

UNIT 2: Linear population models

1 Constructing models

1.1 Dynamical models

Tools to link scales

- Models are what we use to link:
 - Individual-level to population-level processes
 - Short time scales to long time scales
- In both directions

Assumptions

- Models are always simplifications of reality
 - “The map is not the territory”
 - “All models are wrong, but some are useful”
- Models are useful for:
 - linking assumptions to outcomes
 - identifying where assumptions are broken

Dynamical models

- **Dynamical models** describe rules for how a system changes at each point in time
- We will see what these assumptions about how the system *changes* lead to conclusions about what the system *does* over longer time periods

States and state variables

- Our dynamic models imagine that a system has a **state** at any given time, described by one or more **state variables**
- These are the things that follow our rules and change
- Examples:
 - Dandelions: state is population size, described by one state variable (the number of individuals)
 - Bacteria: state is population density, described by one state variable (the number of individuals per ml)
 - Pine trees: state is amount of wood, described by one state variable (tons per hectare)
- Limiting the number of state variables is key to simple models

Parameters

- **Parameters** are the quantities that describe how the rules for our system work
- Examples:
 - Birth rate, death rate, fecundity, survival probability
- Typically *remain constant* while we are simulating a particular scenario
- *Vary* when we compare different scenarios

How do populations change?

- I survey a population in 2009, and again in 2013. I get a different answer the second time.
- Poll: What are some reasons why this answer might change?
 - **Answer:** Birth
 - **Answer:** Death
 - **Answer:** Immigration and emigration
 - **Answer:** Sampling (ie., my counts are not perfectly correct)

Censusing and intermediate variables

- Often, our population models will imagine that the population is **censused** (counted) at particular periods of time
- Calculations of what happens between census times may be part of how we make our population model, without showing up in the main model itself
 - For example, our moth and dandelion examples

Linear population models

- We will focus mostly on births and deaths
- Births and deaths are done by individuals
 - We model the rate of each individual (per capita rates)
 - Total rate is the per capita rate multiplied by population size
- If per capita rates are constant, we say that our population *models* are **linear**
 - Linear models do not usually correspond to linear growth!
 - Poll: What behaviour do we expect from a linear model?
 - * **Answer:** They usually correspond to exponential growth
 - * **Answer:** ...or exponential decline

1.2 Examples

Moth example

- Poll: State variable
 - Answer: Number of moths/ha
- Parameters
 - Answer: Number of eggs
 - Answer: sex ratio
 - Answer: larval survival, pupal survival, adult survival
 - Answer: Time step
- Census time
 - Answer: Annually; use the same time (and stage) each year

Bacteria

- State variables
 - Answer: Number of bacteria/ml
- Poll: Parameters
 - Answer: Division rate, death rate, washout rate
- Census time
 - Answer: Always!

Dandelions

- State variables
 - Answer: Number of dandelions in a field
 - Poll: Are there intermediate variables?
 - * Answer: Number of seeds
- Parameters
 - Answer: Seed production, survival to adulthood, adult survival
- Census time
 - Answer: Annually, before reproduction
 - Answer: When new and returning individuals are most similar

1.3 A simple discrete-time model

Assumptions

- If we have N individuals after T time steps, what determines how many individuals we have after $T + 1$ time steps?
 - A fixed proportion p of the population (on average) survives to be counted at time step $T + 1$
 - Each individual creates (on average) f new individuals that will be counted at time step $T + 1$
- How many individuals do we expect in the next time step?
 - **Answer:** $N_{T+1} = (pN_T + fN_T) = (p + f)N_T$
- Diagram
- Individuals are **independent**: what I do does not depend on how many other individuals are around
- The population is censused at regular time intervals Δt
 - Usually $\Delta t = 1$ yr
- All individuals are the same at the time of census
- Population changes deterministically

Definitions

- p is the **survival probability**
- f is the **fecundity**
- $\lambda \equiv p + f$ is the **finite rate of increase**
 - ... associated with the time step Δt
 - (Δt has units of time)

Model

- Dynamics:

- $N_{T+1} = \lambda N_T$
- $t_{T+1} = t_T + \Delta t$

- Solution:

- $N_T = N_0 \lambda^T$
- $t_T = T \Delta t$

- Poll: How does N behave in this model?

- **Answer:** Increases exponentially (geometrically) when $\lambda > 1$
- **Answer:** Decreases exponentially when $\lambda < 1$

Interpretation

- Assumptions are simplifications based on reality
- We can understand why populations change exponentially sometimes
- We can look for *reasons* when they don't

Examples

- Moths

- $p = 0$, so $\lambda = f$.
- * Moths are **semelparous** (reproduce once); they have an **annual** population

- Dandelions

- If $p > 0$, then the dandelions are **iteroparous**; they are a **perennial** population

1.4 A simple continuous-time model

Assumptions

- If we have N individuals at time t , how does the population change?

- Individuals are giving birth at per-capita rate b
- Individuals are dying at per-capita rate d

- How we describe the population dynamics?

- **Answer:** $\frac{dN}{dt} = (b - d)N$
- **Answer:** That's what calculus is *for* – describing instantaneous rates of change

- Individuals are **independent**: what I do does not depend on how many other individuals are around
- The population can be censused at any time
- Population size changes continuously
- All individuals are the same all the time

Definitions

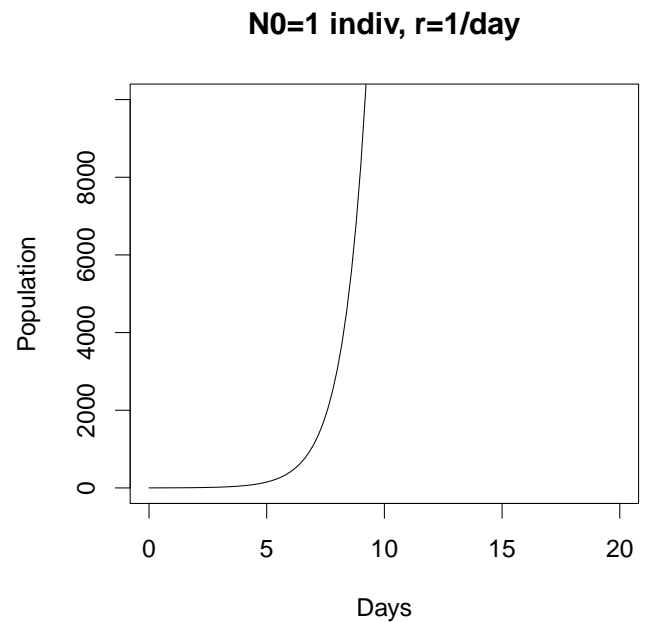
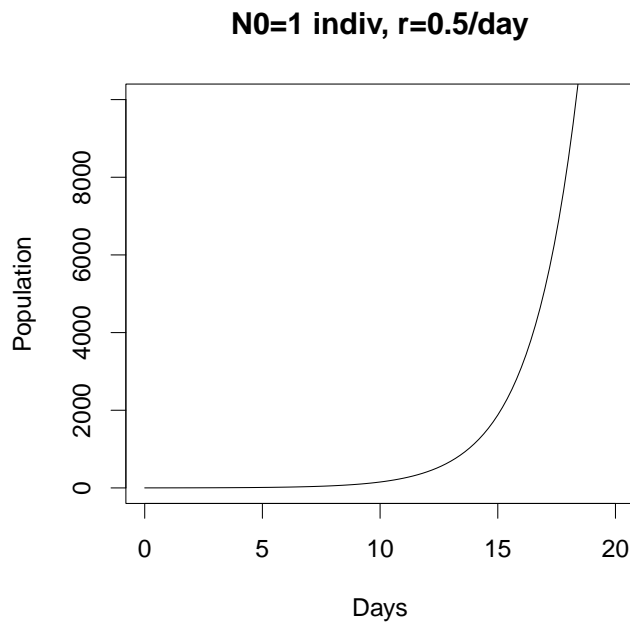
- b is the **birth rate**
- d is the **death rate**
- $r \equiv b - d$ is the **instantaneous rate of increase**.
- These quantities have true units:
 - **Answer:** $1/[\text{time}]$
 - * **Answer:** $\equiv (\text{indiv}/[\text{time}])/\text{indiv}$
- *Comment:* With units, we don't need to mess with "associated with a time period"

Model

- Dynamics:
 - $\frac{dN}{dt} = rN$
- Solution:
 - $N(t) = N_0 \exp(rt)$
- Behaviour
 - **Answer:** Increases exponentially when $r > 0$
 - **Answer:** Decreases exponentially when $r < 0$

Bacteria

- Conceptually, this is just as simple as the dandelions or the moths
 - In fact, simpler
- On the computer, it's a little more complicated to simulate



Summary

- We can construct simple, conceptual models and make them into dynamic models
- If we assume that *individuals* behave independently, then
 - we expect *populations* to grow (or decline) exponentially

2 Units and scaling

Units are our friends

- Keep track of units at all times
- Use units to confirm that your answers make sense
 - Or to find quick ways of getting the answer
- What is $3 \text{ day} \cdot 4 \text{ espressoes/day}$?
 - **Answer:** 12 espressoes
- What is $1 \text{ hr} \cdot 0.2 \text{ cm/day}$?
 - **Answer:** $1 \text{ hr} \cdot 0.2 \text{ cm/day}$
 - **Answer:** $1 \text{ hr} \cdot 0.2 \text{ cm/day} \cdot \frac{1 \text{ day}}{24 \text{ hr}}$
 - **Answer:** 0.0083 cm

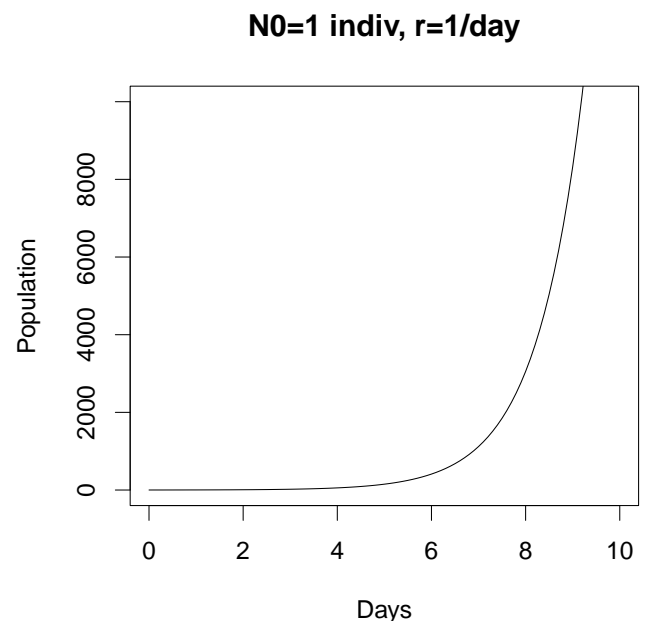
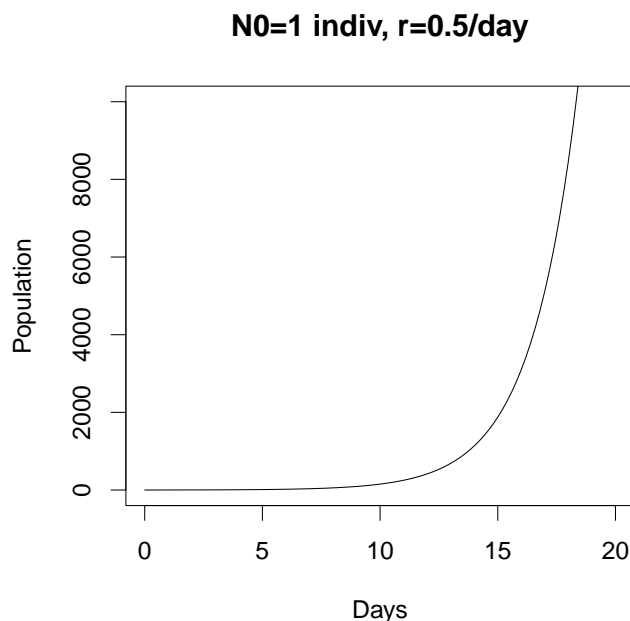
Manipulating units

- We can multiply quantities with different units by keeping track of the units
- We *cannot* add quantities with different units (unless they can be converted to the same units)
- Poll: How many seconds are there in a day?
 - Answer: $\frac{60 \text{ sec}}{\text{min}} \cdot \frac{60 \text{ min}}{\text{hr}} \cdot \frac{24 \text{ hr}}{\text{day}}$
 - Answer: 86400 sec/day
- <http://www.alysion.org/dimensional/fun.htm>

Scaling

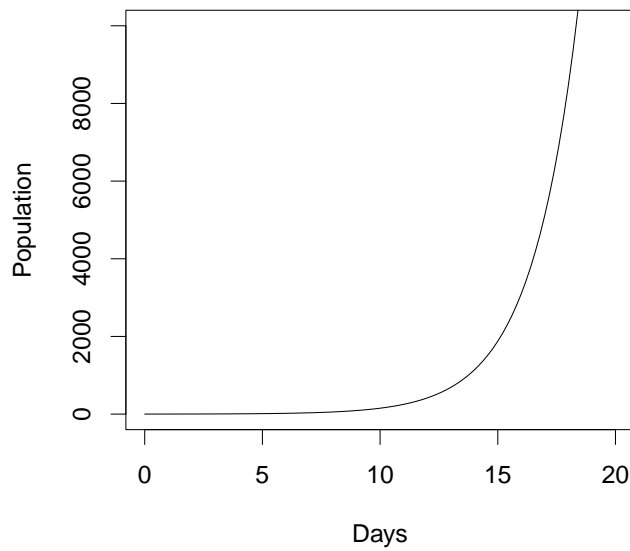
- Quantities with units set scales, which can be changed
 - If I multiply all the quantities with units of time in my model by 10, I should get an answer that looks the same, but with a different time scale
 - If I multiply all the quantities with units of dandelions in my model by 10, I should get an answer that looks the same, but with a different number of dandelions

Scaling time in bacteria

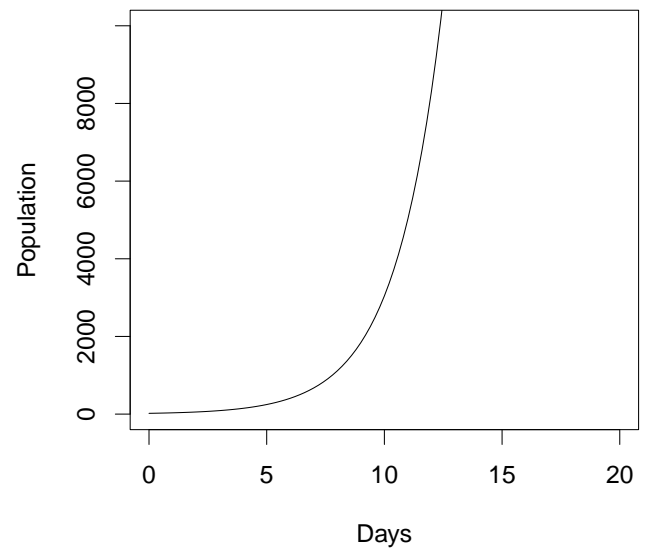


Answer slide: *Scaling population*

$N_0=1$ indiv, $r=0.5/\text{day}$

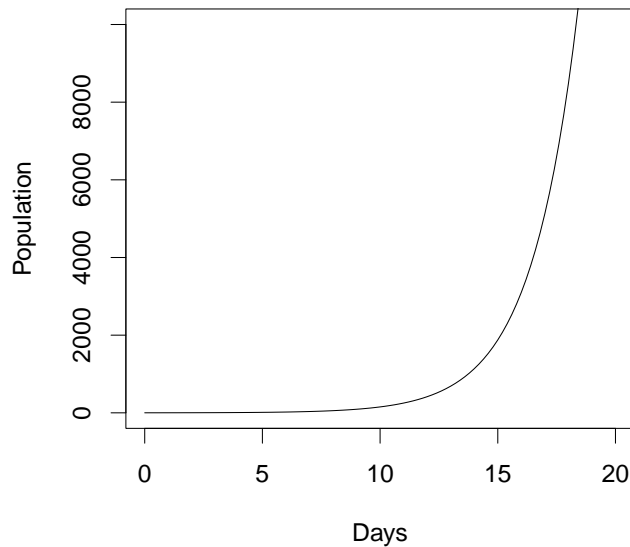


$N_0=20$ indiv, $r=0.5/\text{day}$

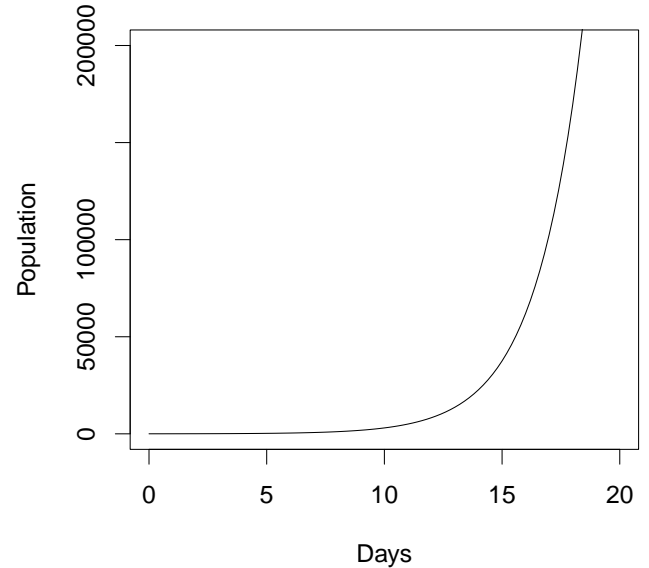


Answer slide: *Scaling population*

$N_0=1$ indiv, $r=0.5/\text{day}$



$N_0=20$ indiv, $r=0.5/\text{day}$



Thinking about units

- Poll: What is 10^3 day ?
 - Answer postponed:
- What is 10^{72} hr ?

- **Answer:** Nonsense! 72 hr means *exactly* the same thing as 3 day – there is no way to resolve this to make sense.
- What is 3 day · 3 day?
 - **Answer:** 9 day² – this *could* make sense, but it’s probably wrong
 - **Answer:** ... very different from 9 day.

Unit-ed quantities

- Quantities with units *scale*
 - If you change everything with the same units by the same factor, you should not change the behaviour of your system
- We typically make sense of quantities with units by comparing them to other quantities with the same units, e.g.:
 - birth rate vs. death rate
 - characteristic time of exponential growth vs. observation time

Unitless quantities

- Quantities in exponents must be unitless
- Quantities with variable exponents (quantities that can be multiplied by themselves over and over) must be unitless
- Quantities that determine *how* a system behaves must have a unitless form
 - Otherwise, they could be scaled
 - Zero works as a unitless quantity:
 - * 0km = 0cm
- What unitless quantities have we already talked about?
 - **Answer:** λ , f and p .
 - **Answer:** These all depend on a time period

Moths

- 600 egg/ rF
- ·0.1 larva/ egg
- ·0.1 pupa/ larva
- ·0.5 A/ pupa
- ·0.5 rA/ A

- Poll: What's the product?
 - **Answer:** $1.5 \text{ rA} / \text{rF}$
 - **Answer:** Not enough information to make a prediction!
 - **Answer:** Need to multiply by something with units rF / rA to close the loop

Closing the loop

- Once we close the loop, it doesn't matter where we start:
 - Reproductive adults to reproductive adults
 - Larvae to larvae
 - Pupae to pupae is common in real studies
 - * **Answer:** Pupae are easy to count
 - * **Answer:** Egg masses, too (depending on species)
- If we don't close the loop, we can't correctly move from step to step

Calculating λ

- $\lambda \equiv p + f$ is the **finite rate of increase**
- If $N_{T+1} = \lambda N_T$, what are the units of λ ?
 - **Answer:** We multiply by λ over and over
 - **Answer:** Therefore λ must be unitless
- Therefore p and f must be unitless
 - example, rA / rA ; seed/seed
 - to do it right, we close the loop

3 Key parameters

3.1 Discrete-time model

- $N_{T+1} = \lambda N_T$
- $\lambda \equiv p + f$

Calculating fecundity

- Fecundity f in our model must be unitless
- Multiply:
 - Probability of surviving from census to reproduction
 - Expected number of offspring when reproducing (maternity)
 - Probability of offspring surviving to census
- Need to end where we started
- Diagram

Calculating survival

- Survival p must be unitless
- Multiply:
 - Probability of surviving from census to reproduction
 - Probability of surviving the reproduction period
 - Probability of surviving until the next census

Finite rate of increase

- Population increases when $\lambda > 1$
- So λ must be unitless
- But it is *associated with* the time step Δt
 - Potentially confusing. It is often better to use \mathcal{R} or r (see below).

Reproductive number

- The reproductive number \mathcal{R} measures the average number of offspring produced by a single individual over the course of its lifetime
- Poll: The population will increase when $\mathcal{R} \dots$:
 - **Answer:** $\mathcal{R} > 1$
- Poll: What are the units of \mathcal{R} ?
 - **Answer:** \mathcal{R} must be unitless

Lifespan

- In this model world, how long do individuals live, on average?
- If p is the proportion of individuals that survive, then the proportion that die is:
 - Answer: $\mu = 1 - p$
- How many time steps do you expect to survive, on average?
 - Answer: $1/\mu$
 - * Answer: Roughly makes sense, and is also right (but I'm not proving it)
 - Answer: Average lifetime is $1/\mu * \Delta t$

Calculating \mathcal{R}

- \mathcal{R} is fecundity multiplied by lifespan
- $\mathcal{R} = f/\mu = f/(1 - p)$
- Why do we multiply by time *steps* instead of lifetime?
 - Answer: Because f is also measured per time step
 - Answer: \mathcal{R} must be unitless

Comparison

Lifetime reproduction

- $\mathcal{R} = f/\mu = f/(1 - p)$
- Unitless
- Population behaviour depends on the **comparison** $\mathcal{R} : 1$
 - Equivalent to $f : \mu$

Reproduction over one time step

- $\lambda = f + p = f + (1 - \mu)$
- Unitless
- Population behaviour depends on the comparison $\lambda : 1$
 - Equivalent to $f : \mu$

Is the population increasing?

- What does λ tell us about whether the population is increasing?
 - **Answer:** Population is increasing each time step when $\lambda > 1$
- What does \mathcal{R} tell us about whether the population is increasing?
 - **Answer:** Population is increasing when $\mathcal{R} > 1$. Each individual is (on average) more than replacing itself over its lifetime
- Therefore, these two criteria must be the same!
 - **Answer:** Both come down to $f > \mu$.

3.2 Continuous-time model

Calculating birth rate

- The birth rate b in the continuous-time model is new individuals per individual per unit time
 - An instantaneous rate
 - Units of $[1/\text{time}]$ – implies what assumption?
 - * **Answer:** New individuals are cancelling with old individuals in the equation
 - * **Answer:** New individuals are being treated the same as old individuals
 - * **Answer:** Not very realistic – a potential problem with our model world

Calculating death rate

- The death rate d in the continuous-time model is deaths per individual per unit time
 - An instantaneous rate
 - Units of $[1/\text{time}]$
- Is there a concern with these units?
 - **Answer:** Not really. The individuals dying are exactly the same ones we're counting.

Instantaneous rate of increase

- Population increases when $r = b - d > 0$
- r is not unitless, units are:
 - **Answer:** $[1/\text{time}]$
- Poll: So how can $r = 0$ be a criterion?
 - **Answer:** Because $0 \times \text{anything}$ is unitless!
 - **Answer:** Does $0\text{km} = 0\text{cm}$?

Calculating \mathcal{R}

- The mean lifespan is $L = 1/d$
 - Equivalent to the characteristic time for the death process
- \mathcal{R} is the average number of births expected over that time frame:
 - $\mathcal{R} = bL = b/d$

Comparison

Lifetime reproduction

- $\mathcal{R} = bL = b/d$
- Unitless
- Population behaviour depends on the comparison $\mathcal{R} : 1$
 - Equivalent to $b : d$

Instantaneous change

- $r = b - d$
- Units $[1/t]$ (a rate)
- Population behaviour depends on the comparison $r : 0$
 - Equivalent to $b : d$

Is the population increasing?

- What does r tell us about whether the population is increasing?
 - **Answer:** Population is increasing at any particular time step when $r > 0$
- What does \mathcal{R} tell us about whether the population is increasing?
 - **Answer:** Population is increasing when $\mathcal{R} > 1$. Each individual is (on average) more than replacing itself over its lifetime
- Therefore, these two criteria must be the same!
 - **Answer:** Both come down to $b > d$.

3.3 Links

- After one time step in a discrete-time model
 - $N_0 \rightarrow N_0 \lambda$
 - $t \rightarrow t + \Delta t$
- In a continuous model
 - $N_0 \rightarrow N_0 \exp(r\Delta t)$ in the same time period
- To link them, we set:
 - $\lambda = \exp(r\Delta t)$
- In the other direction:
 - **Answer:** $r = \log_e(\lambda)/\Delta t$

Characteristic time

- We can now find characteristic times of exponential change:
 - $T_c = 1/r$ for exponential growth when $r > 0$
 - $T_c = -1/r$ for exponential decline when $r < 0$
- Rule of thumb: population changes by a factor of 20 after 3 characteristic times
 - $\exp(3) = 20.1$

4 Growth and regulation

Example: Human population growth

- In the last 50,000 years, the population of **modern humans** has increased from about 1000 to about 7 billion
- What value of r does this correspond to? If we use a time step of 20-year generations, what value of λ does it correspond to?
 - **Answer:** $N(t) = N(0) \exp(rt)$
 - * **Answer:** $r = \log_e(N/N(0))/t$
 - * **Answer:** $r = \log_e(7000000000/1000)/50000 \text{ yr} = 0.0003/\text{yr}$
 - **Answer:** $N_T = N_0 \lambda^T$
 - * **Answer:** $T = t/\Delta t = 50000 \text{ yr}/20 \text{ yr} = 2500$
 - * **Answer:** $\lambda = (N_T/N_0)^{1/T}$
 - * **Answer:** $\lambda = (7000000000/1000)^{1/2500} = 1.006$

Long-term growth rate

- What is the long-term average exponential growth rate (using either r or λ) of:
 - A successful population?
 - * **Answer:** Very close to $r = 0$ or $\lambda = 1$
 - * **Answer:** But a little larger
 - An unsuccessful population?
 - * **Answer:** *Probably* very close to $r = 0$ or $\lambda = 1$
 - * **Answer:** But a little smaller
 - * **Answer:** If more than a little, it would probably be gone by now!

Summary

- We can make simple model worlds where populations are composed of individuals that reproduce and die independently
 - Discrete or continuous time
- We can do structured closed-loop calculations and predict how these populations will change
- If individuals are independent, we expect populations to change exponentially through time
 - **Answer:** The rate at which the population changes is proportional to the size of the population