

UNIT 2: Linear population models

Outline

Constructing models

- Dynamical models

- Examples

- A simple discrete-time model

- A simple continuous-time model

Units and scaling

Key parameters

- Discrete-time model

- Continuous-time model

- Links

Growth and regulation

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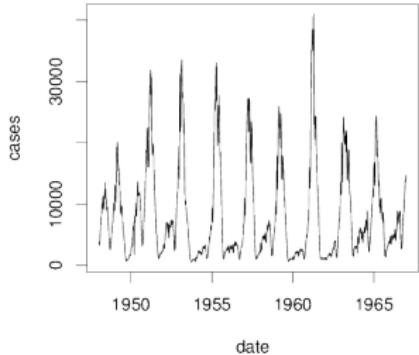
Dynamical models

Tools to link scales

- Models are what we use to link:



Measles reports from England and Wales



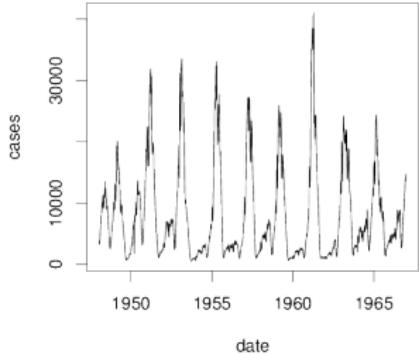
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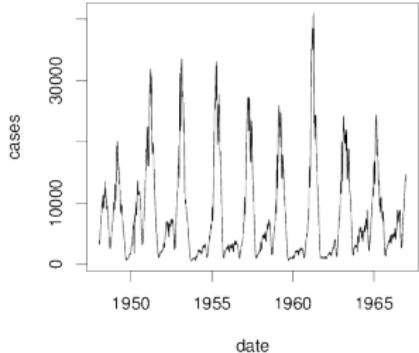
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 - ▶ Short time scales to long time scales



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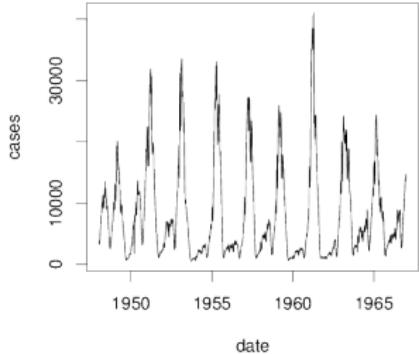
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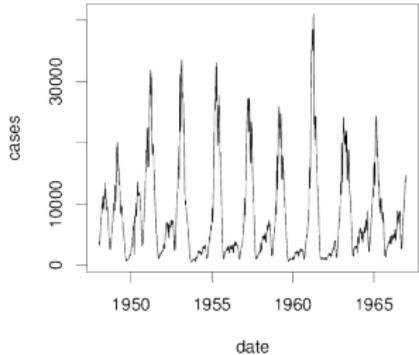
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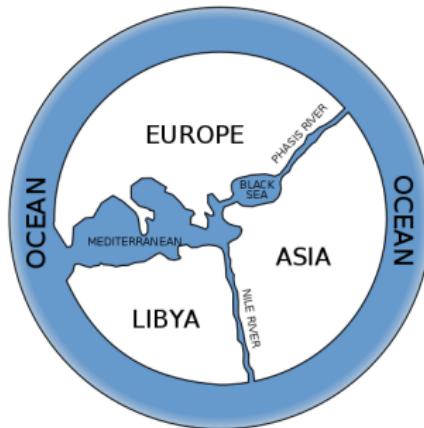


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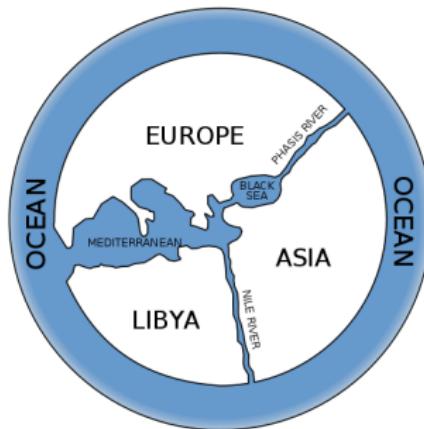
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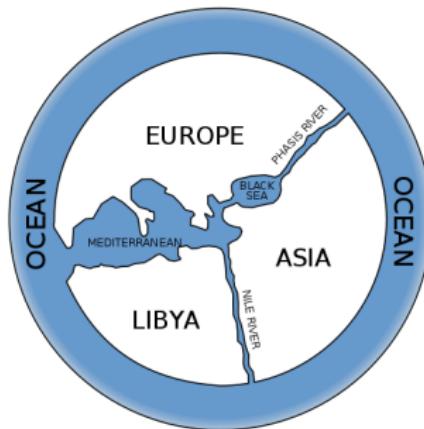
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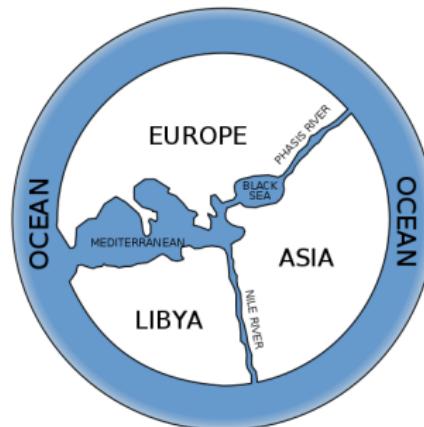
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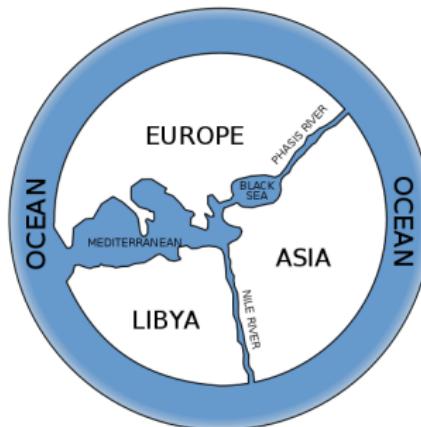
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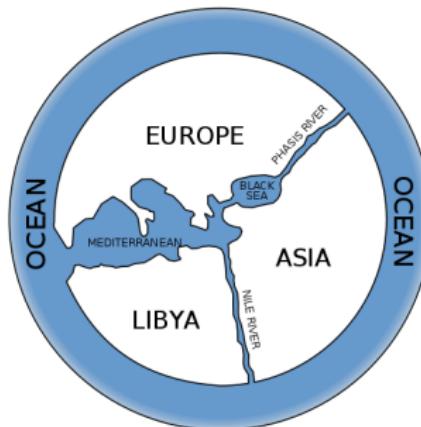
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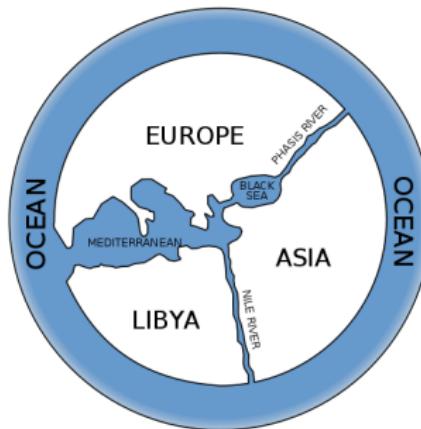
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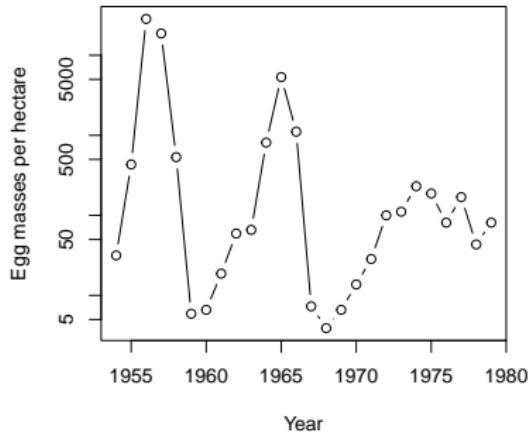
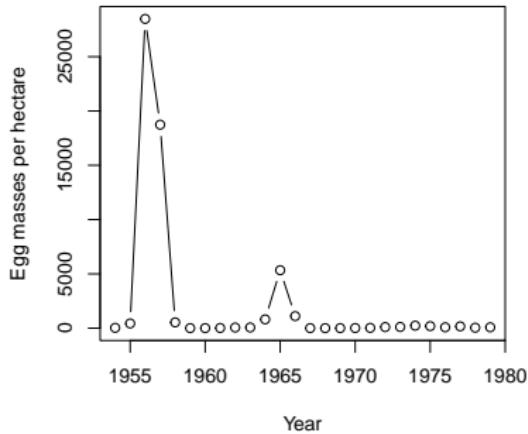


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Gypsy moth populations



Moth example

- State variable



Moth example

- ▶ State variable
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Moth example

- ▶ State variable
 - ▶ * Number of moths/ha



Moth example

- ▶ State variable
 - ▶ * Number of moths/ha
- ▶ Parameters



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Moth example

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Moth example

- ▶ State variable
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Bacteria

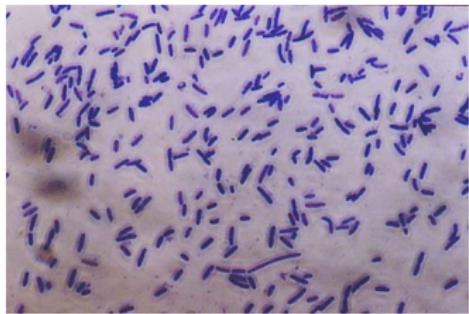
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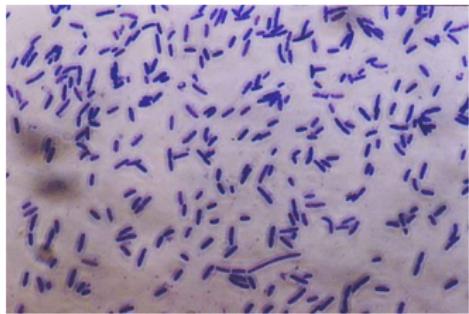
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Bacteria

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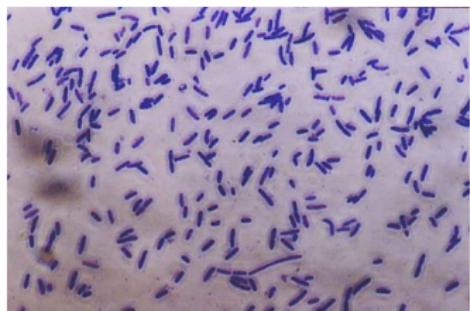
Bacteria

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Dandelions

► State variables



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Dandelions

- ▶ State variables
 - ▶ * Number of dandelions in a field



Dandelions

- ▶ State variables
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 - ▶ Are there intermediate variables?



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- ▶ If we have N individuals after T time steps, what determines how many individuals we have after $T + 1$ time steps?
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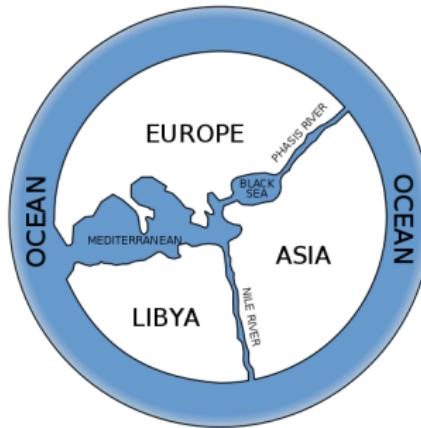
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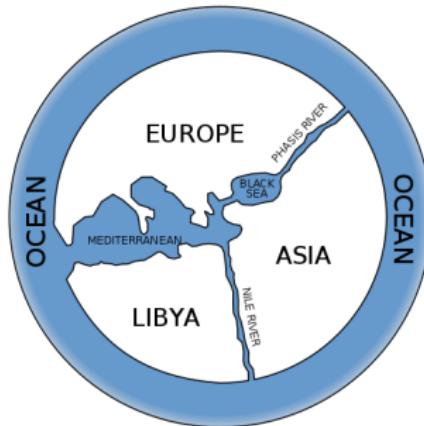
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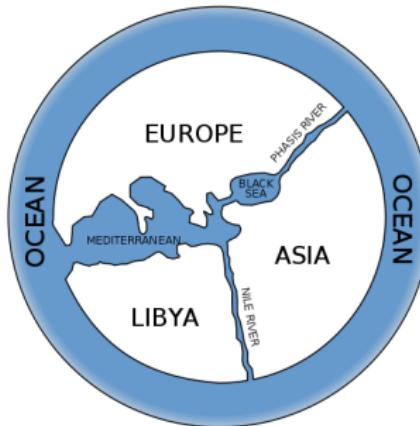
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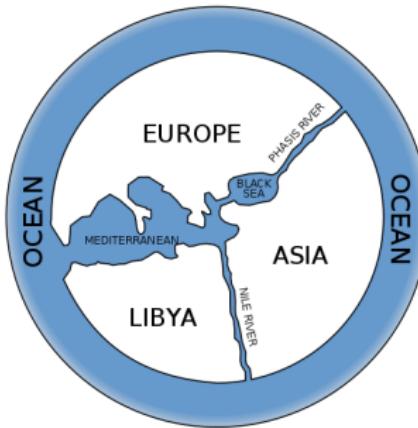
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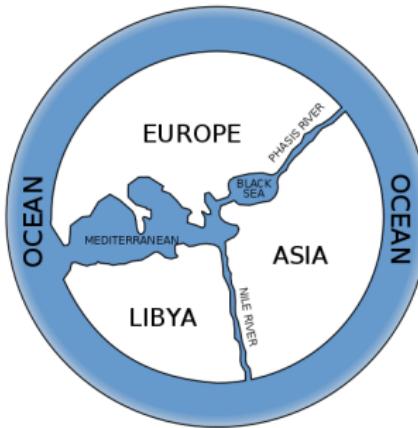
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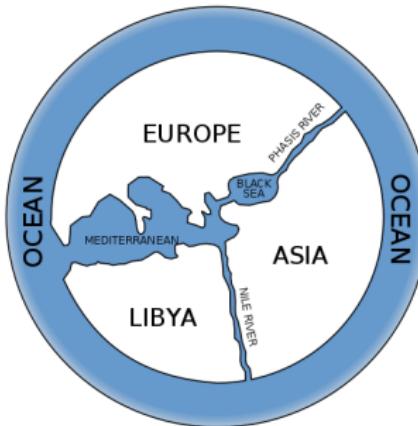
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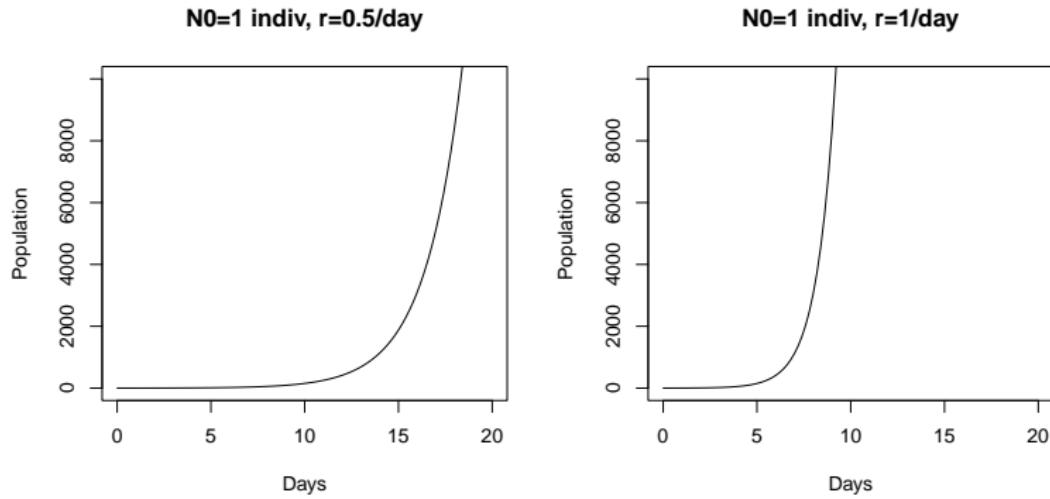
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- ▶ What is $3 \text{ day} \cdot 4 \text{ espressos/day}$?
 - ▶ * 12 espressos
- ▶ What is $1 \text{ hr} \cdot 0.2 \text{ cm/day}$?
 - ▶ * $1 \text{ hr} \cdot 0.2 \text{ cm/day}$
 - ▶ * $1 \text{ hr} \cdot 0.2 \text{ cm/day} \cdot \frac{1 \text{ day}}{24 \text{ hr}}$
 - ▶ * 0.0083 cm



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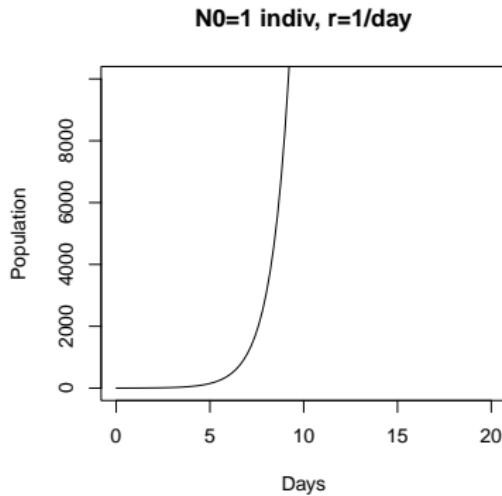
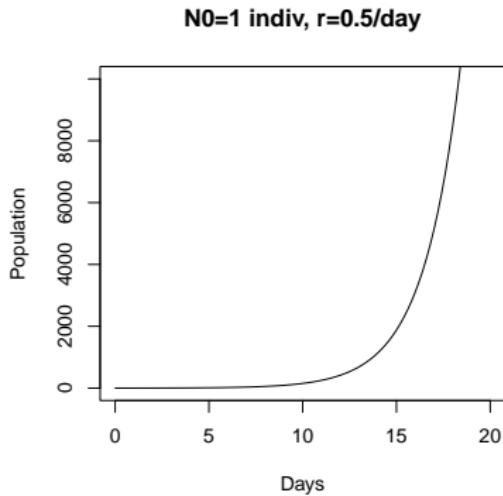
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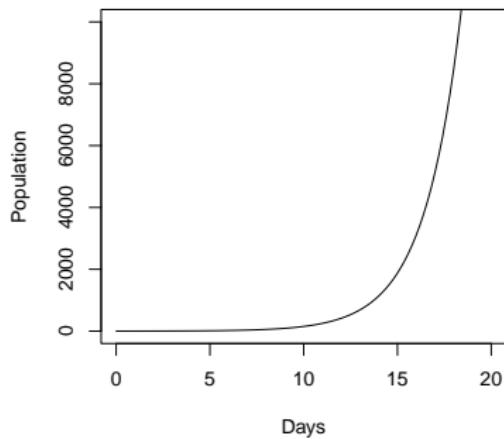
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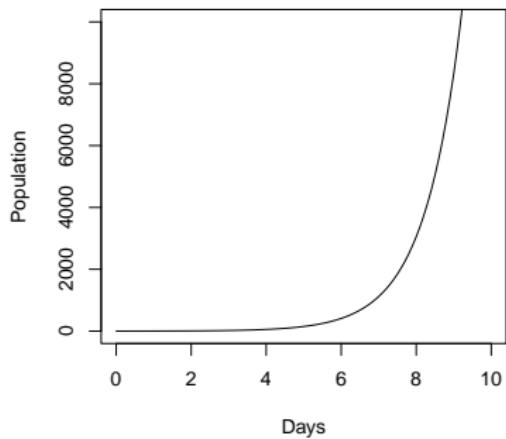


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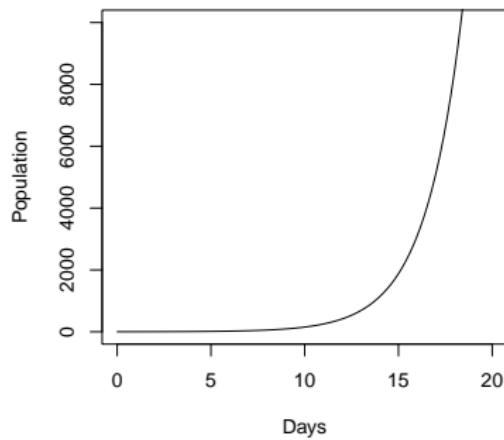


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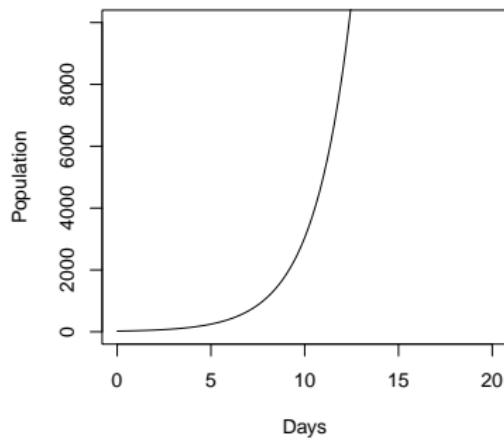


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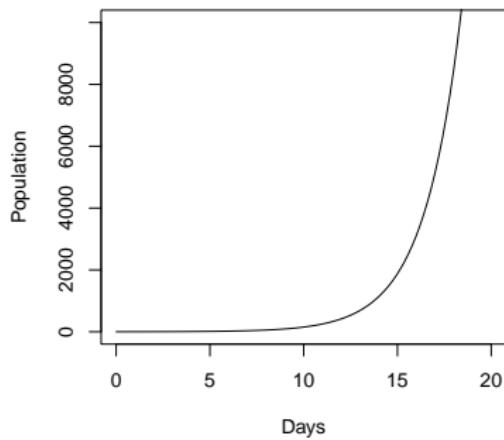


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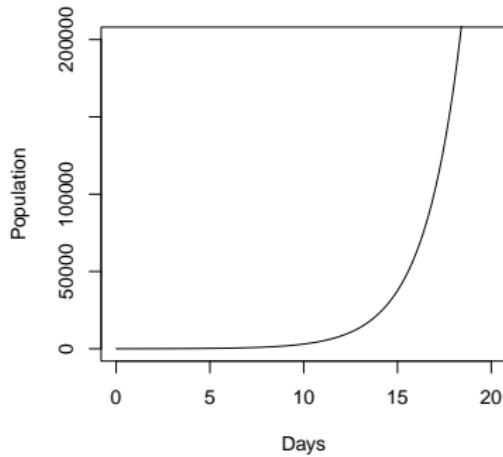


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Outline

Constructing models

Dynamical models

Examples

A simple discrete-time model

A simple continuous-time model

Units and scaling

Key parameters

Discrete-time model

Continuous-time model

Links

Growth and regulation

Outline

Constructing models

Dynamical models

Examples

A simple discrete-time model

A simple continuous-time model

Units and scaling

Key parameters

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- ▶ $r = b - d$
- ▶ Units $[1/t]$ (a rate)
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