Hagen-Poiseuille Law derivation from Navier-Stokes Equation

Hagen-Poiseuille Equation/Law

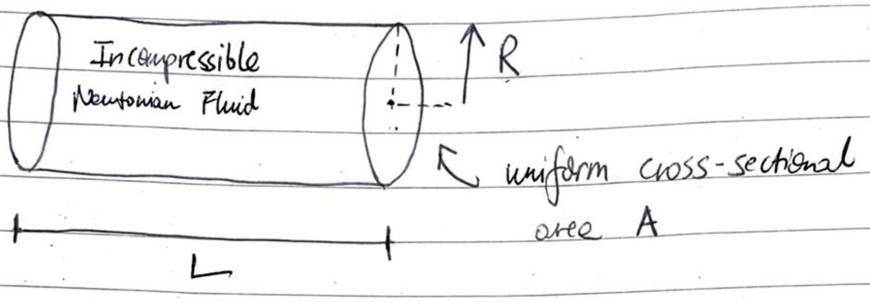
In nonideal fluid dynamics, the Hagen-Poiseuille equation is a physical law that gives the pressure drop in an incompressible and Newtonian fluid in laminar flow flowing through a long cylindrical tube with constant proportion. The Hagen-Poiseuille equation in its standard fluid-kinetics notation is:

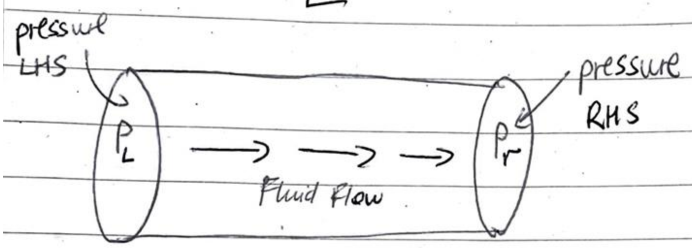
$$\Delta p = \frac{8 \mu L Q}{\pi R^4}$$

where, μ = dynamic viscosity
L = length of tube
Q = Volumetric flow rate
R = pipe radius

Deriving Poiseuille's Law from Navier-Stokes Equation

Poiseuille's Law is a relationship between pressure drop across a pipe, volume flow rate through that pipe, and radius of pipe





* Need to determine velocity distribution in pipe. We need to incorporate both the continuity equation and the Navier-Stokes equation

The continuity equation states that in the case of steady flow, the amount of fluid flowing past one point must be the same as the amount of fluid flowing past another point, or the mass flow rate is constant. It is essentially a law about the conservation of mass.

$$\boxed{\rho_1 A_1 v_1 = \rho_2 A_2 v_2}$$

ρ : density
A : cross-sectional area
v : flow velocity of fluid
The subscripts 1 and 2 indicate different regions in the same tube.

Navier-Stokes equation in fluid mechanics is a partial differential equation that describes the flow of incompressible fluids.
An incompressible fluid is defined as the fluid whose volume or density does not change with pressure.

<u>Continuity + Navier Stokes in cylindrical coordinates</u>

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$ – [continuity equation]

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial r} + \rho g_r + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

Z-direction

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

Y-direction

$$\rho\left(\frac{\partial v}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

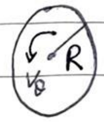
X-direction

μ: viscosity
ρ: fluid density
v: velocity of fluid
P: pressure

Continuity + Navier Stokes

If we assume flow is steady and ignore gravity, then:

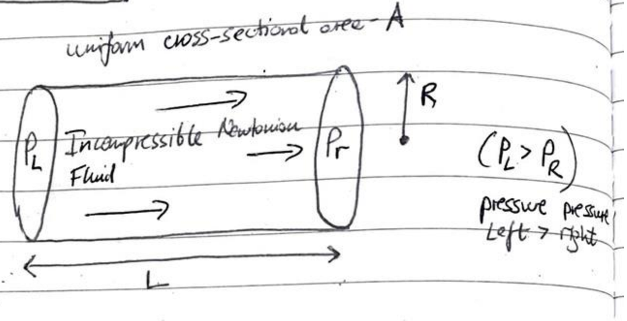
$$\boxed{\begin{array}{ll} v_r = 0 & \text{(radial velocity)} \\ v_\theta = 0 & \text{(angular velocity)} \end{array}}$$



$$\frac{\partial v_z}{\partial z} = 0 \quad \text{(from continuity equation)}$$

$$\frac{\partial p}{\partial r} = 0$$

$$\frac{\partial p}{\partial \theta} = 0$$



- Radial velocity is zero, if we think intuitively, if a fluid is flowing through a stationary and uniform pipe. There is nothing else about the system that makes the fluid attracted to or repelled from the walls of the tube/membrane. There are no radial forces acting on the fluid

- Angular velocity is also zero, if a fluid is going through a uniform and stationary pipe there is no reason that the fluid will be spinning inside the pipe as there is no rotational force that would cause that to happen; as a result the angular velocity is zero throughout the pipe. This however is not true with blood flow since blood vessels are not uniform and stationary with varying diameters throughout the human body.

After simplification, we can cancel out the radial velocity, angular velocity and other variables since their value is zero. This leaves us with a second order ODE (Ordinary differential equation). The reason it is an ODE is because the pressure and velocity both only depend on one variable, since their derivatives with respect to other variables are all zero

$$\boxed{\frac{1}{\mu}\frac{dP}{dz} = \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)}$$

where $\frac{dP}{dz}$ is a constant equal to $\frac{P_r - P_L}{L}$

$$\frac{P_r - P_L}{L} \longrightarrow \frac{r}{\mu}\left(\frac{P_r - P_L}{L}\right) = \frac{d}{dr}\left(r\frac{dv_z}{dr}\right)$$

If we integrate both sides:

$$r\frac{dv_z}{dr} = \left(\frac{P_r - P_L}{L}\right)\frac{r^2}{2\mu} + C_1$$

$$\frac{dv_z}{dr} = \left(\frac{P_r - P_L}{L}\right)\frac{r}{2\mu} + \frac{C_1}{r}$$

Integrate again:

$C_1 = 0$

$$\boxed{v_z(r) = \frac{P_r - P_L}{4\mu L}r^2 + \cancel{C_1 \ln r} + C_2}$$

with boundary conditions
1. At $r=0$, v_z is finite
2. No slip at wall: at $r=R$, $v_z = 0$

Boundary Conditions

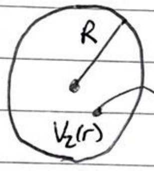
① At $r=0$, v_z is finite
② No slip at wall : at $r=R$, $v_z = 0$

$$v_z(R) = \frac{P_r - P_L}{4\mu L} R^2 + C_2 = 0$$

$$\therefore v_z(r) = \frac{P_r - P_L}{4\mu L}(r^2 - R^2) \rightarrow C_2 = -\left(\frac{P_r - P_L}{4\mu L}\right) R^2$$

The change in pressure $\Delta P = P_L - P_R$ (pressure left - pressure right)

$\therefore V_z(r) = \frac{\Delta P}{4\mu L}(R^2 - r^2)$, and calculating volume flow rate will be:



$$\int dQ = \int v_z \, dA \rightarrow Q = \int V_z \, dA$$

in cylindrical coordinates

$dA = r dr d\theta$

$$Q = \int V_z \, dA$$

$$= \int_0^{2\pi}\int_0^{R} v_z(r) \, r dr d\theta$$

$$= \int_0^{2\pi}\int_0^{R} \frac{\Delta P}{4\mu L}(R^2 - r^2) \, r dr d\theta = \int_0^{2\pi}\int_0^{R} \frac{\Delta P}{4\mu L}(R^2 r - r^3) dr d\theta$$

$$= \int_0^{2\pi} \frac{\Delta P}{4\mu L}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R d\theta = \int_0^{2\pi} \frac{\Delta P}{4\mu L}\frac{R^4}{4} d\theta$$

$$= \frac{\Delta P R^4}{16\mu L}\int_0^{2\pi} d\theta$$

$$\therefore Q = \frac{\pi R^4 \Delta P}{8\mu L}$$

rearrange:

$$\therefore \boxed{\Delta P = \frac{8\mu L Q}{\pi R^4}}$$

Hagen-Poiseuille's Law.