Credit Risk Models

2. Reduced Form Models

Andrew Lesniewski

Baruch College
New York

Fall 2016

Outline

Reduced form models

- In a structural model of credit, a default event is an *endogenous* feature of the model and it results as a consequence of the (stochastic) value of the entity falling below a certain threshold.
- In order to take all factors properly into account, such a description requires a detailed knowledge of the capital structure of the entity, and lead to complicated computational problems.
- In contrast, in the *reduced form* approach, a default event is an *exogenous* occurrence, which is governed exclusively by the assumed probability distribution of a random time τ, namely the *time to default*.
- While structural models offer a better intuitive insight into the nature of the credit of the entity, reduce form models are generally more transparent and numerically tractable.
- We begin by reviewing the basics of the theory of counting processes.

Reduced form models

- Credit events are examples of financial processes with discrete outcomes (unlike, say, stock prices which are usually treated as continuous processes).
- The basic probabilistic set up to model event risk is based on the concept of a random time τ.
- The stochastic process $1_{\tau \leq t}$ defines a filtration $(\mathscr{G}_t)_{t \geq 0}$, which is referred to as the *information set*.
- Of key importance to financial event modeling are *counting processes*.

Counting processes

An stochastic process $N(t)$ is called a counting process if:

- $N(t)$ is defined on a probability space$(\Omega, (\mathscr{G}_t)_{t\geq 0}, \mathrm{P})$.
- $N(t)$ is integer valued, non-negative, with $N(0) = 0$.
- Sample paths of $N(t)$ are right continuous, piecewise constant, may have jumps of size 1.
- Jump times $\tau_1 < \tau_2 < \ldots$ are stopping times at which the process $N(t)$ increases by 1.
- The *intensity* $\lambda(t)$ of $N(t)$ is defined by

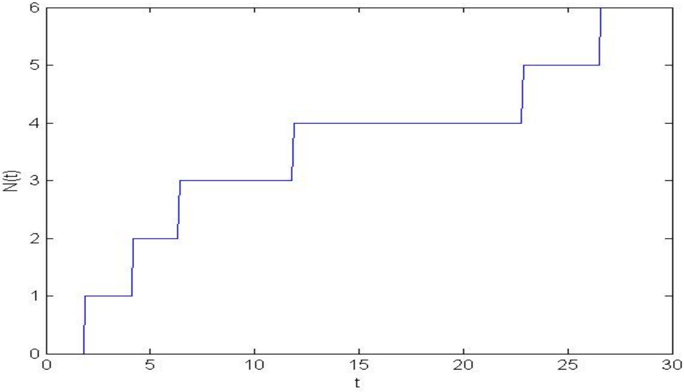
$$\lambda(t)\, dt = \mathrm{P}(N(t+dt) - N(t) = 1 \,|\, \mathscr{G}_{t\_})$$

- Alternatively,

$$\lambda(t)\, dt = \mathrm{E}[N(t+dt) - N(t) \,|\, \mathscr{G}_{t\_}.]$$

Counting processes

The graph below shows a sample path of a counting process.



Counting processes

- The simplest counting process is given by $N(t) \in \{0,1\}$ and constant intensity λ. In this case,

$$\begin{aligned} \mathsf{P}(N(t) = 0) &= \mathsf{P}(\tau \geq t) \\ &= e^{-\lambda t}. \end{aligned}$$

  This probability distribution is called the exponential distribution.

- Indeed,

$$\begin{aligned} \mathsf{P}(\tau \geq t) &= \lim_{n \to \infty} \prod_{j=0}^{n} \Big(1 - \mathsf{P}\Big(\frac{j}{n}\, t \leq \tau \leq \frac{j+1}{n}\, t\Big)\Big) \\ &= \lim_{n \to \infty} \prod_{j=0}^{n} \Big(1 - \lambda \frac{t}{n}\Big) \\ &= \Big(1 - \lambda \frac{t}{n}\Big)^{n} \\ &= e^{-\lambda t}. \end{aligned}$$

Poisson process

- Another important example of a counting process is a *Poisson process*.
- A Poisson process is defined by the two requirements:
  - The probability that given $N(t) = n$, $N(t + dt) = n + 1$ is λdt,
  - The probability that given $N(t) = n$, $N(t + dt) > n + 1$ is 0.
- Here, $\lambda > 0$ is the intensity of the Poisson process.
- Notice that the Poisson process has independent increments and is Markovian.
- We shall show that

$$\mathsf{P}(N(t) = n) = \frac{(\lambda t)^n}{n!}\, e^{-\lambda t}.$$

Poisson process

- Indeed, using $N(0) = 0$,

$$
\begin{aligned}
\mathrm{P}(N(t) = n) &= \lim_{N\to\infty} \binom{N}{n} \Big(\lambda \frac{t}{N}\Big)^n \Big(1 - \lambda \frac{t}{N}\Big)^{N-n} \\
&= \lim_{N\to\infty} \frac{N^N e^{-N}\sqrt{2\pi N}}{n!(N-n)^{N-n} e^{-N+n}\sqrt{2\pi(N-n)}} \Big(\lambda \frac{t}{N}\Big)^n \Big(1 - \lambda \frac{t}{N}\Big)^{N-n} \\
&= \frac{(\lambda t)^n}{n!} \lim_{N\to\infty} \frac{N^{N-n+1/2}}{(N-n)^{N-n+1/2} e^n} \Big(1 - \lambda \frac{t}{N}\Big)^{N-n} \\
&= \frac{(\lambda t)^n}{n!} \lim_{N\to\infty} \frac{1}{(1-\frac{n}{N})^{N-n+1/2} e^n} \Big(1 - \lambda \frac{t}{N}\Big)^{N-n} \\
&= \frac{(\lambda t)^n}{n!} \lim_{N\to\infty} \frac{1}{(1-\frac{n}{N})^{N} e^n} \Big(1 - \lambda \frac{t}{N}\Big)^{N} \\
&= \frac{(\lambda t)^n}{n!} e^{-\lambda t},
\end{aligned}
$$

where we have used Stirling's approximation $n! \approx (n/e)^n \sqrt{2\pi n}$.

Poisson process

- Furthermore, we have
$$E[N(t)] = \lambda t,$$
and
$$\mathrm{Var}[N(t)] = \lambda t.$$
- An important property of the Poisson process is that it is memoryless, meaning that
$$P(N(t+h) = n+m \,|\, N(t) = n) = P(N(h) = m).$$

Poisson process

- Let us consider the time of the first jump (or first arrival) τ_1 in a Poisson process.
- This is simply our first example of a counting process, and so its survival probability is given by the exponential distribution:

$$\mathsf{P}(\tau_1 > t) = e^{-\lambda t},$$

with the first moment

$$\mathsf{E}[\tau_1] = \frac{1}{\lambda}.$$

- The density of τ_1 can be found by differentiation of the cumulative distribution,

$$\mathsf{P}(\tau_1 \in [t, t + dt]) = e^{-\lambda t} \lambda dt.$$

- The distributions of all interarrival times $\tau_2 - \tau_1$, $\tau_3 - \tau_2, \ldots$, are all identical to the distribution of τ_1 (due to the memoryless property).

Inhomogeneous Poisson process

- Another important example of a counting process is an *inhomogeneous Poisson process*.
- It is a generalization of the Poisson procces, in which the intensity is a deterministic function of time $\lambda(t)$.
- The calculations involving inhomogeneous Poisson processes follow the lines of the homogeneous Poisson processes. It is easy to see that all the results stated above for a homogeneous Poisson process hold also for an inhomogeneous Poisson process with λ replaced by the average $\frac{1}{t}\int_0^t \lambda(s)\,ds$.
- In particular, the probability distribution is given by

$$\mathrm{P}(N(t) = n) = \frac{\left(\int_0^t \lambda(s)\,ds\right)^n}{n!}\, e^{-\int_0^t \lambda(s) ds}.$$

Inhomogeneous Poisson process

- Furthermore,

$$\mathsf{E}[N(t)] = \int_0^t \lambda(s)\, ds,$$
$$\mathrm{Var}[N(t)] = \int_0^t \lambda(s)\, ds$$

- For the properties of the first arrival time we have:

$$\mathsf{P}(\tau_1 > t) = e^{-\int_0^t \lambda(s) ds},$$
$$\mathsf{P}(\tau_1 \in [t, t+dt]) = e^{-\int_0^t \lambda(s) ds} \lambda(t)\, dt.$$

- The integral $A(t) = \int_0^t \lambda(s)\, ds$ is called the *compensator* of the process $N(t)$.

Cox processes

- A general class of counting processes are *Cox processes*.
- A Cox process $N(t)$ is associated with an information set of the form $\mathscr{G}_t \vee \mathscr{F}_t$, $t > 0$, where $\mathscr{G}_t$ is the information set pertaining to jump times, and where $\mathscr{F}_t$ is the "background" information set.
- One can think of $\mathscr{F}_t$ as containing the background economic information.
- A point process $N(t)$ is a Cox process if, conditioned on $\mathscr{F}_t$, $N(t)$ is an inhomogeneous Poisson process with a deterministic intensity $\lambda(s)$, $0 \leq s \leq t$.

Cox processes

- Intuitively, a Cox process can be thought of as an inhomogeneous Poisson process whose intensity is stochastic itself.
- We can think about it in the following way: We draw an intensity path first, and then - conditional on this path - we use the properties the inhomogeneous Poisson process.
- The compensator of a Cox process is the integrated intensity process $A(t) = \int_0^t \lambda(s)\, ds$.
- In particular,

$$\mathrm{P}\big(N(t) = n \,|\, \{\lambda(s)\}_{0 \le s \le t}\big) = \frac{A(t)^n}{n!}\, e^{-A(t)},$$

and, taking expectation over all paths $\{\lambda(s)\}$, we get the absolute probability distribution:

$$\mathrm{P}(N(t) = n) = \frac{1}{n!}\, \mathrm{E}\big[A(t)^n\, e^{-A(t)}\big].$$

Cox processes

- As another example, let us find $\mathrm{P}(\tau_1 > t)$. We have

$$\mathrm{P}\left(\tau_1 > t \,|\, \{\lambda(s)\}_{0\leq s\leq t}\right) = e^{-A(t)},$$

and by taking expectation over all paths $\{\lambda(s)\}$, we get the absolute probability:

$$\mathrm{P}(\tau_1 > t) = \mathrm{E}\left[e^{-A(t)}\right].$$

- Finally, the absolute expectation and variance of a Cox process are given by

$$\begin{aligned} \mathrm{E}[N(t)] &= \mathrm{E}[A(t)], \\ \mathrm{Var}[N(t)] &= \mathrm{E}[A(t)], \end{aligned}$$

respectively.

Zero coupons and survival probabilities

- Note that there is a close analogy between the intensity $\lambda(t)$ in a Cox model and the short rate $r(t)$ in an interest rate model.
- Let $S(t, T)$ denote the time t survival probability $S(t, T) = \mathsf{P}(\tau_1 > T)$, and let $P(t, T)$ be the time t discount bond maturing at T. Then,

$$
\begin{aligned}
S(0, T) &= \mathsf{E}\left[e^{-\int_0^T \lambda(s)ds}\right], \\
P(0, T) &= \mathsf{E}\left[e^{-\int_0^T r(s)ds}\right].
\end{aligned}
$$

- More generally, at time $t \geq 0$,

$$
\begin{aligned}
S(t, T) &= 1_{\tau_1 > t}\mathsf{E}\left[e^{-\int_t^T \lambda(s)ds}\right], \\
P(t, T) &= \mathsf{E}\left[e^{-\int_t^T r(s)ds}\right].
\end{aligned}
$$

  Note the presence of the process $1_{\tau_1 > t}$ in the first formula: the time t survival probability for T is zero, if the event has occurred prior to t.

Risky zero coupon

- Consider now a risky zero coupon bond with a zero recovery rate.
- Assuming that the credit intensity follows a Cox process, its price is given by:

$$\begin{aligned}\mathcal{P}(0,T) &= \mathrm{E}\big[e^{-\int_0^T r(s)ds}1_{\tau_1>T}\big]\\ &= \mathrm{E}\big[\mathrm{E}\big[e^{-\int_0^T r(s)ds}1_{\tau_1>T}\,|\,\{\lambda(s)\}_{0\le s\le T}\big]\big]\\ &= \mathrm{E}\big[\mathrm{E}\big[e^{-\int_0^T r(s)ds}e^{-\int_0^T \lambda(s)ds}\,|\,\{\lambda(s)\}_{0\le s\le T}\big]\big]\\ &= \mathrm{E}\big[e^{-\int_0^T (r(s)+\lambda(s))ds}\big].\end{aligned}$$

- Notice that the presence of event risk (i.e. credit) amounts to increasing the discounting rate by the intensity of default. We have already noted this in Lecture Notes 1.

Risky zero coupon

- In particular, if the default probability is *independent* of the rates probability, then,

$$\begin{aligned}\mathcal{P}(0, T) &= \mathsf{E}[e^{-\int_0^T r(s)ds}]\mathsf{E}[1_{\tau_1 > T}] \\ &= P(0, T)S(0, T).\end{aligned}$$

  We have thus recovered the formula for the risky coupon bond discussed in Lecture Notes 1 as a special case of our general framework.
- For any time $t \geq 0$, we have the following expression for the risky zero coupon bond:

$$\mathcal{P}(t, T) = 1_{\tau_1 > t}\mathsf{E}\big[e^{-\int_t^T (r(s)+\lambda(s))ds}\big].$$

  Note the time t value of the bond is zero, if the default happens prior to t.

Gaussian structural model

- As an explicit example, we consider the model in which both the rate and intensity are Gaussian processes:

$$dr(t) = \mu(t)\,dt + \sigma_r dW(t)\,,$$
$$d\lambda(t) = \theta(t)\,dt + \sigma_\lambda dZ(t)\,,$$

  with correlated Brownian motions:

$$dW(t)\,dZ(t) = \rho dt.$$

- Our goal is to compute $\mathcal{P}(0, T)$.

Gaussian structural model

- Note that

$$r(t) = r_0 + \int_0^t \mu(s)\, ds + \sigma_r W(t),$$

$$\lambda(t) = \lambda_0 + \int_0^t \theta(s)\, ds + \sigma_\lambda Z(t),$$

are Gaussian variables with means

$$\mathsf{E}[r(t)] = r_0 + \int_0^t \mu(s)\, ds,$$

$$\mathsf{E}[\lambda(t)] = \lambda_0 + \int_0^t \theta(s)\, ds,$$

and covariance matrix (for $t \leq u$)

$$\mathrm{Cov}(r(t), \lambda(u)) = \begin{pmatrix} \sigma_r^2 t & \rho\sigma_r\sigma_\lambda t \\ \rho\sigma_r\sigma_\lambda t & \sigma_\lambda^2 u \end{pmatrix}.$$

Gaussian structural model

- Hence, the mean and variance of the Gaussian variable $X \triangleq \int_0^T (r(t) + \lambda(t))dt$ are

$$\begin{aligned} \mathrm{E}[X] &= (r_0 + \lambda_0)T + \int_0^T (T - t)(\mu(t) + \theta(t))dt, \\ \mathrm{Var}[X] &= 2(\sigma_r^2 + 2\rho\sigma_r\sigma_\lambda + \sigma_\lambda^2) \iint_{0 \le t \le u \le T} t\, dt\, du \\ &= (\sigma_r^2 + 2\rho\sigma_r\sigma_\lambda + \sigma_\lambda^2)\frac{T^3}{3}. \end{aligned}$$

- Using $\mathrm{E}[\exp(-X)] = \exp(-\mathrm{E}[X] + \frac{1}{2}\mathrm{Var}[X])$ we find

$$\mathcal{P}(0, T) = P(0, T)S(0, T)e^{\rho\sigma_r\sigma_\lambda T^3/3}.$$

- The Gaussian model implies thus that positive correlation between rates and credit implies a higher zero coupon bond price.

Mean reverting model

- A more realistic model is obtained by assuming that the riskless rate and default intensity follow mean reverting processes:

$$dr(t) = \left(\frac{d\mu(t)}{dt} + k_r(\mu(t) - r(t))\right)dt + \sigma_r dW(t),$$
$$d\lambda(t) = \left(\frac{d\theta(t)}{dt} + k_\lambda(\theta(t) - \lambda(t))\right)dt + \sigma_\lambda dZ(t),$$

  with correlated Brownian motions:

$$dW(t)\,dZ(t) = \rho dt.$$

- Here $\mu(t)$ and $\theta(t)$ are the deterministic long term means of the rate process and intensity process, respectively, and k_r and k_λ are the corresponding mean reversion speeds.
- We assume that

$$\mu(0) = r_0,$$
$$\theta(0) = \lambda_0.$$

- This specification is reminiscent of the one-factor Hull-White model used in modeling interest rates.

Mean reverting model

- The solutions to the equations above read

$$
\begin{aligned}
r(t) &= \mu(t) + \sigma_r \int_0^t e^{-k_r(t-u)} dW(u)\,,\\
\lambda(t) &= \theta(t) + \sigma_\lambda \int_0^t e^{-k_\lambda(t-u)} dZ(u)\,.
\end{aligned}
$$

- These are Gaussian variables with means

$$
\begin{aligned}
\mathrm{E}[r(t)] &= \mu(t)\,,\\
\mathrm{E}[\lambda(t)] &= \theta(t)\,,
\end{aligned}
$$

respectively, and covariance matrix (for $t \leq u$)

$$
\mathrm{Cov}(r(t), \lambda(u)) =
$$

$$
\begin{pmatrix}
\sigma_r^2 \, \frac{1-\exp(-2k_r t)}{2k_r} & \rho\sigma_r\sigma_\lambda \, \frac{\exp(-k_\lambda(u-t))-\exp(-k_r t-k_\lambda u)}{k_r+k_\lambda} \\
\rho\sigma_r\sigma_\lambda \, \frac{\exp(-k_\lambda(u-t))-\exp(-k_r t-k_\lambda u)}{k_r+k_\lambda} & \sigma_\lambda^2 \, \frac{1-\exp(-2k_\lambda u)}{2k_\lambda}
\end{pmatrix}.
$$

Mean reverting model

- Hence, the mean and variance of the Gaussian variable $X \triangleq \int_0^T (r(t) + \lambda(t))dt$ are

$$\mathrm{E}[X] = \int_0^T (\mu(t) + \theta(t))dt,$$
$$\mathrm{Var}[X] = \sigma_r^2 \int_0^T h_r(s)^2\, ds + \sigma_\lambda^2 \int_0^T h_\lambda(s)^2\, ds + 2\rho\sigma_r\sigma_\lambda \int_0^T h_r(s)\, h_\lambda(s)\, ds.$$

- The functions $h_r(t)$ and $h_\lambda(t)$ used in the expression above are defined by:

$$h_r(t) = \frac{1 - e^{-k_r t}}{k_r},$$
$$h_\lambda(t) = \frac{1 - e^{-k_\lambda t}}{k_\lambda},$$

respectively.

Mean reverting model

- Using again $\mathrm{E}[\exp(-X)] = \exp(-\mathrm{E}[X] + \frac{1}{2}\mathrm{Var}[X])$, we find that the riskless zero coupon bond and survival probability are given by

$$
\begin{aligned}
P(0,T) &= e^{-\int_0^T \mu(s)ds + \frac{1}{2}\sigma_r^2 \int_0^T h_r(s)^2 ds}, \\
S(0,T) &= e^{-\int_0^T \theta(s)ds + \frac{1}{2}\sigma_\lambda^2 \int_0^T h_\lambda(s)^2 ds},
\end{aligned}
$$

  respectively.
- As a consequence, we find the following relation between the risky zero coupon, riskless zero coupon, and the survival probability in the mean reverting model:

$$
\mathcal{P}(0,T) = P(0,T)S(0,T)e^{\rho\sigma_r\sigma_\lambda \int_0^T h_r(s)h_\lambda(s)ds}.
$$

References

- Andersen, L.: Credit models, lecture notes (2010).
- Lesniewski, A.: Interest rate models, lecture notes (2015).
- O'Kane, D.: *Modelling Single-Name and Mult-Name Credit Derivatives*, Wiley (2008).