

UNIT 3 Non-linear population models

1 Introduction

- In **linear** population models, per capita rates are independent of population size
- In **non-linear** models, not so much
- Why might per capita birth and death rates change with population size?
- What does this imply about population **dynamics**?

The first law of population dynamics

- If individuals are behaving independently:
 - the population-level rate of growth (or decline) is proportional to the population size
 - the population grows (or declines) exponentially

The second law of population dynamics

- Exponential growth (or decline) cannot continue forever
- Something is changing the average rate at which populations we observe grow

The third law of population dynamics

- Exponential growth (or decline) cannot continue forever – *even on average*
- Environmental variation cannot be the only thing that changes growth rates
- Populations are, directly or indirectly, limiting their own growth rates
- This is called **density dependence**

Long-term growth rates

- Populations maintain long-term growth rates very close to $r = 0$
- This is almost certainly because factors affecting their growth rate change with the size of population.

Changing growth rates

- Poll: What is an example of a density-dependent mechanism that affects growth rate?

—
*
—
*
—
*
*
—

Population regulation

- All the populations we see are *regulated*
 - On average, population growth is higher when the population is lower
 - Maybe with a time delay
- Why is this interesting?
 - Lots of populations don't *look like* they are regulated

Sometimes regulation is apparent

- Some species seem to fill a niche (mangroves)
- or deplete their own food resources (gypsy moths)

Sometimes regulation is not apparent

- Other species seem like they could easily be more common (pine trees)
 - May be controlled by cryptic (hard to see) natural enemies (like disease or parasites)
 - May be controlled by limitations that occur only at certain times (e.g., during regular droughts)

Regulation works over the long term

- Not every species is experiencing population regulation at every time
- A species that we see now may be expanding into a niche (e.g., because of climate change)
- Some species are controlled by big outbreaks of disease
- Some species have big outbreaks into marginal habitat, and spend most of their time contracting back to their “core” habitat

How do we know it's regulation?

- Poll: Why don't we believe that population growth is controlled by factors that don't depend on the population itself?

—
—
—
—

1.1 Population Examples

Gypsy moths

- Poll: What are some factors that limit gypsy-moth populations?
- Which are likely to be affected by the moths?
 - Directly or indirectly, in the short or long term?

2 Continuous-time regulation

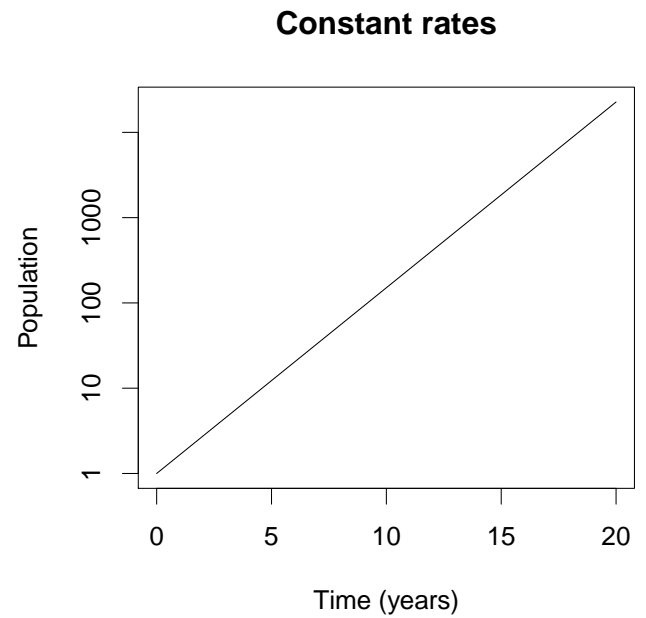
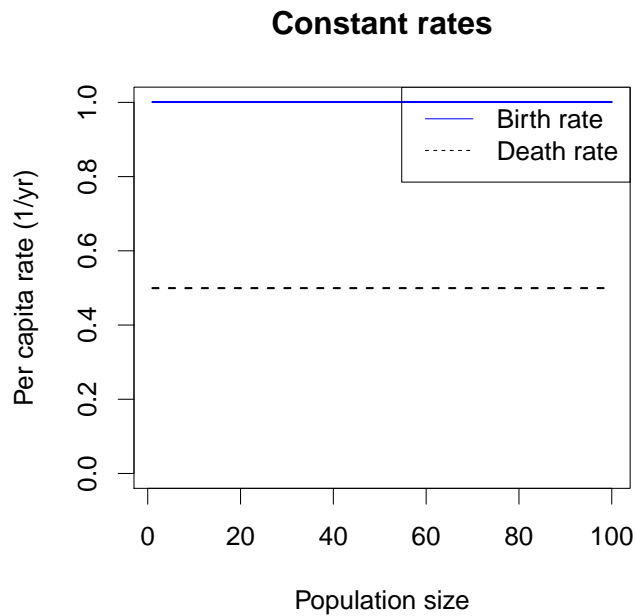
Build on the linear model

- Our linear population model is:

$$- \frac{dN}{dt} = (b - d)N$$

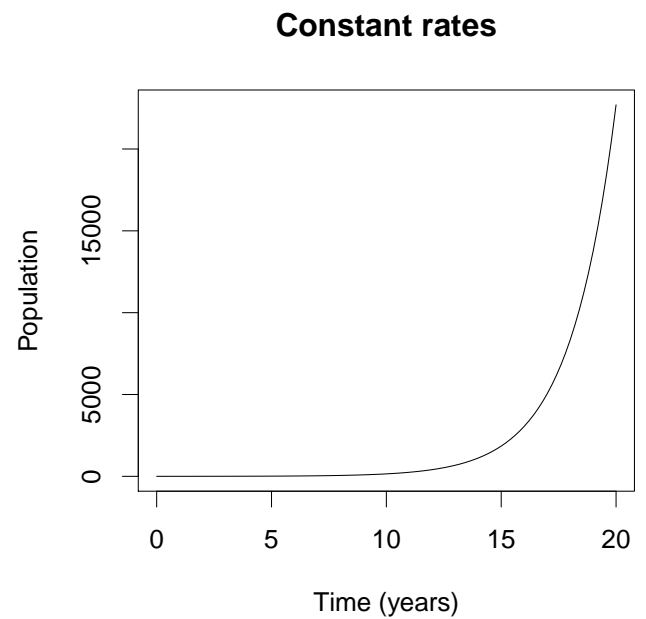
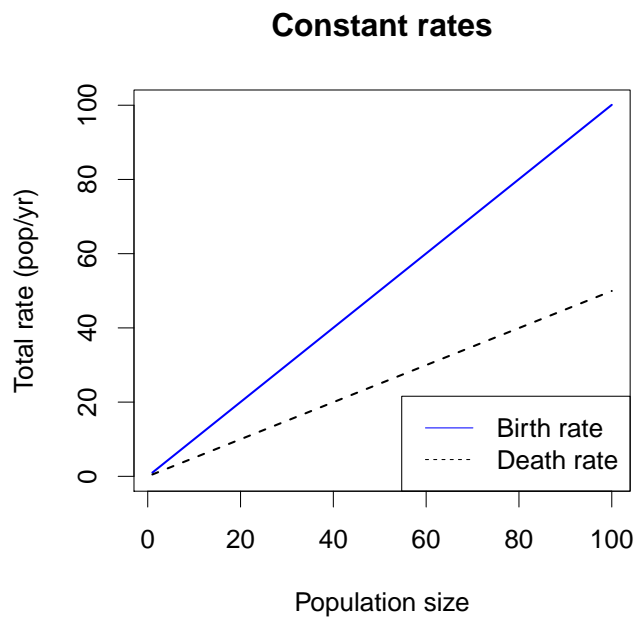
- Per-capita rates are constant
- **Population-level rates are linear**
- Behaviour is exponential

Individual perspective



- Per capita rate shows birth and death per individual
- Corresponds to the time plot showing growth on a log scale
 - On the log scale we see *multiplicative* or *proportional* change

Population perspective



- Total rate shows birth and death for the whole population
- Corresponds to the time plot showing growth on a linear scale
 - On the linear scale we see *additive* or *absolute* change

Non-linear model

- Population has *per capita* birth rate $b(N)$ and death rate $d(N)$
 - Per-capita rates change with the population size
- Our non-linear model is: $\frac{dN}{dt} = (b(N) - d(N))N \equiv r(N)N$
 - Defines how fast the population is changing at any instant

Recruitment

- **Recruitment** is when an organism moves from one life stage to another:
 - Seed \rightarrow seedling \rightarrow sapling \rightarrow tree
 - Egg \rightarrow larva \rightarrow pupa \rightarrow moth
- In simple continuous-time population models, recruitment is included in birth:
 - b is the rate at which adults produce new adults; or seeds produce new seeds – we have to “close the loop”

Birth rates

- When a population is crowded, the birth rate will usually go down
 - Resources are limited: space, food, light
- But it may stay the same
- Or even go up
 - If individuals shift their resources to reproduction instead of survival

Death rates

- When a population is crowded, the death rate will often go up
 - Individuals are starving, or conflict increases
 - But it may stay the same
 - * if reproduction is limited by competition for breeding sites, or by recruitment of juveniles
 - Or even go down
 - * if organisms go into some sort of “resting mode”

Reproductive numbers

- Our model is: $\frac{dN}{dt} = (b(N) - d(N))N \equiv r(N)N$
- Reproductive number now also changes with N :

—

- When the population is crowded, individuals are stressed and the reproductive number will typically go down.

Carrying capacity

- If a population has $\mathcal{R}(N) > 1$ when it's not crowded
- The population should increase
- Eventually, \mathcal{R} will decrease, and eventually cross $\mathcal{R} = 1$
- We call the special value of N where $\mathcal{R}(N) = 1$, the **carrying capacity**, K
 - $\mathcal{R}(K) \equiv 1$
 - $b(K) \equiv d(K)$
- When $N = K$:

—

Logistic model

- A popular model of density-dependent growth is the logistic model
- Per capita instantaneous growth rate r is a function of N
 - $r(N) = r_{\max}(1 - N/K)$
 - Consistent with various assumptions about $b(N)$ and $d(N)$
- Population increases to K and remains there
 - Units of N must match units of K
- Not a linear model, because *population-level* rates are not linear

Exponential-rates model

- In this course, we'll mostly use another simple model:
 - $b(N) = b_0 \exp(-N/N_b)$
 - $d(N) = d_0 \exp(N/N_d)$
- This is the simplest model that is smooth and keeps track of birth and death rates separately
 - Birth rate goes down with characteristic scale N_b
 - Death rate goes up with characteristic scale N_d

Exponential-rates vs. logistic

- The exponential-rates model is conceptually clearer
 - Birth and death rates are clearly defined
- Mathematically nicer
 - Rates always stay positive
- The logistic *looks* less scary

2.1 A simple, continuous-time model

Assumptions

- We model individual-level rates, but individuals are *not* independent: my rates depend on the number (or density) of individuals in the population
- The population can be censused at any time
- Population size changes continuously
- All individuals are the same all the time
- Population changes deterministically

Interpretation

- If we have N individuals at time t , how does the population change?
 - Individuals are giving birth at per-capita rate $b(N)$
 - Individuals are dying at per-capita rate $d(N)$
- Population dynamics follow:

$$- \frac{dN}{dt} = (b(N) - d(N))N \equiv r(N)N$$

States and state variables

- What variable or variables describe the state of this system?
 -
 -
 - *

Parameters

- Poll: What quantities describe the rules for this system?
 -
 -
 -
 -

Characteristic *scale*

- A characteristic scale for density dependence is analogous to a characteristic time
 - It is a *parameter* with the same units as a state variable, and sets a standard by which we measure that variable
 - Nothing (with units) is big or small until it's compared to something with the same units
- For example: $b(N) = b_0 \exp(-N/N_b)$
 - N_b is the characteristic scale of density-dependence in birth rate
 - If $N \ll N_b$, density dependence is linear (and relatively small)
 - If $N \gg N_b$, density dependence is exponential, and very large (virtually no births)

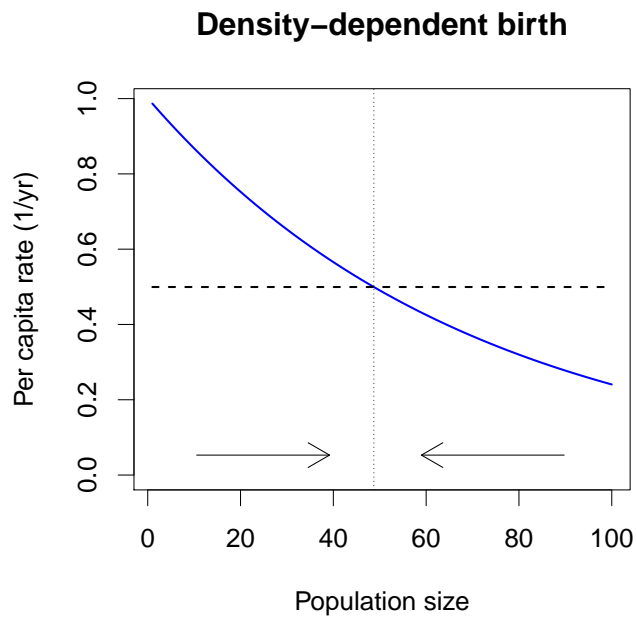
Model

- Dynamics:
 - $\frac{dN}{dt} = (b_0 \exp(-N/N_b) - d_0 \exp(N/N_d))N$
- Exact solution:
 - Insanely complicated
- Behaviour of the solution:
 - Pretty easy!

Dynamics

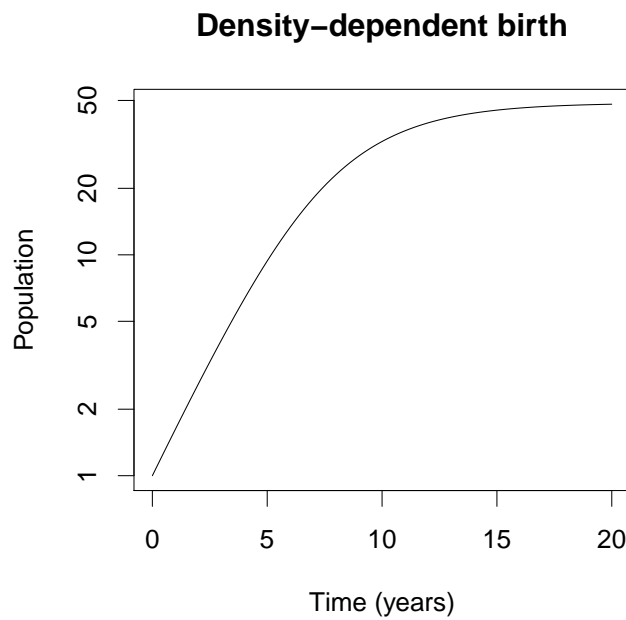
- What sort of **dynamics** do we expect from our conceptual model?
 - I.e., how will it change through time?
- What will the population do if it starts
 - near zero?
 - near the equilibrium?
 - at a high value?

What will this model do?



- Increase when population is below equilibrium
- Decrease when population is above equilibrium
- Converge

Examples



2.2 Simulating model behaviour

Simulations

- We will simulate the behaviour of populations in continuous time using the program R
- This program generates the pictures in this section by implementing our model of how the population changes instantaneously

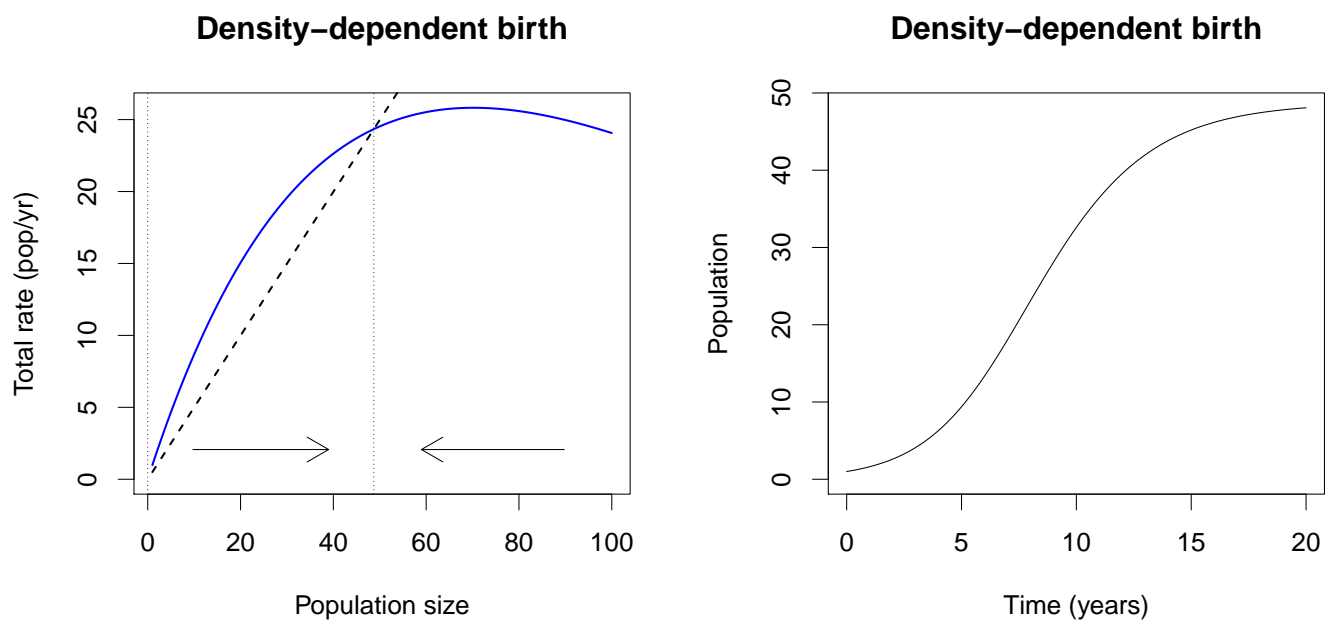
Individual-scale pictures

- We can view graphs of our population assumptions on the individual scale
 - per-capita birth and death rates
 - * units $[1/\text{time}]$
 - what is each individual doing (on average)?
 - corresponds to the dynamics we visualize on a log-scale graph of the population
 - See above

Population-scale pictures

- We can view graphs of our population assumptions on the population scale
 - total birth and death rates
 - * units $[\text{indiv}/\text{time}]$
 - * or $[\text{density}/\text{time}] = [(\text{indiv}/\text{area})/\text{time}]$
 - what is changing at the population level?
 - corresponds to the dynamics we visualize on a linear-scale graph of the population

Population perspective picture



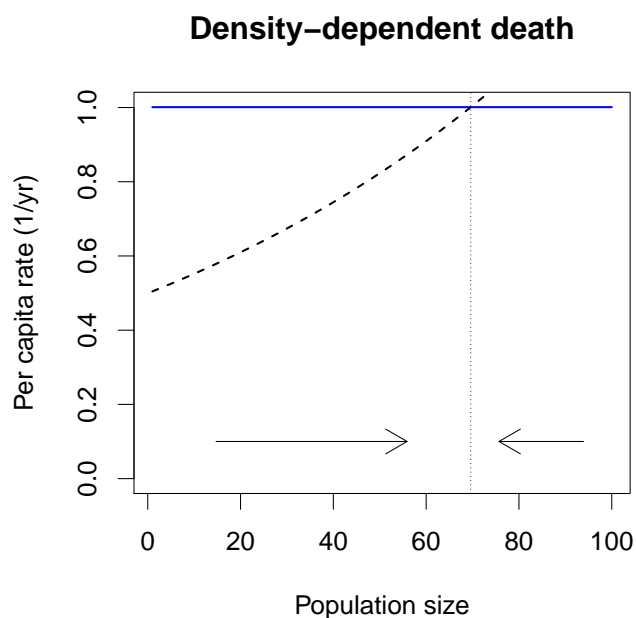
Decreasing birth rate

- Decreasing birth rate (above) might be a good model for organisms that experience density dependence primarily in the recruitment stage
- For example, we might count adult trees, and these might not die more at high density – just fail to recruit younger ones

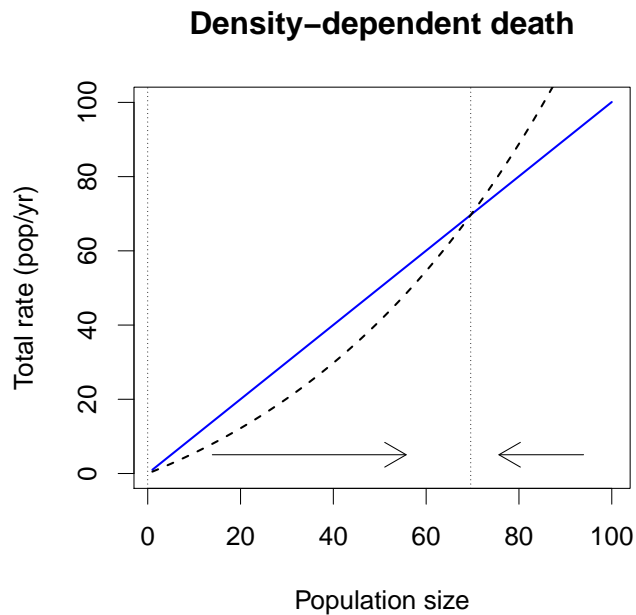
Increasing death rate

- Increasing death rate might be a good model for organisms that experience density dependence primarily as adults
- For example, in some environments, mussel density might be limited by adult crowding. Although juvenile mussels tend to have a hard time, this might not be density dependent

Individual perspective



Population perspective

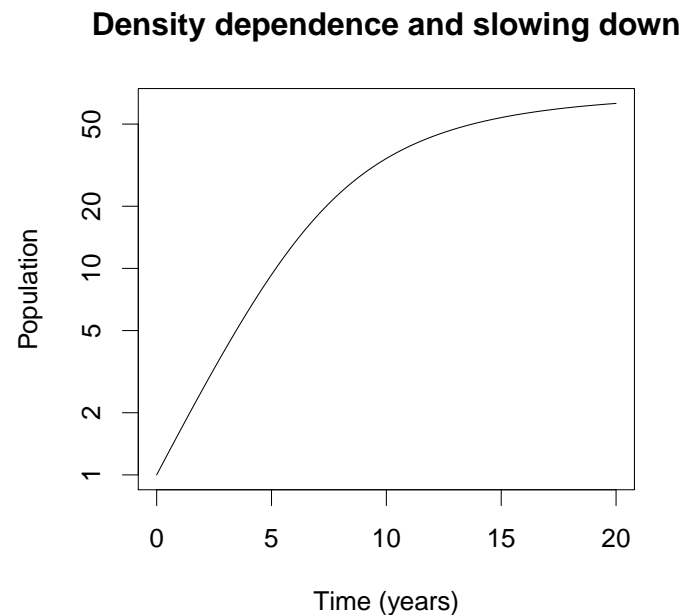
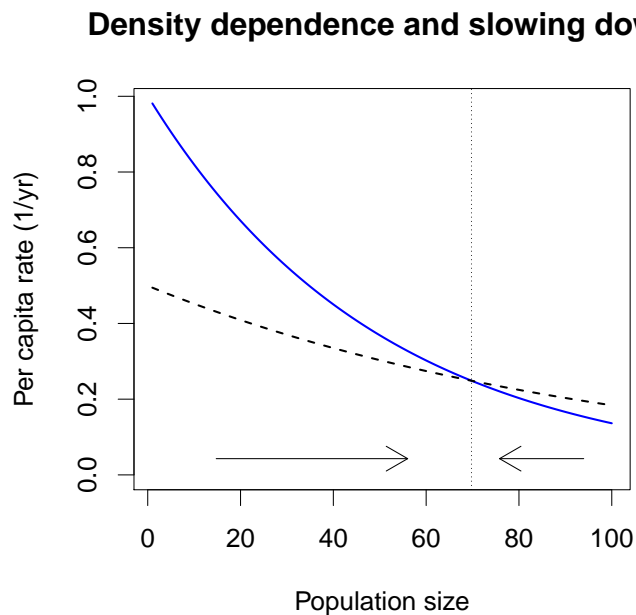


Decreasing death rate

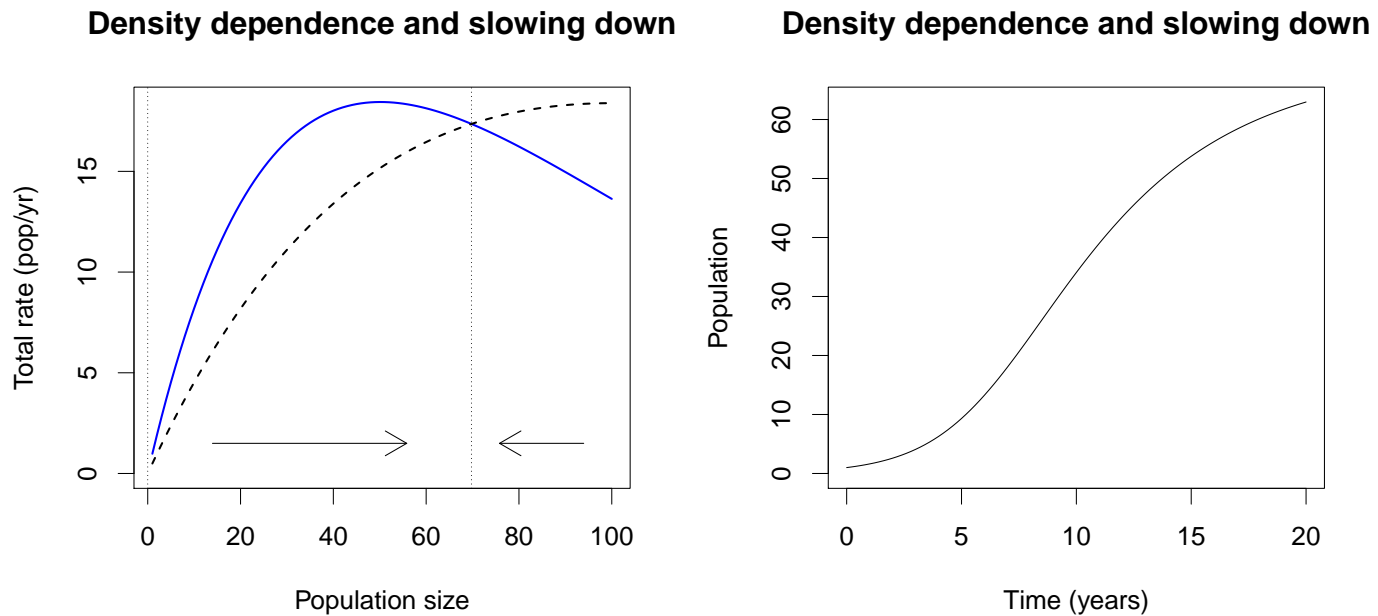
- Some organisms (such as many types of bacteria) slow down their metabolisms under density dependence, so that death rate *decreases*
- How is this consistent with density dependence?

—

Individual perspective



Population perspective



Other examples

- There are two other possible scenarios for density dependence
 - For fun, you can try to think of what they are
- But all of these examples have similar behaviour
 - Increase from low density
 - Decrease from high density
 - Approach carrying capacity

Maximum growth rates

- When does a population in this model have the fastest *per-capita* growth rate?
 -
 -
- When does a population in this model have the fastest *total* growth rate?
 -
 -

2.3 Equilibria and time scales

- We define **equilibrium** as when the population is not changing
- Our simple model is $\frac{dN}{dt} = (b(N) - d(N))N$
- In this simple model, when does equilibrium occur?
 -
 -

Stable and unstable equilibria

- Aren't equilibria always stable?
 - If we are at an equilibrium we expect to stay there
 - (in our simplified model, at least)
- An equilibrium is defined as stable if we expect to approach the equilibrium *when we are near it*.
- An equilibrium is defined as unstable if we expect to move away from the equilibrium *when we are near it*.

What kind of equilibrium?

- How can we tell an equilibrium is stable?
 - If population is just below the equilibrium:
 - *
 - If population is just above the equilibrium:
 - *

Basic reproductive number

- The reproductive number of a population not affected by crowding is called the **basic reproductive number**
 - Written \mathcal{R}_0 or \mathcal{R}_{\max} .
- In this model, when $\mathcal{R}_0 < 1$ the population:
 -
- When $\mathcal{R}_0 > 1$ the population:
 -
 -

- Poll: What is \mathcal{R}_0 in our current model?

—

- Does this make sense?

—

—

*

Invasion

- We say a species can “invade” a system if its rate of change is positive when the population is small.
- In other words, population can invade if the extinction equilibrium is not stable
- In this conceptual model, this is the same as saying $b(0) > d(0)$
- Which is the same as saying $\mathcal{R}_0 > 1$

Invasion examples

- Poll: What are some examples of biological invasions?

—

—

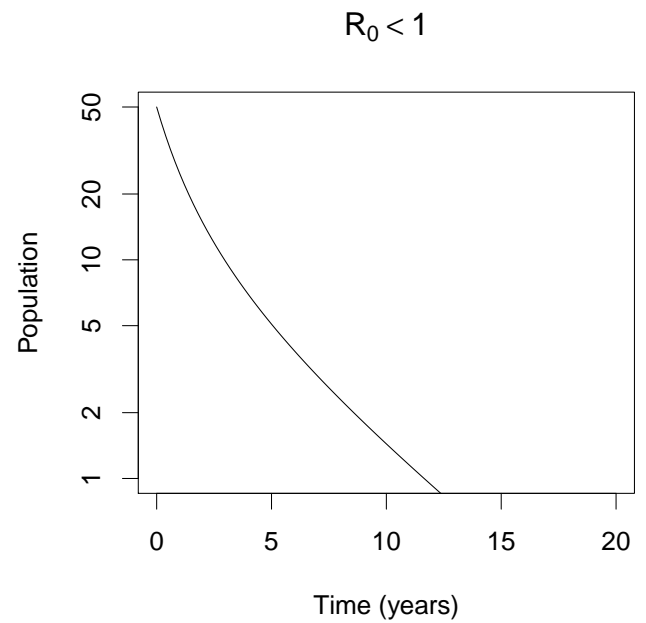
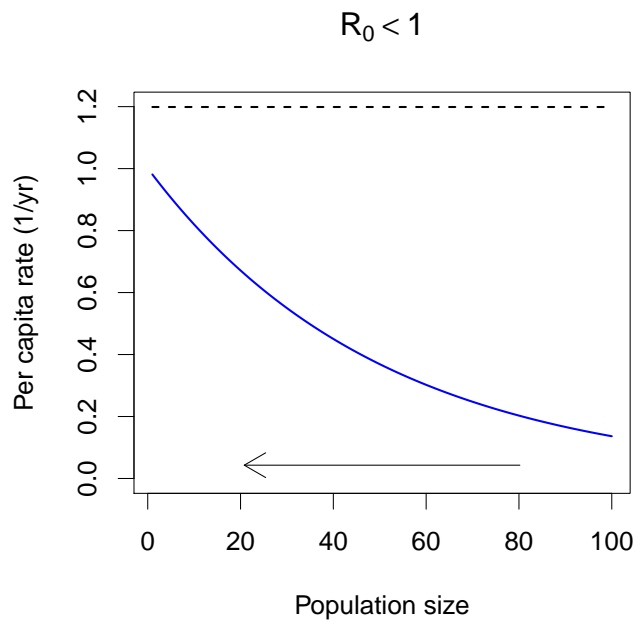
—

—

Different behaviours

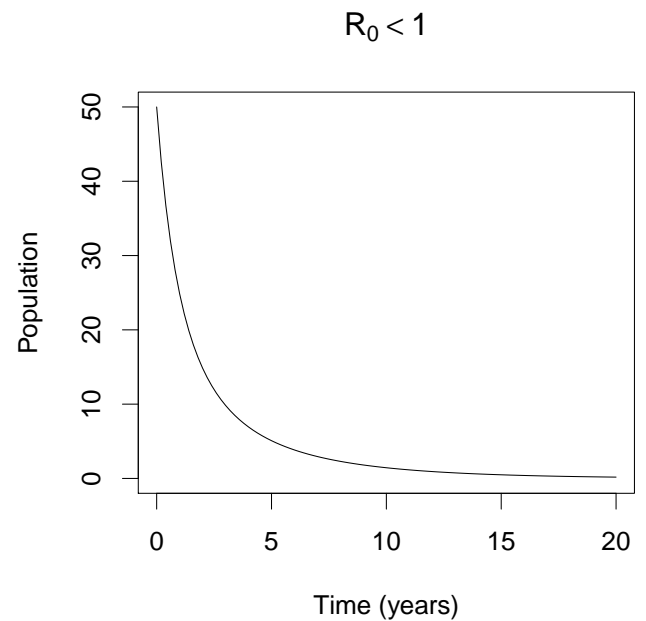
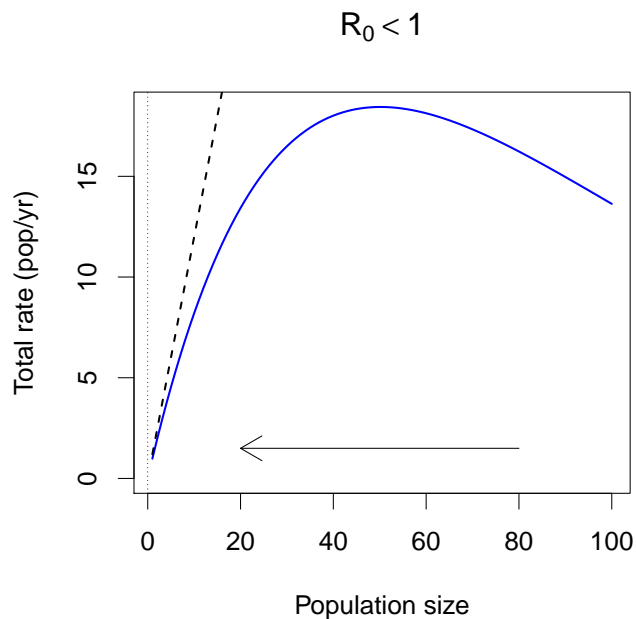
- When $\mathcal{R}_0 > 1$, the population invades
 - Zero equilibrium is unstable, carrying capacity equilibrium is stable
- When $\mathcal{R}_0 < 1$, the population fails to invade
 - Zero equilibrium is stable, carrying capacity equilibrium does not exist

Individual perspective



- When $R_0 < 1$ population always decreases

Population perspective



- When $R_0 < 1$ population always decreases

R_0 and thresholds

- A population with $R_0 < 1$ in general cannot survive in an area

- As conditions get worse for a species in a particular area, or along a particular gradient:
 - It will suddenly disappear at the population level
 - Even while it can still survive and reproduce at an individual level
- This is why there are no white spruce trees in Cootes Paradise
- And no malaria in the mainland United States

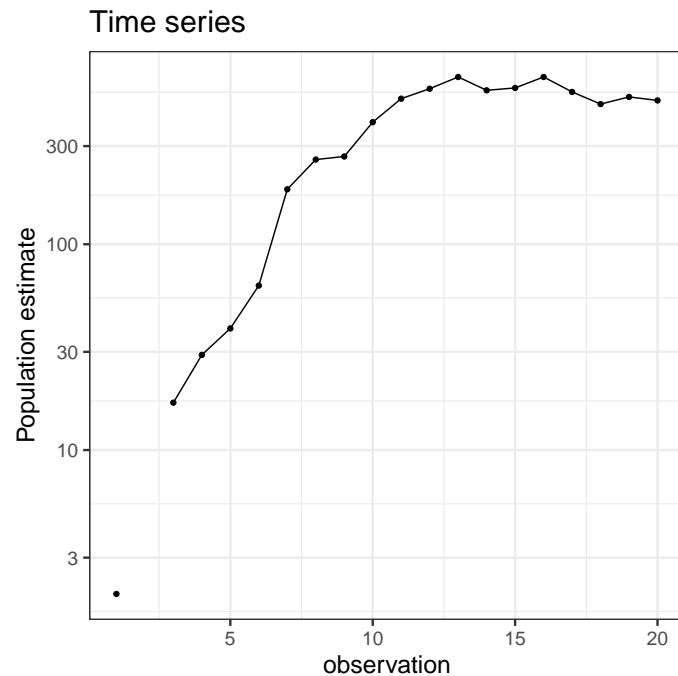
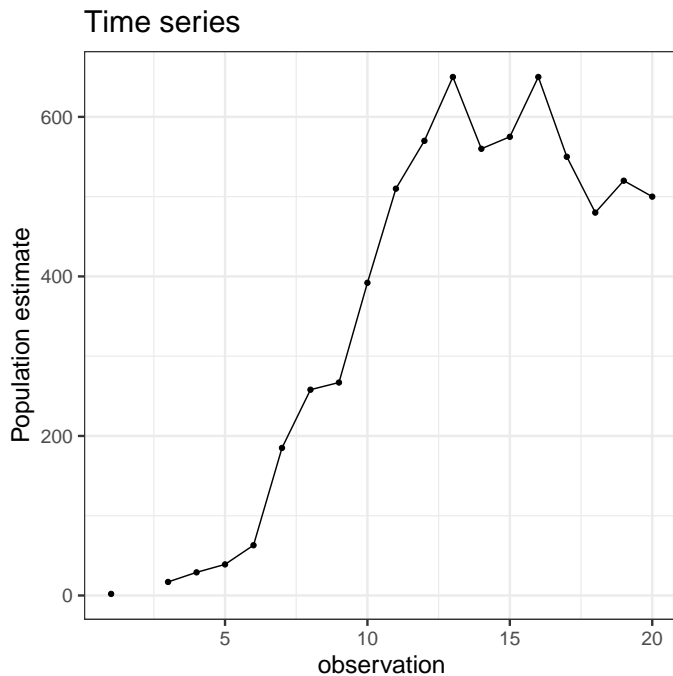
Characteristic times

- Just like in the simple model, an equilibrium will have a characteristic time
- If I'm close to an equilibrium, how long would it take:
 - to go the distance to the equilibrium at my current “speed”
 - to actually get e times closer, or e times farther

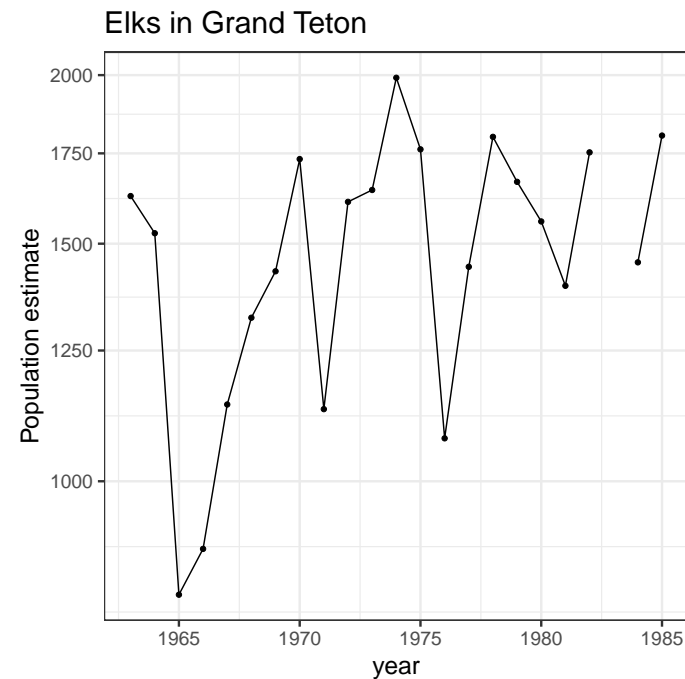
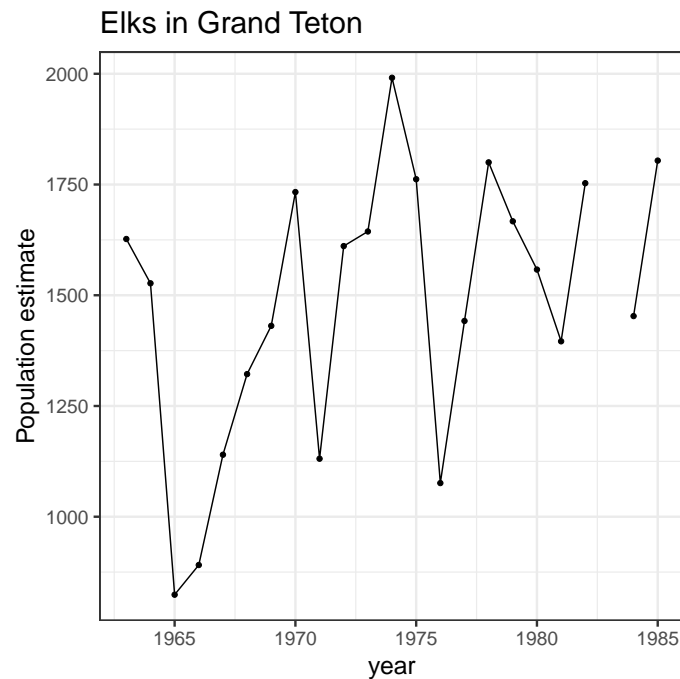
Dynamics of density-dependent populations

- Populations following this model change *smoothly*
 - Equations tell how the population will change at each instant
- They have no memory
 - Birth rate and death rate are determined by population size alone
- Cycling is impossible
 -

Paramecia



Elk



Dynamics of real-world populations

- Initial exponential growth and leveling off frequently observed
- Exponential approach to equilibrium hard to observe
 - Real populations are subject to **stochastic** (random) effects
 - Real populations are subject to changing conditions
- Some species seem to cycle predictably

Continuous-time regulation

- Continuous-time regulation in simple models makes useful predictions:
 - Threshold value for populations to survive
 - Greatest population-level growth at intermediate density
 - Greatest individual-level growth at low density
- Cannot explain complicated dynamics
 - More mechanisms are needed

3 Discrete-time regulation

3.1 A simple, discrete-time model

- We extend our discrete-time model from the previous unit:

- $N_{T+1} = (p + f)N_T \equiv \lambda N_T$
- $t_{T+1} = t_T + \Delta t$ (does not change)

- To:

- $N_{T+1} = (p(N_T) + f(N_T))N_T \equiv \lambda(N_T)N_T$

- This means:

–

Assumptions

- The population is censused at regular time intervals Δt
- All individuals are the same at the time of census
- Population changes deterministically

Specific assumptions

- For our examples, we will assume:
 - $f(N) = f_0 \exp(-N/N_f)$
 - $p(N) = p_0 \exp(-N/N_p)$
- This is the simplest model that is smooth and keeps track of birth and death rates separately
 - Fecundity goes down with characteristic scale N_f
 - Survival goes down with characteristic scale N_p

States and state variables

- What variable or variables describe the state of this system?
 - The same as before: population size (or density)
 - We are still assuming that's all we need to know

Parameters

- What quantities describe the rules for this system?
 -
 -
 -
 -

What is \mathcal{R}_0 ?

- \mathcal{R} is the fecundity multiplied by the lifespan

—
—

- \mathcal{R}_0 is \mathcal{R} in the limit where density is low

—

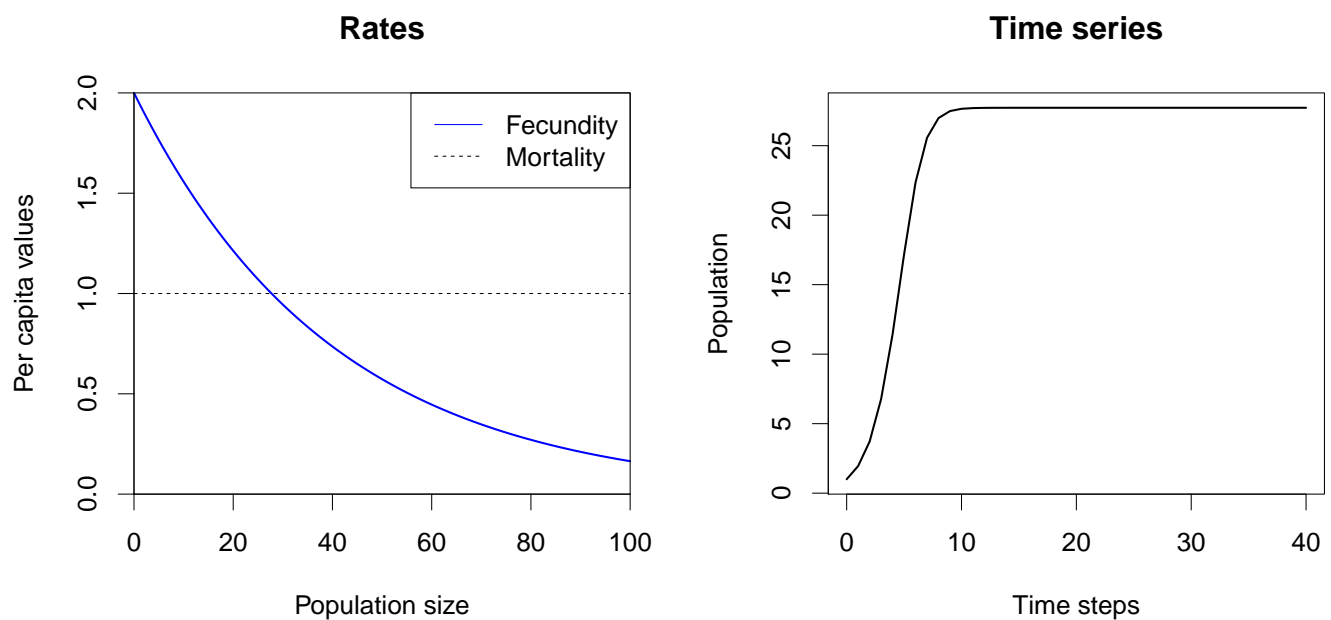
Behaviours

- When $\mathcal{R}_0 < 1$ population always declines
- When $\mathcal{R}_0 > 1$, population can show:
 - Smooth behaviour (like the continuous-time model)
 - Damped oscillations (like the delayed model)
 - Two-year cycles (high \rightarrow low \rightarrow high \rightarrow low)
 - All kinds of other stuff

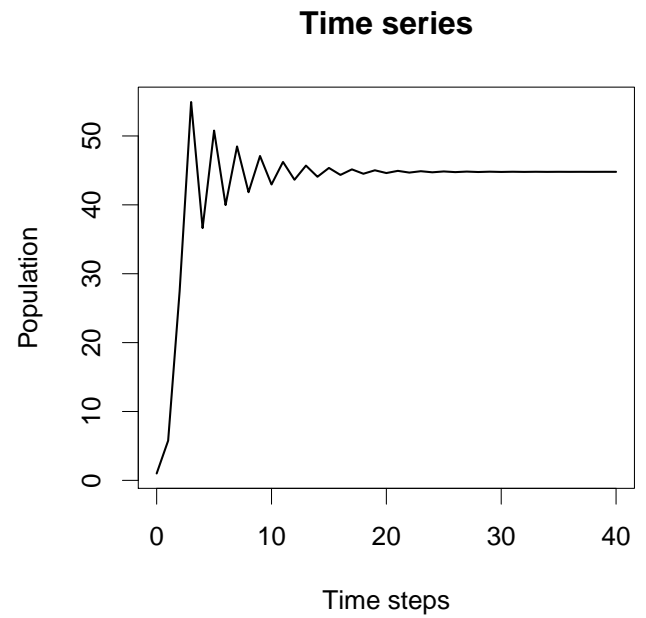
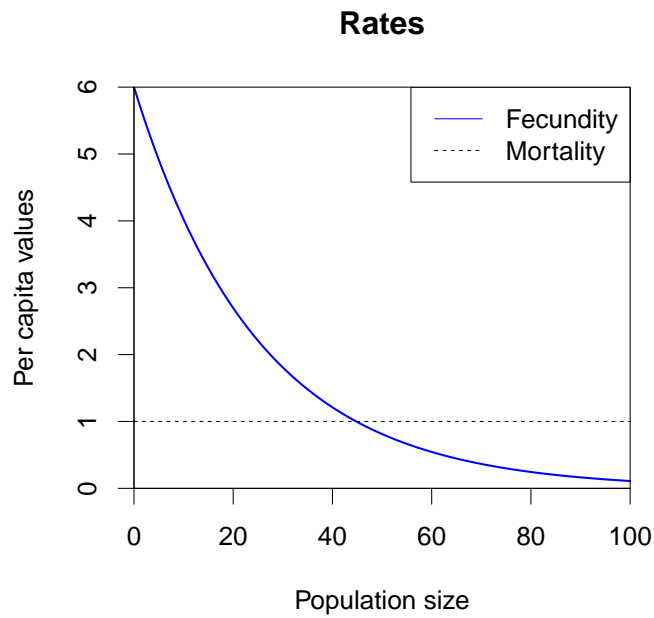
3.2 Simulating this system

- This system can be simulated very easily by following the rule for N_{T+1} as a function of N_T
- We can even do it in the spreadsheet if we have time

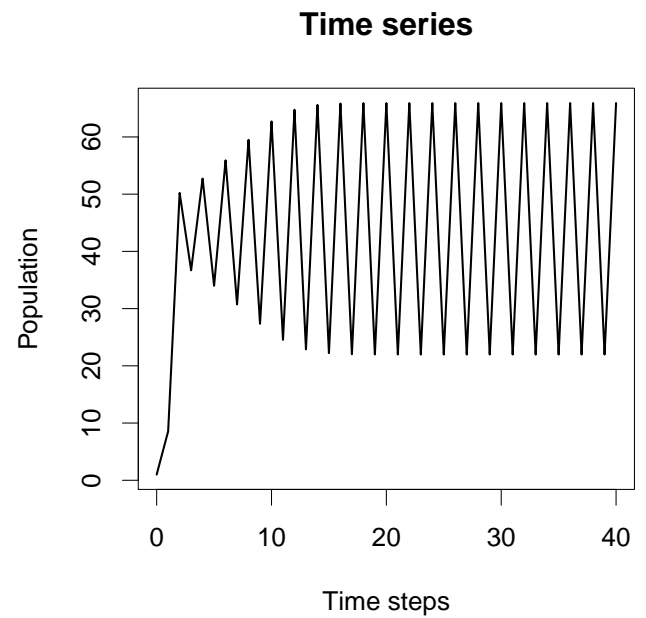
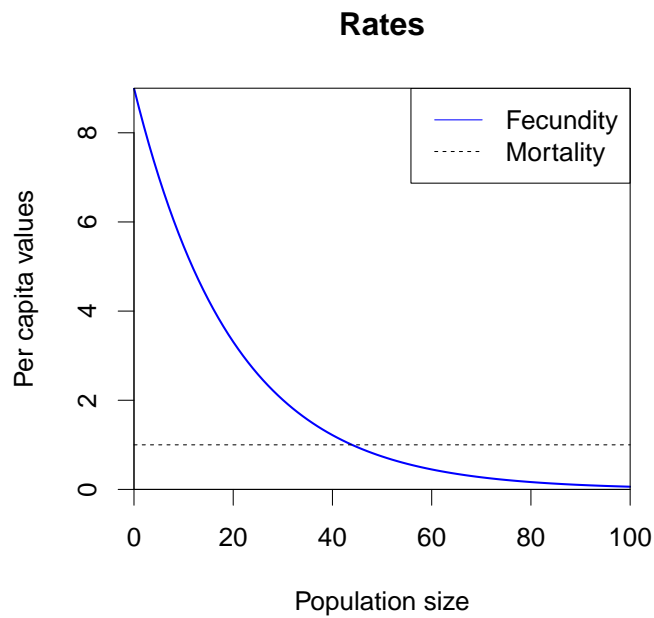
We expect simple dynamics



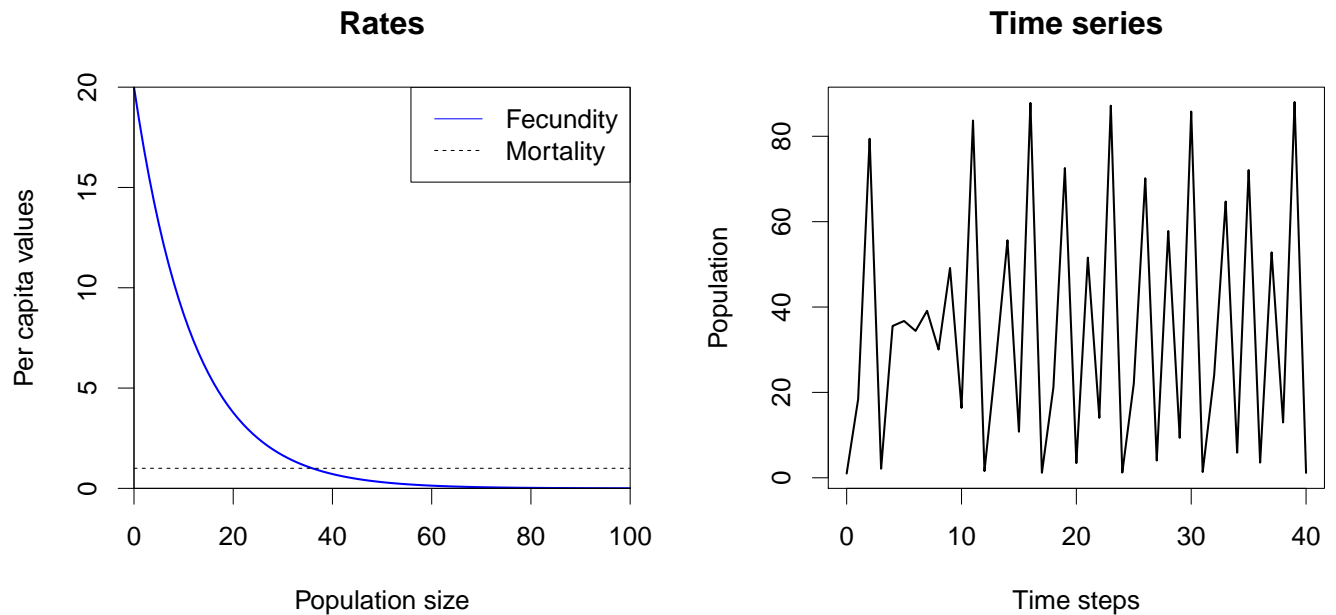
Damped oscillations



Persistent oscillations



Lots of other behaviours



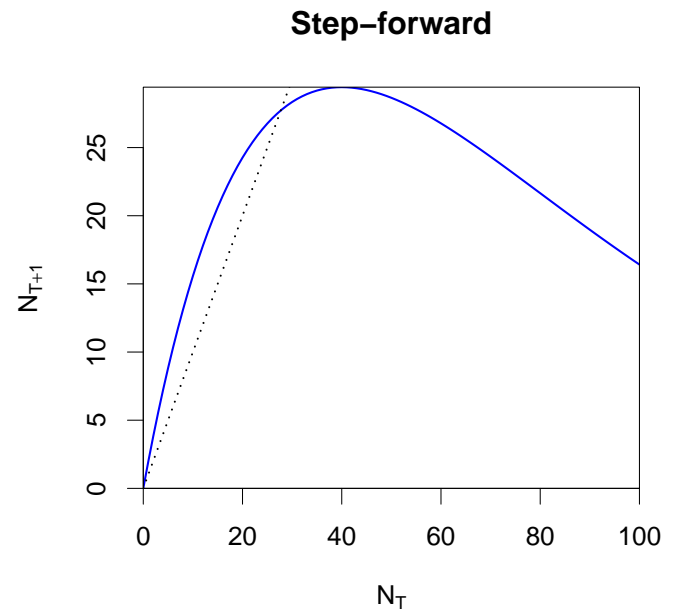
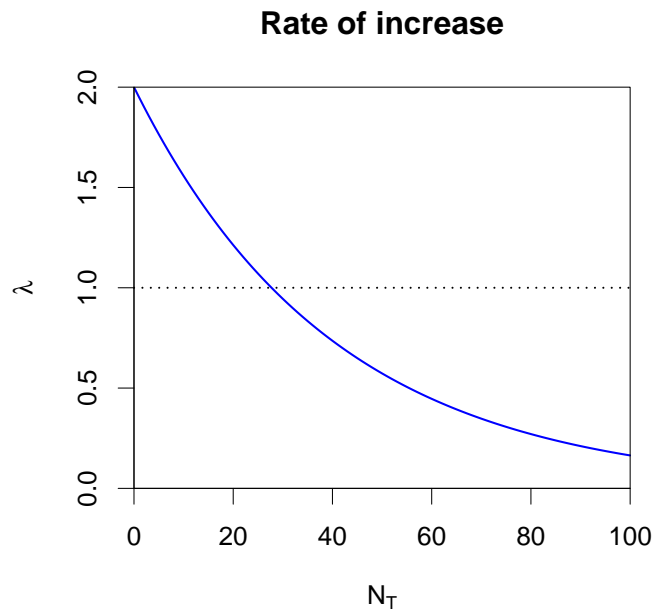
3.3 Interpreting complex behaviour

- In a simple cycle:
 - Low populations this year mean high populations next year
 - and vice versa

Complex behaviour in our simulations

- In our simple models, as N_T increases, what happens to λ ?
 -
- Poll: In our simple models, as N_T increases, what happens to next year's population?
 -
 -
 -

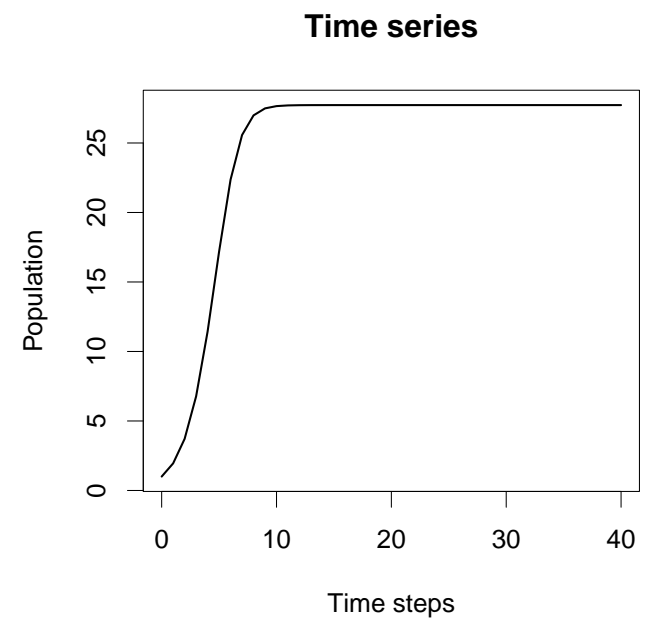
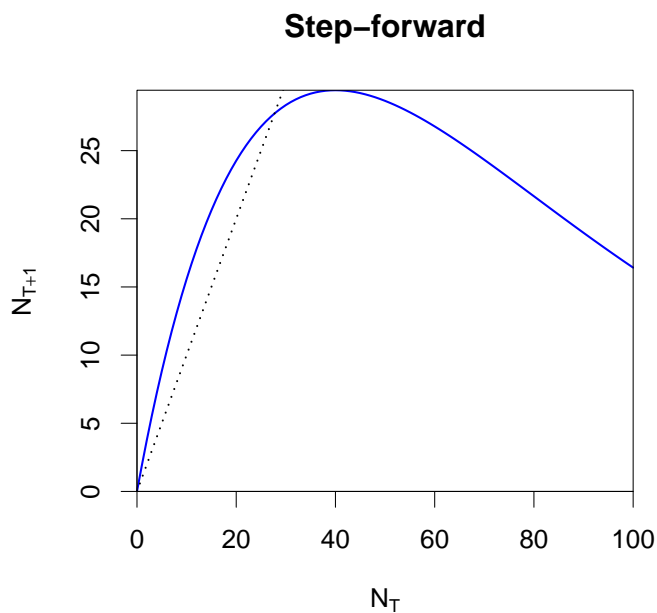
Response to population increase



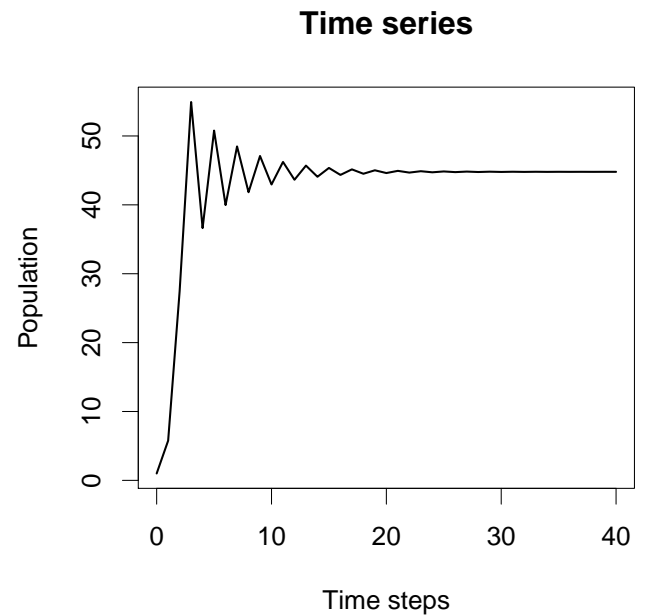
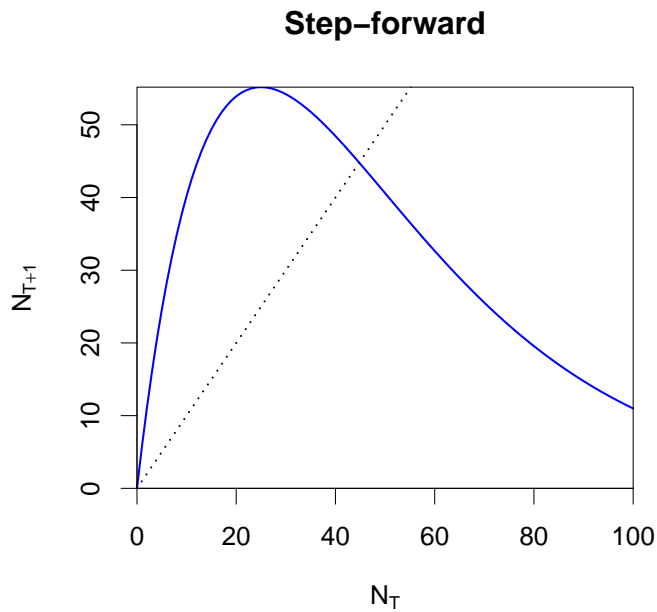
Turnover

- When N_T is small, N_{T+1} increases with N .
- Complex behaviour arises when the relationship between N_T and N_{T+1} **turns over** below the equilibrium value
 - A small population this year leads to a large population next year

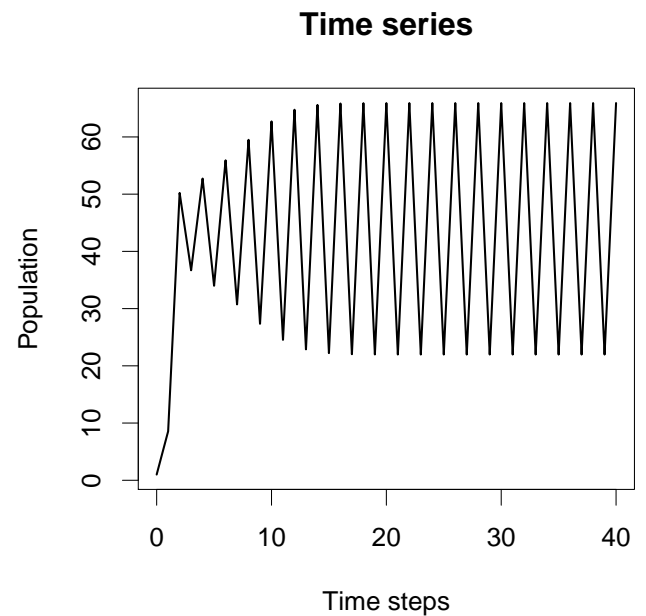
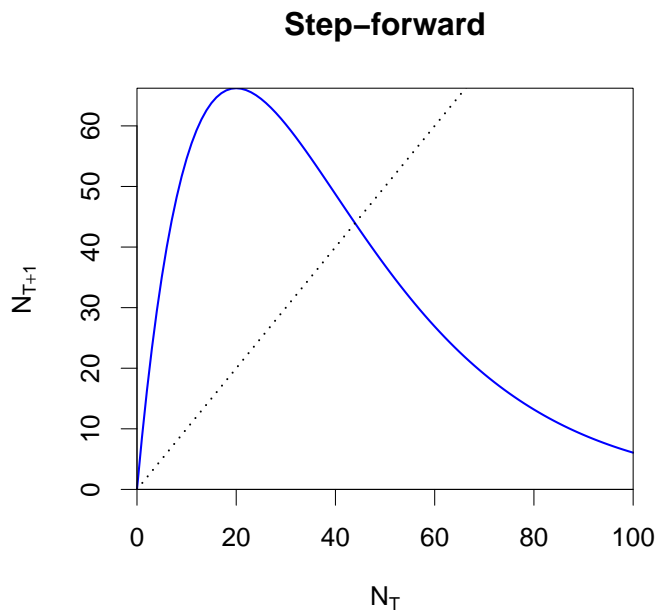
Simple dynamics



Damped oscillations



Persistent oscillations



Complex behaviour in our conceptual model

- Biologically, when might we expect N_{T+1} to “turn over”?

—
—

- When should the mapping *not* turn over?

–
–
–

Scramble competition

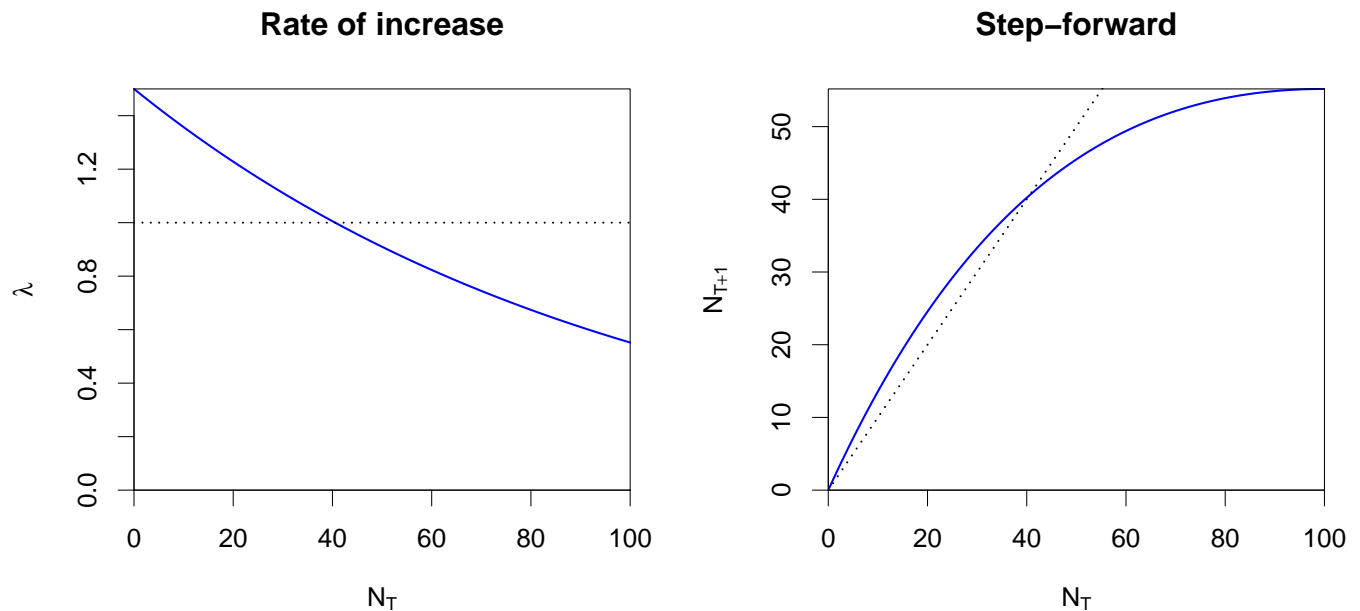
- **Scramble** competition refers to the case where all individuals in a crowded population are gathering resources at similar rates: as the density goes up there is less resource for everyone, and everyone does less well
 - If there is any kind of delay, scramble competition can lead to turning over

Contest competition

- **Contest** competition refers to a case where some individuals successfully control key resources and do well no matter how large the population is
 - Contest competition doesn't usually lead to turning over, even with delay
- How does contest competition square with regulation?

–
–

Contest regulation



Songbirds

- Some songbird populations are limited primarily by competition for breeding sites, whereas others are limited primarily by competition for insects to eat

—
*
—

Plants

- Some plant populations are limited by water, and some by light
- Poll: Which is more likely to work out as a scramble?

—
—

Complex behaviour from a simple model

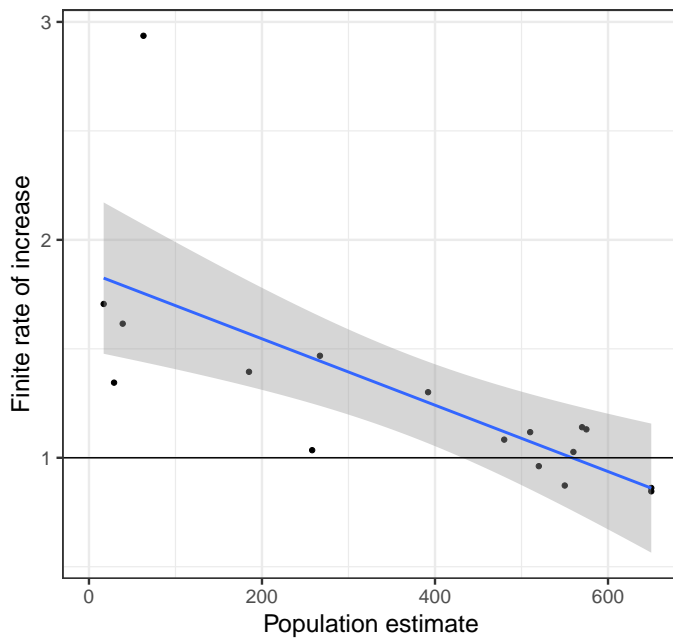
- It's interesting that we can get complicated behaviour from such a dead-simple model
- Complex dynamics may have simple causes
- People always tend to look for specific reasons, but sometimes the changes we observe are just natural dynamics

Complex behaviour in real populations

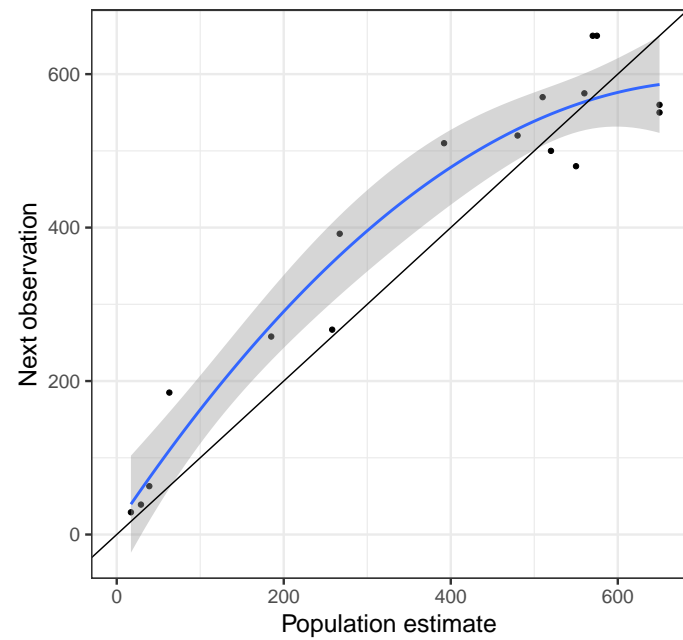
- We can plot λ and N_{T+1} vs. N for real population data
- We expect λ to decrease (on average)
- We're curious about N_{T+1} .

Paramecia

Rate of increase

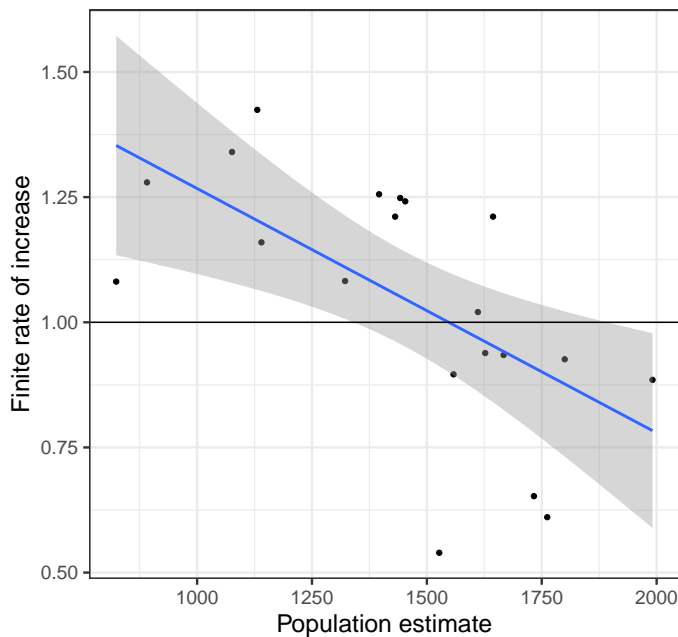


Step-forward

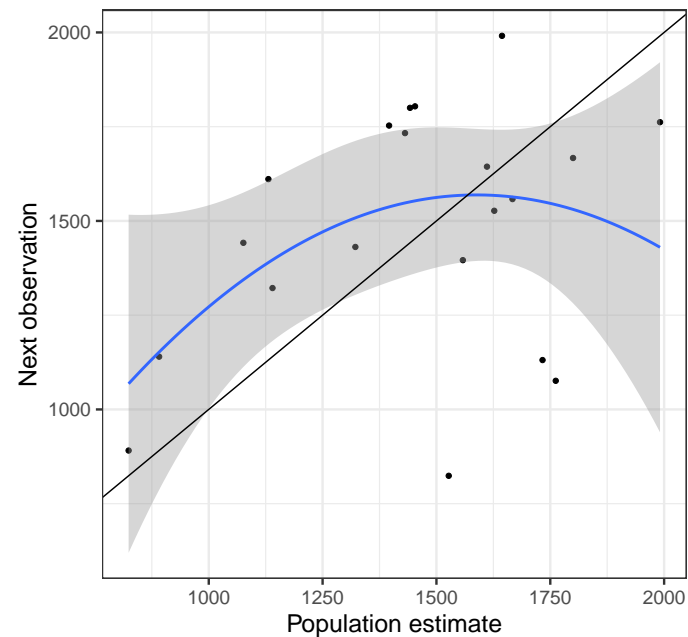


Elk

Rate of increase



Step-forward



Real populations

- It's hard to find examples of turnover from real population data.
- So how do we explain real population cycles?
 - Regulation may happen on a longer time scale
 - May be hard to see because of “noise” – i.e., other sources of variation

- Cycles may be due to more complicated mechanisms
- Time-delayed population models are cool!
 - But not covered in class this year

4 Small populations and stochasticity

Example

- Poll: What would happen if I released one butterfly into a new, highly suitable habitat?
 -
- What about two butterflies?
 -

Small populations

- Population success (reproductive number) may be lower for very small populations
 - We've already assumed reproductive numbers are low for very large populations
- Small populations are likely to be harder to predict
 - More affected by stochasticity

4.1 Allee effects

- Effects which cause small populations to have low per-capita growth rates are called Allee effects
 - Equivalent to saying that medium-sized populations have larger per-capita growth rates than small ones
- Poll: Why might growth rates be low when populations are small?
 -
 -
 -
 -

Types of Allee effect

- Allee effects can affect the (per capita) birth rate

—

- ... or the (per capita) death rate

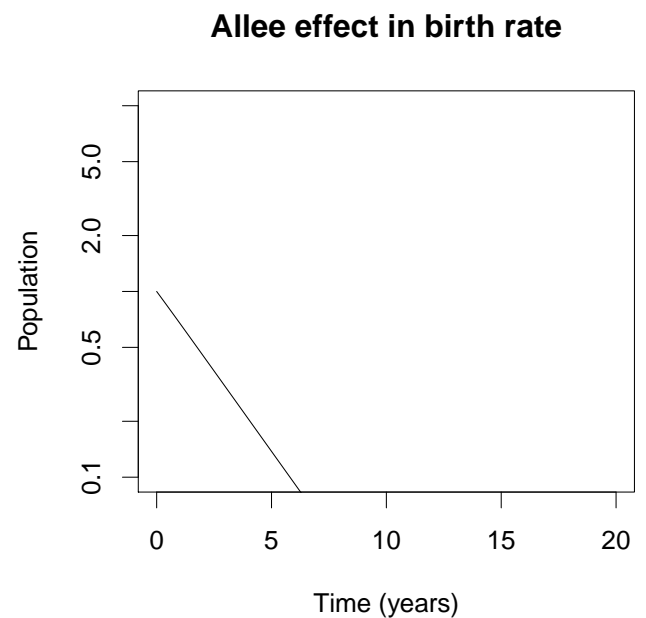
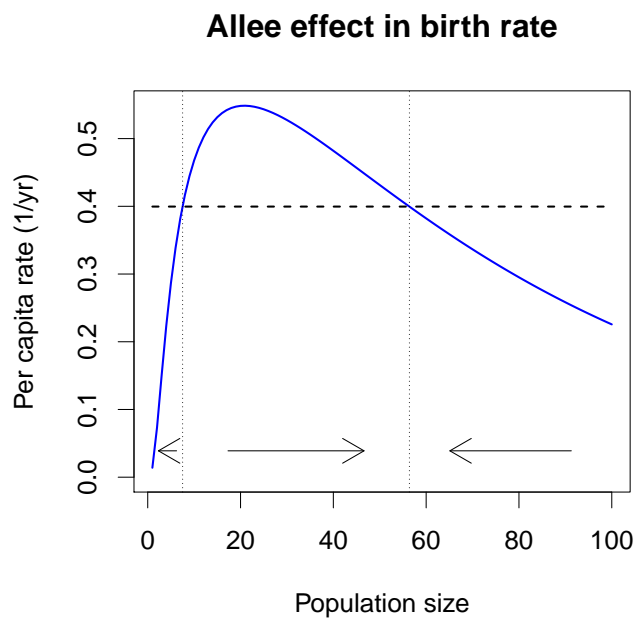
—

Allee effect models

- What will this model do, if the initial population is:

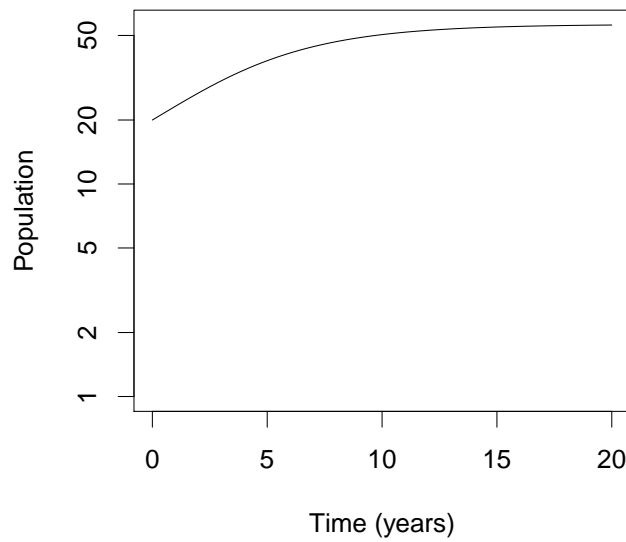
— low, medium or high?

Individual perspective

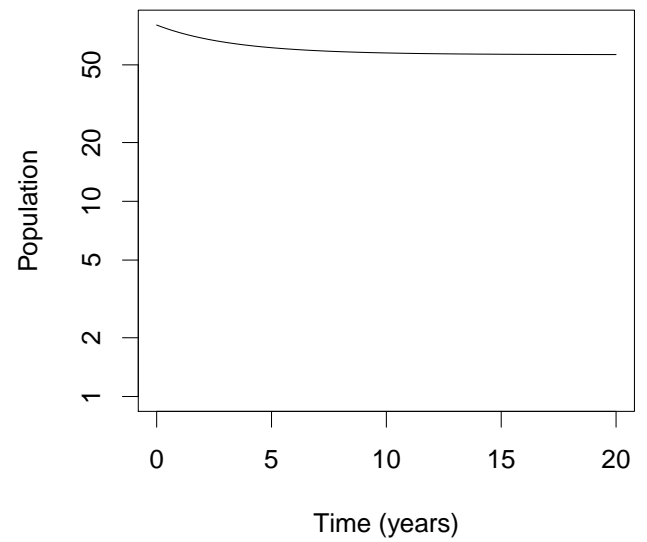


Individual perspective

Allee effect in birth rate

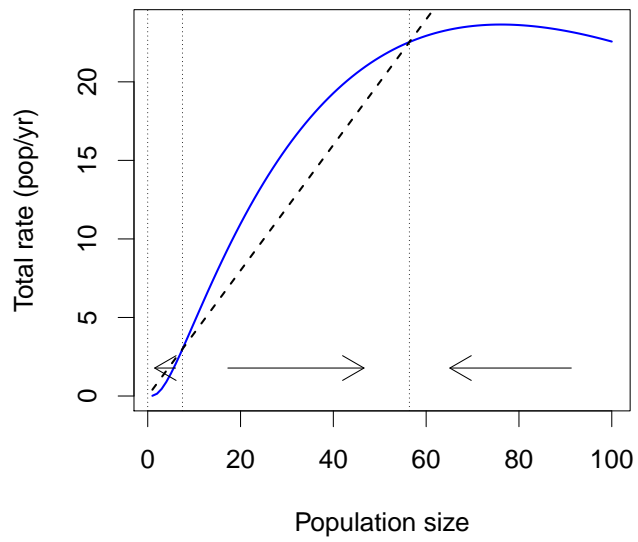


Allee effect in birth rate

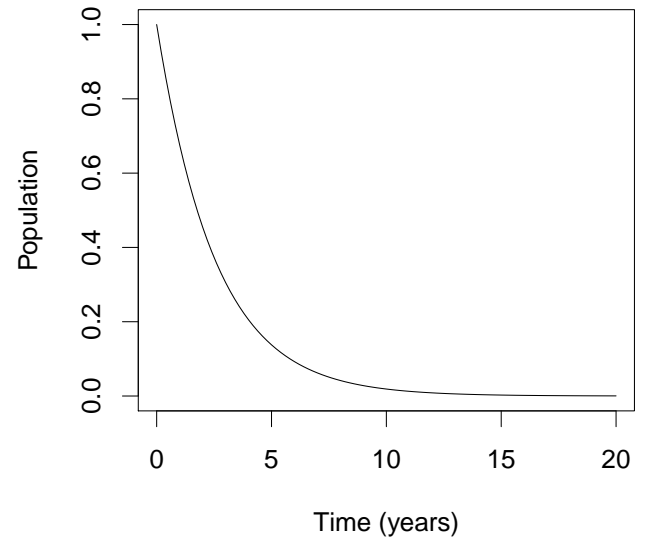


Population perspective

Allee effect in birth rate

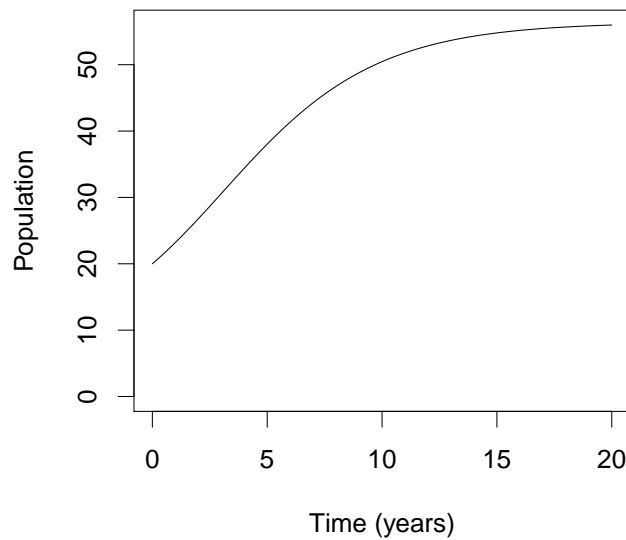


Allee effect in birth rate

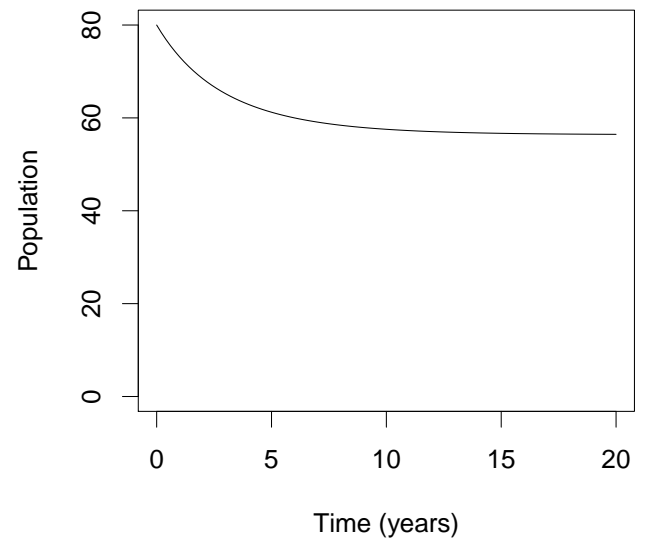


Population perspective

Allee effect in birth rate



Allee effect in birth rate

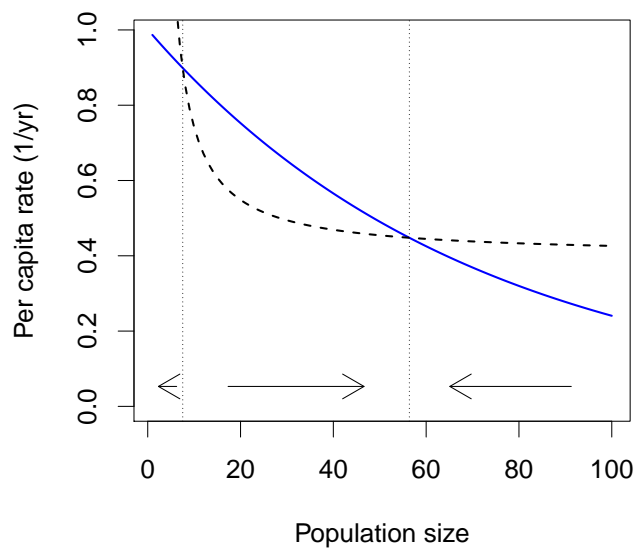


Allee effect in death rate

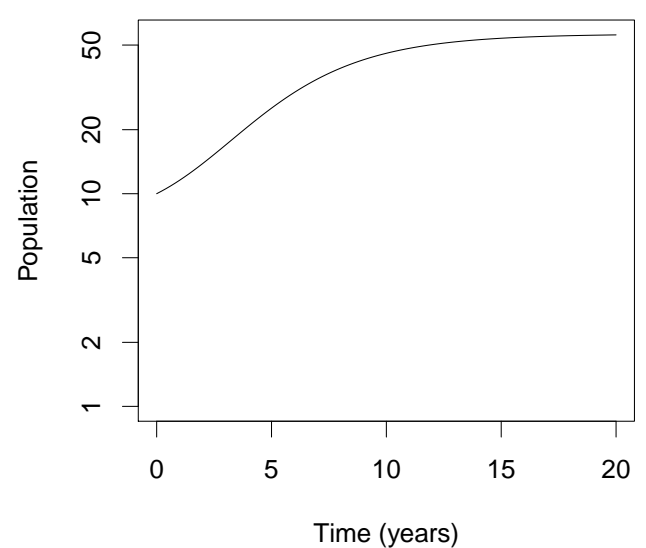
- What is the difference between this example and the previous one?
- What will this model do, if the initial population is:
 - low, medium or high?

Individual perspective

Allee effect in death rate



Allee effect in death rate



More reproductive numbers

- The reproductive number \mathcal{R} means the average lifetime number of offspring per individual
 - Should be unitless, so we consider offspring at the same stage as the individual.
- We can apply \mathcal{R} in general for any set of conditions, or we can distinguish:
 - the **basic reproductive number** \mathcal{R}_0 : \mathcal{R} in the limit when the population is small, and
 - the **maximal reproductive number** \mathcal{R}_{\max} : \mathcal{R} at whatever level is the peak

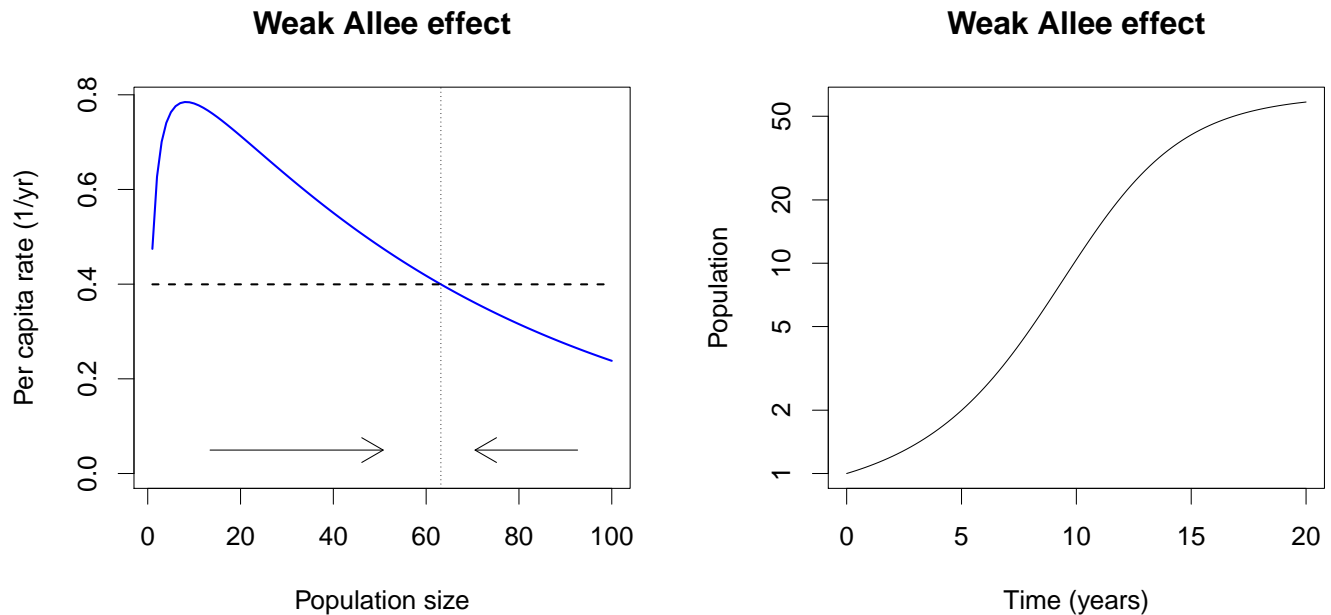
Invasion

- We previously said that when $\mathcal{R}_0 < 1$, the population always went extinct
 - A population that can't invade can never replace itself on average
- When Allee effects are present, it's no longer true that a species that can't invade can't persist
 -

Weak Allee effects

- If birth rates go down or death rates go up at low density, we consider this an Allee effect
- If $\mathcal{R}_0 < 1$ we say it's a *strong* Allee effect
 -
- If $\mathcal{R}_0 > 1$ we say it's a *weak* Allee effect

Individual perspective



Allee effect summary

- Population may go extinct if it drops below a certain threshold
- Poll: How come the population is there in the first place if there's an Allee effect?
-
-
-
-

4.2 Stochastic effects

- The world is complicated and biological populations are not perfectly predictable
- Real populations don't go smoothly to equilibria, instead they bounce around (or sometimes do other wild stuff)
- We divide stochastic (or random) effects into demographic and environmental stochasticity

Example

- Female butterflies of a certain species lay 200 eggs on average, of which:
 - Half are female
 - 50% hatch successfully into larvae

- 10% of larvae successfully pupate
- 60% of pupae become adults
- Half of adult females successfully reproduce
- A single gravid (pregnant) female butterfly is blown away by a freak storm, and lands by chance on a suitable island with no butterflies What do you expect to happen?
-
-

Butterfly example

- Depending on unknown conditions, especially in that first season, all of those probabilities could change dramatically
- Even if we knew the *probabilities*, that would not guarantee an exact result
-
- What if $\lambda < 1$?
-

Demographic stochasticity

- **Demographic** stochasticity is stochasticity that operates at the level of individuals
 - Individuals don't increase gradually, they die or give birth
 - Individuals don't produce 1.2 offspring: they produce 0, 1, 2 or 3 ...
- Even if we control conditions perfectly, we can't exactly predict the dynamics of small populations
- Demographic stochasticity averages out in large populations

Environmental stochasticity

- **Environmental** stochasticity is stochasticity that operates at the level of the population
 - E.g., weather, pollution
- Environmental stochasticity can have large effects on any population
-
- But small populations are the ones in danger of going extinct
-
-

Simulations

- We can simulate stochastic systems very easily
- But if we do the same simulation twice, we can get different answers
- Adds realism
 - But harder to interpret

Summary

- Stochasticity is very important in real populations, but hard to study
 - Mathematical analysis is very difficult
 - Simulations are useful, but hard to interpret
 - * Each time you simulate, you get a different answer
- Ecologists need to learn to recognize and communicate our uncertainty about the future