MTH 9831, FALL 2015. LECTURE 9

ABSTRACT. Connections with PDEs.

1. Further examples on connections with PDEs: pricing of a zero-coupon bond when the interest rate is random.
2. Pricing of barrier options:
   a) Probabilistic approach;
   b) PDE approach.
3. Pricing of Asian options. New ideas: augumentation of the state space and reduction of dimension using the change of numéraire.

1 Stochastic Interest Rate

Assume that the interest rate under $\tilde{\mathbb{P}}$ satisfies

$$dR(t) = \beta(t, R(t))dt + \gamma(t, R(t))d\tilde{B}(t). \tag{1.1}$$

Examples, Vasičeck, Hull-White, CIR models. These are called one-factor short rate models. For examples of two-factor models and more general HJM framework, see Chapter 10 of Shreve II. The discount process

$$dD(t) = -R(t)D(t)dt, \quad D(0) = 1,$$

or equivalently,

$$D(t) = e^{-\int_0^t R(s)ds}.$$

The MMA (Money Market Account) price

$$M(t) = \frac{1}{D(t)} = e^{\int_0^t R(s)ds}$$

satisfies

$$dM(t) = R(t)M(t)dt = \frac{R(t)}{D(t)}dt, \quad M(0) = 1.$$

Denote $B(t,T)$ the time t price of a unit zero coupon bond maturing at T (pays \$1 at T), then

$$B(t,T) = \frac{1}{D(t)}\tilde{\mathbb{E}}\left[D(T)\cdot 1|\mathcal{F}(t)\right] = \tilde{\mathbb{E}}\left[e^{-\int_t^T R(s)ds}\middle|\mathcal{F}(t)\right].$$

Yield over the time interval $[t,T]$ is the *constant* rate over $[t,T]$ which is consistent with the price $B(t,T)$:

$$Y(t,T) \triangleq -\frac{1}{T-t}\log B(t,T),$$

or

$$B(t,T) = e^{-Y(t,T)(T-t)}.$$

Since $R(t)$ is a solution of an SDE, it is a Markove process. We would like to say that

$$B(t,T) = f(t, R(t)) \tag{1.2}$$

for some function $f(t,y)$. Rigorously speaking, $e^{-\int_t^T R(s)ds}$ is not of the form $h(R(T))$, so our definition of a Markove process does not allow to conclude Eq.1.2. But heurisitically, the only way $e^{-\int_t^T R(s)ds}$ depends on the path $R(s)$ for $0 \le s \le t$ is only through its value at $s = t$. To find the PDE for f, we use the fact that

$$D(t)B(t,T) = e^{-\int_0^t R(s)ds}f(t,R(t))$$

is a $\tilde{\mathbb{P}}$-martingale. Hence, for $f = f(t,R(t))$,

$$\begin{aligned}
d\left(e^{-\int_0^t R(s)ds}f(t,R(t))\right) &= e^{-\int_0^t R(s)ds}\left(-R(t)fdt + f_t dt + f_y(\beta dt + \gamma d\tilde{B}(t)) + \frac{1}{2}\gamma^2 f_{yy}dt\right) \\
&= e^{-\int_0^t R(s)ds}\left[\left(-Rf + f_t + \beta f_y + \frac{1}{2}\gamma^2 f_{yy}\right)dt + \gamma f_y d\tilde{B}(t)\right].
\end{aligned}$$

Thus, setting the drift term to zero by martingality,

$$f_t(t,y) + \beta(t,y) f_y(t,y) + \frac{1}{2}\gamma^2(t,y) f_{yy}(t,y) = y f(t,y), \tag{1.3}$$

or in short, $f_t(t,y) + \mathcal{A}_t f(t,y) = y f(t,y)$, $f(T,y) = 1$ for all y. See Example 6.5.1 for a derivation of an explicit formula for the Hull-White model and Example 6.5.2 for CIR model, in Shreve II.

Key idea: these models are *affine yield* models, which means that

$$f(t,y) = e^{-yC_1(t,T) - C_2(t,T)}$$

for some $C_1(t,T)$ and $C_2(t,T)$ to be determined. The name comes from the fact that the yield $Y(t,T)$ can be assumed to be of the form

$$Y(t,T) = y\frac{C_1(t,T)}{T-t} + \frac{C_2(t,T)}{T-t},$$

which is an affine function of y.

Example 6.5.3 shows that the price $c(t,y)$ of a call option on a bond with expiration $0 \le T_1 \le T$ requiresto solve the same PDE 1.3, i.e.,

$$c_t + \beta c_y + \frac{1}{2}\gamma^2 f_{yy} = yc,$$

but with a different terminal condition

$$c(T_1,y) = (f(T_1,y) - K)^+.$$

2 Barrier Options

Knock-out and knock-in barrier options:

- up-and-out;
- down-and-out;
- up-and-in;
- down-and-in.

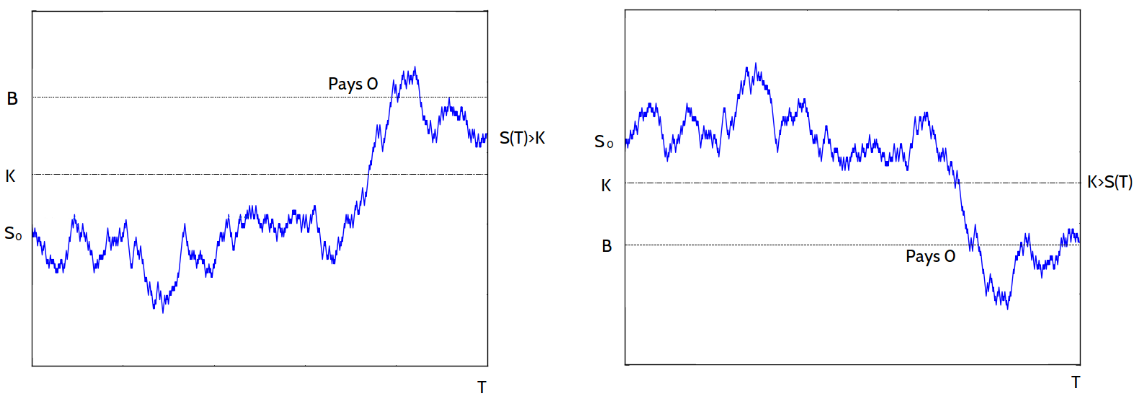

Figure 2.1: Left: Knocked-out call: up-and-out; Right: Knocked out put: down-and-out.

Tools:

- Joint distribution of BM with a drift and its running maximum (see Lecture 4);
- Stopped martingale;
- Risk-neutral pricing formula;

- Knowing how to determine whether a given function of a diffusion process is a martingale.

Framework: Black-Scholes-Merton. Assume that under the risk-neutral measure the stock price satisfies

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{B}(t). \tag{2.1}$$

Option: Up-and-out call with strike K, barrier $B > K$ and expiration T. We assume that $S(0) \leq B$. The goal is to

- Use probabilistic approch to set up an integral which gives the price;
- Write down the PDE and boundary conditions satisfied by the call price.

$$S(t) = S(0)e^{\sigma \tilde{B}(t) + \left(r - \frac{1}{2}\sigma^2\right)t} = S(0)e^{\sigma \hat{B}(t)},$$

where $\hat{B}(t) = \tilde{B}(t) + \alpha t$, and $\alpha = \frac{r - \sigma^2/2}{\sigma}$. Define

$$\hat{B}^*(t) \triangleq \max_{0 \leq s \leq t} \hat{B}(s).$$

Then, since e^x is increasing,

$$\max_{0 \leq t \leq T} S(t) = S(0)e^{\sigma \hat{B}^*(T)}.$$

The payoff of the option is

$$\begin{aligned} V(T) &= \left(S(0)e^{\sigma \hat{B}(T)} - K\right)^+ \mathbb{I}_{\{S(0)e^{\sigma \hat{B}^*(T)} < B\}} \\ &= \left(S(0)e^{\sigma \hat{B}(T)} - K\right) \mathbb{I}_{\{S(0)e^{\sigma \hat{B}(T)} \geq K,\, S(0)e^{\sigma \hat{B}^*(T)} < B\}} \\ &= \left(S(0)e^{\sigma \hat{B}(T)} - K\right) \mathbb{I}_{\{\hat{B}(T) \geq \frac{1}{\sigma} \log \frac{K}{S(0)},\, \hat{B}^*(T) < \frac{1}{\sigma} \log \frac{B}{S(0)}\}}, \end{aligned}$$

where $k \triangleq \frac{1}{\sigma} \log \frac{K}{S(0)} \in \mathbb{R}$ and $b \triangleq \frac{1}{\sigma} \log \frac{B}{S(0)} > 0$. Risk-neutral pricing formula tells us that the value of this option at time $t \in [0, T]$ is equal to

$$V(t) = \mathbb{E}\left[e^{-r(T-t)} V(T) \middle| \mathcal{F}(t)\right], \quad 0 \leq t \leq T.$$

By the tower property, $e^{-rt}V(t)$ is a $\tilde{\mathbb{P}}$-martingale. To compute $V(0)$, we write

$$V(0) = e^{-rT} \tilde{\mathbb{E}}\left[\left(S(0)e^{\sigma \hat{B}(T)} - K\right) \mathbb{I}_{\{\hat{B}(T) \geq k,\, \hat{B}^*(T) < b\}}\right].$$

The joint distribution of $(\hat{B}(T), \hat{B}^*(T))$ was computed in Lecture 6. We have (under $\tilde{\mathbb{P}}$, as it is our reference measure here)

$$\hat{f}(x, a) = \begin{cases} e^{\alpha x - \frac{1}{2}\alpha^2 T} \frac{2(2a - x)}{T\sqrt{2\pi T}} e^{-\frac{1}{2T}(2a - x)^2}, & x \leq a,\ a \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

where the variable x corresponds to $\hat{B}(T)$, and a corresponds to $\hat{B}^*(T)$. Therefore,

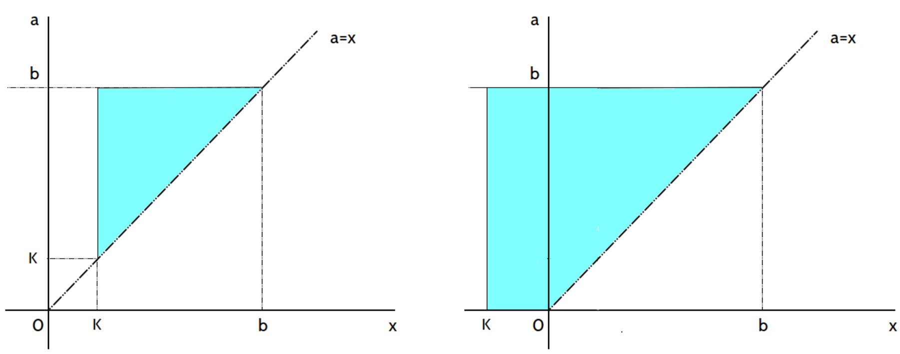

Figure 2.2: Left: $k > 0$; Right: $k < 0$. In this case, a changes from 0 $(=x^+)$ to b when $x < 0$.

$$V(0) = e^{-rT} \int_k^b \int_{x^+}^b (S(0)e^{\sigma x} - K) \frac{2(2a-x)}{T\sqrt{2\pi T}} e^{\alpha x - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2a-x)^2} da dx.$$

This integral can be computed in terms of $N(x)$. This gives a closed form formula. For details, see Shreve II (Page 304-308, §7.3.3).

Let us move to the PDE description.

Theorem 2.1. *Let $v(t,x)$ denote the price at time t of the up-and-out call under the assumption that the call has not knocked out prior to t and $S(t) = x$. Then $v(t,x)$ satisfies the BSM PDE*

$$v_t + rxv_x(t,x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = rv(t,x) \tag{2.2}$$

in the rectangle $\{(t,x) : 0 \le t \le T, 0 \le x \le B\}$ and satisfies boundary conditions

$$v(t,0) = 0, \quad 0 \le t \le T \tag{2.3}$$

$$v(t,B) = 0, \quad 0 \le t < T \tag{2.4}$$

$$v(T,x) = (x-K)^+, \quad 0 \le x \le B. \tag{2.5}$$

Recall that $v(t) = \tilde{\mathbb{E}}[e^{-r(T-t)}V(T)|\mathcal{F}_t]$. It is clear that $v(t)$ cannot be represented $v(t,S(t))$ since $v(t)$ "remembers" whether it has been knocked out or not, and $v(t,S(t))$ "does not remember" anything that happended prior to t.

Define $\rho = \inf\{t \ge 0 : S(t) = B\}$. Then ρ is the knock-out time (since upon reaching B, $S(t)$ will exceed B within any positive additional time increment with probability 1). Since ρ is the first passage time, it is a stopping time, namely, $\{\rho \le t\} \in \mathcal{F}_t$ for each $t \ge 0$. We have that $e^{-rt}V(t)_{0\le t\le T}$ is a $\tilde{\mathbb{P}}$-martingale. We know that a martingale, stopped at a stopping time is again a martingale. Thus, $e^{-r(t\wedge\rho)}V(t\wedge\rho)_{0\le t\le T}$ is a $\tilde{\mathbb{P}}$-martingale. Note: $t \wedge \rho \triangleq \min(t,\rho)$.

Lemma 2.2. *For $0 \le t \le \rho$, $v(t)$ is representable as $v(t,S(t))$. In particular,*

$$e^{-r(t\wedge\rho)}v(t\wedge\rho, S(t\wedge\rho))_{0\le t\le T}$$

is a $\tilde{\mathbb{P}}$-martingale.

On the event $\{\rho > t\}$ the option is alive, so the value of the up-and-out call is the same as the regular call option. Recall that for the regular call option we had

$$V(t) = \tilde{\mathbb{E}}[e^{-r(T-t)} \underbrace{h(S(T))}_{(S(T)-K)^+} |\mathcal{F}_t] \underbrace{=}_{\text{Markov property}} v(t,S(t))$$

for some function $v(t,x)$. Since $e^{-rt}V(t)$ is a martingale, $e^{-rt}v(t,S(t))$ is a martingale. The point here is that the same is true up to the random time ρ, or for all $t \le \rho$. After that we stop our martingale, and the stopped martingale is still a martingale.

Sketch of proof of Theorem 2.1. By Lemma 2.2, $V(t) = v(t,S(t))$ for some v and $0 \le t \le \rho$. Moreover, $e^{-rt}v(t,S(t))$ is a martingale up to time ρ. Therefore, for $0 \le t \le \rho$,

$$\begin{aligned} d\left[e^{-rt}v(t,S(t))\right] &= e^{-rt}\left[-rv(t,S(t)) + v_t(t,S(t)) + v_x(t,S(t))rS(t) + \frac{1}{2}v_{xx}(t,S(t))\sigma^2 S(t)^2\right]dt \\ &\quad + e^{-rt}v_x(t,S(t))\sigma S(t) d\tilde{B}(t). \end{aligned}$$

Setting the dt term to zero we get ($0 \le t \le \rho$):

$$-rv(t,S(t)) + v_t(t,S(t)) + v_x(t,S(t))rS(t) + \frac{1}{2}v_{xx}(t,S(t))\sigma^2 S(t)^2 = 0$$

The process $S(t)_{0\le t\le\rho}$ can reach a neighborhood of any point in $[0,T) \times [0,B]$ with positive probability. This means that the BSM PDE should hold for all $(t,x) \in [0,T) \times [0,B]$:

$$-rv(t,x) + v_t(t,x) + rxv_x(t,x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) = 0.$$

The boundary conditions simply satisfy the conditions set forth by the option. See Shreve II (Page 303-304) for the pitfalls of Δ-hedging strategy for this type of options.