Quantum Theory, Groups and Representations

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Definition 0.1. A representation (π, V) of a group G over a vector space V is an application $\pi : G \to GL(V)$ such that $\forall g_1, g_2 \in G,\ \pi\,(g_1 \cdot g_2) = \pi\,(g_1)\,\pi(g_2)$.

- A representation (π, V) is said *irreducible* if there is no proper subset $W \subset V$ such that $\pi_W : G \to GL(W)$ is a representation.
- Otherwise, (π, V) is said *reducible.*

Definition 0.2. A unitary representation (π, V), where V is a $\mathbb{K}$-EV equiped with an inner product $\langle\ ,\ \rangle : V \times V \to \mathbb{K}$, is a representation which preserves the inner product such that:

$$\forall g \in G,\ \forall v, w \in V,\ \langle v, w\rangle = \langle \pi(g)v, \pi(g)w\rangle\,.$$

If V is finite dimensionnal (of dimension n), we can treat the representation as an element of $\mathcal{M}_{n\times n}\,(\mathbb{K})$, and for a unitary transformation, $\forall g \in G, \pi(g) \in U(n)$.

Definition 0.3. (Direct sum representation). Given representations π_1 and π_2 of dimensions n_1 and n_2, there is a representation of dimension $n_1 + n_2$ called the direct sum of the two representations, denoted $\pi_1 \oplus \pi_2$. This representation is given by the homomorphism:

$$(\pi_1 \oplus \pi_2) : g \in G \mapsto \begin{pmatrix} \pi_1(g) & \mathbf{0} \\ \mathbf{0} & \pi_2(g) \end{pmatrix}.$$

Theorem. *A unitary transformation π over a finite dimensionnal vector space V can be written as $\oplus\pi_i$, where $\{\pi_i\}$ is a set of irreducible representations.*

Proof. If (π, V) is irreducible, then it's obvious.
Otherwise, there exists $W \subset V$ such that (π_W, W) is a representation.
Since $V = W \oplus W^\perp$, we want to check if $(\pi_{W^\perp}, W^\perp)$ is a representation.

- Let $w \in W$ and $v \in W^\perp$, then obviously $v \perp w \Leftrightarrow \langle v, w\rangle = \langle \pi(g)v, \pi(g)w\rangle = 0$. Since (π_W, W) is a representation, $\pi(g)w \in W$ and thus $\forall g \in G,\ \pi(g)v \perp W$.
  This means that $\forall g \in G,\ \forall v \in W^\perp,\ \pi(g)v \in W^\perp$. Thus, $\pi_{W^\perp}, W^\perp$ is a representation.(It respects the group product since (π, V) is a representation.)

Thus:

$$(\pi, V) = (\pi_W, W) \oplus (\pi_{W^\perp}, W^\perp).$$

We can then repeat this process until each subrepresentation is irreducible. ☐

Theorem. *Schur's Lemna: If a* ***complex*** *representation (π, V) is irreducible, then the only application $M : V \to V$ commuting with all the $\pi(g)$ is $M = \lambda\mathbb{I},\ \lambda \in \mathbb{C}$.*

Proof. Let's assume that (π, V) is an irreducible representation and that $\forall g \in G, [M, \pi(g)] = 0$. Since V is a $\mathbb{C}$-vector space, the equation:

$$\det(M - \lambda\mathbb{I}) = 0$$

has n roots (counting the multiplicity) and thus we can find all the eigenspaces:

$$U_\lambda = \{v \in V | Mv = \lambda v\} = \ker(M - \lambda\mathbb{I}) \neq \emptyset.$$

Since π and M commute, $v \in \ker(M - \lambda\mathbb{I}) \implies \forall g \in G\ \pi(g)v \in \ker(M - \lambda\mathbb{I})$. As a consequence, $(\pi, \ker(M - \lambda\mathbb{I}))$ is a representation of G since $\ker(M - \lambda\mathbb{I})$ is stable under the action of π. Since we supposed that (π, V) is irreducible then either:

- $U_\lambda = \ker(M - \lambda\mathbb{I}) = \emptyset$, but it's not possible since U_λ is an eignespace.
- $U_\lambda = \ker(M - \lambda\mathbb{I}) = V \Leftrightarrow \forall v \in V, (M - \lambda\mathbb{I})v = 0$, which implies that $M = \lambda\mathbb{I}$.

□

Theorem. *If G is an abelian group, then all of its irreducible representations are one dimensionnal.*

Proof. Consider an abelian (=commutative) group G and $g \in G$. Any representation (π, V) of G will satisfy:

$$\forall h \in G,\ \pi(g)\pi(h) = \pi(h)\pi(g).$$

and if π is irreducible, by using Schur's Lemna, $\pi(h) = \lambda\mathbb{I} \simeq \lambda$ for $\lambda \in \mathbb{C}$. □