BARUCH, MFE

MTH 9878 Assignment Four Revised *

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*We agree for our work to be posted on the forum.

1 Convexity Correction for LIBOR in Arrears

Prove that

$$\Delta(S,T) = \left(e^{\sigma^2 S} - 1\right) \frac{\delta L_0^2(S,T)}{1+\delta F_0(S,T)}.$$

Proof: The convexity correction term for LIBOR in arreas is written as

$$\Delta(S,T) = \mathbb{E}^{\mathbb{Q}_T}\left[L(S,T)\left(\frac{P_0(S,T)}{P(S,T)} - 1\right)\right].$$

Notice that

$$\begin{aligned} P(S,T) &\triangleq P(S,S,T) = \frac{1}{1+\delta F(S,S,T)} \triangleq \frac{1}{1+\delta F(S,T)}, \\ P_0(S,T) &\triangleq P(0,S,T) = \frac{1}{1+\delta F(0,S,T)} \triangleq \frac{1}{1+\delta F_0(S,T)}, \end{aligned}$$

where the OIS forward rate $F(t,S,T)$ is written as

$$F(t,S,T) = L(t,S,T) - B(S,T)$$

and the LIBOR/OIS spread $B(S,T)$ is assumed to be deterministic and independent of the observation time t. Thus, written entirely in terms of $L(t,S,T)$,

$$\begin{aligned} \Delta(S,T) &= \mathbb{E}^{\mathbb{Q}_T}\left[L(S,T)\left(\frac{P_0(S,T)}{P(S,T)} - 1\right)\right] \\ &= \mathbb{E}^{\mathbb{Q}_T}\left[L(S,T)\left(\frac{1+\delta F(S,T)}{1+\delta F_0(S,T)} - 1\right)\right] \\ &= \delta\mathbb{E}^{\mathbb{Q}_T}\left[L(S,T)\frac{F(S,T)-F_0(S,T)}{1+\delta F_0(S,T)}\right] \\ &= \frac{\delta}{1+\delta F_0(S,T)}\mathbb{E}^{\mathbb{Q}_T}\left[L(S,T)\left(L(S,T)-L_0(S,T)\right)\right] \\ &= \frac{\delta}{1+\delta F_0(S,T)}\left\{\mathbb{E}^{\mathbb{Q}_T}\left[L^2(S,T)\right] - L_0(S,T) \times \underbrace{\mathbb{E}^{\mathbb{Q}_T}\left[L(S,T)\right]}_{=L_0(S,T),\ \text{martingale}}\right\} \\ &= \frac{\delta}{1+\delta F_0(S,T)}\left\{\mathbb{E}^{\mathbb{Q}_T}\left[L^2(S,T)\right] - L_0^2(S,T)\right\} \end{aligned} \tag{1.1}$$

Assume that L follows the Black's model under the T-forward measure $\mathbb{Q}_T$, which is the usual log-normal model,

$$L(S,T) = L_0(S,T)e^{\sigma W(S) - \frac{1}{2}\sigma^2 S},$$

we know

$$\mathbb{E}^{\mathbb{Q}_T}\left[L^2(S,T)\right] = L_0^2(S,T)e^{\sigma^2 S}.$$

Thus,

$$\Delta(S,T) = \left(e^{\sigma^2 S} - 1\right) \frac{\delta L_0^2(S,T)}{1+\delta F_0(S,T)}.$$

2 NUMERICAL CALCULATION OF CMS RATES

Using the market data calculate approximately the following CMS rates:

- 10 year CMS rate settling in 1 year and paying 3 months later.
- 10 year CMS rate settling in 5 years and paying 3 months later.

Implied volatilities for 10 year tails

Mat yrs / bp out	-250	-200	-150	-100	-50	-25	ATM	25	50	100	150	200	250
0.25	48.69%	41.70%	37.59%	35.35%	34.43%	34.31%	34.36%	34.52%	34.77%	35.41%	36.16%	36.92%	37.67%
0.5	45.40%	39.75%	36.29%	34.25%	33.20%	32.95%	32.82%	32.80%	32.86%	33.14%	33.54%	34.00%	34.49%
1	41.10%	36.71%	33.83%	31.99%	30.90%	30.57%	30.35%	30.22%	30.16%	30.21%	30.41%	30.68%	31.01%
2	36.31%	33.05%	30.74%	29.14%	28.11%	27.77%	27.52%	27.35%	27.24%	27.19%	27.29%	27.48%	27.73%
3	33.04%	30.26%	28.18%	26.65%	25.60%	25.22%	24.93%	24.72%	24.58%	24.45%	24.48%	24.63%	24.84%
5	29.69%	27.33%	25.45%	23.99%	22.90%	22.49%	22.15%	21.89%	21.70%	21.47%	21.42%	21.49%	21.64%

Forward swap rates

Mat yrs	Rate
0.25	3.510%
0.5	3.632%
1	3.872%
2	4.313%
3	4.657%
5	5.030%

Use the Black model approach from Lecture Notes. For each of the CMS rates determine the convexity correction, and the portion of the convexity correction that comes from the payment delay.

Solution: Derived in lecture notes, the break-even CMS rate is given by

$$\begin{aligned} C\left(T_0, T|T_p\right) &= S_0\left(T_0, T\right) + \Delta\left(T_0, T|T_p\right) \\ &= S_0\left(T_0, T\right) + \Delta_{\text{corr}}\left(T_0, T\right) + \Delta_{\text{delay}}\left(T_0, T|T_p\right) \\ &= S_0\left(T_0, T\right)\left[1 + \left(\theta_c - \theta_d\right)\left(e^{\sigma^2 T_0} - 1\right)\right], \end{aligned}$$

where T_0 denotes the start date of the reference swap, T denotes the maturity of the reference swap, T_p denotes the payment date of the swaplet, $S_0\left(T_0, T\right)$ denotes the present value of the reference swap rate,

$$\theta_c = 1 - \frac{\frac{S_0}{f}}{1+\frac{S_0}{f}} \cdot \frac{n}{\left(1+\frac{S_0}{f}\right)^n - 1},$$

$$\theta_d = \frac{\frac{S_0}{f}}{1+\frac{S_0}{f}} \cdot \frac{f}{f_{\text{CMS}}},$$

and $f_{\text{CMS}} = 4$ is the payment frequency of the CMS swap, $f = 2$ is the payment frequency of the reference swap, and $n = f \times (T - T_0)$ is the total number of payments in the reference swap. Here we have assumed the typical values given in the lecture notes for f and f_{CMS} such that

$$\frac{f}{f_{\text{CMS}}} = \frac{1}{2}.$$

In addition, we always choose the ATM implied volatility for the following numerical calculations.

- 10 year CMS rate settling in 1 year and paying 3 months later.

$$
\begin{aligned}
&T_0 = 1, \quad T - T_0 = 10, \quad T_p - T_0 = \frac{1}{4}, \quad n = 20 \\
&S_0(T_0, T) \approx 3.8724\%, \quad \sigma \approx 30.3502\%, \quad e^{\sigma^2 T_0} - 1 \approx 0.096289227 \\
&\frac{\frac{S_0}{f}}{1 + \frac{S_0}{f}} \approx 0.018994233, \quad \theta_c \approx 0.187357788, \quad \theta_d \approx 0.009497116 \\
&\Delta_{\text{corr}} \approx 0.018040536 \times S_0, \quad \Delta_{\text{delay}} \approx -0.000914469 \times S_0
\end{aligned}
$$

Thus,

$$C\left(T_0, T | T_p\right) = S_0 + \Delta_{\text{corr}} + \Delta_{\text{delay}} \approx 3.9387\%,$$

where the convexity correction

$$\Delta = \Delta_{\text{corr}} + \Delta_{\text{delay}} \approx 0.017126067 \times S_0 \approx 0.06632\%,$$

and

$$\frac{\Delta_{\text{delay}}}{\Delta_{\text{corr}} + \Delta_{\text{delay}}} \approx -5.34\%.$$

- 10 year CMS rate settling in 5 years and paying 3 months later.

$$
\begin{aligned}
&T_0 = 5, \quad T - T_0 = 10, \quad T_p - T_0 = \frac{1}{4}, \quad n = 20 \\
&S_0(T_0, T) \approx 5.0303\%, \quad \sigma \approx 22.15447\%, \quad e^{\sigma^2 T_0} - 1 \approx 0.278145591 \\
&\frac{\frac{S_0}{f}}{1 + \frac{S_0}{f}} \approx 0.024534422, \quad \theta_c \approx 0.237430482, \quad \theta_d \approx 0.012267211 \\
&\Delta_{\text{corr}} \approx 0.066040241 \times S_0, \quad \Delta_{\text{delay}} \approx -0.00341207 \times S_0
\end{aligned}
$$

Thus,

$$C\left(T_0, T | T_p\right) = S_0 + \Delta_{\text{corr}} + \Delta_{\text{delay}} \approx 5.3453\%,$$

where the convexity correction

$$\Delta = \Delta_{\text{corr}} + \Delta_{\text{delay}} \approx 0.062628171 \times S_0 \approx 0.3150\%,$$

and

$$\frac{\Delta_{\text{delay}}}{\Delta_{\text{corr}} + \Delta_{\text{delay}}} \approx -5.45\%.$$

3 Options on Zero-Coupon Bonds in Hull-White Model

Derive the formula for the valuation of a call option on a zero coupon bond in the one-factor Hull-White model.

Solution: The solution to the one-factor Hull-White model is written as

$$r(t) = r_0(t) + e^{-\lambda t}\int_0^t \sigma(s)e^{\lambda s}dW(s). \tag{3.1}$$

where $W(t)$ is a Brownian motion under probability measure $\mathbb{P}^{\mathcal{N}}$, associated with the risk-neutral numéraire for zero coupon bond pricing

$$\mathcal{N}(t) = e^{\int_0^t r(u)du}.$$

In other words, the price $P(t,T)$ at time t of the zero-coupon bond maturing at time T in the one-factor Hull-White model is evaluated in the following way

$$\frac{P(t,T)}{\mathcal{N}(t)} = \mathbb{E}_t^{\mathcal{N}}\left[\frac{P(T,T)}{\mathcal{N}(T)}\right] = \mathbb{E}_t^{\mathcal{N}}\left[\frac{1}{\mathcal{N}(T)}\right],$$

so that

$$\begin{aligned}
P(t,T) &= \mathbb{E}_t^{\mathcal{N}}\left[\frac{\mathcal{N}(t)}{\mathcal{N}(T)}\right] \\
&= \mathbb{E}_t^{\mathcal{N}}\left[e^{-\int_t^T r(u)du}\right] \\
\text{(derived in lecture notes)} &= e^{-\int_t^T r_0(u)du - h_\lambda(T-t)(r(t)-r_0(t)) + \frac{1}{2}\int_t^T h_\lambda^2(T-v)\sigma^2(v)dv} \\
&= e^{-\int_t^T r_0(u)du + \frac{1}{2}\int_t^T h_\lambda^2(T-v)\sigma^2(v)dv} \times e^{-h_\lambda(T-t)(r(t)-r_0(t))} \\
&= e^{-\int_t^T r_0(u)du + \frac{1}{2}\int_t^T h_\lambda^2(T-v)\sigma^2(v)dv} \times e^{-h_\lambda(T-t)e^{-\lambda t}\int_0^t \sigma(s)e^{\lambda s}dW(s)}
\end{aligned} \tag{3.2}$$

where $h_\lambda(z) = \lambda^{-1}(1-e^{-\lambda z})$, and $r(t)-r_0(t) = e^{-\lambda t}\int_0^t \sigma(s)e^{\lambda s}dW(s) \sim N\left(0, \int_0^t e^{-2\lambda(t-s)}\sigma^2(s)ds\right)$, which is a normal random variable. A direct computation by Itô's lemma gives

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - h_\lambda(T-t)\sigma(t)dW(t),$$

or equivalently in numéraire $\mathcal{N}(t)$,

$$\frac{d\frac{P(t,T)}{\mathcal{N}(t)}}{\frac{P(t,T)}{\mathcal{N}(t)}} = -h_\lambda(T-t)\sigma(t)dW(t).$$

Because $r(0) = r_0(0)$, the time zero price is,

$$P(0,T) = e^{-\int_0^T r_0(u)du + \frac{1}{2}\int_0^T h_\lambda^2(T-v)\sigma^2(v)dv}.$$

A convenient formula that will be useful later is

$$\begin{aligned}
&\frac{P(0,T_{\text{mat}})}{P(0,T)} = e^{-\int_T^{T_{\text{mat}}} r_0(u)du + \frac{1}{2}\int_T^{T_{\text{mat}}} h_\lambda^2(T_{\text{mat}}-v)\sigma^2(v)dv} \times e^{\frac{1}{2}\int_0^T \left[h_\lambda^2(T_{\text{mat}}-v) - h_\lambda^2(T-v)\right]\sigma^2(v)dv} \\
&\Rightarrow e^{-\int_T^{T_{\text{mat}}} r_0(u)du + \frac{1}{2}\int_T^{T_{\text{mat}}} h_\lambda^2(T_{\text{mat}}-v)\sigma^2(v)dv} = \frac{P(0,T_{\text{mat}})}{P(0,T)} e^{-\frac{1}{2}\int_0^T \left[h_\lambda^2(T_{\text{mat}}-v) - h_\lambda^2(T-v)\right]\sigma^2(v)dv}.
\end{aligned} \tag{3.3}$$

For a European call option with strike K and expiration T with

$$\text{Payoff} = (P(T, T_{\text{mat}}) - K)^+,$$

where $P(T, T_{\text{mat}})$ is defined in Equation 3.2, the risk-neutral price at time t is

$$\begin{aligned}
V_{\text{call}}(t) &= \mathcal{N}(t)\mathbb{E}_t^{\mathcal{N}}\left[\frac{(P(T, T_{\text{mat}}) - K)^+}{\mathcal{N}(T)}\right] \\
&= \mathbb{E}_t^{\mathcal{N}}\left[\underbrace{e^{-\int_t^T r(u)du}}_{\text{discounting}}(P(T, T_{\text{mat}}) - K)^+\right] \\
&\triangleq \mathbb{E}_t^{\mathcal{N}}\left[D(t, T)\,(P(T, T_{\text{mat}}) - K)^+\right]
\end{aligned}$$

where we recognize the stochastic discount factor

$$D(t, T) \triangleq e^{-\int_t^T r(u)du}.$$

Notice that both the discount factor $D(t, T)$ and the payoff $(P(T, T_{\text{mat}}) - K)^+$ are stochastic variables, computing the above expectation value could be a tedious if done directly. Instead, we separate $V_{\text{call}}(t)$ into two terms

$$\begin{aligned}
V_{\text{call}}(t) &= \mathbb{E}_t^{\mathcal{N}}\left[\frac{\mathcal{N}(t)}{\mathcal{N}(T)}\,(P(T, T_{\text{mat}}) - K)\,\mathbb{1}_{\{P(T, T_{\text{mat}}) > K\}}\right] \\
&= \underbrace{\mathbb{E}_t^{\mathcal{N}}\left[\frac{\mathcal{N}(t)}{\mathcal{N}(T)}P(T, T_{\text{mat}})\mathbb{1}_{\{P(T, T_{\text{mat}}) > K\}}\right]}_{\text{Term I}} - K\underbrace{\mathbb{E}_t^{\mathcal{N}}\left[\frac{\mathcal{N}(t)}{\mathcal{N}(T)}\mathbb{1}_{\{P(T, T_{\text{mat}}) > K\}}\right]}_{\text{Term II}}.
\end{aligned}$$

The key simplification is to evaluate Term I and Term II in two different numeraires:

- Term I: $\mathcal{M}(t) \triangleq P(t, T_{\max})$, the associated Radon-Nikodym derivative is

$$\frac{d\mathbb{P}^{\mathcal{M}}}{d\mathbb{P}^{\mathcal{N}}} = \frac{P(T, T_{\max})\mathcal{N}(t)}{P(t, T_{\max})\mathcal{N}(T)}$$

- Term II: $\mathcal{L}(t) \triangleq P(t, T)$, the associated Radon-Nikodym derivative is

$$\frac{d\mathbb{P}^{\mathcal{L}}}{d\mathbb{P}^{\mathcal{N}}} = \frac{P(T, T)\mathcal{N}(t)}{P(t, T)\mathcal{N}(T)} = \frac{\mathcal{N}(t)}{P(t, T)\mathcal{N}(T)}$$

Thus,

$$\begin{aligned}
V_{\text{call}}(t) &= \mathbb{E}_t^{\mathcal{N}}\left[\frac{\mathcal{N}(t)}{\mathcal{N}(T)}P(T, T_{\text{mat}})\mathbb{1}_{\{P(T, T_{\text{mat}}) > K\}}\right] - K\mathbb{E}_t^{\mathcal{N}}\left[\frac{\mathcal{N}(t)}{\mathcal{N}(T)}\mathbb{1}_{\{P(T, T_{\text{mat}}) > K\}}\right] \\
&= \mathbb{E}_t^{\mathcal{M}}\left[\frac{\mathcal{N}(t)}{\mathcal{N}(T)}P(T, T_{\text{mat}})\mathbb{1}_{\{P(T, T_{\text{mat}}) > K\}}\frac{d\mathbb{P}^{\mathcal{N}}}{d\mathbb{P}^{\mathcal{M}}}\right] - K\mathbb{E}_t^{\mathcal{L}}\left[\frac{\mathcal{N}(t)}{\mathcal{N}(T)}\mathbb{1}_{\{P(T, T_{\text{mat}}) > K\}}\frac{d\mathbb{P}^{\mathcal{N}}}{d\mathbb{P}^{\mathcal{L}}}\right] \\
&= \mathbb{E}_t^{\mathcal{M}}\left[\frac{\mathcal{N}(t)}{\mathcal{N}(T)}P(T, T_{\text{mat}})\mathbb{1}_{\{P(T, T_{\text{mat}}) > K\}}\frac{\mathcal{N}(T)P(t, T_{\max})}{\mathcal{N}(t)P(T, T_{\max})}\right] - K\mathbb{E}_t^{\mathcal{L}}\left[\frac{\mathcal{N}(t)}{\mathcal{N}(T)}\mathbb{1}_{\{P(T, T_{\text{mat}}) > K\}}\frac{\mathcal{N}(T)P(t, T)}{\mathcal{N}(t)P(T, T)}\right]
\end{aligned}$$

$$= P(t,T_{\max})\mathbb{E}_t^{\mathscr{M}}\left[\mathbb{1}_{\{P(T,T_{\text{mat}})>K\}}\right] - KP(t,T)\mathbb{E}_t^{\mathscr{L}}\left[\mathbb{1}_{\{P(T,T_{\text{mat}})>K\}}\right].$$

The price at time zero is simply

$$V_{\text{call}}(t=0) = P(0,T_{\max})\mathbb{P}^{\mathscr{M}}\{P(T,T_{\text{mat}})>K\} - KP(0,T)\mathbb{P}^{\mathscr{L}}\{P(T,T_{\text{mat}})>K\}. \tag{3.4}$$

The next step is to find the distribution of $P(T,T_{\max})$ under measure $\mathbb{P}^{\mathscr{M}}$ and $\mathbb{P}^{\mathscr{L}}$.

- Term I: $\mathscr{M}(t) \triangleq P(t,T_{\max})$, the associated Radon-Nikodym derivative is

$$\begin{aligned}
\frac{d\mathbb{P}^{\mathscr{M}}}{d\mathbb{P}^{\mathscr{N}}} &= \frac{P(T,T_{\max})\mathscr{N}(t)}{P(t,T_{\max})\mathscr{N}(T)} \\
&= \frac{e^{-\int_t^T r(u)du}}{e^{-\int_t^T r_0(u)du+\frac{1}{2}\int_t^T h_\lambda^2(T_{\max}-v)\sigma^2(v)dv-h_\lambda(T_{\max}-t)(r(t)-r_0(t))+h_\lambda(T_{\max}-T)(r(T)-r_0(T))}} \\
&= e^{-\int_t^T (r(u)-r_0(u))du-\frac{1}{2}\int_t^T h_\lambda^2(T_{\max}-v)\sigma^2(v)dv+h_\lambda(T_{\max}-t)(r(t)-r_0(t))-h_\lambda(T_{\max}-T)(r(T)-r_0(T))} \\
&= e^{-\int_t^T e^{-\lambda u}\int_0^u \sigma(s)e^{\lambda s}dW(s)du-\frac{1}{2}\int_t^T h_\lambda^2(T_{\max}-v)\sigma^2(v)dv+h_\lambda(T_{\max}-t)(r(t)-r_0(t))-h_\lambda(T_{\max}-T)(r(T)-r_0(T))}.
\end{aligned}$$

Note that the first term in the exponent

$$\begin{aligned}
&\int_t^T e^{-\lambda u}\int_0^u \sigma(s)e^{\lambda s}dW(s)du \\
&= \int_0^t \sigma(s)e^{\lambda s}dW(s)\int_t^T e^{-\lambda u}du + \int_t^T \sigma(s)e^{\lambda s}dW(s)\int_s^T e^{-\lambda u}du \\
&= \int_0^t \sigma(s)e^{\lambda s}dW(s)\frac{e^{-\lambda t}-e^{-\lambda T}}{\lambda} + \int_t^T \sigma(s)e^{\lambda s}dW(s)\frac{e^{-\lambda s}-e^{-\lambda T}}{\lambda} \\
&= h_\lambda(T-t)e^{-\lambda t}\int_0^t \sigma(s)e^{\lambda s}dW(s) + \int_t^T h_\lambda(T-s)\sigma(s)dW(s).
\end{aligned} \tag{3.5}$$

Thus,

$$\begin{aligned}
\frac{d\mathbb{P}^{\mathscr{M}}}{d\mathbb{P}^{\mathscr{N}}} &= e^{-h_\lambda(T-t)e^{-\lambda t}\int_0^t \sigma(s)e^{\lambda s}dW(s)-\int_t^T h_\lambda(T-s)\sigma(s)dW(s)-\frac{1}{2}\int_t^T h_\lambda^2(T_{\max}-v)\sigma^2(v)dv} \\
&\quad\times e^{h_\lambda(T_{\max}-t)e^{-\lambda t}\int_0^t \sigma(s)e^{\lambda s}dW(s)-h_\lambda(T_{\max}-T)e^{-\lambda T}\int_0^T \sigma(s)e^{\lambda s}dW(s)} \\
&= e^{[h_\lambda(T_{\max}-t)-h_\lambda(T-t)]\int_0^t \sigma(s)e^{-\lambda(t-s)}dW(s)-h_\lambda(T_{\max}-T)\int_0^T \sigma(s)e^{-\lambda(T-s)}dW(s)-\int_t^T h_\lambda(T-s)\sigma(s)dW(s)} \\
&\quad\times e^{-\frac{1}{2}\int_t^T h_\lambda^2(T_{\max}-v)\sigma^2(v)dv} \\
&= e^{h_\lambda(T_{\max}-T)\int_0^t \sigma(s)e^{-\lambda(T-s)}dW(s)-h_\lambda(T_{\max}-T)\int_0^T \sigma(s)e^{-\lambda(T-s)}dW(s)-\int_t^T h_\lambda(T-s)\sigma(s)dW(s)} \\
&\quad\times e^{-\frac{1}{2}\int_t^T h_\lambda^2(T_{\max}-v)\sigma^2(v)dv} \\
&= e^{-h_\lambda(T_{\max}-T)\int_t^T \sigma(s)e^{-\lambda(T-s)}dW(s)-\int_t^T h_\lambda(T-s)\sigma(s)dW(s)} \times e^{-\frac{1}{2}\int_t^T h_\lambda^2(T_{\max}-v)\sigma^2(v)dv} \\
&= e^{-\int_t^T \left[h_\lambda(T_{\max}-T)e^{-\lambda(T-s)}+h_\lambda(T-s)\right]\sigma(s)dW(s)} \times e^{-\frac{1}{2}\int_t^T h_\lambda^2(T_{\max}-v)\sigma^2(v)dv} \\
&= e^{-\int_t^T h_\lambda(T_{\max}-v)\sigma(v)dW(v)-\frac{1}{2}\int_t^T h_\lambda^2(T_{\max}-v)\sigma^2(v)dv},
\end{aligned}$$

which takes a typical form for Radon-Nikodym derivative in Girsanov's theorem with

$$\Theta(t) \triangleq h_\lambda(T_{\max} - t)\sigma(t).$$

and

$$dW^{\mathscr{M}}(t) \triangleq dW(t) + \Theta(t)dt$$

is a Brownian motion under the probability measure $\mathbb{P}^{\mathscr{M}}$. Thus,

$$\begin{aligned}
P(T, T_{\text{mat}}) &= e^{-\int_T^{T_{\text{mat}}} r_0(u)du + \frac{1}{2}\int_T^{T_{\text{mat}}} h_\lambda^2(T_{\text{mat}}-v)\sigma^2(v)dv} \times e^{-h_\lambda(T_{\text{mat}}-T)e^{-\lambda T}\int_0^T \sigma(s)e^{\lambda s}dW(s)} \\
&= e^{-\int_T^{T_{\text{mat}}} r_0(u)du + \frac{1}{2}\int_T^{T_{\text{mat}}} h_\lambda^2(T_{\text{mat}}-v)\sigma^2(v)dv + h_\lambda(T_{\text{mat}}-T)\int_0^T h_\lambda(T_{\text{mat}}-v)e^{-\lambda(T-v)}\sigma^2(v)dv} \\
&\quad \times e^{\underbrace{-h_\lambda(T_{\text{mat}}-T)\int_0^T \sigma(s)e^{-\lambda(T-s)}dW^{\mathscr{M}}(s)}_{\sim N(0,\hat{\sigma}^2)}} \\
\text{(Equation 3.3)} &= \frac{P(0, T_{\text{mat}})}{P(0,T)} e^{-\frac{1}{2}\int_0^T [h_\lambda^2(T_{\text{mat}}-v) - h_\lambda^2(T-v)]\sigma^2(v)dv + h_\lambda(T_{\text{mat}}-T)\int_0^T h_\lambda(T_{\text{mat}}-v)e^{-\lambda(T-v)}\sigma^2(v)dv} \times e^{-\hat{\sigma}Z} \\
&= \frac{P(0, T_{\text{mat}})}{P(0,T)} e^{-\frac{1}{2}\int_0^T [h_\lambda^2(T_{\text{mat}}-v) - h_\lambda^2(T-v) - 2h_\lambda(T_{\text{mat}}-T)h_\lambda(T_{\text{mat}}-v)e^{-\lambda(T-v)}]\sigma^2(v)dv} \times e^{-\hat{\sigma}Z} \\
&= \frac{P(0, T_{\text{mat}})}{P(0,T)} e^{\frac{1}{2}h_\lambda^2(T_{\text{mat}}-T)\int_0^T e^{-2\lambda(T-v)}\sigma^2(v)dv} \times e^{-\hat{\sigma}Z} \\
&= \frac{P(0, T_{\text{mat}})}{P(0,T)} e^{\frac{1}{2}\hat{\sigma}^2 - \hat{\sigma}Z}
\end{aligned}$$

where $Z \sim N(0,1)$,

$$\hat{\sigma} = \sqrt{\int_0^T e^{-2\lambda(T-v)}\sigma^2(v)dv} \cdot h_\lambda(T_{\text{mat}} - T).$$

and the following algebraic identity is used in the last two equalities

$$h_\lambda^2(T_{\text{mat}} - v) - h_\lambda^2(T - v) - 2h_\lambda(T_{\text{mat}} - T)h_\lambda(T_{\text{mat}} - v)e^{-\lambda(T-v)} = -h_\lambda^2(T_{\text{mat}} - T)\, e^{-2\lambda(T-v)}.$$

Thus we have expressed $P(T, T_{\text{mat}})$ under the probability measure $\mathbb{P}^{\mathscr{M}}$ in a familiar form. Consequently, in the first term in Equation 3.4,

$$\mathbb{P}^{\mathscr{M}}\{P(T, T_{\text{mat}}) > K\} = \mathbb{P}^{\mathscr{M}}\left\{\frac{P(0, T_{\text{mat}})}{P(0,T)} e^{\frac{1}{2}\hat{\sigma}^2 - \hat{\sigma}Z} > K\right\} = \mathbb{P}^{\mathscr{M}}\{Z < d_+\} = N(d_+).$$

where

$$d_+ = \frac{1}{\hat{\sigma}}\left[\log\frac{P_0(0, T_{\text{mat}})}{KP_0(0,T)} + \frac{1}{2}\hat{\sigma}^2\right].$$

- Term II: $\mathscr{L}(t) \triangleq P(t, T)$, the associated Radon-Nikodym derivative is

$$\begin{aligned}
\frac{d\mathbb{P}^{\mathscr{L}}}{d\mathbb{P}^{\mathscr{N}}} &= \frac{P(T,T)\mathscr{N}(t)}{P(t,T)\mathscr{N}(T)} = \frac{\mathscr{N}(t)}{P(t,T)\mathscr{N}(T)} \\
&= \frac{e^{-\int_t^T r(u)du}}{e^{-\int_t^T r_0(u)du - h_\lambda(T-t)(r(t)-r_0(t)) + \frac{1}{2}\int_t^T h_\lambda^2(T-v)\sigma^2(v)dv}}
\end{aligned}$$

$$= e^{-\int_t^T (r(u)-r_0(u))du + h_\lambda(T-t)(r(t)-r_0(t)) - \frac{1}{2}\int_t^T h_\lambda^2(T-v)\sigma^2(v)dv}$$

$$\text{(Equation 3.1)} = e^{-\int_t^T e^{-\lambda u}\int_0^u \sigma(s)e^{\lambda s}dW(s)du + h_\lambda(T-t)e^{-\lambda t}\int_0^t \sigma(s)e^{\lambda s}dW(s) - \frac{1}{2}\int_t^T h_\lambda^2(T-v)\sigma^2(v)dv}$$

$$\text{(Equation 3.5)} = e^{-\int_t^T h_\lambda(T-v)\sigma(v)dW(v) - \frac{1}{2}\int_t^T h_\lambda^2(T-v)\sigma^2(v)dv},$$

which again takes a typical form for Radon-Nikodym derivative in Girsanov's theorem with

$$\Theta(t) \triangleq h_\lambda(T-t)\sigma(t).$$

and

$$dW^{\mathcal{L}}(t) \triangleq dW(t) + \Theta(t)dt$$

is a Brownian motion under the probability measure $\mathbb{P}^{\mathcal{L}}$. Thus,

$$\begin{aligned}
P(T, T_{\text{mat}}) &= e^{-\int_T^{T_{\text{mat}}} r_0(u)du + \frac{1}{2}\int_T^{T_{\text{mat}}} h_\lambda^2(T_{\text{mat}}-v)\sigma^2(v)dv} \times e^{-h_\lambda(T_{\text{mat}}-T)e^{-\lambda T}\int_0^T \sigma(s)e^{\lambda s}dW(s)} \\
&= e^{-\int_T^{T_{\text{mat}}} r_0(u)du + \frac{1}{2}\int_T^{T_{\text{mat}}} h_\lambda^2(T_{\text{mat}}-v)\sigma^2(v)dv + h_\lambda(T_{\text{mat}}-T)\int_0^T h_\lambda(T-v)e^{-\lambda(T-v)}\sigma^2(v)dv} \\
&\quad \times e^{\underbrace{-h_\lambda(T_{\text{mat}}-T)\int_0^T \sigma(s)e^{-\lambda(T-s)}dW^{\mathcal{L}}(s)}_{\sim N(0,\hat{\sigma}^2)}} \\
\text{(Equation 3.3)} &= \frac{P(0,T_{\text{mat}})}{P(0,T)} e^{-\frac{1}{2}\int_0^T [h_\lambda^2(T_{\text{mat}}-v) - h_\lambda^2(T-v)]\sigma^2(v)dv + h_\lambda(T_{\text{mat}}-T)\int_0^T h_\lambda(T-v)e^{-\lambda(T-v)}\sigma^2(v)dv} \times e^{-\hat{\sigma}Z} \\
&= \frac{P(0,T_{\text{mat}})}{P(0,T)} e^{-\frac{1}{2}\int_0^T [h_\lambda^2(T_{\text{mat}}-v) - h_\lambda^2(T-v) - 2h_\lambda(T_{\text{mat}}-T)h_\lambda(T-v)e^{-\lambda(T-v)}]\sigma^2(v)dv} \times e^{-\hat{\sigma}Z} \\
&= \frac{P(0,T_{\text{mat}})}{P(0,T)} e^{-\frac{1}{2}h_\lambda^2(T_{\text{mat}}-T)\int_0^T e^{-2\lambda(T-v)}\sigma^2(v)dv} \times e^{-\hat{\sigma}Z} \\
&= \frac{P(0,T_{\text{mat}})}{P(0,T)} e^{-\frac{1}{2}\hat{\sigma}^2 - \hat{\sigma}Z},
\end{aligned}$$

where the following algebraic identity is used in the last two equalities

$$h_\lambda^2(T_{\text{mat}}-v) - h_\lambda^2(T-v) - 2h_\lambda(T_{\text{mat}}-T)h_\lambda(T-v)e^{-\lambda(T-v)} = h_\lambda^2(T_{\text{mat}}-T)\,e^{-2\lambda(T-v)}.$$

Thus we have expressed $P(T, T_{\text{mat}})$ under the probability measure $\mathbb{P}^{\mathcal{L}}$ in a familiar form. Consequently, in the second term in Equation 3.4,

$$\mathbb{P}^{\mathcal{L}}\{P(T,T_{\text{mat}}) > K\} = \mathbb{P}^{\mathcal{L}}\left\{\frac{P(0,T_{\text{mat}})}{P(0,T)}e^{-\frac{1}{2}\hat{\sigma}^2-\hat{\sigma}Z} > K\right\} = \mathbb{P}^{\mathcal{L}}\{Z < d_-\} = N(d_-).$$

where

$$d_- = \frac{1}{\hat{\sigma}}\left[\log\frac{P_0(0,T_{\text{mat}})}{KP_0(0,T)} - \frac{1}{2}\hat{\sigma}^2\right].$$

In summary, according to Equation 3.4, the risk-neutral price at time zero is

$$\text{PV}_{\text{call}} = P(0,T_{\text{mat}})N(d_+) - KP(0,T)N(d_-),$$

where

$$d_\pm = \frac{1}{\hat{\sigma}}\left[\log\frac{P(0,T_{\text{mat}})}{KP(0,T)} \pm \frac{1}{2}\hat{\sigma}^2\right].$$

Note that notation-wise, in lecture notes, an extra (unnecessary) 0-subscript is placed in $P(0,T_{\text{mat}})$ and $P(0,T)$: $P_0(0,T_{\text{mat}}) \triangleq P(0,T_{\text{mat}})$ and $P_0(0,T) \triangleq P(0,T)$.