

UNIT 2: Linear population models

1 Constructing models

1.1 Dynamical models

Tools to link scales

- Models are what we use to link:
 - Individual-level to population-level processes
 - Short time scales to long time scales
- In both directions

Assumptions

- Models are always simplifications of reality
 - “The map is not the territory”
 - “All models are wrong, but some are useful”
- Models are useful for:
 - linking assumptions to outcomes
 - identifying where assumptions are broken

Dynamical models

- **Dynamical models** describe rules for how a system changes at each point in time
- We will see what these assumptions about how the system *changes* lead to conclusions about what the system *does* over longer time periods

States and state variables

- Our dynamic models imagine that a system has a **state** at any given time, described by one or more **state variables**
- These are the things that follow our rules and change
- Examples:
 - Dandelions: state is population size, described by one state variable (the number of individuals)
 - Bacteria: state is population density, described by one state variable (the number of individuals per ml)
 - Pine trees: state is amount of wood, described by one state variable (tons per hectare)
- Limiting the number of state variables is key to simple models

Parameters

- **Parameters** are the quantities that describe how the rules for our system work
- Examples:
 - Birth rate, death rate, fecundity, survival probability
- Typically *remain constant* while we are simulating a particular scenario
- *Vary* when we compare different scenarios

How do populations change?

- I survey a population in 2009, and again in 2013. I get a different answer the second time.
- Poll: What are some reasons why this answer might change?
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Censusing and intermediate variables

- Often, our population models will imagine that the population is **censused** (counted) at particular periods of time
- Calculations of what happens between census times may be part of how we make our population model, without showing up in the main model itself
 - For example, our moth and dandelion examples

Linear population models

- We will focus mostly on births and deaths
- Births and deaths are done by individuals
 - We model the rate of each individual (per capita rates)
 - Total rate is the per capita rate multiplied by population size
- If per capita rates are constant, we say that our population *models* are **linear**
 - Linear models do not usually correspond to linear growth!
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1.2 Examples

Moth example

- Poll: State variable

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- Parameters

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- Census time

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Bacteria

- State variables

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- Poll: Parameters

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- Census time

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Dandelions

- State variables

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- Poll: Are there intermediate variables?

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- Parameters

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- Census time

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1.3 A simple discrete-time model

Assumptions

- If we have N individuals after T time steps, what determines how many individuals we have after $T + 1$ time steps?
 - A fixed proportion p of the population (on average) survives to be counted at time step $T + 1$
 - Each individual creates (on average) f new individuals that will be counted at time step $T + 1$
- How many individuals do we expect in the next time step?
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- Diagram
- Individuals are **independent**: what I do does not depend on how many other individuals are around
- The population is censused at regular time intervals Δt
 - Usually $\Delta t = 1$ yr
- All individuals are the same at the time of census
- Population changes deterministically

Definitions

- p is the **survival probability**
- f is the **fecundity**
- $\lambda \equiv p + f$ is the **finite rate of increase**
 - ... associated with the time step Δt
 - (Δt has units of time)

Model

- Dynamics:
 - $N_{T+1} = \lambda N_T$
 - $t_{T+1} = t_T + \Delta t$
- Solution:
 - $N_T = N_0 \lambda^T$
 - $t_T = T \Delta t$
- Poll: How does N behave in this model?
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Interpretation

- Assumptions are simplifications based on reality
- We can understand why populations change exponentially sometimes
- We can look for *reasons* when they don't

Examples

- Moths
 - $p = 0$, so $\lambda = f$.
 - * Moths are **semelparous** (reproduce once); they have an **annual** population
- Dandelions
 - If $p > 0$, then the dandelions are **iteroparous**; they are a **perennial** population

1.4 A simple continuous-time model

Assumptions

- If we have N individuals at time t , how does the population change?
 - Individuals are giving birth at per-capita rate b
 - Individuals are dying at per-capita rate d
- How we describe the population dynamics?
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- Individuals are **independent**: what I do does not depend on how many other individuals are around
- The population can be censused at any time
- Population size changes continuously
- All individuals are the same all the time

Definitions

- b is the **birth rate**
- d is the **death rate**
- $r \equiv b - d$ is the **instantaneous rate of increase**.
- These quantities are not associated with a time period, but they have units:

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Model

- Dynamics:

$$- \frac{dN}{dt} = rN$$

- Solution:

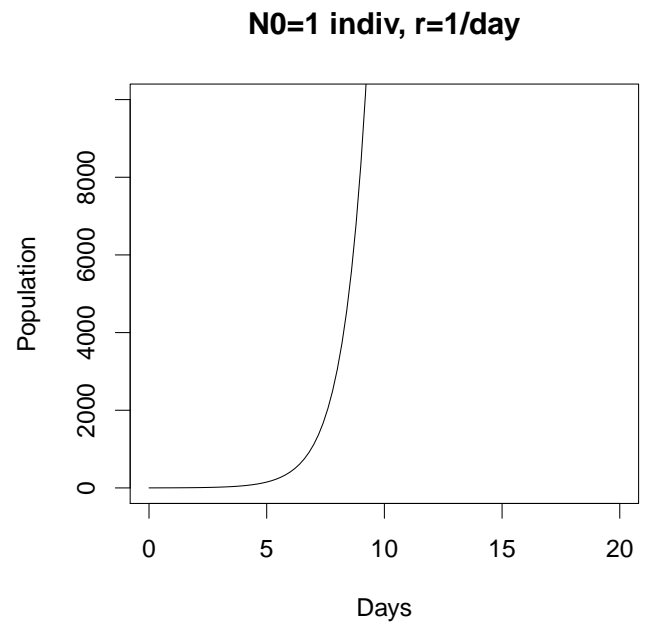
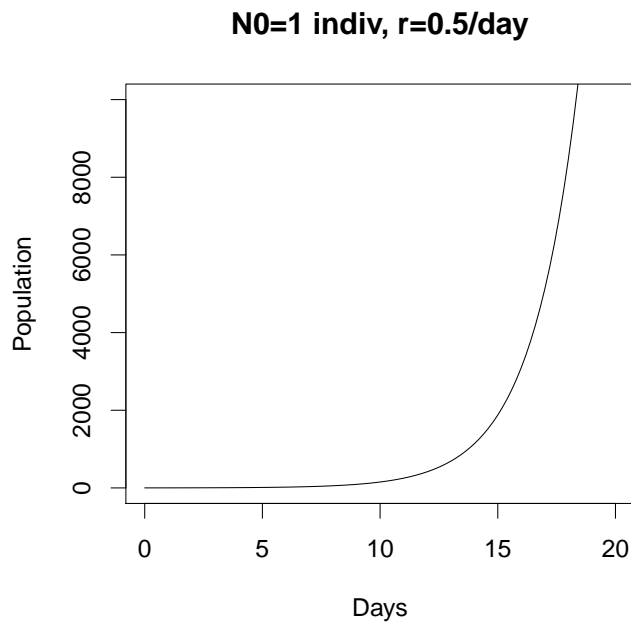
$$- N(t) = N_0 \exp(rt)$$

- Behaviour

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Bacteria

- Conceptually, this is just as simple as the dandelions or the moths
 - In fact, simpler
- But we can't do an infinite number of simulation steps on the computer
 - We need fancier methods



Summary

- We can construct simple, conceptual models and make them into dynamic models
- If we assume that *individuals* behave independently, then
 - we expect *populations* to grow (or decline) exponentially

2 Units and scaling

Units are our friends

- Keep track of units at all times
- Use units to confirm that your answers make sense
 - Or to find quick ways of getting the answer
- What is $3 \text{ day} \cdot 4 \text{ espressoes/day}$?
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- What is $1 \text{ hr} \cdot 0.2 \text{ cm/day}$?
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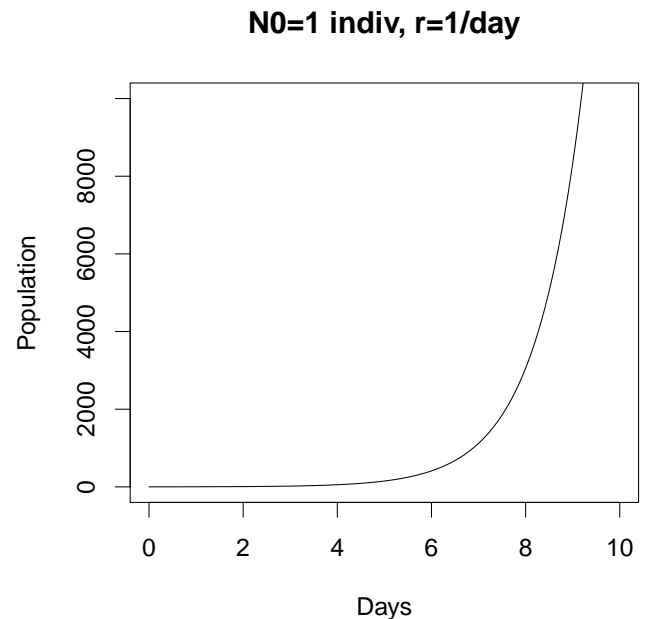
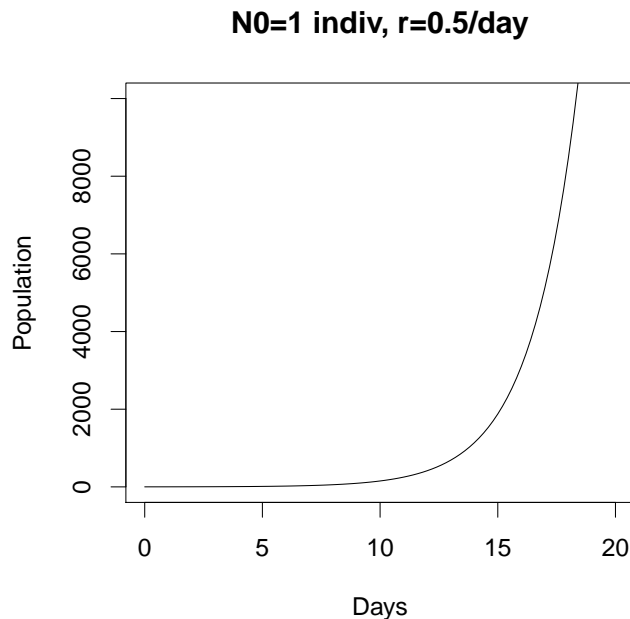
Manipulating units

- We can multiply quantities with different units by keeping track of the units
- We *cannot* add quantities with different units (unless they can be converted to the same units)
- Poll: How many seconds are there in a day?
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- <http://www.alysion.org/dimensional/fun.htm>

Scaling

- Quantities with units set scales, which can be changed
 - If I multiply all the quantities with units of time in my model by 10, I should get an answer that looks the same, but with a different time scale
 - If I multiply all the quantities with units of dandelions in my model by 10, I should get an answer that looks the same, but with a different number of dandelions

Scaling time in bacteria



Thinking about units

- Poll: What is 10^3day ?

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- What is 10^{72}hr ?

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- What is $3\text{ day} \cdot 3\text{ day}$?

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Unit-ed quantities

- Quantities with units *scale*
 - If you change everything with the same units by the same factor, you should not change the behaviour of your system
- We typically make sense of quantities with units by comparing them to other quantities with the same units, e.g.:
 - birth rate vs. death rate
 - characteristic time of exponential growth vs. observation time

Unitless quantities

- Quantities in exponents must be unitless
- Quantities with variable exponents (quantities that can be multiplied by themselves over and over) must be unitless
- Quantities that determine *how* a system behaves must have a unitless form
 - Otherwise, they could be scaled
 - Zero works as a unitless quantity:
 - * $0\text{km} = 0\text{cm}$
- What unitless quantities have we already talked about?

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Moths

- 600 egg/rF
- $\cdot 0.1\text{ larva/egg}$
- $\cdot 0.1\text{ pupa/ larva}$
- $\cdot 0.5\text{ A/ pupa}$
- $\cdot 0.5\text{ rA/ A}$

- Poll: What's the product?

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Closing the loop

- Once we close the loop, it doesn't matter where we start:
 - Reproductive adults to reproductive adults
 - Larvae to larvae
 - Pupae to pupae is common in real studies
- If we don't close the loop, we can't correctly move from step to step

Calculating λ

- $\lambda \equiv p + f$ is the **finite rate of increase**
- If $N_{T+1} = \lambda N_T$, what are the units of λ ?
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- Therefore p and f must be unitless
 - example, rA/rA; seed/seed
 - to do it right, we close the loop

3 Key parameters

3.1 Discrete-time model

- $N_{T+1} = \lambda N_T$
- $\lambda \equiv p + f$

Calculating fecundity

- Fecundity f in our model must be unitless
- Multiply:
 - Probability of surviving from census to reproduction
 - Expected number of offspring when reproducing (maternity)
 - Probability of offspring surviving to census
- Need to end where we started
- Diagram

Calculating survival

- Survival p must be unitless
- Multiply:
 - Probability of surviving from census to reproduction
 - Probability of surviving the reproduction period
 - Probability of surviving until the next census

Finite rate of increase

- Population increases when $\lambda > 1$
- So λ must be unitless
- But it is *associated with* the time step Δt
 - This means it is potentially confusing. It is often better to use \mathcal{R} or r (see below).

Reproductive number

- The reproductive number \mathcal{R} measures the average number of offspring produced by a single individual over the course of its lifetime
- Poll: The population will increase when $\mathcal{R} \dots$:
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- Poll: What are the units of \mathcal{R} ?
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Lifespan

- In this model world, how long do individuals live, on average in this model?
- If p is the proportion of individuals that survive, then the proportion that die is:

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- How many time steps do you expect to survive, on average?

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Calculating \mathcal{R}

- \mathcal{R} is fecundity multiplied by lifespan
- $\mathcal{R} = f/\mu = f/(1 - p)$
- Why do we multiply by time *steps* instead of lifetime?

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Comparison

Lifetime reproduction

- $\mathcal{R} = f/\mu = f/(1 - p)$
- Unitless
- Population behaviour depends on the **comparison** $\mathcal{R} : 1$
 - Equivalent to $f : \mu$

Reproduction over one time step

- $\lambda = f + p = f + (1 - \mu)$
- Unitless
- Population behaviour depends on the comparison $\lambda : 1$
 - Equivalent to $f : \mu$

Is the population increasing?

- What does λ tell us about whether the population is increasing?
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- What does \mathcal{R} tell us about whether the population is increasing?
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- Therefore, these two criteria must be the same!
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3.2 Continuous-time model

Calculating birth rate

- The birth rate b in the continuous-time model is new individuals per individual per unit time
 - An instantaneous rate
 - Units of $[1/\text{time}]$ – implies what assumption?
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Calculating death rate

- The death rate d in the continuous-time model is deaths per individual per unit time
 - An instantaneous rate
 - Units of $[1/\text{time}]$
- Is there a concern with these units?
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Instantaneous rate of increase

- Population increases when $r = b - d > 0$
- r is not unitless, units are:
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- So how can $r = 0$ be a criterion?
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Calculating \mathcal{R}

- The mean lifespan is $L = 1/d$
 - Equivalent to the characteristic time for the death process
- \mathcal{R} is the average number of births expected over that time frame:
 - $\mathcal{R} = bL = b/d$

Comparison

Lifetime reproduction

- $\mathcal{R} = bL = b/d$
- Unitless
- Population behaviour depends on the comparison $\mathcal{R} : 1$
 - Equivalent to $b : d$

Instantaneous change

- $r = b - d$
- Units $[1/t]$ (a rate)
- Population behaviour depends on the comparison $r : 0$
 - Equivalent to $b : d$

Is the population increasing?

- What does r tell us about whether the population is increasing?
 -
- What does \mathcal{R} tell us about whether the population is increasing?
 -
- Therefore, these two criteria must be the same!
 -

3.3 Links

- After one time step in a discrete-time model
 - $N_0 \rightarrow N_0 \lambda$
 - $t \rightarrow t + \Delta t$
- In a continuous model
 - $N_0 \rightarrow N_0 \exp(r\Delta t)$ in the same time period
- To link them, we set:
 - $\lambda = \exp(r\Delta t)$
- In the other direction:
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Characteristic time

- We can now find characteristic times of exponential change:
 - $T_c = 1/r$ for exponential growth when $r > 0$
 - $T_c = -1/r$ for exponential decline when $r < 0$
- Rule of thumb: population changes by a factor of 20 after 3 characteristic times
 - $\exp(3) = 20.1$

4 Growth and regulation

Example: Human population growth

- In the last 50,000 years, the population of **modern humans** has increased from about 1000 to about 7 billion
- What value of r does this correspond to? If we use a time step of 20-year generations, what value of λ does it correspond to?
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Long-term growth rate

- What is the long-term average exponential growth rate (using either r or λ) of:
 - A successful population?
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 - An unsuccessful population?
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Time scales

- Estimated characteristic time scales for exponential growth or decay are usually a few (or a few tens) of generations
 - years to a few kiloyears
- Species typically persist for far longer
 - many kiloyears to megayears

Balance

- If populations grow and shrink proportionally to their size, why don't they go exponentially to zero or infinity?
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- How is this possible?
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Changing growth rates

- Poll: What sort of factors can make species growth rates change?
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Regulation

- What do we expect to happen if a population's growth rate is affected only by seasons and climate?
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- What sort of mechanism could keep a population in a reasonable range for a long time?
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- This is even true for modern humans!