FK & IK

Kinematics Analysis

> **Status**: Source Note (Level 3)
> **Reference**: Coord definitions, System_Parameters

0. Notation

To maintain consistency with the derivation notes in Coord definitions, we adopt the following symbolic conventions:

- **Position (Point):** $[x, y, z]_F^T$ (using square brackets).
- **Direction (Vector/Axis):** $\langle x, y, z \rangle_F$ (using angle brackets).
- **Transformation:** $[x, y, z, 1]_B^T = \mathbf{T}_A^B \cdot [x, y, z, 1]_A^T$.
- **Actuated Configuration:** $\mathbf{q} = [\theta, \beta, \gamma]^T$.

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1. Forward Kinematics (FK)

The Forward Kinematics maps the actuated joint space (θ, β, γ) and the contact selection parameters (α, w) to the task space coordinates in the robot root frame.

1.1 Sagittal Plane Mapping: $[x, y, w]_L^T$

The continuous contact surface of the foot rim is parameterized by α and the lateral width w. The position of a point on the rim in the Leg Frame {L} is:

$$[x, y, w]_L^T = \begin{cases} \begin{bmatrix} [x, y]_{U_R}^T + R \cdot \mathbf{rot}(\alpha - \psi_{FG}) \frac{\langle x,y \rangle_{U_R \to J_R}}{|\langle x,y \rangle_{U_R \to J_R}|} \\ w \end{bmatrix} & , \alpha > \psi_{FG} \\ \begin{bmatrix} [x, y]_{O_r}^T + R \cdot \mathbf{rot}(\alpha) \frac{\langle x,y \rangle_{O_r \to G}}{|\langle x,y \rangle_{O_r \to G}|} \\ w \end{bmatrix} & , -\psi_{FG} \le \alpha \le \psi_{FG} \\ \begin{bmatrix} [x, y]_{U_L}^T + R \cdot \mathbf{rot}(\alpha + \psi_{FG}) \frac{\langle x,y \rangle_{U_L \to J_L}}{|\langle x,y \rangle_{U_L \to J_L}|} \\ w \end{bmatrix} & , \alpha < -\psi_{FG} \end{cases}$$

Where:

- $w \in [-T/2, T/2]$ is the lateral coordinate.
- $[x, y]_{U_L}^T, [x, y]_{U_R}^T, [x, y]_{O_r}^T$ are the centers of the circular axes.

1.2 Coordinate Transformation Hierarchy

The total transformation from the leg contact point to the robot body is:

Step 1: Leg to Module ($[x, y, z, 1]_{M_i}^T = \mathbf{T}_{L_i}^{M_i} \cdot [x, y, z, 1]_{L_i}^T$)

Accounts for wheel axial offset d_{wheel} and side-specific mounting.

- **Left Legs ($i \in \{0, 3\}$):**

$$\mathbf{T}_{L_i}^{M_i} = \begin{bmatrix} 0 & 0 & 1 & d_{\text{wheel}} \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Right Legs ($i \in \{1, 2\}$):**

$$\mathbf{T}_{L_i}^{M_i} = \begin{bmatrix} 0 & 0 & -1 & d_{\text{wheel}} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2: Module to Robot ($[x, y, z, 1]_R^T = \mathbf{T}_{M_i}^R \cdot [x, y, z, 1]_{M_i}^T$)

Accounts for hip offset $[x, y, z]_{M_i \text{ in } R}^T$ and active ABAD rotation γ.

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2. Inverse Kinematics (IK)

The Inverse Kinematics solves for $\mathbf{q} = [\theta, \beta, \gamma]^T$ given a target $[x, y, z]_{\text{target}}^T$ and selection parameters (α, w).

2.1 Problem Formulation

We define the error function $\mathbf{E}(\mathbf{q})$ using the homogeneous transformation chain:

$$\mathbf{E}(\mathbf{q}) = \text{FK}(\mathbf{q}, \alpha, w) - [x, y, z]_{\text{target}}^T = 0$$

2.2 Gauss–Newton Iteration

To find the configuration $\mathbf{q}$ that minimizes the squared tracking error $|||\mathbf{E}(\mathbf{q})|||^2$, we employ the Gauss-Newton method.

1. **Construct Jacobian**: $\mathbf{J}(\mathbf{q})$ via numerical perturbation.
2. **Compute Update**:

$$\Delta\mathbf{q} = -(\mathbf{J}^T\mathbf{J})^{-1}\mathbf{J}^T\mathbf{E}(\mathbf{q})$$

3. **Update State**: $\mathbf{q}_{n+1} = \mathbf{q}_n + \Delta\mathbf{q}$

2.3 Numerical Jacobian Construction

Since an analytical Jacobian is complex to derive for the 3D mechanism with rolling contact, we approximate $\mathbf{J}(\mathbf{q})$ using the **Finite Difference Method** by applying small perturbations δ to each joint.

Let $\mathbf{FK}(\mathbf{q}) = [x, y, z]^T$. The 3×3 Jacobian matrix is expanded as:

$$\mathbf{J}(\mathbf{q}) = \begin{bmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \beta} & \frac{\partial x}{\partial \gamma} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \beta} & \frac{\partial y}{\partial \gamma} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \beta} & \frac{\partial z}{\partial \gamma} \end{bmatrix} \approx \begin{bmatrix} \frac{\Delta x_\theta}{\delta} & \frac{\Delta x_\beta}{\delta} & \frac{\Delta x_\gamma}{\delta} \\ \frac{\Delta y_\theta}{\delta} & \frac{\Delta y_\beta}{\delta} & \frac{\Delta y_\gamma}{\delta} \\ \frac{\Delta z_\theta}{\delta} & \frac{\Delta z_\beta}{\delta} & \frac{\Delta z_\gamma}{\delta} \end{bmatrix}$$

Where the columns are computed by perturbing each joint individually:

- **Column 1 (θ perturbation):**

$$\mathbf{J}_{col1} \approx \frac{\mathbf{FK}(\theta + \delta, \beta, \gamma) - \mathbf{FK}(\theta, \beta, \gamma)}{\delta}$$

- **Column 2 (β perturbation):**

$$\mathbf{J}_{col2} \approx \frac{\mathbf{FK}(\theta, \beta + \delta, \gamma) - \mathbf{FK}(\theta, \beta, \gamma)}{\delta}$$

- **Column 3 (γ perturbation):**

$$\mathbf{J}_{col3} \approx \frac{\mathbf{FK}(\theta, \beta, \gamma + \delta) - \mathbf{FK}(\theta, \beta, \gamma)}{\delta}$$