October 21, 2015

MTH 9831. Solutions to Quiz 6.

Required number of points is 10. Possible number of points is 12.

Let $(B(t))_{t\geq 0}$ be a standard Brownian motion under $\mathbb{P}$, $(\mathcal{F}(t))_{t\geq 0}$ be a the filtration generated by this Brownian motion, and $(\Theta(t))_{t\geq 0}$, be a stochastic process adapted to this filtration and such that $\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T \Theta^2(t)\,dt\right)\right) < \infty$.

(1) (3 points) Check formally that the process

$$Z(t) = \exp\left(-\int_0^t \Theta(s)\,dB(s) - \frac{1}{2}\int_0^t \Theta^2(s)\,ds\right), \quad 0 \leq t \leq T,$$

is a an $((\mathcal{F}(t))_{0\leq t\leq T}, \mathbb{P})$ martingale. Hint: Use Itô's formula for Itô processes and compute $dZ(t)$.

Solution. $Z(t)$ is of the form $e^{X(t)}$ where $X(t)$ is an Itô process. Therefore,

$$\begin{aligned} dZ(t) &= Z(t)\left(dX(t) + \frac{1}{2}d[X]_t\right) \\ &= Z(t)\left(-\Theta(t)dB(t) - \frac{1}{2}\Theta^2(t)\,dt\right) + \frac{1}{2}\Theta^2(t)dt = -\Theta(t)Z(t)dB(t), \end{aligned}$$

and the desired conclusion follows.

(2) (2 points) Let $\tilde{\mathbb{P}}$ be defined by $\tilde{\mathbb{P}}(A) = \int_A Z(T)\,d\mathbb{P}$, where $Z(T)$ is from problem (1). Which process is a standard Brownian motion under $\tilde{\mathbb{P}}$? Answer: the process

$$\tilde{B}(t) = B(t) + \int_0^t \Theta(u)\,du$$

is a standard Brownian motion under $\tilde{\mathbb{P}}$.

(3) (3 points) Show that $1/Z(t)$ is a martingale under $\tilde{\mathbb{P}}$. Hint: rewrite $1/Z(t)$ in terms of the the process $\tilde{B}(t)$ (get rid of $B(t)$) and apply problem (1).

Solution. We write

$$\begin{aligned} \frac{1}{Z(t)} &= \exp\left(\int_0^t \Theta(s)\,dB(s) + \frac{1}{2}\int_0^t \Theta^2(s)\,ds\right) \\ &= \exp\left(\int_0^t \Theta(s)\,[d\tilde{B}(s) - \Theta(s)\,ds] + \frac{1}{2}\int_0^t \Theta^2(s)\,ds\right) \\ &= \exp\left(\int_0^t \Theta(s)\,d\tilde{B}(s) - \frac{1}{2}\int_0^t \Theta^2(s)\,ds\right), \end{aligned}$$

which is problem 1 with $-\Theta(t)$ instead of $\Theta(t)$ and $\tilde{B}(t)$ instead of $B(t)$. Applying the result of problem 1 we get

$$d(1/Z(t)) = (1/Z(t))\Theta(t)d\tilde{B}(t).$$

We conclude that $1/Z(t)$ is a martingale under $\tilde{\mathbb{P}}$.

(4) (4 points) A chooser option gives its owner the right at time t_0 to choose either the call or the put (same strike K and expiration $T > t_0$). What is the time t_0 value of the chooser option? What is the time 0 price of the chooser option (you may assume that prices of all vanilla options are given)?

Solution. The time t_0 value of the chooser option is

$$\begin{aligned}\max\{C(t_0), P(t_0)\} &= C(t_0) + \max\{0, P(t_0) - C(t_0)\} \\ &= C(t_0) + \max\{0, e^{-r(T-t_0)}K - S(t_0)\},\end{aligned}$$

where $C(t_0)$ and $P(t_0)$ are the time t_0 values of the call and put mentioned above. In the last equation we used the put-call parity. Therefore the time 0 price of the chooser option is

$$\begin{aligned}A(0) &= \tilde{E}(\tilde{E}(D(T)V(T) \,|\, \mathcal{F}(t_0))) \\ &= \tilde{E}\left(e^{-rt_0}\left[C(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right]\right) \\ &= \tilde{E}\left[e^{-rt_0}C(t_0)\right] + \tilde{E}\left[e^{-rt_0}\left(e^{-r(T-t_0)}K - S(t_0)\right)^+\right] = C(0) + P^*(0),\end{aligned}$$

where $C(0)$ is the time 0 price of a call with expiration T and strike K, and $P^*(0)$ is the time 0 price of a put with expiration t_0 and strike $e^{-r(T-t_0)}K$.