

## UNIT 3 Non-linear population models

### 1 Introduction

- In **linear** population models, per capita rates are independent of population size
- In **non-linear** models, not so much
- Why might per capita birth and death rates change with population size?
- What does this imply about population **dynamics**?

#### The first law of population dynamics

- If individuals are behaving independently:
  - the population-level rate of growth (or decline) is proportional to the population size
  - the population grows (or declines) exponentially

#### The second law of population dynamics

- Exponential growth (or decline) cannot continue forever
- Something is changing the average rate at which populations we observe grow

#### The third law of population dynamics

- Exponential growth (or decline) cannot continue forever – *even on average*
- Environmental variation cannot be the only thing that changes growth rates
- Populations are, directly or indirectly, limiting their own growth rates
- This is called **density dependence**

#### Long-term growth rates

- Populations maintain long-term growth rates very close to  $r = 0$
- This is not possible just by chance, or averaging:
  - Population size must affect its own growth rate
  - Directly or indirectly
  - Long term or short term

## Changing growth rates

- Poll: What is an example of a density-dependent mechanism that affects growth rate?

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## Population regulation

- All the populations we see are *regulated*
  - On average, population growth is higher when the population is lower
  - Maybe with a time delay
- Why is this interesting?
  - Lots of populations don't *look like* they are regulated

## Sometimes regulation is apparent

- Some species seem to fill a niche (mangroves)
- or deplete their own food resources (gypsy moths)

## Sometimes regulation is not apparent

- Other species seem like they could easily be more common (pine trees)
  - May be controlled by cryptic (hard to see) natural enemies (like disease or parasites)
  - May be controlled by limitations that occur only at certain times (e.g., during regular droughts)

## Regulation works over the long term

- Not every species is experiencing population regulation at every time
- A species that we see now may be expanding into a niche (e.g., because of climate change)
- Some species are controlled by big outbreaks of disease
- Some species have big outbreaks into marginal habitat, and spend most of their time contracting back to their “core” habitat

## How do we know it's regulation?

- Poll: Why don't we believe that population growth is controlled by factors that don't depend on the population itself?

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### 1.1 Population Examples

#### Gypsy moths

- What are some factors that limit gypsy-moth populations?
- Which are likely to be affected by the moths?
  - Directly or indirectly, in the short or long term?

## 2 Continuous-time regulation

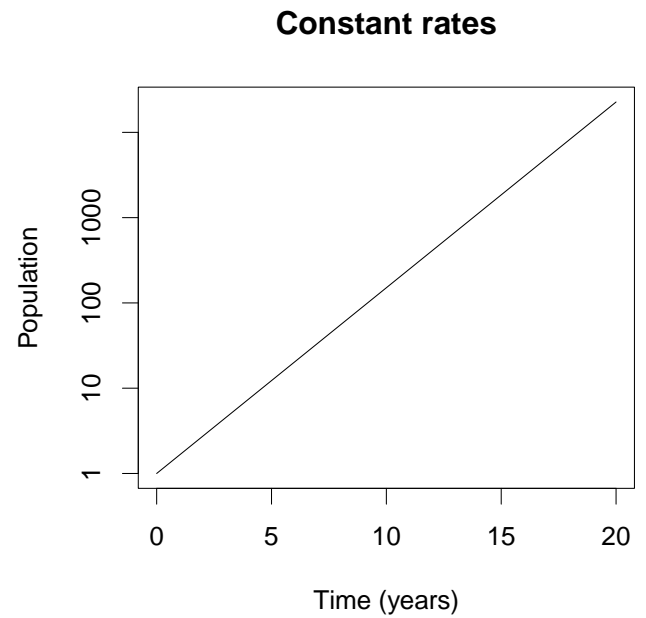
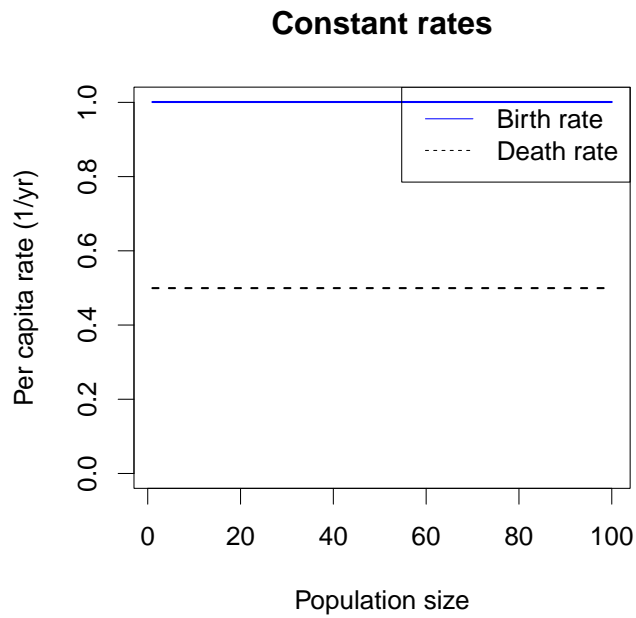
### Build on the linear model

- Our linear population model is:

$$- \frac{dN}{dt} = (b - d)N$$

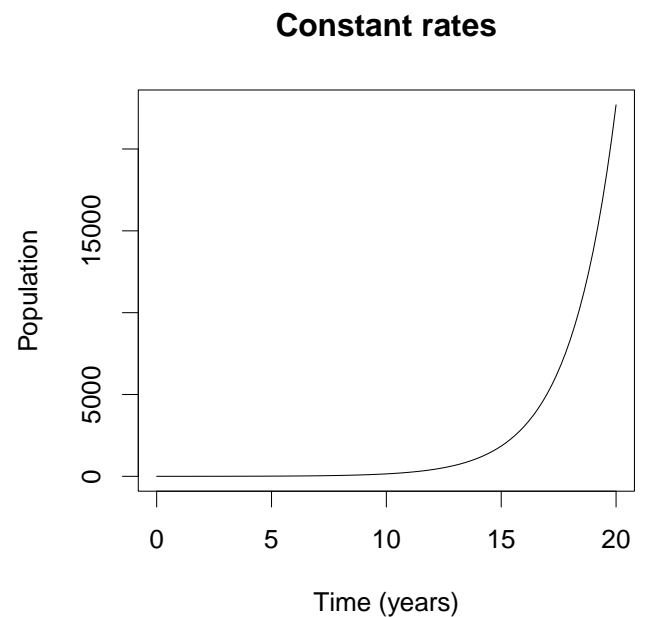
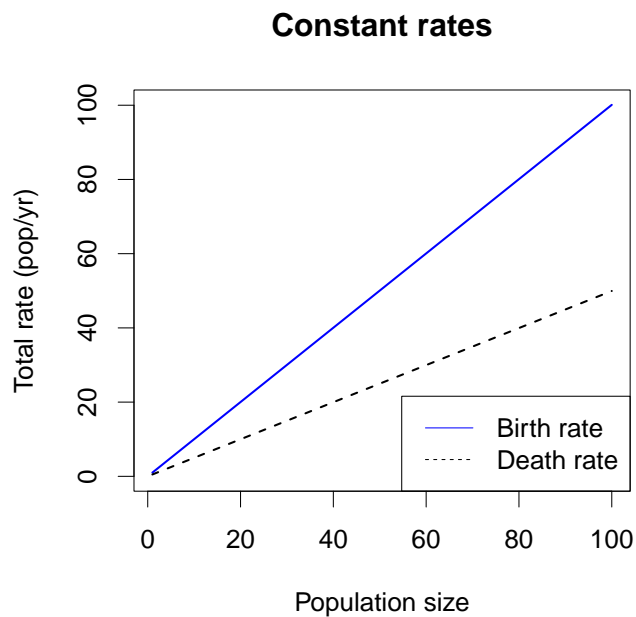
- Per-capita rates are constant
- **Population-level rates are linear**
- Behaviour is exponential

## Individual perspective



- Per capita rate shows birth and death per individual
- Corresponds to the time plot showing growth on a log scale
  - On the log scale we see *multiplicative* or *proportional* change

## Population perspective



- Total rate shows birth and death for the whole population
- Corresponds to the time plot showing growth on a linear scale
  - On the linear scale we see *additive* or *absolute* change

## Non-linear model

- Population has *per capita* birth rate  $b(N)$  and death rate  $d(N)$ 
  - Per-capita rates change with the population size
- Our non-linear model is:  $\frac{dN}{dt} = (b(N) - d(N))N \equiv r(N)N$ 
  - Defines how fast the population is changing at any instant

## Birth rates

- When a population is crowded, the birth rate will usually go down
  - Resources are limited: space, food, light
- But it may stay the same
- Or even go up
  - If individuals shift their resources to reproduction instead of survival

## Death rates

- When a population is crowded, the death rate will often go up
  - Individuals are starving, or conflict increases
  - But it may stay the same
    - \* if reproduction is limited by competition for breeding sites, or by recruitment of juveniles
  - Or even go down
    - \* if organisms go into some sort of “resting mode”

## Reproductive numbers

- Our model is:  $\frac{dN}{dt} = (b(N) - d(N))N \equiv r(N)N$
- Reproductive number now also changes with  $N$ :
  -
- When the population is crowded, individuals are stressed and the reproductive number will typically go down.

## Carrying capacity

- If a population has  $\mathcal{R}(N) > 1$  when it's not crowded
- The population should increase
- Eventually,  $\mathcal{R}$  will decrease, and eventually cross  $\mathcal{R} = 1$
- We call the special value of  $N$  where  $\mathcal{R}(N) = 1$ , the **carrying capacity**,  $K$ 
  - $\mathcal{R}(K) \equiv 1$
  - $b(K) \equiv d(K)$
- When  $N = K$ :
  -

## Logistic model

- A popular model of density-dependent growth is the logistic model
- Per capita instantaneous growth rate  $r$  is a function of  $N$ 
  - $r(N) = r_{\max}(1 - N/K)$
  - Consistent with various assumptions about  $b(N)$  and  $d(N)$
- Population increases to  $K$  and remains there
  - Units of  $N$  must match units of  $K$
- Not a linear model, because *population-level* rates are not linear

## Exponential-rates model

- In this course, we'll mostly use another simple model:
  - $b(N) = b_0 \exp(-N/N_b)$
  - $d(N) = d_0 \exp(N/N_d)$
- This is the simplest model that is smooth and keeps track of birth and death rates separately
  - Birth rate goes down with characteristic scale  $N_b$
  - Death rate goes up with characteristic scale  $N_d$

## Exponential-rates vs. logistic

- The exponential-rates model is conceptually clearer
  - Birth and death rates are clearly defined
- Mathematically nicer
  - Rates always stay positive
- The logistic *looks* less scary

## 2.1 A simple, continuous-time model

### Assumptions

- We model individual-level rates, but individuals are *not* independent: my rates depend on the number (or density) of individuals in the population
- The population can be censused at any time
- Population size changes continuously
- All individuals are the same all the time
- Population changes deterministically

### Interpretation

- If we have  $N$  individuals at time  $t$ , how does the population change?
  - Individuals are giving birth at per-capita rate  $b(N)$
  - Individuals are dying at per-capita rate  $d(N)$
- Population dynamics follow:

$$- \frac{dN}{dt} = (b(N) - d(N))N \equiv r(N)N$$

### States and state variables

- What variable or variables describe the state of this system?
  - 
  - 
  - \*

## Parameters

- Poll: What quantities describe the rules for this system?
  - 
  - 
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## Characteristic *scale*

- A characteristic scale for density dependence is analogous to a characteristic time
  - It is a *parameter* with the same units as a state variable, and sets a standard by which we measure that variable
  - Nothing (with units) is big or small until it's compared to something with the same units
- For example:  $b(N) = b_0 \exp(-N/N_b)$ 
  - $N_b$  is the characteristic scale of density-dependence in birth rate
  - If  $N \ll N_b$ , density dependence is linear (and relatively small)
  - If  $N \gg N_b$ , density dependence is exponential, and very large (virtually no births)

## Model

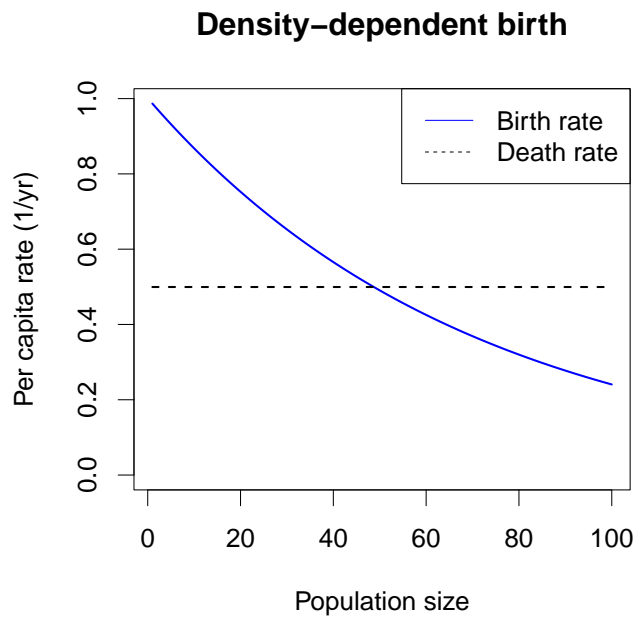
- Dynamics:
  - $\frac{dN}{dt} = (b_0 \exp(-N/N_b) - d_0 \exp(N/N_d))N$
- Exact solution:
  - Insanely complicated
- Behaviour of the solution:
  - Pretty easy!

## Dynamics

- What sort of **dynamics** do we expect from our conceptual model?
  - I.e., how will it change through time?
- What will the population do if it starts
  - near zero?
  - near the equilibrium?
  - at a high value?

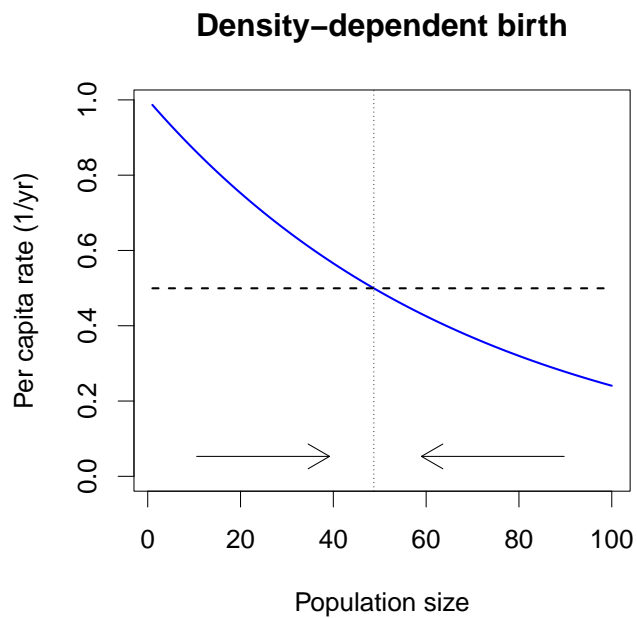


What is the characteristic scale?



- No density dependence in death rate
- Characteristic scale for birth rate is about 70 indiv

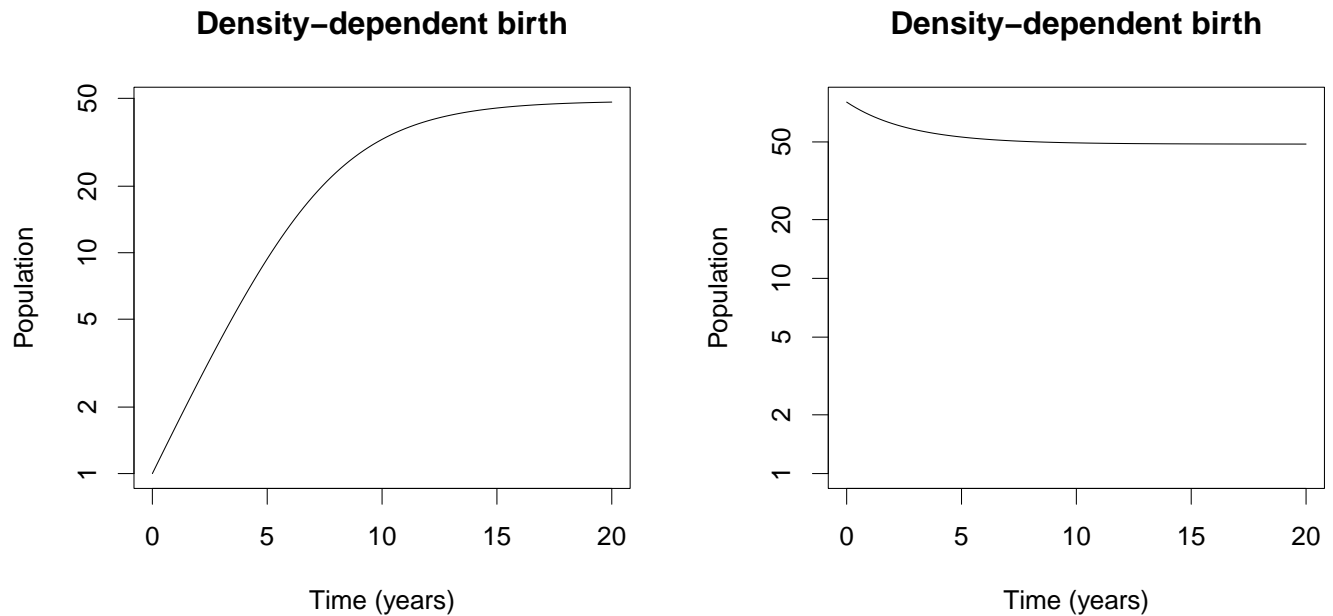
What will this model do?



- Increase when population is below equilibrium
- Decrease when population is above equilibrium

- Converge

## Examples



## 2.2 Simulating model behaviour

### Simulations

- We will simulate the behaviour of populations in continuous time using the program R
- This program generates the pictures in this section by implementing our model of how the population changes instantaneously

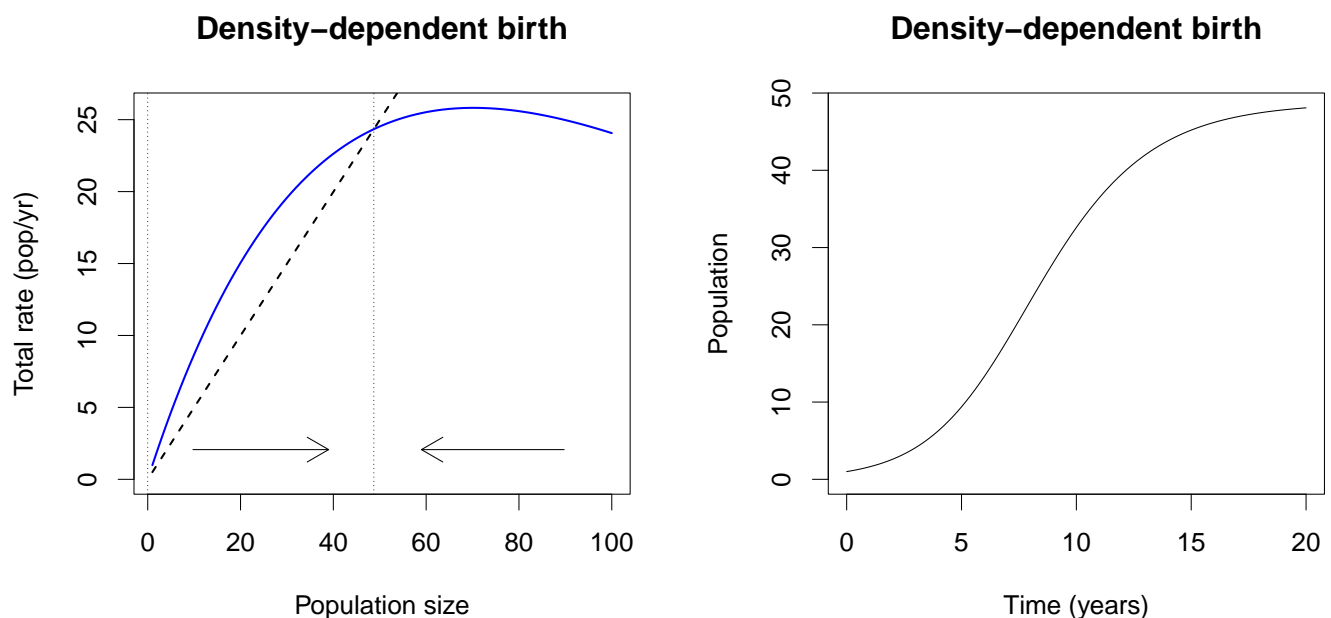
### Individual-scale pictures

- We can view graphs of our population assumptions on the individual scale
  - per-capita birth and death rates
    - \* units  $[1/\text{time}]$
  - what is each individual doing (on average)?
  - corresponds to the dynamics we visualize on a log-scale graph of the population
  - See above

## Population-scale pictures

- We can view graphs of our population assumptions on the population scale
  - total birth and death rates
    - \* units [indiv/time]
    - \* or [density/time] = [(indiv/area)/time]
  - what is changing at the population level?
  - corresponds to the dynamics we visualize on a linear-scale graph of the population

## Population perspective picture



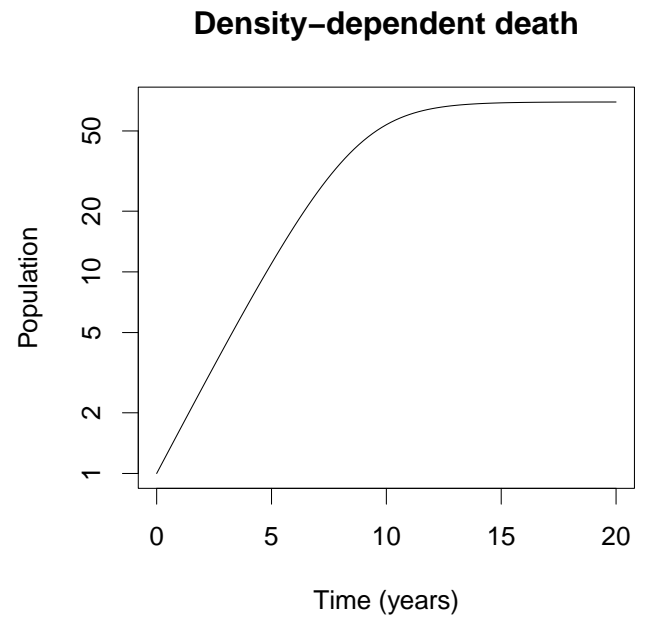
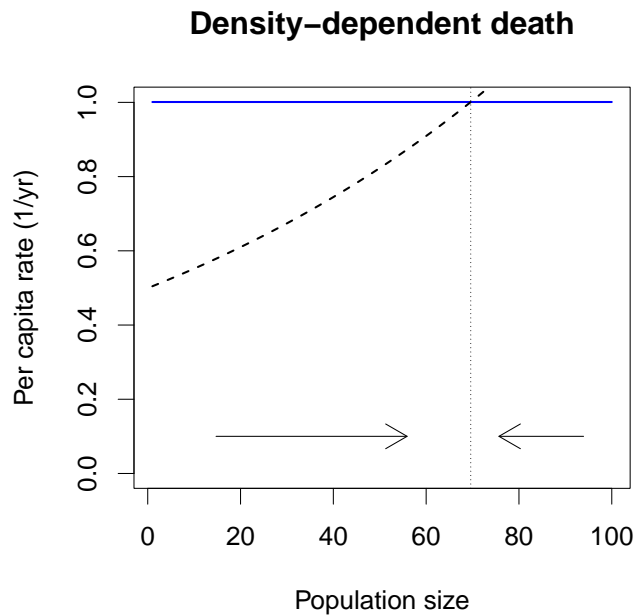
## Decreasing birth rate

- Decreasing birth rate (above) might be a good model for organisms that experience density dependence primarily in the recruitment stage
- For example, we might count adult trees, and these might not die more at high density – just fail to recruit younger ones

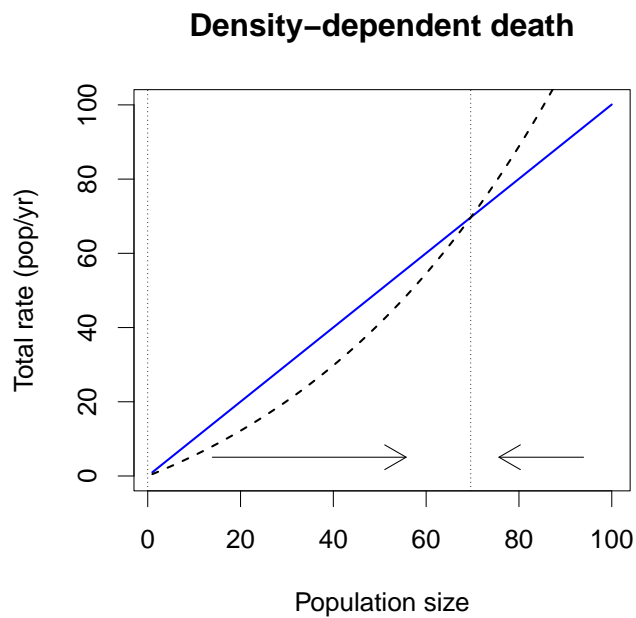
## Increasing death rate

- Increasing death rate might be a good model for organisms that experience density dependence primarily as adults
- For example, in some environments, mussel density might be limited by adult crowding. Although juvenile mussels tend to have a hard time, this might not be density dependent

## Individual perspective



## Population perspective



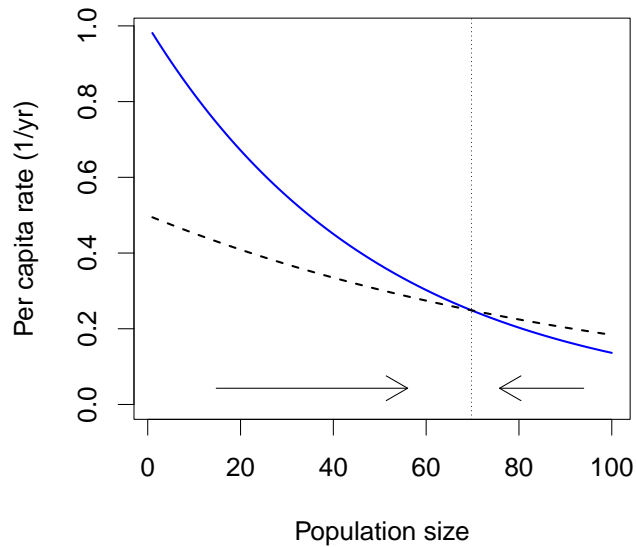
## Decreasing death rate

- Some organisms (such as many types of bacteria) slow down their metabolisms under density dependence, so that death rate *decreases*
- How is this consistent with density dependence?

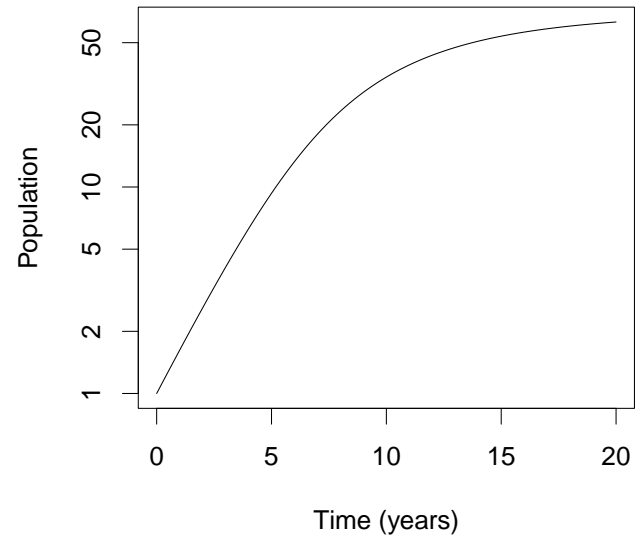
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## Individual perspective

Density dependence and slowing down

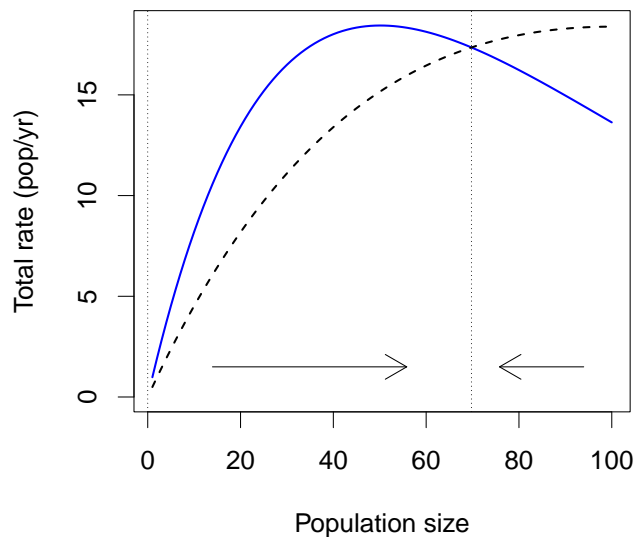


Density dependence and slowing down

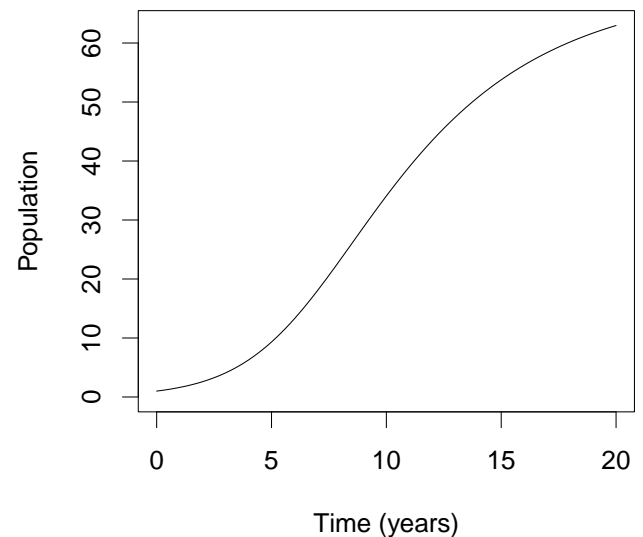


## Population perspective

Density dependence and slowing down



Density dependence and slowing down



## Other examples

- There are two other possible scenarios for density dependence
  - For fun, you can try to think of what they are
- But all of these examples have similar behaviour

- Increase from low density
- Decrease from high density
- Approach carrying capacity

## Maximum growth rates

- When does a population in this model have the fastest *per-capita* growth rate?
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  -
- When does a population in this model have the fastest *total* growth rate?
  - 
  -

## 2.3 Equilibria

- We define **equilibrium** as when the population is not changing
- Our simple model is  $\frac{dN}{dt} = (b(N) - d(N))N$
- In this simple model, when does equilibrium occur?
  - 
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## Stable and unstable equilibria

- Aren't equilibria always stable?
  - If we are at an equilibrium we expect to stay there
  - (in our simplified model, at least)
- An equilibrium is defined as stable if we expect to approach the equilibrium *when we are near it*.
- An equilibrium is defined as unstable if we expect to move away from the equilibrium *when we are near it*.

## What kind of equilibrium?

- How can we tell an equilibrium is stable?
  - If population is just below the equilibrium:  
\*
  - If population is just above the equilibrium:  
\*

## Basic reproductive number

- The reproductive number of a population not affected by crowding is called the **basic reproductive number**
  - Written  $\mathcal{R}_0$  or  $\mathcal{R}_{\max}$ .
- In this model, when  $\mathcal{R}_0 < 1$  the population:
  -
- When  $\mathcal{R}_0 > 1$  the population:
  - 
  -
- Poll: What is  $\mathcal{R}_0$  in our current model?
  -
- How do we interpret  $b(0)$  and  $d(0)$ ?
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## Invasion

- We say a species can “invade” a system if its rate of change is positive when the population is small.
- In other words, population can invade if the extinction equilibrium is not stable
- In this conceptual model, this is the same as saying  $b(0) > d(0)$
- Which is the same as saying  $\mathcal{R}_0 > 1$

## Invasion examples

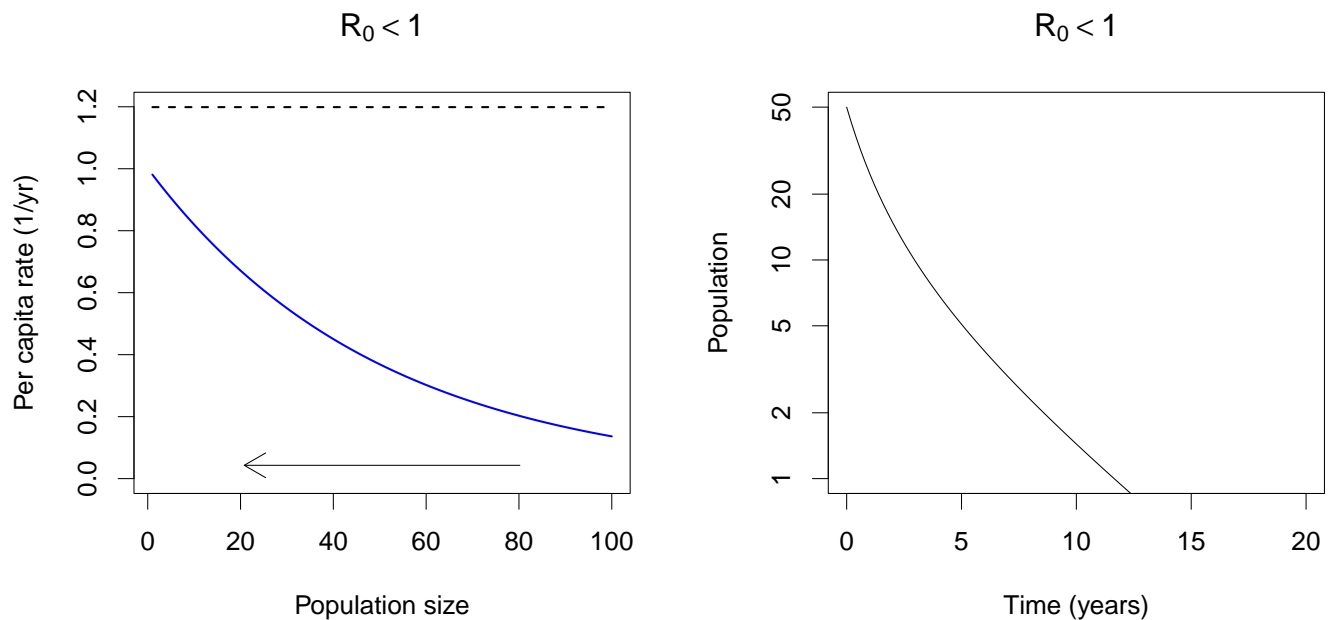
- Poll: What are some examples of biological invasions?

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## Different behaviours

- When  $\mathcal{R}_0 > 1$ , the population invades
  - Zero equilibrium is unstable, carrying capacity equilibrium is stable
- When  $\mathcal{R}_0 < 1$ , the population fails to invade
  - Zero equilibrium is stable, carrying capacity equilibrium does not exist

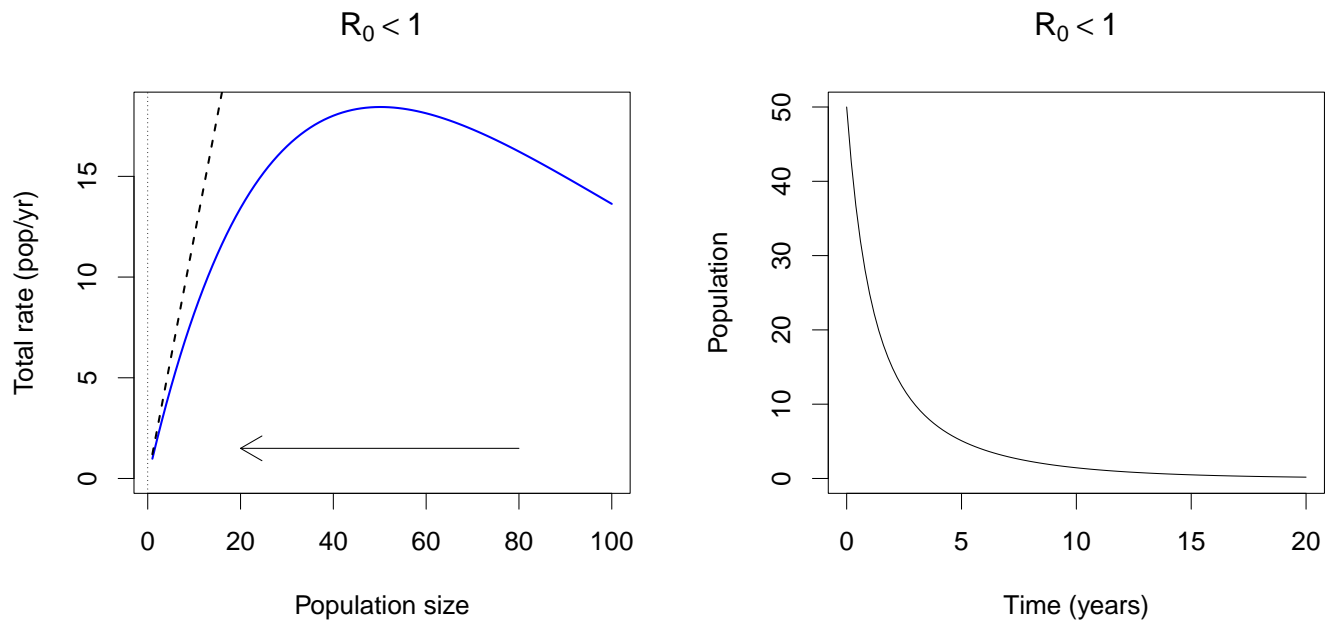
## Individual perspective



- When  $\mathcal{R}_0 < 1$  population always decreases



## Population perspective



- When  $\mathcal{R}_0 < 1$  population always decreases

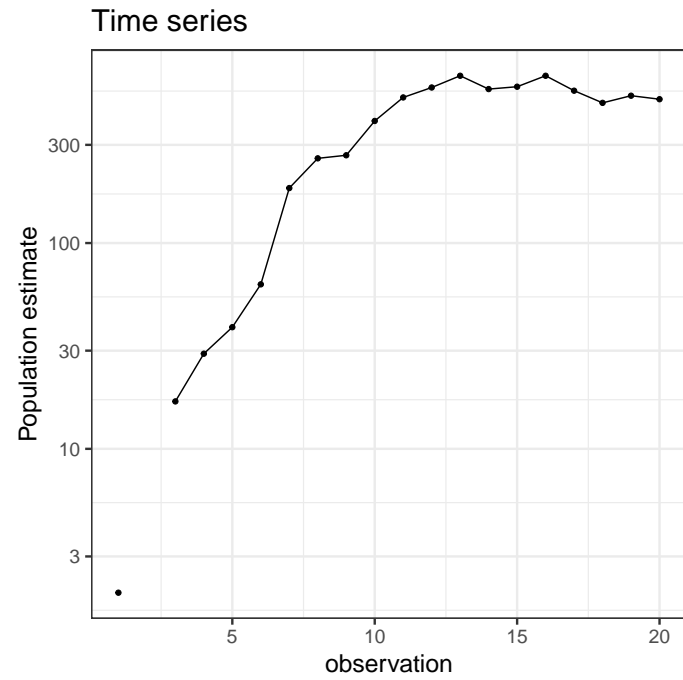
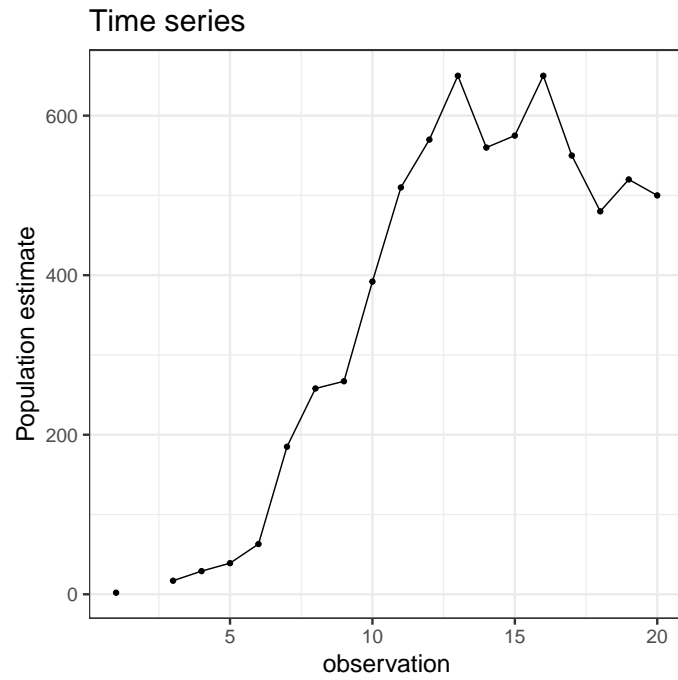
## $\mathcal{R}_0$ and thresholds

- A population with  $\mathcal{R}_0 < 1$  in general cannot survive in an area
- As conditions get worse for a species in a particular area, or along a particular gradient:
  - It will suddenly disappear at the population level
  - Even while it can still survive and reproduce at an individual level
- This is why there are no white spruce trees in Cootes Paradise
- And no malaria in the mainland United States

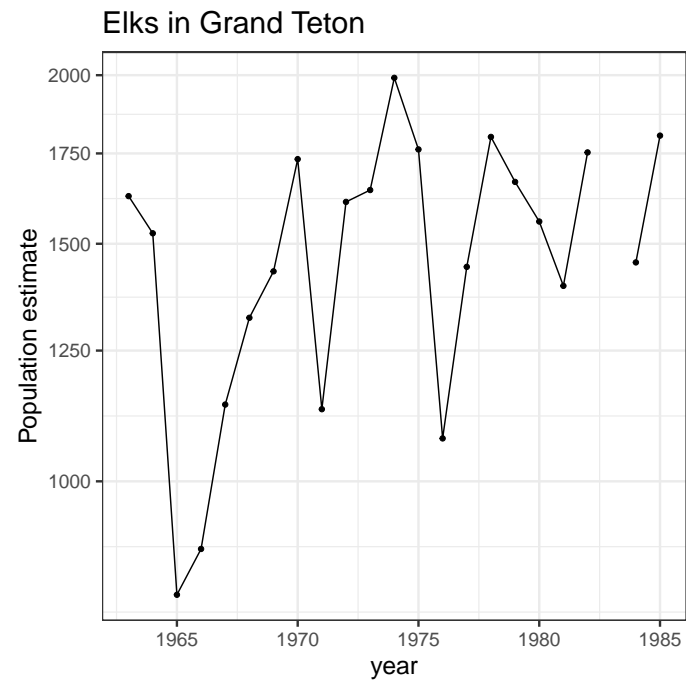
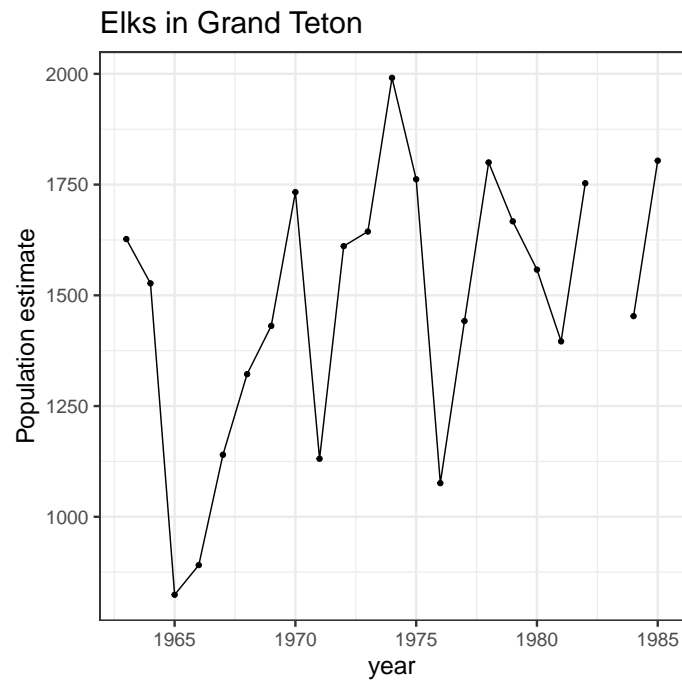
## Smooth dynamics

- Populations following this model change *smoothly*
  - Equations tell how the population will change at each instant
- They have no memory
  - Birth rate and death rate are determined by population size alone
- Cycling is impossible
  -

## Paramecia



## Elk



## Dynamics of real-world populations

- Initial exponential growth and leveling off frequently observed
- Exponential approach to equilibrium hard to observe
  - Real populations are subject to **stochastic** (random) effects
  - Real populations are subject to changing conditions

- Some species seem to cycle predictably

## Summary

- Continuous-time regulation in simple models makes useful predictions:
  - Threshold value for populations to survive
  - Greatest population-level growth at intermediate density
  - Greatest individual-level growth at low density
- Cannot explain complicated dynamics
  - More mechanisms are needed

## 3 Discrete-time regulation

### 3.1 A simple, discrete-time model

- We extend our discrete-time model from the previous unit:
  - $N_{T+1} = (p + f)N_T \equiv \lambda N_T$
  - $t_{T+1} = t_T + \Delta t$  (does not change)
- To:
  - $N_{T+1} = (p(N_T) + f(N_T))N_T \equiv \lambda(N_T)N_T$
- This means:
  -

## Assumptions

- The population is censused at regular time intervals  $\Delta t$
- All individuals are the same at the time of census
- Population changes deterministically

## Specific assumptions

- For our examples, we will assume:
  - $f(N) = f_0 \exp(-N/N_f)$
  - $p(N) = p_0 \exp(-N/N_p)$
- This is the simplest model that is smooth and keeps track of birth and death rates separately
  - Fecundity goes down with characteristic scale  $N_f$
  - Survival goes down with characteristic scale  $N_p$

## States and state variables

- What variable or variables describe the state of this system?
  - The same as before: population size (or density)
  - We are still assuming that's all we need to know

## Parameters

- What quantities describe the rules for this system?
  - 
  - 
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## What is $\mathcal{R}_0$ ?

- $\mathcal{R}$  is the fecundity multiplied by the lifespan
  - 
  -
- $\mathcal{R}_0$  is  $\mathcal{R}$  in the limit where density is low
  -

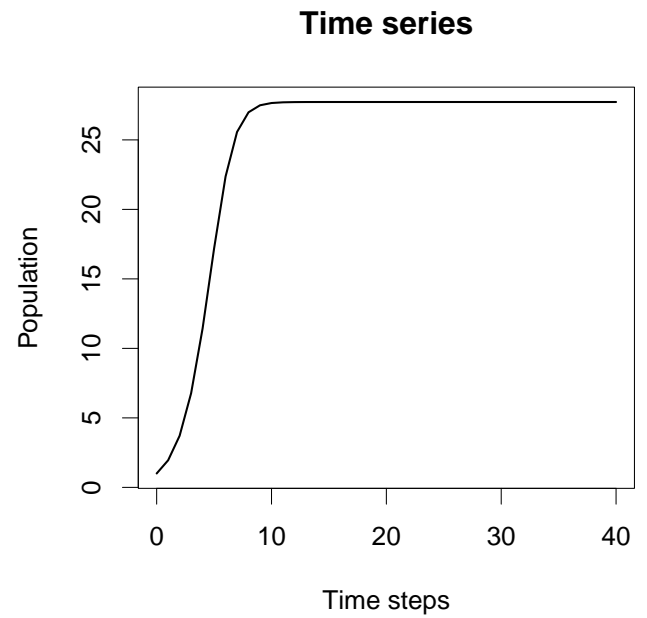
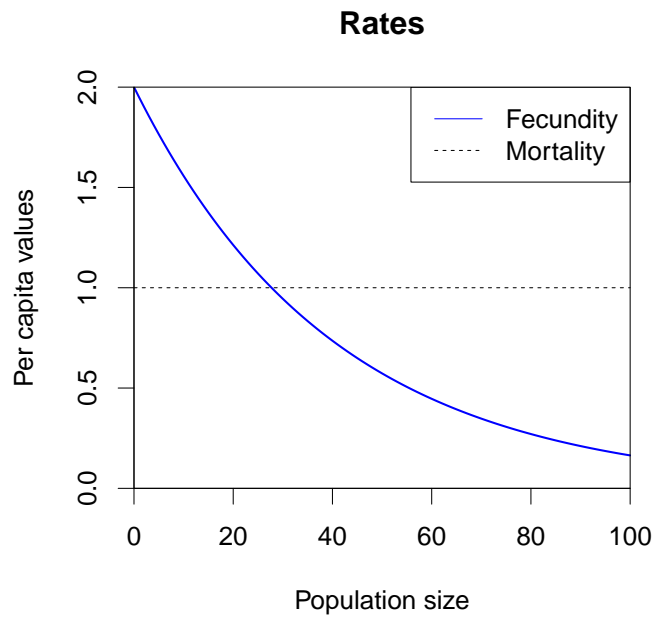
## Behaviours

- When  $\mathcal{R}_0 < 1$  population always declines
- When  $\mathcal{R}_0 > 1$ , population can show:
  - Smooth behaviour (like the continuous-time model)
  - Damped oscillations (like the delayed model)
  - Two-year cycles (high  $\rightarrow$  low  $\rightarrow$  high  $\rightarrow$  low)
  - All kinds of other stuff

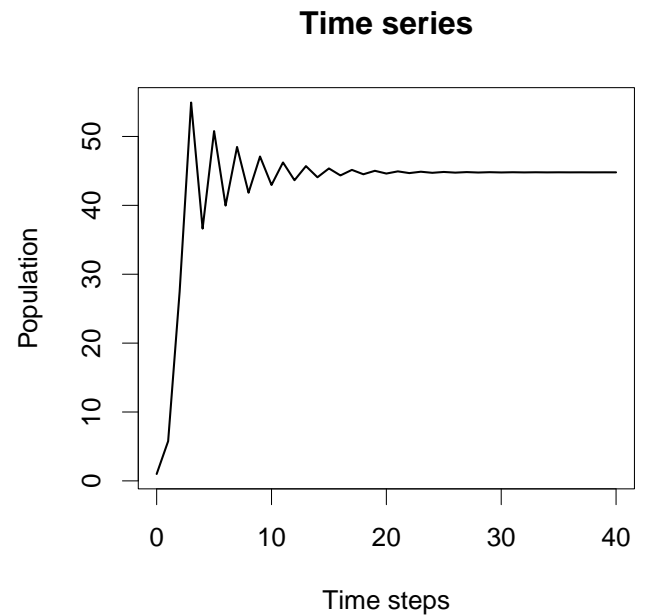
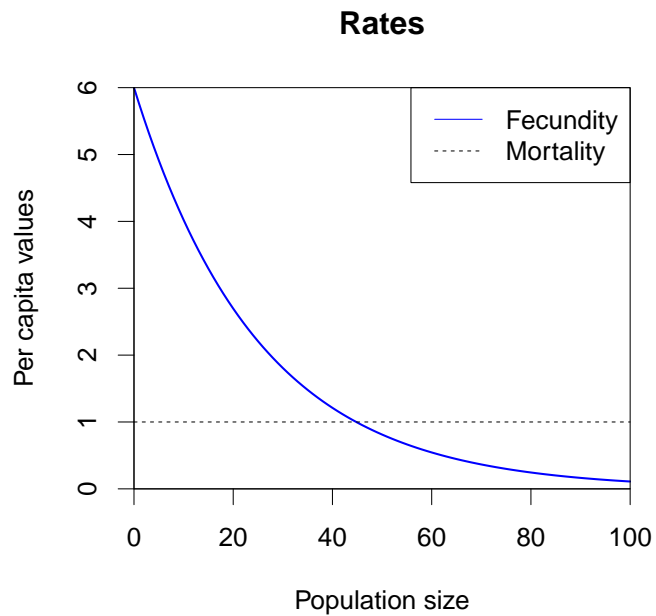
## 3.2 Simulating this system

- This system can be simulated very easily by following the rule for  $N_{T+1}$  as a function of  $N_T$
- We can even do it in the spreadsheet if we have time

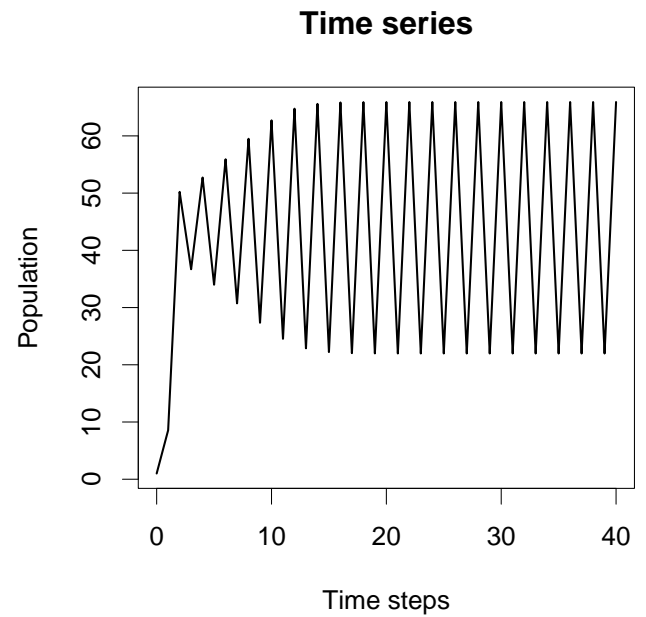
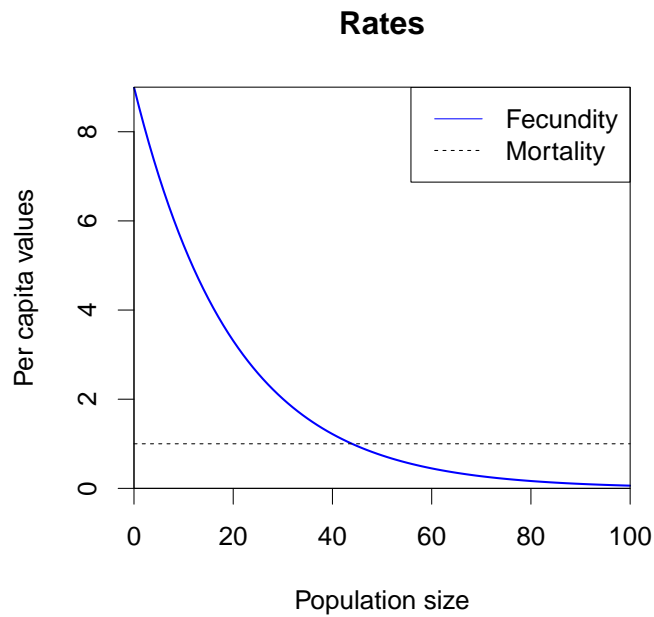
We expect simple dynamics



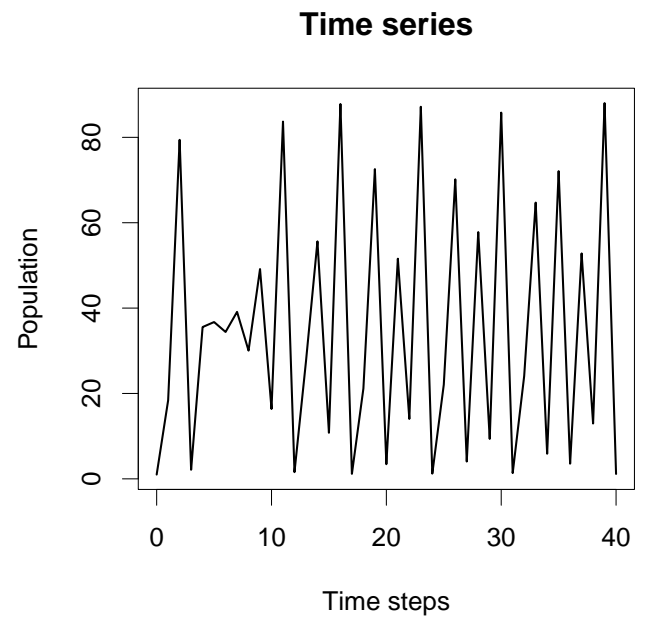
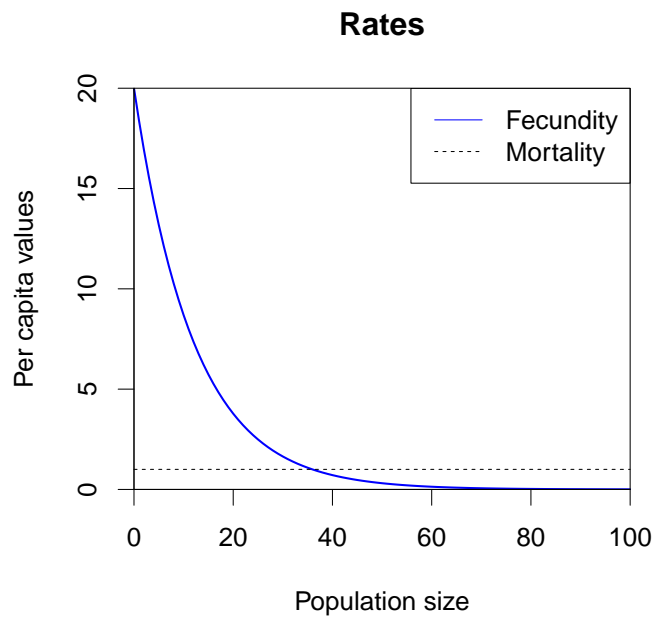
Damped oscillations



Persistent oscillations



**Lots of other behaviours**



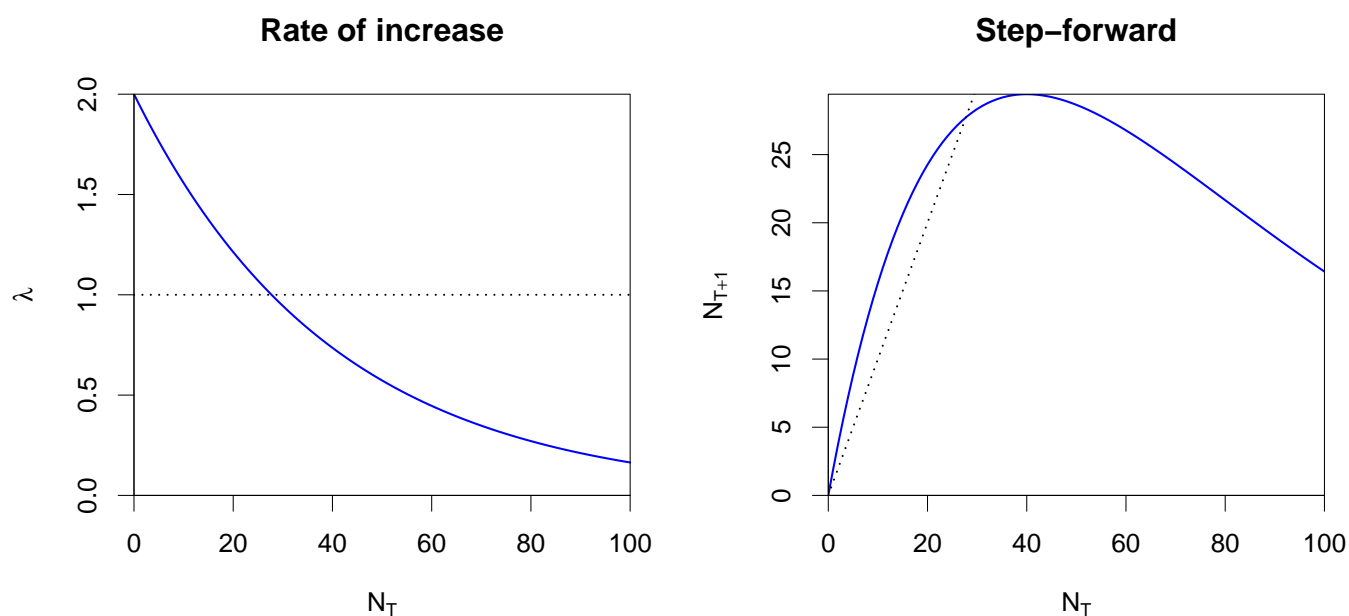
### 3.3 Interpreting complex behaviour

- In a simple cycle:
  - Low populations this year mean high populations next year
  - and vice versa

## Complex behaviour in our simulations

- In our simple models, as  $N_T$  increases, what happens to  $\lambda$ ?
  -
- Poll: In our simple models, as  $N_T$  increases, what happens to next year's population?
  - 
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## Response to population increase

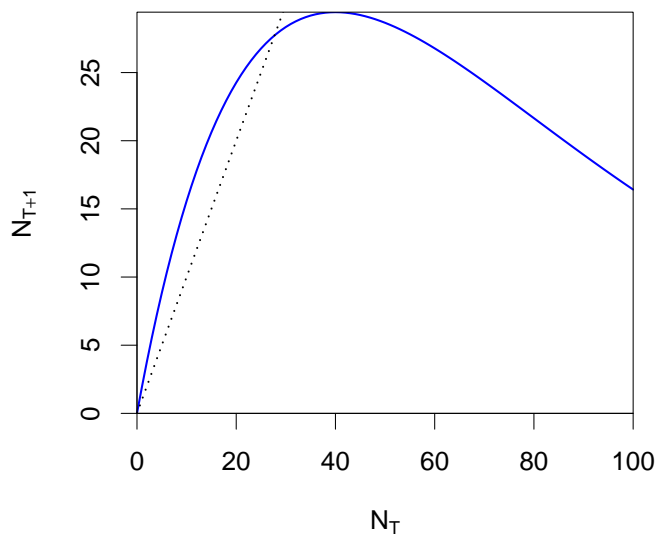


## Turnover

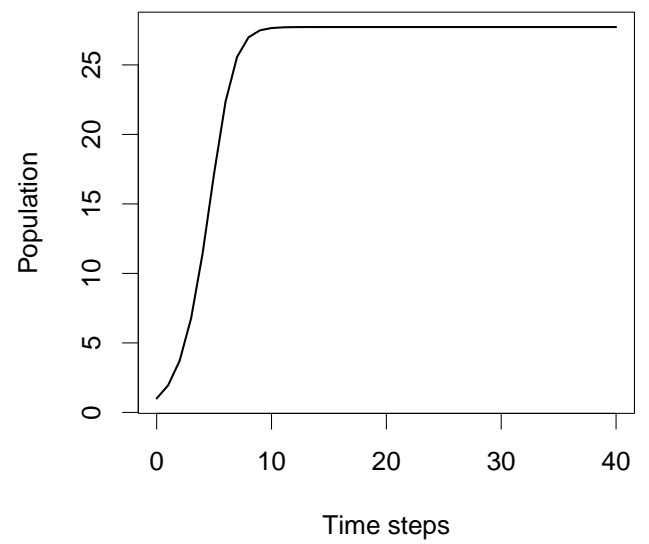
- When  $N_T$  is small,  $N_{T+1}$  increases with  $N$ .
- Complex behaviour arises when the relationship between  $N_T$  and  $N_{T+1}$  **turns over** below the equilibrium value
  - A small population this year leads to a large population next year

## Simple dynamics

**Step-forward**

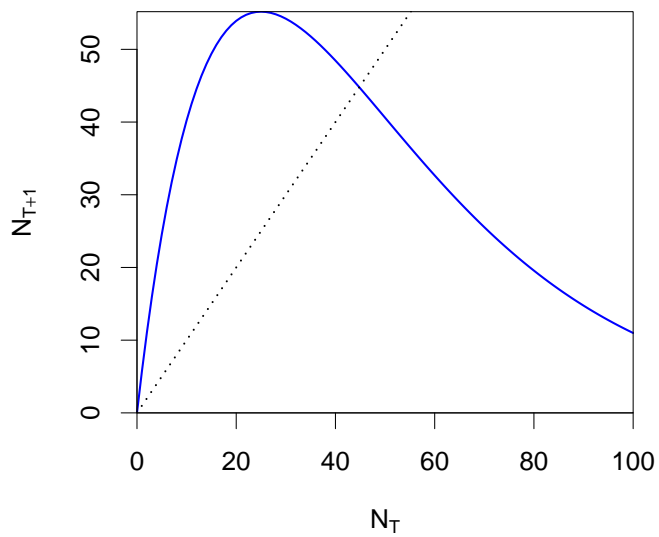


**Time series**

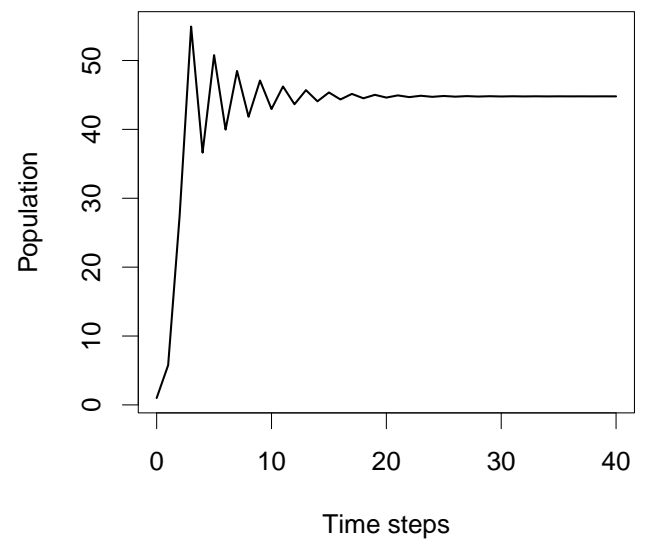


## Damped oscillations

**Step-forward**

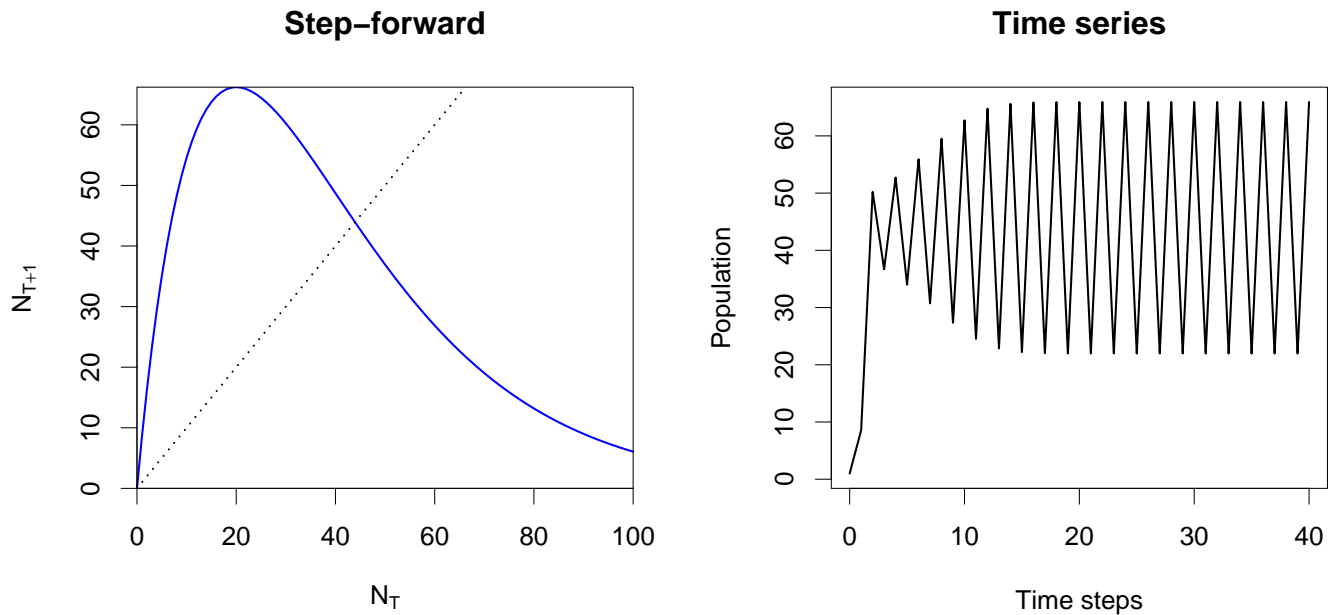


**Time series**



## Persistent oscillations





## Complex behaviour in our conceptual model

- Biologically, when might we expect  $N_{T+1}$  to “turn over”?

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- When should the mapping *not* turn over?

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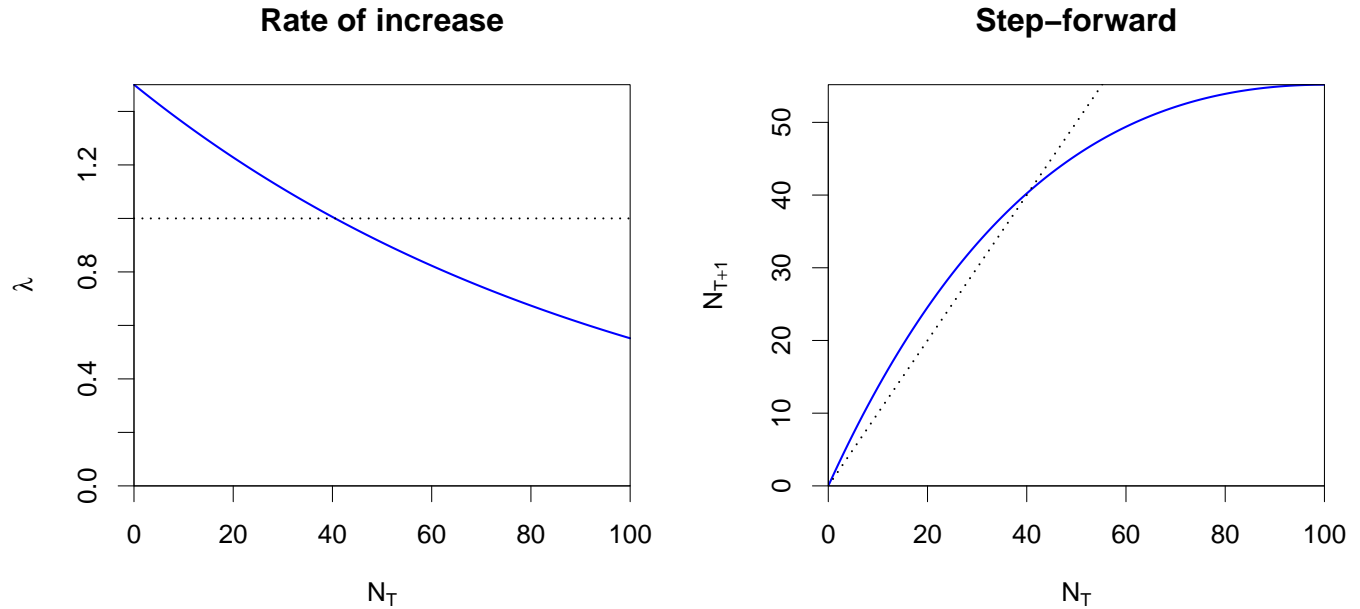
## Scramble competition

- **Scramble** competition refers to the case where all individuals in a crowded population are gathering resources at similar rates: as the density goes up there is less resource for everyone, and everyone does less well
  - If there is any kind of delay, scramble competition can lead to turning over

## Contest competition

- **Contest** competition refers to a case where some individuals successfully control key resources and do well no matter how large the population is
  - Contest competition doesn’t usually lead to turning over, even with delay
- How does contest competition square with regulation?

## Contest regulation



## Songbirds

- Some songbird populations are limited primarily by competition for breeding sites, whereas others are limited primarily by competition for insects to eat

## Plants

- Some plant populations are limited by water, and some by light
- Poll: Which is more likely to work out as a scramble?

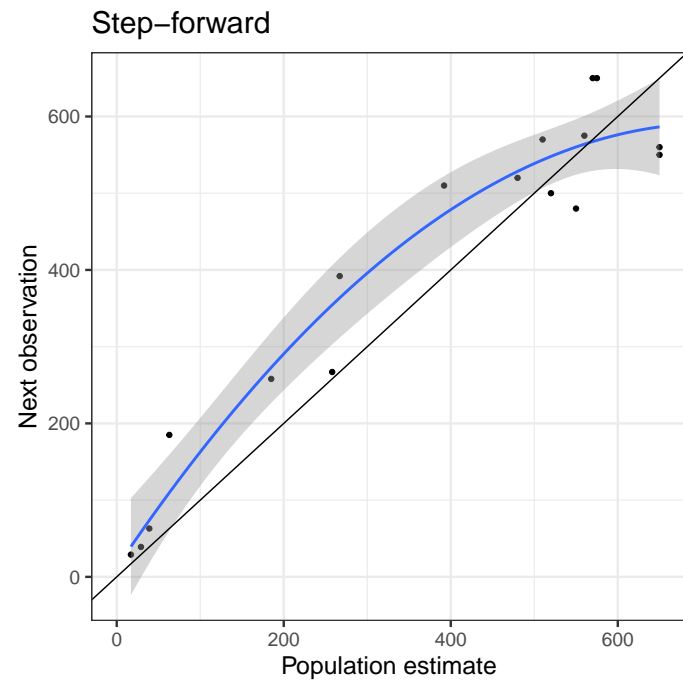
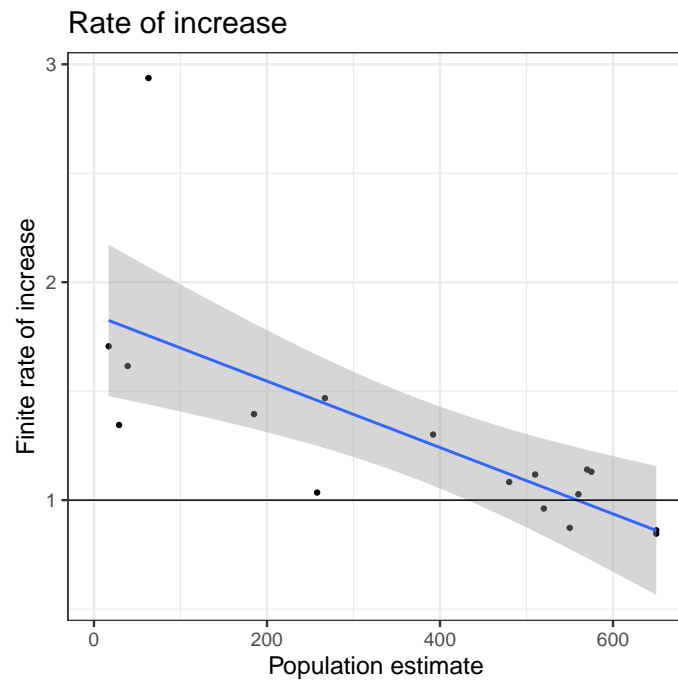
## Complex behaviour from a simple model

- It's interesting that we can get complicated behaviour from such a dead-simple model
- Complex dynamics may have simple causes
- People always tend to look for specific reasons, but sometimes the changes we observe are just natural dynamics

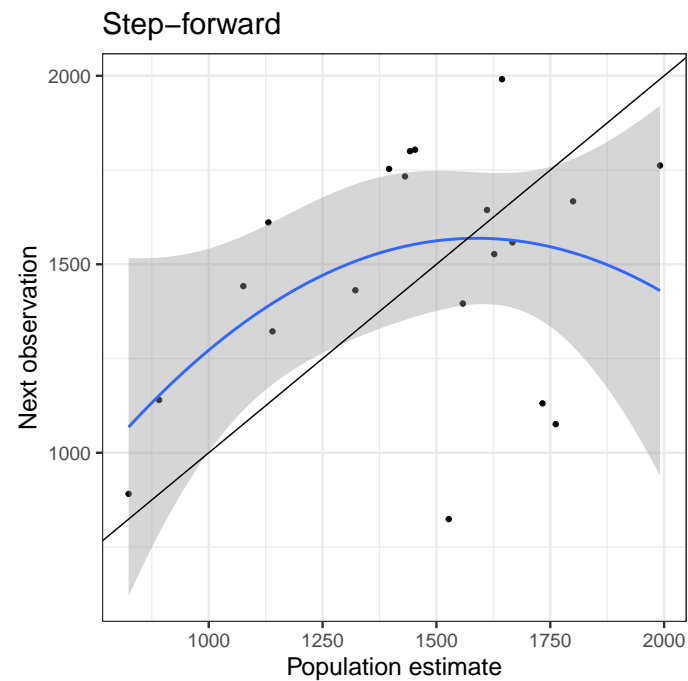
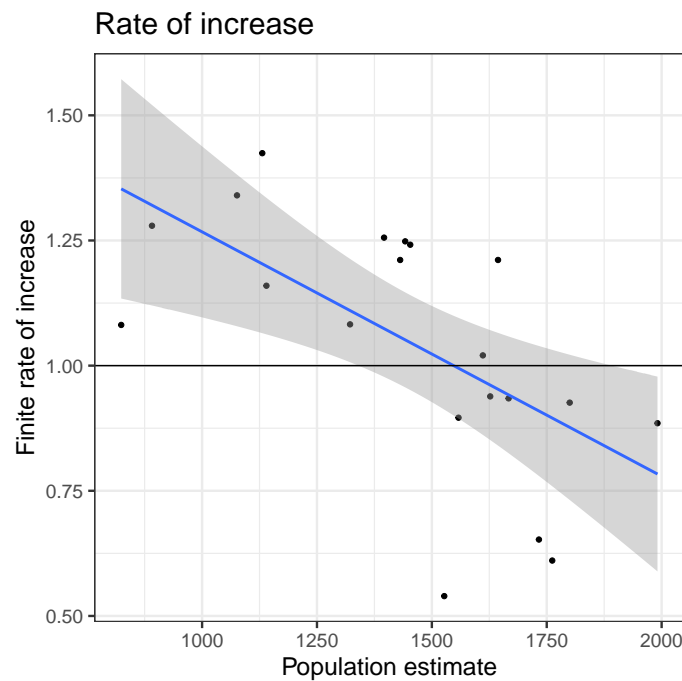
## Complex behaviour in real populations

- We can plot  $\lambda$  and  $N_{T+1}$  vs.  $N$  for real population data
- We expect  $\lambda$  to decrease (on average)
- We're curious about  $N_{T+1}$ .

### Paramecia



### Elk



## Real populations

- It's hard to find examples of turnover from real population data.
- So how do we explain real population cycles?
  - Regulation may happen on a longer time scale
  - May be hard to see because of “noise” – i.e., other sources of variation
  - Cycles may be due to more complicated mechanisms
- Time-delayed population models are cool!
  - But not covered in class this year

## 4 Small populations

### Example

- Poll: What would happen if I released one butterfly into a new, highly suitable habitat?
  -
- What about two butterflies?
  -

### Small populations

- Population success (reproductive number) may be lower for very small populations
  - We've already assumed reproductive numbers are low for very large populations
- Small populations are likely to be harder to predict
  - More affected by stochasticity

#### 4.1 Allee effects

- Effects which cause small populations to have low per-capita growth rates are called Allee effects
  - Equivalent to saying that medium-sized populations have larger per-capita growth rates than small ones

- Poll: Why might growth rates be low when populations are small?

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—  
—  
—

## Types of Allee effect

- Allee effects can affect the (per capita) birth rate

—

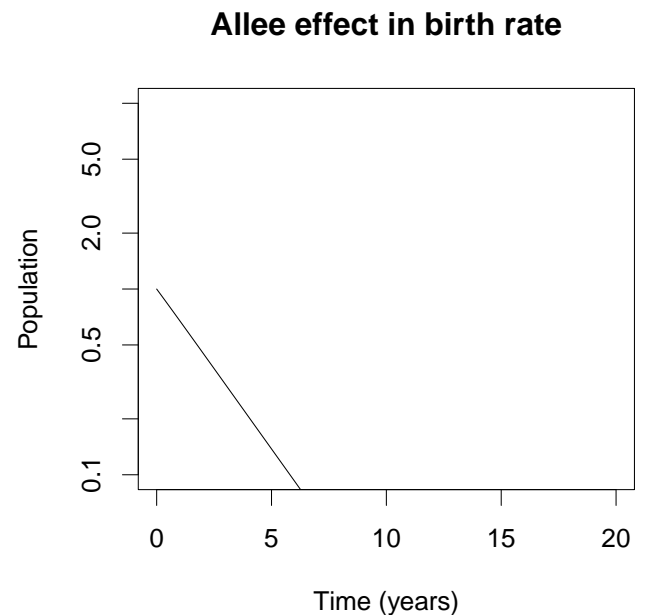
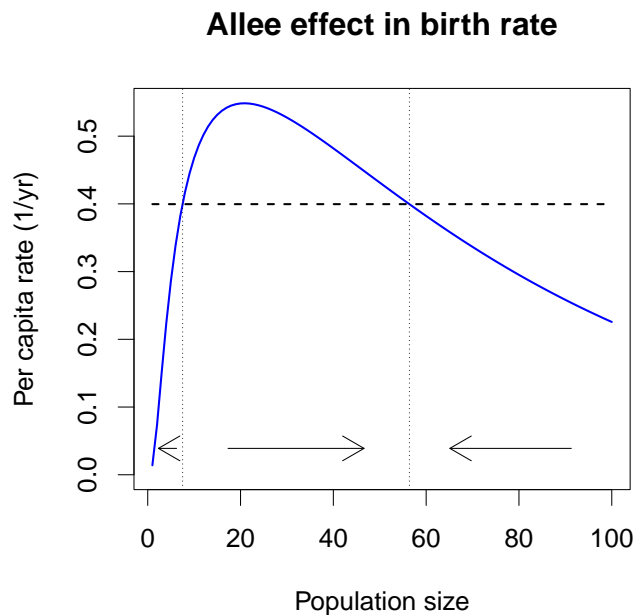
- ... or the (per capita) death rate

—

## Allee effect models

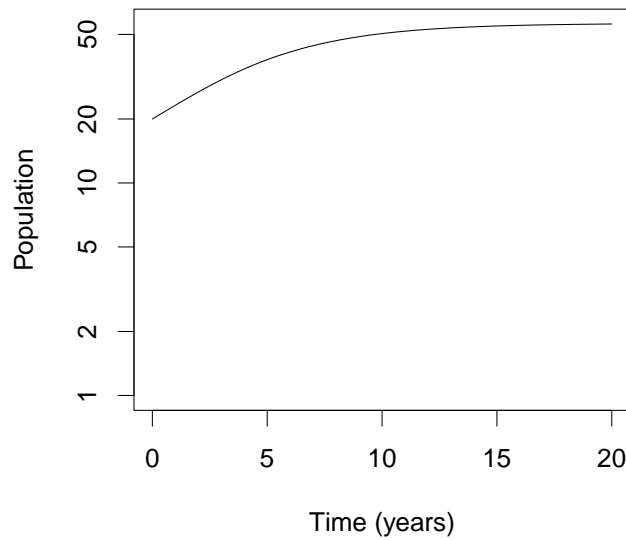
- What will this model do, if the initial population is:
  - low, medium or high?

## Individual perspective

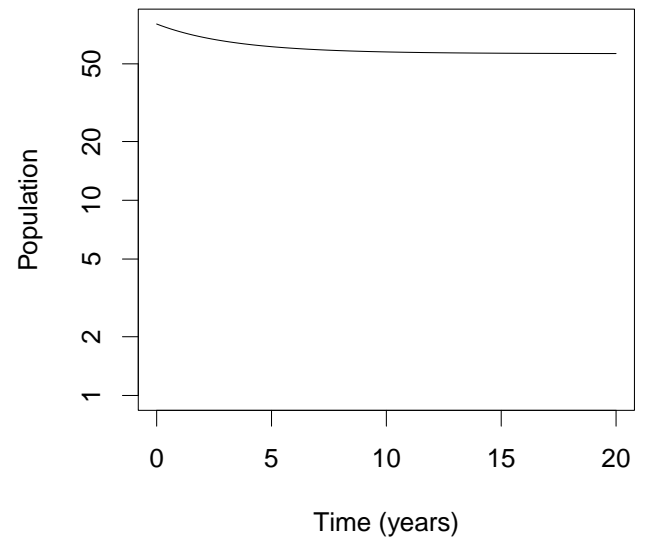


## Individual perspective

**Allee effect in birth rate**

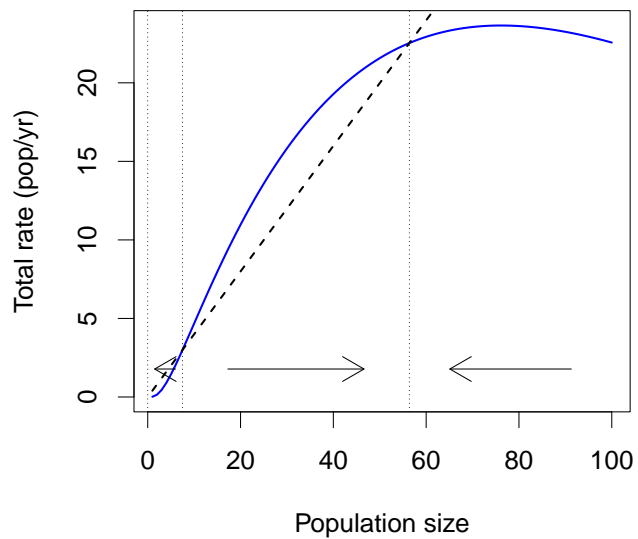


**Allee effect in birth rate**

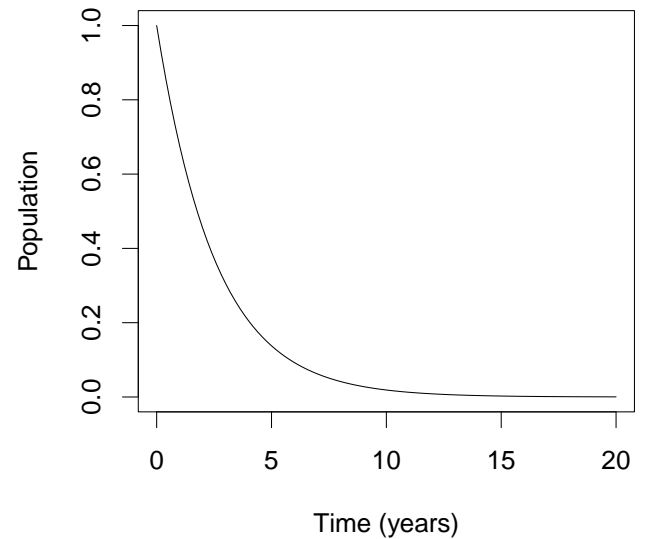


## Population perspective

**Allee effect in birth rate**

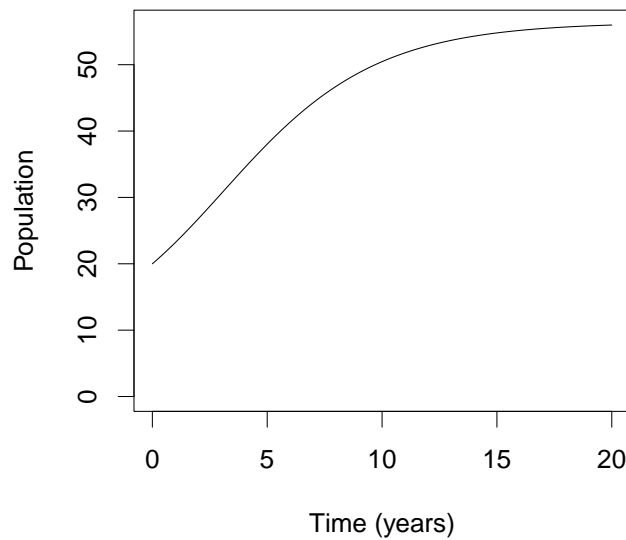


**Allee effect in birth rate**

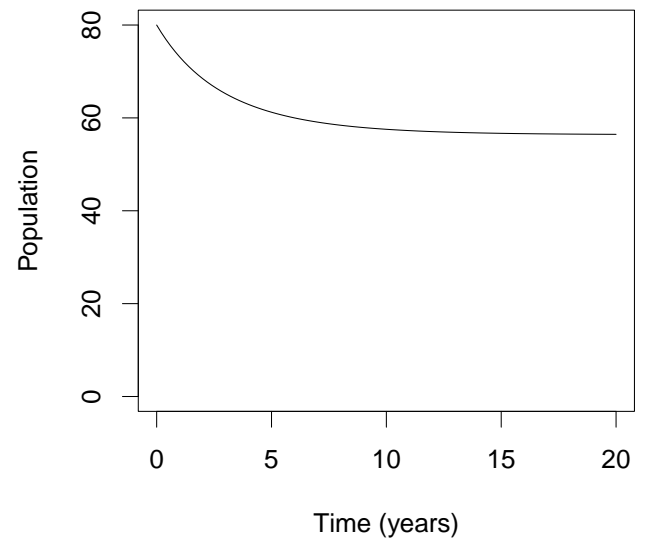


## Population perspective

**Allee effect in birth rate**



**Allee effect in birth rate**

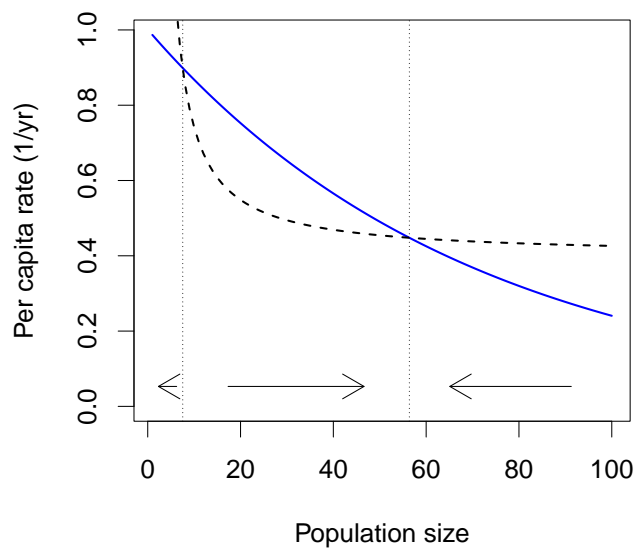


## Allee effect in death rate

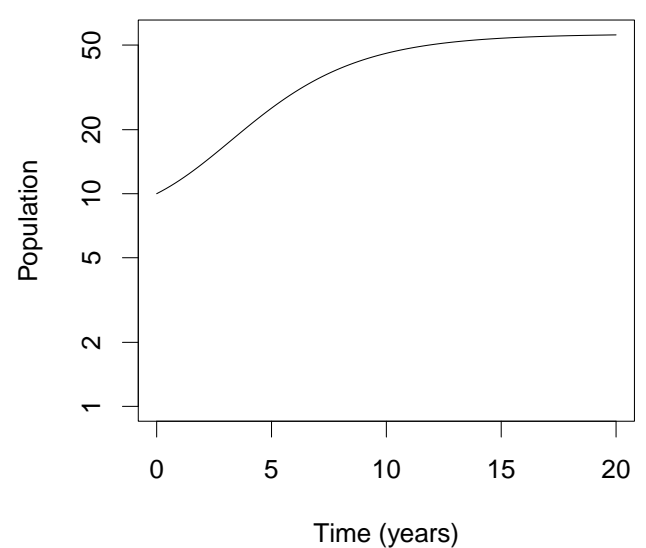
- What is the difference between this example and the previous one?
- What will this model do, if the initial population is:
  - low, medium or high?

## Individual perspective

**Allee effect in death rate**



**Allee effect in death rate**



## More reproductive numbers

- The reproductive number  $\mathcal{R}$  means the average lifetime number of offspring per individual
  - Should be unitless, so we consider offspring at the same stage as the individual.
- We can apply  $\mathcal{R}$  in general for any set of conditions, or we can distinguish:
  - the **basic reproductive number**  $\mathcal{R}_0$ :  $\mathcal{R}$  in the limit when the population is small, and
  - the **maximal reproductive number**  $\mathcal{R}_{\max}$ :  $\mathcal{R}$  at whatever level is the peak

## Invasion

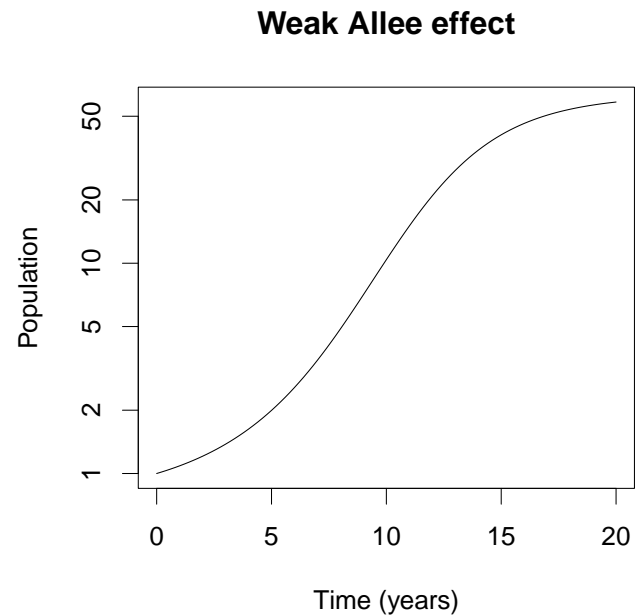
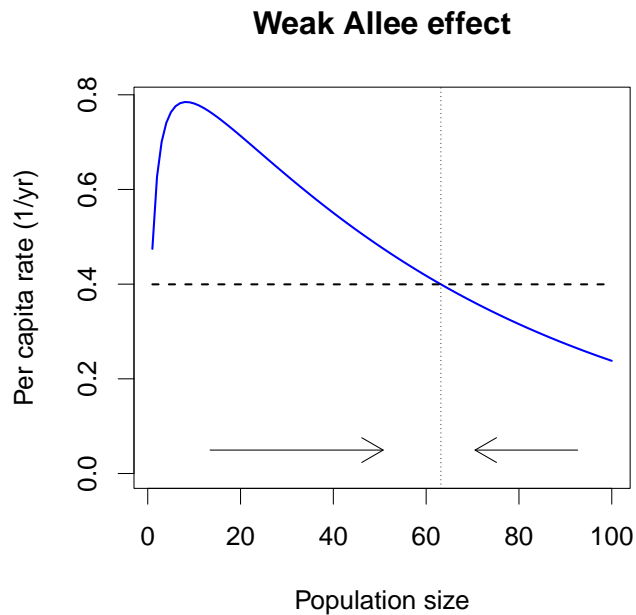
- We previously said that when  $\mathcal{R}_0 < 1$ , the population always went extinct
  - A population that can't invade can never replace itself on average
- When Allee effects are present, it's no longer true that a species that can't invade can't persist
  -

## Weak Allee effects

- If birth rates go down or death rates go up at low density, we consider this an Allee effect
- If  $\mathcal{R}_0 < 1$  we say it's a **strong** Allee effect
  -
- If  $\mathcal{R}_0 > 1$  we say it's a **weak** Allee effect

## Individual perspective





## Allee effect summary

- Population may go extinct if it drops below a certain threshold
- Poll: How come the population is there in the first place if there's an Allee effect?
- 
- 
- 
- 

## 4.2 Stochastic effects

- The world is complicated and biological populations are not perfectly predictable
- Real populations don't go smoothly to equilibria, instead they bounce around (or sometimes do other wild stuff)
- We divide stochastic (or random) effects into demographic and environmental stochasticity

## Example

- Female butterflies of a certain species lay 200 eggs on average, of which:
  - Half are female
  - 50% hatch successfully into larvae

- 10% of larvae successfully pupate
- 60% of pupae become adults
- Half of adult females successfully reproduce
- A single gravid (pregnant) female butterfly is blown away by a freak storm, and lands by chance on a suitable island with no butterflies What do you expect to happen?
- 
- 

## Butterfly example

- Depending on unknown conditions, especially in that first season, all of those probabilities could change dramatically
- Even if we knew the *probabilities*, that would not guarantee an exact result
- 
- What if  $\lambda < 1$ ?
- 

## Demographic stochasticity

- **Demographic** stochasticity is stochasticity that operates at the level of individuals
  - Individuals don't increase gradually, they die or give birth
  - Individuals don't produce 1.2 offspring: they produce 0, 1, 2 or 3 ...
- Even if we control conditions perfectly, we can't exactly predict the dynamics of small populations
- Demographic stochasticity averages out in large populations

## Environmental stochasticity

- **Environmental** stochasticity is stochasticity that operates at the level of the population
  - E.g., weather, pollution
- Environmental stochasticity can have large effects on any population
- 
- But small populations are the ones in danger of going extinct
- 
-

## Simulations

- We can simulate stochastic systems very easily
- But if we do the same simulation twice, we can get different answers
- Adds realism
  - But harder to interpret

## Summary

- Stochasticity is very important in real populations, but hard to study
  - Mathematical analysis is very difficult
  - Simulations are useful, but hard to interpret
    - \* Each time you simulate, you get a different answer
- Ecologists need to learn to recognize and communicate our uncertainty about the future