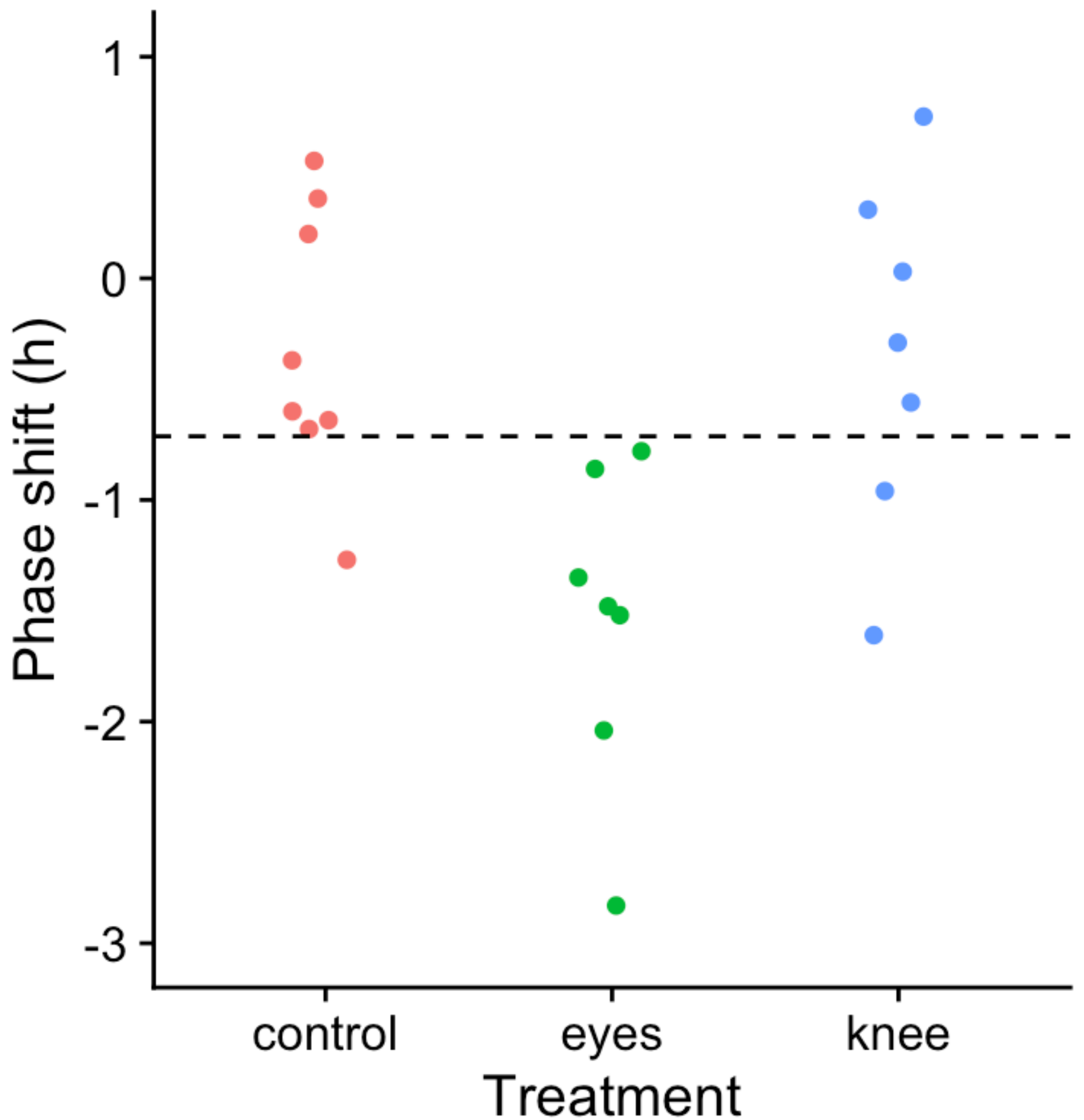
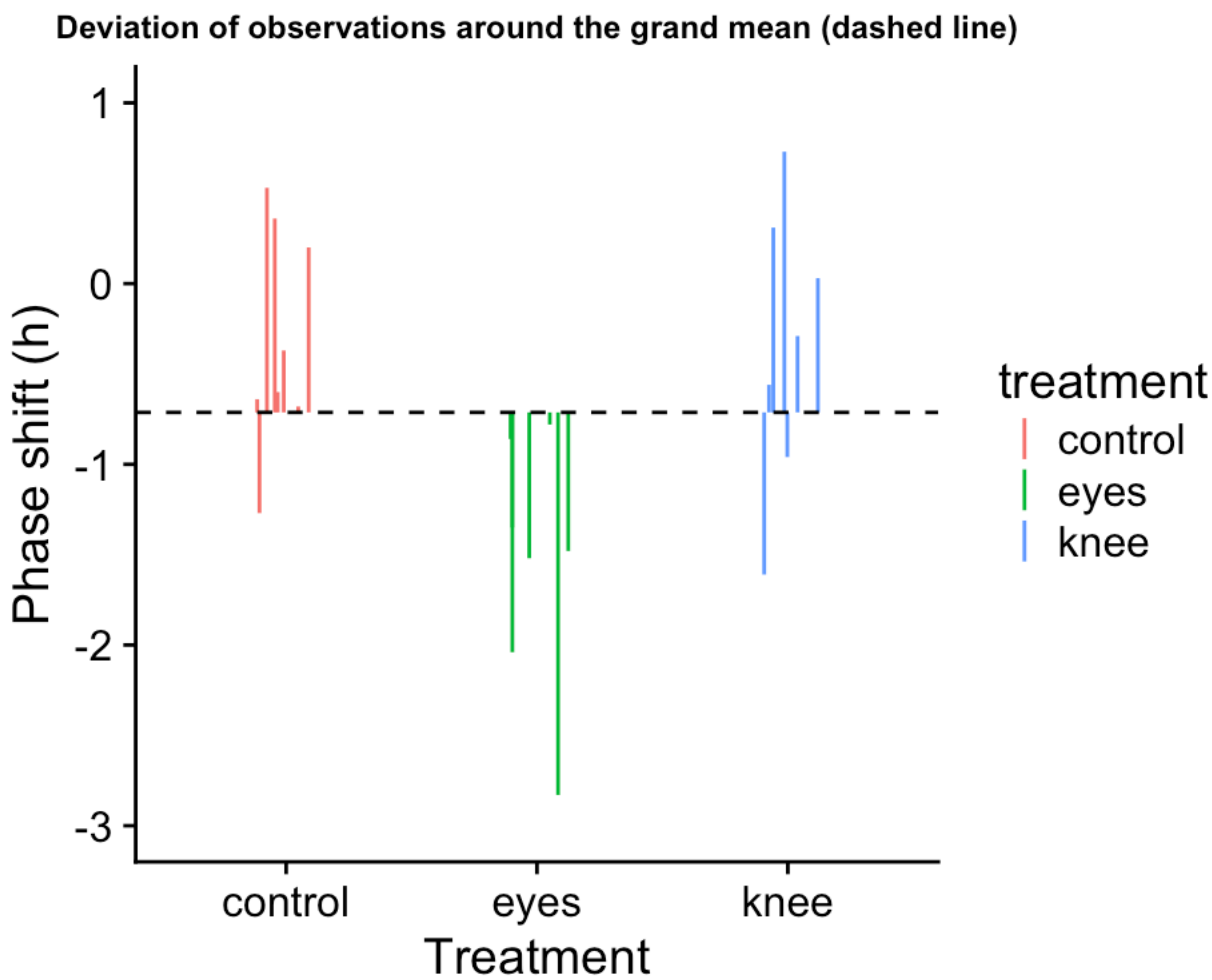
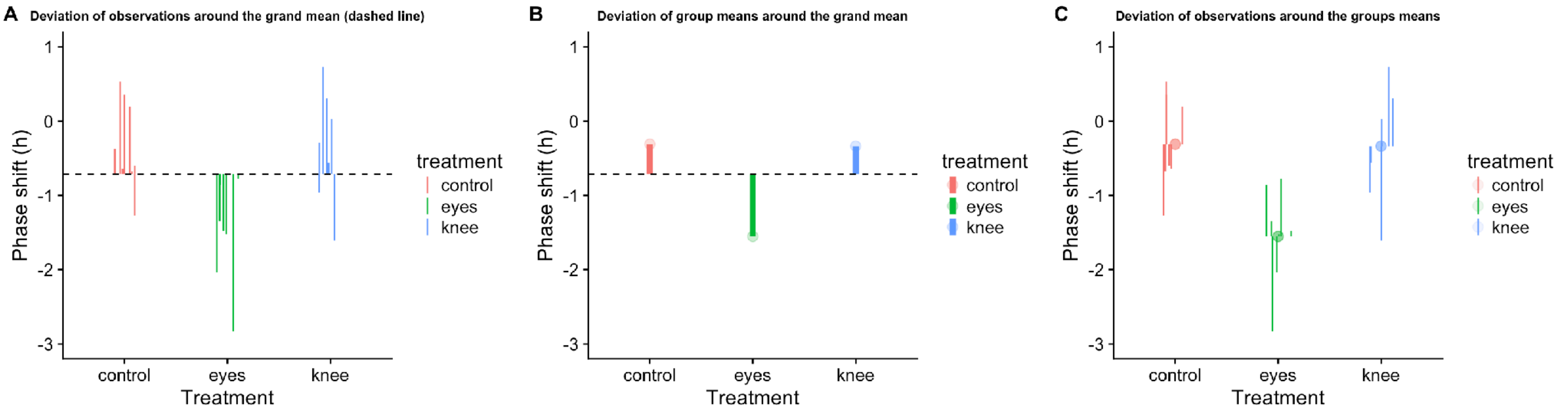
Analysis of Variance (ANOVA)

Does the mean of at least one of the groups differ from the others?




Deviation of observations around the grand mean (dashed line)
1
0
-1
-2
-3
Phase shift (h)
control
eyes
knee
Treatment
treatment
control
eyes
knee

Partitioning of variation into "between groups" (group) and "within groups" (error) components



ANOVA test statistic

$$F = \frac{\text{between group variation}}{\text{within group variation}} = \frac{\text{MS}_{\text{group}}}{\text{MS}_{\text{error}}}$$

Two-group ANOVA as Regression

We can also use a geometric perspective to test whether the mean of a variable differs between two groups of subjects.

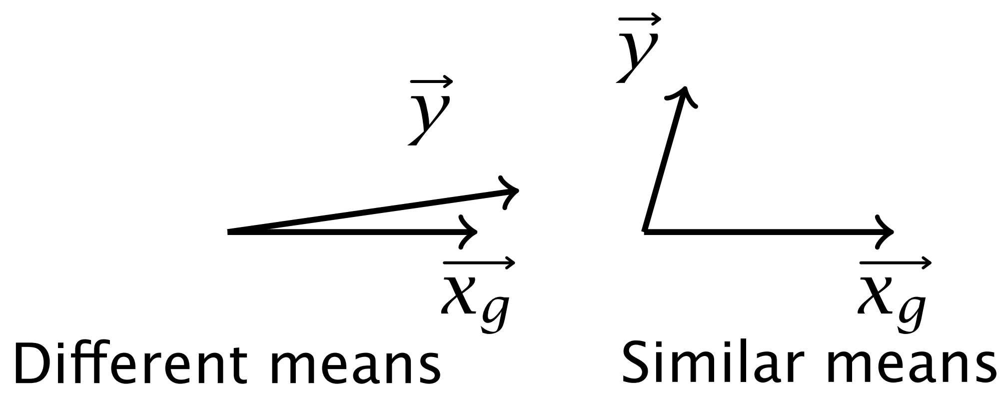
- Setup a 'dummy variable' as the predictor X_g. We assign all subjects in group 1 the value 1 and all subjects in group 2 the value -1 on the dummy variable. We then regress the variable of interest, Y, on X_g.

$$\mathsf{y} = \mathsf{X}_g \mathsf{b} + \mathsf{e}$$

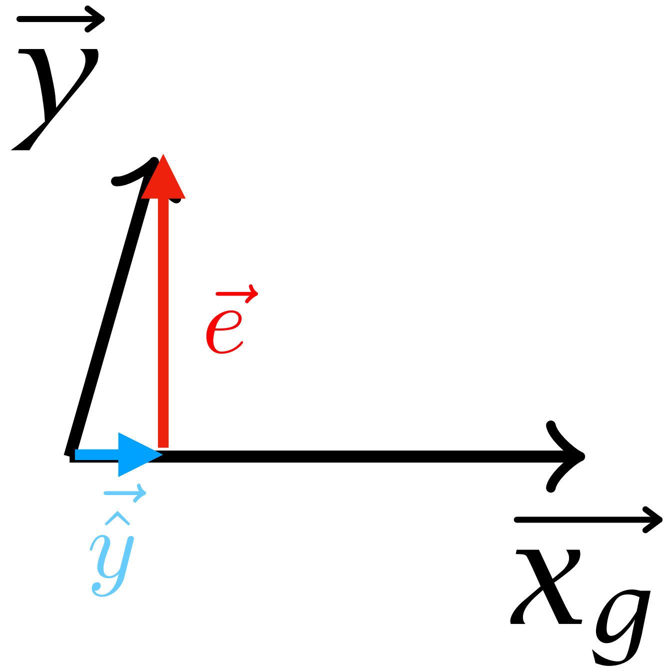
	Raw		Centered	
Group	Y_i	X_i	y_i	x_i
1	2	-1	-3	$-\frac{4}{3}$
	3	-1	-1	$-\frac{4}{3}$
2	5	1	0	$\frac{2}{3}$
	6	1	1	$\frac{2}{3}$
	6	1	1	$\frac{2}{3}$
	7	1	2	$\frac{2}{3}$
Mean	5	$\frac{1}{3}$	0	0

Two-group ANOVA as Regression, cont

- When the means are different in the two groups, X_g will be a good predictor of the variable of interest, hence $\vec{y}$ and $\overrightarrow{x_g}$ will have a small angle between them.
- When the means in the two groups are similar, the dummy variable will not be a good predictor. Hence the angle between $\vec{y}$ and $\overrightarrow{x_g}$ will be large.



Visual representation of F-statistic



$$MS_{\text{error}} = \frac{|\vec{e}|^2}{\dim(V_e)}$$

$$MS_{\text{group}} = \frac{|\vec{\hat{y}}|^2}{\dim(V_x)}$$

$$F = \frac{MS_{\text{group}}}{MS_{\text{error}}}$$

Multi-way ANOVA as Regression

- Exactly the same idea applies to g groups, except now instead of one grouping variable, we define $g-1$ grouping variables, $\dim(X_g) = g-1$.
- Then we calculate the multiple regression as we did before:

$$\mathsf{y} = \mathsf{X}\mathsf{b} + e$$

$$\mathsf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}; \mathsf{X} = \begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1g} \\ 1 & x_{21} & x_{22} & \cdots & x_{2g} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{ng} \end{bmatrix};$$

Estimate b as:

$$\mathsf{b} = (\mathsf{X}^T\mathsf{X})^{-1}\mathsf{X}^T\mathsf{y}$$

How Do We Construct the Grouping Matrix, X_g?

Two common methods are:

1. Dummy coding – define a set of g grouping variables, where values take either 0 or 1, depending on group membership, but *use only the first* $g - 1$ columns:

$$U_j = \begin{cases} 1, & \text{for every subject in group } j, \\ 0, & \text{for all other subjects.} \end{cases}$$

   and

$$X_g = [U_1, U_2, \cdots, U_{g-1}]$$

2. Effect coding – define the U_j as above, and set:

$$X_g = [U_1 - U_g, U_2 - U_g, \cdots, U_{g-1} - U_g]$$

In general, effect coding is more similar to standard ANOVA contrasts.

ANOVA: Example Data Set

	g_1	g_2	g_3	g_4	
	20	21	17	8	
	17	16	16	11	
	17	14	15	8	
$M_{g.}$	18	17	16	9	$M_{..} = 15$

$$y = \begin{bmatrix} 20 \\ 17 \\ 17 \\ 21 \\ 16 \\ 14 \\ 17 \\ 16 \\ 15 \\ 8 \\ 11 \\ 8 \end{bmatrix}, \qquad \mathsf{X} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

ANOVA: Example Data Set, cont

Solving for b we find:

$$\mathrm{b} = \begin{bmatrix} 15 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \quad |\hat{\mathrm{y}}|^2 = 150, \; |\mathrm{e}|^2 = 40$$

Since, $\dim(\mathcal{V}_x) = 3$, and $\dim(\mathcal{V}_e) = 8$, we get:

$$F = \frac{\dim(\mathcal{V}_e)\,|\vec{\hat{y}}|^2}{\dim(\mathcal{V}_x)\,|\vec{e}|^2} = 10$$

Here's the more conventional ANOVA table for the same data:

Source	df	SS	MS	F	$\Pr(F)$
Experimental	3	150	50	10	.0044
Error	8	40	5		
Total	11	190			