BARUCH, MFE

MTH 9876 Assignment Two

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October 2, 2016

1 Valuation of Constant Intensity CDS

Consider a credit model with a constant intensity λ, and a constant discounting rate r. Consider a new CDS starting on a roll date and maturing T years from now (T is an integer) . For simplicity, assume that the roll dates are equally spaced with all day count fractions equal 0.25. The recovery rate is assumed to be R.

1.1 Protection Leg

Solution: The present value of receiving the protection $(1-R)$ at time τ is given by

$$\begin{aligned}
\mathscr{V}(T) &= \mathbb{E}\left[(1-R)e^{-\int_0^\tau r(t)dt}\mathbb{1}_{\tau\le T}\right] \\
&= (1-R)\int_0^T \mathbb{E}\left[e^{-\int_0^s r(t)dt}\mathbb{1}_{\tau\in[s,s+ds]}\right] \\
&= (1-R)\int_0^T \mathbb{E}\left[\mathbb{E}\left[e^{-\int_0^s r(t)dt}\mathbb{1}_{\tau\in[s,s+ds]}\middle|\mathscr{G}_s\right]\right] \\
&= (1-R)\int_0^T \mathbb{E}\left[e^{-\int_0^s r(t)dt}\mathbb{E}\left[\mathbb{1}_{\tau\in[s,s+ds]}\middle|\mathscr{G}_s\right]\right] \\
&= (1-R)\int_0^T \mathbb{E}\left[e^{-\int_0^s r(t)dt}e^{-\int_0^s \lambda(t)dt}\lambda(s)ds\right] \\
&= (1-R)\int_0^T \mathbb{E}\left[\lambda(s)e^{-\int_0^s (r(t)+\lambda(t))dt}\right]ds.
\end{aligned}$$

Now set the rates and intensities to constant $r(t)=r$, $\lambda(t)=\lambda$, according to the question,

$$\mathscr{V}(T) = \lambda(1-R)\int_0^T e^{-(r+\lambda)s}ds = (1-R)\left(\frac{\lambda}{r+\lambda}\right)\left(1-e^{-(r+\lambda)T}\right).$$

1.2 Risk Annuity

Solution: The present value of \$1 paid on T_j is given by

$$\mathscr{P}(0,T_j) = \mathbb{E}\left[e^{-\int_0^{T_j} r(s)ds}\mathbb{1}_{\tau>T_j}\right] = \mathbb{E}\left[e^{-\int_0^{T_j}(r(s)+\lambda(s))ds}\right] = e^{-(r+\lambda)T_j}.$$

Summing over all coupon payments on roll dates,

$$\sum_j \delta_j \mathscr{P}(0,T_j) = \frac{1}{4}\sum_{j=1}^{4T} e^{-\frac{(r+\lambda)}{4}j} = \frac{e^{-\frac{r+\lambda}{4}}}{4}\times\frac{1-e^{-(r+\lambda)T}}{1-e^{-\frac{r+\lambda}{4}}}.$$

The other part of risk annuity is the premium accrued from the previous payment date to the default time. Suppose the default time falls within the time interval $[T_{j-1}, T_j]$,

$$\begin{aligned}
-\int_{T_{j-1}}^{T_j} (s - T_{j-1}) P(0,s)\, dS(0,s) &= \lambda \int_{T_{j-1}}^{T_j} (s - T_{j-1}) e^{-(r+\lambda)s} ds \\
&= \lambda e^{-(r+\lambda)T_{j-1}} \int_0^{\delta_j} s e^{-(r+\lambda)s} ds \\
&= e^{-(r+\lambda)T_{j-1}} \frac{\lambda}{(r+\lambda)^2} \left[1 - e^{-\frac{r+\lambda}{4}} \left(1 + \frac{r+\lambda}{4}\right)\right].
\end{aligned}$$

Summing over all time intervals between roll dates,

$$\begin{aligned}
\sum_j -\int_{T_{j-1}}^{T_j} (s - T_{j-1}) P(0,s)\, dS(0,s) &= \frac{\lambda}{(r+\lambda)^2} \left[1 - e^{-\frac{r+\lambda}{4}} \left(1 + \frac{r+\lambda}{4}\right)\right] \times \sum_{j=1}^{4T} e^{-(r+\lambda)T_{j-1}} \\
&= \frac{\lambda}{(r+\lambda)^2} \left[1 - e^{-\frac{r+\lambda}{4}} \left(1 + \frac{r+\lambda}{4}\right)\right] \times \frac{1 - e^{-(r+\lambda)T}}{1 - e^{-\frac{r+\lambda}{4}}}.
\end{aligned}$$

Combining the two contributions to risk annuity, we get

$$\begin{aligned}
\mathscr{A}(T) &= \frac{e^{-\frac{r+\lambda}{4}}}{4} \times \frac{1 - e^{-(r+\lambda)T}}{1 - e^{-\frac{r+\lambda}{4}}} + \frac{\lambda}{(r+\lambda)^2} \left[1 - e^{-\frac{r+\lambda}{4}} \left(1 + \frac{r+\lambda}{4}\right)\right] \times \frac{1 - e^{-(r+\lambda)T}}{1 - e^{-\frac{r+\lambda}{4}}} \\
&= \left\{\frac{1}{4} e^{-\frac{r+\lambda}{4}} + \frac{\lambda}{(r+\lambda)^2} \left[1 - e^{-\frac{r+\lambda}{4}} \left(1 + \frac{r+\lambda}{4}\right)\right]\right\} \times \frac{1 - e^{-(r+\lambda)T}}{1 - e^{-\frac{r+\lambda}{4}}}.
\end{aligned}$$

1.3 Par Spread

Solution: The par credit spread

$$C_0 = \frac{\mathscr{V}(T)}{\mathscr{A}(T)} = (1-R) \left(\frac{\lambda}{r+\lambda}\right) \left(1 - e^{-\frac{r+\lambda}{4}}\right) \left\{\frac{1}{4} e^{-\frac{r+\lambda}{4}} + \frac{\lambda}{(r+\lambda)^2} \left[1 - e^{-\frac{r+\lambda}{4}} \left(1 + \frac{r+\lambda}{4}\right)\right]\right\}^{-1}.$$

This expression reduces to the credit triangle $C_0 \approx (1-R)\lambda$ when $\lambda \ll 1$ and $r \ll 1$.

2 VALUATION OF STOCHASTIC INTENSITY CDS

Repeat Problem 1 assuming a constant discounting rate r and the stochastic intensity $\lambda(t)$

$$d\lambda(t) = \theta dt + \sigma dW(t),$$

with constant θ and $\lambda(t=0) = \lambda_0$.
Solution: From Homework Assignment #1, we have $\lambda(t) = \lambda_0 + \theta t + \sigma W(t)$,

$$\int_0^t \lambda(s)ds = \lambda_0 t + \frac{1}{2}\theta t^2 + \sigma \int_0^t W(s)ds \sim N\left(\lambda_0 t + \frac{1}{2}\theta t^2, \frac{1}{3}\sigma^2 t^3\right),$$

and the survival probability

$$S(0,t) = \mathbb{E}\left[e^{-\int_0^t \lambda(s)ds}\right] = e^{-\lambda_0 t - \frac{1}{2}\theta t^2 + \frac{1}{6}\sigma^2 t^3}.$$

2.1 PROTECTION LEG

The present value of receiving the protection $(1-R)$ at time τ is given by

$$\begin{aligned}
\mathscr{V}(T) &= (1-R)\int_0^T \mathbb{E}\left[\lambda(s)e^{-\int_0^s (r(t)+\lambda(t))dt}\right]ds \\
&= (1-R)\int_0^T e^{-rs}\mathbb{E}\left[\lambda(s)e^{-\int_0^s \lambda(t)dt}\right]ds \\
&= -(1-R)\int_0^T e^{-rs}d\mathbb{E}\left[e^{-\int_0^s \lambda(t)dt}\right] \\
&= -(1-R)\int_0^T e^{-rs}dS(0,s) \\
&= (1-R)\left[1 - e^{-rT}S(0,T) - r\int_0^T S(0,s)e^{-rs}ds\right] \\
&= (1-R)\left[1 - e^{-(r+\lambda_0)T - \frac{1}{2}\theta T^2 + \frac{1}{6}\sigma^2 T^3} - r\int_0^T e^{-(r+\lambda_0)s - \frac{1}{2}\theta s^2 + \frac{1}{6}\sigma^2 s^3}ds\right]
\end{aligned}$$

2.2 RISK ANNUITY

The present value of \$1 paid on T_j is given by

$$\mathscr{P}(0,T_j) = \mathbb{E}\left[e^{-\int_0^{T_j}(r(s)+\lambda(s))ds}\right] = e^{-rT_j}\mathbb{E}\left[e^{-\int_0^{T_j}\lambda(s)ds}\right] = e^{-(r+\lambda_0)T_j - \frac{1}{2}\theta T_j^2 + \frac{1}{6}\sigma^2 T_j^3}$$

Summing over all coupon payments on roll dates,

$$\sum_j \delta_j \mathscr{P}(0,T_j) = \frac{1}{4}\sum_{j=1}^{4T} e^{-\frac{1}{4}(r+\lambda_0)j - \frac{1}{32}\theta j^2 + \frac{1}{384}\sigma^2 j^3}.$$

The premium accrued from the previous payment date to the default time within the time interval $[T_{j-1}, T_j]$,

$$
\begin{aligned}
-\int_{T_{j-1}}^{T_j}(s-T_{j-1})P(0,s)dS(0,s) &= \int_{T_{j-1}}^{T_j}\left(s-T_{j-1}\right)\left(\lambda_0+\theta s-\frac{1}{2}\sigma^2 s^2\right)e^{-(r+\lambda_0)s-\frac{1}{2}\theta s^2+\frac{1}{6}\sigma^2 s^3}ds \\
&= \int_{\frac{j-1}{4}}^{\frac{j}{4}}\left(s-\frac{j-1}{4}\right)\left(\lambda_0+\theta s-\frac{1}{2}\sigma^2 s^2\right)e^{-(r+\lambda_0)s-\frac{1}{2}\theta s^2+\frac{1}{6}\sigma^2 s^3}ds
\end{aligned}
$$

Summing over all time intervals between roll dates,

$$
\sum_j -\int_{T_{j-1}}^{T_j}(s-T_{j-1})P(0,s)dS(0,s) = \sum_{j=1}^{4T}\int_{\frac{j-1}{4}}^{\frac{j}{4}}\left(s-\frac{j-1}{4}\right)\left(\lambda_0+\theta s-\frac{1}{2}\sigma^2 s^2\right)e^{-(r+\lambda_0)s-\frac{1}{2}\theta s^2+\frac{1}{6}\sigma^2 s^3}ds.
$$

Combining the two contributions to risk annuity, we get

$$
\mathscr{A}(T) = \sum_{j=1}^{4T}\left[\frac{1}{4}e^{-\frac{1}{4}(r+\lambda_0)j-\frac{1}{32}\theta j^2+\frac{1}{384}\sigma^2 j^3}+\int_{\frac{j-1}{4}}^{\frac{j}{4}}\left(s-\frac{j-1}{4}\right)\left(\lambda_0+\theta s-\frac{1}{2}\sigma^2 s^2\right)e^{-(r+\lambda_0)s-\frac{1}{2}\theta s^2+\frac{1}{6}\sigma^2 s^3}ds\right].
$$

2.3 Par Spread

By definition, the par credit spread is given by

$$
C_0 = \frac{\mathscr{V}(T)}{\mathscr{A}(T)} = \frac{(1-R)\left[1-e^{-(r+\lambda_0)T-\frac{1}{2}\theta T^2+\frac{1}{6}\sigma^2 T^3}-r\int_0^T e^{-(r+\lambda_0)s-\frac{1}{2}\theta s^2+\frac{1}{6}\sigma^2 s^3}ds\right]}{\sum_{j=1}^{4T}\left[\frac{1}{4}e^{-\frac{1}{4}(r+\lambda_0)j-\frac{1}{32}\theta j^2+\frac{1}{384}\sigma^2 j^3}+\int_{\frac{j-1}{4}}^{\frac{j}{4}}\left(s-\frac{j-1}{4}\right)\left(\lambda_0+\theta s-\frac{1}{2}\sigma^2 s^2\right)e^{-(r+\lambda_0)s-\frac{1}{2}\theta s^2+\frac{1}{6}\sigma^2 s^3}ds\right]}.
$$