# Quantum Theory, Groups and Representations

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**Definition 0.1.** A representation $(\pi, V)$ of a group $G$ over a vector space $V$ is an application $\pi : G \to GL(V)$ such that $\forall g_1, g_2 \in G,\ \pi\left(g_1 \cdot g_2\right) = \pi\left(g_1\right) \pi(g_2)$.

- A representation $(\pi, V)$ is said *irreducible* if there is no proper subset $W \subset V$ such that $\pi_W : G \to GL(W)$ is a representation.
- Otherwise, $(\pi, V)$ is said *reducible*.

**Definition 0.2.** A unitary representation $(\pi, V)$, where $V$ is a $\mathbb{K}$-EV equiped with an inner product $\langle\ ,\ \rangle : V \times V \to \mathbb{K}$, is a representation which preserves the inner product such that:

$$\forall g \in G,\ \forall v, w \in V,\ \langle v, w \rangle = \langle \pi(g)v, \pi(g)w \rangle\,.$$

If $V$ is finite dimensionnal (of dimension $n$), we can treat the representation as an element of $\mathcal{M}_{n\times n}\left(\mathbb{K}\right)$, and for a unitary transformation, $\forall g \in G, \pi(g) \in U(n)$.

**Definition 0.3.** (Direct sum representation). Given representations $\pi_1$ and $\pi_2$ of dimensions $n_1$ and $n_2$, there is a representation of dimension $n_1 + n_2$ called the direct sum of the two representations, denoted $\pi_1 \oplus \pi_2$. This representation is given by the homomorphism:

$$(\pi_1 \oplus \pi_2) : g \in G \mapsto \begin{pmatrix} \pi_1(g) & \mathbf{0} \\ \mathbf{0} & \pi_2(g) \end{pmatrix}.$$

**Theorem.** *A unitary transformation $\pi$ over a finite dimensionnal vector space $V$ can be written as $\oplus \pi_i$, where $\{\pi_i\}$ is a set of irreducible representations.*

*Proof.* If $(\pi, V)$ is irreducible, then it's obvious.
Otherwise, there exists $W \subset V$ such that $(\pi_W, W)$ is a representation.
Since $V = W \oplus W^\perp$, we want to check if $(\pi_{W^\perp}, W^\perp)$ is a representation.

- Let $w \in W$ and $v \in W^\perp$, then obviously $v \perp w \Leftrightarrow \langle v, w \rangle = \langle \pi(g)v, \pi(g)w \rangle = 0$. Since $(\pi_W, W)$ is a representation, $\pi(g)w \in W$ and thus $\forall g \in G,\ \pi(g)v \perp W$.
  This means that $\forall g \in G,\ \forall v \in W^\perp,\ \pi(g)v \in W^\perp$. Thus, $\pi_{W^\perp}, W^\perp$ is a representation.(It respects the group product since $(\pi, V)$ is a representation.)

Thus:

$$(\pi, V) = (\pi_W, W) \oplus (\pi_{W^\perp}, W^\perp).$$

We can then repeat this process until each subrepresentation is irreducible. □

**Theorem.** *Schur's Lemna: If a **complex** representation $(\pi, V)$ is irreducible, then the only application $M : V \to V$ commuting with all the $\pi(g)$ is $M = \lambda\mathbb{I},\ \lambda \in \mathbb{C}$.*

*Proof.* Let's assume that $(\pi, V)$ is an irreducible representation and that $\forall g \in G, [M, \pi(g)] = 0$. Since $V$ is a $\mathbb{C}$-vector space, the equation:

$$\det(M - \lambda\mathbb{I}) = 0$$

has $n$ roots (counting the multiplicity) and thus we can find all the eigenspaces:

$$U_\lambda = \{v \in V | Mv = \lambda v\} = \ker(M - \lambda\mathbb{I}) \neq \emptyset.$$

Since $\pi$ and $M$ commute, $v \in \ker(M - \lambda\mathbb{I}) \implies \forall g \in G \ \pi(g)v \in \ker(M - \lambda\mathbb{I})$. As a consequence, $(\pi, \ker(M - \lambda\mathbb{I}))$ is a representation of $G$ since $\ker(M - \lambda\mathbb{I})$ is stable under the action of $\pi$. Since we supposed that $(\pi, V)$ is irreducible then either:

- $U_\lambda = \ker(M - \lambda\mathbb{I}) = \emptyset$, but it's not possible since $U_\lambda$ is an eignespace.
- $U_\lambda = \ker(M - \lambda\mathbb{I}) = V \Leftrightarrow \forall v \in V, (M - \lambda\mathbb{I})v = 0$, which implies that $M = \lambda\mathbb{I}$.

□

**Theorem.** *If $G$ is an abelian group, then all of its irreducible representations are one dimensionnal.*

*Proof.* Consider an abelian (=commutative) group $G$ and $g \in G$. Any representation $(\pi, V)$ of $G$ will satisfy:

$$\forall h \in G, \ \pi(g)\pi(h) = \pi(h)\pi(g).$$

and if $\pi$ is irreducible, by using Schur's Lemna, $\pi(h) = \lambda\mathbb{I} \simeq \lambda$ for $\lambda \in \mathbb{C}$. □