Unit 2. Linear programming. Introduction

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Preliminary

This course is strongly based on the monography on Operations Research by Carter, Price and Rabadi [1], and in material obtained from different sources (quoted when needed through the slides).

Learning outcomes

- Learn how to formulate a linear programming model
- Get familiar with the terms feasible region, feasiblke solutions and optimal feasible solutions
- Managing graphical solution of the linear programming model
- Identify models with unique, multiple or no optimal feasible solutions
- Introducing the Simplex method

Summary

The linear programming model

Linear programming

Linear programs have a linear objective function and linear constraints, which may include both equalities and inequalities. The feasible set is a polytope, that is, a convex, connected set with flat, polygonal faces. The contours of the objective function are planar. [6]

A wide variety of applications can be modeled with linear programming.

Variables, function and bounds

Steps:

1. identify the controllable decision variables x_i, with $i = 1, \dots, n$.
2. establish the objective criterion: to either maximize or minimize some function of the form:

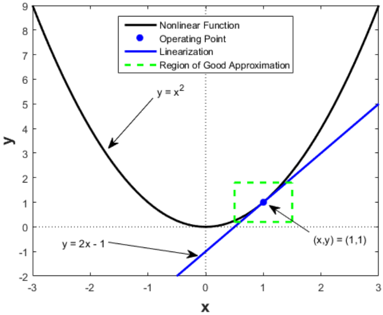
$$z = c_1 x_1 + \cdots + c_n x_n = \sum_{i=1}^{n} c_i x_i$$

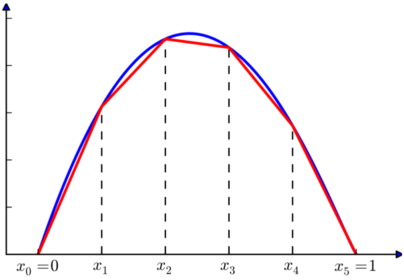
   where c_i represents problem dependent constants.

3. resource limitations and bounds on decision variables as linear equations or inequalities like

$$a_1 x_1 + \cdots + a_n x_n \leq b$$

Linearization of problems





Matlab example

Problem formulation I

Exercise 1

A manufacturer of computer system components assembles two models of wireless routers, model A and model B. The amounts of materials and labor required for each assembly, and the total amounts available, are shown in the following table. The profits that can be realized from the sale of each router are \$22 and \$28 for models A and B, respectively, and we assume there is a market for as many routers as can be manufactured.[1]

	Resources per Unit A	Resources B per Unit B	Resources available
Materials	8	10	3400
Labor	2	3	960

How to maximize the benefits?

Problem formulation II

$$
\begin{array}{ll}
\text{maximize} & z = 22x_A + 28x_B \\
\text{subject to} & \begin{array}{rcl}
8x_A + 10x_B & \leq & 3400 \\
2x_A + 3x_B & \leq & 960 \\
x_A & \geq & 0 \\
x_B & \geq & 0
\end{array}
\end{array}
$$

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Problem formulation III

Exercise 2

A company wishes to minimize its combined costs of production and inventory over a four-week time period. An item produced in a given week is available for consumption during that week, or it may be kept in inventory for use in later weeks. Initial inventory at the beginning of week 1 is 250 units. The minimum allowed inventory carried from one week to the next is 50 units. Unit production cost is \$15, and the cost of storing a unit from one week to the next is \$3. The following table shows production capacities and the demands that must be met during each week.[1]

Period	Production capacity	Demand
1	800	900
2	700	600
3	600	800
4	800	600

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Problem formulation IV

A minimum production of 500 items per week must be maintained. Inventory costs are not applied to items remaining at the end of the fourth production period, nor is the minimum inventory restriction applied after this final period.

Problem formulation V

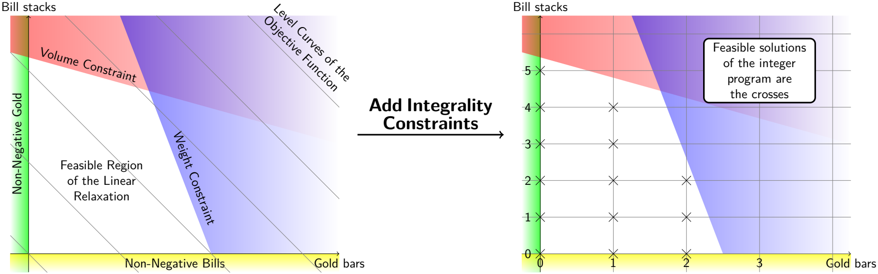
Let x_i be the number of units produced during the i^{th} week, for $i = 1, , 4$. The formulation is somewhat more manageable if we let A_i denote the number of items remaining at the end of each week (accounting for those held over from previous weeks, those produced during the current week, and those consumed during the current week). Note that the A_i values are not decision variables, but merely serve to simplify our written formulation. Thus,

$$\begin{aligned} A_1 &= 250 + x_1 - 900 \\ A_2 &= A_1 + x_2 - 600 \\ A_3 &= A_2 + x_3 - 800 \\ A_4 &= A_3 + x_4 - 600 \end{aligned}$$

Problem formulation VI

$$
\begin{array}{ll}
\text{minimize} & z = 15 \cdot (x_1 + x_2 + x_3 + x_4) + 3 \cdot (A_1 + A_2 + A_3) \\
 & 500 \leq x_1 \leq 700 \\
 & 500 \leq x_2 \leq 700 \\
\text{subject to} & 500 \leq x_3 \leq 600 \\
 & 500 \leq x_4 \leq 800 \\
 & x_i \geq 0,\ i = 1, 2, 3, 4
\end{array}
$$

Integer problems



from Lê Nguyên Hoang's http://www.science4all.org

Summary

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General issues

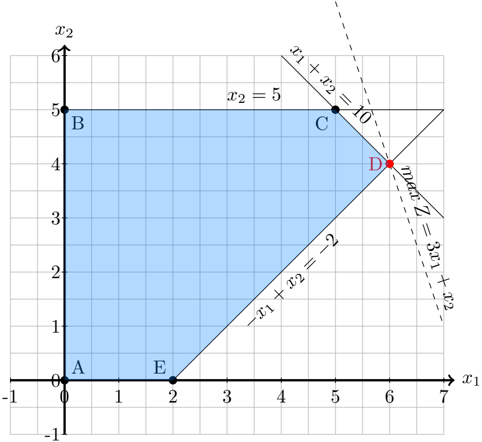
- Remember: An optimal feasible solution is a point in the feasible space that is as effective as any other point in achieving the specified goal.
- The solution of linear programming problems with only two decision variables can be illustrated graphically.
- If an optimal feasible solution exists, it occurs at one of the extreme points of the feasible space.
- We illustrate this for problems with one, with multiple, and with none optimal feasible solutions.

Unique optimal feasible solution I

Exercise 3

$$
\begin{array}{ll}
\text{maximize} & z = 3x_1 + x_2 \\
\text{subject to} & \begin{array}{rcl} x_2 & \leq & 5 \\ x_1 + x_2 & \leq & 10 \\ -x_1 + x_2 & \geq & -2 \\ x_1, x_2 & \geq & 0 \end{array}
\end{array}
$$

Unique optimal feasible solution II

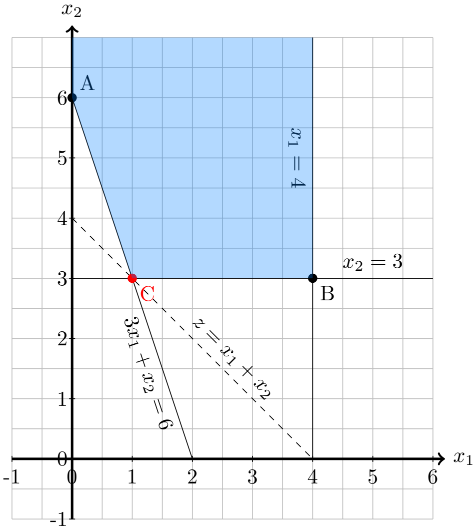


Unique optimal feasible solution III

Exercise 4

$$
\begin{array}{ll}
\text{minimize} & z = x_1 + x_2 \\
\text{subject to} & \begin{array}{rcl} 3x_1 + x_2 & \geq & 6 \\ x_2 & \geq & 3 \\ x_1 & \leq & 4 \\ x_1, x_2 & \geq & 0 \end{array}
\end{array}
$$

Unique optimal feasible solution IV

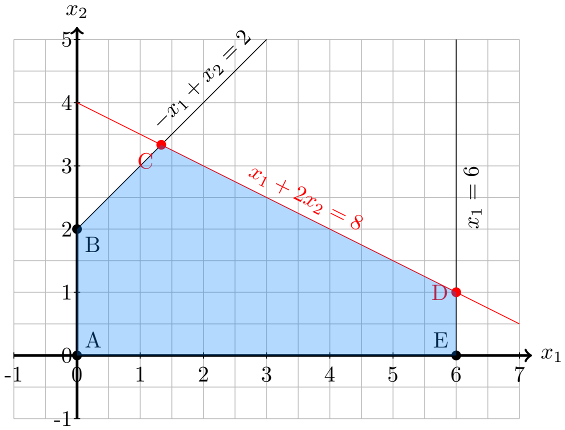


Multiple optimal feasible solutions I

Exercise 5

$$
\begin{array}{ll}
\text{maximize} & z = x_1 + 2x_2 \\
\text{subject to} & \begin{aligned} -x_1 + x_2 &\leq 2 \\ x_1 + 2x_2 &\leq 8 \\ x_1 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}
\end{array}
$$

Multiple optimal feasible solutions II

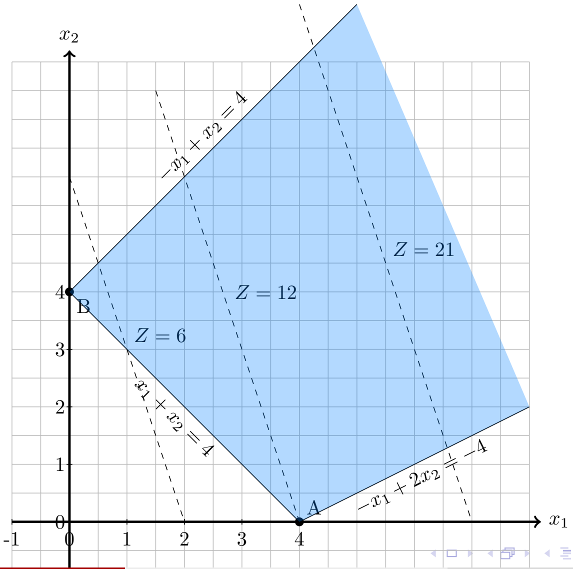


No optimal feasible solutions I

Exercise 6

$$
\begin{array}{ll}
\text{maximize} & z = 3x_1 + x_2 \\
\text{subject to} & \begin{array}{rcl} x_1 + x_2 & \geq & 4 \\ -x_1 + x_2 & \leq & 4 \\ -x_1 + 2x_2 & \geq & -4 \\ x_1, x_2 & \geq & 0 \end{array}
\end{array}
$$

No optimal feasible solutions II

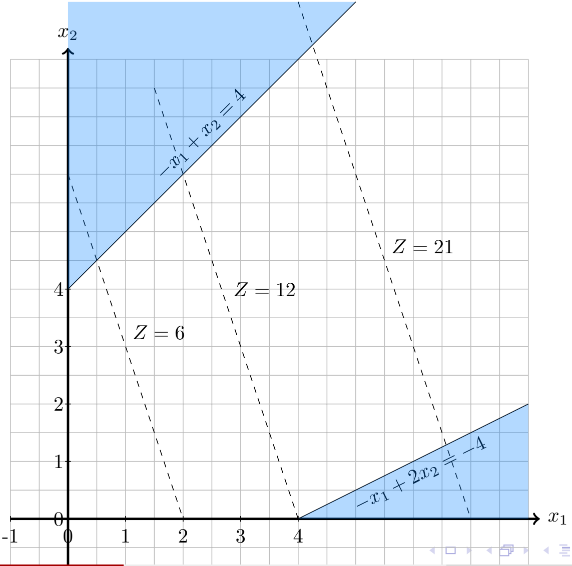


No feasible solutions I

Exercise 7

$$
\begin{array}{lrcl}
\text{maximize} & z = 3x_1 + x_2 & & \\
 & -x_1 + x_2 & \geq & 4 \\
\text{subject to} & -x_1 + 2x_2 & \leq & -4 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$

No feasible solutions II



E4. LP program

Programming exercise

E4. Build a Python code using Google OR tools' MPSolver interface to solve an arbitrary LP problem. Test it with the different exercises in this presentation.

Summary

- If an optimal solution exists, it occurs at an extreme point of the feasible region.
- Only the finitely many extreme points need be examined (rather than all the points in the feasible region).
- Thus, an optimal solution may be found systematically by considering the objective function values at the extreme points.
- In fact, in actual practice, only a small subset of the extreme points need be examined.

This is the foundation for a **general solution method** of a LP problem called the **Simplex method**.

Summary

References

[1] Michael W. Carter, Camille C. Price, and Ghaith Rabadi. Operations Research, 2nd Edition. CRC Press.

[2] David Harel, with Yishai Feldman. Algorithmics: the spirit of computing, 3rd Edition. Addison-Wesley.

[3] Ronald L. Rardin. Optimization in Operations Research, 2nd Edition. Pearson.

[4] J. Hefferon. Linear algebra (4th Ed).

[5] K.F. Riley, M.P. Hobson, S.J. Bence. Mathematical Methods for Physics and Engineering (2nd Ed). McGraw Hill.

[6] J. Nocedal, S. J. Wright. Numerical Optimization (2nd Ed). Springer.

[7] Kenneth J. Beers. Numerical methods for chemical engineering: applications in Matlab. Cambridge University Press.

[8] D. Barber. Bayesian reasoning and machine learning. Cambridge University Press.