Unit 3. Linear programming

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Preliminary

This course is strongly based on the monography on Operations Research by Carter, Price and Rabadi [1], and in material obtained from different sources (quoted when needed through the slides).

Introduction to linear programming: Summary

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Learning outcomes

- Learn how to formulate a linear programming model
- Get familiar with the terms feasible region, feasible solutions and optimal feasible solutions
- Managing graphical solution of the linear programming model
- Identify models with unique, multiple or no optimal feasible solutions
- Introducing the Simplex method

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The linear programming model

Linear programming

Linear programs consist of a linear objective function and linear constraints, which can include both equalities and inequalities. The feasible region forms a polytope —a convex, connected set with flat, polygonal faces. The contours of the objective function are flat planes. [6]

A wide variety of applications can be modeled with linear programming.

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Variables, function and bounds

Steps:

1. identify the controllable decision variables x_i, with $i = 1, \ldots, n$.

2. establish the objective criterion: to either maximize or minimize some function of the form:
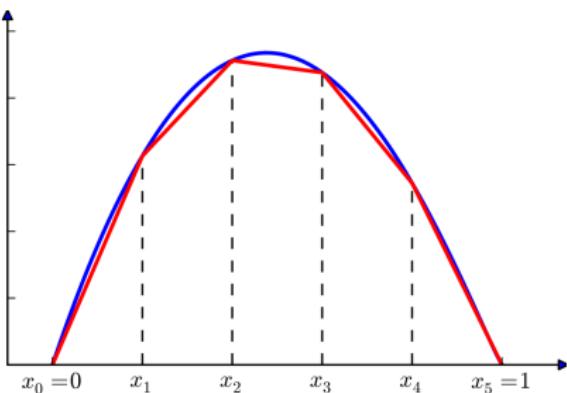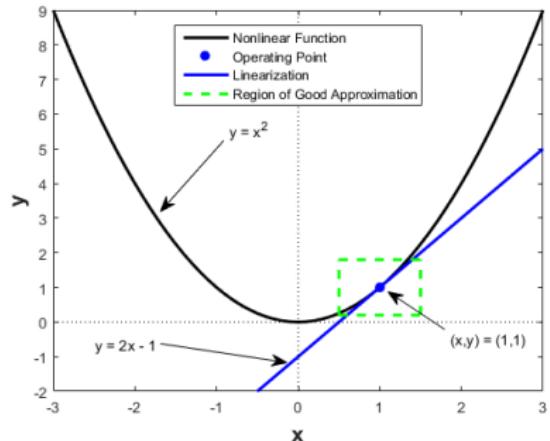
$$z = c_1 x_1 + \cdots + c_n x_n = \sum_{i=1}^{n} c_i x_i$$

where c_i represents problem dependent constants.

3. resource limitations and bounds on decision variables as linear equations or inequalities like

$$a_1 x_1 + \cdots + a_n x_n \leq b$$

Linearization of problems

Matlab example

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

Linearization of $f(x) = x^2$ around $x = 1$: $f(x) \approx 1 + 2(x - 1)$

Problem formulation I

Exercise 1 A manufacturer of computer system components assembles

two models of wireless routers, model A and model B. The amounts of materials and labor required for each assembly, and the total amounts available, are shown in the following table. The profits that can be realized from the sale of each router are $22 and $28 for models A and B, respectively, and we assume there is a market for as many routers as can be manufactured.[1]

	Resources per Unit A	Resources B per Unit B	Resources available
Materials	8	10	3400
Labor	2	3	960

How to maximize the benefits?

Problem formulation II

$$\text{maximize} \quad z = 22x_A + 28x_B$$

$$\text{subject to} \quad \begin{array}{rcl} 8x_A + 10x_B & \leq & 3400 \\ 2x_A + 3x_B & \leq & 960 \\ x_A & \geq & 0 \\ x_B & \geq & 0 \end{array}$$

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Problem formulation III

Exercise 2 A company wishes to minimize its combined costs of

production and inventory over a four-week time period. An item produced in a given week is available for consumption during that week, or it may be kept in inventory for use in later weeks. Initial inventory at the beginning of week 1 is 250 units. The minimum allowed inventory carried from one week to the next is 50 units. Unit production cost is $15, and the cost of storing a unit from one week to the next is $3. The following table shows production capacities and the demands that must be met during each week.[1]

Period	Production capacity	Demand
1	800	900
2	700	600
3	600	800
4	800	600

Problem formulation IV

A minimum production of 500 items per week must be maintained.
Inventory costs are not applied to items remaining at the end of the fourth
production period, nor is the minimum inventory restriction applied after
this final period.

Problem formulation V

Let x_i be the number of units produced during the ith week, for $i = 1, \ldots, 4$. The formulation is somewhat more manageable if we let A_i denote the number of items remaining at the end of each week (accounting for those held over from previous weeks, those produced during the current week, and those consumed during the current week). Note that the A_i values are not decision variables, but merely serve to simplify our written formulation. Thus,

$$
\begin{aligned}
A_1 &= 250 + x_1 - 900 \\
A_2 &= A_1 + x_2 - 600 \\
A_3 &= A_2 + x_3 - 800 \\
A_4 &= A_3 + x_4 - 600
\end{aligned}
$$

Problem formulation VI

$$\text{minimize} \quad z = 15 \cdot (x_1 + x_2 + x_3 + x_4) + 3 \cdot (A_1 + A_2 + A_3)$$
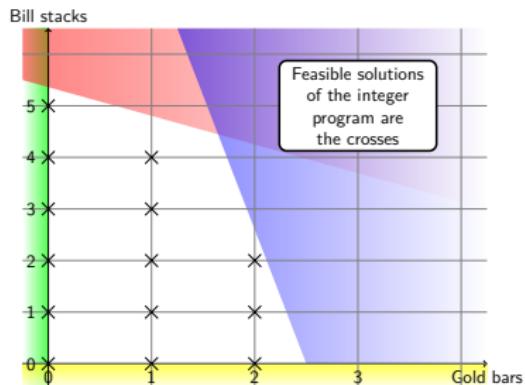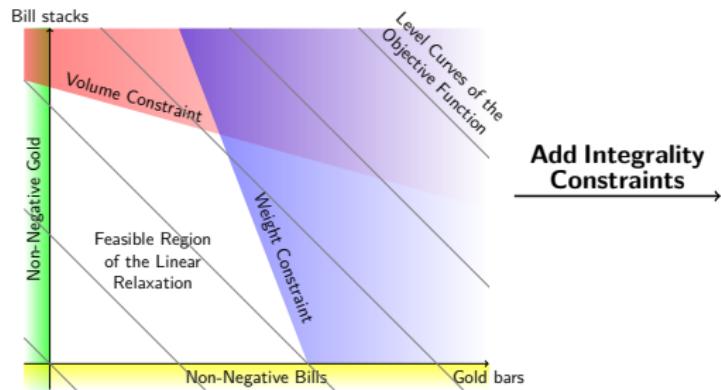
$$\begin{aligned}
&500 \leq x_1 \leq 700 \\
&500 \leq x_2 \leq 700 \\
\text{subject to} \quad &500 \leq x_3 \leq 600 \\
&500 \leq x_4 \leq 800 \\
&x_i \geq 0, \; i = 1, 2, 3, 4
\end{aligned}$$

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Integer problems

from Lê Nguyên Hoang's http://www.science4all.org

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General issues

- Remember: An optimal feasible solution is a point in the feasible space that is as effective as any other point in achieving the specified goal.
- The solution of linear programming problems with only two decision variables can be illustrated graphically.
- If an optimal feasible solution exists, it occurs at one of the extreme points of the feasible space.
- We illustrate this for problems with one, with multiple, and with none optimal feasible solutions.

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Unique optimal feasible solution I
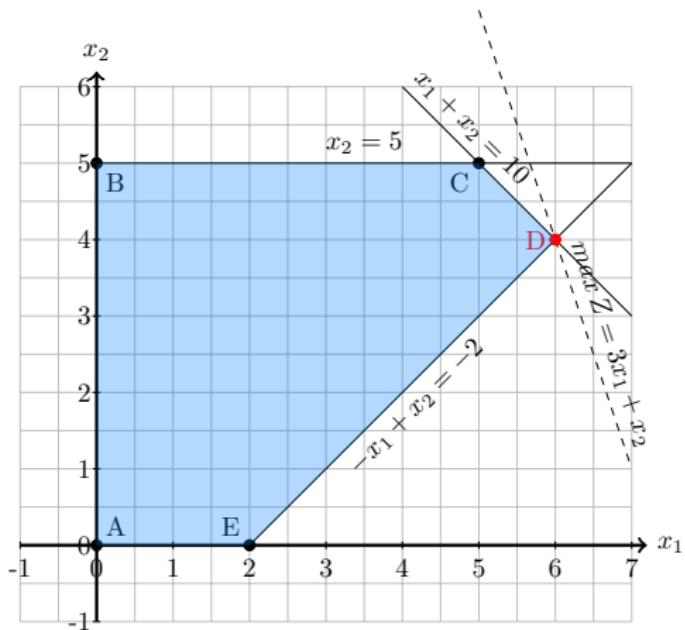
Exercise 3

$$\text{maximize} \quad z = 3x_1 + x_2$$

$$\text{subject to} \quad \begin{aligned} x_2 &\leq 5 \\ x_1 + x_2 &\leq 10 \\ -x_1 + x_2 &\geq -2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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Unique optimal feasible solution II

Unique optimal feasible solution III
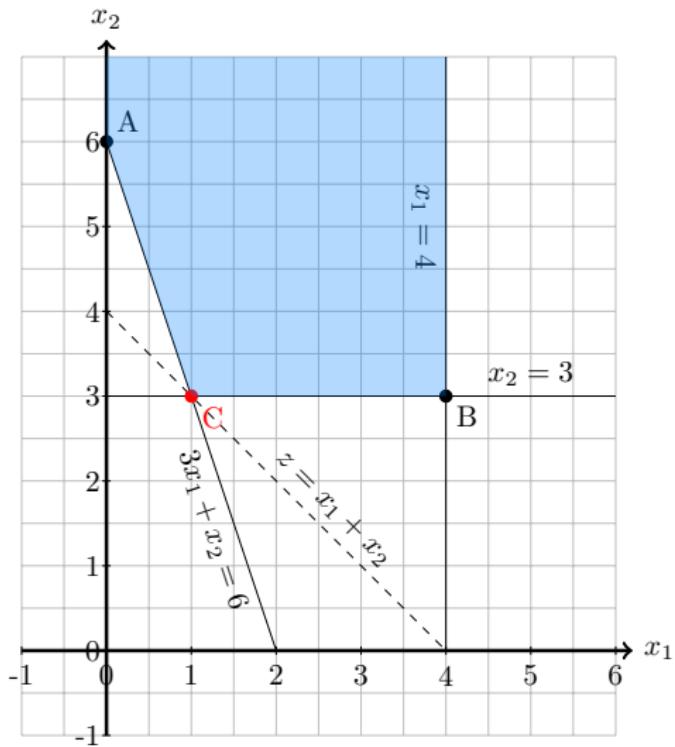
Exercise 4

$$\text{minimize} \quad z = x_1 + x_2$$

$$\text{subject to} \quad \begin{aligned} 3x_1 + x_2 &\geq 6 \\ x_2 &\geq 3 \\ x_1 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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Unique optimal feasible solution IV

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Multiple optimal feasible solutions I
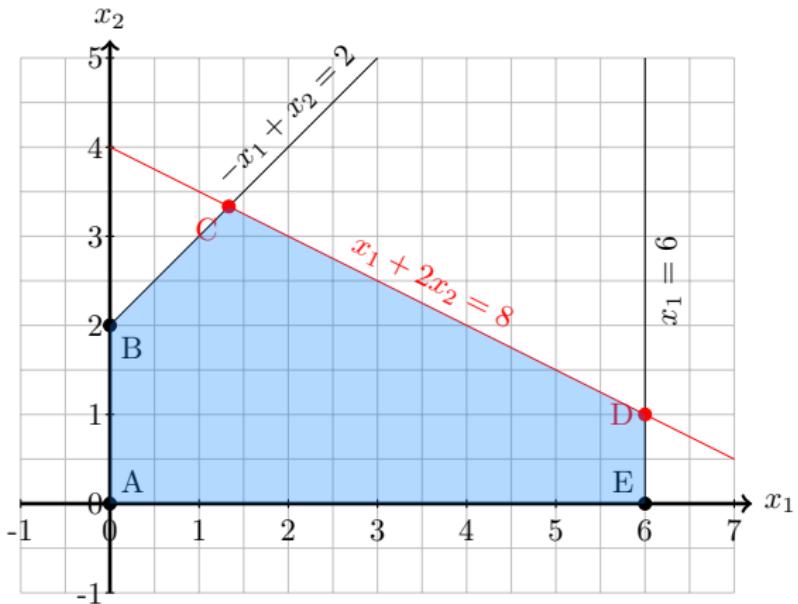
Exercise 5

$$\text{maximize} \quad z = x_1 + 2x_2$$

$$\text{subject to} \quad \begin{array}{rcl} -x_1 + x_2 & \leq & 2 \\ x_1 + 2x_2 & \leq & 8 \\ x_1 & \leq & 6 \\ x_1, x_2 & \geq & 0 \end{array}$$

Multiple optimal feasible solutions II

No optimal feasible solutions I
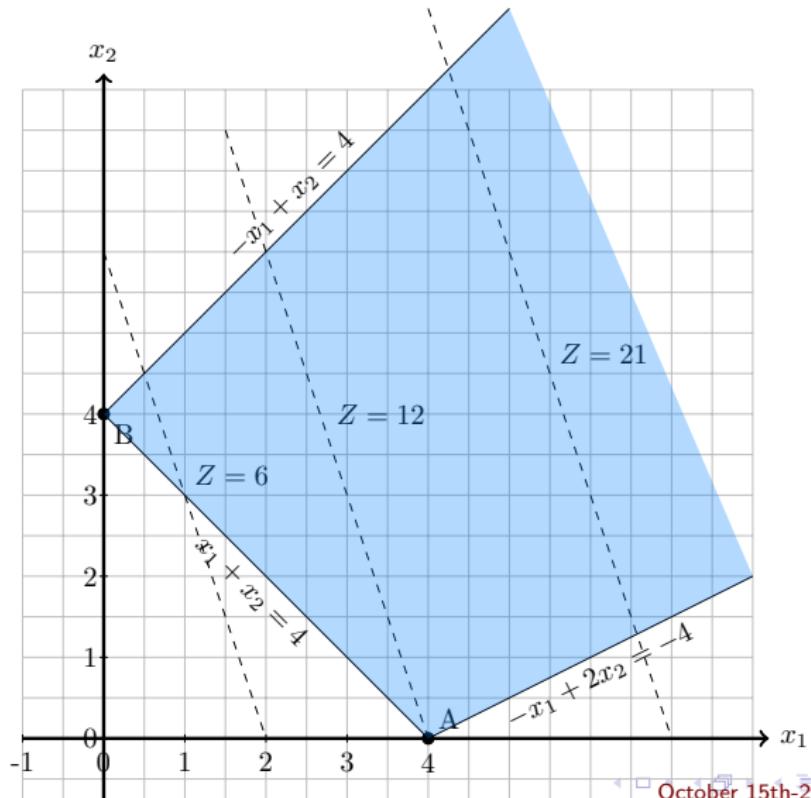
Exercise 6

$$\text{maximize} \quad z = 3x_1 + x_2$$

$$\text{subject to} \quad \begin{array}{rcl} x_1 + x_2 & \geq & 4 \\ -x_1 + x_2 & \leq & 4 \\ -x_1 + 2x_2 & \geq & -4 \\ x_1, x_2 & \geq & 0 \end{array}$$

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No optimal feasible solutions II

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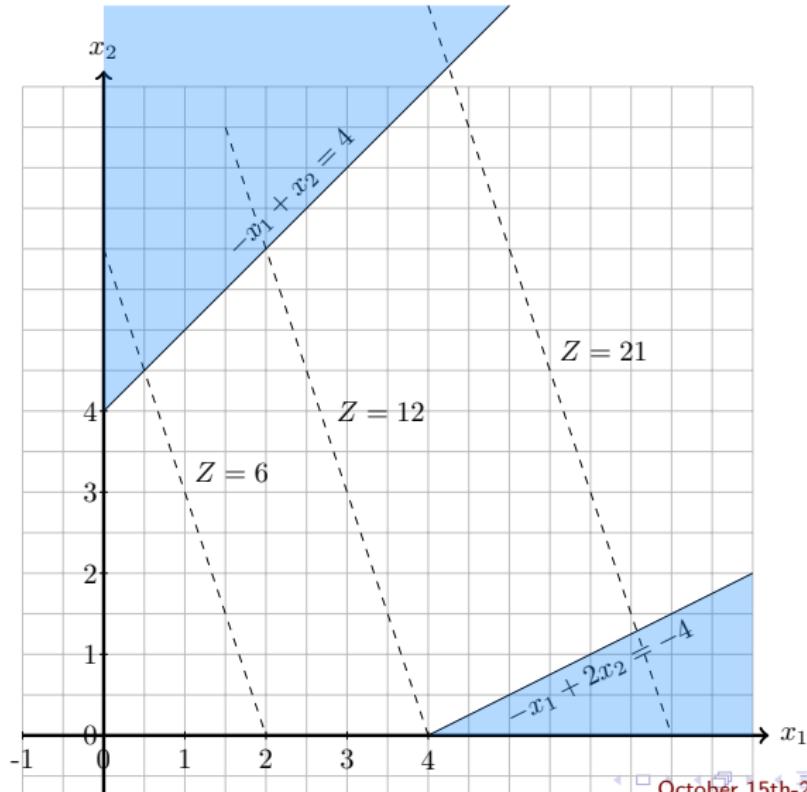
No feasible solutions I

Exercise 7

$$\text{maximize} \quad z = 3x_1 + x_2$$

$$\text{subject to} \quad \begin{array}{rcl} -x_1 + x_2 & \geq & 4 \\ -x_1 + 2x_2 & \leq & -4 \\ x_1, x_2 & \geq & 0 \end{array}$$

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No feasible solutions II

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General solution method of an LP problem

- If an optimal solution exists, it occurs at an extreme point of the feasible region.
- Only the finitely many extreme points need be examined (rather than all the points in the feasible region).
- Thus, an optimal solution may be found systematically by considering the objective function values at the extreme points.
- In fact, in actual practice, only a small subset of the extreme points need be examined.

This is the foundation for a **general solution method** of a LP problem called the **Simplex method**.

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Introduction to the Simplex method for LP: Summary
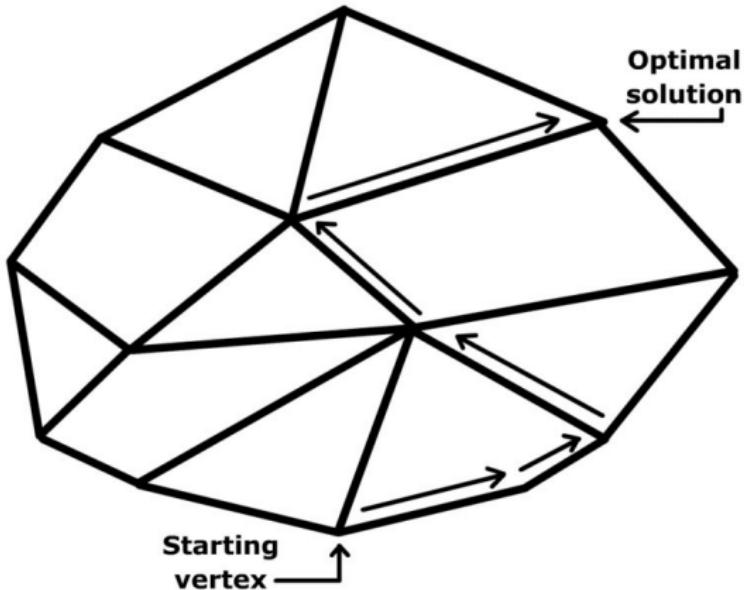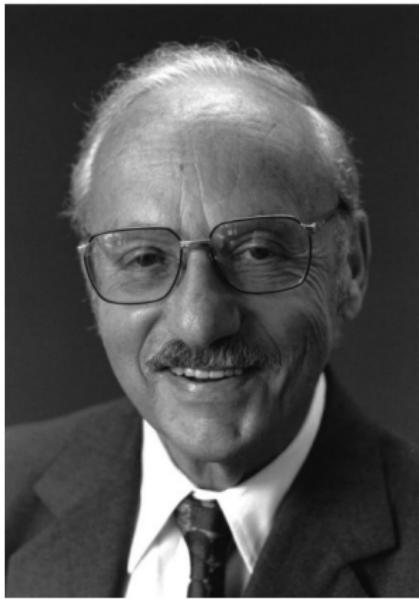
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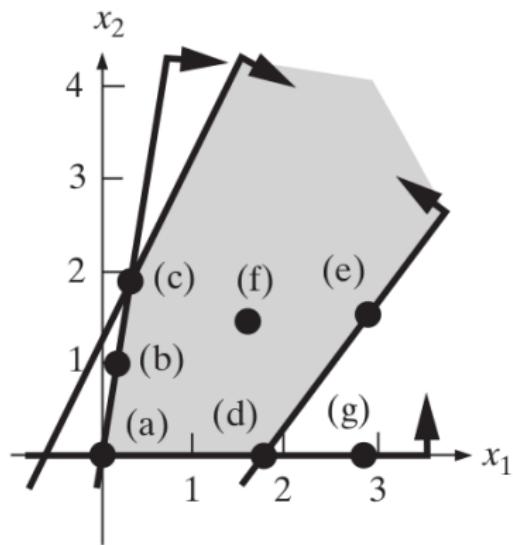
Learning outcomes

- Understanding the rational behind the Simplex method for LP.
- Understanding and practicing the algorithm in 2 variables.
- Recognizing the different types of results one can achieve in LP from the Simplex Method algorithm.
- Getting familiar with slack, surplus and artificial variables.

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Left: George Dantzig (1914-2005), the American mathematical scientist who devised linear programming and the simplex algorithm. Right: The simplex algorithm moves along the edges of the polytope until it reaches the optimum solution [Image Wikimedia Commons].

Classification of solutions in an LP problem

(a), (c), and (d) are both extreme points and boundary points.

(b) and (e) are boundary points that are not extreme.

(f) is interior because no constraint is active.

(g) is neither interior nor boundary (nor extreme) because it is infeasible.[3]

The standard form

Standard form

For a LP with n variables and m constraints, the standard form is given by:

$$\text{maximize} \quad z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$\text{subject to} \quad \begin{array}{rcl} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n & = & b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n & = & b_2 \\ \vdots & = & \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n & = & b_m \end{array}$$

where $x_1, \ldots, x_n \geq 0$ and $b_1, \ldots, b_m \geq 0$

Thus:

$$\text{maximize} \quad z = cx$$

$$\text{subject to} \quad \begin{array}{c} Ax = b \\ x \geq 0 \\ b \geq 0 \end{array}$$

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Of course, not always the system is proposed in this standard form, so[1]:

- We need to leave a maximization problem. If the problem is to minimize the objective function, we simply multiply it by (-1).

- In order to change inequality constraints into equality constraints we use *slack* variables for $\leq$ inequalities:

$$3x_1 + 4x_2 \leq 7 \rightarrow 3x_1 + 4x_2 + s_1 = 7$$

or *surplus* variables for $\geq$ inequalities:

$$x_1 + 3x_2 \geq 10 \rightarrow x_1 + 3x_2 - s_2 = 10$$

- All variables should be non-negative. If some variable x_1 needs to be considered as unrestricted in sign, we will replace it by $x_1' - x_1''$, with $x_1', x_1'' \geq 0$

- The system of linear equations, $Ax = b$, consists of m equations and n unknowns (the original decision variables plus the rest introduced to get a standard form). If the system has any solution, then $m \leq n$.
- If $m = n$ (and if $rank(A) = m$ and A is nonsingular), then $x = A^{-1}b$. No optimization needed!
- If $m < n$, there are infinitely many solutions and $n - m$ degrees of freedom.
- A **basic solution** is obtained by setting $n - m$ of the variables to zero (**non-basic variables**), and solving for the remaining (**basic**) m variables. The number of basic solutions is $\binom{n}{n-m} = \binom{n}{m} = \frac{n!}{m!(n-m)!}$
- Basic solutions that do not satisfy all problem constraints and non-negativity constraints are **infeasible solutions**.
- An **optimal basic feasible solution** is a basic feasible solution that optimizes the objective function.

We define two extreme points of the feasible region (or two basic feasible solutions) as being adjacent if all but one of their basic variables are the same. Thus, *a transition from one basic feasible solution to an adjacent basic feasible solution can be thought of as exchanging the roles of one basic variable and one non-basic variable*. The Simplex method performs a sequence of such transitions and thereby examines a succession of adjacent extreme points. A transition to an adjacent extreme point will be made only if by doing so the objective function is improved (or stays the same). It is a property of linear programming problems that this type of search will lead us to the discovery of an optimal solution (if one exists). The Simplex method is not only successful in this sense, but it is remarkably efficient because it succeeds after examining only a fraction of the basic feasible solutions.[1]

The algorithm consists in:

1. obtaining an initial feasible solution,
2. make transitions to a better basic feasible solution,
3. recognizing an optimal solution.

From any basic feasible solution, we have the assurance that, if a better solution exists at all, then there is an adjacent solution that is better than the current one.[1]

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The algorithm

1. Convert the system of inequalities to equations (using slack or surplus variables).

2. Set the objective function z to zero.

3. Create the Simplex tableau and label active and basic variables.

4. Select the pivot column (the one with the most negative coefficient in the zeroed objective function). Linked to the **entering variable**.

5. Select the pivot row (once divided the entry in the constant column by the ceffient in that row in the pivot column, we choose the smallest ratio). Linked to the **leaving variable**.

6. The pivot is the intersection between the pivot row and pivot column.

7. Use the pivot value to make zeros in the rest of elements in the column.

8. Repeat the process from step 4, until the z row is all non-negative.

Example. Standard form

$$\text{maximize} \quad z = 8x_1 + 5x_2$$

$$\text{subject to} \quad \begin{array}{rcl} x_1 & \leq & 150 \\ x_2 & \leq & 250 \\ 2x_1 + x_2 & \leq & 500 \\ x_1, x_2 & \geq & 0 \end{array}$$

We build first the standard form:

$$\begin{array}{rcl} -8x_1 - 5x_2 - 0s - 0t - 0u + z & = & 0 \\ x_1 + s & = & 150 \\ x_2 + t & = & 250 \\ 2x_1 + x_2 + u & = & 500 \end{array}$$

Example. The Simplex Tableau

We write the coefficients matrix and we identify the basic ($m = 3$) and
non-basic ($n - m = 5 - 3 = 2$ degrees of freedom) variables:

$$
\begin{array}{c c|c c c c c c|c}
 & & z & x_1 & x_2 & s & t & u & b \\
 & z & 1 & -8 & -5 & 0 & 0 & 0 & 0 \\ \hline
\rightarrow & s & 0 & 1 & 0 & 1 & 0 & 0 & 150 \\
 & t & 0 & 0 & 1 & 0 & 1 & 0 & 250 \\
 & u & 0 & 2 & 1 & 0 & 0 & 1 & 500 \\
\text{basic} & & & \uparrow & & & & &
\end{array}
\tag{1}
$$

Note that the basic variables are those for which each column is a
collection of 1 and zero. We will arbitrarily assign zeros to the non-basic
variables. So a possible solution is $\boxed{P_A = (x_1 = 0, x_2 = 0) \Rightarrow z = 0}$.
The arrow marks the pivot column. We will take the pivot row by
considering which is the lowest value among $150/1$ and $500/2$. So, the
pivot row corresponds to the s basic variable.

Example. Entering/leaving variables
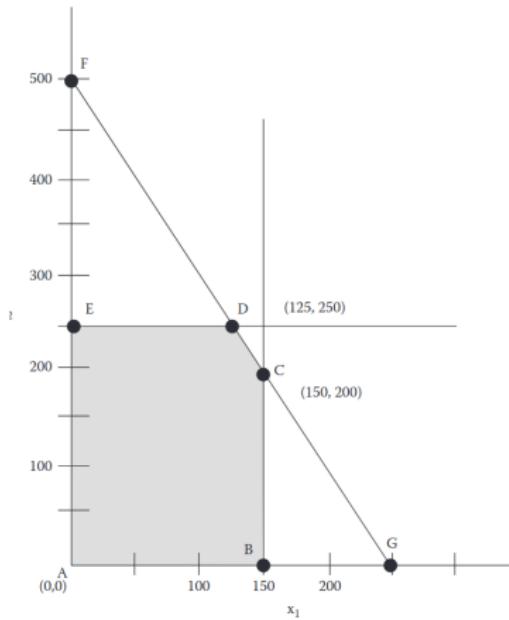
		z	x_1	x_2	s	t	u	b
$R_1 + 8R_2$	z	1	0	-5	8	0	0	1200
	x_1	0	1	0	1	0	0	150
	t	0	0	1	0	1	0	250
$\rightarrow R_4 - 2R_2$	u	0	0	1	-2	0	1	200
	basic			$\uparrow$				

$P_B = (150, 0)$ and $z(P_B) = 1200$ with $t = 250$, $u = 200$, $s = 0$.

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		z	x_1	x_2	s	t	u	b
$R_1 + 5R_4$	z	1	0	0	-2	0	5	2200
	x_1	0	1	0	1	0	-1	150
$\rightarrow R_3 - R_4$	t	0	0	0	2	0	1	50
	x_2	0	0	1	-2	0	1	200
	basic				$\uparrow$			

		z	x_1	x_2	s	t	u	b
$R_1 + 5R_4$	z	1	0	0	0	1	4	2250
	x_1	0	1	0	0	$-1/2$	$1/2$	125
$\rightarrow R_3 - R_4$	s	0	0	0	1	$1/2$	$-1/2$	25
	x_2	0	1	0	1	0	0	250
	basic				$\uparrow$			

$\boxed{P_D = (125, 250)}$ and $\boxed{z(P_D) = 2250}$ with $t, u = 0$ (binding constraints),
$s = 25$ (non-binding constraint).

Simplex steps for the given example (taken from [1]).

Exercises

Solve, using the Simplex method, the Exercises 3 through 7 of the previous session. Check also different methods for optimizing LP problems in python.

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Artificial variables

When adding slack variables, we end up finding an initial feasible set of basic variables. But what happens when not all constraints are of the form "$\leq$"?[1]

1. Make all right hand sides b_i of the constraint (in)equalities positive. If they are not, multiply the expression by (-1).

2. If we end up just with slack variables, they can be directly used as initial basic variables. If the expressions are of the type $\geq$ the selection of initial basic variables is not trivial.

3. We then create **artificial variables**, as a trick to create a starting basic solution

$$\text{maximize} \quad z = x_1 + 3x_2$$

$$\text{subject to} \quad \begin{aligned} 2x_1 - x_2 &\leq -1 \\ x_1 + x_2 &= 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

By leaving the right hand side of the constraints positive and adding a surplus variable, we end up having

$$\text{subject to} \quad \begin{array}{rcl} -2x_1 + x_2 - s_1 & = & 1 \\ x_1 + x_2 & = & 3 \end{array}$$

and now we introduce the artificial variables to create an initial solution:

$$\text{subject to} \quad \begin{array}{rcl} -2x_1 + x_2 - s_1 + R_1 & = & 1 \\ x_1 + x_2 + R_2 & = & 3 \end{array}$$

with all variables non-negative. Recall that, in fact, $R_1 = R_2 = 0$, so they will be just used temporarily.

There are two main methods to solve the problem: the two-phase method, explained next, and the big-M method, that we are not explaining here.

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The Two-Phase method I

1. First we will use the Simplex method to minimize the values of R_1 and R_2.

2. If they reach the value of zero, there is a solution for the original problem. If not, there is no feasible solution to the original problem.

In the above example, we want to minimize the function $z_R = R_1 + R_2$ or, what is the same, we want to maximize the function $z_R = -R_1 - R_2$

	x_1	x_2	s_1	R_1	R_2	Solution
z_R	0	0	0	1	1	0
R_1	-2	1	-1	1	0	1
R_2	1	1	0	0	1	3

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The Two-Phase method II

Next, we transform the problem in order to get basic variables (columns of one 1 and the rest zeros). For example, by using the second and third row to make appear 0 in the first row for the artificial variables:

	x_1	x_2	s_1	R_1	R_2	Solution
z_R	1	-2	1	0	0	-4
R_1	-2	1	-1	1	0	1
R_2	1	1	0	0	1	3

We can solve this problem with the Simplex method, obtaining a tableau:

	x_1	x_2	s_1	R_1	R_2	Solution
z_R	0	0	0	1	1	0
x_2	0	1	$-1/3$	$1/3$	$2/3$	$7/3$
x_1	1	0	$1/3$	$-1/3$	$1/3$	$2/3$

The Two-Phase method III

which provides the optimal solution for Phase 1. This means that the original problem has solution (as $R_1 = R_2 = 0$). In Phase 2, we will replace the first row by the original maximization problem and leave out the artificial variables, solving the now standard Simplex problem:

	x_1	x_2	s_1	Solution
z	-1	-3	0	0
x_2	0	1	$-1/3$	$7/3$
x_1	1	0	$1/3$	$2/3$

that leads to

	x_1	x_2	s_1	Solution
z	2	0	0	9
x_2	1	1	0	3
s_1	3	0	1	2

Draw the graphical solution of this problem and interpret the results.

Information in the tableau

Some special cases:

Multiple optimal solutions If a zero appears in the objective function row
corresponding to a non-basic variable, then that non-basic
variable can enter the basis without changing the value of
the objective function. So, two adjacent points have the
same objective function value.
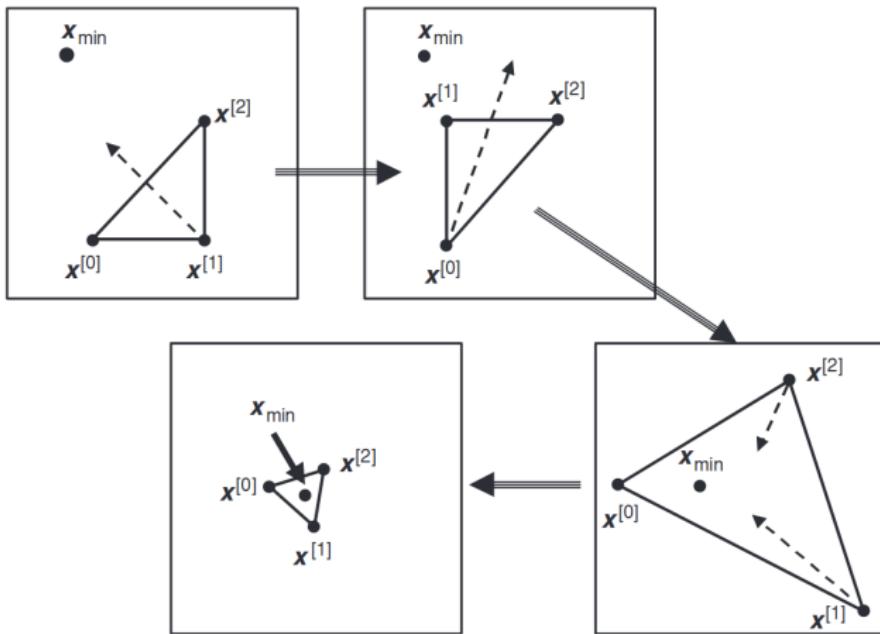
Unbounded solution If in any tableau the constraint coefficients
corresponding to a non-basic variable are all either negative
or zero, then that non-basic variable can be increased
arbitrarily without violating any constraint. Thus, the
feasible region is unbounded in the direction of that variable.

Degenerate solutions A solution to a linear programming problem is said
to be degenerate if one or more of the basic variables has a
value of zero. This shows redundancy in the constraints

Application of SIMPLEX to non-linear problems

Algorithm (see example in the next page):

1. In $\mathbf{R}^2$, we form a triangle with vertices at $x^{[0]}$ and the two points $x^{[1]} = x^{[0]} + l_1 e^{[1]}$ and $x^{[2]} = x^{[0]} + l_2 e^{[2]}$. In $\mathbf{R}^N$ the triangle becomes a simplex with $N + 1$ vertices.

2. We aim at moving $x^{[0]}, x^{[1]}, x^{[2]}, ...$ so that the triangle comes to enclose a local minimum $x_{\min}$ while shrinking in size. First, the vertex of highest cost function value is moved in the direction of the **triangle's center** towards a region where we expect the cost function values to be lower. After a sequence of such moves, the triangle eventually contains in its interior a local minimum.

3. The size of the triangle is progressively reduced until it tightly bounds the local minimum, as the vertex moves are most likely to succeed when the triangle is small enough that $F(x)$ varies linearly over the triangle region.

The simplex method moves the vertices of the simplex (in two dimensions, a triangle) until it contains the local minimum, and shrinks the simplex to find the local minimum (taken from [7]).

Duality: Summary

Learning outcomes

- Understanding Dual and Primal problems in LP
- Economic interpretation
- Conditions of optimality
- Resolution of the dual by the primal and penalty method

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A first example I

Linear programming problems come in pairs!

$$\text{maximize} \quad 4x_1 + x_2 + 5x_3 + 3x_4$$

$$\text{subject to} \quad \left\{ \begin{array}{rcl} x_1 - x_2 - x_3 + 3x_4 & \leq & 1 \\ 5x_1 + x_2 + 3x_3 + 8x_4 & \leq & 55 \\ -x_1 + x_2 + 3x_3 - 5x_4 & \leq & 3 \\ x_1, x_2, x_3 & \geq & 0 \end{array} \right.$$

note that

$$\begin{array}{rcl} y_1(x_1 - x_2 - x_3 + 3x_4) & + & \\ y_2(5x_1 + x_2 + 3x_3 + 8x_4) & + & \\ y_3(-x_1 + x_2 + 3x_3 - 5x_4) & \leq & y_1 + 55y_2 + 3y_3 \end{array}$$

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Primal and Dual

We see that maximizing the **Primal objective function**
$4x_1 + x_2 + 5x_3 + 3x_4$ is equivalent to minimize the **Dual objective function** $y_1 + 55y_2 + 3y_3$:

$$\text{minimize} \quad y_1 + 55y_2 + 3y_3$$

$$\text{subject to} \quad \left\{ \begin{array}{rcl} y_1 + 5y_2 - y_3 & \geq & 4 \\ -y_1 + y_2 + y_3 & \geq & 1 \\ -y_1 + 3y_2 + 3y_3 & \geq & 5 \\ 3y_1 + 8y_2 - 5y_3 & \geq & 3 \\ y_1, y_2, y_3, y_4 & \geq & 0 \end{array} \right.$$

Generalization

In general, if we can write the LP problem in its **normal formulation**:

$$\left.\begin{array}{ll} \min & \mathbf{c}^t\mathbf{x} \\ \text{subject to} & A\mathbf{x} \geq \mathbf{b}, \forall x_i \geq 0 \end{array}\right\} \text{PRIMAL}$$

$$\left.\begin{array}{ll} \max & \mathbf{b}^t\mathbf{y} \\ \text{subject to} & A^t\mathbf{y} \leq \mathbf{c}, \forall y_i \geq 0 \end{array}\right\} \text{DUAL}$$

The dual problem is a transposition of the primal problem. Since any LP can be written in the standard form above, any LP has a dual.

Theorem 1

In the case of linear programming, the dual of the dual is the primal.

Problem transformations

1. if you are asked to minimize $f(x)$, this is equivalent to maximize $-f(x)$;

2. add slack/surplus variables if you want to transform an inequality into an equality constraint;

3. $a \cdot x = b$ is equivalent to having, simultaneously, $a \cdot x \leq b$ and $a \cdot x \geq b$;

4. replace an unconstrained variable x_i by $u_i - v_i$ and impose that $u_i, v_i \geq 0$

Trick: Leave equality constraints to use the simplex algorithm. Use the inequality constraints for the duality theorem to be used.

Exercise 8 Find the dual problem of

$$\text{maximize} \quad 3x_1 + 2x_2$$

$$\text{subject to} \quad \left\{ \begin{array}{rcl} 2x_1 + x_2 & \leq & 4 \\ 2x_1 + 3x_2 & \leq & 6 \\ x_1, x_2 & \geq & 0 \end{array} \right.$$

Solve it graphically and using Simplex.

Exercise 9 Find the dual problem of

$$\text{minimize} \quad 4x_1 + 4x_2 + x_3$$

$$\text{subject to} \quad \begin{cases} x_1 + x_2 + x_3 & \leq & 2 \\ 2x_1 + x_2 & = & 3 \\ 2x_1 + x_2 + 3x_3 & \geq & 3 \\ x_1, x_2, x_3 & \geq & 0 \end{cases}$$

Trick: normalize the problem first: eg $2x_1 + x_2 = 3 \rightarrow \begin{cases} 2x_1 + x_2 \geq 3 \\ 2x_1 + x_2 \leq 3 \end{cases}$.

Meaning of Duality I

Let us retake the example we saw in the last session:

$$\text{maximize} \quad z = 8x_1 + 5x_2$$

$$\text{subject to} \quad \begin{array}{rcl} x_1 & \leq & 150 \\ x_2 & \leq & 250 \\ 2x_1 + x_2 & \leq & 500 \\ x_1, x_2 & \geq & 0 \end{array}$$

We found out that, from the initial tableau:

		z	x_1	x_2	s	t	u	b
	z	1	−8	−5	0	0	0	0
→	s	0	1	0	1	0	0	150
	t	0	0	1	0	1	0	250
	u	0	2	1	0	0	1	500
basic			↑					

Meaning of Duality II

and after applying the different steps of the Simplex method we ended up
with the final tableau:

		z	x_1	x_2	s	t	u	b
$R_1 + 5R_4$	z	1	0	0	0	1	4	2250
	x_1	0	1	0	0	$-1/2$	$1/2$	125
$\rightarrow R_3 - R_4$	s	0	0	0	1	$1/2$	$-1/2$	25
	x_2	0	1	1	0	0	0	250
	basic							

Now, if we multiply the original availability of each resource (shown in the
original tableau) by its marginal worth (taken from the final tableau) and
get the sum, we obtain the optimal objective function value:

$$z* = 2250 = 0(150) + 1(250) + 4(500)$$

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Duality properties

Theorem 2 (Weak Duality)

In a max LP, the value of primal objective function for any feasible solution is bounded from above by any feasible solution to its dual:

$$\bar{z} \leq \bar{w}$$

The statement is analogous to a minimization problem.

Theorem 3 (Unboundness property)

If primal (dual) problem has an unbounded solution, then the dual (primal) is unfeasible.

$$\bar{z} \leq \infty$$

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Duality properties I

Theorem 4 (Strong Duality)

If the primal problem has an optimal solution,

$$x^* = (x_1^*, \ldots, x_n^*)$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, \ldots, y_m^*)$$

and

$$z^* := \sum_j c_j x_j^* = \sum_i b_i y_i^* := w^*$$

Duality properties II

Thus: if feasible objective function values are found for a primal and dual pair of problems, and if these values are equal to each other, then both of the solutions are optimal solutions.

The Shadow prices that appear at the top of the optimal tableau of the primal problem are precisely the optimal values of the dual variables!

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A useful table

For any two given LP problems:

Primal / Dual	not feasible	unbounded	has a solution
not feasible	possible	possible	no
unbounded	possible	no	no
has a solution	no	no	same values

The simplex algorithm solves the primal and dual problems simultaneously.

Cases with no optimal solutions for primal and dual

Exactly one of the following mutually exclusive cases always occurs:

- Both primal and dual problems are feasible, and both have optimal (and equal) solutions.
- Both primal and dual problems are infeasible (have no feasible solution).
- The primal problem is feasible but unbounded, and the dual problem is infeasible.
- The dual problem is feasible but unbounded, and the primal problem is infeasible.

Complementary slackness I

Because each decision variable in a primal problem is associated with a constraint in the dual problem, each such variable is also associated with a slack or surplus variable in the dual.

Variables in one problem are complementary to constraints in the other.

In any solution, if the primal variable is basic (with value ≥ 0, hence having slack, hence being non-binding), then the associated dual variable is non-basic ($= 0$, hence having no slack, hence being binding), and viceversa.

Theorem 5 (Complementary slackness)

If in an optimal solution to a LP problem an inequality constraint is not binding, then the dual variable corresponding to that constraint has a value of zero in any optimal solution to the dual problem.

Complementary slackness II

Let us see how this can be better understood. Let us take a general primal/dual case

$$
\left.\begin{array}{ll}
\max & \mathbf{c}^t\mathbf{x} \\
\text{subject to} & A\mathbf{x} \leq \mathbf{b}, \forall x_i \geq 0
\end{array}\right\} \text{PRIMAL}
$$

$$
\left.\begin{array}{ll}
\min & \mathbf{b}^t\mathbf{y} \\
\text{subject to} & A^t\mathbf{y} \geq \mathbf{c}, \forall y_i \geq 0
\end{array}\right\} \text{DUAL}
$$

Complementary slackness III

Let vectors x_0, y_0 be feasible solutions for the (max) and the (min) problems, respectively. The weak duality theorem 2 states that (we remove the transpose where it is obvious, for simplicity)

$$\max \quad c \cdot x_0 \leq \min \quad b \cdot y_0$$

Then we can say that (we will not demonstrate it here)

$$b \cdot y_0 - c \cdot x_0 = \underbrace{(b - Ax_0)}_{u} \cdot y_0 + \underbrace{(A^t y_0 - c)}_{v} \cdot x_0$$

where u is the vector of slack variables in the primal problem and v is the vector of the slack variables in the dual.

Complementary slackness IV

According to Theorem 4, when $x_0, y_0 = x^*, y^*$ are optimal solutions,

$$b \cdot y^* - c \cdot x^* = 0$$

This means that in such optimal solutions

$$\underbrace{(b - Ax^*)}_{u \geq 0} \cdot y^* + \underbrace{(A^t y^* - c)}_{v \geq 0} \cdot x^* = 0$$

The fact that all terms are nonnegative implies that

$$(b - Ax^*) \cdot y^* = (A^t y^* - c) \cdot x^* = 0 \qquad (2)$$

Complementary slackness V

Now let us haver a look at one of these terms:

$$(b - Ax^*) \cdot y^* \equiv u \cdot y^* = 0$$

As the components of the two vectors are nonnegative aswell:
$u_1, \ldots, u_m \geq 0$ and $y_1^*, \ldots, y_m^* \geq 0$,

$$\underbrace{u_1 y_1^*}_{\geq 0} + \cdots \underbrace{u_m y_m^*}_{\geq 0} = 0$$

So, each term should be zero!. Now, for each $i = 1, \ldots, m$ we must have at least one of $u_i, y_i^* = 0$. So, if $u_i \neq 0 \Rightarrow y_i^* = 0$ and the other way around. The same occurs to the other term in Eq.5.

Complementary slackness VI

Thus, Theorem 5 can be reformulated as:

Theorem 6 (Complementary slackness (rewritten))

Suppose we have a primal and dual problems with variables x_0, y_0 for the (max) and (min), respectively. Then, $x_0 \equiv x^*$ and $y_0 \equiv y^*$ if and only if

$$(b - Ax^*) \cdot y^* = 0 \tag{3}$$

and

$$(A^t y^* - c) \cdot x^* = 0 \tag{4}$$

Complementary slackness VII

In other words:

- if a variable is positive, then the associated dual constraint must be binding; or
- if a constraint fails to bind, than the associated variable must be zero; but recall that
- it is possible for a primal constraint to be binding while the associated dual variable is equal to zero (no slack in both primal and dual).

There cannot be slack in both a constraint and the associated dual variable.

Example of the use of CST I

Let us show an example. Imagine that we have the primal problem

$$\min \quad 12x_1 + 5x_2 + 10x_3$$

$$\text{subject to} \quad \begin{aligned} x_1 - x_2 + 2x_3 &\geq 10 \\ -3x_1 + x_2 + 4x_3 &\geq -9 \\ -x_1 + 2x_2 + 3x_3 &\geq 1 \\ 2x_1 - 3x_2 &\geq -2 \\ 7x_1 - x_2 - 5x_3 &\geq 34 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

and we are said that the optimal solution is $x^* = (7, 0, 3)$. Find the optimal solution to the dual y^*.

Example of the use of CST II

We build first the dual problem:

$$\max \quad 10y_1 + 9y_2 + y_3 - 2y_4 + 34y_5$$

$$\text{subject to} \quad \begin{array}{rcl} y_1 - 3y_2 - y_3 + 2y_4 + 7y_5 & \leq & 12 \\ -y_1 + y_2 + 2y_3 - 3y_4 - y_5 & \leq & 5 \\ 2y_1 + 4y_2 + 3y_3 - 5y_5 & \leq & 10 \\ y_1, y_2, y_3, y_4, y_5 & \geq & 0 \end{array} \tag{5}$$

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Example of the use of CST III

1. Find the slack of (min) by replacing the values in the primal problem constraints:

$$\begin{aligned}
(7 - 0 + 2 \cdot 3) - 10 &= 3 \\
(-3 \cdot 7 + 0 + 4 \cdot 3) + 9 &= 0 \\
(-7 + 2 \cdot 0 + 3 \cdot 3) - 1 &= 1 \\
(2 \cdot 7 - 3 \cdot 0) + 2 &= 16 \\
(7 \cdot 7 - 0 - 5 \cdot 3) - 34 &= 0
\end{aligned}$$

2. From Eq.3, $\begin{pmatrix} 3 \\ 0 \\ 1 \\ 16 \\ 0 \end{pmatrix} \cdot y^* = 0$, which gives $y_1^* = y_3^* = y_4^* = 0$ and

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Example of the use of CST IV

3. from Eq.4, since $x^* = (7, 0, 3)$ I know that the slack of the dual should be zero for its first and third constraints. So, the constraints in the dual (EQ.5) end up being:

$$
\begin{aligned}
-3y_2^* + 7y_5^* &= 12 \\
4y_2^* - 5y_5^* &= 10
\end{aligned}
$$

  leading to $y^* = (0, 10, 0, 0, 6)$.

Follow up: Verify that $c \cdot x^* = b \cdot y^*$.

Example: The diet problem I

Exercise 10 You are feeding animals in a farm. Your animals have some

basic requirements for nutrients, and you want to minimize the cost fulfilling those requirements. You need to know:

1. What foods do I have avaliable and which is each ones cost?

2. What nutrients are needed and which is the nutrients content per food?

Formulate the problem in an abstract way.

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Example: The diet problem II

Exercise 11 Come back to the previous problem, but now, there is a

seller that tells you she has the pills you need to ensure the nutrients, forgetting about specific food. Reformulate the problem in terms of the maximum revenue the seller company wants. What is the price per pill to maximize her benefits, subject to the constraint that the pills are cheaper than the actual food.

Exercise 12 Rationalize the complementarity slackness theorem within

the context of the previous exercise. What does it mean to buy a positive quantity of the first food (primal LP problem) in terms of the constraint related to the first pill (dual problem)?

Using duality and CS to solve LP problems

Exercise 13 Just by checking the feasibility of the primal and dual

problem for

$$\text{maximize} \quad z = x_1 - x_2$$

$$\text{subject to} \quad \begin{array}{rcl} -2x_1 + x_2 & \leq & 2 \\ x_1 - 2x_2 & \leq & 2 \\ x_1 + x_2 & \leq & 5 \\ x_1, x_2 & \geq & 0 \end{array}$$

find what is the solution.

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