_a then the reflected process will be again Brownian motion. This and the equality $\tau_a = \widetilde{\tau}_a$, where $\widetilde{\tau}_a$ is the hitting time of a for $\widetilde{B}$, imply that the last probability above is the same as

$$P(\widetilde{\tau}_a \le t, \widetilde{B}(t) \ge 2a - x) \overset{\text{note: } x \le a}{=} P(\widetilde{B}(t) \ge 2a - x) = 1 - N\left(\frac{2a - x}{\sqrt{t}}\right). \qquad \square$$

Exercise 2. *Find the joint density of $(B^*(t), B(t))$. Sketch the region where this joint density is non-zero.*

2. Scaling properties of Brownian motion.

Theorem 2.1. *Let $(B(t))_{t\ge0}$ be a Brownian motion. Then*

(i) *for every $c \in \mathbb{R} \setminus \{0\}$ the process $(cB(c^{-2}t))_{t\ge0}$ is a Brownian motion.*
(ii) *for every fixed $s > 0$ the process $(B(s) - B(s - t))_{0\le t\le s}$ is a Brownian motion on $[0, s]$.*
(iii) *$(tB(t^{-1})_{t\ge0})$ (defined to be 0 at $t = 0$) is a Brownian motion.*

Proof. Parts (i) and (ii) are contained in (6) of Assignment 1, the proofs amount to a straightforward check of Definition 2.1. We shall prove (iii). It is just as easy except for one detail: continuity at 0. One has to show that the paths are continuous on $[0, \infty)$ a.s.. There is no problem with continuity on $(0, \infty)$ as the function t^{-1} is continuous on $(0, \infty)$ (t is continuous everywhere).

Notice that

$$\lim_{t\to0+} tB(t^{-1}) = \lim_{s\to\infty} \frac{B(s)}{s}.$$

The next lemma supplies the necessary result.

Lemma 2.2. *Let $(B(s))_{s\ge0}$ be a Brownian motion. Then*[2]

$$\lim_{s\to\infty} \frac{B(s)}{s} = 0 \quad a.s..$$

Let us assume this lemma (its proof is given right after the proof of this theorem) and continue. Now that we know that the process in (iii) a.s. has continuous paths we can proceed. We shall use Theorem 2.2(i) from Lecture 1. It is obvious that for all $m \in \mathbb{N}$ and $t_0 = 0 < t_1 < \cdots < t_m$ the random variables $t_1B(1/t_1)$, $t_2B(1/t_2)$, $\dots$, $t_mB(1/t_m)$ are jointly normally distributed and have mean 0. We only need to check the covariance function. For all $s, t > 0$ the case $t = 0$ or $s = 0$ is trivial) the covariance of $sB(1/s)$ and $tB(1/t)$ is

$$E(tsB(1/t)B(1/s)) = ts\left(\frac{1}{t} \wedge \frac{1}{s}\right) = t \wedge s. \qquad \square$$

Proof of Lemma 2.2. This fact is essentially a consequence of the strong law of large numbers. Write $B(n) = \sum_{k=1}^n (B(k) - B(k-1))$. Since $(B(k) - B(k-1))_{k\ge1}$ are i.i.d. mean zero standard normal random variables, the strong law of large numbers implies that $B(n)/n \to 0$ a.s. as $n \to \infty$. We have

$$\frac{B(s)}{s} = \frac{B(s) - B([s])}{s} + \frac{B([s])}{[s]}\,\frac{[s]}{s}.$$

By the above observation, $B([s])/[s] \to 0$ as $s \to \infty$ a.s.. Moreover, $[s]/s \to 1$ as $s \to \infty$, and we only need to deal with the term $(B(s) - B([s]))/s$. Note that for $n \le s < n + 1$ we have

$$\left|\frac{B(s) - B([s])}{s}\right| \le \frac{1}{n} \max_{n\le s<n+1} |B(s) - B(n)|.$$

[2] In fact, a much more precise statement about the behavior of Brownian motion at infinity is known to hold, namely, $\limsup_{s\to\infty} \frac{B(s)}{\sqrt{2s\ln\ln s}} = 1$ a.s.. This statement is called the *law of iterated logarithm*.

Therefore,

$$\lim_{s\to\infty}\left|\frac{B(s)-B([s])}{s}\right|\le \lim_{n\to\infty}\frac{X_n}{n},\quad \text{where } X_n:=\max_{n\le s<n+1}|B(s)-B(n)|,$$

where random variables X_n, $n\in\mathbb{N}$, have the same distribution as $X:=\max_{0\le t<1}|B(t)|$. We want to show that $X_n/n\to 0$ a.s. as $n\to\infty$. This will finish the proof of the lemma.

The key to the proof of the last statement is the identity

$$\left\{\lim_{n\to\infty}\frac{X_n}{n}\ne 0\right\}=\bigcup_{m=1}^{\infty}\left(\bigcap_{N=1}^{\infty}\bigcup_{n\ge N}\left\{\frac{X_n}{n}\ge\frac{1}{m}\right\}\right).$$

To show that the probability of the last union is zero, it is sufficient to show that for each m the probability of the event in parentheses is zero. Fix an arbitrary $m\in\mathbb{N}$ and let $A_{n,m}:=\{X_n/n\ge 1/m\}$. Note that

$$P\left(\bigcap_{N=1}^{\infty}\bigcup_{n\ge N}A_{n,m}\right)=\lim_{N\to\infty}P\left(\bigcup_{n\ge N}A_{n,m}\right)\le\sum_{n=N}^{\infty}P(A_{n,m}).$$

The last sum is the tail of a convergent series:

$$\sum_{n=1}^{\infty}P(A_{n,m})=\sum_{n=1}^{\infty}P(X_n/n\ge 1/m)=\sum_{n=1}^{\infty}P(mX\ge n)<\infty.$$

The series converges by Exercise 1 and problem (4) of refresher HW2. We conclude that $\sum_{n=N}^{\infty}P(A_{n,m})$ vanishes as $N\to\infty$, and, since this is true for every m, we are done. $\square$

Part (iii) of Theorem 2.1 allows us to translate properties of Brownian motion at infinity into properties at zero and the other way around. The next exercise illustrates this point.

Exercise 3. Let $(B(t))_{t\ge 0}$ be a Brownian motion. Use the law of iterated logarithm (see the footnote on the previous page) and part (iii) of Theorem 2.1 to show that

$$\limsup_{t\to 0+}\frac{|B(t)|}{\sqrt{2t\ln\ln(1/t)}}=1.$$

3. Continuous time martingales. Martingales associated to the Brownian motion. Lévy's characterization of Brownian motion.

Definition 3.1. *A stochastic process $(M(t))_{t\ge0}$, on $(\Omega,\mathcal{F},P)$ is said to be a martingale with respect to filtration $(\mathcal{F}_t)_{t\ge0}$ if*

(i) *$E(|M(t)|)<\infty$ for all $t\ge0$;*
(ii) *$M(t)$ is $\mathcal{F}_t$-measurable for all $t\ge 0$;*
(iii) *$E(M(t)\,|\,\mathcal{F}(s))=M(s)$ a.s. for all $t\ge s\ge 0$.*

Theorem 3.2. *Let $(B(t))_{t\ge0}$ be a Brownian motion and $(\mathcal{F}(t))_{t\ge0}$ be a filtration for this Brownian motion. The following processes are martingales relative to $(\mathcal{F}(t))_{t\ge0}$:*

(a) *$(B(t))_{t\ge0}$;*
(b) *$(B^2(t)-t)_{t\ge0}$;*
(c) *$\left(e^{\sigma B(t)-\sigma^2t/2}\right)_{t\ge0}$ (for each $\sigma\in\mathbb{R}$).*

Proof. Parts (a) and (b) are simple and are left as exercises. Fix $\sigma\ne0$ (when $\sigma=0$ the process in (c) is identically equal to 1 so it is trivially a martingale) and set $M(t)=e^{\sigma B(t)-\sigma^2t/2}$, $t\ge0$. From the tail behavior of Brownian motion we know that all moments of $M(t)$ are finite. Also, simply by construction, $M(t)$ is $\mathcal{F}_t$-measurable. Let $0 \le s \le t$. Then

$$E(M(t)\,|\,\mathcal{F}(s)) = E\left(e^{\sigma(B(t)-B(s))+\sigma B(s)-\sigma^2 t/2}\,|\,\mathcal{F}(s)\right) = e^{\sigma B(s)-\sigma^2 t/2} E\left(e^{\sigma(B(t)-B(s)}\right)$$
$$= e^{\sigma B(s)-\sigma^2 t/2}\, e^{\sigma^2(t-s)/2} = e^{\sigma B(s)-\sigma^2 s/2} = M(s).$$

Here we used the fact that $B(t)-B(s)$ is independent of $\mathcal{F}(s)$ and $\sigma(B(t)-B(s)) \sim \text{Normal}(0,\sigma^2(t-s))$ so that $E\left(e^{\sigma(B(t)-B(s)}\right) = E\left(e^{\sigma\sqrt{t-s}\,Z}\right) = e^{\sigma^2(t-s)/2}$ (Z is a standard normal random variable). $\square$

As an application, we shall prove a useful fact about the Laplace transform[3] of the first hitting time by a standard Brownian motion of a given level a.

Theorem 3.3. *Let $(B(t))_{t\ge 0}$ be a Brownian motion and $\tau_a := \inf\{t \ge 0 : B(t) = a\}$. Then $E(e^{-\lambda\tau_a}) = e^{-|a|\sqrt{2\lambda}}$.*

Proof. Without loss of generality we shall assume that $a > 0$ (the case $a = 0$ is trivial and the case $a < 0$ follows by symmetry). Let $M(t) = e^{\sigma B(t)-\sigma^2 t/2}$ for some $\sigma > 0$. Consider the process $(M(\tau_a \wedge t))_{t\ge 0}$. This process is also a martingale. Since $\tau_a \wedge t \le t$, we can apply the continuous analog of the optional stopping theorem A from refresher Lecture 5.[4] We get

$$1 = E(M(0) = E(M(\tau_a \wedge t)) = E\left(e^{\sigma B(\tau_a\wedge t)-\sigma^2(\tau_a\wedge t)/2}\right)$$
$$= E\left(e^{\sigma B(\tau_a\wedge t)-\sigma^2(\tau_a\wedge t)/2}; \tau_a \le t\right) + E\left(e^{\sigma B(\tau_a\wedge t)-\sigma^2(\tau_a\wedge t)/2}; \tau_a > t\right).$$

Notice that on the event $\{\tau_a > t\}$ we have $B(t) \le a$. Therefore,

$$E\left(e^{\sigma B(\tau_a\wedge t)-\sigma^2(\tau_a\wedge t)/2}; \tau_a > t\right) \le E\left(e^{\sigma a-\sigma^2 t/2}; \tau_a > t\right) \le e^{\sigma a} P(\tau_a > t) \to 0 \text{ as } t \to \infty.$$

Finally, we claim that

$$E\left(e^{\sigma B(\tau_a\wedge t)-\sigma^2(\tau_a\wedge t)/2}; \tau_a \le t\right) = E\left(e^{\sigma a-\sigma^2\tau_a/2}; \tau_a \le t\right) \to 1 \text{ as } t \to \infty.$$

Indeed the random function $e^{\sigma a-\sigma^2\tau_a/2} 1_{\{\tau_a\le t\}}$ is non-decreasing in t and converges to $e^{\sigma a-\sigma^2\tau_a/2}$ as $t \to \infty$, since $P(\tau_a < \infty) = 1$ (see Exercise 1). By the monotone convergence theorem, $E(e^{\sigma a-\sigma^2\tau_a/2} 1_{\{\tau_a\le t\}})$ converges to $E\left(e^{\sigma a-\sigma^2\tau_a/2}\right)$. We conclude that $E\left(e^{\sigma a-\sigma^2\tau_a/2}\right) = 1$ which is the same as $E\left(e^{-\sigma^2\tau_a/2}\right) = e^{-\sigma a}$. Denoting $\sigma^2/2$ by λ we obtain the statement of the theorem. $\square$

We finish this section with a powerful and somewhat surprising result due to Paul Lévy. The proof is best done using stochastic calculus and will be sketched later in the semester.

Theorem 3.4 (Lévy's characterization of Brownian motion)**.** *Let $(M(t))_{t\ge 0}$ be a stochastic process with continuous paths and such that $M(0) = 0$ and both $(M(t))_{t\ge 0}$ and $(M^2(t) - t)_{t\ge 0}$ are martingales relative to filtration $(\mathcal{F}(t))_{t\ge 0}$. Then $(M(t))_{t\ge 0}$ is a standard Brownian motion.*

[3] Laplace transform $\mathcal{L}(\lambda)$ of a non-negative random variable X is defined as $E(e^{-\lambda X})$ for $\lambda \ge 0$. Its properties are very similar to those of the moment generating function.

[4] Let $(M(t))_{t\ge 0}$ be a martingale with right-continuous paths which is adapted to a right-continuous filtration $(\mathcal{F}(t))_{t\ge 0}$ (i.e. such that $\cap_{t>s} F(t) = \mathcal{F}(s)$ for all $s \ge 0$). If $\tau \le K$ a.s. for some $K > 0$ then $E(M(\tau)) = E(M(0))$.

4. Stochastic representation of solutions of the heat equation.

Let $f(x)$ be a continuous bounded function on $\mathbb{R}$. Let

$$u(t,x)=E(f(B(t))\,|\,B(0)=x)=:E_x(f(B(t))=\int_{\mathbb{R}}f(y)p(t,x,y)\,dy, \tag{4.1}$$

where

$$p(t,x,y)=\frac{1}{\sqrt{2\pi\,t}}e^{-(y-x)^2/(2t)}$$

is the so-called transition probability density of the Brownian motion.[5] Here we start the Brownian motion from x instead of 0 and still call the process $(B(t))_{t\geq 0}$.

Theorem 4.1. *Let f be a bounded continuous function with bounded continuous first and second derivatives*[6] *and $u(t,x)$ be defined by (4.1). Then $u(t,x)$ solves the heat equation $u_t=\frac{1}{2}\,u_{xx}$ with the initial condition $u(0,x)=f(x)$.*

Proof. Under our assumption we can differentiate under the integral sign up to two times. By changing the variable of integration, $z:=(y-x)/\sqrt{t}$, and then integrating by parts we get

$$u(t,x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}f(x+\sqrt{t}z)e^{-z^2/2}\,dz;\quad u_{xx}(t,x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}f''(x+\sqrt{t}z)e^{-z^2/2}\,dz;$$

$$\begin{aligned}u_t(t,x)&=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}f'(x+\sqrt{t}z)\frac{z}{2\sqrt{t}}e^{-z^2/2}\,dz=-\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}f'(x+\sqrt{t}z)\frac{1}{2\sqrt{t}}\,de^{-z^2/2}\\&=-\frac{1}{\sqrt{2\pi}2\sqrt{t}}f'(x+\sqrt{t}z)e^{-z^2/2}\Big|_{-\infty}^{\infty}+\frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}}f''(x+\sqrt{t}z)e^{-z^2/2}\,dz=\frac{1}{2}u_{xx}(t,x).\end{aligned}$$

It is left to check that $\lim_{t\to 0+}u(t,x)=f(x)$. By the mean value theorem, $f(x+\sqrt{t}\,z)-f(x)=f'(\xi)\sqrt{t}\,z$, where $\xi\in[x,x+\sqrt{t}\,z]$. Since we assumed that f' is a bounded function, $|f(x+\sqrt{t}\,z)-f(x)|\leq M\sqrt{t}\,|z|$ for some constant M and all $t\geq 0$, $x,z\in\mathbb{R}$. Then

$$|u(t,x)-f(x)|\leq\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|f(x+\sqrt{t}z)-f(x)|e^{-z^2/2}\,dz\leq\frac{M\sqrt{t}}{\sqrt{2\pi}}\int_{\mathbb{R}}|z|e^{-z^2/2}\,dz\leq C\sqrt{t}$$

for some constant C. The last expression goes to 0 as $t\to 0+$. □

In view of the above theorem formula (4.1) can be referred to as a stochastic representation of a solution to the initial value problem for the heat equation. This representation, in particular, gives "for free" several important properties of solutions. For example, it is completely obvious from (4.1) that if $f(x)\geq 0$ for all $x\in\mathbb{R}$ then the solution $u(t,x)\geq 0$ for all $t>0,\ x\in\mathbb{R}$. It is also immediate that $u(t,x)\in[\min_{x\in\mathbb{R}}f(x),\max_{x\in\mathbb{R}}f(x)]$ for all t.

Representation (4.1) can also be used for solving the heat equation by Monte-Carlo simulation. Since these ideas extend to a large class of linear partial differential equations (where the underlying process is not a Brownian motion but a more general diffusion process), we shall say two words about Monte-Carlo simulation.

In finance one usually solves the terminal rather than the initial value problem. Namely, let us fix a finite time horizon $T>0$ and consider $v(t,x)=u(T-t,x)$, $t\in[0,T]$. Then $v(t,x)$ solves the backward heat equation $v_t+\frac{1}{2}\,v_{xx}=0$ with the terminal condition $v(T,x)=f(x)$. By Theorem 4.1 we have

$$v(t,x)=u(T-t,x)=E(f(B(T-t))\,|\,B(0)=x)=\int_{\mathbb{R}}f(y)p(T-t,x,y)\,dy. \tag{4.2}$$

[5] Informally speaking $p(t,x,y)dy$ is the probability that the Brownian motion, which is currently at x will be in the interval $(y,y+dy)$ after time t.

[6] The requirements on f can be significantly relaxed. Since my goal was to discuss main ideas with a minimal number of technicalities, I made no attempt to state the best possible result.

Observe also that $E(f(B(T-t)) \mid B(0)=x) = E(f(B(T) \mid B(t)=x)$, so in order to visualize the meaning of the formula for $v(t,x)$ we can (recall the picture drawn in class)

(1) start a Brownian motion at point x at time t,
(2) run it up to time T and compute $f(B(T))$,
(3) repeat the first two steps many-many times and take the average.

5. Geometric Brownian motion. Applications.

Given $\mu \in \mathbb{R}$ and $\sigma > 0$ the geometric Brownian motion (GBM) with parameters μ and σ which starts at a given point $S(0) > 0$ is the process

$$S(t) = S(0)e^{\mu t + \sigma B(t)},\ t \geq 0,$$

where $(B(t))_{t\geq 0}$ is the standard Brownian motion. We denote by $(\mathcal{F}(t))_{t\geq 0}$ a filtration for Brownian motion as defined in Lecture 1. Geometric Brownian motion satisfies the following properties which are very useful for computations and are often taken as a definition of the GBM.

(i) GBM has continuous paths a.s.;
(ii) $\dfrac{S(t+y)}{S(t)}$ is independent of $\mathcal{F}(t)$ for all $t, y \geq 0$;
(iii) $\ln \dfrac{S(t+y)}{S(t)}$ has normal distribution with mean μy and variance $\sigma^2 y$.

Properties (i)-(iii) follow from properties of Brownian motion.

Many questions about GBM can be converted into questions about Brownian motion. Suppose that we are interested in finding the distribution of the time when GBM hits a given level $A > S(0)$,[7] i.e. in finding $P(\tau_A^S \leq T) = P(\max_{0\leq t\leq T} S(t) \geq A)$. This probability is equal to

$$P(S(0)e^{\max_{0\leq t\leq T}(\mu t + \sigma B(t))} \geq A) = P\left(\max_{0\leq t\leq T}\left(\frac{\mu}{\sigma}t + B(t)\right) \geq \frac{1}{\sigma}\ln\frac{A}{S(0)}\right).$$

If we set $\tau_{a,b} := \inf\{t \geq 0 : B(t) = a + bt\} = \inf\{t \geq 0 : -bt + B(t) = a\}$, then the probability which we are interested in is the probability that $\tau_{a,b} \leq T$, where $a = \sigma^{-1}\ln(A/S(0))$ and $b = -\mu/\sigma$. The following theorem gives information about the distribution of $\tau_{a,b}$ through its Laplace transform. Just like the characteristic and moment generating functions, the Laplace transform of a random variable uniquely characterizes its distribution.[8]

Theorem 5.1. *For all $\lambda > 0$, $a > 0$, and $b \in \mathbb{R}$*

$$E\left(e^{-\lambda \tau_{a,b}}\right) = e^{-a(b+\sqrt{b^2+2\lambda})}.$$

An outline of the proof of this formula is given in parts (i)–(iii) of Exercise 3.7 (p. 119) of the textbook which will naturally be included in HW2. The detailed proof of Theorem 5.1 is also contained in the textbook, see Theorem 8.3.2, pp. 346–348.

Example 5.2. As a simple application of Theorem 5.1 we shall find the price of a (fictitious) perpetual option which pays \$1 at the time when the stock price $S(t)$ hits a given level $A > S(0)$. We shall assume that the interest rate is 0 and that under the risk-neutral measure the price follows the GBM $S(t) = S(0)e^{-\sigma^2 t/2 + \sigma B(t)}$, $t \geq 0$.

It might seem at the first glance that the price should be equal to \$1, since there is no discounting and we just have to wait until the stock price hits level A to get \$1. But we have to be aware of

[7] This question is relevant for pricing some exotic options in the setting of Black-Scholes model.

[8] One can find the density of $\tau_{a,b}$ without inverting the Laplace transform. This can be done using a clever but elementary method outlined in Exercise 14, p. 69, A. Etheridge, A Course in Financial Calculus. I highly recommend this exercise, since it brings together several parts of this lecture. The density of $\tau_{a,b}$ can also be found as an application of Girsanov's theorem which we shall study later. If interested, look at (7.2.6) in the textbook. Noticing that $P(\tau_{a,b} > T) = P(\max_{0\leq t\leq T}(-bt + B(t)) < a)$, where the right hand side is known by (7.2.6), and differentiating the last equation with respect to T we get the density of $\tau_{a,b}$ (with the minus sign).

the possibility that the price might never hit A, i.e. that it could be that $P(\tau_A^S = \infty) > 0$, where $\tau_A^S = \inf\{t \ge 0 \,:\, S(t) = A\}$. According to the risk-neutral pricing paradigm the option price is equal to the risk-neutral expectation of the discounted payoff. There is no discounting here, therefore, the price is equal to $\$1 \cdot P(\tau_A^S < \infty)$. Converting the question to the question about Brownian motion we see that $P(\tau_A^S < \infty) = P(\tau_{a,b} < \infty)$ with $a = \sigma^{-1}\ln(A/S(0) > 0$ and $b = \sigma/2$. Theorem 5.1 tells us that for all $\lambda > 0$

$$E(e^{-\lambda\tau_{a,b}}) = e^{-a(b+\sqrt{b^2+2\lambda})}.$$

Noting that $e^{-\lambda\tau_{a,b}} = e^{-\lambda\tau_{a,b}}\mathbb{1}_{\tau_{a,b}<\infty}$ and letting $\lambda \downarrow 0$ we conclude by the monotone convergence theorem that

$$P(\tau_{a,b} < \infty) = \lim_{\lambda\downarrow 0} E(e^{-\lambda\tau_{a,b}}\mathbb{1}_{\tau_{a,b}<\infty}) = e^{-a(b+|b|)} = \begin{cases} 1, & \text{if } b \le 0; \\ e^{-2ab}, & \text{if } b > 0. \end{cases}$$

Substituting our values of a and b we find that the price is equal to $S(0)/A$.

Exercise 4. Compute the price of the above option when $A < S(0)$ and $r \in [0, \sigma^2/2]$ (in this case, under the risk-neutral measure, $S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma B(t)}$, $t \ge 0$).