MTH 9831 Lecture 11

Note Title

1. Two words about a finite expiration American put.
2. Poisson process (see Lectur 6 of Refresher)
3. Compound Poisson process.
4. Jump processes as integrators.
5. Quadratic and cross variation of jump processes.
6. Ito-Doeblin formula for jump processes $(d=1)$

Return for a moment to the perpetual American put option. Its price, $v(x)$ when $S(0)=x$, does not depend on time and the optimal exercise time is τ_{L_*}, where L_* is an explicitly given level, that depends only on K, σ, and r.

$$\mathcal{C} = \{(t,x): t\geq 0, x\geq 0, v(x) > (K-x)_+\}$$

$$rv(x) - rxv'(x) - \tfrac{1}{2}\sigma^2x^2v''(x) = 0 \qquad (1)$$

$x = L_*$

$$\mathcal{S} = \{(t,x): t\geq 0, x\geq 0, v(x) = (K-x)_+\}$$

$$rv(x) - rxv'(x) - \tfrac{1}{2}\sigma^2x^2v''(x) = rK \qquad (2)$$

$\mathcal{S}$ – stopping set; $\mathcal{C}$ – continuation set

The inclusion of the time variable in the description of $\mathcal{S}$ & $\mathcal{C}$ is superficial, as everything is determined by the value of x. But for the finite expiration american

put option the price, $v(t,x)$, and both, S & $\mathcal{C}$, will depend on t.
The equations (1) & (2) will be replaced by parabolic equations, and L_* will depend on the time to expiration.

1. Consider an American put with strike K and expiration T. Assume BSM model.

<u>Def</u>. 1. Let $t \in [0,T]$ and $x \geq 0$ be given and $S(t) = x$. For each $u \in [t,T]$ denote by $\mathcal{F}_u^{(t)}$ the

σ-algebra generated by the price process

$(S(v))_{t \leq v \leq u}$ and let $\mathcal{T}_{t,T}$ be the set of

all stopping times for the filtration $\{\mathcal{F}_u^{(t)}\}_{t \leq u \leq T}$

taking values in $[t,T] \cup \{\infty\}$.

The price at time t of the American put option with strike K and expiration T is defined to be

$$(*) \quad v(t,x) = \max_{\tau \in \mathcal{T}_{t,T}} \widetilde{\mathbb{E}}^{t,x}\left[e^{-r(\tau - t)} (K - S(\tau)) \mathbb{1}_{\{\tau < \infty\}} \right]$$

a) Analytical characterization of $v(t,x)$; no proofs will be given.

It is not difficult to believe that (by the analogy with the perpetual american put) the following statement holds.

Theorem 1. Let $v(t,x)$ be the time t price of the American put with strike K and expiration T. Then $v(t,x)$ satisfies the linear complementarity conditions

- $v(t,x) \geq (K-x)_+ , \quad t \in [0,T], \; x \geq 0; \qquad (3)$

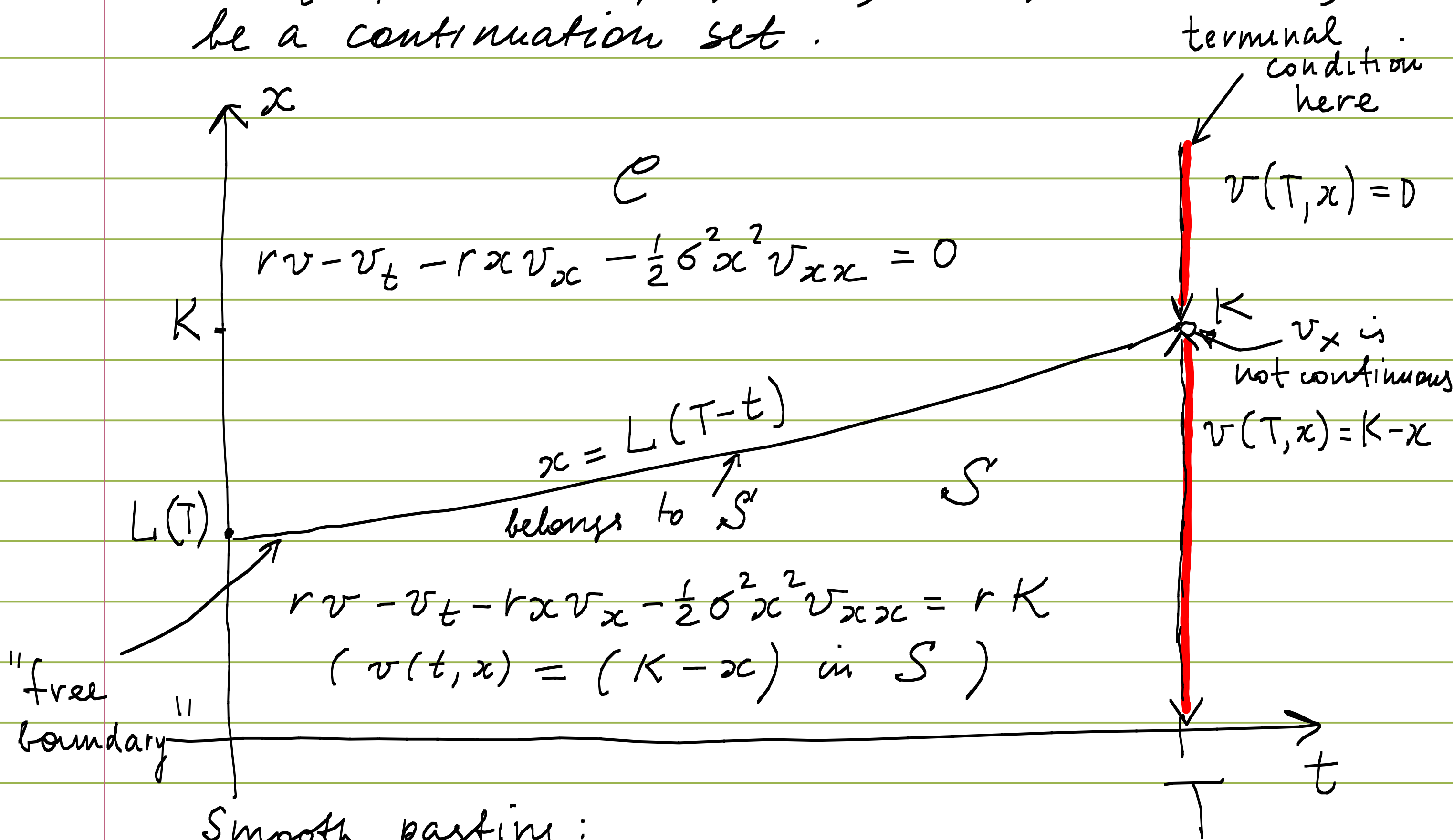
- $r v(t,x) - v_t(t,x) - r x v_x(t,x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) \geq 0,$
  $t \in [0,T), \; x \geq 0 \; ; \qquad (4)$

- for each $t \in [0,T)$ and $x \geq 0$, equality holds in either (3) or (4). (5)

The idea is that the holder of the put has to wait until the stock price falls to a certain level below K. This level, L, will now depend on the time left to expiration, i.e. $L = L(T-t)$. Clearly, $\lim_{T\to\infty} L(T) = L_*$ and $L(0) = K$.

It is also plausible that $L(T-t)$ should be non-decreasing in t: the less time is left the higher is the level at which the owner should be willing to exercise. No formula for $L(T-t)$ is known.

Let $S = \{ (t,x) : t \in [0,T], x \geq 0, v(t,x) = (K-x)_+ \}$ be a stopping set and
$C = \{ (t,x) : t \in [0,T], x \geq 0, v(t,x) > (K-x)_+ \}$ be a continuation set.



Smooth pasting:

(6) $\quad v(t, L(T-t)+) = v(t, L(T-t)-)$

(v is continuous across $x = L(T-t)$)

(7) $\quad v_x(t, L(T-t)+) = v_x(t, L(T-t)-) = -1$

(v_x is continuous across $x = L(T-t)$ for $0 \leq t < T$)

(v_t and v_{xx} are not continuous across the free boundary.)

<u>Theorem 2</u>. There is a unique bounded function $v(t,x)$ on $t \in [0,T]$, $x \geq 0$, and a curve $x = L(T-t)$, $t \in [0,T]$, that satisfy:

$$rv - v_t - rxv_x - \frac{1}{2}\sigma^2 x^2 v_{xx} = 0, \quad x \geq L(T-t)$$

$$v(t,x) = (K-x), \quad x \in [0, L(T-t)],$$

(6), (7), the terminal condition $L(0) = K$ and $v(T,x) = (K-x)_+$, and the asymptotic condition

$$\lim_{x\to\infty} v(t,x) = 0.$$

(b) Probabilistic characterization of $v(t,x)$.

Theorem 3. Let $(S(u))_{t\le u\le T}$ be the stock price process starting at $S(t) = x$ with the stopping set

$$\mathcal{S} := \{(t,x) : t\in[0,T], x\geq 0, v(t,x) = (K-x)_+\}.$$

Let

$$\tau_* = \min\{u\in[t,T] : (u, S(u)) \in \mathcal{S}\},$$

where $\tau_* := \infty$ if $(u, S(u))$ does not enter $\mathcal{S}$ for any $u\in[t,T]$. Then

$$\left(e^{-ru} v(u, S(u))\right)_{t\le u\le T} \quad \text{is a supermartingale}$$

under $\widetilde{\mathbb{P}}$ relative to $(\mathcal{F}_u^{(t)})_{t\le u\le T}$ and the stopped process $\left(e^{-r(u\wedge\tau_*)} v(u\wedge\tau_*, S(u\wedge\tau_*))\right)_{t\le u\le T}$

is a martingale.

2. Poisson process (review refresher Lecture 6 or textbook Section 11.2).

3 Compound Poisson process.

Def. 2. Let $\{N(t), t\geq 0\}$ be a Poisson process and $Y_1, Y_2, \ldots$ be i.i.d. random variables independent of $\{N(t), t\geq 0\}$. The compound Poisson process

$$Q(t)=0 \quad \text{if } N(t)=0, \quad Q(t)=\sum_{i=1}^{N(t)} Y_i \quad \text{if } N(t)\geq 1.$$

This process also has stationary and independent increments, but the distribution of $Q(t+s)-Q(t)$ is not Poisson, it depends on the distribution of Y_1.

The following theorem generalizes Lemma 1.5 (refresher Lecture 6) to compound Poisson processes.

Theorem 4. Let $Q(t)$ be a [com]pound Poisson process. Set $\beta := EY_1$; $\sigma^2 := \operatorname{Var} Y_1$, $M_Y(u) = E(e^{uY_1})$. Then

(i) $E(Q(t)) = \beta\lambda t$;

(ii) $\operatorname{Var}(Q(t)) = (\sigma^2+\beta^2)\lambda t = (EY_1)^2\lambda t$.

(iii) $M_{Q(t)}(u) = \exp(\lambda t(M_Y(u)-1))$

(iv) The compensated compound Poisson process $Q(t)-\beta\lambda t$ is a martingale (w.r.t. its natural filtration).

Proof is left as an exercise.

Theorems 1.9 and 1.10 from refresher Lecture 6 have the following generalizations.

Theorem 5. Let $y_1, \ldots, y_M \in \mathbb{R}\setminus\{0\}$ and $p_1, \ldots, p_M \geq 0$, $\sum_{i=1}^{M} p_i = 1$. Let $\lambda > 0$ be given and

$N_1(t), \dots, N_M(t)$ be independent Poisson processes with intensities $\lambda p_1, \dots, \lambda p_M$ respectively.
Then $Q(t) := \sum_{m=1}^{M} y_m N_m(t)$, $t \geq 0$, is a compound

Poisson process whose jump size distribution is given by $P(Y_1 = y_i) = p_i$ and the times of jumps coinside with those of $N(t) := \sum_{m=1}^{M} N_m(t)$.

In other words, the process Q has the same law as $\overline{Q}(t) := \sum_{i=1}^{N(t)} \overline{Y}_i$, $t \geq 0$, where

$\overline{Y}_1, \overline{Y}_2, \dots$ are i.i.d. and $P(\overline{Y}_1 = y_m) = p_m$, $m = 1, 2, \dots, M$.

For proof see Shreve II, Th. 11.3.3.

<u>Theorem</u> 6. Let $y_1, \dots, y_M \in \mathbb{R} \setminus \{0\}$ and $p_1, \dots, p_M \geq 0$, $\sum_{i=1}^{M} p_i = 1$. Let $Y_1, Y_2, \dots$, be i.i.d. $P(Y_1 = y_m) = p_m$, $m = 1, 2, \dots, M$. Let $N(t)$ be a Poisson process with intensity λ and $Q(t) := \sum_{i=1}^{N(t)} Y_i$, $t \geq 0$.

For $m = 1, 2, \dots, M$ let $N_m(t)$ denote the number of jumps of Q of size y_m up to time t inclusively.
Then
$$N(t) = \sum_{m=1}^{M} N_m \; ; \; Q(t) = \sum_{m=1}^{M} y_m N_m(t)$$

Moreover, the processes $N_1, \dots, N_M$ are independent Poisson processes with intensities $\lambda p_1, \dots, \lambda p_M$ respectively.

See Shreve II, Corollary 11.3.4.

4 Jump processes as integrators

<u>Def.3.</u> Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}(t)$, $t \geq 0$, be a filtration on it. We say that

(i) B is a BM relative to $\{\mathcal{F}(t)\}_{t\geq 0}$ if $B(t)$ is $\mathcal{F}(t)$-measurable $\forall t \geq 0$ and for $0 \leq s < t$ $B(t) - B(s)$ is independent of $\mathcal{F}(s)$.

(i) N is a Poisson process relative to $\{\mathcal{F}(t)\}_{t\geq 0}$ if $N(t)$ is $\mathcal{F}(t)$-measurable $\forall t \geq 0$ and for $0 \leq s < t$ $N(t) - N(s)$ is independent of $\mathcal{F}(s)$.

(ii) Q is a compound Poisson process relative to $\{\mathcal{F}(t)\}_{t\geq 0}$ if $Q(t)$ is $\mathcal{F}(t)$-measurable $\forall t \geq 0$ and for $0 \leq s < t$ $Q(t) - Q(s)$ is independent of $\mathcal{F}(s)$.

<u>Def.4</u> Jump process with starting point $X(0)$ is

$$X(t) = X(0) + I(t) + R(t) + J(t), \text{ where}$$

$X(0)$ is a constant; I is an Ito integral $I(t) = \int_0^t \Gamma(s)\, dB(s)$, $\Gamma(\cdot)$ is adapted to $\mathcal{F}(\cdot)$.

R is a Riemann integral $R(t) = \int_0^t \Theta(s)\, ds$, $\Theta(\cdot)$ is adapted to $\mathcal{F}(\cdot)$, and

J is a right continuous pure jump process with $J(0) = 0$. In particular, $J(t) = \lim_{s \downarrow t} J(s)$, $J(t-) = \lim_{s \uparrow t} J(s)$, $\Delta J(t) = J(t) - J(t-)$ - jump size at time t.

We assume that there is • no jump at time 0,
that • in any finite time interval $[0,T]$ there are only finitely many jumps, and
that • the process is constant between jumps.

$X^c(t) := X(0) + I(t) + R(t)$ is called a continuous part of $X(t)$. It is just an Ito process.

$J(t)$ is a pure jump part of $X(t)$

$\Delta X(t) = X(t) - X(t-) = J(t) - J(t-) = \Delta J(t)$.

($X(0-) := X(0)$; $\Delta X(t) = 0$ when there is no jump at time t.)

<u>Def</u>. 5 Let X be a jump process and Φ be process adapted to $(\mathcal{F}(t))_{t\geq 0}$. Then

$$\int_0^t \Phi(s)\, dX(s) := \int_0^t \Phi(s)\Gamma(s)\, dB(s) + \int_0^t \Phi(s)\Theta(s)\, ds + \sum_{0<s\leq t} \Phi(s)\Delta J(s),$$

or in differential notation

$$\Phi(t)\, dX(t) = \underbrace{\Phi(t)\Gamma(t)\, dB(t) + \Phi(t)\Theta(t)\, dt}_{\Phi(t)\, dX^c(t)} + \Phi(t)\, dJ(t)$$

<u>Example 1</u> Let $\Phi(t) = \Delta N(t)$; $X(t) = N(t) - \lambda t$

$$\int_0^t \Phi(s)\, dX(s) = -\lambda \int_0^t \Delta N(s)\, ds + \int_0^t \Delta N(s)\, dN(s)$$

$$= 0 + \sum_{0<s\leq t} (\Delta N(s))^2 = N(t).$$

The integrator $X(t)$ is a martingale but the integral is not a martingale!

Theorem 7. Assume that $X(t)$ is a martingale and $\Phi(t)$ is left-continuous and adapted, and

$$E\int_0^t \Gamma^2(s)\Phi^2(s)\,ds < \infty \quad \forall t \geq 0.$$

Then $\int_0^t \Phi(s)\,dX(s)$, $t \geq 0$, is a martingale w.r.t $\{\mathcal{F}(t)\}_{t\geq 0}$.

Remark: left-continuity is not really needed, but it is easy to check. Left-continuity can be replaced by predictabiley: $\Phi(t)$ is $\mathcal{F}(t-)$-measurable, where $\mathcal{F}(t-)$ is the σ-algebra generated by $\bigcup_{s<t} \mathcal{F}(s)$ (the union of σ-algebras is not a σ-algebra, in general).

5 Quadratic and cross variation of jump processes. Let X and Y be jump processes and Π be a finite partition of $[0, t]$: $0 = t_0 < t_1 < \dots < t_n = t$. Define

$$Q_\Pi(X) := \sum_{i=0}^{n-1} (X(t_{j+1}) - X(t_j))^2$$

$$C_\Pi(X,Y) := \sum_{i=0}^{n-1} (X(t_{j+1}) - X(t_j))(Y(t_{j+1}) - Y(t_j))$$

If these quantities have an L^2-limit for all $t \geq 0$ as $\|\Pi\| \to 0$ then the limiting processes are denoted $[X,X](t)$ and $[X,Y](t)$ respectively and called the quadratic variation of X and cross variation of X and Y on $[0,t]$ (respectively).

We state without a proof the following fact.

Theorem 8. Let $X_i(t)$, $t \geq 0$, $i = 1, 2$, be jump processes as defined above. Then

$$[X_i, X_j](t) = [X_i^c, X_j^c](t) + [J_i, J_j](t)$$

$$= \int_0^t \Gamma_i(s)\Gamma_j(s)\,ds + \sum_{0<s\le t} \Delta J_i(s)\,\Delta J_j(s)$$

$(i,j=1,2)$

The important new developments:

- the quadratic variation of a pure jump process on $(0,t]$ is the sum of the squares of jumps up to time t inclusively)
- the cross-variation of a continuous process X_i^c and a pure jump process J_j (subject to all conditions we imposed on it) is zero $(i,j=1,2)$.

<u>Corollary 1.</u> Let X be a jump process and Φ be an adapted process such that all integrals below are well-defined.

Set $Y(t) = \int_0^t \Phi(s)\,dX(s) + Y(0)$, where $Y(0)$ is a constant

Then $[Y,Y](t) = \int_0^t \Phi^2(s)\,d[X,X](s)$

$$= \int_0^t \Phi^2(s)\,\Gamma^2(s)\,ds + \sum_{0<s\le t} \Phi^2(s)(\Delta J(s))^2$$

6 Ito-Doeblin formula for jump processes.

<u>Theorem 9.</u> Let $X(t)$ be a jump process and $f\in C^2(\mathbb{R})$. Then

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))\,dX^c(s) + \frac{1}{2}\int_0^t f''(X(s))\,d[X^c,X^c](s)$$

$$+ \sum_{0<s\le t}\big(f(X(s)) - f(X(s-))\big)$$

Read the proof of Theorem 11.5.1 in Shreve II

Remark. $\int_0^t f'(X(s))dX^c(s)$ can be replaced with $\int_0^t f(X(s-))\,dX^c(s)$;
$\int_0^t f''(X(s))d[X^c,X^c](s)$ with $\int_0^t f''(X(s-))\,d[X^c,X^c](s)$.

Example 2. (Geometric Poisson process)

$$S(t) = S(0)\,e^{-\lambda\sigma t}(1+\sigma)^{N(t)}, \quad t \geq 0$$

where $\sigma > -1$ is a constant.

If $\sigma > 0$ then the process jumps up and moves down between the jumps; if $\sigma \in (-1,0)$ the the process jumps down and moves up between the jumps; if $\sigma = 0$ then $S(t) \equiv S(0)$.

Geometric Poisson process is a martingale.
It satisfies $dS(t) = \sigma S(t-)\,dM(t)$, where $M(t) = N(t) - \lambda t$.

$S(t) = S(0) f(X(t))$, where $f(x) = e^x$, $f'(x) = f(x)$

$X(t) = -\lambda\sigma t + N(t)\ln(1+\sigma)$ (no Ito part!)

$dX(t) = -\lambda\sigma\,dt + \ln(1+\sigma)\,dN(t)$.

$$S(t) = S(0) - \lambda\sigma\int_0^t S(u)\,du + \sum_{0<u\leq t}\big(S(u) - S(u-)\big)$$

$$S(u) - S(u-) = S(0)\,e^{-\lambda\sigma u}\Big((1+\sigma)^{N(u)} - (1+\sigma)^{N(u-)}\Big)$$
$$= S(0)\,e^{-\lambda\sigma u}(1+\sigma)^{N(u-)}\Big((1+\sigma)^{N(u)-N(u-)} - 1\Big)$$
$$= S(u-)\,\sigma\,(N(u) - N(u-)) = S(u-)\,\sigma\,\Delta N(u)$$

$$\sum_{0<u\leq t}\big(S(u) - S(u-)\big) = \sigma\sum_{0<u\leq t} S(u-)\,\Delta N(u) = \sigma\int_0^t S(u-)\,dN(u).$$

$$S(t) = S(0) - \lambda\sigma \int_0^t S(u-)\,du + \sigma \int_0^t S(u-)\,dN(u)$$

$$= S(0) + \sigma \int_0^t S(u-)\,dM(u) \quad , \text{ or}$$

$$dS(t) = \sigma S(t-)\,dM(t) \,, \quad M(t) = N(t) - \lambda t .$$

<u>Example 3</u>. (Doléans-Dade exponential)
Let $X(t)$ be a jump process. The Doléans-Dade exponential of $X(t)$ is

$$Z(t) := e^{X^c(t) - \frac{1}{2}[X^c, X^c](t)} \prod_{0<s\le t} (1 + \Delta X(s)) \quad (8)$$

This is a generalization of a previous example:

$$X(t) = \sigma M(t) = -\sigma\lambda t + \sigma N(t) = X^c(t) + J(t)$$

$$(1 + \Delta X(s)) = (1 + \Delta J(s)) = (1 + \sigma \Delta N(s))$$

$$\prod_{0<s\le t} (1 + \Delta X(s)) = \prod_{0<s\le t} (1 + \sigma \Delta N(s)) = (1+\sigma)^{N(t)}$$

$$\frac{S(t)}{S(0)} = e^{-\lambda\sigma t} (1+\sigma)^{N(t)} .$$

$Z(t)$ solves:

$$dZ(t) = Z(t-)\,dX(t) . \quad (9)$$

$$Z(t) = Z(0) + \underbrace{\int_0^t Z(s-)\,dX^c(s)}_{Z^c(t)} + \underbrace{\sum_{0<s\le t} Z(s-)\,\Delta X(s)}_{J_Z(t)}$$

We shall show that if Z satisfies (9) then it is given by (8).

$$\log Z(t) = \log Z(0) + \int_0^t \frac{1}{Z(s-)}\,dZ^c(s) - \frac{1}{2} \int_0^t \frac{1}{Z^2(s-)}\,d[Z^c, Z^c](s)$$

$$+\sum_{0<s\le t}\left(\log Z(s)-\log Z(s-)\right)=\log Z(0)+\int_0^t dX^c(s)$$

$$-\frac{1}{2}\int_0^t d[X^c,X^c](s)+\sum_{0<s\le t}\log\left(1+\Delta X(s)\right)$$

since $\log Z(s)-\log Z(s-)=\log\dfrac{Z(s)}{Z(s-)}=\log\left(1+\Delta X(s)\right)$

Thus $\log Z(t)=\log Z(0)+X^c(t)-\frac{1}{2}[X^c,X^c](t)$

$$+\sum_{0<s\le t}\log\left(1+\Delta X(s)\right),\text{ i.e.}$$

$$Z(t)=Z(0)\,e^{X^c(t)-\frac{1}{2}[X^c,X^c](t)}\prod_{0<s\le t}\left(1+\Delta X(s)\right).$$

If $X(t)$ is a martingale then by theorem 7

$$Z(t)=Z(0)+\int_0^t Z(s-)\,dX(t)$$

is also a martingale since the integrand is left-continuous.

<u>Example 4.</u> (Merton's jump diffusion model)

Let $X(t)=\mu t+\sigma B(t)+Q(t)$, where $Q(t)$ is a compound Poisson process $Q(t)=\sum_{i=1}^{N(t)}Y_i$. The stock price dynamics is

$$S(t)=S(0)+\int_0^t S(u)\,dX^c(u)+\sum_{0<u\le t}S(u-)\Delta X(u).$$