# UNIT 2 Non-linear population models

# Outline

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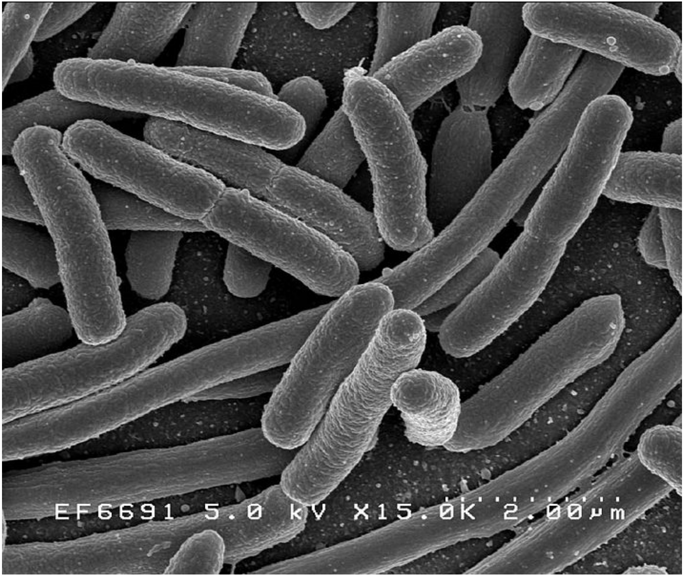

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Subsection 1

## Population Examples

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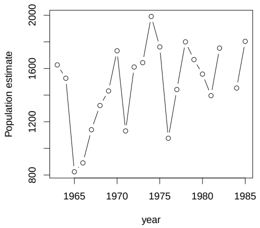
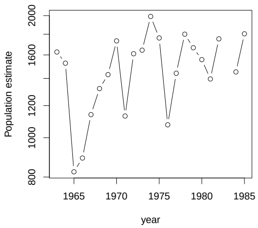


Population estimate
800
1200
1600
2000
1965
1970
1975
1980
1985
year

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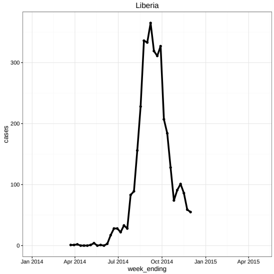
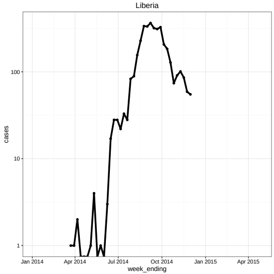


Liberia
cases
300
200
100
0
Jan 2014
Apr 2014
Jul 2014
Oct 2014
Jan 2015
Apr 2015
week_ending

Liberia
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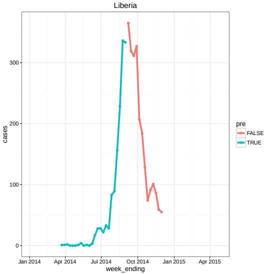
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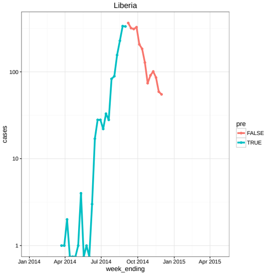
week_ending
pre
FALSE
TRUE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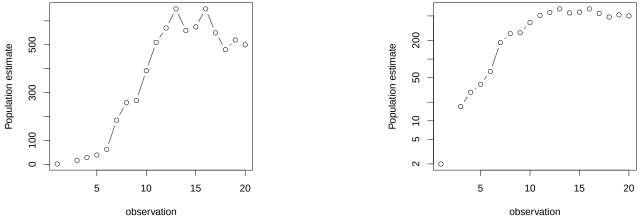
# Paramecia


Population estimate
0
100
300
500
5
10
15
20
observation
Population estimate
2
5
10
50
200
5
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15
20
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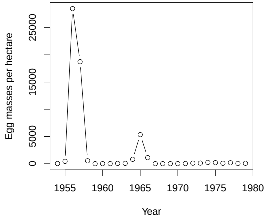
# Gypsy moths



Egg masses per hectare
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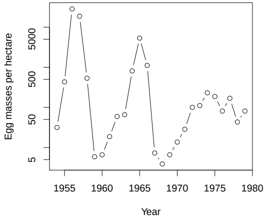

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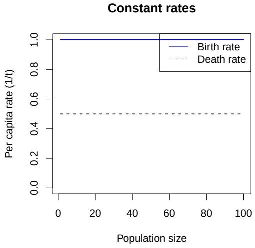


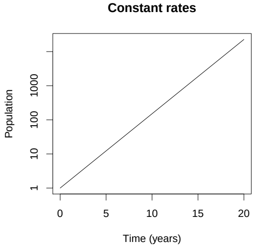

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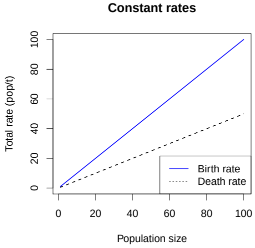


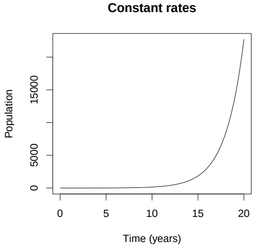

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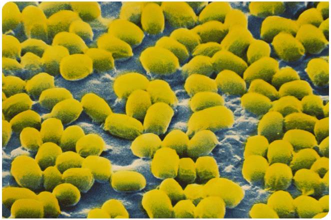

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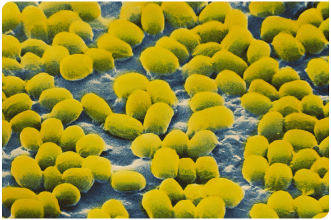

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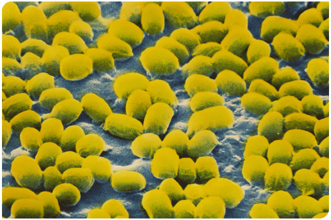

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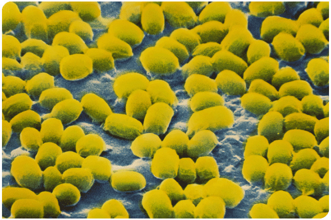

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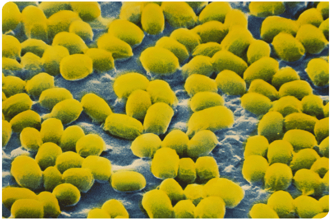

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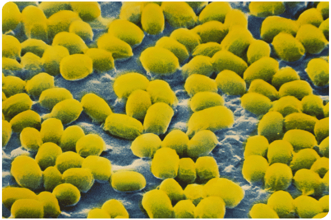

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Subsection 1

## A simple, continuous-time model

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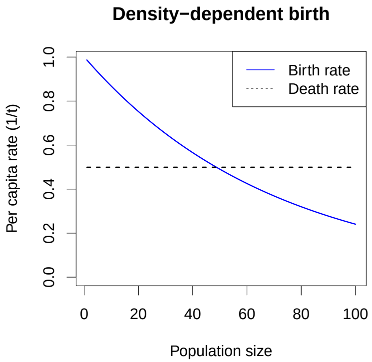

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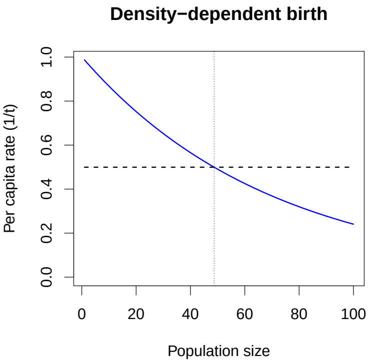

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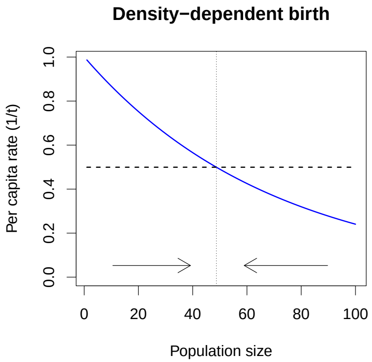

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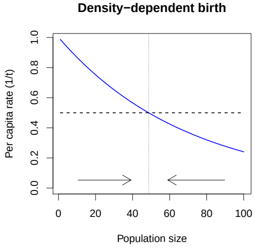


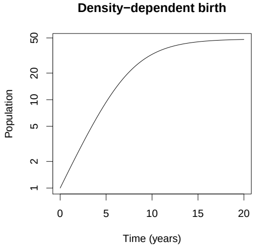

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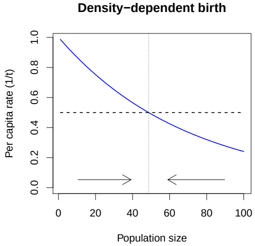


## Subsection 2

## Simulating model behaviour

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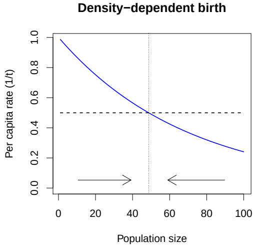


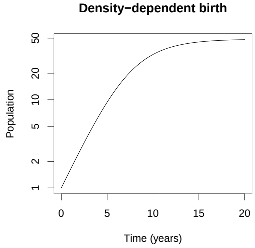

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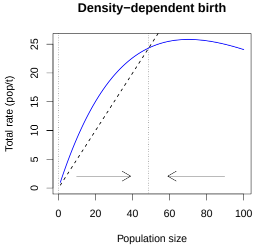


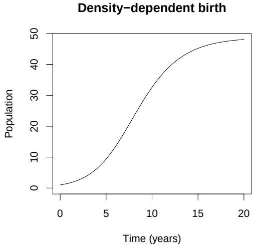

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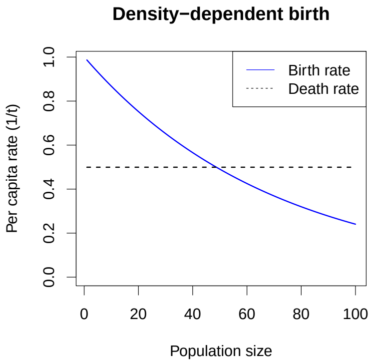

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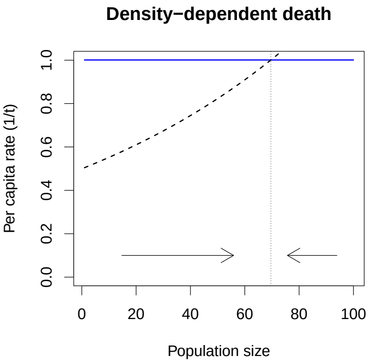

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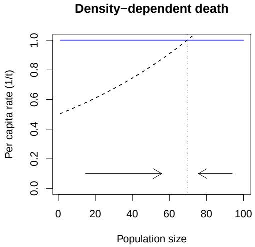


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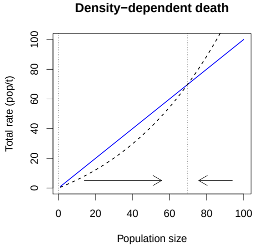


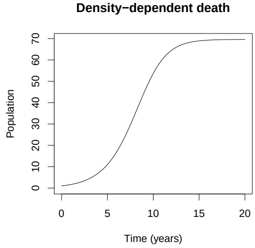

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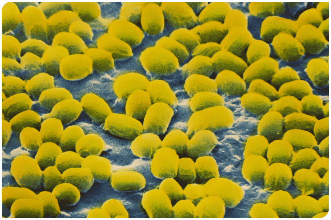

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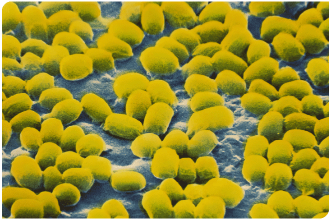

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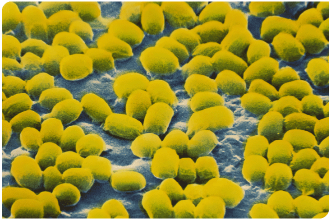

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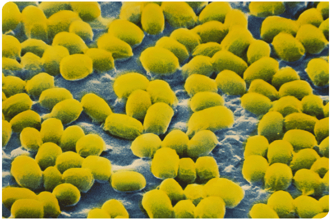

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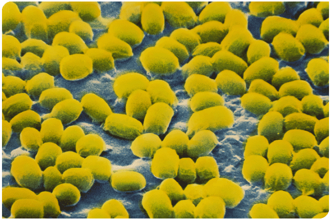

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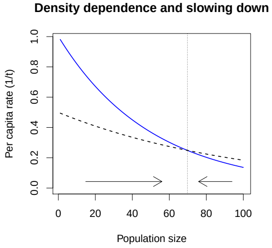


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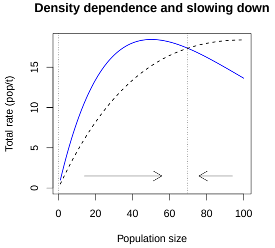


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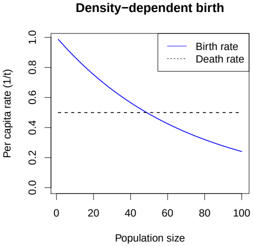


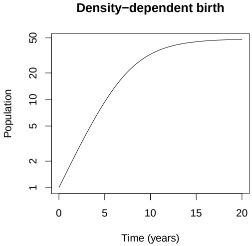

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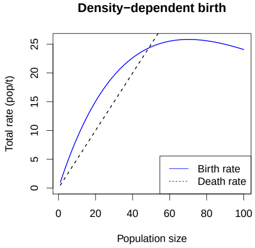


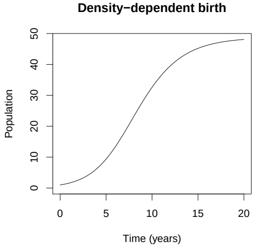

Subsection 3

## Equilibria and time scales

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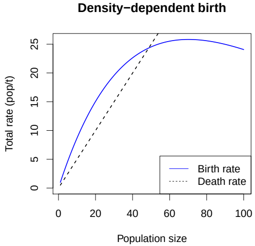


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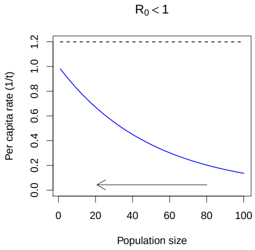


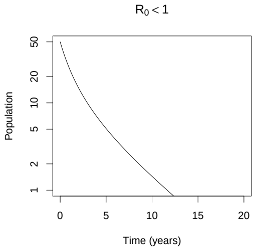

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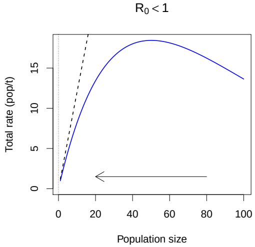


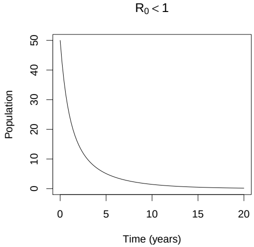

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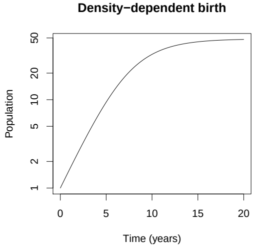


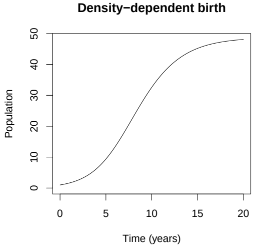

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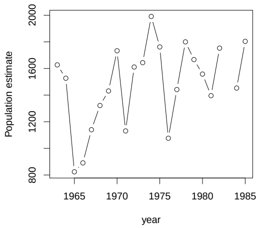

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1200
1600
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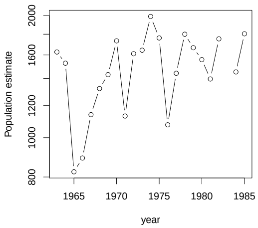

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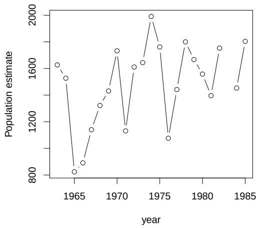

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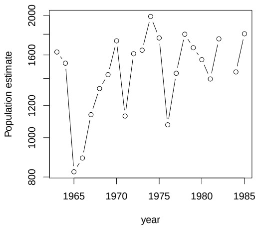

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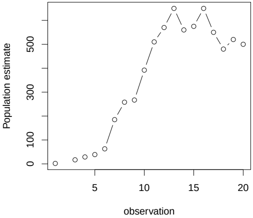
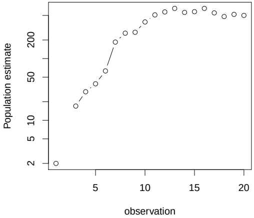
year

# Paramecia


Population estimate
0
100
300
500
5
10
15
20
observation


Population estimate
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5
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200
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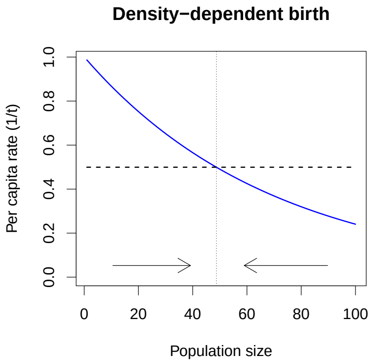

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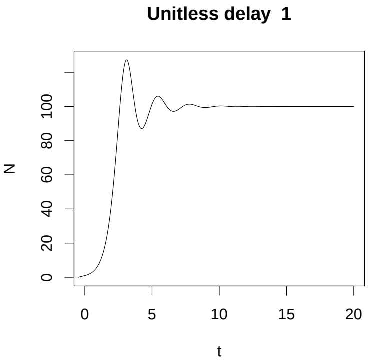

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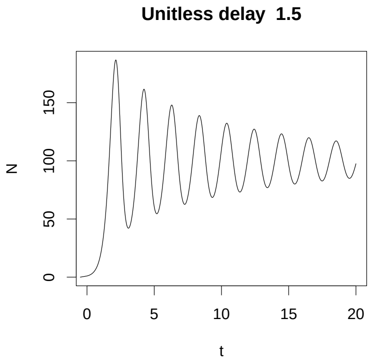

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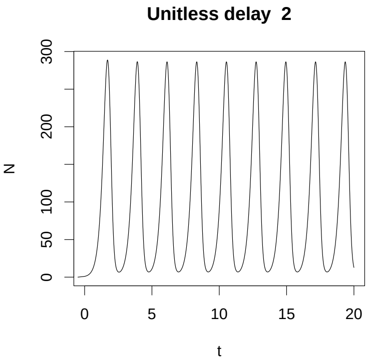

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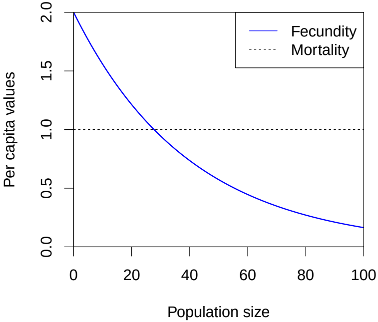

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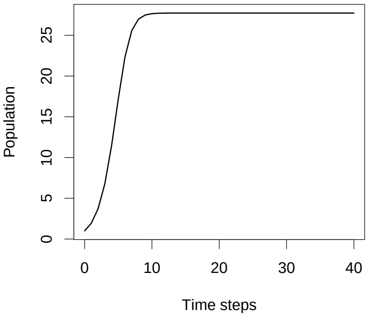
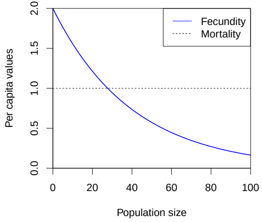
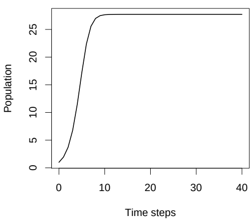


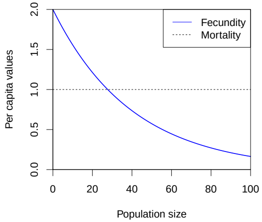
# We expect simple dynamics

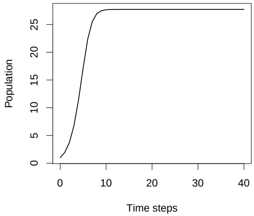


# *What dynamics do we get?*

# Simple dynamics



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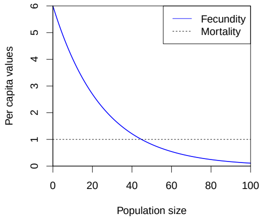


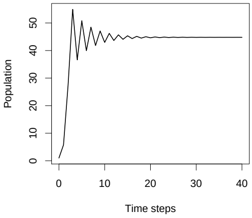

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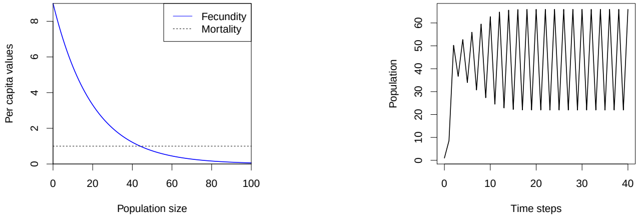
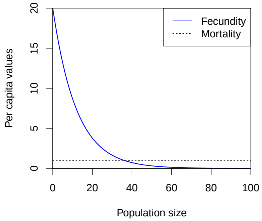
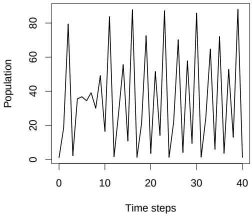


# Lots of other behaviours



Subsection 3

## Interpreting complex behaviour

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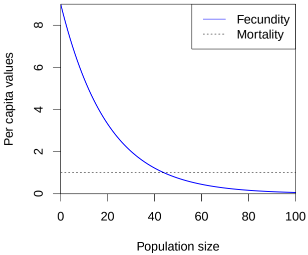
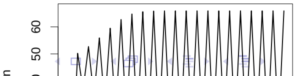



- In a simple cycle:



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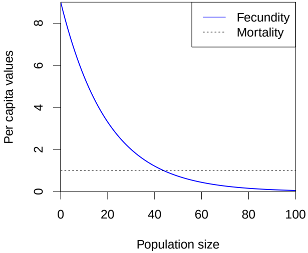


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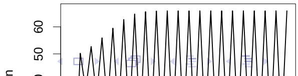

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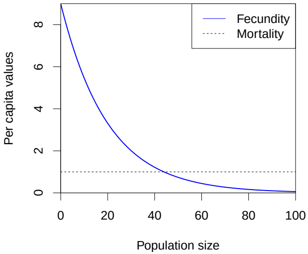


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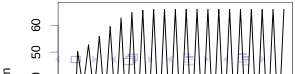

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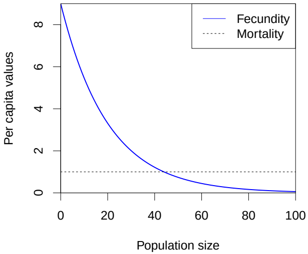


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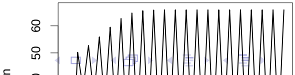

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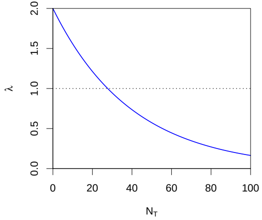
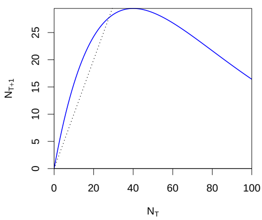
# Response to population increase



# Turnover

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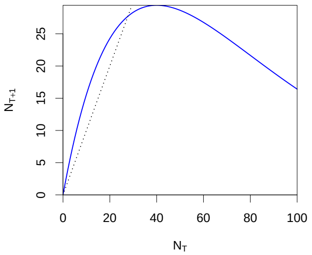

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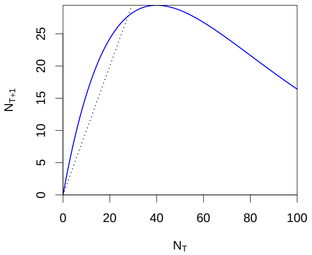

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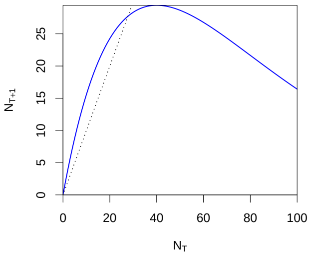

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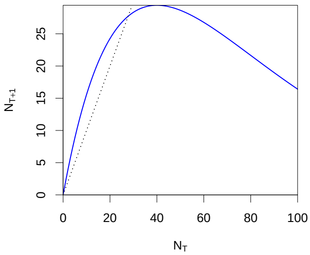

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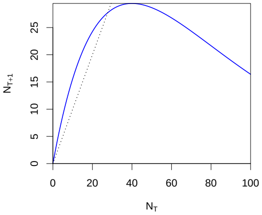


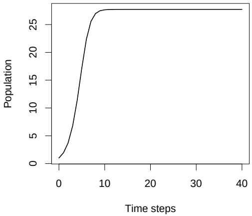

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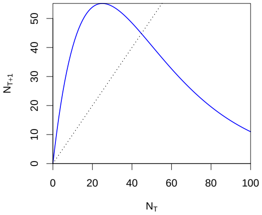


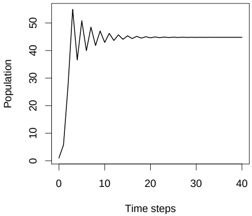

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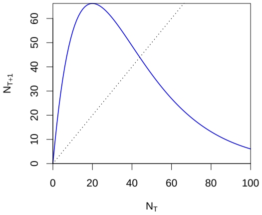


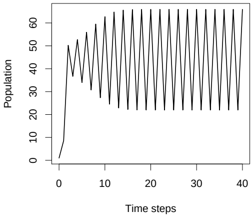

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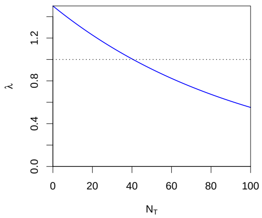


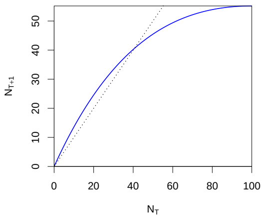

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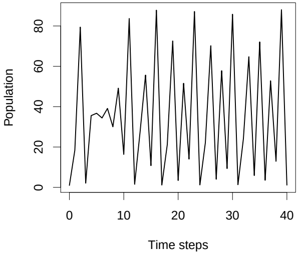

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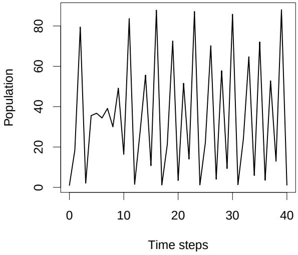

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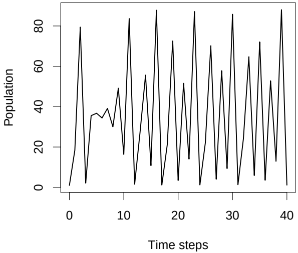

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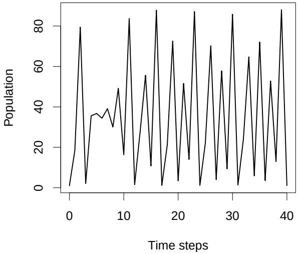

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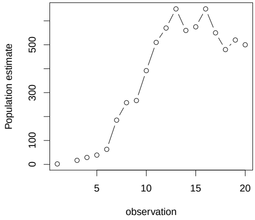
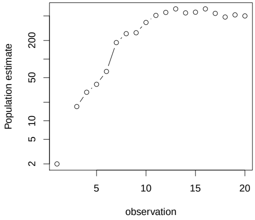

Population estimate
0
100
300
500
5
10
15
20
observation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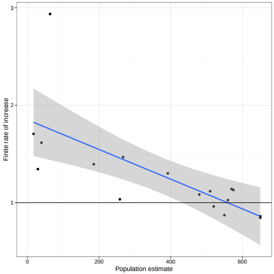
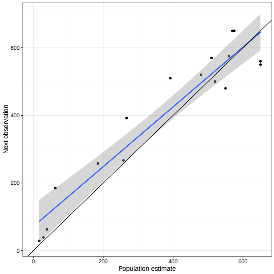
observation

# Paramecia

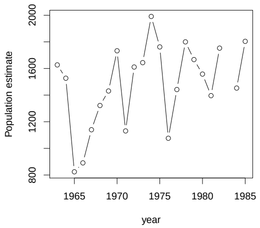


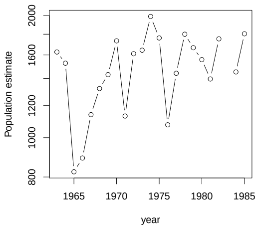
Finite rate of increase
3
2
1
0
200
400
600
Population estimate

Next observation
600
400
200
0
0
200
400
600
Population estimate

# Elk


Population estimate
800
1200
1600
2000
1965
1970
1975
1980
1985
year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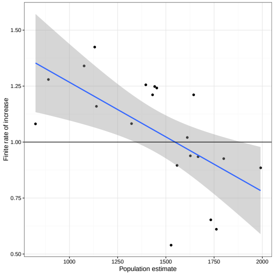
1.50
1.25
1.00
0.75
0.50
Finite rate of increase
1000
1250
1500
1750
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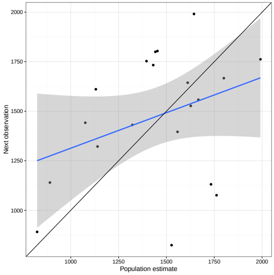
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Subsection 1

## Allee effects

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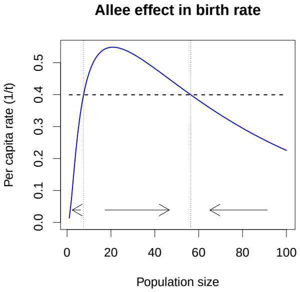

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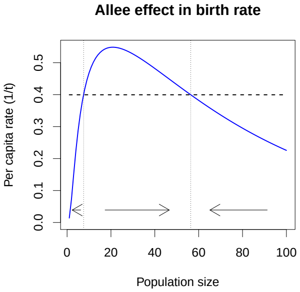

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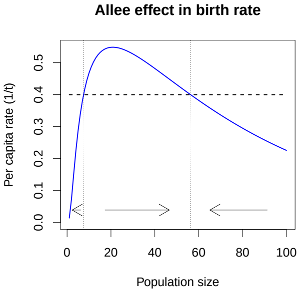

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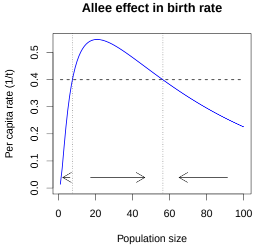


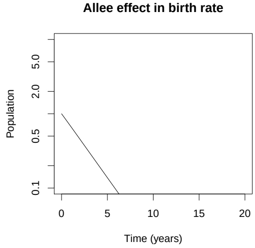

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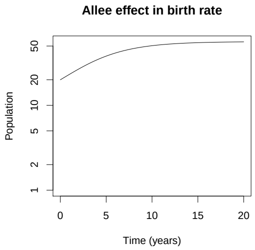


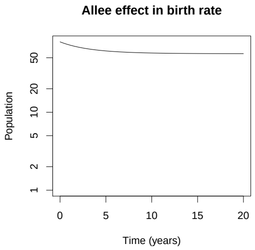

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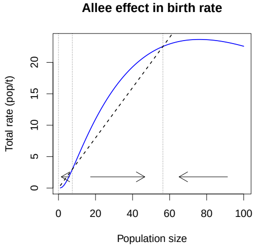


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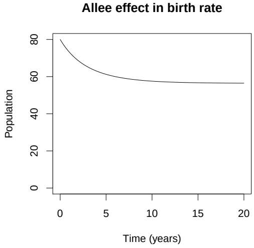

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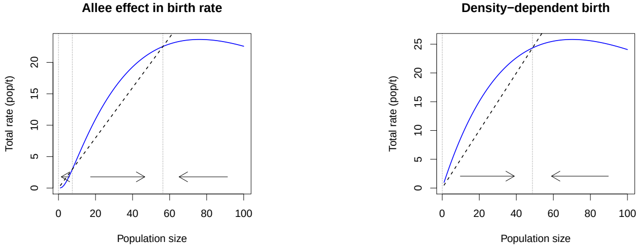

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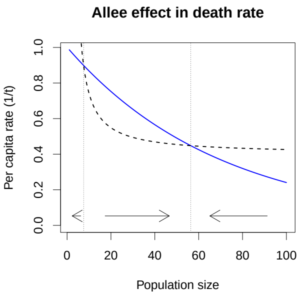

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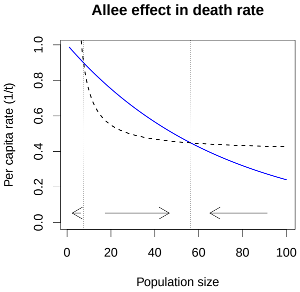

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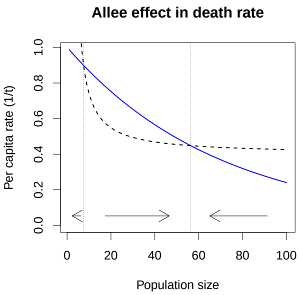

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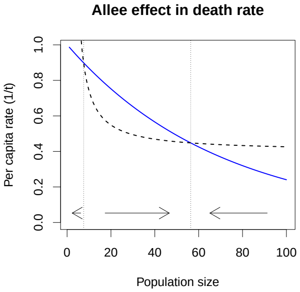

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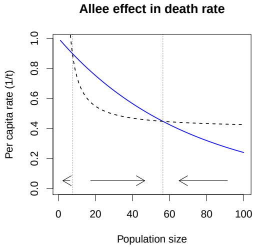


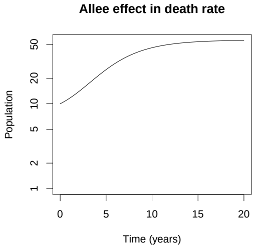

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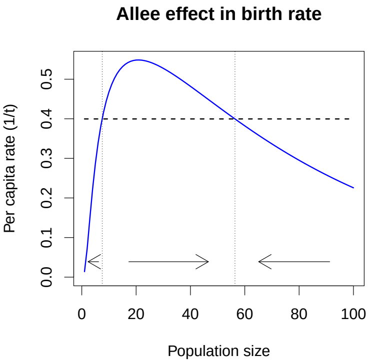

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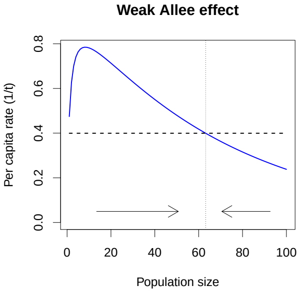

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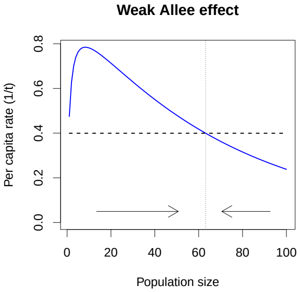

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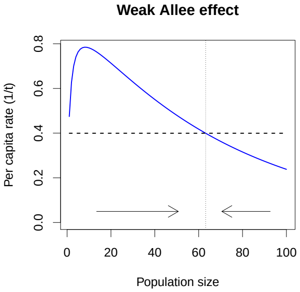

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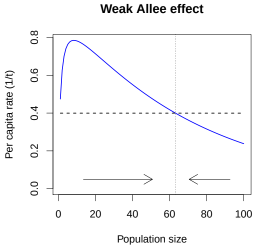


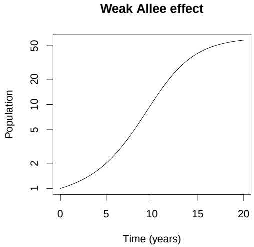

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Subsection 2

## Stochastic effects

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  - 10% of larvae successfully pupate
  - 60% of pupae become adults
  - Half of adult females successfully reproduce
- A single gravid (pregnant) female butterfly is blown away by a freak storm, and lands by chance on a suitable island with no butterflies What do you expect to happen?
  - * $\lambda = 1.5$ (remember not to multiply by the sex ratio twice!)
  - * Almost anything can happen

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- Female butterflies of a certain species lay 200 eggs on average, of which:
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# Simulations

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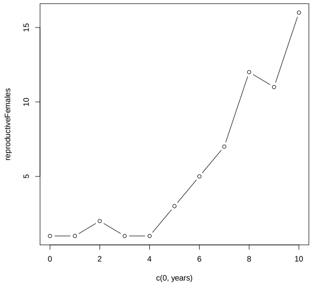


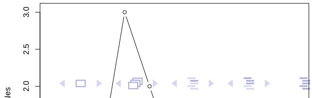

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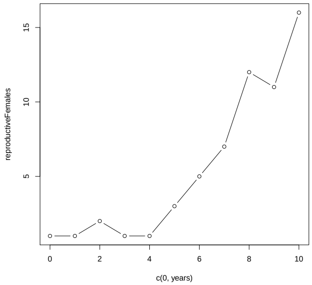


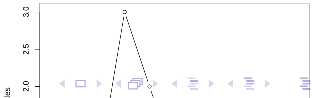

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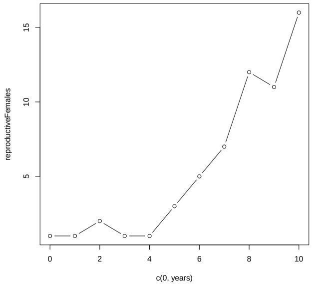


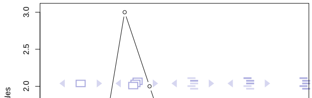

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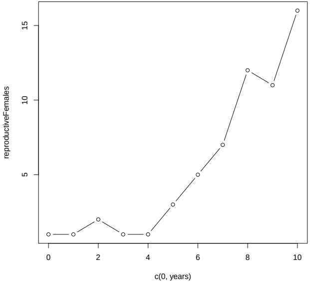


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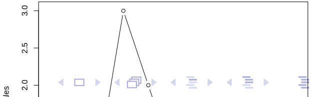

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