

# Solved Exercises

## Optimization and Operations Research



Jordi Villà-Freixa

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\*e.mail: [jordi.villa@uvic.cat](mailto:jordi.villa@uvic.cat)

# 1 Non-linear optimization

## 1.1 Lagrange multipliers

**Exercise 1** — Optimization of  $f(x, y) = x^2 - y$  subject to  $x^2 + y^2 = 4$

**Solution (Exercise 1)** — We will use the method of Lagrange multipliers.

**Step 1: Define the Lagrange Function** Define the constraint as  $g(x, y) = x^2 + y^2 - 4 = 0$ . The Lagrange function is then:

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda \cdot g(x, y) = x^2 - y + \lambda(x^2 + y^2 - 4).$$

**Step 2: Compute the Partial Derivatives** We now compute the partial derivatives of  $\mathcal{L}(x, y, \lambda)$  with respect to  $x$ ,  $y$ , and  $\lambda$ :

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= 2x + \lambda \cdot 2x = 2x(1 + \lambda) = 0, \\ \frac{\partial \mathcal{L}}{\partial y} &= -1 + \lambda \cdot 2y = 0 \quad \Rightarrow \quad \lambda = \frac{1}{2y}, \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= x^2 + y^2 - 4 = 0.\end{aligned}$$

**Step 3: Solve the Equations**

**From**  $\frac{\partial \mathcal{L}}{\partial x} = 0$ :

$$2x(1 + \lambda) = 0.$$

This gives two possibilities:

- $x = 0$ , or
- $1 + \lambda = 0 \quad \Rightarrow \quad \lambda = -1.$

**Case 1:  $x = 0$**  Substitute  $x = 0$  into the constraint equation  $x^2 + y^2 = 4$ :

$$0^2 + y^2 = 4 \quad \Rightarrow \quad y^2 = 4 \quad \Rightarrow \quad y = \pm 2.$$

For  $y = 2$ , substitute into  $f(x, y) = x^2 - y$ :

$$f(0, 2) = 0^2 - 2 = -2.$$

For  $y = -2$ , substitute into  $f(x, y)$ :

$$f(0, -2) = 0^2 - (-2) = 2.$$

**Case 2:**  $\lambda = -1$  Substitute  $\lambda = -1$  into  $\lambda = \frac{1}{2y}$ :

$$-1 = \frac{1}{2y} \Rightarrow y = -\frac{1}{2}.$$

Now, substitute  $y = -\frac{1}{2}$  into the constraint equation  $x^2 + y^2 = 4$ :

$$x^2 + \left(-\frac{1}{2}\right)^2 = 4 \Rightarrow x^2 + \frac{1}{4} = 4 \Rightarrow x^2 = \frac{15}{4} \Rightarrow x = \pm \frac{\sqrt{15}}{2}.$$

Now, calculate  $f(x, y) = x^2 - y$  for  $y = -\frac{1}{2}$  and  $x = \pm \frac{\sqrt{15}}{2}$ :

For  $x = \frac{\sqrt{15}}{2}$ :

$$f\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) = \left(\frac{\sqrt{15}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{15}{4} + \frac{1}{2} = \frac{17}{4}.$$

For  $x = -\frac{\sqrt{15}}{2}$ :

$$f\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) = \left(-\frac{\sqrt{15}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{15}{4} + \frac{1}{2} = \frac{17}{4}.$$

**Step 4: Compare the Results** We now compare the function values:

- For  $(0, 2)$ :  $f(0, 2) = -2$ .
- For  $(0, -2)$ :  $f(0, -2) = 2$ .
- For  $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$  and  $\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$ :  $f = \frac{17}{4} \approx 4.25$ .

## Conclusion

- The maximum value is  $f = \frac{17}{4} \approx 4.25$  at  $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$  and  $\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$ .
- The minimum value is  $f = -2$  at  $(0, 2)$ .

For a Python implementation check this file.

## 1.2 Karush-Kuhn-Tucker theorem

**Exercise 2** — Maximize  $f(x, y) = xy$  subject to  $100 \geq x + y$  and  $x \leq 40$  and  $(x, y) \geq 0$ .

**Solution (Exercise 2)** — The Karush Kuhn Tucker (KKT) conditions for optimality are a set of necessary conditions for a solution to be optimal in a mathematical optimization problem. They are necessary and sufficient conditions for a local minimum in nonlinear programming problems. The KKT conditions consist of the following elements:

For an optimization problem in its standard form:

$$\begin{aligned} \max f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) - b_i \leq 0 \quad i = 1, \dots, k \\ & g_i(\mathbf{x}) - b_i = 0 \quad i = k + 1, \dots, m \end{aligned}$$

There are 4 KKT conditions for optimal primal ( $\mathbf{x}$ ) and dual ( $\lambda$ ) variables. If  $\mathbf{x}^*$  denotes optimal values:

1. Primal feasibility: all constraints must be satisfied:  $g_i(\mathbf{x}^*) - b_i$  is feasible. Applies to both equality and non-equality constraints.
2. Gradient condition or No feasible descent: No possible improvement at the solution:

$$\nabla f(\mathbf{x}^*) - \sum_{i=1}^m \lambda_i^* \nabla g_i(\mathbf{x}^*) = 0$$

3. Complementarity slackness:

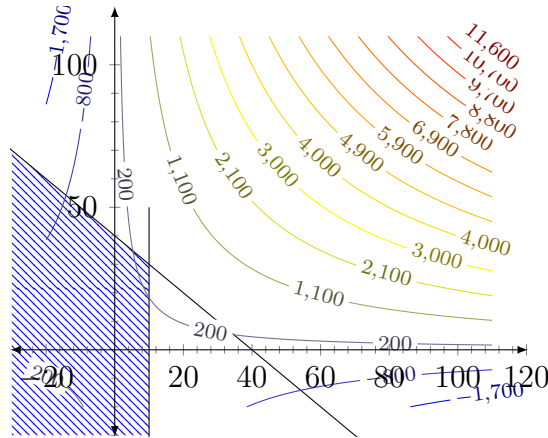
$$\lambda_i^* (g_i(\mathbf{x}^*) - b_i) = 0$$

4. Dual feasibility:  $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active ( $=0$ ) and a zero Lagrange multiplier when the constraint is inactive ( $>0$ ).

To solve our problem, first we will put it in its standard form:

$$\begin{aligned}
 \max f(x, y) &= xy \\
 \text{s.t.} \quad g_1(x, y) &= x + y - 100 \leq 0 \\
 g_2(x, y) &= x - 40 \leq 0 \\
 g_3(x, y) &= -x \leq 0 \\
 g_4(x, y) &= -y \leq 0
 \end{aligned}$$



We will go through the different conditions:

- on the gradient:

$$\left( \frac{\partial f}{\partial x} \right) - \lambda_1 \left( \frac{\partial g_1}{\partial x} \right) - \lambda_2 \left( \frac{\partial g_2}{\partial x} \right) - \lambda_3 \left( \frac{\partial g_3}{\partial x} \right) - \lambda_4 \left( \frac{\partial g_4}{\partial x} \right) = 0$$

which, in this example, resolves into:

$$y - (\lambda_1 + \lambda_2 - \lambda_3) = 0 \quad (1)$$

$$x - (\lambda_1 - \lambda_4) = 0 \quad (2)$$

- on the complementary slackness:

$$\lambda_1(x + y - 100) = 0 \quad (3)$$

$$\lambda_2(x - 40) = 0 \quad (4)$$

$$\lambda_3 x = 0 \quad (5)$$

$$\lambda_4 y = 0 \quad (6)$$

• on the constraints:

$$x + y \leq 100 \quad (7)$$

$$x \leq 40 \quad (8)$$

$$-x \leq 0 \quad (9)$$

$$-y \leq 0 \quad (10)$$

plus  $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ .

We will start by checking Eq. 3:

–Let us see what occurs if  $\lambda_1 = 0$ . Then, from Eq. 2,  $x + \lambda_4 = 0$  which implies that  $x = \lambda_4 = 0$ <sup>1</sup>. But, then, from Eq. 4 we obtain that  $\lambda_2 = 0$  which, using Eq. 1 gives  $y + \lambda_3 = 0 \Rightarrow y = \lambda_3 = 0$ . Indeed, the KKT conditions are satisfied when all variables and multipliers are zero, but it is not a maximum of the function (see figure above).

–So, let us see what happens if  $x + y - 100 = 0$  and consider the two possibilities for  $x$ :

**Case  $x = 0$ :** Then,  $y = 100$ , which would lead (Eq. 6) to  $\lambda_4 = 0$  and (Eq. 2) to  $x = \lambda_1 = 0$ , that was discussed in the previous item. So, we need to explore the other possibility for  $x$ .

**Case  $x > 0$ :** From Eq. 5  $\lambda_3 = 0$  and, from Eqs. 1 and 2:

$$\begin{cases} y = \lambda_1 + \lambda_2 \\ x = \lambda_1 + \lambda_4 \end{cases}$$

let us try what happens if, e.g.,  $\lambda_2 \neq 0$  (or, said in other words, if constraint 8 is active):  $x = 40$ . As we know we do not want  $\lambda_1 = 0$ , from Eq. 3 we obtain  $x + y - 100 = 0 \Rightarrow y = 60$ .

The point  $(x, y) = (40, 60)$  fulfills the KKT conditions and is a maximum in the constrained maximization problem.

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<sup>1</sup>Recall that both variables and multipliers must be positive or zero, so, the only possibility for the equation to fulfill is that both are zero.

## 2 Duality

**Exercise 3 —** Consider the linear programming problem:

$$\begin{array}{ll} \max & x_1 + 4x_2 + 2x_3 \\ \text{s.t.} & \begin{array}{ll} 5x_1 + 2x_2 + 2x_3 & \leq 145 \\ 4x_1 + 8x_2 - 8x_3 & \leq 260 \\ x_1 + x_2 + 4x_3 & \leq 190 \\ x_1, x_2, x_3 & \geq 0 \end{array} \end{array}$$

Find  $x_1$ ,  $x_2$  and  $x_3$  to solve it.

**Solution (Exercise 3)** — material: complementary slackness.pdf

**Exercise 4 —** Consider the linear programming problem:

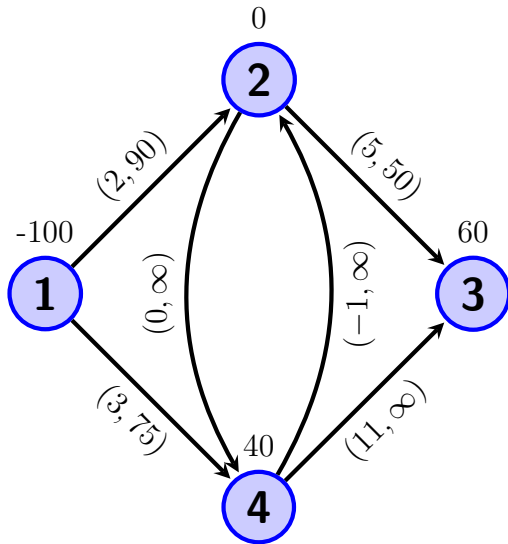
$$\begin{array}{ll} \max & 2x_1 + 16x_2 + 2x_3 \\ \text{s.t.} & \begin{array}{ll} 2x_1 + x_2 - x_3 & \leq -3 \\ -3x_1 + x_2 + 2x_3 & \leq 12 \\ x_1, x_2, x_3 & \geq 0 \end{array} \end{array}$$

Check whether each of the following is an optimal solution, using complementary slackness:

**Solution (Exercise 4)** — material: compslack.pdf

## 3 Network analysis

**Exercise 5 —** The figure shows a network on four nodes, including net demands on the vertex,  $b_k$ , and cost and capacity on the edges,  $(c_{i,j}, u_{i,j})$ . (Adapted from [?])



1. Formulate the corresponding minimum cost network flow model

2. Classify the nodes as *source*, *sink* or *transshipment*

**Solution (Exercise 5)** — In this problem, vertex and edges are:

$$V = \{1, 2, 3, 4\}$$

$$A = \{(1, 2), (1, 4), (2, 3), (2, 4), (4, 2), (4, 3)\}$$

we can use the variables  $x_{i,j}$  to represent the flows in the different members of set  $A$ . Thus, the formulation of the problem is:

$$\begin{aligned} \min \quad & 2x_{1,2} + 3x_{1,4} + 5x_{2,3} - x_{4,2} + 11x_{4,3} \\ \text{subject to} \quad & \left\{ \begin{array}{lcl} -x_{1,2} - x_{1,4} & & = -100 \\ x_{1,2} & - x_{2,3} - x_{2,4} & + x_{4,2} = 0 \\ & x_{2,3} & + x_{4,3} = 60 \\ & x_{1,4} & + x_{2,4} - x_{4,3} - x_{4,2} = 40 \\ x_{1,2} & & \leq 90 \\ & x_{1,4} & \leq 75 \\ & x_{2,3} & \leq 50 \end{array} \right. \end{aligned}$$

and  $x_{i,j} \geq 0$ .

There are 4 KKT conditions for optimal primal ( $x$ ) and dual ( $\lambda$ ) variables. If  $x^*$  denotes optimal values:



1. Primal feasibility: all constraints must be satisfied:  $g_i(x^*) - b_i$  is feasible. Applies to both equality and non-equality constraints.
2. Gradient condition or No feasible descent: No possible improvement at the solution:

$$\nabla f(x^*) - \sum_{i=1}^m \lambda_i^* \nabla g_i(x^*) = 0$$

3. Complementarity slackness:

$$\lambda_i^* (g_i(x^*) - b_i) = 0$$

4. Dual feasibility:  $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active ( $=0$ ) and a zero Lagrange multiplier when the constraint is inactive ( $>0$ ).

to solve our problem, first we will put it in its standard form:

$$\begin{aligned} \min f(x, y) &= -xy \\ \text{subject to} \quad & -x - y + 100 \geq 0 \\ & -x - 40 \geq 0 \end{aligned}$$

We will go through the different conditions:

1. Primal feasibility:  $g_i(x^*) - b_i$  is feasible.

$$-x^* - y^* + 100 = 0$$

$$-x^* - 40 = 0$$

2. Gradient condition or No feasible descent:

$$\left( \frac{\partial f}{\partial x} \right) - \lambda_1 \left( \frac{\partial g_1}{\partial x} \right) - \lambda_2 \left( \frac{\partial g_2}{\partial x} \right) = 0$$

which, in this example, resolves into:

$$\begin{cases} -y + \lambda_1 + \lambda_2 = 0 \\ -x - \lambda_1 = 0 \end{cases}$$

3. Complementarity slackness:

$$\lambda_1^* (-x^* - y^* + 100) = 0$$

$$\lambda_2^* (-x^* - 40) = 0$$

4. Dual feasibility:  $\lambda_1, \lambda_2 \geq 0$

We can put the resulting 5 expressions for conditions 1 and 2 into matrix form:

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -100 \\ 40 \\ 0 \\ 0 \end{pmatrix}$$