

Quantum Theory, Groups and Representations

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1 Representation theory

Definition 1.1. A representation (π, V) of a group G over a vector space V is an application $\pi : G \rightarrow GL(V)$ such that $\forall g_1, g_2 \in G, \pi(g_1 \cdot g_2) = \pi(g_1)\pi(g_2)$.

- A representation (π, V) is said *irreducible* if there is no proper subset $W \subset V$ such that $\pi_W : G \rightarrow GL(W)$ is a representation.
Another way to express that is that the only invariant¹ subvector spaces of the representation are $\{0\}$ and V .
- Otherwise, (π, V) is said *reducible*.

Definition 1.2. A unitary representation (π, V) , where V is a \mathbb{K} -EV equipped with an inner product $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{K}$, is a representation which preserves the inner product such that:

$$\forall g \in G, \forall v, w \in V, \langle v, w \rangle = \langle \pi(g)v, \pi(g)w \rangle.$$

If V is finite dimensionnal (of dimension n), we can treat the representation as an element of $\mathcal{M}_{n \times n}(\mathbb{K})$, and for a unitary transformation, $\pi : G \rightarrow U(n)$.

Definition 1.3. (Direct sum representation). Given representations π_1 and π_2 of dimensions n_1 and n_2 , there is a representation of dimension $n_1 + n_2$ called the direct sum of the two representations, denoted $\pi_1 \oplus \pi_2$. This representation is given by the homomorphism:

$$(\pi_1 \oplus \pi_2) : g \in G \mapsto \begin{pmatrix} \pi_1(g) & \mathbf{0} \\ \mathbf{0} & \pi_2(g) \end{pmatrix}.$$

Theorem 1.1. A unitary transformation π over a finite dimensionnal vector space V can be written as $\oplus \pi_i$, where $\{\pi_i\}$ is a set of irreducible representations.

Proof. If (π, V) is irreducible, then it's obvious.

Otherwise, there exists $W \subset V$ such that (π_W, W) is a representation.

Since $V = W \oplus W^\perp$, we want to check if (π_{W^\perp}, W^\perp) is a representation.

- Let $w \in W$ and $v \in W^\perp$, then obviously $v \perp w \Leftrightarrow \langle v, w \rangle = \langle \pi(g)v, \pi(g)w \rangle = 0$. Since (π_W, W) is a representation, $\pi(g)w \in W$ and thus $\forall g \in G, \pi(g)v \perp W$.
This means that $\forall g \in G, \forall v \in W^\perp, \pi(g)v \in W^\perp$. Thus, π_{W^\perp}, W^\perp is a representation. (It respects the group product since (π, V) is a representation.)

Thus:

$$(\pi, V) = (\pi_W, W) \oplus (\pi_{W^\perp}, W^\perp).$$

We can then repeat this process until each subrepresentation is irreducible. □

¹A subspace $V' \subseteq V$ is called invariant if $\pi(g)V' \subseteq V'$ for all $g \in G$.

Theorem 1.2. *Schur's Lemna: If a **complex** representation (π, V) is irreducible, then the only application $M : V \rightarrow V$ commuting with all the $\pi(g)$ is $M = \lambda \mathbb{I}$, $\lambda \in \mathbb{C}$.*

Proof. Let's assume that (π, V) is an irreducible representation and that $\forall g \in G, [M, \pi(g)] = 0$. Since V is a \mathbb{C} -vector space, the equation:

$$\det(M - \lambda \mathbb{I}) = 0$$

has n roots (counting the multiplicity) and thus we can find all the eigenspaces:

$$U_\lambda = \{v \in V \mid Mv = \lambda v\} = \ker(M - \lambda \mathbb{I}) \neq \emptyset.$$

Since π and M commute, $v \in \ker(M - \lambda \mathbb{I}) \implies \forall g \in G \pi(g)v \in \ker(M - \lambda \mathbb{I})$. As a consequence, $(\pi, \ker(M - \lambda \mathbb{I}))$ is a representation of G since $\ker(M - \lambda \mathbb{I})$ is stable under the action of π . Since we supposed that (π, V) is irreducible then either:

- $U_\lambda = \ker(M - \lambda \mathbb{I}) = \emptyset$, but it's not possible since U_λ is an eigenspace.
- $U_\lambda = \ker(M - \lambda \mathbb{I}) = V \Leftrightarrow \forall v \in V, (M - \lambda \mathbb{I})v = 0$, which implies that $M = \lambda \mathbb{I}$.

□

Theorem 1.3. *If G is an abelian group, then all of its irreducible representations are one dimensional.*

Proof. Consider an abelian (=commutative) group G and $g \in G$. Any representation (π, V) of G will satisfy:

$$\forall h \in G, \pi(g)\pi(h) = \pi(h)\pi(g).$$

and if π is irreducible, by using Schur's Lemna, $\pi(h) = \lambda \mathbb{I} \simeq \lambda$ for $\lambda \in \mathbb{C}$.

□

2 Lie Algebra

The main idea is that a representation π is homomorphic and this property allows to characterize π only in term of its derivative π' . π' is a linear map from the tangent space to G at the identity to the tangent space of $U(n)$ at the identity. The tangent space to G at the identity will carry some extra structure coming from the group law and this *vector space* with this structure will be called the Lie algebra of G , while π' will be an example of a Lie algebra representation.

Definition 2.1 (Lie Group). A Lie group G is a differentiable manifold equipped with a group law.

Definition 2.2 (Lie Algebra). The Lie Algebra of a Lie group G is the tangent space of G at the identity.

Definition 2.3 (Lie Algebra of a matrix group G). For G a Lie group of $n \times n$ invertible matrices, the Lie algebra of G (written $Lie(G)$ or \mathfrak{g}) is the space X of $n \times n$ matrices such that $e^{tX} \in G$ for $t \in \mathbb{R}$.

We have defined the exponential of the matrices through its power serie

$$e^A = \mathbf{1} + A + \frac{1}{2}A^2 + \dots + \frac{1}{n!}A^n + \dots \quad (2.1)$$

which can be shown to converge for any matrix A .

Proof. We define the *matrix absolute value*² of $A = (a_{ij})$ to be

$$|A| = \sqrt{\sum_{i,j} |a_{ij}|^2}.$$

For an $n \times n$ real matrix A the absolute value $|A|$ is the distance from the origin O in \mathbb{R}^{n^2} of the point

$$(a_{11}, a_{12}, \dots, a_{1n}, a_{21}, a_{22}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn}).$$

If A has complex entries, and if we interpret each copy of \mathbb{C} as \mathbb{R}^2 , then $|A|$ is the distance from O of the corresponding point in $\mathbb{R}^{(2n)^2}$. Similarly, if A has quaternion entries, then $|A|$ is the distance from O of the corresponding point in $\mathbb{R}^{(4n)^2}$.

In all cases, $|A - B|$ is the distance between the matrices A and B , and we say that a sequence A_1, A_2, A_3, \dots of $n \times n$ matrices has *limit* A if, for each $\varepsilon > 0$, there is an integer M such that

$$m > M \implies |A_m - A| < \varepsilon.$$

The key property of the matrix absolute value is the following inequality, a consequence of the triangle inequality (which holds in the plane and hence in any \mathbb{R}^k) and the Cauchy-Schwarz inequality (For all $u, v \in E$ a vector space with a scalar product, we have: $|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle$).

Submultiplicative property. For any two real $n \times n$ matrices A and B ,

$$|AB| \leq |A||B|.$$

Proof. If $A = (a_{ij})$ and $B = (b_{ij})$, then it follows from the definition of matrix product that

$$\begin{aligned} |(i, j)\text{-entry of } AB| &= |a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}| \\ &\leq |a_{i1}b_{1j}| + |a_{i2}b_{2j}| + \dots + |a_{in}b_{nj}| \end{aligned}$$

by the triangle inequality

$$= |a_{i1}||b_{1j}| + |a_{i2}||b_{2j}| + \dots + |a_{in}||b_{nj}|$$

by the multiplicative property of absolute value

$$\leq \sqrt{|a_{i1}|^2 + \dots + |a_{in}|^2} \sqrt{|b_{1j}|^2 + \dots + |b_{nj}|^2}$$

by the CauchySchwarz inequality.

Now, summing the squares of both sides, we get

$$\begin{aligned} |AB|^2 &= \sum_{i,j} |(i, j)\text{-entry of } AB|^2 \\ &\leq \sum_{i,j} (|a_{i1}|^2 + \dots + |a_{in}|^2)(|b_{1j}|^2 + \dots + |b_{nj}|^2) \\ &= \sum_i (|a_{i1}|^2 + \dots + |a_{in}|^2) \sum_j (|b_{1j}|^2 + \dots + |b_{nj}|^2) \\ &= |A|^2 |B|^2, \quad \text{as required.} \quad \square \end{aligned}$$

²One can show that it is a norm just by identifying the isomorphism between $M_n(\mathbb{K})$ and \mathbb{K}^{n^2} equipped with the canonical norm, and since we are in a finite dimensionnal vector space, all norms are equivalent.

It follows from the submultiplicative property that $|A^m| \leq |A|^m$. Along with the *triangle inequality* $|A + B| \leq |A| + |B|$, the submultiplicative property enables us to test convergence of matrix infinite series by comparing them with series of real numbers. In particular, we have:

Convergence of the exponential series. *If A is any $n \times n$ real matrix, then*

$$1 + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots, \quad \text{where } 1 = n \times n \text{ identity matrix,}$$

is convergent in \mathbb{R}^{n^2} .

Proof. It suffices to prove that this series is absolutely convergent, that is, to prove the convergence of

$$|1| + \frac{|A|}{1!} + \frac{|A|^2}{2!} + \frac{|A|^3}{3!} + \dots$$

This is a series of positive real numbers, whose terms (except for the first) are less than or equal to the corresponding terms of

$$1 + \frac{|A|}{1!} + \frac{|A|^2}{2!} + \frac{|A|^3}{3!} + \dots$$

by the submultiplicative property. The latter series is the series for the real exponential function $e^{|A|}$; hence the original series is convergent. \square

Notice that while the group G determines the Lie algebra \mathfrak{g} , the Lie algebra does not determine the group. For example $O(n)$ and $SO(n)$ admit the same tangent space at identity and thus the same Lie algebra, but elements in $O(n)$ not in the component of the identity (ie with determinant -1) cannot be written in the form e^{tX} since then you could make a path of matrixes connecting such element to the identity by shrinking t to zero. Note also that for a given X , different values of t could give the same group element.

Definition 2.4 (Adjoint representation). The adjoint representation $(\text{Ad}, \mathfrak{g})$ is given by the homomorphism

$$\text{Ad} : g \in G \rightarrow \text{Ad}(g) \in GL(\mathfrak{g})$$

where $\text{Ad}(g)$ acts on $X \in \mathfrak{g}$ by

$$(\text{Ad}(g))(X) = gXg^{-1}$$

To show that this is well defined, one needs to check that $gXg^{-1} \in \mathfrak{g}$ when $X \in \mathfrak{g}$ but it is guaranteed by the identity

$$e^{tgXg^{-1}} = ge^{tX}g^{-1}$$

obtained directly from the serie representation of the exponential. The homomorphic property of Ad is also easy to show.

A Lie algebra is not just a *real* vector space, it comes with an additional structure:

Definition 2.5 (Lie Bracket). The Lie bracket operation on \mathfrak{g} is the bilinear antisymmetric map given by the commutator of matrices

$$[\cdot, \cdot] : (X, Y) \in \mathfrak{g} \times \mathfrak{g} \rightarrow [X, Y] = XY - YX \in \mathfrak{g}$$

We now have to check that it is well defined, ie that $[X, Y] \in \mathfrak{g}$.

Theorem 2.1. *If $X, Y \in \mathfrak{g}$, $[X, Y] = XY - YX \in \mathfrak{g}$.*

Proof. Since $X \in \mathfrak{g}$, we have $e^{tX} \in G$ and we can act on $Y \in \mathfrak{g}$ with the adjoint representation

$$\text{Ad}(e^{tX})Y = e^{tX}Y e^{-tX} \in \mathfrak{g}$$

As t varies, this gives us a parametrized curve in \mathfrak{g} , whose derivative is also in \mathfrak{g} :

$$\frac{d}{dt}(e^{tX}Y e^{-tX}) \in \mathfrak{g}.$$

Computing the derivative with the product rule gives:

$$\frac{d}{dt}(e^{tX}Y e^{-tX}) = X e^{tX}Y e^{-tX} - e^{tX}Y e^{-tX} X$$

and evaluating it at $t = 0$, we end up with:

$$\frac{d}{dt}(e^{tX}Y e^{-tX})|_{t=0} = XY - YX \in \mathfrak{g}.$$

□

Remark. The relation

$$\left. \frac{d}{dt}(e^{tX}Y e^{-tX}) \right|_{t=0} = [X, Y]$$

used in this proof is continually useful to relate the Lie group and the Lie algebra.

Definition 2.6 (Structure constants). Since \mathfrak{g} is a vector space of dimension n , one can choose a basis X_1, \dots, X_n and use the fact that the Lie bracket can be written as:

$$[X_i, X_j] = \sum_{k=1}^n c_{ijk} X_k.$$

The set of c_{ijk} is the set of structure constants of the algebra, that are antisymmetric in $i \rightleftharpoons j$ due to the antisymmetry of the Lie bracket.

Remark (Structure constants of $\mathfrak{su}(2)$). $\mathfrak{su}(2)$ has a basis X_1, X_2, X_3 satisfying

$$[X_i, X_j] = \sum_{k=1}^3 \epsilon_{ijk} X_k,$$

where ϵ_{ijk} is the Levi-Cevita antisymmetric tensor.

We will now focus ourself on the groups of linear transformations that preserve an inner product: The unitary and orthogonal groups. These groups are subgroups of respectively $GL(n, \mathbb{C})$ and $GL(n, \mathbb{R})$, consisting of elements Ω satisfying:

$$\Omega \Omega^{\dagger, T} = 1 \tag{2.2}$$

2.1 Lie Algebra of the Unitary/Orthogonal Group $U(n)$ and $O(n)$

Let X be an element of the Lie algebra $\mathfrak{u}(n)$ such that $\Omega = e^{tX}$ for some t . The condition 2.2 then becomes:

$$e^{tX} e^{tX^\dagger} = 1.$$

Taking the derivative of the last equation and evaluating it at $t = 0$ gives:

$$X + X^\dagger = 0 \Leftrightarrow X = -X^\dagger.$$

Definition 2.7 ($\mathfrak{u}(n)$). The lie Algebra $\mathfrak{u}(n)$ is the set of $n \times n$ skew-adjoint complex matrices.

Definition 2.8 ($\mathfrak{o}(n)$). The Lie Algebra $\mathfrak{o}(n)$ is the set of anti-symmetric $n \times n$ real matrices $X = -X^T \quad \forall X \in \mathfrak{o}(n)$.

Regarding the special orthogonal/unitary group, we can use the following identity:

$$\det e^X = e^{\text{Tr}(X)}.$$

Since the determinant of the matrices of $SU(n)$ is 1, it add the conditions of tracelessness for the lie Algebra $\mathfrak{su}(n)$. It is slightly different for $\mathfrak{so}(n)$ since matrices X of $\mathfrak{o}(n)$ are already traceless because of $X^T = -X$ implying that the diagonal coefficients are zero.

2.2 To remember

- The group of linear application $GL(n, \mathbb{K})$ admit $\mathfrak{gl}(n, \mathbb{K}) = M_{n \times n}(\mathbb{K})$ the space of n by n \mathbb{K} matrices as its Lie Algebra.
- The Lie algebra $\mathfrak{sl}(n, \mathbb{K})$ of $SL(n, \mathbb{K})$ the group of inversible \mathbb{K} -matrices of determinant 1 is the set of traceless \mathbb{K} -matrices.
- The orthogonal group $O(n) \subset GL(n, \mathbb{R})$ is the group of $n \times n$ real matrices Ω satisfying $\Omega^T = -\Omega$. Its Lie algebra $\mathfrak{o}(n)$ is the Lie algebra of $n \times n$ real matrices X such as $X^T = -X$.
- The special orthogonal group $SO(n) \subset SL(n, \mathbb{R})$ is the subgroup of $O(n)$ with determinant 1. It has the same Lie algebra as $O(n)$: $\mathfrak{so}(n) = \mathfrak{o}(n)$.
- The unitary group $U(n) \subset GL(n, \mathbb{C})$ is the group of $n \times n$ complex matrices Ω such that $\Omega^\dagger = -\Omega$. Its Lie algebra $\mathfrak{u}(n)$ is the Lie algebra of $n \times n$ complex anti-Hermitian ³ matrices X , those satisfying $X^\dagger = -X$.
- The special unitary group $SU(n) \subset SL(n, \mathbb{C})$ is the subgroup of $U(n)$ matrices with determinant 1. Its Lie algebra $\mathfrak{su}(n)$ is the Lie algebra of $n \times n$ anti-Hermitian traceless matrices X .

³skew = anti