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ShengQuan Zhou

1 Variance of Two Assets Exchange Warrant

Assume

$$\frac{dS_1(t)}{S_1(t)} = rdt + \sigma_1 dW_1(t),$$
$$\frac{dS_2(t)}{S_2(t)} = rdt + \sigma_2 dW_2(t),$$

where $\mathbb{E}[dW_1 dW_2] = \rho dt$. Then

$$S_1(t) = S_1(0)e^{\left(r-\frac{\sigma_1^2}{2}\right)t+\sigma_1 W_1(t)} = S_1(0)e^{\left(r-\frac{\sigma_1^2}{2}\right)t+\sigma_1 B_1(t)},$$
$$S_2(t) = S_2(0)e^{\left(r-\frac{\sigma_2^2}{2}\right)t+\sigma_2 W_2(t)} = S_2(0)e^{\left(r-\frac{\sigma_2^2}{2}\right)t+\sigma_2\rho B_1(t)+\sigma_2\sqrt{1-\rho^2}B_2(t)},$$

written in terms of independent Brownian motions $\{B_1(t), B_2(t)\}$

$$W_1(t) = B_1(t),$$
$$W_2(t) = \rho B_1(t) + \sqrt{1-\rho^2}B_2(t).$$

The exchange rate

$$\frac{S_2(t)}{S_1(t)} = \frac{S_2(0)}{S_1(0)}\exp^{(\rho\sigma_2-\sigma_1)B_1(t)+\sigma_2\sqrt{1-\rho^2}B_2(t)+\frac{\sigma_1^2-\sigma_2^2}{2}t}.$$

Let

$$\tilde{B}_1(t) = B_1(t) - \sigma_1 t,$$
$$\tilde{B}_2(t) = B_2(t),$$

then

$$\begin{aligned}\frac{S_2(t)}{S_1(t)} &= \frac{S_2(0)}{S_1(0)}\exp^{(\rho\sigma_2-\sigma_1)(\tilde{B}_1(t)+\sigma_1 t)+\sigma_2\sqrt{1-\rho^2}\tilde{B}_2(t)+\frac{\sigma_1^2-\sigma_2^2}{2}t} \\ &= \frac{S_2(0)}{S_1(0)}\exp^{(\rho\sigma_2-\sigma_1)\tilde{B}_1(t)+\sigma_2\sqrt{1-\rho^2}\tilde{B}_2(t)-\frac{1}{2}(\rho\sigma_2-\sigma_1)^2 t-\frac{1}{2}(1-\rho^2)\sigma_2^2 t}.\end{aligned}$$

Let

$$\tilde{\sigma}_1 = \rho\sigma_2 - \sigma_1,$$
$$\tilde{\sigma}_2 = \sigma_2\sqrt{1-\rho^2},$$

then

$$\frac{S_2(t)}{S_1(t)} = \frac{S_2(0)}{S_1(0)}\exp^{\tilde{\sigma}_1\tilde{B}_1(t)+\tilde{\sigma}_2\tilde{B}_2(t)-\frac{1}{2}\tilde{\sigma}_1^2 t-\frac{1}{2}\tilde{\sigma}_2^2 t}.$$

Thus, define Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{\sigma_1 B_1(T)-\frac{1}{2}\sigma_1^2 T},$$

the process $\{\tilde{B}_1(t), \tilde{B}_2(t)\}$ are two-dimensional Brownian motions under $\tilde{\mathbb{P}}$. Moreover, let

$$\tilde{\sigma} = \sqrt{\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2} = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2},$$

we get

$$\frac{S_2(t)}{S_1(t)} = \frac{S_2(0)}{S_1(0)}\exp^{\tilde{\sigma}\tilde{B}(t)-\frac{1}{2}\tilde{\sigma}^2 t},$$

where $\tilde{B}(t) = \frac{\tilde{\sigma}_1\tilde{B}_1(t)+\tilde{\sigma}_2\tilde{B}_2(t)}{\sqrt{\tilde{\sigma}_1^2+\tilde{\sigma}_2^2}}$ is again a one-dimensional Brownian motion.

2 Pricing Vanilla Call Option on a Generic Index

Given:

- Index is newly created: not much of historical index prices is available.
- Index components have been traded for a while and do have historical prices.
- Feel free to use earlier discussed GBI-EM index as example to make question more detailed.
- There is no option market for this index

Questions:

- Discuss what model we could use to price this option. Can we start with Black-Sholes?
- Discuss how could we extract or derive parameters for this model: Index volatility?
- If we are to deviate from Black-Sholes a bit and to introduce some simple Local Vol as vol for strike, how could we mark this smile and where from?
- How would we hedge this option if ever traded?

Answers:

- One can use the available historical prices of the underlying components to calculate (or estimate) historical index price. If the resulting historical estimates of the index is close to log normal distribution, one can start with Black-Scholes; otherwise, one needs to look at the higher moments (skew/kurtosis) of the distribution. Admittedly, physical distribution is different from risk neutral distribution.
- Historical realized volatility can be evaluated from the estimated time series of the index aggregated from its components. Implied volatility, however, is usually higher than realized volatility. There are several possible ways to estimate implied volatility.
  1. If index components have volatility market, one can observe from the market how much its implied volatility is higher than realized volatilit and apply this spread to the historical realized volatility.
  2. Presumably, the index will have higher volatility when its price goes down. One can evaluate the historical realized volatility in the worst-case scenario and use it as a proxy for the offer price of implied volatility.
  3. Refer to the implied volatility of some other similar indices.
- Several approaches can be considered
  1. Refer to the smile of implied volatility surface of the components if that is available.
  2. Calibrate the implied volatility smile to match the higher moments, e.g. skew/kurtosis, of the index realized volatility distribution in physical measure. The higher moments are assumed to be invariant under a change of probability measure.
  3. Simulate the cost of delta hedge. One can use historical distribution to generate simulated future prices and the associated delta hedges. The cost of delta hedge should be reflected in the price.
- If the index itself is readily tradable, the delta hedges can be carried out in the usual sense; otherwise, the delta hedages have to be done through trading the underlying components of the index. The same reasoning applies for volatility hedging depending on the existence of volatility market.