Unit 2. Linear programming. Duality

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Preliminary

This course is strongly based on the monography on Operations Research by Carter, Price and Rabadi [Carter, 2019], and in material obtained from different sources (quoted when needed through the slides).

Learning outcomes

- Understanding Dual and Primal problems in LP
- Economic interpretation
- Conditions of optimality
- Resolution of the dual by the primal and penalty method

Summary

A first example

$$
\begin{array}{ll}
\text{maximize} & 4x_1 + x_2 + 5x_3 + 3x_4 \\
\text{subject to} & \left\{
\begin{array}{rcl}
x_1 - x_2 - x_3 + 3x_4 & \leq & 1 \\
5x_1 + x_2 + 3x_3 + 8x_4 & \leq & 55 \\
-x_1 + x_2 + 3x_3 - 5x_4 & \leq & 3 \\
x_1, x_2, x_3 & \geq & 0
\end{array}
\right.
\end{array}
$$

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A first example

note that

$$
\begin{aligned}
y_1(x_1 - x_2 - x_3 + 3x_4) + \\
y_2(5x_1 + x_2 + 3x_3 + 8x_4) + \\
y_3(-x_1 + x_2 + 3x_3 - 5x_4) \\
\leq \\
y_1 + 55y_2 + 3y_3
\end{aligned}
$$

Primal and Dual

We see that maximizing the **Primal objective function** $4x_1 + x_2 + 5x_3 + 3x_4$ is equivalent to minimize the **Dual objective function** $y_1 + 55y_2 + 3y_3$:

$$
\begin{array}{ll}
\text{minimize} & y_1 + 55y_2 + 3y_3 \\
\text{subject to} & \left\{ \begin{array}{rcl}
y_1 - 5y_2 - y_3 & \geq & 4 \\
-y_1 + y_2 + 2y_3 & \geq & 1 \\
-y_1 + 3y_2 + 3y_3 & \geq & 5 \\
3y_1 + 8y_2 - 5y_3 & \geq & 3 \\
y_1, y_2, y_3, y_4 & \geq & 0
\end{array} \right.
\end{array}
$$

Generalization

In general, if we can write the LP problem in its **normal formulation**:

$$\left.\begin{array}{ll} \min & \mathbf{c}^t\mathbf{x} \\ \text{subject to} & A\mathbf{x} \geq \mathbf{b}, \forall x_i \geq 0 \end{array}\right\} \text{PRIMAL}$$

$$\left.\begin{array}{ll} \max & \mathbf{b}^t\mathbf{y} \\ \text{subject to} & A^t\mathbf{y} \leq \mathbf{c}, \forall y_i \geq 0 \end{array}\right\} \text{DUAL}$$

The dual problem is a transposition of the primal problem.

Exercise 1

Find the dual problem of

$$\text{maximize} \quad 3x_1 + 2x_2$$

$$\text{subject to} \quad \begin{cases} 2x_1 + x_2 & \leq & 4 \\ 2x_1 + 3x_2 & \leq & 6 \\ x_1, x_2 & \geq & 0 \end{cases}$$

Solve it graphically and using Simplex.

Exercise 2

Find the dual problem of

$$
\begin{array}{ll}
\text{minimize} & 4x_1 + 4x_2 + x_3 \\
\text{subject to} & \left\{ \begin{array}{rcl} x_1 + x_2 + x_3 & \leq & 2 \\ 2x_1 + x_2 & = & 3 \\ 2x_1 + x_2 + 3x_3 & \geq & 3 \\ x_1, x_2, x_3 & \geq & 0 \end{array} \right.
\end{array}
$$

Trick: normalize the problem first: eg $2x_1 + x_2 = 3 \to \begin{cases} 2x_1 + x_2 \geq 3 \\ 2x_1 + x_2 \leq 3 \end{cases}$.

Summary

Dual of the dual

Theorem

In the case of linear programming, the dual of the dual is the primal.

Proof.

$$\left.\begin{array}{ll} \max & \mathbf{b}^t\mathbf{y} \\ \text{subject to} & A^t\mathbf{y} \leq \mathbf{c}, \forall y_i \geq 0 \end{array}\right\} \text{DUAL}$$

is equivalent to

$$\left.\begin{array}{ll} \min & -\mathbf{b}^t\mathbf{y} \\ \text{subject to} & -A^t\mathbf{y} \geq -\mathbf{c}, \forall y_i \geq 0 \end{array}\right\} \text{DUAL}$$

where, by taking again the Dual, leads to the original Primal. □

Meaning of Duality I

Let us retake the example we saw in the last session:

$$
\begin{array}{ll}
\text{maximize} & z = 8x_1 + 5x_2 \\
\text{subject to} & \begin{array}{rcl} x_1 & \leq & 150 \\ x_2 & \leq & 250 \\ 2x_1 + x_2 & \leq & 500 \\ x_1, x_2 & \geq & 0 \end{array}
\end{array}
$$

We found out that, from the initial tableau:

$$
\begin{array}{cc|c|ccccc|c}
 & & z & x_1 & x_2 & s & t & u & b \\
\hline
 & z & 1 & -8 & -5 & 0 & 0 & 0 & 0 \\
\hline
\rightarrow & s & 0 & 1 & 0 & 1 & 0 & 0 & 150 \\
 & t & 0 & 0 & 1 & 0 & 1 & 0 & 250 \\
 & u & 0 & 2 & 1 & 0 & 0 & 1 & 500 \\
\text{basic} & & & \uparrow & & & & &
\end{array}
$$

Meaning of Duality II

and after applying the different steps of the Simplex method we ended up with the final tableau:

		z	x_1	x_2	s	t	u	b
$R_1 + 5R_4$	z	1	0	0	0	1	4	2250
	x_1	0	1	0	0	$-1/2$	$1/2$	125
$\rightarrow R_3 - R_4$	s	0	0	0	1	$1/2$	$-1/2$	25
	x_2	0	1	0	1	0	0	250
	basic				$\uparrow$			

Now, if we multiply the original availability of each resource (shown in the original tableau) by its marginal worth (taken from the final tableau) and get the sum, we obtain the optimal objective function value:

$$z* = 2250 = 0(150) + 1(250) + 4(500)$$

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Duality properties

Theorem (Weak Duality)

In a max LP, the value of primal objective function for any feasible solution is bounded from above by any feasible solution to its dual:

$$max \qquad \bar{z} \leq \bar{w}$$

The statement is analogous to a minimization problem.

Theorem (Unboundness property)

If primal (dual) problem has an unbounded solution, then the dual (primal) is unfeasible.

$$max \qquad \bar{z} \leq \infty$$

Duality properties

Theorem (Strong Duality)

If the primal problem has an optimal solution,

$$x^* = (x_1^*, \ldots, x_n^*)$$

the the dual also has an optimal solution,

$$y^* = (y_1^*, \ldots, y_m^*)$$

and

$$z^* := \sum_j c_j x_j^* = \sum_i b_i y_i^* := w^*$$

Thus: if feasible objective function values are found for a primal and dual pair of problems, and if these values are equal to each other, then both of the solutions are optimal solutions.

The Shadow prices that appear at the top of the optimal tableau of the

Cases with no optimal solutions for primal and dual

Exactly one of the following mutually exclusive cases always occurs:

- Both primal and dual problems are feasible, and both have optimal (and equal) solutions.
- Both primal and dual problems are infeasible (have no feasible solution).
- The primal problem is feasible but unbounded, and the dual problem is infeasible.
- The dual problem is feasible but unbounded, and the primal problem is infeasible.

Complementary slackness

Because each decision variable in a primal problem is associated with a constraint in the dual problem, each such variable is also associated with a slack or surplus variable in the dual.
In any solution, if the primal variable is basic (with value ≥ 0, hence having slack), then the associated dual variable is non-basic ($= 0$, hence having no slack), and viceversa.

Theorem

If in an optimal solution to a LP problem and inequality constraint is not binding, then the dual variable corresponding to that constraint has a value of zero in any optimal solution to the dual problem. In other words: suppose x_0 and y_0 are feasible solutions of (max) and (min) representations. Then, x_0, y_0 are optimal solutions if and only if

$$(b - Ax_0) \cdot y_0 = 0$$
$$(A^t y_0 - c) \cdot x_0 = 0$$

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