Emerging Markets and Inflation

Lecture 3. Linear Rates and FX Intro. Part 2: Rates

Fall 2017
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Agenda for Today

Lecture 3. Linear Rates and FX Intro. Part 2: Rates

Linear Rates modeling aspects important in Emerging Markets

- **Considered Linear FX, now we turn to Interest Rates:**

1. **Fixed Income Instruments and Curves: Swaps, Deliverable and Non-Deliverable**

2. **Interest Rate (Yield) Curve Bootstrapping: variety of patterns**

3. **Differential Discounting: multiple Collaterals, special cases of collaterized EM trading**

Agenda for Today

Lecture 2. Linear Rates and FX Intro. Part 2: Rates

Linear Rates modeling aspects important in Emerging Markets

1. **Fixed Income Instruments and Curves**
   - **Bond**
   - **Deposit, FRA and Cash Money Market Rate**
   - **Interest Rate Swap**
   - **Asset Swap and Swap vs. Bond curve in Emerging Markets**
   - **Cross Currency Swaps**
2. **Interest Rate (Yield) Curve Bootstrapping**
3. **Differential Discounting**

- **Stay in Linear domain**
  - **Define *Linear* as products with no vol dependency**
  - **Review products contributing to yield curve construction**

- **Bond**
  - **Recall Zero Coupon Bond with Fixed cash flow at time *T* pricing:**

$$\boxed{\begin{aligned} &Z(0,T) = DF(0,T) \\ &Z(t,t) = 1 \end{aligned}}$$

  - **Define *y* as simple rate of return on investment of the amount *Z(t,T)* that becomes 1 at time *T* and Exponential rates representation to write as in [Wilmott 2000]**

$$\boxed{Z(t,T) = e^{-y(T-t)}}$$

Bond (continued)

- **For coupon baring bond rate of return defined via *PV* or *Dirty Price*:**

$$PV(t,T) = P \cdot e^{-y(T-t)} + \sum_{i}^{N} C_i e^{-y(t_i - t)}$$

P **- bond's principal**

N **- number of coupons**

C_i **- coupon paid at time** t_i

- **More common semi-annual Bond Day Count convention:**

$$PV(t, t_N = T) = \frac{P}{\left(1 + \frac{1}{2} y\right)^{(T-t)}} + \sum_{i}^{N} \frac{C_i}{\left(1 + \frac{1}{2} y\right)^{(t_i - t)}}$$

Bond (continued)

- **Graph of yield-to-maturity vs. maturity for bonds' collection is the simplest Yield Curve example:**

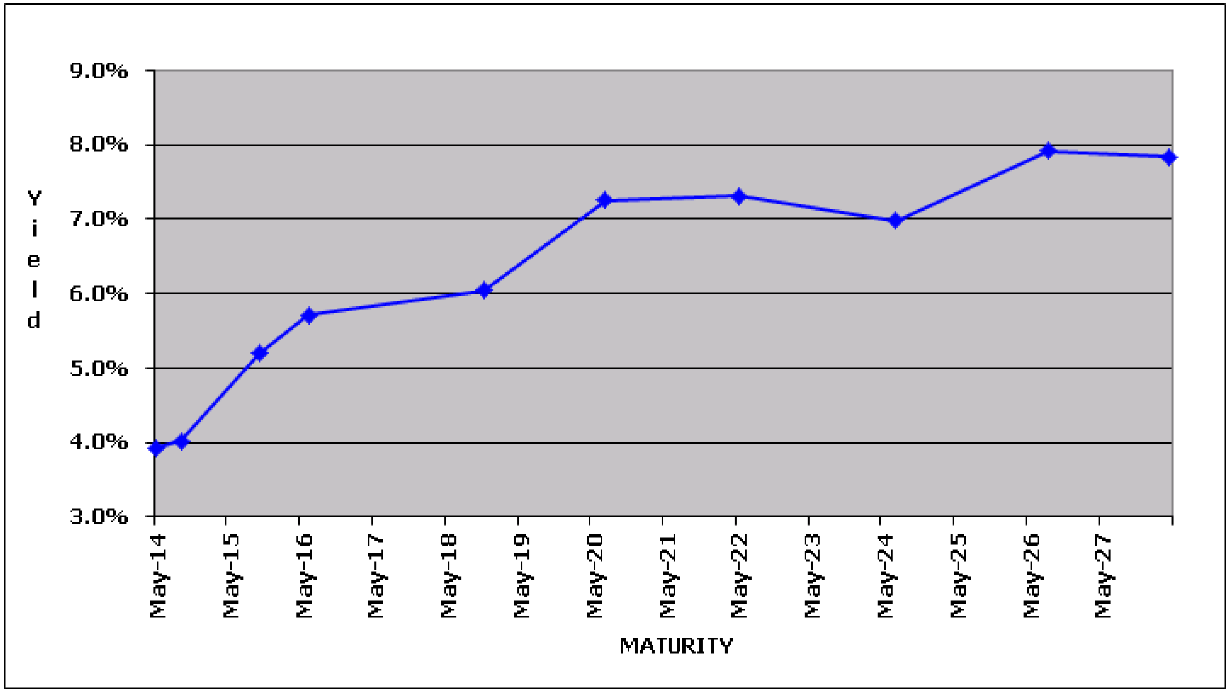

Figure 1. Yield to Maturity of Colombia Local TES bonds as of November 2013

Bond (continued)

- So given collection of bonds, we know how to bootstrap a curve, extract collection of Discount Factors to price derivatives
  - Ignore for now actual technical bootstrapping mechanism
- But there is a problem: can we really use bond curve?
  - Bonds is associated with an issuer
  - Non-zero probability of default
  - Bond carries credit risk
  - Bond curve is not really risk-free
  - Keep that thought for now…

Deposit, FRA and Cash Money Market

- **Recalling again from Lecture 2, instrument that pays pre-agreed rate *L* for pre-agreed period of time *τ* has Future Value of**

$$\boxed{FV(T, T+\tau) = 1 + \tau \cdot L}$$

- ***Discount Factor* or price of Zero Coupon Bond:**

$$\boxed{\begin{aligned} 1 &= (1+\tau \cdot L) \cdot Z(T, T+\tau) \\ Z(T, T+\tau) &= \frac{1}{1+\tau \cdot L} \end{aligned}}$$

- ***Deposit* or *Money Market* starts today, lasts till maturity**

Deposit, FRA and Cash Money Market

- **Forward Rate Agreement (FRA) locks rate for future time**
- **FRA carries exchange of funds with pre-defined strike *K***

$$\boxed{FV(T, T+\tau) = \tau \cdot (K - L)}$$

- **FRA rate K and Discount Factors are connected as:**

$$\boxed{K = L = \frac{Z(t,T) - Z(t, T+\tau)}{\tau \cdot Z(t, T+\tau)}}$$

IR Swap

- **Collection of cash flows**
- **FRA-type exchanging series of Fixed and Floating payments**
- **The most common Interest Rate instrument in the market**

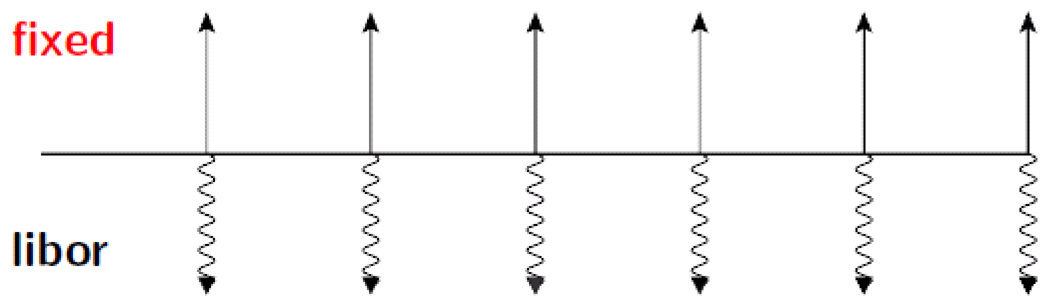

Figure 2. Standard Interest Rate Swap receiving Fixed and paying Floating LIBOR [Fujii, 2010]

IR Swap (continued)

- **Present Value would be a collection of FRA PV's:**

$$PV(t) = \sum_{i=0}^{N-1} \tau \cdot (L_i - K) \cdot Z(t, T_{i+1}) \tag{1}$$

- **Introduce Annuity and Par Swap Rate that prices swap at 0:**

$$Ann(t) \overset{def}{=} \sum_{i=0}^{N-1} \tau_i \cdot Z(t, T_{i+1})$$

$$S_{PAR}(t) \overset{def}{=} \frac{\sum_{i=0}^{N-1} \tau_i \cdot L_i \cdot Z(t, T_{i+1})}{Ann(t)}$$

$$PV(t) = Ann(t) \cdot [K - S(t)] \tag{2}$$

- **Swap has Fixed and Floating Leg**

- **Fixed Leg equivalent to *K*-coupon paying bond**

Asset Swap

- **Back to comment about building Zero Curve from Bonds: is problematic due to risky nature of bonds!**
- **Assume LIBOR based Swap curve risk-free for now**
  - **Will discuss comparison between LIBOR (3m) and OIS (O/N) later today**
- ***Asset Swap Spread* as difference between Bond and Swap curves**
- **Party A (Investor) with risky bond swaps it for flow of LIBOR's plus fixed spread S_A passing the default risk to Counterparty B:**

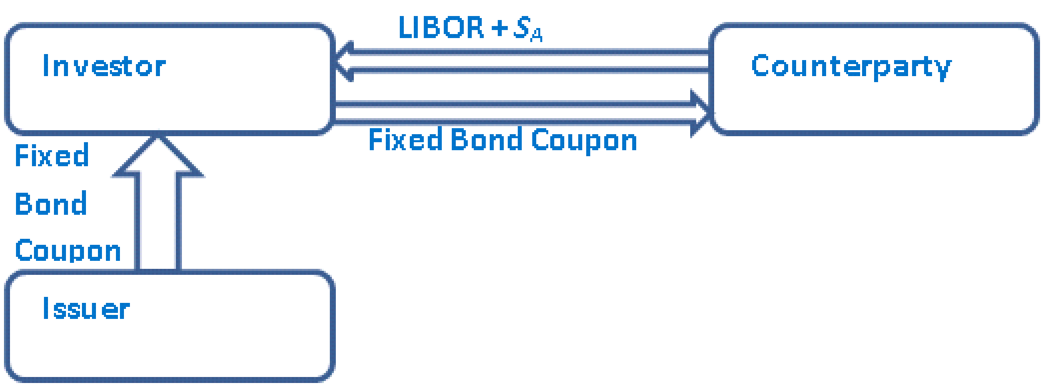

Figure 3. Asset Swap with Asset Swap Spread S_A

Asset Swap (continued)

- History of financial derivatives showing how Interest Rate Swap market grows out of matured and long traded Bond market
- Happened few times in Emerging Markets over last few years
- Bonds lack flexibility of a swap, so as market matures, need of a Swap becomes more important
- Look at example of “predicting” a Swap curve from a Bond one using real case study in Latin America
- Assume collection of liquidly trading bonds to provide us with a Curve
- Present bond’s cash flows as collection of risky flows

Derive Swap Curve from a risky Bond Curve

- **For price of a risky bond using Credit notations**
- **Weight payment by probability of no default**
- **Add default Recovery weighted by forward default probabilities**

$$DP(t,T)=\sum_{i=1}^{N}\left\{CF_i \cdot Z_S(t,t_i)\cdot p_{nD}(t,t_i)+\left[C_i+R\right]\cdot\frac{Z_S(t,t_{i-1})+Z_S(t,t_i)}{2}\cdot p_{FwdD}(t,t_i)\right\} \qquad \textbf{(3)}$$

$$CF_i=\begin{cases} C_i, & i<N \\ P, & i=N \end{cases}$$

- ***DP(t,T)*** **is a time *t* Dirty Price of a bond maturing at time *T***
- **C_i is bond's coupon payment at time t_i**
- **$p_{nD}(t,t_i)$ is time *t* probability of no default at time t_i**
- **$p_{FwdD}(t,t_i)$ is time *t* forward default probability at time t_i**
- ***R*** **is recovery of bond's notional at time of default**
- **$Z_S(t,t_i)$ is "riskless" or Swap-based Discount Factor**

Derive Swap Curve from a risky Bond Curve (continued)

- **Start with classic Merton's jump diffusion model [Merton, 1976]:**

$$\frac{dS}{S} = \left(r - q - \lambda \mathbf{E}[\eta - 1]\right) + \sigma dW + (\eta - 1)dq$$

- **Here:**

$$dq = \begin{cases} 0, & \text{with probability } \left[1 - \lambda(t)dt\right] \\ 1, & \text{with probability } \left[\lambda(t)dt\right] \end{cases}$$

  - $\eta = e^{\alpha(t)+\beta(t)\cdot\varepsilon}$ **with** $\varepsilon \propto \mathrm{N}(0,1)$
  - **σ - lognormal vol**
  - **λ - intensity of the jump process, or expected jump annual frequency**
  - **α and β are jump's mean and standard deviation respectively**
  - **Poisson process *dq* and Brownian *dW* are assumed independent**

Derive Swap Curve from a risky Bond Curve (continued)

- **Recall Spot FX dynamics from Lecture 2, and reduce Merton to Spot FX dynamics with jump coinciding with credit event:**

$$\boxed{\frac{dS_t}{S_t} = (r_t^D - r_t^A + \lambda K)dt + \sigma dW_t - Kdq} \qquad (4)$$

- **Here:**
  - **Superscript *D*: Denominated**
  - **r^D : Denominated CCY Interest rates**
  - **Superscript *A* : Asset or USD here**
  - **r^A : US LIBOR rates**
  - ***K* : jump size**
  - **λ : jump intensity or the clean CDS spread**
  - **If λ=*Const*, at default Spot FX goes from S_o to S_{wD} as** $\boxed{S_{wD} = S_o e^K}$

Derive Swap Curve from a risky Bond Curve (continued)

- **Assume FX jump simultaneous with default**
- **Re-write Eq.(3) for a cash flow at t_i**

$$\boxed{Z_{wD} = Z_{nD} \cdot p_{nD} + R \cdot Z_{nD} \, p_{FwdD}} \qquad (5)$$

- **Here:**
  - **Z_{wD}: composite Zero Coupon bond price with default**
  - **Z_{nD} : price of riskless Zero Coupon bond (equivalent to Swap)**

- **Assuming, or rather defining here**

$$\boxed{\begin{aligned} Z_{nD} &\equiv Z_S \\ p_{nD} &= \exp\left(-\int \lambda dt\right) \end{aligned}}$$

Derive Swap Curve from a risky Bond Curve (continued)

- **Re-write Eq.(5) a bit:**

$$Z_{wD} = Z_{nD} \cdot p_{nD} \left[1 + R \frac{p_{FwD}}{p_{nD}} \right]$$

- **From Eq.(4) default adds factor λK to domestic rates**
- **For USD denominated (off-shore) bonds recovery is FX adjusted:**

$$R \cdot \frac{PPL^D}{S_o} \overset{Default}{\Rightarrow} R \cdot \frac{PPL^D}{S_{wD}} = R \cdot \frac{PPL^D}{S_o} \cdot e^{-K}$$

- **Equivalent to a substitution:**

$$\begin{aligned} Z_{wD} &:= Z_{wD} e^{-K \int \lambda dt} = Z_{wD} \left(p_{nD} \right)^K \\ R &:= e^{-K} \cdot R \end{aligned}$$

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Derive Swap Curve from a risky Bond Curve (continued)

- **Combine it all together**

$$Z_{wD}\left(p_{nD}\right)^{K} = Z_{nD} \cdot p_{nD}\left[1+e^{-K} \cdot R\frac{p_{FwdD}}{p_{nD}}\right]$$

- **And come to the final expression:**

$$Z_{nD}\left(t,t_i\right)=\frac{Z_{wD}\left(t,t_i\right)\left[p_{nD}\left(t,t_i\right)\right]^{K-1}}{\left[1+e^{-K}\cdot R\cdot\frac{p_{FwdD}\left(t,t_i\right)}{p_{nD}\left(t,t_i\right)}\right]} \qquad (6)$$

- **Simple intuitive derivation**
- **Where and how do we source parameters?**
- **How can we practically use it?**

Derive Swap Curve from a risky Bond Curve (continued)

- Z_{wD} : risky Zero Coupon Bond price from the market
- p_{nD} and p_{FwdD} : default related probabilities from the CDS market
- R : bond's recovery rate. Assume same for all. Not directly observed in the market, but almost agreed upon between market participants
- K : expected FX devaluation:
  - Not on the market
  - Some historical observations from other markets, but in different countries and under different economics
  - Still can calibrate to few other LatAm countries
  - Economics is very important as will impact default
  - Source from market participants (traders) and Economic Research

Derive Swap Curve from a risky Bond Curve (continued)

- **Practical consideration with 25% recovery and *K=0.7* ($e^{0.7} \approx 2$):**

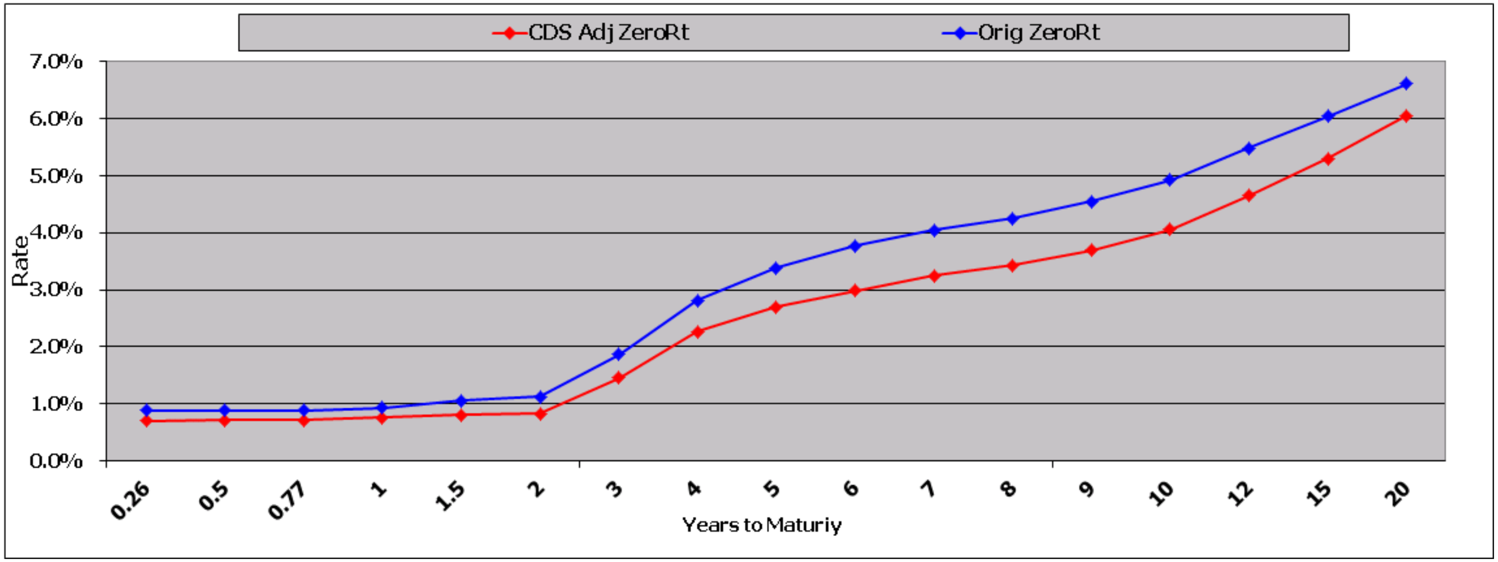


Figure 5. Riskless (CDS Adjusted) Curve derived from traded Risky Bonds Curve

- **Swap curve below Bond curve indicating "cost of risk" and in line with historical observations**

HW1: Use market data provided and the formulae above to study dependency of the asset-swap spread on observable and non-observable market parameters. Discuss a calibration approach to test parameters required. Discuss back-testing study to assess validity of the modeling approach.

Cross Currency Swaps

- **Type 1: Float - Float Cross Currency Basis Swap**
- **Exchange of floating cash flows in 2 Ccys:**

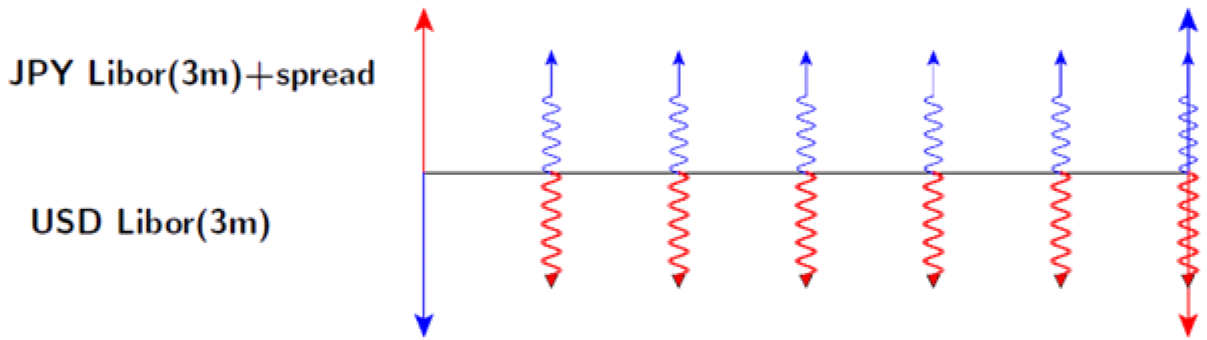


Figure 6. Cross Currency Float - Float Basis Swap

- **Notionals connected via Spot FX reset**
- **Traded in Developed Market with Notional resetting every coupon. Do we need Convexity Adjustments here?**
- **Traded in Emerging Markets mostly in cash-settling or non-deliverable flavor ➔ arguments considered earlier stand!**

Cross Currency Swaps (continue with Type 1)

- **Example before of 3m JPY LIBOR + Spread for 3m USD LIBOR**
- **Define PV and Par Spread setting price of swap at zero for non-resetting notional swap as**

$$pv(t, t_N = T) = \left(\sum_i \left(L_i^{CCY} + S^{CCY} \right) \cdot \tau \cdot Z_i^{CCY} + Z_T^{CCY} \right) - \left(\sum_i L_i^{\$} \cdot \tau \cdot Z_i^{\$} + Z_T^{\$} \right)$$

$$S_{PAR}^{CCY} \overset{def}{=} \frac{\left(\sum_i L_i^{\$} \cdot \tau \cdot Z_i^{\$} + Z_T^{\$} \right) - \left(\sum_i L_i^{CCY} \cdot \tau \cdot Z_i^{CCY} + Z_T^{CCY} \right)}{Ann^{CCY}} \qquad (7)$$

- **Emerging Markets swap usually have spread on the USD side and are cash settling. Example: CLP – USD Cross Currency Swap**

Cross Currency Swaps

- **Type 2: Fixed - Float Cross Currency Basis Swap**
- **More common in Emerging Markets**
- **Fixed on EM side due to no liquid Floating Swaps trading**
- **Cash settled**
- **Present Value and Par Swap rated defined as:**

$$pv(t, t_N = T) = \left(\sum_i K^{CCY} \cdot \tau \cdot Z_i^{CCY} + Z_T^{CCY} \right) - \left(\sum_i L_i^{\$} \cdot \tau \cdot Z_i^{\$} + Z_T^{\$} \right)$$

$$K_{PAR}^{CCY} \stackrel{def}{=} \frac{\left(\sum_i L_i^{\$} \cdot \tau \cdot Z_i^{\$} + Z_T^{\$} \right) - Z_T^{CCY}}{Ann^{CCY}}$$

Lecture 2. Linear Rates and FX Intro. Part 2: Rates

Linear Rates modeling aspects important in Emerging Markets

1. Fixed Income Instruments and Curves

2. Interest Rate (Yield) Curve Bootstrapping
   - Calibration
   - Markets and Patterns
   - New Instrument Derivation

3. Differential Discounting

Interest Rate (Yield) Curve Bootstrapping

- **Covered most instruments used in Zero Curve Construction**
- **FX and Rates**
- **Discuss Curve construction itself in Developed and Emerging markets**
- **Define Zero Curve as function *f(t)* to give us info on instantaneous forward rates at some time *t***
- **Write Zero Coupon Bond price more accurately:**

$$\boxed{Z(t,T) = \exp\left(-\int_t^T f(s)\,ds\right)}$$

Calibration

- **Restrict *f(t)* on *Smoothness*, *Uniqueness* and *Parametrization***
- **Must price back original instruments (*Benchmarks*) via minimization**

$$\min\left[\sum_i \left(B_i(\mathbf{C}) - \hat{B}_i\right)^2\right]$$

Here

- $\hat{B}_i$ **: market observable benchmark rate (considered earlier)**
- $\mathbf{C}$ **: generalized vector representation of the curve**
- $\hat{B}_i(\mathbf{C})$ **: price of *i*-th benchmark off the curve**

Calibration (continued)

- Ideal case all benchmarks are linear, consecutive and non-overlapping
- Result is collection of nodes or anchor points for Zero Curve:

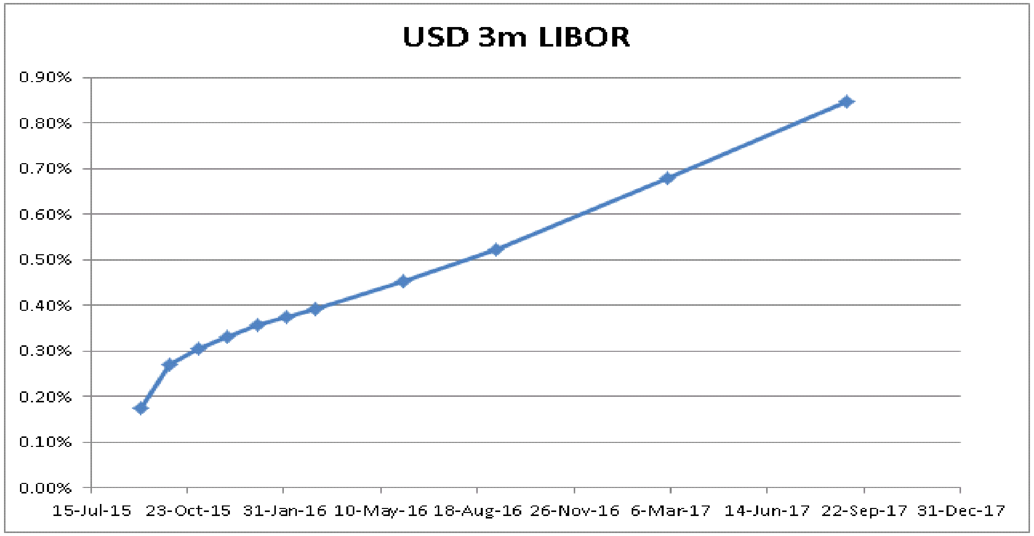


Figure 7. Front end of the 3m USD LIBOR curve. Market benchmark rates

- Now interpolation: start with Linear, see that it is not smooth
- So, must respect economic restrictions, smooth, no risk spillage, etc…
- Suggest Splines: Exponential, tension…

Markets and Patterns

- **Summarize most common Zero Curve building blocks:**
  - **FX Forward or NDF:**
    - **Equivalent to Cash Flows in 2 currencies**
    - **Main instrument for short end of Cross Currency curve (up to 2Y?)**
  - **Single Currency Instruments:**
    - **Cash and Money Market rates**
    - **IR Futures in Developed Markets**
    - **Single Currency Interest Rate Swap**
  - **Cross Currency Instruments:**
    - **Float - Float Basis Swap**
    - **Fixed - Float Basis Swap**

Markets and Patterns (continue with Patterns)

Pattern I. STD10: Standard G10

- FX Forward, Single Currency IR Futs and Swaps
- Cross Currency Float – Float Basis Swaps
- Examples: EUR, JPY

Pattern II. AdvEM: Advanced EM

- More developed Emerging Markets
- Non-Deliverable FX Forwards
- Single Currency IRS with offshore settlement
- Cross Currency Float – Float Basis Swaps cross USD
- Examples: Chile, Colombia

Pattern III. EM1: Emerging Markets 1

- FX Forwards
- Cross Currency Float – Float and Fixed – Float Basis Swaps
- Examples: Turkey

Markets and Patterns (continue with Patterns)

Pattern IV. AsiaEM:

- Mostly used in Emerging Asia
- FX Forwards
- Single Currency IRS
- Cross Currency Fixed – Float Basis Swaps
- Examples: Malaysia, South Korea

Pattern V. LowEM:

- Least developed Emerging Markets with no IRS trading
- Non-Deliverable FX Forwards
- Cross Currency Fixed – Float Basis Swaps cross USD
- Examples: Ukraine, Peru

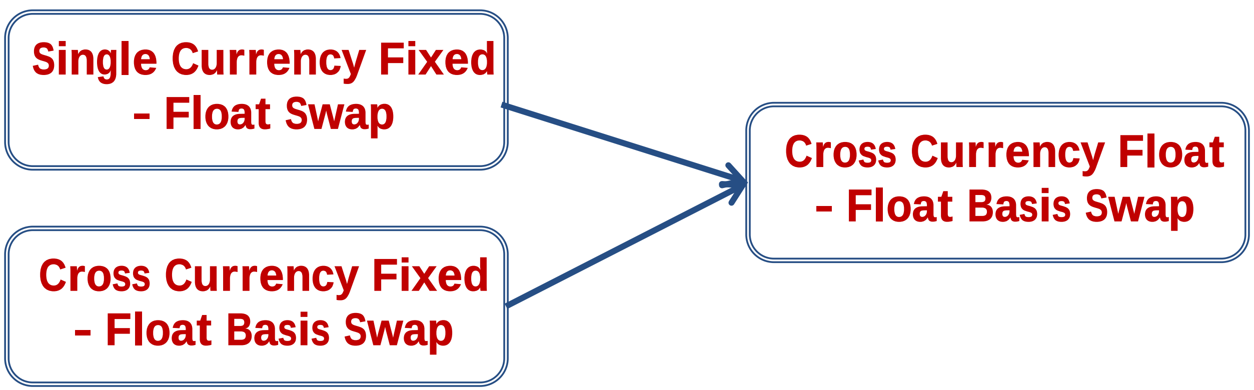
Markets and Patterns (continue with Patterns)

- **Most common observation from 5 patterns?**
- **Only one comes from Developed Markets and four from Emerging!**

Note: **Not mentioned curves package! Most must have Projection and Discounting separated from each other! More on that to follow**

New Instruments Derivation

- **Emerging Market emerges in front of us**
- **Price non-existing instrument effectively upgrading a Pattern**
- **Real life example of market moving from Pattern IV to Pattern II:**



New Instruments Derivation (continued)

- **Two steps in the process**
- **Step 1: derive theoretical Basis Swap Par Spread S_T from swap legs that we expect to see in the market:**

$$PV^{\$} = \sum_i \left(L_i^{\$} + S_T\right)\cdot \tau \cdot Z_{t_i}^{\$} + Z_T^{\$}$$

$$PV^{CCY} = \sum_i L_i^{CCY} \cdot \tau \cdot Z_{t_i}^{CCY} + Z_T^{CCY}$$

- **Note offshore discounting that may be questionable in onshore swap**
- **Assuming them equal in a par swap:**

$$S_T = \frac{\left(\sum_i L_i^{CCY} \cdot \tau \cdot Z_{t_i}^{CCY} + Z_T^{CCY}\right) - \left(\sum_i L_i^{\$} \cdot \tau \cdot Z_{t_i}^{\$} + Z_T^{\$}\right)}{Ann^{CCY}} \qquad (8)$$

New Instruments Derivation (continued)

- **Step 2: extract Cross Currency discounting from Cross Currency Fixed - Float Basis Swap:**

$$\boxed{\sum_i L_i^{\$} \cdot \tau \cdot Z_{t_i}^{\$} + Z_T^{\$} = K \cdot Ann^{CCY} + Z_T^{CCY}} \qquad (9)$$

- **Eq.(8) and (9) are classic bootstrapping problem**
- **Could solve consecutively extracting discounting, solving for spread**
- **Reality of overlapping instruments solving *2xN* equations with *2xN* unknowns plus restrictions…**
- **How can we test that before the market actually appears?**

Agenda for Today

Lecture 2. Linear Rates and FX Intro. Part 2: Rates

Linear Rates modeling aspects important in Emerging Markets

1. Fixed Income Instruments and Curves
2. Interest Rate (Yield) Curve Bootstrapping
3. **Differential Discounting**
   - **Multiple CSA Discounting Curves Intro**
   - **Special Cases of Local Collaterized Trading in Emerging Markets**
     - **EM Currencies collaterized in USD**
     - **EM Currencies collaterized in Off-Market same Ccy**

Differential Discounting

- **Recognition of role of Collateral in discounting after turmoil of 2008**
- **Collaterized trades valuation must depend on Collateral**
- **Markets moving away from LIBOR discounting**
- **Spread between perceived as risk-free LIBOR and truly risk-free OIS widening**
- **Market participants agree with concept of Funding-equivalent discounting**
- **Recognize difference between secured and unsecured borrowing as in [Whittall, 2010]**

Differential Discounting

Multiple CSA Discounting Curves Intro

- **Price of Par Interest Rate Swap is zero at inception**
- **Not anymore once market moves**
- **Counterparty with negative Mark to Market (MTM) posts Collateral in pre-agreed denomination to protect other side against default**
- **Posted Collateral must be earning interest at overnight risk-free rate of collateral currency**
- **Thus no-arbitrage says Future Value discounting must use appropriate OIS and not LIBOR**

Multiple CSA Discounting Curves Intro

- **Consider bank A with a future obligation of $1mm to bank B in 1 year**
- **B is exposed to possible loss of this future payment, so**
- **A posts $1mm collateral to B, B earns interest on it and returns with interest at maturity**

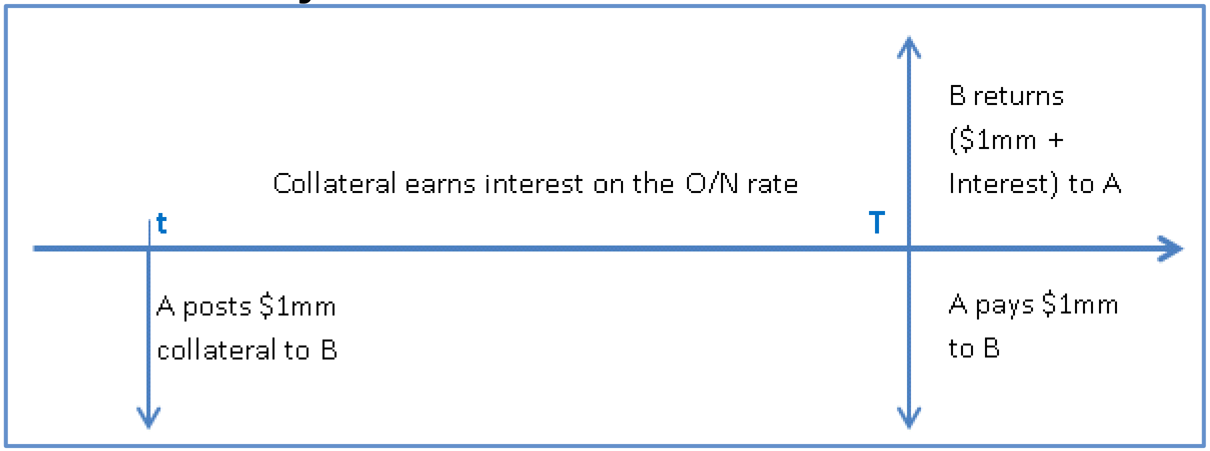


Figure 8. Future Cash Flow and Collateral example

- **Earn interest at most risk-free curve → the OIS rate**
- **Arbitrage-free tells us that same curve must be used for Discounting!**

Multiple CSA Discounting Curves Intro (continued)

- **Look back at IR curve bootstrapping story**
- **We now have more than one curve to account for!**
- **Generally defined and connected Projection and Discounting**
- **How many of each? ➔**
- **Arbitrage free again requires to have only one Projection ➔**
- **But there could be more than one Discounting! Even now we know at least two: with and without Collateral**

Multiple CSA Discounting Curves Intro

- **USD example above could be easily extended to EUR with EURIBOR and EONIA overnight rates and so forth…**

- **Introduce two cases:**

1. **Single and Same Currency Collateral: default or On-Market mode**

2. **Collateral in different currency or a choice of more than one currency: Off-Market mode**

On-Market Discounting

- **USD market as example again looking at 3m LIBOR Swap Floating leg**

$$PV_{FLT} = Z_{\$|\$}(t_N) + \sum_{i=1}^{N} L_i \cdot \tau \cdot Z_{\$|\$}(t_i)$$

$$S_{\$|\$} \overset{def}{=} \frac{\sum_{i=1}^{N} L_i \cdot Z_{\$|\$}(t_i)}{Ann_{\$|\$}} \qquad \textbf{(10)}$$

Here CCY1|CCY2 is for CCY1 rate or price collaterized in CCY2

- **OIS funding rate expressed as spread to LIBOR must price to par:**

$$Z_{\$|\$}(t_N) + \sum_{i=1}^{N} (L_i + OIS_\$) \cdot \tau \cdot Z_{\$|\$}(t_i) = 1 \qquad \textbf{(11)}$$

- **Eq. (10) and (11) uniquely define Projection and Discount curve**

On-Market Discounting (continue)

- **Extend same for case of Cross Currency Float - Float Basis Swap**

- **Two currency leg, still one collateral currency USD**

- **Same JPY - USD Basis Swap as we considered earlier:**

$$Z_{JPY|\$}(t_N)+\sum_{i=1}^{N}\left(L^{JPY}{}_i+S\right)\cdot\tau\cdot Z_{JPY|\$}(t_i)=Z_{\$|\$}(t_N)+\sum_{i=1}^{N}L^{\$}{}_i\cdot\tau\cdot Z_{\$|\$}(t_i)$$

- **Still can uniquely solve using Eq. (10) and (11), so is also On-Market case**

Off-Market Discounting

- For other Ccy collateral assuming market for B_{CCY} funding spread over LIBOR

$$Z_{\$|CCY}(t_N)+\sum_{i=1}^{N}(L_i+B_{CCY})\cdot\tau\cdot Z_{\$|CCY}(t_i)=1$$

- Single equation to solve for unknown discounting

- More complex with Option to switch collateral: USD or EUR

$$Z_{\$|(USD,EUR)}(t_N)+\sum_{i=1}^{N}\left(L_i+B_{(USD,EUR)}\right)\cdot\tau\cdot Z_{\$|(USD,EUR)}(t_i)=1$$

- With funding spread $B_{(USD,EUR)}$ not available in the market, so we need a model to derive it

Off-Market Discounting: FX invariance

- **FX Forward invariance comes from simple no-arbitrage:**
  - **Trade 1: receive *$Q* at *T* with CSA collateral during the trade**

$$\boxed{PV_1(t) = Q_\$ \cdot Z_{\$|CSA}(t,T)}$$

  - **Trade 2: receive *$Q* at *T* in JPY under same collateral during**

$$\boxed{PV_2(t) = \frac{1}{S}\left[Q_\$ \cdot F(T) \cdot Z_{JPY|CSA}(t,T)\right]}$$

Same as before, *S* is FX Spot and *F(T)* is forward FX at *T*

  - **Economically same trade, so:**

$$PV_1(t) = PV_2(t)$$

$$Q_\$ \cdot Z_{\$|CSA}(t,T) = \frac{1}{S}\left[Q_\$ \cdot F(T) \cdot Z_{JPY|CSA}(t,T)\right]$$

$$F(T) = S\,\frac{Z_{\$|CSA}(t,T)}{Z_{JPY|CSA}(t,T)}$$

Off-Market Discounting: FX invariance (continued)

- **FX Forward invariance gives easy recipe of collateral switch:**

$$\frac{Z_{\$|CSA}(t,T)}{Z_{JPY|CSA}(t,T)} = \frac{Z_{\$|\$}(t,T)}{Z_{JPY|\$}(t,T)}$$

$$Z_{JPY|CSA}(t,T) = Z_{JPY|\$}(t,T) \cdot \frac{Z_{\$|CSA}(t,T)}{Z_{\$|\$}(t,T)}$$

- **Or simply**

$$Z_{JPY|CSA}(t,T) = Z_{JPY|JPY}(t,T) \cdot \frac{Z_{\$|CSA}(t,T)}{Z_{\$|JPY}(t,T)}$$

- **Strictly speaking this logic only stands in case of deterministic rates. A natural assumption in short-dated world of FX derivatives, requires a stochastic IR adjustment for longer maturities**

HW2: Assigned reading. Follow [Fujii, 2010] for a more rigorous derivation of FX invariance under different Collaterals

Special Cases of Local Collateral in Emerging Markets

- Above we assumed collateral in deliverable and acceptable Ccy: G10
- Not the case in Emerging Markets. Tow special considerations
- Case 1: EM Currencies collaterized in USD
  - USD funded trading in EM currencies
  - Even applied to on-shore Single Currency Fixed - Float IRS
  - Write again Eq. (10) and (11) replacing US Libor with Ccy:

$$\sum_{i=1}^{N}\left[L_{CCY}(t_i)+OIS_{CCY|\$}\right]\cdot\tau\cdot Z_{CCY|\$}(t_i)=1-Z_{CCY|\$}(t_N)$$

$$S_{CCY|\$}\overset{def}{=}\frac{\sum_{i=1}^{N}L_{CCY}(t_i)\cdot Z_{CCY|\$}(t_i)}{Ann_{CCY|\$}} \qquad (12)$$

EM Currencies collaterized in USD (continued)

- **So for Single Currency Fixed – Float IRS it would be:**

$$\sum_{i=1}^{N} L_{CCY}(t_i) \cdot Z_{CCY|\$}(t_i) = K_{CCY|\$} \cdot Ann_{CCY|\$}$$

- **And for offshore Cross Currency Float – Float Basis Swap:**

$$\sum_{i=1}^{N} \left[L_{\$}(t_i) + B_{CCY|\$}\right] \cdot \tau \cdot Z_{\$|\$}(t_i) + Z_{\$|\$}(t_N) = \sum_{i=1}^{N} L_{CCY}(t_i) \cdot \tau \cdot Z_{CCY|\$}(t_i) + Z_{CCY|\$}(t_N) \qquad \textbf{(13)}$$

- **Combining Eq. (12) and (13) for discount curve expression only depending on observable market quantities:**

$$S_{CCY|\$} \cdot Ann_{CCY|\$} + Z_{CCY|\$}(t_N) = \sum_{i=1}^{N} \left[L_{\$}(t_i) + B_{CCY|\$}\right] \cdot \tau \cdot Z_{\$|\$}(t_i) + Z_{\$|\$}(t_N)$$

EM Currencies collaterized in Off-Market same Ccy

- **Default collateral of Single Currency Fixed - Float IRS is USD, but local Ccy funding is available ➔ qualifies it as Off-Market case**

- **Apply FX invariance to switch collateral:**

$$\boxed{\begin{array}{l} F_{FX/USD|\$}(T) = F_{FX|CSA}(T) \\ \dfrac{Z_{CCY|\$}(t,T)}{Z_{\$|\$}(t,T)} = \dfrac{Z_{CCY|CCY}(t,T)}{Z_{\$|CCY}(t,T)} \end{array}}$$

Above:

- **Terms with USD collateral $Z_{CCY|\$}(t,T)$ and $Z_{\$|\$}(t,T)$ are observable;**
- **Local discount factor with local collateral $Z_{CCY|CCY}(t,T)$ is local cost funding, also expected to be observable**
- **Single unknown to derive:**

$$\boxed{Z_{\$|CCY}(t,T) = Z_{CCY|CCY}(t,T) \cdot \frac{Z_{\$|\$}(t,T)}{Z_{CCY|\$}(t,T)}}$$

References

[Wilmott 2000] Wilmott, P. (2000). *Quantitative Finance*. John Wiley & Sons Ltd.

[Andersen & Piterbarg, 2010] Andersen, L. B., & Piterbarg, V. V. (2010). *Interest Rate Modeling*. Atlantic Financial Press

[Whittall, 2010] Whittall. (2010). *The price is wrong*. Risk.net

[Fujii, 2010] Fujii, M., Shimada, Y., & Takahashi, A. (2010). *On the Term Structure of Interest Rates with Basis Spreads, Collateral and Multiple Currencies*. International Workshop on Mathematical Finance

Homework

1. **Use market data provided and the formulae around Eq. (6) to study dependency of the asset-swap spread on observable and non-observable market parameters. Discuss a calibration approach to test parameters required. Discuss back-testing study to assess validity of the modeling approach.**

2. **Follow [Fujii 2010] for a more rigorous derivation of FX invariance under different Collaterals. No submission is expected here**

	Bond Recovery	25%
	K	70%
Tenor, [Yrs]	Exp Bond Zero Rate (Risky)	Prob Default
0.25	0.88%	0.25%
0.5	0.88%	0.50%
0.75	0.88%	0.75%
1	0.95%	1.00%
1.5	1.00%	2.10%
2	1.10%	3.30%
3	1.90%	7.00%
4	2.80%	12.00%
5	3.40%	18.25%
6	3.80%	25.00%
7	4.00%	30.00%
8	4.25%	35.00%
9	4.50%	40.00%
10	5.00%	45.00%
12	5.50%	52.00%
15	6.00%	61.00%
20	6.50%	73.00%