Probability and Stochastic Processes in Finance I (MTH 9831).

Solutions to the Midterm Examination.

Problem 1. Let X, Y, Z be independent standard normal random variables. Find the mean and variance of X given that $X+Y+Z=1$. Hint: a simple way to do this problem is to use the decomposition method taught in class.

Solution. The first observation is that since the joint distribution of $(X, X+Y+Z)$ is normal,[1] the conditional on the second coordinate is also normal. All we need is the mean and variance of that conditional distribution.

We find the distribution of X given that $W_1 = (X+Y+Z)/\sqrt{3} = 1/\sqrt{3}$. Notice that X and W_1 are standard normals with correlation $1/\sqrt{3}$. Thus, we can write

$$X = \frac{1}{\sqrt{3}} W_1 + \sqrt{\frac{2}{3}} W_2,$$

where W_2 is standard normal independent of W_1. Given $W_1 = 1/\sqrt{3}$, we conclude that X has mean $1/3$ and variance $2/3$.

You should realize that $(X, X+Y+Z) \stackrel{\mathrm{d}}{=} (B(1), B(3))$ where $B(t),\ t \geq 0$, is a standard Brownian motion. This type of calculation is a part of the Lévy's construction of Brownian motion. It is a subset of Problem (3) of HW1 ($\alpha = 1/3,\ t = 3$). The same type of question was a part of Quiz 1. I repeat below the solution once more.

Let us try to write X as a sum of $W = X+Y+Z$ (which is normal with mean 0 and variance 3) and U, where U is independent of W.[2]

$$X = aW + U \quad \Rightarrow \quad \mathrm{Cov}(X, W) = a \,\mathrm{Var}(W) = 3a \quad \Rightarrow \quad a = 1/3.$$

Therefore, $X = W/3 + U$. To find the variance of U we use the independence of W and U and write

$$1 = \mathrm{Var}(X) = \frac{1}{9}\mathrm{Var}(W) + \mathrm{Var}(U) \quad \Rightarrow \quad \mathrm{Var}(U) = \frac{2}{3}.$$

We are given that $W = 1$. Thus, conditioned on $W = 1$, we have that $X = 1/3 + U$, where U is a normal random variable with mean 0 and variance $2/3$. We conclude that, given $X+Y+Z=1$, X has mean $1/3$ and variance $2/3$.

Problem 2. Let $(X(t))_{t\geq 0}$ be the solution to the following SDE

$$dX(t) = (1 - e^{-t} - X(t))\,dt + dB(t), \quad X(0) = 0.$$

(a) Solve the above equation.

(b) Compute $E[X(t)]$ and $\mathrm{Var}(X(t))$. Write down the density of $X(t)$.

[2] Since the vector (X, Y, Z) is multivariate normal, for independence of U and W it is enough to ensure that U and W are uncorrelated.

Solution. This is a variation on the Ornstein-Uhlenbeck "theme".

(a) Multiply through by e^t and combine $e^t dX(t) + e^t X(t) dt$ into $d(e^t X(t))$ to get

$$d(e^t X(t) = (e^t - 1)\, dt + e^t\, dB(t).$$

Writing in the integral form we obtain

$$e^t X(t) - 0 = \int_0^t (e^u - 1)\, du + \int_0^t e^u\, dB(u) = e^t - 1 - t + \int_0^t e^u\, dB(u).$$

Dividing through by e^t we find the answer

$$X(t) = 1 - e^{-t} - te^{-t} + e^{-t} \int_0^t e^u\, dB(u).$$

(b) All randomness of $X(t)$ is in the last integral. The integrand is deterministic. Therefore the distribution of $X(t)$ is normal. We find

$$E[X(t)] = 1 - e^{-t} - te^{-t}, \quad \mathrm{Var}(X(t)) = e^{-2t} \int_0^t e^{2u}\, du = \frac{1}{2}\,(1 - e^{-2t}).$$

Problem 3. Compute the mean and variance of the process given by the equation

$$X(t) = 1 - \int_0^t X(u)\, du + 2 \int_0^t \sqrt{X(u)}\, dB(u).$$

Solution. Let $m(t) = E(X(t))$. Then taking the expectation of both parts of the integral equation we get

$$m(t) = 1 - \int_0^t m(u)\, du.$$

This is equivalent to the equation $m' = -m$ with the initial condition $m(0) = 1$. Therefore, $m(t) = e^{-t}$.

To compute the variance, consider the SDE for $X^2(t)$:

$$dX^2(t) = 2X(t)\, dX(t) + d[X]_t = -2X^2(t)\, dt + 4X(t)\sqrt{X(t)}\, dB(t) + 4X(t)\, dt.$$

Integrating, taking the expectation of both sides, and denoting $E(X^2(t))$ by $v(t)$ we get

$$v(t) - v(0) = -2 \int_0^t v(u)\, du + 4 \int_0^t m(u)\, du.$$

This leads to the ODE

$$v' = -2v + 4m, \quad v(0) = 1.$$

I shall use the variation of constant method. Set $v(t) = C(t)e^{-2t}$. Then substituting to the equation we find after cancellations that

$$C'(t) = 4e^{2t} m(t) = 4e^t, \quad C(t) = 4e^t + \mathrm{const}.$$

Since $v(0) = C(0) = 1$, we get that $C(t) = 4e^t - 3$. Finally, $v(t) = (4e^t - 3)e^{-2t}$, and the variance of $X(t)$ is equal to

$$v(t) - m^2(t) = 4(1 - e^{-t})e^{-t}.$$

You will get a solution a tiny bit quicker if you start with $d(X(t) - m(t))^2$ right away.

Problem 4. Let $B_1(t),\ B_2(t)$ be standard independent one-dimensional Brownian motions. Determine which of the following processes are standard Brownian motions (dimension 1 or 2). Justify your answer by stating an appropriate theorem.

(a) $X(t) = \dfrac{1}{\sqrt{2}} \displaystyle\int_0^t \operatorname{sign}(B_1(u))\, dB_1(u) + \dfrac{1}{\sqrt{2}} \displaystyle\int_0^t \operatorname{sign}(B_2(u))\, dB_2(u)$.

Assume that $\operatorname{sign}(x)$ is equal to 1 for $x \geq 0$ and is equal to -1 for $x < 0$.

(b) $W(t) = (W_1(t), W_2(t))$, where

$$W_1(t) = \frac{1}{\sqrt{3}} B_1(t) + \frac{\sqrt{2}}{\sqrt{3}} B_2(t) \quad \text{and} \quad W_2(t) = \frac{1}{\sqrt{2}} B_1(t) - \frac{1}{\sqrt{2}} B_2(t).$$

Solution. (a) The process X is one-dimensional. We shall use the one-dimensional Lévy's characterization of Brownian motion. It states that a continuous martingale with respect to some filtration $(\mathcal{F}(t))_{t\geq 0}$ which starts from 0 and has t as its quadratic variation process is a standard Brownian motion.

First of all, $X(0) = 0$. We know that stochastic integrals are continuous with respect to the upper limit. Take $(\mathcal{F}(t))_{t\geq 0}$ to be the natural filtration of the two-dimensional Brownian motion $(B_1(t), B_2(t))$, $t \geq 0$. Since B_1 and B_2 are independent Brownian motions, $(\mathcal{F}(t))_{t\geq 0}$ satisfies the conditions for the filtration associated to each of B_1 and B_2. Therefore each of the stochastic integrals is a martingale relative to $(\mathcal{F}(t))_{t\geq 0}$, and X, as the sum of two martingales relative to the same filtration, is a martingale relative to that filtration.

Let us compute $[X]_t$. Again by independence of B_1 and B_2 we find that

$$d[X]_t = \frac{1}{2}(\operatorname{sign}(B_1(t)))^2\, dt + \frac{1}{2}(\operatorname{sign}(B_2(t)))^2\, dt = dt.$$

We conclude that X is a standard Brownian motion.

(b) W is two-dimensional. We consider applying the two-dimensional version of Lévy's characterization of Brownian motion. One of the conditions requires that with probability one $[W_1, W_2]_t \equiv 0$. This condition is not satisfied, since

$$d[W_1, W_2](t) = \frac{1}{\sqrt{6}}\, dt - \frac{1}{\sqrt{3}}\, dt,$$

which implies that the cross-variation is not zero. We conclude that W is not a standard 2-dimensional Brownian motion.

Problem 5. Let $(B(t))_{t\geq 0}$ be a standard Brownian motion. List at least 3 martingales of the form $f(t, B(t))$, $t \geq 0$. Justify your answer. Is any of them useful for computing $E(\tau_a)$ where $\tau_a = \inf\{t \geq 0 : B(t) = a\}$? Find $E(\tau_a)$.

Solution. For the first part of this question I refer to Theorem 3.2 of Lecture 2 (as a possible solution). I shall not type up the proof.[3]

Even though martingales can be used to find the expectation of τ_a (see the remark below), there is a more straightforward way to get $E(\tau_a)$ (considering the limited time given for the exam). First, if $a = 0$ then $\tau_a = 0$ and $E(\tau_a) = 0$. This is the trivial case. Assume now that $a > 0$. Without any martingales, merely by the reflection principle, we know that

$$P(\tau_a \leq t) = P(\max_{0\leq s\leq t} B(s) \geq a) = 2P(B(t) \geq a) = 2(1 - N(a\sqrt{t})).$$

Thus, we know the density of τ_a

$$f_{\tau_a}(t) = \frac{a}{\sqrt{2\pi t^3}}\, e^{-a^2/(2t)} 1_{(0,\infty)}(t).$$

Estimating the expectation we get (observe that $e^{-a^2/(2t)} \geq e^{-a^2/2}$ for all $t \geq 1$)

$$E(\tau_a) = \int_0^\infty t f_{\tau_a}(t)\, dt \geq \frac{ae^{-a^2/2}}{\sqrt{2\pi}} \int_1^\infty \frac{1}{\sqrt{t}}\, dt = \infty.$$

If $a < 0$, we notice that by symmetry (i.e. by the fact that $-B$ is also a Brownian motion) $\tau_a \overset{\mathrm{d}}{=} \tau_{|a|}$, and conclude that $E(\tau_a) = \infty$ in this case as well.

Remark. One can also utilize the quadratic or exponential martingales but it takes a bit more effort to give rigorous arguments. For example, we can use the quadratic martingale. Assume without loss of generality that $a < 0$. Let $b > 0$ and define $\tau_{a,b} := \tau_a \wedge \tau_b$. We know how to use the martingale $M(t) = B^2(t) - t$, $t \geq 0$, to show that $E(\tau_{a,b}) = |a|b$. Then we can let $b \to \infty$, argue that $\tau_{a,b} \uparrow \tau_a$ a.s., and use the monotone convergence theorem to conclude that $E(\tau_a) = \infty$. We can also use the exponential martingale to show that $E(e^{-\lambda\tau_a}) = e^{-|a|\sqrt{2\lambda}}$ for all $\lambda \geq 0$ (see Theorem 3.3 of Lecture 2). Then differentiate both parts with respect to λ and let $\lambda \downarrow 0$ to conclude that $E(\tau_a) = \infty$ (for $a \neq 0$). Again one needs to justify the differentiation under the expectation and taking the limit. Therefore, a simple solution based on the reflection principle, to me, is the winner.

Problem 6. Let T be a fixed positive time, $(\Omega, \mathcal{F}(T), \mathbb{P})$ be a probability space, $(\mathcal{F}(t))_{0\leq t\leq T}$ be a filtration generated by a two-dimensional Brownian motions with correlated components $(W_1(t))_{0\leq t\leq T}$ and $(W_2(t))_{0\leq t\leq T}$,

$$d[W_1, W_2](t) = \rho\, dt,\ \ \rho \in [-1, 1].$$

[3] Many other solutions are possible. In particular, if $f(t, x)$ is any smooth function which satisfies $f_t + f_{xx}/2 = 0$ (the backward heat equation), and $f,\ f_t,\ f_x,\ f_{xx}$ grow not too fast as $|x| \to \infty$ then $f(t, B(t))$, $t \geq 0$, is a martingale. This can be easily verified by Itô-Doeblin formula. The growth condition is needed to ensure that all required integrability conditions hold.

Consider a market model with a MMA that pays interest at a constant continuously compounded rate r and 2 stocks with prices $S_1(t)$ and $S_2(t)$, $0 \le t \le T$, which satisfy the following equations:

$$dS_1(t) = \alpha_1 S_1(t)dt + \sigma_1 S_1(t)dW_1(t)$$
$$dS_2(t) = \alpha_2 S_2(t)dt + \sigma_2 S_2(t)dW_2(t),$$

where $\alpha_i \in \mathbb{R}$, $\sigma_i > 0$, $i = 1, 2$ are constants.

(a) Derive the market price of risk equations. Determine when there is a risk neutral measure and find it (when it exists) by stating its Radon-Nikodym derivative with respect to $\mathbb{P}$.

(b) Give a definition of arbitrage. For which values of parameters the model admits arbitrage?

Solution. (a) At first, we replace the correlated Brownian motion $(W_1(t), W_2(t))$, $t \ge 0$, by a linear transformation of a standard two-dimensional Brownian motion, since Girsanov's theorem is stated for the standard d-dimensional Brownian motion ($d = 2$ in our case). We write $W_1(t) = B_1(t)$ and $W_2(t) = \rho B_1(t) + \sqrt{1-\rho^2}B_2(t)$, where $(B_1(t), B_2(t)$, $t \ge 0$, is a two-dimensional standard Brownian motion.[4] If $|\rho| = 1$ then one Brownian motion $B(t) = B_1(t)$ is enough. The equations for the stock prices become

$$dS_1(t) = \alpha_1 S_1(t)dt + \sigma_1 S_1(t)dB_1(t)$$
$$dS_2(t) = \alpha_2 S_2(t)dt + \sigma_2 S_2(t)(\rho\, dB_1(t) + \sqrt{1-\rho^2}\, dB_2(t)).$$

In particular, the volatility matrix is equal to $\begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{pmatrix}$. The equations for the discounted stock prices are

$$d(D(t)S_1(t)) = (\alpha_1 - r)D(t)S_1(t)dt + \sigma_1 D(t)S_1(t)dB_1(t$$
$$d(D(t)S_2(t)) = (\alpha_2 - r)D(t)S_2(t)dt + \sigma_2 D(t)S_2(t)(\rho\, dB_1(t) + \sqrt{1-\rho^2}\, dB_2(t)).$$

We wish that the equations look as follows:

$$d(D(t)S_1(t)) = \sigma_1 D(t)S_1(t)d\tilde{B}_1(t)$$
$$d(D(t)S_2(t)) = D(t)S_2(t)(\rho\sigma_2\, d\tilde{B}_1(t) + \sqrt{1-\rho^2}\sigma_2\, d\tilde{B}_2(t), \tag{1}$$

where $\tilde{B}_1(t) = B_1(t) + \theta_1 t$ and $\tilde{B}_2(t) = B_2(t) + \theta_2 t$ for some θ_1 and θ_2. Comparing the two systems we find the market price of risk equations

$$\alpha_1 - r = \sigma_1\theta_1$$
$$\alpha_2 - r = \rho\sigma_2\theta_1 + \sqrt{1-\rho^2}\sigma_2\theta_2. \tag{2}$$

If $|\rho| \ne 1$ then this system has a solution

$$\theta_1 = \frac{\alpha_1 - r}{\sigma_1}, \quad \theta_2 = \frac{\alpha_2 - r}{\sigma_2\sqrt{1-\rho^2}} - \frac{\rho(\alpha_1 - r)}{\sigma_1\sqrt{1-\rho^2}},$$

and we can change the measure so that $(\tilde{B}_1(t), \tilde{B}_2(t))_{0\le t\le T}$ becomes a standard two-dimensional Brownian motion. Define

$$Z = \exp\left(-\theta_1 B_1(T) - \theta_2 B_2(T) - \frac{1}{2}(\theta_1^2 + \theta_2^2)T\right)$$

and set

$$\tilde{P}(A) = \int_A Z\,dP, \quad A \in \mathcal{F}(T).$$

Then according to (1) under $\tilde{P}$ the discounted stock prices are martingales. Thus, $\tilde{P}$ is a risk-neutral measure.

Consider the case $|\rho| = 1$. Then the stock prices are driven by the same Brownian motion $B(t) = B_1(t)$. The system (2) is inconsistent unless

$$\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\rho\sigma_2}. \tag{3}$$

If the last equation holds then we set $\theta_1 = (\alpha_1 - r)/\sigma_1$ and use the one-dimensional Girsanov's theorem. Let

$$Z = \exp\left(-\theta_1 B(T) - \frac{1}{2}\,\theta_1^2 T\right) \quad \text{and} \quad \frac{d\tilde{P}}{dP} = Z.$$

$\tilde{P}$ is a risk-neutral measure, since under $\tilde{P}$ the discounted stock prices are martingales.

(b) Recall the definition from Lecture 7. An arbitrage is a portfolio value process $X(t)$ satisfying the following conditions: (i) $X(0) = 0$; (ii) $P(X(T) \ge 0) = 1$; (iii) $P(X(T) > 0) > 0$.

The first fundamental theorem of asset pricing asserts that the model admits no arbitrage if and only if there exists a risk-neutral measure. By the analysis in part (a), if $|\rho| \ne 1$ or if $|\rho| = 1$ and the equation (3) is satisfied then there is a risk-neutral measure, i.e. no arbitrage is possible. In Example 1.3 of Lecture 7 (or Example 5.4.4 of the textbook) we constructed an arbitrage explicitly for $\rho = 1$ when (3) failed. The example for $\rho = -1$ can be constructed in exactly the same way. Thus, the answer is: the given model admits arbitrage if and only if $|\rho| = 1$ and (3) does not hold.

Problem 7. Consider the model of Problem 6 with $\rho = 1$[5] and $\sigma_1 \ne \sigma_2$. Assume that the market price of risk equations are satisfied so that there is a risk-neutral measure $\tilde{P}$ under which for a standard Brownian motion $(\tilde{B}(t))_{0\le t\le T}$

$$d(D(t)S_1(t)) = \sigma_1 D(t)S(t)_1 d\tilde{B}(t), \quad d(D(t)S_2(t)) = \sigma_2 D(t)S_2(t)d\tilde{B}(t), \quad 0 \le t \le T.$$

(a) Choose the second asset as a numéraire and set $S_1^{(S_2)}(t) := S_1(t)/S_2(t)$. State which measure is the risk-neutral measure $\tilde{P}^{(S_2)}$ for the new numéraire and define a new risk-neutral Brownian motion $(\tilde{B}^{(S_2)}(t))_{0\le t\le T}$. Write the equation for $dS_1^{(S_2)}(t)$ in terms of $(\tilde{B}^{(S_2)}(t))_{0\le t\le T}$.

[5] This assumption is made only to simplify your work on the exam. It is true that with this assumption part (b) can be also easily solved without changing the numéraire. In the case $\rho \in (-1, 1)$ the change of numéraire for this type of option (i.e. with the payoff of the form $S_2(T)f(S_2(T)/S_1(T))$) gives a simple formula while the straightforward computation is much more cumbersome.

(b) Use the new risk-neutral measure to find the price at time 0 of the option which expires at time T and pays $S_2(T)$ if $S_2(T) \le S_1(T)$ and nothing otherwise.

Solution. (a) We have derived all necessary formulas in class. Here I shall derive them for this particular problem. Solving the given equations we get that

$$S_i(t) = S_i(0)\, e^{\sigma_i \tilde{B}(t) + (r - \sigma_i^2/2)t}, \quad 0 \le t \le T, \quad i = 1, 2.$$
$$S_1^{(S_2)}(t) = S_1^{(S_2)}(0)\, e^{(\sigma_1 - \sigma_2)\tilde{B}(t) - (\sigma_1^2 - \sigma_2^2)t/2}.$$

If we want the last expression to be a martingale under the new measure $\tilde{P}^{(S_2)}$, then we should find θ such that

$$S_1^{(S_2)}(t) = S_1^{(S_2)}(0) \exp\Big((\sigma_1 - \sigma_2)\overbrace{(\tilde{B}(t) + \theta t)}^{\tilde{B}^{(S_2)}(t)} - (\sigma_1 - \sigma_2)^2 t/2\Big).$$

Equating the exponents in the two expressions for $S_1^{(S_2)}(t)$ we obtain $\theta = -\sigma_2$. If we set $\tilde{B}^{(S_2)}(t) = \tilde{B}(t) - \sigma_2 t$ then $S_1^{(S_2)}(t)$ satisfies

$$dS_1^{(S_2)}(t) = (\sigma_1 - \sigma_2) S_1^{(S_2)}(t)\, d\tilde{B}^{(S_2)}(t). \tag{4}$$

Thus, we have got the required equation.

Now let us spell out the change of measure. If we define $\tilde{P}^{(S_2)}$ in such a way that the process $\tilde{B}^{(S_2)}(t) = \tilde{B}(t) - \sigma_2 t$, $0 \le t \le T$, is a martingale under $\tilde{P}^{(S_2)}$ then we obtain the risk-neutral measure for $S_1^{(S_2)}(t)$, $0 \le t \le T$. By Girsanov's theorem the Radon-Nikodym derivative of $\tilde{P}^{(S_2)}$ with respect to $\tilde{P}$ should equal to

$$Z(T) = e^{\sigma_2 \tilde{B}(T) - \sigma_2^2 T/2} = \frac{S(T)D(T)}{S(0)}.$$

Therefore, the following change of measure ensures that $\tilde{B}(t) - \sigma_2 t$, $0 \le t \le T$, is a standard Brownian motion under $\tilde{P}^{(S_2)}$ and that $S_1^{(S_2)}(t)$, $0 \le t \le T$ is a martingale:

$$\tilde{P}^{(S_2)}(A) = \int_A Z(T) d\tilde{P} = \frac{1}{S_2(0)} \int_A D(T) S_2(T) d\tilde{P}, \quad A \in \mathcal{F}(T). \tag{5}$$

(b) By the risk-neutral pricing formula, the price of this option is

$$\tilde{E}\left(D(T)S_2(T)1_{\{S_1(t) \ge S_2(t)\}}\right) \overset{(5)}{=} S_2(0)\, \tilde{E}^{(S_2)}\left(1_{\{S_1^{(S_2)}(T) \ge 1\}}\right) = S_2(0)\, \tilde{P}^{(S_2)}(S_1^{(S_2)}(T) \ge 1).$$

From the equation (4) we know that $S_1^{(S_2)}$ is a geometric Brownian motion, so the computation is the same as for a digital call with strike 1 in the BSM model. The only little twist here is that $\sigma_1 - \sigma_2$ need not be positive, but $\tilde{B}^{(S_2)}(T)(\sigma_1 - \sigma_2)$ is still normally distributed with mean 0 and standard deviation $\sqrt{T}|\sigma_1 - \sigma_2|$. Continuing with the price calculation we get

$$S_2(0)\tilde{P}^{(S_2)}(\ln S_1^{(S_2)}(T) \ge 0) = S_2(0)\tilde{P}^{(S_2)}(Z \ge -a) = S_2(0)N(a),$$

where Z is a standard normal random variable, $N(x)$ is its CDF, and

$$a = \frac{\ln(S_1(0)/S_2(0)) - (\sigma_1 - \sigma_2)^2 T/2}{\sqrt{T}|\sigma_1 - \sigma_2|}.$$