4. Pricing Instruments with Credit Risk

Risky cash flows

So far, we have confined ourselves primarily to instruments whose cash flows are risk-free, but of course this is an idealization to which real bonds do not often conform. For a real fixed-income instrument, the observed market price is usually influenced by the fact that the cash flows are risky—that is, they may not occur as promised.

Given the importance of the subject, naturally a great variety of approaches to it are taken. Some attempt to explain the reason for credit risk by explicitly modeling the capital structure of risky issuers; others take characteristics such as agency ratings or market-based estimates of credit quality into account and attempt to forecast default behaviors based on the historical experience of comparable issues.

Here, we present a reduced-form approach to the valuation of risky cash flows, meaning that we make no effort to explain why default occurs, but instead focus on concerns directly relevant to pricing in the presence of credit risk. This approach has the advantages of being simple and, in many ways, mathematically appealing. It is capable of translating price information in the bond market for a wide variety of instruments into explicit and comprehensible statements of the credit component of the price; it also forms the basis on which credit derivatives are analyzed and traded.

A natural place to begin is the case of a risky zero-coupon bond. If a real obligor promises to pay some amount N at a future time T, then there is a possibility that it may default on this obligation. Due to the legal nature of default, we consider it what is called an "absorbing" state—that is, default occurs once and, once it has occurred, the entity remains in default. If the time of default τ falls before T, then for now we make the simplifying assumption that no cash flow of any kind is received. Alternatively, τ may fall after the time of the cash flow, an outcome whose present value is associated with the familiar price $V_{risk\text{-}free} = NP_T$.

We suppose then that the cash flow may have two possible present values as seen at time zero, conditional on the default time:

$$V_\tau = \begin{cases} 0 & \tau \in (0,T] \\ NP_T & \tau \in (T,+\infty) \end{cases}$$

That is, the cash flow's value has a random component. To account for this, we must do something that seems natural but is fundamentally different from the approaches we have seen so far. We must price the cash flow in expectation:

$$V = E[V_\tau] = 0P(\tau \le T) + NP_T P(\tau > T) = V_{risk-free} P(\tau > T)$$

We may see this risky value as the discounted value of the cash flow, discounted further by the probability of its being received—usually called the survival probability of the obligor to time T.

A simple default model

The amount of additional discounting required depends upon the distribution of the default time τ. For a 1-year period beginning today, suppose we know the obligor's probability of default $0 < p_d < 1$ over that interval. Define the quantity…

$$\lambda = \frac{P(default)}{P(survival)} = \frac{p_d}{1 - p_d}$$

…which we will call the hazard rate. Mechanically, this transforms a default probability on the interval (0, 1) into a positive number. Conceptually, this would be over a large number of trials the ratio of the number of defaults observed to the number of survivals observed; a bookmaker would call this the odds of the obligor surviving one year.

We can of course also express the survival probability p_s in terms of the hazard rate:

$$\lambda = \frac{1 - p_s}{p_s}$$

$$p_s = \frac{1}{1 + \lambda}$$

Thus, for a cash flow occurring at 1 year:

$$V = V_{risk-free}(1 + \lambda)^{-1}$$

To extend this to a multi-period model, we make the assumption that the hazard rate is constant in all subsequent years—that is, assuming the obligor survives to year n, the probability of default between then and year n+1 is also p_d. So for a cash flow occurring at the end of two years:

$$V = V_{risk-free}(1 + \lambda)^{-2}$$

…reflecting the fact that in order for the cash flow to be received, the obligor must survive year one, and then go on to survive year two. We may extend this for any integer number of years. Thus, the additional discounting we must do for a cash flow at the end of year T is…

$$(1 + \lambda)^{-T}$$

…which is a familiar-looking expression.

The distribution of τ in this model

Since we are working with a discrete-time model, τ can only take on a limited set of values. In this case, we will choose the domain 0, 1, 2,…, corresponding to the number of years the obligor survives before default occurs. Using this choice, we can see that:

$$P(\tau = n) = p_s^n p_d$$

This is the probability mass function of a geometric distribution. Determining the expected time to default for this distribution demonstrates one of its more useful properties.

At time zero, the expected time to default is the value of interest $E[\tau]$. In the first year, one of two things will happen: Either with probability p_d the obligor will default, in which case our formulation of this discrete-time problem assigns the value $\tau = 0$; otherwise, with probability p_s the obligor will survive this period. If we assume that the obligor survives, then at time 1 the obligor is once again in the same state it was initially. This means that the two-year survival probability *conditional on* surviving the first year is the same as the probability of surviving the first year. Thus, the process we are modeling is "memoryless": Past survivals have no influence on the probability of surviving an additional period.

Conditional on survival for the first year, then, the expected additional time to default must be the same expectation we seek, $E[\tau]$. As seen at time zero, this means that the expected default time conditional on the obligor surviving the first year is $E[\tau] + 1$. Thus our expectation satisfies the equation:

$$E[\tau] = p_d(0) + p_s(E[\tau]+1)$$

$$E[\tau](1-p_s) = p_s$$

$$E[\tau] = \frac{p_s}{p_d} = \frac{1}{\lambda}$$

This result sheds rather more light on our initial choice of the definition of the hazard rate, which we now see is the inverse of the expected time to default.

Choosing a finer discretization

It seems reasonable to take the simple annual model above and extend it by subdividing each year into m segments of equal length. This gives rise to new default and survival probabilities over the intervals, and therefore a new single-period hazard rate λ_m. We have already seen how we may take a model of this sort and extend it to multiple periods. The value of a cash flow at time T, providing it falls at some integer number of time periods, can be expressed as:

$$V = V_{risk-free}(1+\lambda_m)^{-mT}$$

All that remains is to choose the new hazard rate. The possible default times are now 0, $1/m$, $2/m$,…, meaning that we now have:

$$P\left(\tau = \frac{n}{m}\right) = P(m\tau = n) = p_{s,m}^{n} p_{d,m}$$

As above, this leads to the expectation:

$$E[m\tau] = \frac{1}{\lambda_m}$$

$$mE[\tau] = \frac{1}{\lambda_m}$$

$$E[\tau] = \frac{1}{m\lambda_m}$$

If we imagine beginning with an annual model and increasing the parameter *m*, then a sensible way to choose each new hazard rate is to insist that the expected default time remain constant:

$$\frac{1}{\lambda} = \frac{1}{m\lambda_m}$$

$$\lambda_m = \frac{\lambda}{m}$$

We conclude, then, that the additional factor that must be applied to account for default risk in this finer-grained model is given by:

$$\left(1 + \frac{\lambda}{m}\right)^{-mT}$$

This is a familiar expression indeed, and it should come as no surprise that we are interested in passing to the limit as our discretization becomes infinitely fine:

$$\lim_{m \to \infty} \left(1 + \frac{\lambda}{m}\right)^{-mT} = e^{-\lambda T}$$

…which is an appealing expression for the additional discounting necessary to adjust a given cash flow's present value to account for credit risk.

In light of our initial discussion of interest rates, it may seem odd that the value of the parameter λ is preserved throughout the process of making *m* arbitrarily large. After all, in the case of interest rates we saw that increasing the compounding frequency causes a given discount factor to be associated with lower and lower interest rates.

Observe, however, that in that case our goal was to preserve the *discount factor* as we passed to the limit, necessitating a change in the rate; in this case, our goal is to preserve the *expected time to default* of the obligor being modeled. In the annual model, all defaults for the first year are assigned the value $n = 0$; in the semiannual model, defaults in the first year may take on values $n = 0$ or $n = 0.5$. Thus, preserving the expected time to default definitely does not preserve the survival probabilities themselves: The total probability of default in year one for the semiannual model is not the same as it is in the annual model, or the model with any other frequency.

Since the continuous-time model permits default at any time on the interval from zero to infinity, we will choose this method of calculating survival probabilities. We may thus say that, under the simple model we have chosen, the value of a risky cash flow is given by:

$$V = V_{risk-free} e^{-\lambda T}$$

Yield of a zero-coupon bond revisited

Recall that our original treatment of yield led to the pricing expression for a zero-coupon bond:

$$V = Ne^{-yT}$$

This is, until now, the only setting in which we have so far encountered risky cash flows. Intuitively, we understood that such a cash flow had less value, and therefore a higher yield, the riskier the cash flow was. If we expand this expression to incorporate r_T, the risk-free zero rate to this maturity, then we say that…

$$y = r_T + z_T$$

…where the value z_T is called the z-spread to this maturity. The present value of a zero-coupon bond can thus be written:

$$V = Ne^{-(r_T + z_T)T} = Ne^{-r_T T} e^{-z_T T} = V_{risk-free} e^{-z_T T}$$

As we have just seen, in the simple credit model where the hazard rate is constant and no cash flow is received in the event of default…

$$V = V_{risk-free} P(\tau > T) = V_{risk-free} e^{-\lambda T}$$

…meaning that, under this model, the z-spread is equal to the hazard rate to all maturities.

Distribution of τ in continuous time

Given the evident utility of this simple model, it seems sensible to examine the properties of the resulting distribution of default time. Since we know an expression for the survival probability for any time t, we can determine the cumulative distribution function of the default time τ:

$$F_\tau(t) = P(\tau \le t) = 1 - P(\tau > t) = 1 - e^{-\lambda t}$$

…and thus its density function:

$$f_\tau(t) = P(\tau = t) = F_\tau'(t) = \lambda e^{-\lambda t}$$

This is an exponential distribution, which describes the distribution of the waiting time to the first arrival of a Poisson process. Many natural processes can be approximated well with distributions of this sort. We see that this continuous-time analog of the geometric distribution preserves the property of memorylessness if we consider the question of the conditional distribution of τ.

Our expression above describes the unconditional distribution of default time as seen at time zero. If we choose a time $t > 0$, then in order to determine what the distribution of τ will look like when this time is reached, we must account for the possibility of a default before t. Obviously, this distribution is only interesting if the obligor survives at least to t, so we seek the distribution of τ conditional on survival to that time.

We write the conditional cumulative density function for a time $s > t > 0$:

$$P(\tau \le s \mid \tau > t) = \frac{P(\tau \in (t,s])}{P(\tau > t)} = \frac{\int_t^s \lambda e^{-\lambda u} du}{e^{-\lambda t}} = \frac{e^{-\lambda t} - e^{-\lambda s}}{e^{-\lambda t}} = 1 - e^{-\lambda(s-t)} = 1 - e^{-\lambda \Delta t}$$

This once again is the cumulative of an exponential distribution with the same parameter lambda. Should we arrive at t, we discover that the distribution of default time is the same as it was initially, again under the assumption that the hazard rate does not change over time.

It seems sensible to also consider cases when the hazard rate may vary. To suggest a way of doing this, consider the conditional probability density of the default time:

$$P(\tau = t \mid \tau \geq t) = \frac{P(\tau = t, \tau \geq t)}{P(\tau \geq t)} = \frac{P(\tau = t)}{P(\tau \geq t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

This is, conditional on survival up to time t, the default probability density at that instant. It is not, despite the common notation adopted above, a probability in the discrete-time sense—the probability of default at any particular single time is of course zero in continuous time. Instead, it is the instantaneous rate of growth of the cumulative default probability, rescaled by the probability that the obligor has not defaulted at any prior time. For any small finite period, the value $\lambda\,\Delta t$ provides a first-order approximation of the conditional probability of default in that first tiny interval of time. In the infinitesimal limit $\lambda\,dt$, this approximation becomes "exact."

Note also that the expression above can be written in terms of the survival probability function $S_\tau(t)$:

$$S_\tau(t) = P(\tau > t) = e^{-\lambda t}$$

$$\frac{-S_\tau'(t)}{S_\tau(t)} = \lambda$$

This is the continuous-time analog of our original definition of the hazard rate. This expression also makes it most convenient to derive the survival probability function in the case where we allow the hazard rate to vary with time:

$$\frac{-S_\tau'(t)}{S_\tau(t)} = \lambda(t)$$

We think of the function lambda here as representing the term structure of instantaneous hazard rates. This was one of the first equations you solved in your study of differential equations. Solutions are of the form:

$$S_\tau(t) = ce^{f(t)}$$

…where we may choose any function f satisfying:

$$f'(t) = -\lambda(t)$$

Given the nature of the problem, it is most suitable to choose the definite integral of the instantaneous hazard rate from time zero to t. Since we know that the survival probability at time zero is 1, c must also be 1, and we obtain the result:

$$S_\tau(t) = e^{-\int_0^t \lambda(s)\,ds}$$

From this, we can easily determine both the cumulative distribution and the density functions for the default time τ in this more general case…

$$F_\tau(t) = P(\tau < t) = 1 - S_\tau(t) = 1 - e^{-\int_0^t \lambda(s)ds}$$

$$f_\tau(t) = F'(t) = \lambda(t)e^{-\int_0^t \lambda(s)ds}$$

…which, as can be easily seen, reduces in the case of a constant hazard rate to the exponential distribution we have already seen.

Moreover, the fact that we can so nicely express survival probabilities in the time-varying case allows us to determine an explicit expression for the z-spread as well. Recall that without any particular choice of distribution of the default time we may say…

$$V_{risky} = Ne^{-(r_T + z_T)T} = Ne^{-r_T T}e^{-z_T T}$$

…and in the case where we assume no recovery in the event of default we have found that…

$$V_{risky} = Ne^{-r_T T}P(\tau > T) = Ne^{-r_T T}S_\tau(T)$$

…meaning in the case of the time-varying hazard rate that:

$$e^{-\int_0^T \lambda(s)ds} = e^{-z_T T}$$

$$z_T = \frac{1}{T}\int_0^T \lambda(s)ds$$

That is, under this model the z-spread is the average value of the hazard rate over the time from zero to *T*. This is plainly analogous to the result we saw in interest rates relating instantaneous forward rates to zero rates. It thus appears that the term structure of z-spreads can be dealt with in much the same way that we handled the term structure of zero rates.

Bootstrapping a term structure of z-spreads

If we have access to reliable pricing information for several bonds issued by the same obligor, the above suggests that we should be able to use our simple credit model to determine the term structure of that obligor's z-spreads. Such an approach may be useful to assist in valuing an illiquid obligation by the same issuer, or determining what financing rate is reasonable for that obligor for any new credit obligations it may choose to take on.

We begin with a series of bonds and their known market prices:

maturity	coupon	frequency	price (dirty)
0.25	0.07	2	103.18
1	0.065	2	104.74
2	0.06	2	107.38
5	0.04	2	105.83
10	0.035	2	100.41

Additionally, we have a risk-free curve extracted from other market instruments:

time	df	r (continuous)
0	1	n/a
0.25	0.997503122	0.01
0.5	0.994017964	0.012
1	0.986097544	0.014
2	0.960789439	0.02
5	0.886920437	0.024
10	0.740818221	0.03

For this illustration we choose to interpolate in the discount factors using piecewise constant forward rates.

Observe that the first instrument has only one remaining cash flow, and thus is in effect a zero-coupon bond. Evidently it is a bond that was issued some time ago and is about to mature. Oddly, such bonds may be noticeably more liquid than bonds with, say, one year to maturity, since the short maturity allows certain kinds of investors to purchase them who would otherwise be prohibited from doing so.

The price quote we are given for the first bond is indicated as "dirty," meaning simply that it is the present value of the bond's future cash flows. This is the price at which you would have to transact if you wished to buy the bond. (By contrast, the "clean" price would reduce the bond's value by the coupon amount accrued since the last payment date.)

Since the instrument consists only of a single cash flow, we can directly calculate the z-spread appropriate for this bond:

$$V = N\left(1+\frac{c}{m}\right)P_T e^{-z_T T}$$

$$z_T = -\frac{1}{T}\ln\left[\frac{V}{NP_T}\left(1+\frac{c}{m}\right)^{-1}\right]$$

Our initial discussion of the relationship between z-spreads and instantaneous hazard rates (under the assumption of zero recovery) makes it seem reasonable to interpolate in z-spreads using the assumption that λ is constant over each interval. Therefore, we take the initial points on our z-spread curve to be:

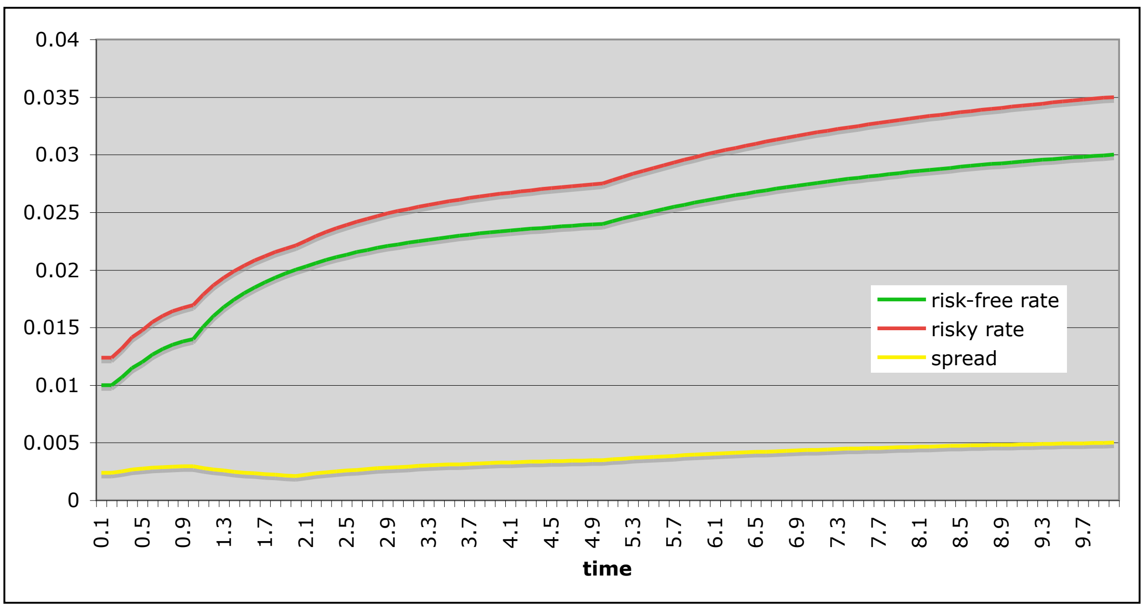
time	z
0	0.002386308
0.25	0.002386308

At the 1-year point we have a bond paying a fixed semiannual coupon. Since its cash flows occur at times 0.5 and 1, we need to know the z-spread appropriate for these two points. As with our previous example with bootstrapping discount factors, we take the z-spread at 1 year as our variable of interest and interpolate between 0.25 and this point to determine the z-spread applicable to the cash flow at 0.5. This is once again an equation requiring a nonlinear solver, and applying the method we have previously seen allows us to extend our term structure to the 1-year point:

time	z
0	0.002386308
0.25	0.002386308
1	0.002957417

Extending the same method to the remaining bonds leads to the following term structure:

time	z
0	0.002386308
0.25	0.002386308
1	0.002957417
2	0.002118431
5	0.003489154
10	0.005000733



With this information in hand, we can price other commitments from the same obligor in a manner that is consistent with their traded securities. Similarly, we can use this information to estimate arbitrage-free forward financing rates for the obligor, which is valuable for pricing instruments such as callable bonds whose cash flows depend upon the obligor's future choices about whether and when to refinance outstanding debt.

Moreover, the fact that these spreads are extracted directly from market prices means that other features of the instruments aside from default probability are captured in these spreads—recoveries in the event of default, for example, and liquidity—without requiring an explicit model for these factors. Since they are directly tied to price, z-spreads are useful for expressing and investigating the market risk of fixed-income instruments.

Recovery and Loss Given Default (LGD)
As we have repeatedly mentioned, however, the direct interpretation of z-spreads as expressions of the obligor's default probability relies on an unrealistic assumption. While z-spreads offer a good way of simply modeling the real financing spreads of the obligor in the market, translating these values into default probabilities under the assumption of a total loss in the event of default assuredly does not provide an accurate picture of default probability by itself.

In reality, the traded price of a bond accounts for the fact that, in the event of default, the bond will retain at least some value. This means that z-spreads extracted from bond prices, if naively converted to hazard rates, will assuredly understate the actual probability of default over the period of interest.

The value that a bond has at the point its issuer defaults is referred to as a recovery. Typically, this is expressed as a rate R between 0 and 1 and is thought of as applying to the principal alone, not to any interest payments that may have accrued at the time of default. Thus we may extend our initial expression of the conditional value of a risky zero-coupon bond as follows:

$$V_\tau = \begin{cases} NRP_\tau & \tau \in (0,T] \\ NP_T & \tau \in (T,+\infty) \end{cases}$$

The work we have previously done in the case of no recovery can, fortunately, be extended without too much difficulty to this case. We return to the original model assumption of a constant hazard rate λ between now and T; furthermore, to simplify this presentation, we will assume a constant continuously compounded zero rate r over this period as well.

Under these assumptions, the value of the cash flow becomes:

$$V = E[V_\tau] = \int_0^T RNe^{-rt}P(\tau = t)dt + Ne^{-rT}P(\tau > T) = N\lambda R\int_0^T e^{-(r+\lambda)t}dt + Ne^{-(r+\lambda)T}$$

$$V = Ne^{-(r+\lambda)T} + N\frac{\lambda R}{r+\lambda}\left(1 - e^{-(r+\lambda)T}\right)$$

Clearly, in the case where $R = 0$ this reduces to the valuation method we have been using all along. However, in other cases there is no convenient expression for the z-spread corresponding to this price in terms of the variables we have defined here. Assuredly one can be solved for numerically given a particular set of parameters, but it is evident that the fixed recovery amount leads to a model that is not completely reducible to the zero-recovery model we initially explored.

There is, however, an interesting relationship that comes to light when we imagine adding a coupon strip to this risky zero-coupon bond. To make the relationship clear, we abstract the concept of a fixed coupon somewhat and pretend that the bond pays its coupon rate *c* continuously, so that in an infinitesimal time increment, the amount of interest paid is *Nc dt*.

Our stylized instrument has the same pricing features as our zero-coupon bond above, with the addition of the continuous coupon strip:

$$V_{bond} = V_{rec} + V_{bullet} + V_{strip}$$

$$V_{bond} = \int_0^T RNe^{-rt} P(\tau = t)dt + Ne^{-rT} P(\tau > T) + \int_0^T Nce^{-rt} P(\tau > t)dt$$

We have already found an expression for the first two terms, so we consider the value of the strip alone…

$$V_{strip} = \int_0^T Nce^{-rt} P(\tau > t)dt = Nc\int_0^T e^{-(r+\lambda)t}dt = N\frac{c}{r+\lambda}\left(1 - e^{-(r+\lambda)T}\right)$$

…making the final expression of the bond's value:

$$V_{bond} = Ne^{-(r+\lambda)T} + N\frac{c + \lambda R}{r+\lambda}\left(1 - e^{-(r+\lambda)T}\right)$$

Using this, we can determine the fair coupon of a risky bond of this sort:

$$N = V_{bond}$$

$$N = Ne^{-(r+\lambda)T} + N\frac{c^* + \lambda R}{r+\lambda}\left(1 - e^{-(r+\lambda)T}\right)$$

$$1 - e^{-(r+\lambda)T} = \frac{c^* + \lambda R}{r+\lambda}\left(1 - e^{-(r+\lambda)T}\right)$$

$$c^* = r + \lambda - \lambda R = r + \lambda(1 - R)$$

$$c^* = r + \lambda(LGD)$$

We conclude that, quite sensibly, the coupon that causes our stylized bond to price to par is the risk-free rate of interest plus an amount that compensates for the expected loss, which has two components: The rate at which losses arrive on average, and the loss given default—that is, the proportion of the principal that is not recovered when default occurs.

Moreover, using an extension of the same argument from our initial discussion of par yield, we may make another observation about this bond: Since this is the coupon that causes it to price to par, this must then also be the yield of such a bond, expressed with continuous compounding. Of course, if we go back to our initial assumption of no recovery in the event of default, the risky par yield becomes simply $r + \lambda$ for this stylized bond under the assumption of constant interest rates.

Bootstrapping a default term structure

In the case of real bonds, the fact that the bond pays periodic rather than continuous coupons, combined with the fact that market interest rates have a term structure, means that our analytic expressions above cannot be directly used for pricing. However, the methods outlined above can easily be extended to this case.

Consider, for example, the set of bond prices for a risky obligor we saw above:

maturity	coupon	frequency	price (dirty)
0.25	0.07	2	103.18
1	0.065	2	104.74
2	0.06	2	107.38
5	0.04	2	105.83
10	0.035	2	100.41

Given a term structure of hazard rates and an assumed recovery rate, it is not difficult to price one of these bonds. After all, we have already seen that for a given cash flow under the assumption of zero recovery…

$$V_{c,risky} = c_t P_t e^{-\int_0^t \lambda(s)ds} = c_t P_t e^{-\bar{\lambda}_t t}$$

…where the hazard rate used is the average value of the instantaneous hazard rate between time zero and the time of the cash flow. Calculating the portion of the bond's value due to the risky cash flows is thus a matter of pricing (and interpolating) the same way we initially did using z-spreads.

All that remains, then, is to price the recovery. For a bond with maturity *T*, as we have seen the present value of the recovery is given by…

$$V_{rec} = \int_0^T NRP_t P(\tau = t)dt = \int_0^T NRP_t \lambda(t) e^{-\int_0^t \lambda(s)ds} dt$$

This integral is particularly amenable to approximation. If we subdivide the time to maturity *T* into *M* equal segments, then we have a discretization consisting of times $t_i = iT / M$, $i = 0, 1, 2, \ldots, M$. We assume that, if default occurs in the i^{th} period ($i > 0$), then the recovery occurs at time t_i. The present value of this recovery outcome is then…

$$NRP_{t_i}$$

…and the probability of default in this interval is…

$$\Delta P_i = P(\tau > t_{i-1}) - P(\tau > t_i) = e^{-\bar{\lambda}_{t_{i-1}} t_{i-1}} - e^{-\bar{\lambda}_{t_i} t_i}$$

…allowing us to approximate the value of the recovery as:

$$V_{rec} \approx \sum_{i=1}^{M} NRP_{t_i} \Delta P_i$$

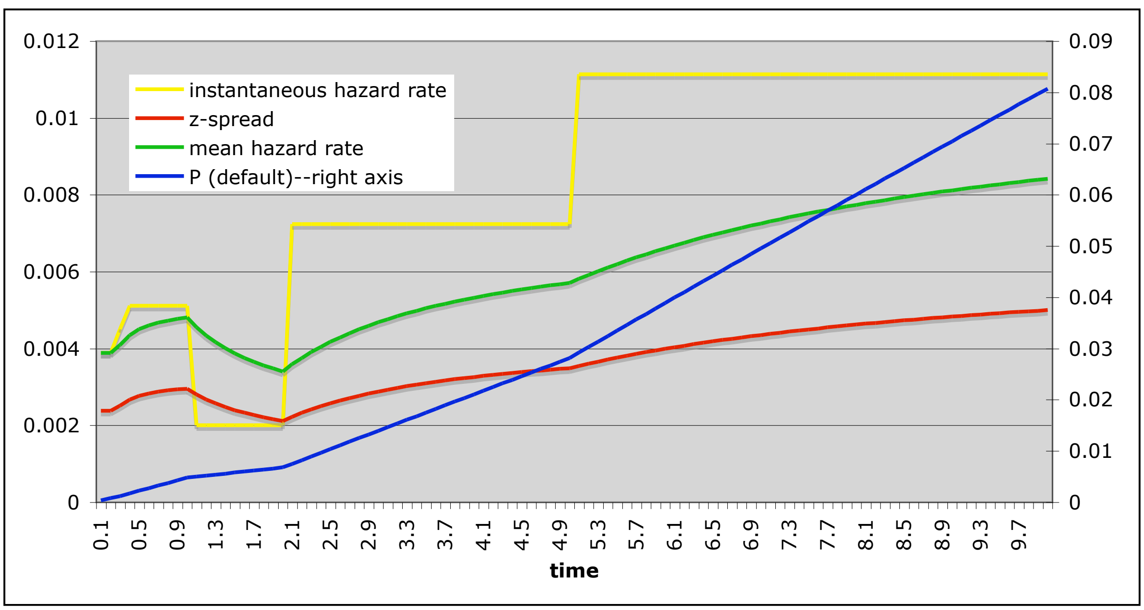
The total value of any bond under this model is thus the sum of the value of the risky cash flows appropriately weighted by survival probabilities, plus the value of the recovery.

If we assume a constant recovery rate for all of the bonds (0.4 is a common choice), then the problem of bootstrapping the term structure of hazard rates is in essence the same as our earlier problem of bootstrapping the term structure of z-spreads. In this case, there is no longer an analytic expression for the hazard rate that applies for the first bond, so even a bond with a single cash flow requires a nonlinear solver to find the appropriate hazard rate.

Here as before we assume a constant hazard rate between knot points on our curve. The resulting term structure of hazard rates illustrates our earlier claim that the hazard rate, taking recovery into account, is in fact higher than the corresponding z-spread that correctly prices the cash flows:

time	mean hazard rate	z-spread
0	0.003890839	0.002386308
0.25	0.003890839	0.002386308
1	0.004806312	0.002957417
2	0.003406838	0.002118431
5	0.005706109	0.003489154
10	0.008419146	0.005000733

Moreover, the term structure of hazard rates can of course be used to calculate default probabilities:



This, then, offers a way of transforming prices of a risky issuer's bonds, under some assumption of recovery rates, into a term structure of default probabilities.

Yield spread and approximation of market-implied hazard rates

While the above provides an appealing method for assimilating all the information concerning a risky obligor into a single default curve, it is computationally intensive.

Moreover, established habits in fixed-income trading tend to favor expressing bond price information in yield terms, and it is therefore natural in such a context to desire some method that expresses credit information in these terms as well.

Under the reduced-form model we have been discussing so far, there is no exact expression that translates bond yield into information concerning the likelihood of default. However, we have already seen that, for a stylized bond paying a continuous coupon at fixed rate c, the coupon c^* that causes a risky bond to price to par—and, therefore, also the par yield y^* of that bond—is given by…

$$c^* = y^* = r + \lambda(LGD)$$

…with all rates expressed with continuous compounding.

Furthermore, we have previously seen that for a real bond in a practical context, price information is most often rendered in yield terms, and this yield is most often expressed with compounding m when the bond pays m coupons per year. If we take $y^*_{rf,m}$ to be the risk-free par yield to the same maturity with the same coupon payment frequency, then we may define the yield spread…

$$s = y_m - y^*_{rf,m}$$

Since this is the excess return that the bond offers over a risk-free bond priced at par and paying with the same frequency, it is a quantity of clear interest to traders. If we make the approximation…

$$y_m \approx y^*$$

…then it at least seems reasonable to obtain the following approximate expression for the hazard rate to the maturity of the bond:

$$y_m = y^*_{rf,m} + \lambda_{approx}(LGD)$$

$$\lambda_{approx} = \frac{y_m - y^*_{rf,m}}{1-R} = \frac{s}{1-R}$$

Aside from the approximation of the risky issuer's par yield with the yield of an actual bond, this expression also contains some implicit abuse of compounding frequencies. Nevertheless, it is a convenient approximation of the market-implied hazard rate that can be calculated from readily available information.

It is worth considering just how large the error in this approximation may be, and as our previous investigations have led us to expect, there is no simple analytic expression to use in answering this question. For that reason, we use the sample bonds considered above in our bootstrapping examples to get a sense, if we can, of when this approximation is suitable.

Using the methods we discussed earlier, we can determine the yield of each bond, the risk-free par yield appropriate to each (both expressed with the natural compounding frequency), and yield spread as defined above.

maturity	coupon	frequency	price (dirty)	s/a yield	s/a risk-free	yield spread

					yield	
0.25	0.07	2	103.18	0.012424742	0.01001251	0.002412232
1	0.065	2	104.74	0.016994977	0.014042065	0.002952911
2	0.06	2	107.38	0.022076149	0.020034693	0.002041456
5	0.04	2	105.84	0.027421244	0.024014546	0.003406698
10	0.035	2	100.41	0.034511697	0.029686005	0.004825692

We can then compare the average hazard rate obtained by three calculation methods: via our bootstrapping procedure, via an approximation using our bootstrapped z-spread term structure, and via an approximation from the yield spreads above. All assume a recovery rate of 40%, with the z-spread approximation calculated by:

$$z_T \approx \lambda(1 - R)$$

$$\lambda \approx \frac{z_T}{1 - R}$$

For the sake of comparison we have slightly recalculated the yield spread as a difference in the continuously compounded risky / risk-free yields, and expressed all hazard rates to the nearest basis point:

bond #	T	price	bootstrapped	via z-spread	via yield spread
1	0.25	103.18	0.0039	0.0040	0.0040
2	1	104.74	0.0048	0.0049	0.0049
3	2	107.38	0.0034	0.0035	0.0034
4	5	105.84	0.0057	0.0058	0.0056
5	10	100.41	0.0084	0.0083	0.0079

The approximation is evidently good for relatively small spreads at relatively short maturities. Even at the ten-year maturity and the largest spread, the z-spread approximation remains good for a bond near par, while the yield spread approximation diverges a bit.

All three methods, however, are quite good enough for making relative or comparative statements of credit risk, and given that the recovery rate may be difficult to estimate in any case, each of these methods can be useful depending upon the use to which it is put.

Can credit spreads be negative?

Credit spread, as we have seen, is primarily a description of the difference between a traded bond's price and the price of an equivalent risk-free instrument. In terms of z-spread, if we simplify the question to the case of a single cash flow, then the issue is whether the expression…

$$V = Ne^{-(r_T + z_T)T}$$

…admits the choice of a negative rate z_T, and of course plainly it does. Indeed, we must choose a negative z-spread for any bond of this variety for which $V > NP_T$, or more generally for any bond whose market price is greater than the price of the same instrument as determined using the risk-free rates.

However, in this circumstance we are at a loss as to how to interpret such a result. Clearly default probabilities cannot be negative, and so in the zero-recovery case, just as clearly we cannot have survival probabilities greater than one. The inclusion of recoveries in our model cannot explain such a result, either, unless one admits the possibility of a recovery rate R that is greater than one; although such an outcome is not completely impossible with regard to some kinds of credit, certainly it is overwhelmingly rare.

The key element to recall in answering this question is our initial determination of the "risk-free" rate. This concept is a convenient abstraction, but as we have noted before it is not for the most part a financial reality. For example, it is common to calculate the risk-free rates for use in pricing corporate bonds based on either LIBOR or its local equivalent—which is a value derived by surveying the rates at which banks say they can borrow—or else on instruments such as term deposits, bankers' acceptances, and so on. In other words, the most common choices for curve construction all pertain to banks, and as such market perceptions of banks' creditworthiness have an influence on the level of the rate. Thus it is often the case—particularly in times of market stress—that the highest-quality short-term debt of non-financial companies may trade at a negative spread to the "risk-free" rate.

It is important to be aware, then, that the market-implied default probabilities arrived at using these techniques are, in the end, relative measures, and that the benchmark used in the process may itself encompass some credit risk.