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Probability and Stochastic Processes in Finance II (MTH 9862).

Solutions to the Midterm Examination.

Problem 1. Let $\alpha(t)$ and $\sigma(t)$ be continuous bounded deterministic functions on $[0,\infty)$.

(a) Find a closed form solution of the equation

$$dX(t) = \alpha(t)X(t)\,dt + \sigma(t)X(t)\,dB(t).$$

(b) Find the equation satisfied by $\ln(X(t))$. Determine the distribution of $\ln(X(t))$.

(c) Given $p \neq 0$, find $E(X^p(t))$. What is the distribution of $X^p(t)$?

Solution. (a) $X(t)$ is a common factor in the right-hand side. This is a hint to compute $d\ln(X(t))$.

Remark. In a regular calculus when you solve the equation $dm(t) = a(t)m(t)dt$, i.e. $m'(t) = a(t)m(t)$, with some initial condition $m(t_0) = m_0 > 0$, you often rewrite it as

$$\frac{dm(t)}{m(t)} = a(t)dt, \quad \text{or} \quad \frac{d\ln m(t)}{dt} = a(t),$$

and then integrate from t_0 to t to get

$$\ln m(t) - \ln m_0 = \int_{t_0}^t a(u)\,du, \quad m(t) = m_0 \exp\left(\int_{t_0}^t a(u)\,du\right).$$

Here the idea is the same, but the difference is that

$$d\ln(X(t)) \neq \frac{dX(t)}{X(t)},$$

since you have an additional "Ito term"in the right-hand side:

$$d\ln X(t) = \frac{dX(t)}{X(t)} - \frac{1}{2}\frac{1}{X^2}d[X,X](t) = \left(\alpha(t) - \frac{\sigma^2(t)}{2}\right)dt + \sigma(t)dB(t). \tag{1}$$

From this equation we get

$$\ln(X(t)) = \ln(X(0)) + \int_0^t \left(\alpha(u) - \frac{\sigma^2(u)}{2}\right)du + \int_0^t \sigma(u)dB(u) \tag{2}$$

and finally

$$X(t) = X(0)\exp\left(\int_0^t \left(\alpha(u) - \frac{\sigma^2(u)}{2}\right)du + \int_0^t \sigma(u)dB(u)\right). \tag{3}$$

(b) See equation (1). From (2) and the fact that $\alpha(t)$ and $\sigma(t)$ are deterministic we see that the random variable $\ln(X(t))$ for a fixed t is a sum of a constant and an Ito integral with a deterministic integrand. We conclude that $\ln(X(t))$ is **normal**

with **mean** $\ln(X(0)) + \int_0^t \left(\alpha(u) - \frac{\sigma^2(u)}{2}\right) du$ and **variance** $\int_0^t \sigma^2(u) du.$

(c) We know $X(t)$ explicitly, so we know $X^p(t)$ as well:

$$X^p(t) = X^p(0) \exp\left(p \int_0^t \left(\alpha(u) - \frac{\sigma^2(u)}{2}\right) du + p \int_0^t \sigma(u) dB(u)\right).$$

The process $X^p(t)$ has the same structure as $X(t)$. It simply has different parameters. We conclude immediately that the distribution is lognormal. For a lognormal distribution one usually specifies the mean and the variance of the logarithm. We have

$$\ln(X^p(t)) = \ln(X^p(0)) + p \int_0^t \left(\alpha(u) - \frac{\sigma^2(u)}{2}\right) du + p \int_0^t \sigma(u) dB(u).$$

Just as before,

$$\ln(X^p(t)) \sim \text{Norm}\left(\ln(X^p(0)) + p \int_0^t \left(\alpha(u) - \frac{\sigma^2(u)}{2}\right) du,\ p^2 \int_0^t \sigma^2(u)\, du\right).$$

Remark. We know from basic probability that

$$\text{if} \quad \ln Y \sim \text{Norm}(\mu, \sigma^2) \ \Leftrightarrow\ \ln Y \overset{\mathrm{d}}{=} \mu + \sigma Z, \ \text{then} \ \ E(Y) = E e^{\mu + \sigma Z} = e^{\mu + \sigma^2/2}.$$

Here Z is a standard normal random variable.

Therefore,

$$E(X^p(t)) = X^p(0) \exp\left(\int_0^t p\alpha(u) - \frac{p\sigma^2(u)}{2} + \frac{p^2\sigma^2(u)}{2}\, du\right).$$

Problem 2. Let $S(0) = x$ and for some fixed $a, \mu \geq 0$ and $\sigma > 0$

$$dS(t) = (\mu - aS(t))dt + \sigma\, dB(t).$$

(a) Find the expectation and variance of $S(t)$.

(b) Find a closed form solution of the above equation. Then calculate the expectation and variance of $S(t)$ directly from the closed form solution.

Solution. (a) Write the equation in the integral form and take expectations

$$S(t) = S(0) + \int_0^t \mu - aS(u))du + \int_0^t \sigma\, dB(u)$$
$$E(S(t)) = S(0) + \int_0^t \mu - aE(S(u))du + \int_0^t \sigma\, dB(u)$$

Let $m(t) = E(S(t))$. Then $m(0) = S(0)$ and $m'(t) = \mu - am(t)$, and

$$m(t) = xe^{-at} + \frac{\mu}{a}(1 - e^{-at}).$$

Remark. There are several ways to solve a linear non-homogeneous equation. I shall give only one way ("variation of a constant"), but if you are used to a different approach, use the one you know.

Step 1. Find the general solution of the homogeneous equation $m_h'(t) = -am_h(t)$. Get $m_h(t) = Ce^{-at}$, where C is an arbitrary constant.

Step 2. Try to find a solution of the original equation in the form $m(t) = C(t)e^{-at}$, $m(0) = C(0) = x$. Substitution gives

$$m'(t) = C'(t)e^{-at} - aC(t)e^{-at} = \mu - aC(t)e^{-at} \Rightarrow C'(t) = \mu e^{at},\ C(0) = x.$$

Therefore, $C(t) = x + \frac{\mu}{a}(e^{at} - 1)$ and we get the above formula for $m(t)$.

A standard approach for finding the variance is to find the equation for $dS^2(t)$ using Ito's formula and proceed in the same way as for the expectation to compute $E(S^2(t))$. Then we have to subtract the square of the expectation to get the variance. But by the definition the variance is the expectation of $(S(t) - m(t))^2$, which we shall try to compute directly.

$$d(S(t) - m(t))^2 = 2(S(t) - m(t))d(S(t) - m(t)) + d[S - m, S - m](t),$$

where $d[S - m, S - m](t) = d[S, S](t) = \sigma^2 dt$ and

$$\begin{aligned} d(S(t) - m(t)) &= (\mu - aS(t))dt + \sigma\, dB(t) - m'(t)dt \\ &= (\mu - aS(t))dt + \sigma\, dB(t) - (\mu - am(t))dt \\ &= -a(S(t) - m(t))\, dt + \sigma\, dB(t). \end{aligned}$$

Therefore,

$$d(S(t) - m(t))^2 = -2a(S(t) - m(t))^2\, dt + 2\sigma(S(t) - m(t))\, dB(t) + \sigma^2 dt.$$

Setting $v(t) := E(S(t) - m(t))^2$ we get that $v(0) = 0$ and

$$v(t) = -2a\int_0^t v(u)\, du + \sigma^2 t.$$

This equation is of the same type as the one for $m(t)$. Applying the last remark we obtain

$$v(t) = \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

(b) We see that the equation has the term $-aS(t)dt$. In other words, $S(t)$ gets "discounted". This tells us that that undiscounted process might look simpler and we consider the process $e^{at}S(t)$ instead (this approach comes from ODEs, we are finding an "integrating factor"). We have

$$d(e^{at}S(t)) = e^{at}((\mu - aS(t))dt + \sigma\, dB(t)) + ae^{at}S(t)\, dt = e^{at}(\mu\, dt + \sigma\, dB(t)). \tag{4}$$

Then

$$e^{at} S(t) = x + \mu \int_0^t e^{au}\, du + \sigma \int_0^t e^{au}\, dB(t) = x + \frac{\mu}{a}\,(e^{at} - 1) + \sigma \int_0^t e^{au}\, dB(u).$$

and

$$S(t) = xe^{-at} + \frac{\mu}{a}\,(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{au}\, dB(t).$$

In the same way as in problem 1(b) we see that

$$S(t) \sim \text{Norm}\left(xe^{-at} + \frac{\mu}{a}\,(1 - e^{-at}),\; \frac{\sigma^2}{2a}\,(1 - e^{-2at}) \right).$$

Problem 3. Let

$$I(t) = \int_0^t \sin u\, dB(u) + \frac{1}{2} \int_0^t \cos^2 u\, du.$$

(a) Determine if $e^{I(t)}$ is a martingale.

(b) Find $[I, e^I](t)$, i.e. the cross-variation of $I(t)$ and $e^{I(t)}$.

Solution. (a) We apply Ito's formula for Ito processes.

$$de^{I(t)} = e^{I(t)} dI(t) + \frac{1}{2}\, e^{I(t)} d[I, I](t) = e^{I(t)} \left(\sin t\, dB(t) + \frac{1}{2} \cos^2 t\, dt + \frac{1}{2} \sin^2 t\, dt \right)$$
$$= e^{I(t)} \left(\sin t\, dB(t) + \frac{1}{2}\, dt \right).$$

This process is not a martingale. It it were a martingale then, in particular, $E(e^{I(0)}) = 1$ would be equal to $E(e^{I(t)})$ for all $t > 0$, but here

$$E(e^{I(t)}) = 1 + \frac{1}{2} \int_0^t e^{I(t)} dt > 1 \text{ for all } t > 0.$$

(b) Since we know $dI(t)$ and $de^{I(t)}$ we can find

$$d[I, e^I](t) = e^{I(t)} \sin^2 t\, dt, \quad \text{and} \quad [I, e^I](t) = \int_0^t e^{I(u)} \sin^2 u\, du.$$

Problem 4. Let $\mu \in \mathbb{R},\ r, a \geq 0$ and $S(t)$ satisfy under $\mathbb{P}$

$$dS(t) = (\mu - aS(t))dt + \sigma\, dB(t), \quad 0 \leq t \leq T.$$

(a) Find the probability measure $\tilde{\mathbb{P}}$ such that the process $e^{-(r-a)t} S(t)$, $0 \leq t \leq T$, is a martingale under $\tilde{\mathbb{P}}$. Give explicitly the Radon-Nikodym derivative $d\tilde{\mathbb{P}}/d\mathbb{P}$.

(b) Find the equation for $S(t)$ under $\tilde{\mathbb{P}}$.

Solution. Our goal is to eliminate the drift for the process $e^{-(r-a)t}S(t)$.

$$
\begin{aligned}
d(e^{-(r-a)t}S(t)) &= e^{-(r-a)}(-(r-a)S(t)\,dt + dS(t)) \\
&= e^{-(r-a)}(-(r-a)S(t)\,dt + (\mu - aS(t))dt + \sigma\, dB(t)) \\
&= e^{-(r-a)}((\mu - rS(t))dt + \sigma\, dB(t)) = e^{-(r-a)}\sigma \left(\frac{\mu - rS(t)}{\sigma}dt + dB(t)\right).
\end{aligned}
$$

Now it is clear that we can set

$$
\theta(t) = \frac{\mu - rS(t)}{\sigma}, \quad \tilde{B}(t) = B(t) + \int_0^t \theta(u)\,du.
$$

Define $\tilde{\mathbb{P}}$ on $\mathcal{F}(T)$ by its Radon-Nikodym derivative

$$
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(-\int_0^T \theta(u)\,dB(u) - \frac{1}{2}\int_0^T \theta^2(u)\,du\right).
$$

Then by Girsanov's theorem (assuming that all conditions check out) under $\tilde{\mathbb{P}}$ the process $\tilde{B}(t)$ is a standard Brownian motion, and by construction $e^{-(r-a)t}S(t)$ is a martingale under $\tilde{\mathbb{P}}$.

(b) Under $\tilde{P}$ we have $dB(t) = d\tilde{B}(t) - \theta(t)\,dt$. Substituting this in the equation for $dS(t)$ we get

$$
\begin{aligned}
dS(t) = (\mu - aS(t))dt + \sigma\,(d\tilde{B}(t) - \theta(t)\,dt) &= (\mu - aS(t))dt + \sigma d\tilde{B}(t) - (\mu - rS(t))\,dt \\
&= (r-a)S(t)\,dt + \sigma\, d\tilde{B}(t).
\end{aligned}
$$

This equation is of the same type as in problem 2 (take $\mu = 0$ and replace a with $a - r$).

Problem 5. Let $B(t) = (B_1(t),\ B_2(t))^T$ be a standard two-dimensional Brownian motion and $R(t) = \sqrt{B_1^2(t) + B_2^2(t)}$. Determine which of the following processes are standard Brownian motions (dimension 1 or 2). Clearly state and check all the required conditions.

(a) $dX(t) = R^{-1}(t)(B_1(t)dB_1(t) + B_2(t)dB_2(t))$.

(b) $W(t) = (W_1(t), W_2(t))^T$, where

$$
W(t) = \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\beta & \sin\beta \end{pmatrix} B(t),
$$

and $\alpha,\ \beta \in [0, 2\pi]$ are fixed numbers.

Solution. (a) $X(t)$ is a process with one-dimensional state space. We assume that it starts from 0 (it is neither given nor implied by the equation). $X(t)$ is a sum of Ito integrals, therefore it has continuous paths (a.s.). For the same reason it is a martingale with respect to the filtration generated by $(B(t))_{t\geq 0}$. The only thing we really need to check is that $[X, X](t) \equiv t$ a.s.. We compute

$$
d[X, X](t) = R^{-2}(t)(B_1^2(t)\,dt + B_2^2(t)\,dt) = dt, \quad [X, X](t) = t,\ t \geq 0,\ \text{a.s.}.
$$

Conclusion: $(X(t))_{t\geq 0}$ is a standard Brownian motion.

(b) $W(t)$ has two-dimensional state space. Here we see from the definition that $W(0) = (0,0)^T$. The path continuity is inherited from Brownian motion, since the process is just a linear transformation of $(B(t))_{t\geq 0}$. We need to check whether the covariance matrix is equal to tI_2. We can write

$$dW(t) = \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\beta & \sin\beta \end{pmatrix} dB(t) =: AdB(t)$$

Then

$$\begin{aligned} d[W,W](t) &= Ad[B,B](t)A^T = AI_2A^T dt = AA^T dt \\ &= \begin{pmatrix} \sin^2\alpha + \cos^2\alpha & -\sin\alpha\cos\beta + \cos\alpha\sin\beta \\ -\cos\beta\sin\alpha + \sin\beta\cos\alpha & (-\cos\beta)^2 + \sin^2\beta \end{pmatrix} dt \\ &= \begin{pmatrix} 1 & \sin(\beta-\alpha) \\ \sin(\beta-\alpha) & 1 \end{pmatrix} dt \end{aligned}$$

Therefore, the answer is

$$\begin{cases} \text{no}, & \text{if } \beta-\alpha \notin \{-2\pi, -\pi, 0, \pi, 2\pi\} \\ \text{yes}, & \text{if } \beta-\alpha \in \{-2\pi, -\pi, 0, \pi, 2\pi\}. \end{cases}$$

Problem 6. Let $(B(t))_{t\geq 0}$ be a standard Brownian motion, $(\mathcal{F}(t))_{t\geq 0}$ be its natural filtration, and $T > 0$ be a fixed time. Define $M(t) := E(B^3(T) \,|\, \mathcal{F}(t))$, $0 \leq t \leq T$. Then we know (by the tower property of conditional expectations) that $(M(t))_{0\leq t\leq T}$ is a martingale. This problem will guide you through finding an explicit representation for this martingale, i. e. finding a stochastic process $\Gamma(t)$ such that

$$M(t) = \int_0^t \Gamma(u)\, dB(u), \quad 0 \leq t \leq T.$$

(a) Use one of the standard inequalities to show that $M(t)$ is square integrable for each $t \in [0,T]$.

(b) Compute $dB^3(t)$ and write $B^3(T)$ as a sum of a stochastic and a regular integral.

(c) Integrate your regular integral by parts. Write the result as a single stochastic integral.

(d) Use parts (b) and (c) together with one of the basic properties of Ito integral to write $E(B^3(T) \,|\, \mathcal{F}(t))$ as a single stochastic integral. State explicitly your answer, i.e. $\Gamma(t) = \dots$.

Solution. (a) We use the conditional Jensen inequality $\phi(E(X|\mathcal{F})) \leq E(\phi(X)|\mathcal{F})$ a.s. for the convex function $\phi(x) = x^2$ and then take the expectation:

$$\begin{aligned} M^2(T) = (E(B^3(T) \,|\, \mathcal{F}(t)))^2 \leq E(B^6(T) \,|\, \mathcal{F}(t)) \;\text{ a.s. } \Rightarrow \\ E(M^2(t)) \leq E(E(B^6(T) \,|\, \mathcal{F}(t))) = E(B^6(T))) = 15T^3 < \infty. \end{aligned}$$

(b) We obtain

$$dB^3(t) = 3B^2(t)\,dB(t) + 3B(t)\,dt;\;\; B^3(T) = 3\int_0^T B^2(u)\,dB(u) + 3\int_0^T B(u)\,du.$$

(c) Integrating the last integral by parts we get

$$\int_0^T B(u)\,du = B(T)T - \int_0^T u\,dB(u) = \int_0^T T\,dB(u) - \int_0^T u\,dB(u) = \int_0^T (T-u)\,dB(u).$$

(d) Therefore,

$$B^3(T) = 3\int_0^T (B^2(u) + (T-u))dB(u).$$

Using the martingale property of the Ito integral we get for $0 \le t \le T$

$$M(t) = E(B^3(T)\,|\,\mathcal{F}(t)) = 3\int_0^t (B^2(u) + (T-u))dB(u), \text{ and } \Gamma(t) = 3(B^2(t) + (T-t)).$$

Problem 7. Let $(W_1(t), W_2(t))_{t\ge 0}$ be a Brownian motion with correlated coordinates so that $dW_1(t)dW_2(t) = \rho\,dt$ for some constant $\rho \in [-1,1]$. Consider a market model which consists of 2 stocks and a MMA. Assume that the (continuously compounded) interest rate is $r \ge 0$ and that stock prices $S_1(t)$ and $S_2(t)$, $0 \le t \le T$, satisfy the following equations under the risk-neutral measure $\tilde{\mathbb{P}}$:

$$\begin{aligned} dS_1(t) &= rS_1(t)dt + \sigma_1 S_1(t)dW_1(t), \quad S_1(0) = s_1; \\ dS_2(t) &= rS_2(t)dt + \sigma_2 S_2(t)dW_2(t), \quad S_2(0) = s_2. \end{aligned}$$

where σ_i, s_i, $i = 1, 2$, are positive constants.

(a) Suppose that for some deterministic function $u(t, x_1, x_2)$ the process $e^{-rt}u(t, S_1(t), S_2(t))$, $t \ge 0$, is a martingale under the risk-neutral measure. Then the function $u(t, x_1, x_2)$ should satisfy some partial differential equation. Find this equation. Hint: compute $d(e^{-rt}u(t, S_1(t), S_2(t)))$.

(b) Find the correlation between $S_1(t)$ and $S_2(t)$. Is it true that when $\rho = 1$ then the correlation between $S_1(t)$ and $S_2(t)$ is also equal to 1?

Solution. To make notation more compact we set $u := u(t, S_1(t), S_2(t))$, $u_t := u_t(t, S_1(t), S_2(t))$, $u_i := u_{x_i}(t, S_1(t), S_2(t))$, and $u_{ij} := u_{x_i x_j}(t, S_1(t), S_2(t))$, $i = 1, 2$. Then

$$\begin{aligned} d(e^{-rt}u) &= e^{-rt}(-ru\,dt + u_t\,dt + u_1 dS_1(t) + u_2\,dS_2(t) \\ &\qquad + \frac{1}{2}u_{11}d[S_1, S_1](t) + \frac{1}{2}u_{22}d[S_2, S_2](t) + u_{12}d[S_1, S_2](t)) \\ &= e^{-rt}(-ru\,dt + u_t\,dt + u_1(rS_1(t)dt + \sigma_1 S_1(t)dW_1(t)) + u_2(rS_2(t)dt + \sigma_2 S_2(t)dW_2(t)) \\ &\qquad + \frac{\sigma_1^2}{2}u_{11}S_1^2(t)dt + \frac{\sigma_2^2}{2}u_{22}S_2^2(t)dt + \rho\sigma_1\sigma_2 u_{12}S_1(t)S_2(t)dt) \end{aligned}$$

We need to find the coefficient of the dt term. This coefficient is equal to

$$e^{-rt}(-ru + u_t + rS_1(t)u_1 + rS_2(t)u_2 + \frac{\sigma_1^2}{2}S_1^2(t)u_{11} + \frac{\sigma_2^2}{2}S_2^2(t)u_{22} + \rho\sigma_1\sigma_2 S_1(t)S_2(t)u_{12}).$$

If we require that $u(t, x_1, x_2)$ satisfy

$$-ru + u_t + rx_1u_{x_1} + rx_2u_{x_2} + \frac{1}{2}\sigma_1^2x_1^2u_{x_1x_1} + \frac{1}{2}\sigma_1^2x_2^2u_{x_2x_2} + \rho\sigma_1\sigma_2x_1x_2u_{x_1x_2} = 0$$

for all $t \in [0, T),\ x_1, x_2 > 0$, then we can be sure that no matter what the values of $S_1(t)$ and $S_2(t)$ are (they are always positive in this model) the dt term vanishes. The converse also holds but it is harder to prove. In any case, the desired PDE for $u(t, x_1, x_2)$ is

$$u_t + rx_1u_{x_1} + rx_2u_{x_2} + \frac{1}{2}\sigma_1^2x_1^2\,u_{x_1x_1} + \frac{1}{2}\sigma_1^2x_2^2\,u_{x_2x_2} + \rho\sigma_1\sigma_2x_1x_2\,u_{x_1x_2} = ru.$$

After the break we shall consider connections with PDEs in more detail.

(b) Let us first make a simplifying remark. Set $\tilde{S}_i(t) := e^{-rt}S_i(t),\ i = 1, 2$. Then

$$\frac{\mathrm{Cov}(S_1(t), S_2(t))}{\sqrt{\mathrm{Var}(S_1(t))\,\mathrm{Var}(S_2(t))}} = \frac{\mathrm{Cov}(\tilde{S}_1(t), \tilde{S}_2(t))}{\sqrt{\mathrm{Var}(\tilde{S}_1(t))\,\mathrm{Var}(\tilde{S}_2(t))}}. \tag{5}$$

The advantage of using the discounted prices is that they are martingales. We have

$$d\tilde{S}_1(t) = \sigma_1\tilde{S}_1(t)dW_1(t), \quad \tilde{S}_1(0) = s_1;$$
$$d\tilde{S}_2(t) = \sigma_2\tilde{S}_2(t)dW_2(t), \quad \tilde{S}_2(0) = s_2.$$

To compute the covariance matrix (we need the covariance and both variances) we use the usual method. For $i, j \in \{1, 2\}$

$$d(\tilde{S}_i(t)\tilde{S}_j(t)) = \tilde{S}_i(t)d\tilde{S}_j(t) + \tilde{S}_j(t)d\tilde{S}_i(t) + d[\tilde{S}_i, \tilde{S}_j](t).$$

The stochasic integral with respect to a martingale is a martingale, therefore the first two terms will make no contribution to the expectation. Moreover,

$$d[\tilde{S}_i, \tilde{S}_j](t) = \sigma_i\sigma_j\tilde{S}_i(t)\tilde{S}_j(t)d[W_i, W_j](t) = \sigma_i\sigma_j\tilde{S}_i(t)\tilde{S}_j(t)\rho_{ij}dt,$$

where $\rho_{ii} := 1,\ \rho_{ij} := \rho\ (i \neq j),\ i, j \in \{1, 2\}$, and

$$E(\tilde{S}_i(t)\tilde{S}_j(t)) = s_is_j + \sigma_i\sigma_j\rho_{ij}\int_0^t E(\tilde{S}_i(u)\tilde{S}_j(u))du.$$

We get a simple ODE and conclude that $E(\tilde{S}_i(t)\tilde{S}_j(t)) = s_is_je^{\sigma_i\sigma_j\rho_{ij}t}$. Since $\tilde{S}_i(t)$ is a martingale, its expectation does not depend on t and is equal to $\tilde{S}_i(0) = s_i$. Therefore,

$$E(\tilde{S}_i(t)\tilde{S}_j(t)) - E(\tilde{S}_i(t))E(\tilde{S}_j(t)) = s_is_j(e^{\sigma_i\sigma_j\rho_{ij}t} - 1).$$

Substituting this in (5) we obtaine the correlation between $S_1(t)$ and $S_2(t)$

$$\frac{s_1s_2(e^{\sigma_1\sigma_2\rho t} - 1)}{\sqrt{s_1^2(e^{\sigma_1^2t} - 1)s_2^2(e^{\sigma_2^2t} - 1)}} = \frac{e^{\sigma_1\sigma_2\rho t} - 1}{\sqrt{(e^{\sigma_1^2t} - 1)(e^{\sigma_2^2t} - 1)}}.$$

Even if $\rho = 1$ this correlation is not equal to one unless $\sigma_1 = \sigma_2$.