Quantum Theory, Groups and Representations

February 6, 2025

1 Representation theory

Definition 1.1. A representation (π, V) of a group G over a vector space V is an application $\pi : G \to GL(V)$ such that $\forall g_1, g_2 \in G,\ \pi(g_1 \cdot g_2) = \pi(g_1)\pi(g_2)$.

- A representation (π, V) is said *irreducible* if there is no proper subset $W \subset V$ such that $\pi_W : G \to GL(W)$ is a representation.
  Another way to express that is that the only invariant[1] subvector spaces of the representation are $\{0\}$ and V.
- Otherwise, (π, V) is said *reducible.*

Definition 1.2. A unitary representation (π, V), where V is a $\mathbb{K}$-EV equiped with an inner product $\langle\ ,\ \rangle : V \times V \to \mathbb{K}$, is a representation which preserves the inner product such that:

$$\forall g \in G,\ \forall v, w \in V,\ \langle v, w\rangle = \langle \pi(g)v, \pi(g)w\rangle\,.$$

If V is finite dimensionnal (of dimension n), we can treat the representation as an element of $\mathcal{M}_{n\times n}(\mathbb{K})$, and for a unitary transformation, $\pi : G \to U(n)$.

Definition 1.3. (Direct sum representation). Given representations π_1 and π_2 of dimensions n_1 and n_2, there is a representation of dimension $n_1 + n_2$ called the direct sum of the two representations, denoted $\pi_1 \oplus \pi_2$. This representation is given by the homomorphism:

$$(\pi_1 \oplus \pi_2) : g \in G \mapsto \begin{pmatrix} \pi_1(g) & \mathbf{0} \\ \mathbf{0} & \pi_2(g) \end{pmatrix}.$$

Theorem 1.1. *A unitary transformation π over a finite dimensionnal vector space V can be written as $\oplus\pi_i$, where $\{\pi_i\}$ is a set of irreducible representations.*

Proof. If (π, V) is irreducible, then it's obvious.
Otherwise, there exists $W \subset V$ such that (π_W, W) is a representation.
Since $V = W \oplus W^\perp$, we want to check if $(\pi_{W^\perp}, W^\perp)$ is a representation.

- Let $w \in W$ and $v \in W^\perp$, then obviously $v \perp w \Leftrightarrow \langle v, w\rangle = \langle \pi(g)v, \pi(g)w\rangle = 0$. Since (π_W, W) is a representation, $\pi(g)w \in W$ and thus $\forall g \in G,\ \pi(g)v \perp W$.
  This means that $\forall g \in G,\ \forall v \in W^\perp,\ \pi(g)v \in W^\perp$. Thus, $\pi_{W^\perp}, W^\perp$ is a representation.(It respects the group product since (π, V) is a representation.)

Thus:

$$(\pi, V) = (\pi_W, W) \oplus (\pi_{W^\perp}, W^\perp).$$

We can then repeat this process until each subrepresentation is irreducible. ☐

[1] A subspace $V' \subseteq V$ is called invariant if $\pi(g)V' \subseteq V'$ for all $g \in G$.

Theorem 1.2. *Schur's Lemna: If a **complex** representation (π, V) is irreducible, then the only application $M : V \to V$ commuting with all the $\pi(g)$ is $M = \lambda\mathbb{I},\ \lambda \in \mathbb{C}$.*

Proof. Let's assume that (π, V) is an irreducible representation and that $\forall g \in G, [M, \pi(g)] = 0$. Since V is a $\mathbb{C}$-vector space, the equation:

$$\det(M - \lambda\mathbb{I}) = 0$$

has n roots (counting the multiplicity) and thus we can find all the eigenspaces:

$$U_\lambda = \{v \in V | Mv = \lambda v\} = \ker(M - \lambda\mathbb{I}) \neq \emptyset.$$

Since π and M commute, $v \in \ker(M - \lambda\mathbb{I}) \implies \forall g \in G\ \pi(g)v \in \ker(M - \lambda\mathbb{I})$. As a consequence, $(\pi, \ker(M - \lambda\mathbb{I}))$ is a representation of G since $\ker(M - \lambda\mathbb{I})$ is stable under the action of π. Since we supposed that (π, V) is irreducible then either:

- $U_\lambda = \ker(M - \lambda\mathbb{I}) = \emptyset$, but it's not possible since U_λ is an eignespace.
- $U_\lambda = \ker(M - \lambda\mathbb{I}) = V \Leftrightarrow \forall v \in V, (M - \lambda\mathbb{I})v = 0$, which implies that $M = \lambda\mathbb{I}$.

□

Theorem 1.3. *If G is an abelian group, then all of its irreducible representations are one dimensional.*

Proof. Consider an abelian (=commutative) group G and $g \in G$. Any representation (π, V) of G will satisfy:

$$\forall h \in G,\ \pi(g)\pi(h) = \pi(h)\pi(g).$$

and if π is irreducible, by using Schur's Lemna, $\pi(h) = \lambda\mathbb{I} \simeq \lambda$ for $\lambda \in \mathbb{C}$. □

2 Lie Algebra

The main idea is that a representation π is homomorphic and this proporty allows to charachterize π only in term of its derivative π'. π' is a linear map from the tangent space to G at the identity to the tangent space of $U(n)$ at the identity. The tangent space to G at the identity will carry some extra structure coming from the group law and this *vector space* with this structure will be called the Lie algebra of G, while π' will be an example of a Lie algebra representation.

Definition 2.1 (Lie Group)**.** A Lie group G is a differentiable manifold equiped with a group law.

Definition 2.2 (Lie Algebra)**.** The Lie Algebra of a Lie group G is the tangent space of G at the identity.

Definition 2.3 (Lie Algebra of a matrix group G)**.** For G a Lie group of $n \times n$ invertible matrices, the Lie algebra of G (written $Lie(G)$ or $\mathfrak{g}$) is the space X of $n \times n$ matrices such that $e^{tX} \in G$ for $t \in \mathbb{R}$.

We have defined the exponential of the matrices throug its power serie

$$e^A = \mathbf{1} + A + \frac{1}{2}A^2 + \cdots + \frac{1}{n!}A^n + \cdots \tag{2.1}$$

which can be shown to converge for any matrix A.

Proof. We define the *matrix absolute value*[2] of $A = (a_{ij})$ to be

$$|A| = \sqrt{\sum_{i,j} |a_{ij}|^2}.$$

For an $n \times n$ real matrix A the absolute value $|A|$ is the distance from the origin O in $\mathbb{R}^{n^2}$ of the point

$$(a_{11}, a_{12}, \dots, a_{1n}, a_{21}, a_{22}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn}).$$

If A has complex entries, and if we interpret each copy of $\mathbb{C}$ as $\mathbb{R}^2$, then $|A|$ is the distance from O of the corresponding point in $\mathbb{R}^{(2n)^2}$. Similarly, if A has quaternion entries, then $|A|$ is the distance from O of the corresponding point in $\mathbb{R}^{(4n)^2}$.

In all cases, $|A - B|$ is the distance between the matrices A and B, and we say that a sequence $A_1, A_2, A_3, \dots$ of $n \times n$ matrices has *limit* A if, for each $\varepsilon > 0$, there is an integer M such that

$$m > M \implies |A_m - A| < \varepsilon.$$

The key property of the matrix absolute value is the following inequality, a consequence of the triangle inequality (which holds in the plane and hence in any $\mathbb{R}^k$) and the Cauchy-Schwarz inequality (For all $u, v \in E$ a vector space with a scalar product, we have :$|\langle u, v\rangle|^2 \leq \langle u, u\rangle \langle v, v\rangle$).

Submultiplicative property. *For any two real $n \times n$ matrices A and B,*

$$|AB| \leq |A||B|.$$

Proof. If $A = (a_{ij})$ and $B = (b_{ij})$, then it follows from the definition of matrix product that

$$\begin{aligned}|(i,j)\text{-entry of } AB| &= |a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}| \\ &\leq |a_{i1}b_{1j}| + |a_{i2}b_{2j}| + \dots + |a_{in}b_{nj}|\end{aligned}$$

by the triangle inequality

$$= |a_{i1}||b_{1j}| + |a_{i2}||b_{2j}| + \dots + |a_{in}||b_{nj}|$$

by the multiplicative property of absolute value

$$\leq \sqrt{|a_{i1}|^2 + \dots + |a_{in}|^2}\sqrt{|b_{1j}|^2 + \dots + |b_{nj}|^2}$$

by the CauchySchwarz inequality.

Now, summing the squares of both sides, we get

$$|AB|^2 = \sum_{i,j} |(i,j)\text{-entry of } AB|^2$$

$$\leq \sum_{i,j} (|a_{i1}|^2 + \dots + |a_{in}|^2)(|b_{1j}|^2 + \dots + |b_{nj}|^2)$$

$$= \sum_{i} (|a_{i1}|^2 + \dots + |a_{in}|^2) \sum_{j} (|b_{1j}|^2 + \dots + |b_{nj}|^2)$$

$$= |A|^2|B|^2, \quad \text{as required.} \quad \square$$

[2] One can show that it is a norm just by identifying the isomorphism between $M_n(\mathbb{K})$ and $\mathbb{K}^{n^2}$ equiped with the canonical norm, and since we are in a finite dimensionnal vector space, all norms are equivalent.

It follows from the submultiplicative property that $|A^m| \leq |A|^m$. Along with the *triangle inequality* $|A+B| \leq |A| + |B|$, the submultiplicative property enables us to test convergence of matrix infinite series by comparing them with series of real numbers. In particular, we have:

Convergence of the exponential series. *If A is any $n \times n$ real matrix, then*

$$1 + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots, \quad \text{where } 1 = n \times n \text{ identity matrix,}$$

is convergent in $\mathbb{R}^{n^2}$.

Proof. It suffices to prove that this series is absolutely convergent, that is, to prove the convergence of

$$|1| + \frac{|A|}{1!} + \frac{|A^2|}{2!} + \frac{|A^3|}{3!} + \ldots.$$

This is a series of positive real numbers, whose terms (except for the first) are less than or equal to the corresponding terms of

$$1 + \frac{|A|}{1!} + \frac{|A|^2}{2!} + \frac{|A|^3}{3!} + \ldots$$

by the submultiplicative property. The latter series is the series for the real exponential function $e^{|A|}$; hence the original series is convergent. □

Notice that while the group G determines the Lie algebra $\mathfrak{g}$, the Lie algebra does not determine the group. For example $O(n)$ and $SO(n)$ admit the same tangent space at identity and thus the same Lie algebra, but elements in $O(n)$ not in the component of the identity (ie with determinant -1) cannot be written in the form e^{tX} since then you could make a path of matrixes connecting such element to the identity by shrinking t to zero. Note also that for a given X, different values of t could give the same group element.

Definition 2.4 (Adjoint representation)**.** The adjoint representation (Ad, $\mathfrak{g}$) is given by the homomorphism

$$Ad : g \in G \to Ad(g) \in GL(\mathfrak{g})$$

where $Ad(g)$ acts on $X \in \mathfrak{g}$ by

$$(Ad(g))(X) = gXg^{-1}$$

To show that this is well defined, one needs to check that $gXg^{-1} \in mathfrakg$ when $X \in mathfrakg$ but it is guarenteed by the identity

$$e^{tgXg^{-1}} = ge^{tX}g^{-1}$$

obtained directly from the serie representation of the exponential. The homomorphic property of Ad is also easy to show.

A Lie algebra is not just a *real* vector space, it comes with an additional structure:

Definition 2.5 (Lie Bracket)**.** The Lie bracket operation on $\mathfrak{g}$ is the bilinear antisymetric map given by the commutator of matrices

$$[\cdot,\cdot] : (X,Y) \in \mathfrak{g} \times \mathfrak{g} \to [X,Y] = XY - YX \in \mathfrak{g}$$

We now have to check that it is well defined, ie that $[X,Y] \in \mathfrak{g}$.

Theorem 2.1. *If $X, Y \in \mathfrak{g}, [X,Y] = XY - YX \in \mathfrak{g}$.*

Proof. Since $X \in \mathfrak{g}$, we have $e^{tX} \in G$ and we can act on $Y \in \mathfrak{g}$ with the adjoint representation

$$Ad(e^{tX})Y = e^{tX} Y e^{-tX} \in \mathfrak{g}$$

As t varies, this gives us a parametrized curve in $\mathfrak{g}$, whose derivative is also in $\mathfrak{g}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tX} Y e^{-tX}\right) \in \mathfrak{g}.$$

Computing the derivative with the product rule gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tX} Y e^{-tX}\right) = X e^{tX} Y e^{-tX} - e^{tX} Y e^{-tX} X$$

and evaluating it a $t = 0$, we end up with:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tX} Y e^{-tX}\right)|_{t=0} = XY - YX \in \mathfrak{g}.$$

$\square$

Remark. *The relation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tX} Y e^{-tX}\right)\Big|_{t=0} = [X, Y]$$

used in this proof is continually useful to relate the Lie group and the Lie algebra.

Definition 2.6 (Structure constants)**.** Since $\mathfrak{g}$ is a vector space of dimension n, one can choose a basis $X_1, \cdots, X_n$ and use the fact that the Lie bracket can be written as:

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k.$$

The set of c_{ijk} is the set of structure constants of the algebra, that are antisymetric in $i \leftrightarrows j$ due to the antisymetry of the Lie bracket.

Remark (Structure constants of $\mathfrak{su}(2)$)**.** $\mathfrak{su}(2)$ *has a basis* X_1, X_2, X_3 *satisfying*

$$[X_i, X_j] = \sum_{k=1}^{3} \epsilon_{ijk} X_k,$$

where ϵ_{ijk} *is the Levi-Cevita antisymetric tensor.*

We will now focus ourself on the groups of linear transformations that preserve an inner product: The unitary and orthogonal groups. These groups are subgroups of respectively $GL(n, \mathbb{C})$ and $GL(n, \mathbb{R})$, consisting of elements Ω satisfying:

$$\Omega \Omega^{\dagger, T} = 1 \tag{2.2}$$

2.1 Lie Algebra of the Unitary/Orthogonal Group $U(n)$ and $O(n)$

Let X be an element of the Lie algebra $\mathfrak{u}(n)$ such that $\Omega = e^{tX}$ for some t. The condition 2.2 then becomes:

$$e^{tX} e^{tX^\dagger} = 1.$$

Taking the derivative of the last equation and evaluating it at $t = 0$ gives:

$$X + X^\dagger = 0 \Leftrightarrow X = -X^\dagger.$$

Definition 2.7 ($\mathfrak{u}(n)$)**.** The lie Algebra $\mathfrak{u}(n)$ is the set of $n \times n$ skew-adjoint complex matrices.

Definition 2.8 ($\mathfrak{o}(n)$)**.** The Lie Algebra $\mathfrak{o}(n)$ is the setis the set of anti-symmetric $n \times n$ real matrices $X = -X^T \quad \forall X \in \mathfrak{o}(n)$.

Regarding the special orthogonal/unitary group, we can use the following identity:

$$\det e^X = e^{\mathrm{Tr}(X)}.$$

Since the determinant of the matrices of $SU(n) \supset SO(n)$ is 1, it add the conditions of tracelessness for the lie Algebra $\mathfrak{su}(n) \supset \mathfrak{so}(n)$.

2.2 To remember

- The group of linear application $GL(n, \mathbb{K})$ admit $\mathfrak{gl}(n, \mathbb{K}) = M_{n \times n}(\mathbb{K})$ the space of n by n $\mathbb{K}$ matrices as its Lie Algebra.
- The Lie algebra $\mathfrak{sl}(n, \mathbb{K})$ of $SL(n, \mathbb{K})$ the group of inversibe $\mathbb{K}$-matrices of determinant 1 is the set of traceless $\mathbb{K}$-matrices.