

# Quantum Theory, Groups and Representations

January 31, 2025

## 1 Representation theory

**Definition 1.1.** A representation  $(\pi, V)$  of a group  $G$  over a vector space  $V$  is an application  $\pi : G \rightarrow GL(V)$  such that  $\forall g_1, g_2 \in G, \pi(g_1 \cdot g_2) = \pi(g_1)\pi(g_2)$ .

- A representation  $(\pi, V)$  is said *irreducible* if there is no proper subset  $W \subset V$  such that  $\pi_W : G \rightarrow GL(W)$  is a representation.  
Another way to express that is that the only invariant<sup>1</sup> subvector spaces of the representation are  $\{0\}$  and  $V$ .
- Otherwise,  $(\pi, V)$  is said *reducible*.

**Definition 1.2.** A unitary representation  $(\pi, V)$ , where  $V$  is a  $\mathbb{K}$ -EV equipped with an inner product  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{K}$ , is a representation which preserves the inner product such that:

$$\forall g \in G, \forall v, w \in V, \langle v, w \rangle = \langle \pi(g)v, \pi(g)w \rangle.$$

If  $V$  is finite dimensionnal (of dimension  $n$ ), we can treat the representation as an element of  $\mathcal{M}_{n \times n}(\mathbb{K})$ , and for a unitary transformation,  $\pi : G \rightarrow U(n)$ .

**Definition 1.3.** (Direct sum representation). Given representations  $\pi_1$  and  $\pi_2$  of dimensions  $n_1$  and  $n_2$ , there is a representation of dimension  $n_1 + n_2$  called the direct sum of the two representations, denoted  $\pi_1 \oplus \pi_2$ . This representation is given by the homomorphism:

$$(\pi_1 \oplus \pi_2) : g \in G \mapsto \begin{pmatrix} \pi_1(g) & \mathbf{0} \\ \mathbf{0} & \pi_2(g) \end{pmatrix}.$$

**Theorem 1.1.** A unitary transformation  $\pi$  over a finite dimensionnal vector space  $V$  can be written as  $\oplus \pi_i$ , where  $\{\pi_i\}$  is a set of irreducible representations.

*Proof.* If  $(\pi, V)$  is irreducible, then it's obvious.

Otherwise, there exists  $W \subset V$  such that  $(\pi_W, W)$  is a representation.

Since  $V = W \oplus W^\perp$ , we want to check if  $(\pi_{W^\perp}, W^\perp)$  is a representation.

- Let  $w \in W$  and  $v \in W^\perp$ , then obviously  $v \perp w \Leftrightarrow \langle v, w \rangle = \langle \pi(g)v, \pi(g)w \rangle = 0$ . Since  $(\pi_W, W)$  is a representation,  $\pi(g)w \in W$  and thus  $\forall g \in G, \pi(g)v \perp W$ .  
This means that  $\forall g \in G, \forall v \in W^\perp, \pi(g)v \in W^\perp$ . Thus,  $\pi_{W^\perp}, W^\perp$  is a representation. (It respects the group product since  $(\pi, V)$  is a representation.)

Thus:

$$(\pi, V) = (\pi_W, W) \oplus (\pi_{W^\perp}, W^\perp).$$

We can then repeat this process until each subrepresentation is irreducible. □

---

<sup>1</sup>A subspace  $V' \subseteq V$  is called invariant if  $\pi(g)V' \subseteq V'$  for all  $g \in G$ .

**Theorem 1.2.** *Schur's Lemna: If a **complex** representation  $(\pi, V)$  is irreducible, then the only application  $M : V \rightarrow V$  commuting with all the  $\pi(g)$  is  $M = \lambda \mathbb{I}$ ,  $\lambda \in \mathbb{C}$ .*

*Proof.* Let's assume that  $(\pi, V)$  is an irreducible representation and that  $\forall g \in G, [M, \pi(g)] = 0$ . Since  $V$  is a  $\mathbb{C}$ -vector space, the equation:

$$\det(M - \lambda \mathbb{I}) = 0$$

has  $n$  roots (counting the multiplicity) and thus we can find all the eigenspaces:

$$U_\lambda = \{v \in V | Mv = \lambda v\} = \ker(M - \lambda \mathbb{I}) \neq \emptyset.$$

Since  $\pi$  and  $M$  commute,  $v \in \ker(M - \lambda \mathbb{I}) \implies \forall g \in G \pi(g)v \in \ker(M - \lambda \mathbb{I})$ . As a consequence,  $(\pi, \ker(M - \lambda \mathbb{I}))$  is a representation of  $G$  since  $\ker(M - \lambda \mathbb{I})$  is stable under the action of  $\pi$ . Since we supposed that  $(\pi, V)$  is irreducible then either:

- $U_\lambda = \ker(M - \lambda \mathbb{I}) = \emptyset$ , but it's not possible since  $U_\lambda$  is an eigenspace.
- $U_\lambda = \ker(M - \lambda \mathbb{I}) = V \Leftrightarrow \forall v \in V, (M - \lambda \mathbb{I})v = 0$ , which implies that  $M = \lambda \mathbb{I}$ .

□

**Theorem 1.3.** *If  $G$  is an abelian group, then all of its irreducible representations are one dimensional.*

*Proof.* Consider an abelian (=commutative) group  $G$  and  $g \in G$ . Any representation  $(\pi, V)$  of  $G$  will satisfy:

$$\forall h \in G, \pi(g)\pi(h) = \pi(h)\pi(g).$$

and if  $\pi$  is irreducible, by using Schur's Lemna,  $\pi(h) = \lambda \mathbb{I} \simeq \lambda$  for  $\lambda \in \mathbb{C}$ .

□

## 2 Lie Algebra

The main idea is that a representation  $\pi$  is homomorphic and this property allows to characterize  $\pi$  only in term of its derivative  $\pi'$ .  $\pi'$  is a linear map from the tangent space to  $G$  at the identity to the tangent space of  $U(n)$  at the identity. The tangent space to  $G$  at the identity will carry some extra structure coming from the group law and this *vector space* with this structure will be called the Lie algebra of  $G$ , while  $\pi'$  will be an example of a Lie algebra representation.

**Definition 2.1** (Lie Group). A Lie group  $G$  is a differentiable manifold equipped with a group law.

**Definition 2.2** (Lie Algebra). The Lie Algebra of a Lie group  $G$  is the tangent space of  $G$  at the identity.

**Definition 2.3** (Lie Algebra of a matrix group  $G$ ). For  $G$  a Lie group of  $n \times n$  invertible matrices, the Lie algebra of  $G$  (written  $Lie(G)$  or  $\mathfrak{g}$ ) is the space  $X$  of  $n \times n$  matrices such that  $e^{tX} \in G$  for  $t \in \mathbb{R}$ .

We have defined the exponential of the matrices through its power serie

$$e^A = \mathbf{1} + A + \frac{1}{2}A^2 + \dots + \frac{1}{n!}A^n + \dots \quad (2.1)$$

which can be shown to converge for any matrix  $A$ .

*Proof.* We define the *matrix absolute value*<sup>2</sup> of  $A = (a_{ij})$  to be

$$|A| = \sqrt{\sum_{i,j} |a_{ij}|^2}.$$

For an  $n \times n$  real matrix  $A$  the absolute value  $|A|$  is the distance from the origin  $O$  in  $\mathbb{R}^{n^2}$  of the point

$$(a_{11}, a_{12}, \dots, a_{1n}, a_{21}, a_{22}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn}).$$

If  $A$  has complex entries, and if we interpret each copy of  $\mathbb{C}$  as  $\mathbb{R}^2$ , then  $|A|$  is the distance from  $O$  of the corresponding point in  $\mathbb{R}^{(2n)^2}$ . Similarly, if  $A$  has quaternion entries, then  $|A|$  is the distance from  $O$  of the corresponding point in  $\mathbb{R}^{(4n)^2}$ .

In all cases,  $|A - B|$  is the distance between the matrices  $A$  and  $B$ , and we say that a sequence  $A_1, A_2, A_3, \dots$  of  $n \times n$  matrices has *limit*  $A$  if, for each  $\varepsilon > 0$ , there is an integer  $M$  such that

$$m > M \implies |A_m - A| < \varepsilon.$$

The key property of the matrix absolute value is the following inequality, a consequence of the triangle inequality (which holds in the plane and hence in any  $\mathbb{R}^k$ ) and the Cauchy-Schwarz inequality (For all  $u, v \in E$  a vector space with a scalar product, we have:  $|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle$ ).

**Submultiplicative property.** For any two real  $n \times n$  matrices  $A$  and  $B$ ,

$$|AB| \leq |A||B|.$$

**Proof.** If  $A = (a_{ij})$  and  $B = (b_{ij})$ , then it follows from the definition of matrix product that

$$\begin{aligned} |(i, j)\text{-entry of } AB| &= |a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}| \\ &\leq |a_{i1}b_{1j}| + |a_{i2}b_{2j}| + \dots + |a_{in}b_{nj}| \end{aligned}$$

by the triangle inequality

$$= |a_{i1}||b_{1j}| + |a_{i2}||b_{2j}| + \dots + |a_{in}||b_{nj}|$$

by the multiplicative property of absolute value

$$\leq \sqrt{|a_{i1}|^2 + \dots + |a_{in}|^2} \sqrt{|b_{1j}|^2 + \dots + |b_{nj}|^2}$$

by the CauchySchwarz inequality.

Now, summing the squares of both sides, we get

$$\begin{aligned} |AB|^2 &= \sum_{i,j} |(i, j)\text{-entry of } AB|^2 \\ &\leq \sum_{i,j} (|a_{i1}|^2 + \dots + |a_{in}|^2)(|b_{1j}|^2 + \dots + |b_{nj}|^2) \\ &= \sum_i (|a_{i1}|^2 + \dots + |a_{in}|^2) \sum_j (|b_{1j}|^2 + \dots + |b_{nj}|^2) \\ &= |A|^2 |B|^2, \quad \text{as required.} \quad \square \end{aligned}$$

---

<sup>2</sup>One can show that it is a norm just by identifying the isomorphism between  $M_n(\mathbb{K})$  and  $\mathbb{K}^{n^2}$  equipped with the canonical norm, and since we are in a finite dimensionnal vector space, all norms are equivalent.

It follows from the submultiplicative property that  $|A^m| \leq |A|^m$ . Along with the *triangle inequality*  $|A + B| \leq |A| + |B|$ , the submultiplicative property enables us to test convergence of matrix infinite series by comparing them with series of real numbers. In particular, we have:

**Convergence of the exponential series.** *If  $A$  is any  $n \times n$  real matrix, then*

$$1 + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots, \quad \text{where } 1 = n \times n \text{ identity matrix,}$$

*is convergent in  $\mathbb{R}^{n^2}$ .*

**Proof.** It suffices to prove that this series is absolutely convergent, that is, to prove the convergence of

$$|1| + \frac{|A|}{1!} + \frac{|A|^2}{2!} + \frac{|A|^3}{3!} + \dots$$

This is a series of positive real numbers, whose terms (except for the first) are less than or equal to the corresponding terms of

$$1 + \frac{|A|}{1!} + \frac{|A|^2}{2!} + \frac{|A|^3}{3!} + \dots$$

by the submultiplicative property. The latter series is the series for the real exponential function  $e^{|A|}$ ; hence the original series is convergent.  $\square$

Notice that while the group  $G$  determines the Lie algebra  $\mathfrak{g}$ , the Lie algebra does not determine the group. For example  $O(n)$  and  $SO(n)$  admit the same tangent space at identity and thus the same Lie algebra, but elements in  $O(n)$  not in the component of the identity (ie with determinant  $-1$ ) cannot be written in the form  $e^{tX}$  since then you could make a path of matrixes connecting such element to the identity by shrinking  $t$  to zero. Note also that for a given  $X$ , different values of  $t$  could give the same group element.

**Definition 2.4** (Adjoint representation). The adjoint representation  $(\text{Ad}, \mathfrak{g})$  is given by the homomorphism

$$\text{Ad} : g \in G \rightarrow \text{Ad}(g) \in GL(\mathfrak{g})$$

where  $\text{Ad}(g)$  acts on  $X \in \mathfrak{g}$  by

$$(\text{Ad}(g))(X) = gXg^{-1}$$

To show that this is well defined, one needs to check that  $gXg^{-1} \in \mathfrak{g}$  when  $X \in \mathfrak{g}$  but it is guaranteed by the identity

$$e^{tgXg^{-1}} = ge^{tX}g^{-1}$$

obtained directly from the serie representation of the exponential. The homomorphic property of  $\text{Ad}$  is also easy to show.

A Lie algebra is not just a *real* vector space, it comes with an additional structure:

**Definition 2.5** (Lie Bracket). The Lie bracket operation on  $\mathfrak{g}$  is the bilinear antisymmetric map given by the commutator of matrices

$$[\cdot, \cdot] : (X, Y) \in \mathfrak{g} \times \mathfrak{g} \rightarrow [X, Y] = XY - YX \in \mathfrak{g}$$

We now have to check that it is well defined, ie that  $[X, Y] \in \mathfrak{g}$ .

**Theorem 2.1.** *If  $X, Y \in \mathfrak{g}$ ,  $[X, Y] = XY - YX \in \mathfrak{g}$ .*

*Proof.* Since  $X \in \mathfrak{g}$ , we have  $e^{tX} \in G$  and we can act on  $Y \in \mathfrak{g}$  with the adjoint representation

$$\text{Ad}(e^{tX})Y = e^{tX}Y e^{-tX} \in \mathfrak{g}$$

As  $t$  varies, this gives us a parametrized curve in  $\mathfrak{g}$ , whose derivative is also in  $\mathfrak{g}$ :

$$\frac{d}{dt}(e^{tX}Y e^{-tX}) \in \mathfrak{g}.$$

Computing the derivative with the product rule gives:

$$\frac{d}{dt}(e^{tX}Y e^{-tX}) = X e^{tX}Y e^{-tX} - e^{tX}Y e^{-tX} X$$

and evaluating it at  $t = 0$ , we end up with:

$$\left. \frac{d}{dt}(e^{tX}Y e^{-tX}) \right|_{t=0} = XY - YX \in \mathfrak{g}.$$

□

**Remark.** *The relation*

$$\left. \frac{d}{dt}(e^{tX}Y e^{-tX}) \right|_{t=0} = [X, Y]$$

*used in this proof is continually useful to relate the Lie group and the Lie algebra.*

**Definition 2.6** (Structure constants). Since  $\mathfrak{g}$  is a vector space of dimension  $n$ , one can choose a basis  $X_1, \dots, X_n$  and use the fact that the Lie bracket can be written as:

$$[X_i, X_j] = \sum_{k=1}^n c_{ijk} X_k.$$

The set of  $c_{ijk}$  is the set of structure constants of the algebra, that are antisymmetric in  $i \rightleftharpoons j$  due to the antisymmetry of the Lie bracket.

**Remark** (Structure constants of  $\mathfrak{su}(2)$ ).  $\mathfrak{su}(2)$  has a basis  $X_1, X_2, X_3$  satisfying

$$[X_i, X_j] = \sum_{k=1}^3 \epsilon_{ijk} X_k,$$

where  $\epsilon_{ijk}$  is the Levi-Cevita antisymmetric tensor.