*Department of Mathematics* *March 21, 2013*

# *Probability and Stochastic Processes in Finance II (MTH 9862).*

## *Midterm Examination.*

**Instructions:** Please **print** your name below. Write all solutions in a blue book. This is a closed book test. Required number of points is 100; the rest will be counted as extra credit. **Good luck!**

**Student name:**

| **Grade** |
|---|
| |

| Problem | Out of | Score | Comments |
|---|---|---|---|
| 1 | 20 | | 7+6+7 |
| 2 | 20 | | |
| 3 | 10 | | |
| 4 | 20 | | |
| 5 | 15 | | 7+8 |
| 6 | 20 | | |
| 7 | 25 | | 10+15 |
| Total | 130 | | |

**Problem 1.** Let $\alpha(t)$ and $\sigma(t)$ be continuous bounded deterministic functions on $[0, \infty)$.

(a) Find a closed form solution[1] of the equation

$$dX(t) = \alpha(t)X(t)\,dt + \sigma(t)X(t)\,dB(t).$$

(b) Find the equation satisfied by $\ln(X(t))$. Determine the distribution of $\ln(X(t))$.

(c) Given $p \neq 0$, find $E(X^p(t))$. What is the distribution of $X^p(t)$?

[1] If you simply remember the answer then you still have to show that your answer is correct. No check - no credit.

**Problem 2.** Let $S(0) = x$ and for some fixed $a, \mu \geq 0$ and $\sigma > 0$

$$dS(t) = (\mu - aS(t))dt + \sigma\, dB(t).$$

(a) Find the expectation and variance of $S(t)$.

(b) Find a closed form solution of the above equation. Then calculate the expectation and variance of $S(t)$ directly from the closed form solution.

**Problem 3.** Let

$$I(t) = \int_0^t \sin u \, dB(u) + \frac{1}{2} \int_0^t \cos^2 u \, du.$$

(a) Determine if $e^{I(t)}$ is a martingale.

(b) Find $[I, e^I](t)$, i.e. the cross-variation of $I(t)$ and $e^{I(t)}$.

**Problem 4.** Let $\mu \in \mathbb{R},\ r, a \geq 0$ and $S(t)$ satisfy under $\mathbb{P}$

$$dS(t) = (\mu - aS(t))dt + \sigma\, dB(t), \quad 0 \leq t \leq T.$$

(a) Find the probability measure $\tilde{\mathbb{P}}$ such that the process $e^{-(r-a)t}S(t)$, $0 \leq t \leq T$, is a martingale under $\tilde{\mathbb{P}}$. Give explicitly the Radon-Nikodym derivative $d\tilde{\mathbb{P}}/d\mathbb{P}$.

(b) Find the equation for $S(t)$ under $\tilde{\mathbb{P}}$.

**Problem 5.** Let $B(t) = (B_1(t),\ B_2(t))^T$ be a standard two-dimensional Brownian motion and $R(t) = \sqrt{B_1^2(t) + B_2^2(t)}$. Determine which of the following processes are standard Brownian motions (dimension 1 or 2). Clearly state and check all the required conditions.

(a) $dX(t) = R^{-1}(t)(B_1(t)dB_1(t) + B_2(t)dB_2(t))$.

(b) $W(t) = (W_1(t), W_2(t))^T$, where

$$W(t) = \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\beta & \sin\beta \end{pmatrix} B(t),$$

and $\alpha,\ \beta \in [0, 2\pi]$ are fixed numbers.

**Problem 6.** Let $(B(t))_{t\geq 0}$ be a standard Brownian motion, $(\mathcal{F}(t))_{t\geq 0}$ be its natural filtration, and $T > 0$ be a fixed time. Define $M(t) := E(B^3(T)\,|\,\mathcal{F}(t))$, $0 \leq t \leq T$. Then we know (by the tower property of conditional expectations) that $(M(t))_{0\leq t\leq T}$ is a martingale. This problem will guide you through finding an explicit representation for this martingale, i. e. finding a stochastic process $\Gamma(t)$ such that

$$M(t) = \int_0^t \Gamma(u)\,dB(u), \quad 0 \leq t \leq T.$$

(a) Use one of the standard inequalities to show that $M(t)$ is square integrable for each $t \in [0, T]$.

(b) Compute $dB^3(t)$ and write $B^3(T)$ as a sum of a stochastic and a regular integral.

(c) Integrate your regular integral by parts. Write the result as a single stochastic integral.

(d) Use parts (b) and (c) together with one of the basic properties of Ito integral to write $E(B^3(T)\,|\,\mathcal{F}(t))$ as a single stochastic integral. State explicitly your answer, i.e. $\Gamma(t) = \ldots$.

**Problem 7.** Consider a market model which consists of 2 stocks and a MMA. Assume that the (continuously compounded) interest rate is $r \geq 0$ and that stock prices $S_1(t)$ and $S_2(t)$, $0 \leq t \leq T$, satisfy the following equations under the risk-neutral measure $\tilde{\mathbb{P}}$:

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)d\tilde{W}_1(t), \quad S_1(0) = s_1;$$
$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)d\tilde{W}_2(t), \quad S_2(0) = s_2.$$

where $\sigma_i, s_i$, $i = 1, 2$, are positive constants and $(\tilde{W}_1(t))_{t\geq 0}$ and $(\tilde{W}_2(t))_{t\geq 0}$ are standard Brownian motions such that $d[\tilde{W}_1, \tilde{W}_2](t) = \rho\, dt$ for some constant $\rho \in [-1, 1]$.

(a) Suppose that for some deterministic function $u(t, x_1, x_2)$ the process $e^{-rt}u(t, S_1(t), S_2(t))$, $t \geq 0$, is a martingale under the risk-neutral measure. Then the function $u(t, x_1, x_2)$ should satisfy some partial differential equation. Find this equation. Hint: compute $d(e^{-rt}u(t, S_1(t), S_2(t)))$.

(b) Find the correlation between $S_1(t)$ and $S_2(t)$. Is it true that when $\rho = 1$ then the correlation between $S_1(t)$ and $S_2(t)$ is also equal to 1?