## UNIT 4: Structured populations

# 1 Introduction

- Up until now we've tracked populations with a single state variable (population size or population density)
- Poll: What assumption are we making?
  - **<u>Answer</u>:** All individuals can be counted the same.
    - **<u>Answer</u>:** Always (continuous time)
    - **<u>Answer</u>:** At census time (discrete time)
- What are some organisms for which this seems like a good approximation?
  - **<u>Answer</u>:** Dandelions, bacteria, insects
- What are some organisms that don't work so well?
  - **<u>Answer</u>:** Trees, people, codfish

## Structured populations

- If we think age or size is important to understanding a population, we might model it as an **structured** population
- Instead of just keeping track of the total number of individuals in our population ...
  - Keeping track of how many individuals of each age
    - or size
    - or developmental stage

## 1.1 Example: biennial dandelions

- Imagine a population of dandelions
  - Adults produce 80 seeds each year
  - 1% of seeds survive to become adults
  - 50% of first-year adults survive to reproduce again
  - Second-year adults never survive
- Will this population increase or decrease through time?

## How to study this population

- Choose a census time
  - Before reproduction or after
  - Since we have complete cycle information, either one should work
- Figure out how to predict the population at the next census

## Census choices

- Before reproduction
  - All individuals are adults
  - We want to know how many adults we will see next year
- After reproduction
  - Seeds, one-year-olds and two-year-olds
  - Two-year-olds have already produced their seeds; once these seeds are counted, the two-year-olds can be ignored, since they will not reproduce or survive again

## What determines $\lambda$?

- If we have 20 adults before reproduction, how many do we expect to see next time?
- $\lambda = p + f$ is the total number of individuals per individual after one time step
- Poll: What is $f$ in this example?
  - **Answer:** 0.8
- Poll: What is $p$ in this example?
  - **Answer:** 0.5 for 1-year-olds and 0 for 2-year-olds.
  - **Answer:** We can't take an average, because we don't know the population structure

## What determines $\mathcal{R}$?

- $\mathcal{R}$ is the average total number of offspring produced by an individual over their lifespan
- Can start at any stage, but need to close the loop
- Poll: What is the reproductive number?
- **Answer:** If you become an adult you produce (on average)
  - **Answer:** 0.8 adults your first year
  - **Answer:** 0.4 adults your second year
- **Answer:** $\mathcal{R} = 1.2$

## What does $\mathcal{R}$ tell us about $\lambda$?

- **Answer:** Population increases when $\mathcal{R} > 1$, so $\lambda > 1$ exactly when $\mathcal{R} > 1$
- If $\mathcal{R} = 1.2$, then $\lambda$
  - **Answer:** $> 1$ – the population is increasing
  - **Answer:** $< 1.2$ – the life cycle takes more than 1 year, so it should take more than one year for the population to increase 1.2 times

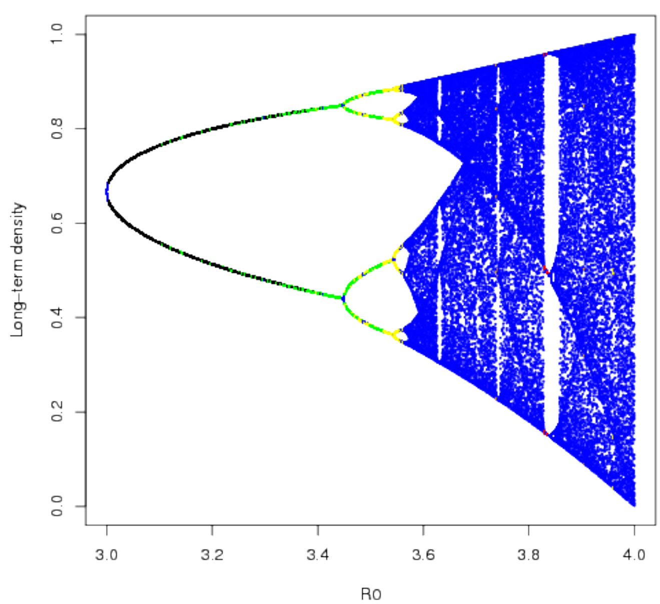
## 1.2 Modeling approach

- In this unit, we will construct *simple* models of structured populations
  - To explore how structure might affect population dynamics
  - To investigate how to interpret structured data

## Regulation

- *Simple* population models with regulation can have extremely complicated dynamics
- *Structured* population models with regulation can have insanely complicated dynamics
- Here we will focus on understanding structured population models *without regulation*:
  - **Answer:** Individuals behave independently, or (equivalently)
  - **Answer:** Average per capita rates do not depend on population size

## Comment slide: *Complexity*



### Age-structured models

- The most common approach is to structure by age
- In age-structured models we model how many individuals there are in each "age class"
  - Typically, we use age classes of one year
  - Example: salmon live in the ocean for roughly a fixed number of years; if we know how old a salmon is, that strongly affects how likely it is to reproduce

### Stage-structured models

- In stage-structured models, we model how many individuals there are in different stages
  - Ie., newborns, juveniles, adults
  - More flexible than an age-structured model
  - Example: forest trees may survive on very little light for a long time before they have the opportunity to recruit to the sapling stage

### Discrete vs. continuous time

- Structured models can be done in either discrete or continuous time
- Continuous-time models are structurally simpler (and smoother)
- Poll: How do population characteristics affect the choice?
  - **Answer:** Populations with continuous reproduction (e.g. bacteria), may be better suited to continuous-time models
  - **Answer:** Populations with **synchronous** reproduction (e.g., moths) may be better suited to discrete-time models
- Adding age structure is conceptually simpler with discrete time
  - **Answer:** So we'll do that.

## 2 Constructing a model

- This section will focus on **linear, discrete-time, age-structured** models
- State variables: how many individuals of each age at any given time
- Parameters: $p$ and $f$ *for each age that we are modeling*

## When to count

- We will choose a census time that is appropriate for our study
  - Before reproduction, to have the fewest number of individuals
  - After reproduction, to have the most information about the population processes
  - Some other time, for convenience in counting
    * **Answer:** A time when individuals gather together
    * **Answer:** A time when they are easy to find (insect pupae)

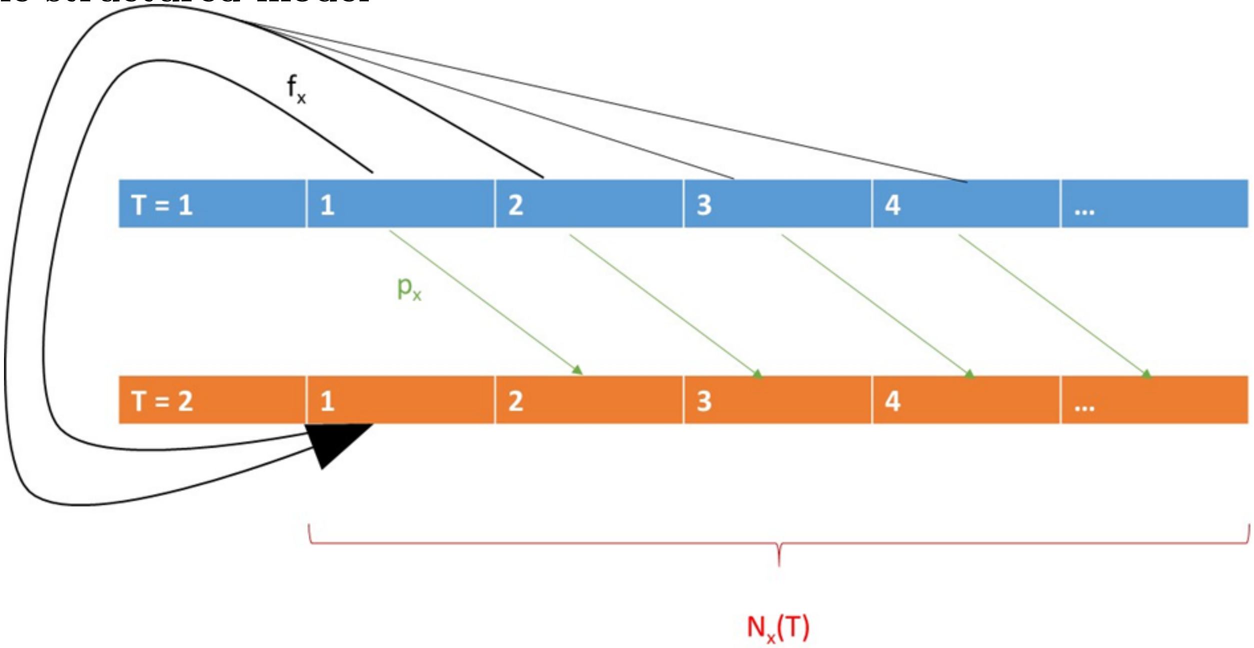
## The conceptual model

- Once we choose a census time, we imagine we know the population for each age $x$ after time step $T$.
  - We call these values $N_x(T)$
- Now we want to calculate the expected number of individuals in each age class at the next time step
  - We call these values $N_x(T+1)$
- Poll: What do we need to know?
  - **Answer:** The survival probability of each age group: $p_x$
  - **Answer:** The average fecundity of each age group: $f_x$

## Closing the loop

- $f_x$ and $p_x$ must close the loop back to the census time, so we can use them to simulate our model:
  - $f_x$ has units [new indiv (at census time)]/[age $x$ indiv (at census time)]
  - $p_x$ has units [age $x+1$ indiv (at census time)]/[age $x$ indiv (at census time)]

## The structured model




## 2.1 Model dynamics

### Short-term dynamics

- This model's short-term dynamics will depend on parameters ...
  - It is more likely to go up if fecundities and survival probabilities are high
- ... and starting conditions
  - If we start with mostly very old or very young individuals, it might go down; with lots of reproductive adults it might go up

### Long-term dynamics

- If a population follows a model like this, it will tend to reach
  - a **stable age distribution**:
    - the *proportion* of individuals in each age class is constant
  - a stable value of $\lambda$
    - if the proportions are constant, then we can average over $f_x$ and $p_x$, and the system will behave like our simple model
- Poll: What are the long-term dynamics of such a system?
  - **Answer:** Exponential growth or exponential decline

## Exception

- Populations with **independent cohorts** do not tend to reach a stable age distribution
  - A **cohort** is a group that enters the population at the same time
  - We say my cohort and your cohort interact if my children might be in the same cohort as your children
  - or my grandchildren might be in the same cohort as your great-grandchlidren
  - ...
- As long as all cohorts interact (none are independent), then the unregulated model leads to a stable age distribution (SAD)

## Independent cohorts

- Some populations might have independent cohorts:
  - If salmon reproduce *exactly* every four years, then:
    - the 2015 cohort would have offspring in 2019, 2023, 2027, 2031, ...
    - the 2016 cohort would have offspring in 2020, 2024, 2028, 2032, ...
    - in theory, they could remain independent – distribution would not converge
- Examples could include 17-year locusts, century plants, ...

# 3 Life tables

- People often study structured models using **life tables**
- A life table is made *from the perspective of a particular census time*
- It contains the information necessary to project to the next census:
  - How many survivors do we expect at the next census for each individual we see at this census? ($p_x$ in our model)
  - How many offspring do we expect at the next census for each individual we see at this census? ($f_x$ in our model)

## Cumulative survivorship

- The first key to understanding how much each organism will contribute to the population is **survivorship**
- In the field, we estimate the probability of survival from age $x$ to age $x+1$: $p_x$
  - This is the probability you will be *counted* at age $x+1$, given that you were *counted* at age $x$.

- To understand how individuals contribute to the population, we are also interested in the overall probability that individuals survive to age $x$: $\ell_x$.
  - **Answer:** $\ell_x = p_1 \times \dots p_{x-1}$
  - **Answer:** $\ell_x$ measures the probability that an individual survives to be counted at age $x$, given that it is ever counted at all (ie., it survives to its first census)

## Calculating $\mathcal{R}$

- We calculate $\mathcal{R}$ by figuring out the estimated contribution at each age group, *per individual who was ever counted*
  - We figure out expected contribution given you were ever counted by multiplying:
  - **Answer:** $f_x \times \ell_x$

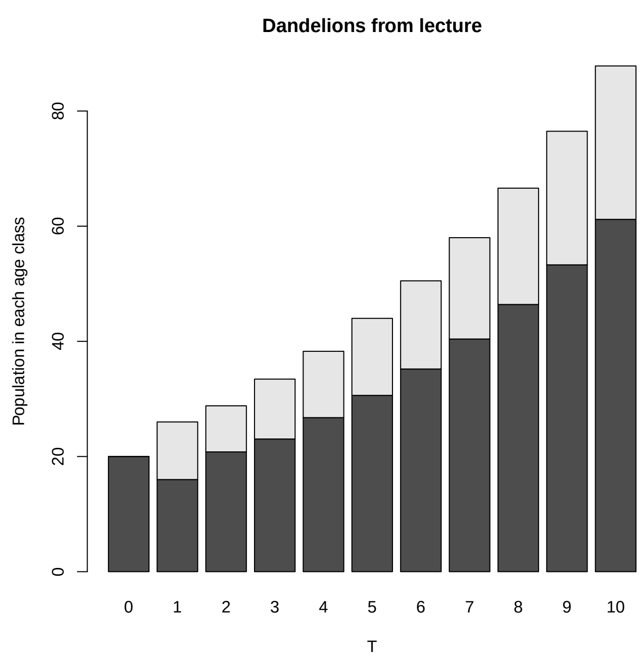
## 3.1 Examples

### Dandelion example

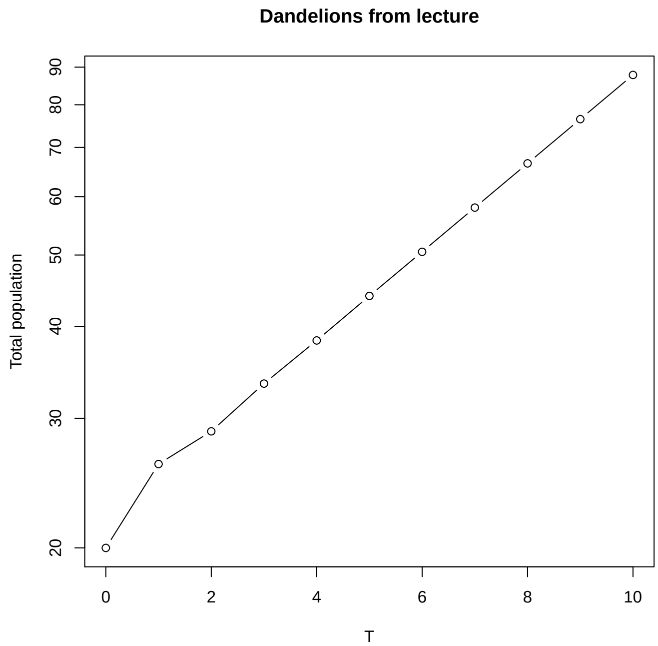
**Answer slide:** *Dandelion life table*

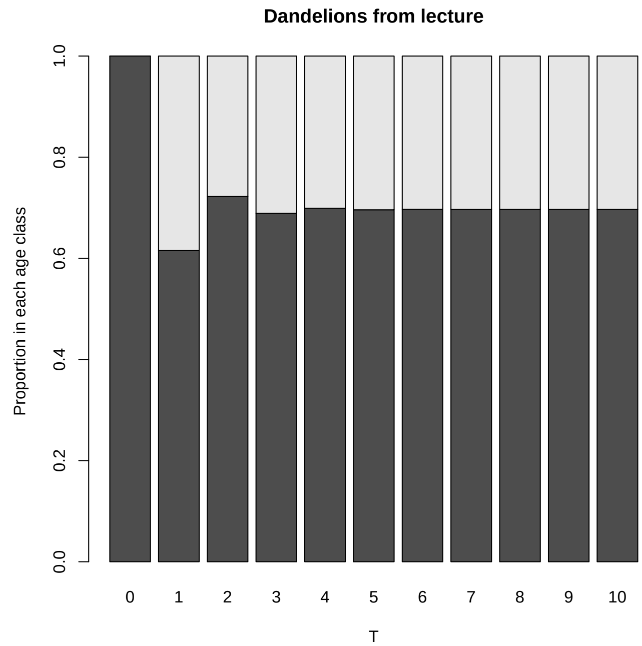
| $x$ | $f_x$ | $p_x$ | $\ell_x$ | $\ell_x f_x$ |
|---|---|---|---|---|
| 1 | 0.8 | 0.5 | 1.000 | 0.800 |
| 2 | 0.8 | 0 | 0.500 | 0.400 |
| R | | | | 1.200 |

### Dandelion dynamics



## Dandelion dynamics



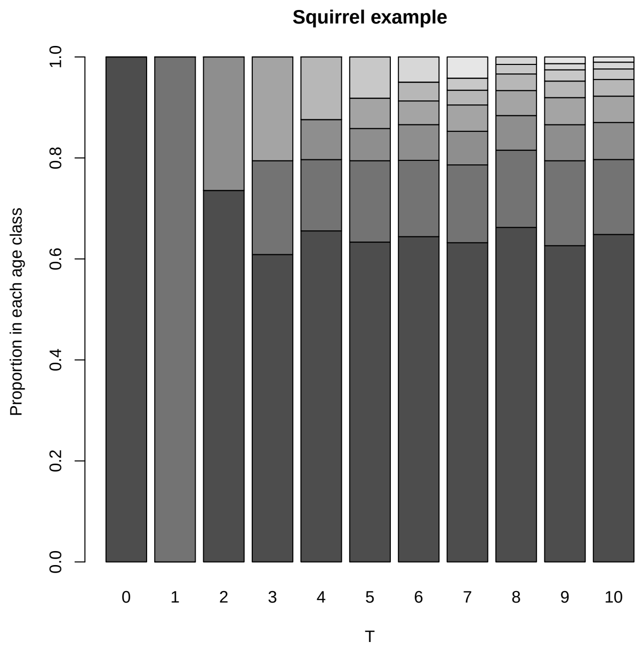

## Squirrel example
## Squirrel observations

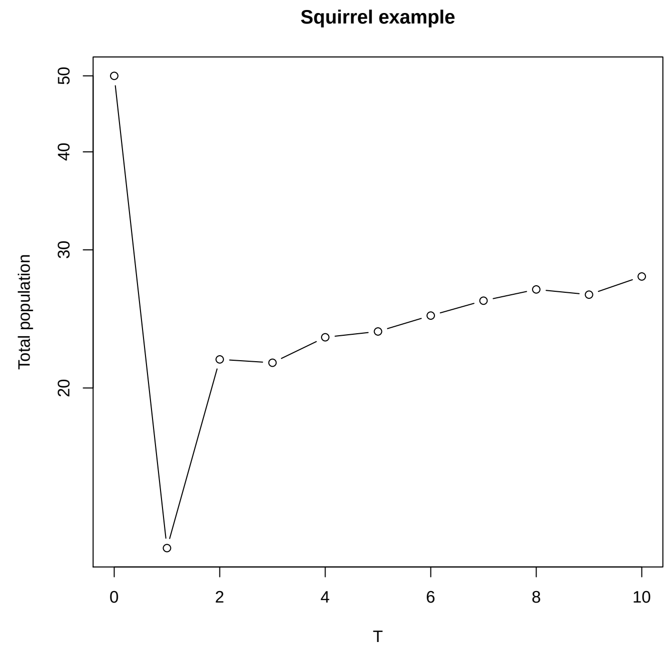
- Poll: Do you notice anything strange about the squirrel life table?
  - **<u>Answer</u>:** Older age groups seem to be grouped for fecundity.
  - **<u>Answer</u>:** Strange pattern in survivorship; do we really believe nobody survives past the last year?
  - **<u>Answer</u>:** Might be better to use a model where they keep track of 1 year, 2 year, and "adult" – not much harder.

## **<u>Answer slide</u>:** *Gray squirrel population example*

| $x$ | $f_x$ | $p_x$ | $\ell_x$ | $\ell_x f_x$ |
|---|---|---|---|---|
| 1 | 0 | 0.25 | 1.000 | 0.000 |
| 2 | 1.28 | 0.46 | 0.250 | 0.320 |
| 3 | 2.28 | 0.77 | 0.115 | 0.262 |
| 4 | 2.28 | 0.65 | 0.089 | 0.202 |
| 5 | 2.28 | 0.67 | 0.058 | 0.131 |
| 6 | 2.28 | 0.64 | 0.039 | 0.088 |
| 7 | 2.28 | 0.88 | 0.025 | 0.056 |
| 8 | 2.28 | 0.0 | 0.022 | 0.050 |
| R | | | | 1.109 |

## Gray squirrel dynamics



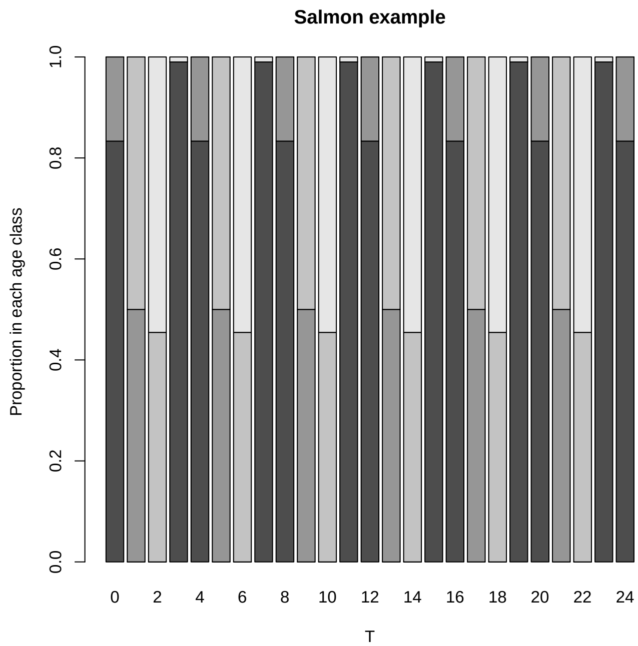

## Salmon example

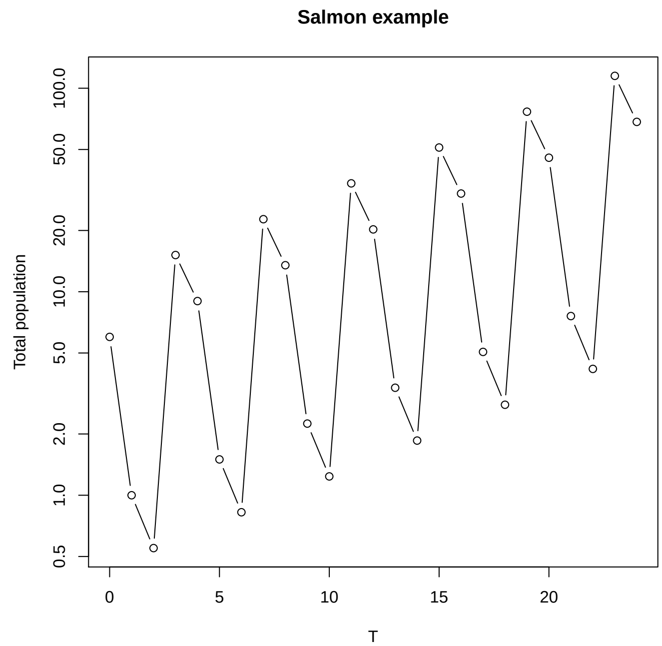
- What happens when a population has independent cohorts?
  - Does not necessarily converge to a SAD

## Answer slide: *Salmon example*

| $x$ | $f_x$ | $p_x$ | $\ell_x$ | $\ell_x f_x$ |
|---|---|---|---|---|
| 1 | 0 | 0.1 | 1.000 | 0.000 |
| 2 | 0 | 0.5 | 0.100 | 0.000 |
| 3 | 0 | 0.6 | 0.050 | 0.000 |
| 4 | 50 | 0 | 0.030 | 1.500 |
| R | | | | 1.500 |

## Salmon dynamics




## 3.2 Calculation details

### $f_x$ vs. $m_x$

- Here we focus on $f_x$ – the number of offspring seen at the next census (next year) per organism of age $x$ seen at this census
- An alternative perspective is $m_x$: the total number of offspring per reproducing individual of age $x$
- Poll: How would I calculate one from the other?
  - **Answer:** To get $f_x$ we multiply $m_x$ by one or more survival terms, depending on when the census is
  - **Answer:** $f_x$ needs to close the loop from one census to the next

### When do we start counting?

- Is the first age class called 0, or 1?
  - In this course, we will start from age class 1
  - If we count right *after* reproduction, this means we are calling newborns age class 1. Don't get confused.

### **Answer slide:** *Dandelion life table*

| $x$ | $f_x$ | $p_x$ | $\ell_x$ | $\ell_x f_x$ |
|---|---|---|---|---|
| 1 | 0.8 | 0.5 | 1.000 | 0.800 |
| 2 | 0.8 | 0 | 0.500 | 0.400 |
| R | | | | 1.200 |

**Answer slide:** *Counting* after *reproduction*

| $x$ | $f_x$ | $p_x$ | $\ell_x$ | $\ell_x f_x$ |
|---|---|---|---|---|
| 1 | 0.8 | 0.01 | 1.000 | 0.800 |
| 2 | 40 | 0 | 0.010 | 0.400 |
| R | | | | 1.200 |

REMARK Explain two-line and three-line versions. Take your time

## Calculating $\mathcal{R}$

- The reproductive number $\mathcal{R}$ gives the average lifetime reproduction of an individual, and is a valuable summary of the information in the life table
  - $\mathcal{R} = \sum_x \ell_x f_x$
  - If $\mathcal{R} > 1$ in the long (or medium) term, the population will increase
  - If $\mathcal{R}$ is persistently $< 1$, the population is in trouble
- We can ask (for example):
  - Which ages have a large *contribution* to $\mathcal{R}$?
  - Poll: Which values of $p_x$ and $f_x$ is $\mathcal{R}$ sensitive to?
    * **Answer:** The $p$s for young individuals affect all the $\ell$s.

## The effect of old individuals

- Estimating the effects of old individuals on a population can be difficult, because both $f$ and $\ell$ can be extreme
  - The contribution of an age class to $\mathcal{R}$ is $\ell_x f_x$
  - Poll: Extreme how?
  - **Answer:** In most populations $\ell$ can be very small for large $x$
  - **Answer:** In many populations, $f$ can be very large for large $x$
- Reproductive potential of old individuals *may* or *may not* be important
  - **Answer:** In many tree populations, most individuals don't survive to get huge, but the huge trees may have most of the total reproduction
  - **Answer:** In many bird populations, old birds produce fairly well, but not nearly enough to outweigh the low probability of being old.

**Comment slide:** *Old individuals*

## 3.3 Measuring growth rates

- In a constant population, each age class replaces itself:
  - $\mathcal{R} = \sum_x \ell_x f_x = 1$
- In an exponentially changing population, each year's **cohort** is a factor of $\lambda$ bigger (or smaller) than the previous one
  - $\lambda$ is the finite rate of increase, like before
- Looking back in time, the cohort $x$ years ago is $\lambda^{-x}$ as large as the current one
- The existing cohorts need to make the next one:
  - $\sum_x \ell_x f_x \lambda^{-x} = 1$

## The Euler equation

- If the life table doesn't change, then $\lambda$ is given by $\sum_x \ell_x f_x \lambda^{-x} = 1$
- We basically ask, if the population has the structure we would expect from growing at rate $\lambda$, would it continue to grow at rate $\lambda$.
- On the left-side each cohort started as $\lambda$ times smaller than the one after it
  - Then got multiplied by $\ell_x$.
- Under this assumption, is the next generation $\lambda$ times bigger again?
- Example from spreadsheet

## $\lambda$ and $\mathcal{R}$

- If the life table doesn't change, then $\lambda$ is given by $\sum_x \ell_x f_x \lambda^{-x} = 1$
  - What's the relationship between $\lambda$ and $\mathcal{R}$?
- When $\lambda = 1$, the left hand side is just $\mathcal{R}$.
  - If $\mathcal{R} > 1$, the population more than replaces itself when $\lambda = 1$. We must make $\lambda > 1$ to decrease LHS and balance.
  - If $\mathcal{R} < 1$, the population fails to replace itself when $\lambda = 1$. We must make $\lambda < 1$ to increase LHS and balance.
- So $\mathcal{R}$ and $\lambda$ tell the same story about whether the population is increasing

## Time scales

- $\lambda$ gives the number of individuals per individual *every year*
- $\mathcal{R}$ gives the number of individuals per individual *over a lifetime*
- Poll: What relationship do we expect for an annual population (individuals die every year)?
  - **Answer:** $\mathcal{R} = \lambda$; each organism observed reproduces $\mathcal{R}$ offspring on average, all in one time step
- Poll: For a long-lived population?
  - **Answer:** The $\mathcal{R}$ offspring are produced slowly, so population changes slowly
    - **Answer:** $\lambda$ should be closer to 1 than $\mathcal{R}$ is.
    - **Answer:** But on the same side (same answer about whether population is growing)

## Studying population growth

- $\lambda$ and $\mathcal{R}$ give similar information about your population
- $\mathcal{R}$ is easier to calculate, and more generally useful
- But $\lambda$ gives the actual rate of growth
  - More useful in cases where we expect the life table to be constant with exponential growth or decline for a long time

## Growth and decline

- If we think a particular period of growth or decline is important, we might want to study how factors affect $\lambda$
  - Complicated, but well-developed, theory
  - In a growing population, what happens early in life is more important to $\lambda$ than to $\mathcal{R}$.
  - In a declining population, what happens late in life is more important to $\lambda$ than to $\mathcal{R}$.
- Which is likely to be more important to ecology and evolution?
  - **Answer:** The two phases (growth and decline) will be roughly balanced
  - **Answer:** Because otherwise the population would go to zero or infinity

# 4 Life-table patterns

## 4.1 Survivorship

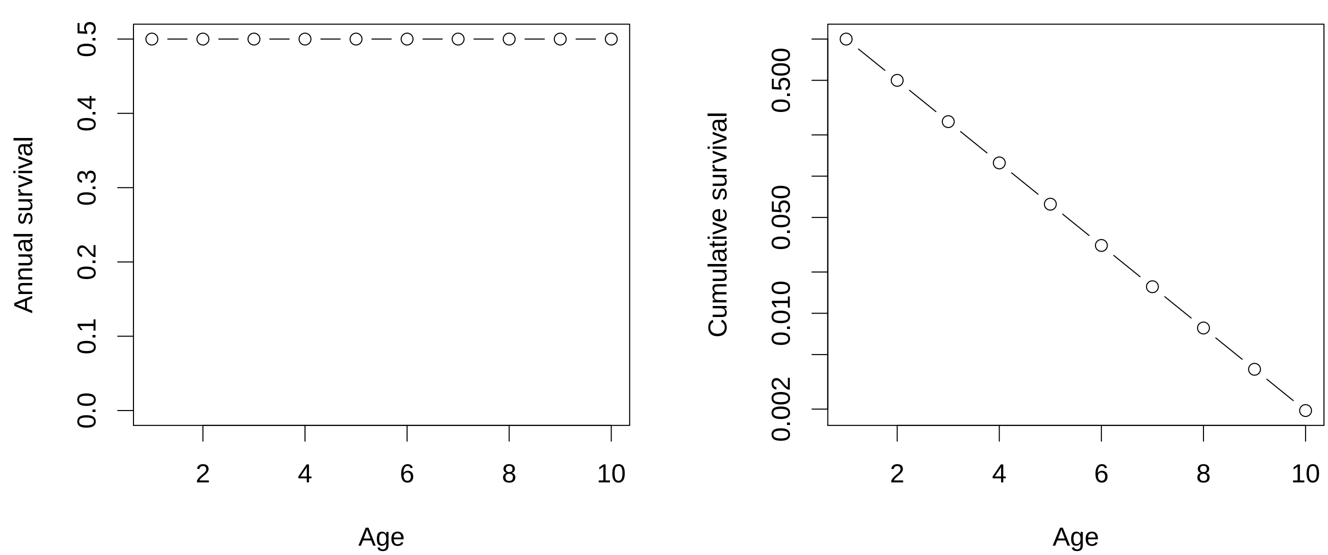
### Patterns of survivorship

- Poll: What sort of patterns do you expect to see in $p_x$?
  - **Answer:** Younger individuals usually have lower survivorship
  - **Answer:** Older individuals often have lower survivorship
- What about $\ell_x$?
  - **Answer:** It goes down
  - **Answer:** But sometimes faster and sometimes slower
  - **Answer:** Best understood on a log scale

### Starting off

- Recall: we always start from age *class* 1
  - If we count newborns, we still call them class 1.
- Poll: What is $\ell_1$ when we count before reproduction?
  - **Answer:** 1
  - **Answer:** $\ell_1$ is the probability you're counted at age class 1, *given* that you're counted at age class 1.
  - **Answer:** We don't count individuals that we don't count

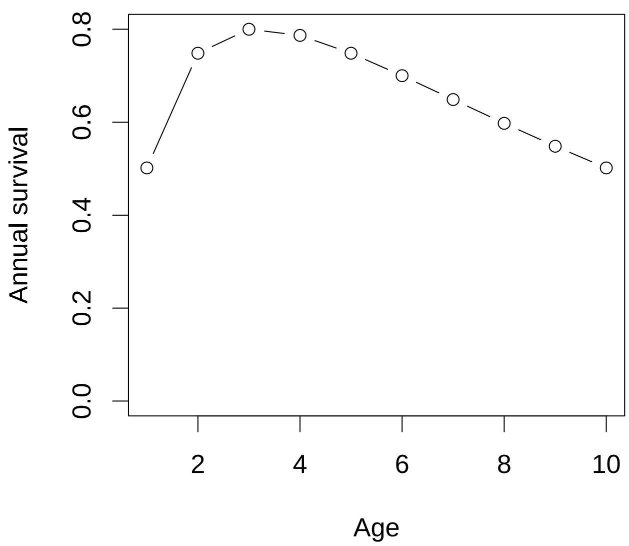
### Constant survivorship

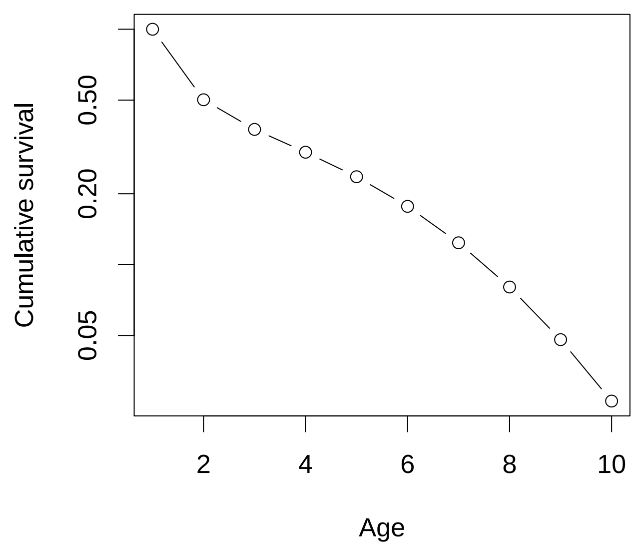


## “Types” of survivorship

- There is a history of defining survivorship as:
  - Type I, II or III depending on whether it increases, stays constant or decreases with age *(don't memorize this, just be aware in case you encounter it later in life).*
  - Real populations tend to be more complicated
- Most common pattern is: high mortality at high and low ages, with less mortality between

## Changing survivorship




## 4.2 Fecundity

- Just as in our simple population growth models, we define fecundity as the expected number of offspring at the next census produced by an individual observed at this census
  - Parent must survive from counting to reproduction
  - Parent must give birth
  - Offspring must survive from birth to counting
- Remember to think clearly about gender when necessary
  - Are we tracking females, or everyone?

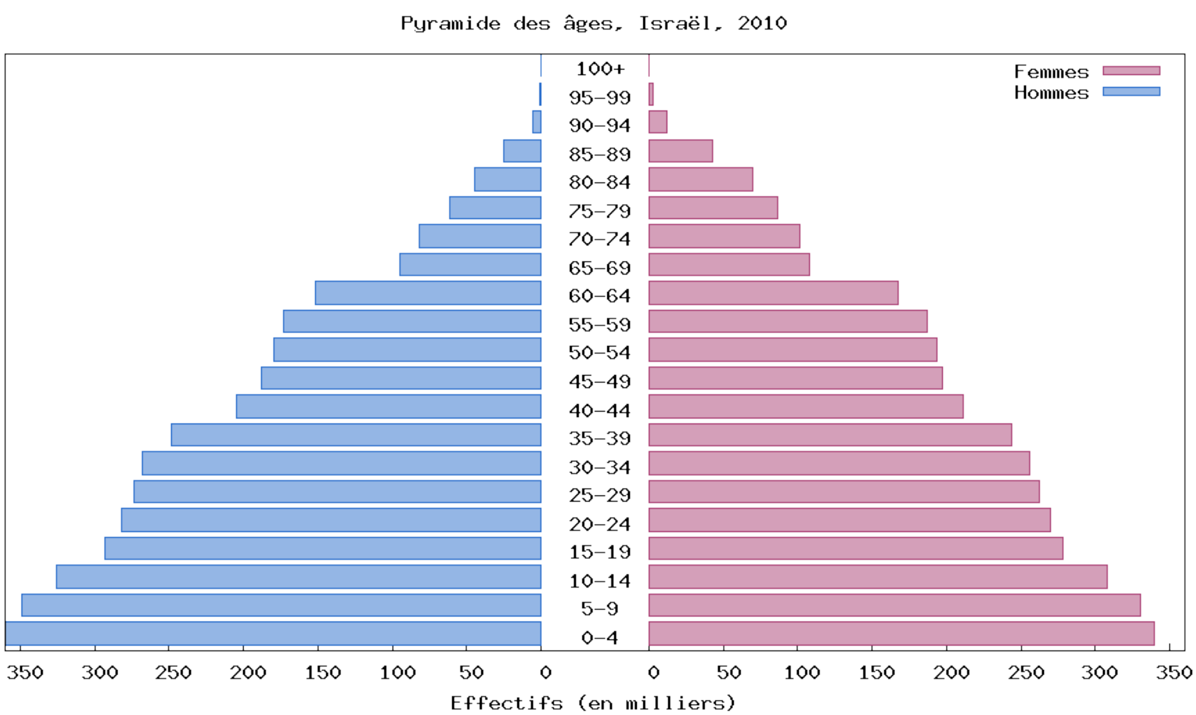
### Fecundity patterns

- $f_x$ is the average number of new individuals *counted* next census per individual in age class $x$ *counted* this census
- Fecundity often goes up early in life and then remains constant
  - **Answer:** Most birds, many large mammals
- It may also go up and then come down
  - **Answer:** people
- It may also go up and up as organisms get older
  - **Answer:** Many fish, many trees

# 5 Age distributions

- `http://www.gapminder.org/population/tool/`
- `https://en.wikipedia.org/wiki/Population_pyramid`

## Comment slide: *Age distributions*



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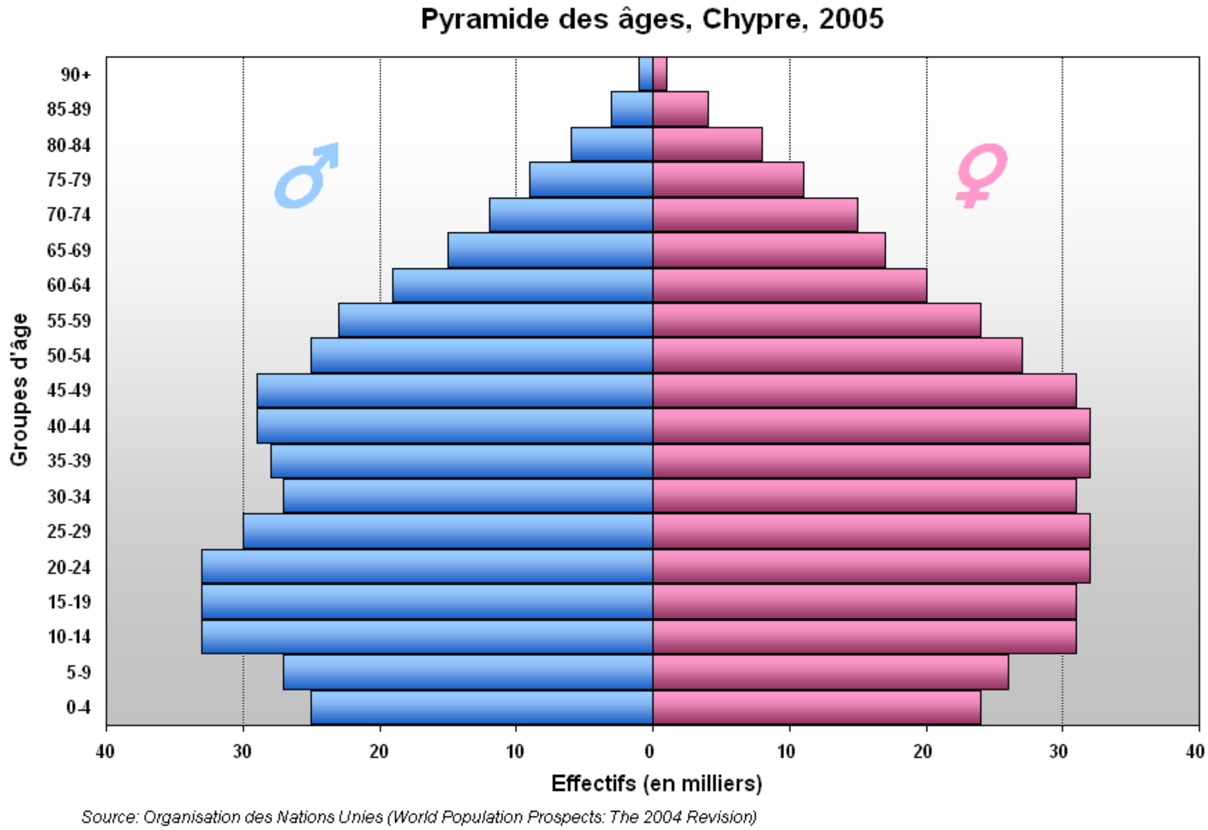


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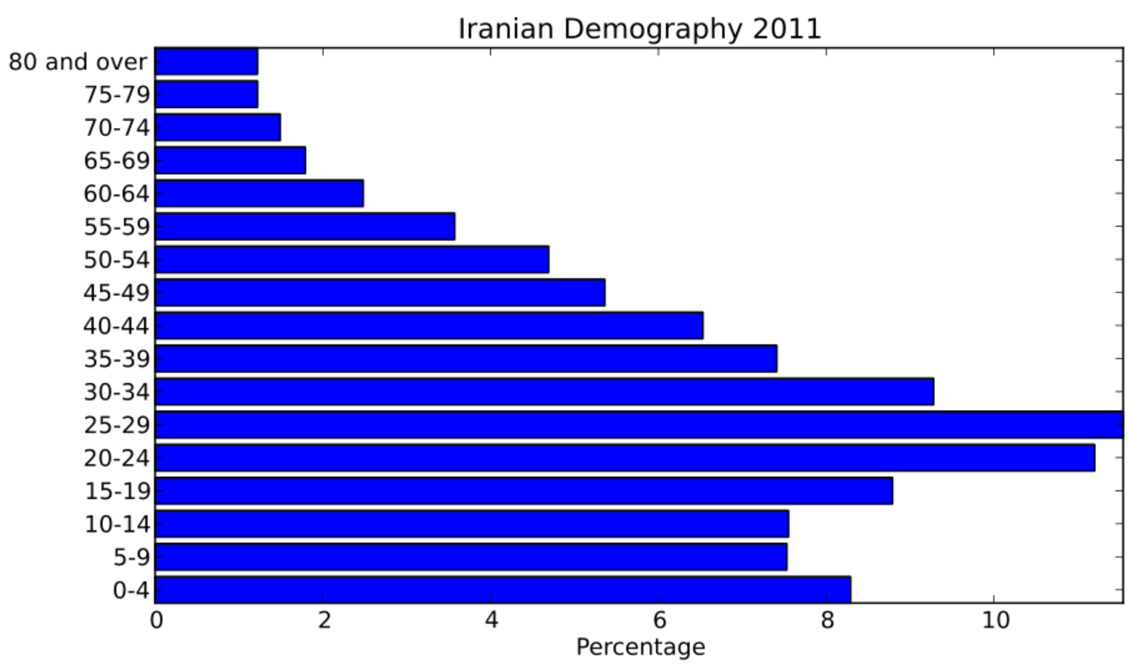
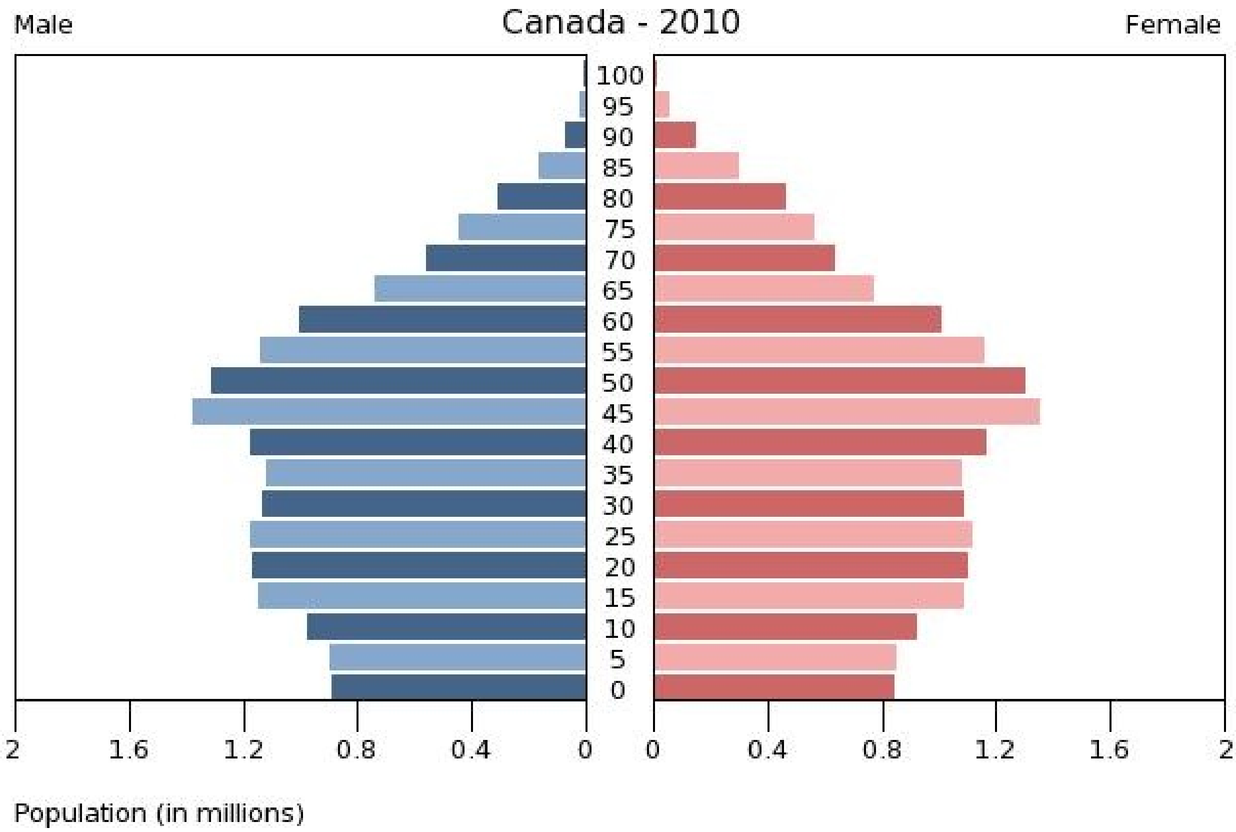


**Comment slide:** *Age distributions*

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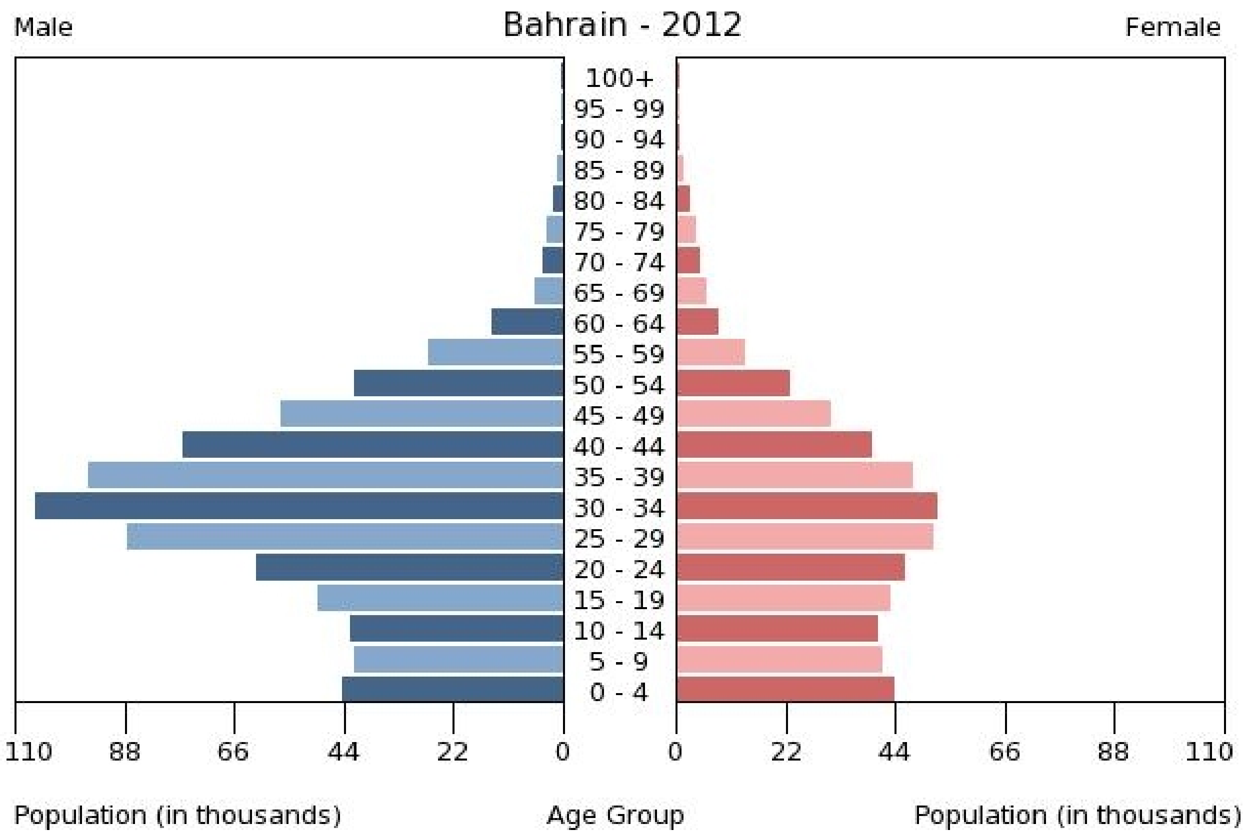


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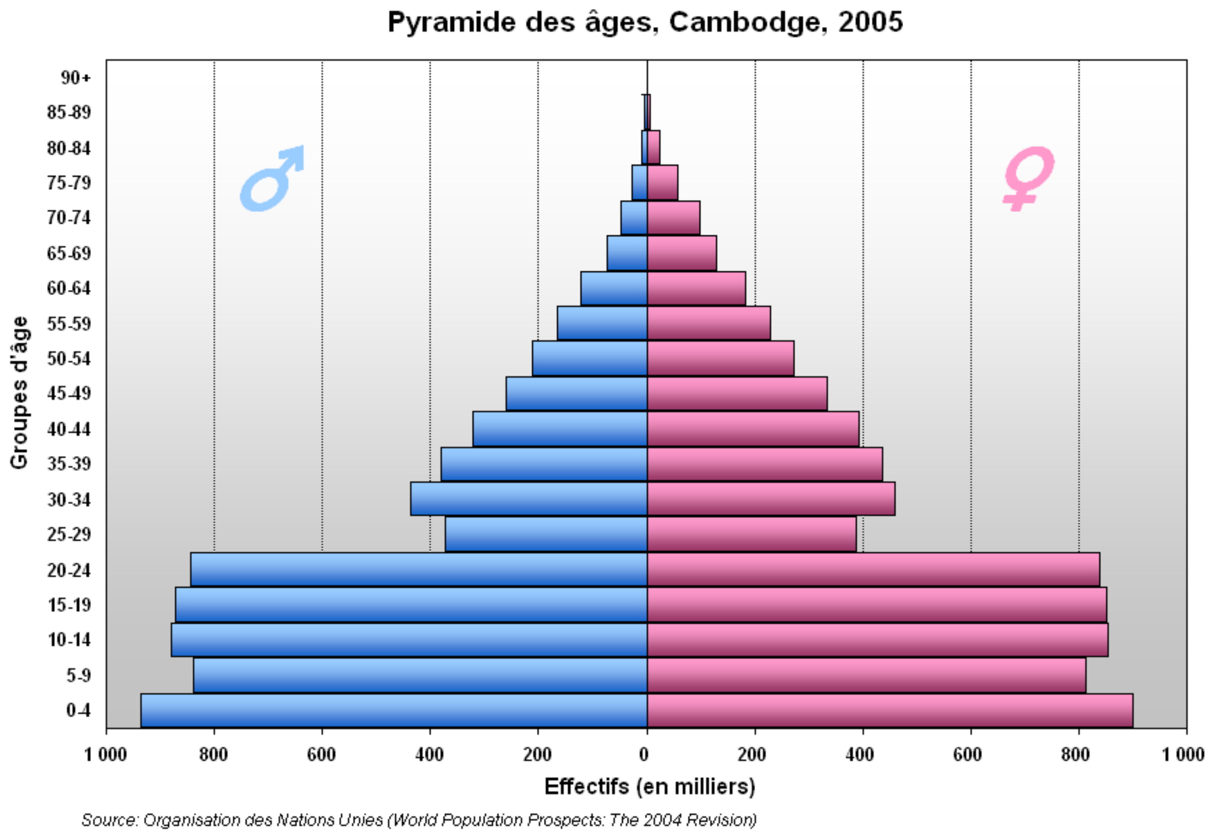



## Learning from the model

- If a population has constant size (ie., the same number of individuals are born every year), what determines the proportion of individuals in each age class?
  - **Answer:** Distribution should be proportional to $\ell_x$
- What if population is growing?
  - **Answer:** We expect proportionally more individuals in younger age classes
    - **Answer:** Number of births in more recent cohorts is larger

## Stable age distribution

- If a population has reached a SAD, and is increasing at rate $\lambda$ (given by the Euler equation):
  - the $x$ year old cohort, born $x$ years ago originally had a size $\lambda^{-x}$ relative to the current one
  - a proportion $\ell_x$ of this cohort has survived
  - thus, the relative size of cohort $x$ is $\lambda^{-x}\ell_x$
  - SAD depends only on survival distribution $\ell_x$ and $\lambda$.

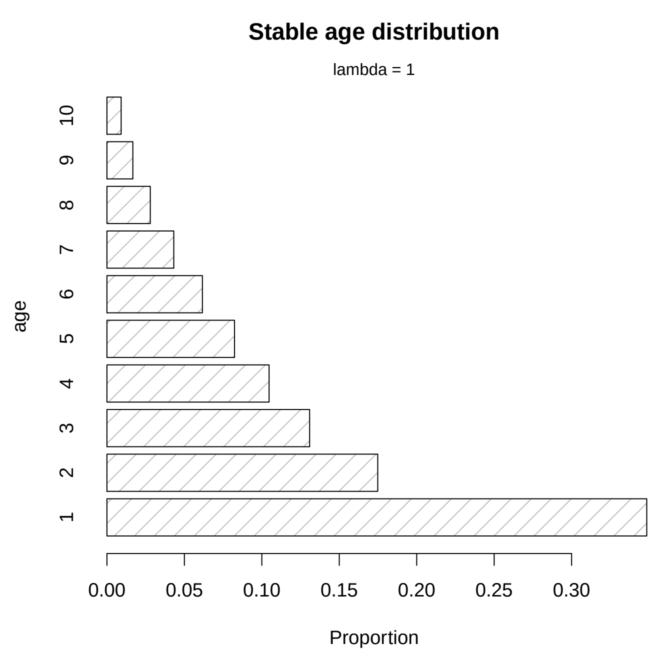
## Patterns

- Populations tend to be bottom-heavy (more individuals at lower age classes)
  - Many individuals born, few survive to older age classes
- If population is growing, this increases the lower classes further
  - More individuals born more recently
- If population is *declining*, this shifts the age distribution in the opposite direction
  - Results can be complicated
  - Declining populations may be bottom-heavy, top-heavy or just jumbled

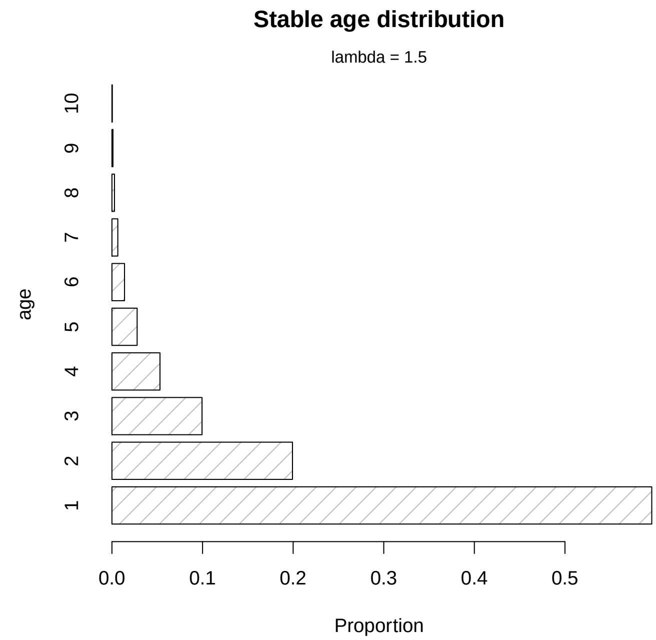
## University cohorts

- McMaster accepts only first-year students. For any given stage (e.g., end of third year) the same proportion drop out each year
- What can you say about the relative size of the classes if:
  - The same number of students is admitted each year?
    * **Answer:** The lower classes are larger
  - Poll: More students are admitted each year?
    * **Answer:** The lower classes are larger (even more so)
  - Poll: Fewer students are admitted each year?
    * **Answer:** Anything could happen (drop outs and size change are operating in different directions)

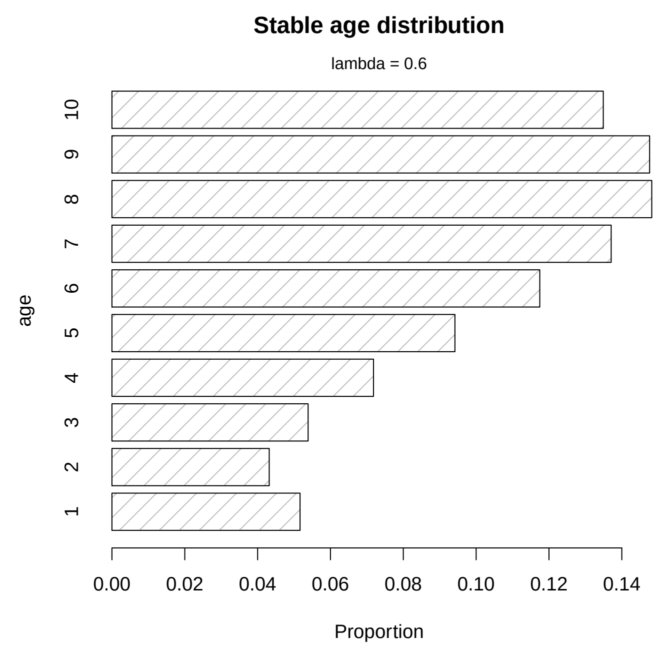
## Age distributions



## Age distributions




## Age distributions




# 6 Other structured models

## Forest example

- Forests have obvious population structure

- They also seem to remain stable for long periods of time
- Populations are presumably *regulated* at some time scale

## Forest size classes

- When we go to an apparently stable forest ecosystem, it seems to be dominated by large trees, not small ones. What up?
- Poll: How is it possible that these systems are dominated by large trees?
  - **Answer:** Large trees are larger
  - **Answer:** Population may be declining
  - **Answer:** Trees may spend longer in some size classes than in others
  - **Answer:** Life table may not be constant (smaller trees may recruit at certain times and places)

## 6.1 Stage structure

- Stage structure works just like age structure, except that what stage you are in is not strictly predicted by how old you are
  - Age-structured models need fecundity, and survival probability
  - **Answer:** In stage-structured models survival is typically broken into:
    * **Answer:** Survival into same stage
    * **Answer:** Survival with recruitment (ie., to the next larger class of individuals)
  - More complicated models are also possible

## Advantages

- Stage structured models don't need a maximum age
- Nor one box for every single age class

## Unregulated growth

- What happens if you have a constant stage table (no regulation)?
  - Fecundity, and survival and recruitment probabilities are constant
- Similar to constant life table
  - Can calculate $\mathcal{R}$ and $\lambda$ (will be consistent with each other)
  - Can calculate a stable stage distribution
  - *Comment:* $\mathcal{R}$ is about the same as in age structured model
- Unregulated growth cannot persist

## Summary

- If the life table remains constant (no regulation or stochasticity):
  - Reach a stable age (or stage) distribution
  - Grow or decline with a constant $\lambda$
  - Factors behind age distribution can be understood

## 6.2 Regulated growth

- Our models up until now have assumed that individuals are independent
- In this case, we expect populations to grow (or decline) exponentially
- We do not expect that the long-term average value of $\mathcal{R}$ or $\lambda$ will be exactly 1.

## The value of simple models

- There is a lot of mathematical study of unregulated, age-structured populations, but it should be taken with a grain of salt
  - **Answer:** We know that real populations are regulated
  - **Answer:** Populations can't increase or decrease exponentially for very long
- Understanding this behaviour is helpful:
  - interpreting age structures in real populations
  - beginning to understand more complicated systems

## Regulation and structure

- We expect real populations to be regulated
- The long-term average value of $\lambda$ under regulation *could* be exactly 1
- There is also likely to be substantial variation from year to year, due to changing conditions and other random-seeming forces

## Dynamics

- We expect to see smooth behaviour in many cases
- Cycles and complex behaviour should arise for reasons similar to our unstructured models:
  - Delays in the system
  - Strong population response to density

- Age distribution will be determined by:
  - $\ell_x$, and
  - whether the population has been growing or declining recently