

Solved Exercises

Optimization and Operations Research



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1 Non-linear optimization

1.1 Lagrange multipliers

Exercise 1 — Optimization of $f(x, y) = x^2 - y$ subject to $x^2 + y^2 = 4$

Solution (Exercise 1) — We are tasked with optimizing the function

$$f(x, y) = x^2 - y$$

subject to the constraint

$$x^2 + y^2 = 4.$$

We will use the method of Lagrange multipliers.

Step 1: Define the Lagrange Function Define the constraint as $g(x, y) = x^2 + y^2 - 4 = 0$. The Lagrange function is then:

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda \cdot g(x, y) = x^2 - y + \lambda(x^2 + y^2 - 4).$$

Step 2: Compute the Partial Derivatives We now compute the partial derivatives of $\mathcal{L}(x, y, \lambda)$ with respect to x , y , and λ :

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= 2x + \lambda \cdot 2x = 2x(1 + \lambda) = 0, \\ \frac{\partial \mathcal{L}}{\partial y} &= -1 + \lambda \cdot 2y = 0 \quad \Rightarrow \quad \lambda = \frac{1}{2y}, \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= x^2 + y^2 - 4 = 0. \end{aligned}$$

Step 3: Solve the Equations

From $\frac{\partial \mathcal{L}}{\partial x} = 0$:

$$2x(1 + \lambda) = 0.$$

This gives two possibilities:

- $x = 0$, or
- $1 + \lambda = 0 \quad \Rightarrow \quad \lambda = -1.$

Case 1: $x = 0$ Substitute $x = 0$ into the constraint equation $x^2 + y^2 = 4$:

$$0^2 + y^2 = 4 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2.$$

For $y = 2$, substitute into $f(x, y) = x^2 - y$:

$$f(0, 2) = 0^2 - 2 = -2.$$

For $y = -2$, substitute into $f(x, y)$:

$$f(0, -2) = 0^2 - (-2) = 2.$$

Case 2: $\lambda = -1$ Substitute $\lambda = -1$ into $\lambda = \frac{1}{2y}$:

$$-1 = \frac{1}{2y} \Rightarrow y = -\frac{1}{2}.$$

Now, substitute $y = -\frac{1}{2}$ into the constraint equation $x^2 + y^2 = 4$:

$$x^2 + \left(-\frac{1}{2}\right)^2 = 4 \Rightarrow x^2 + \frac{1}{4} = 4 \Rightarrow x^2 = \frac{15}{4} \Rightarrow x = \pm \frac{\sqrt{15}}{2}.$$

Now, calculate $f(x, y) = x^2 - y$ for $y = -\frac{1}{2}$ and $x = \pm \frac{\sqrt{15}}{2}$:

For $x = \frac{\sqrt{15}}{2}$:

$$f\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) = \left(\frac{\sqrt{15}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{15}{4} + \frac{1}{2} = \frac{17}{4}.$$

For $x = -\frac{\sqrt{15}}{2}$:

$$f\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) = \left(-\frac{\sqrt{15}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{15}{4} + \frac{1}{2} = \frac{17}{4}.$$

Step 4: Compare the Results We now compare the function values:

- For $(0, 2)$: $f(0, 2) = -2$.
- For $(0, -2)$: $f(0, -2) = 2$.
- For $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$: $f = \frac{17}{4} \approx 4.25$.

Conclusion

- The maximum value is $f = \frac{17}{4} \approx 4.25$ at $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$.
- The minimum value is $f = -2$ at $(0, 2)$.

1.2 Karush-Kuhn-Tucker theorem

Exercise 2 — Maximize $f(x, y) = xy$ subject to $100 \geq x + y$ and $x \leq 40$

Solution (Exercise 2) — The Karush Kuhn Tucker (KKT) conditions for optimality are a set of necessary conditions for a solution to be optimal in a mathematical optimization problem. They are necessary and sufficient conditions for a local minimum in nonlinear programming problems. The KKT conditions consist of the following elements:

For an optimization problem in its standard form:

$$\begin{aligned} \max f(x) \\ \text{s.t.} \quad & g_i(x) - b_i \leq 0 \quad i = 1, \dots, k \\ & g_i(x) - b_i = 0 \quad i = k + 1, \dots, m \end{aligned}$$

There are 4 KKT conditions for optimal primal (x^*) and dual (λ) variables. If x^* denotes optimal values:

- (A) Primal feasibility: all constraints must be satisfied: $g_i(x^*) - b_i$ is feasible. Applies to both equality and non-equality constraints.
- (B) Gradient condition or No feasible descent: No possible improvement at the solution:

$$\nabla f(x^*) - \sum_{i=1}^m \lambda_i^* \nabla g_i(x^*) = 0$$

- (C) Complementarity slackness:

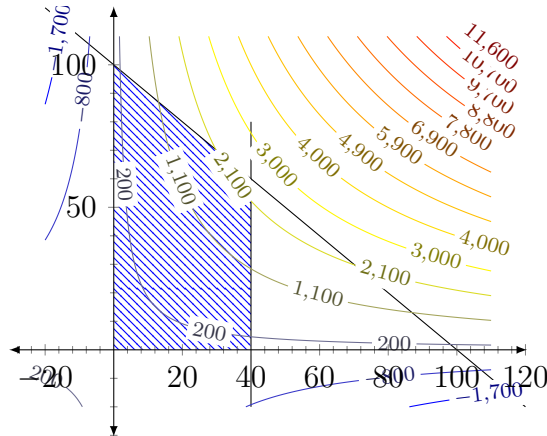
$$\lambda_i^* (g_i(x^*) - b_i) = 0$$

- (D) Dual feasibility: $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active ($=0$) and a zero Lagrange multiplier when the constraint is inactive (>0).

To solve our problem, first we will put it in its standard form:

$$\begin{aligned}
 \max f(x, y) &= xy \\
 \text{s.t.} \quad &g_1(x, y) = x + y - 100 \leq 0 \\
 &g_2(x, y) = x - 40 \leq 0 \\
 &g_3(x, y) = -x \leq 0 \\
 &g_4(x, y) = -y \leq 0
 \end{aligned}$$



We will go through the different conditions:

- on the gradient:

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} - \lambda_1 \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_1}{\partial y} \end{pmatrix} - \lambda_2 \begin{pmatrix} \frac{\partial g_2}{\partial x} \\ \frac{\partial g_2}{\partial y} \end{pmatrix} - \lambda_3 \begin{pmatrix} \frac{\partial g_3}{\partial x} \\ \frac{\partial g_3}{\partial y} \end{pmatrix} - \lambda_4 \begin{pmatrix} \frac{\partial g_4}{\partial x} \\ \frac{\partial g_4}{\partial y} \end{pmatrix} = 0$$

which, in this example, resolves into:

$$y - (\lambda_1 + \lambda_2 - \lambda_3) = 0 \quad (1)$$

$$x - (\lambda_1 - \lambda_4) = 0 \quad (2)$$

- on the complementary slackness:

$$\lambda_1(x + y - 100) = 0 \quad (3)$$

$$\lambda_2(x - 40) = 0 \quad (4)$$

$$\lambda_3 x = 0 \quad (5)$$

$$\lambda_4 y = 0 \quad (6)$$

- on the constraints:

$$x + y \leq 100 \quad (7)$$

$$x \leq 40 \quad (8)$$

$$-x \leq 0 \quad (9)$$

$$-y \leq 0 \quad (10)$$

plus $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$.

We will start by checking Eq. 3:

–Let us see what occurs if $\lambda_1 = 0$. Then, from Eq. 2, $x + \lambda_4 = 0$ which implies that $x = \lambda_4 = 0$ ¹. But, then, from Eq. 4 we obtain that $\lambda_2 = 0$ which, using Eq. 1 gives $y + \lambda_3 = 0 \Rightarrow y = \lambda_3 = 0$. Indeed, the KKT conditions are satisfied when all variables and multipliers are zero, but it is not a maximum of the function (see figure above).

–So, let us see what happens if $x + y - 100 = 0$ and consider the two possibilities for x :

Case $x = 0$: Then, $y = 100$, which would lead (Eq. 6) to $\lambda_4 = 0$ and (Eq. 2) to $x = \lambda_1 = 0$, that was discussed in the previous item. So, we need to explore the other possibility for x .

Case $x > 0$: From Eq. 5 $\lambda_3 = 0$ and, from Eqs. 1 and 2:

$$\begin{cases} y = \lambda_1 + \lambda_2 \\ x = \lambda_1 + \lambda_4 \end{cases}$$

let us try what happens if, e.g., $\lambda_2 \neq 0$ (or, said in other words, if constraint 8 is active): $x = 40$. As we know we do not want $\lambda_1 = 0$, from Eq. 3 we obtain $x + y - 100 = 0 \Rightarrow y = 60$.

The point $(x, y) = (40, 60)$ fulfills the KKT conditions and is a maximum in the constrained maximization problem.

¹Recall that both variables and multipliers must be positive or zero, so, the only possibility for the equation to fulfill is that both are zero.

2 Duality

Exercise 3 — Consider the linear programming problem:

$$\begin{aligned} \max \quad & x_1 + 4x_2 + 2x_3 \\ \text{s.t.} \quad & 5x_1 + 2x_2 + 2x_3 \leq 145 \\ & 4x_1 + 8x_2 - 8x_3 \leq 260 \\ & x_1 + x_2 + 4x_3 \leq 190 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Find x_1 , x_2 and x_3 to solve it.

Solution (Exercise 3) — material: complementary slackness.pdf

Exercise 4 — Consider the linear programming problem:

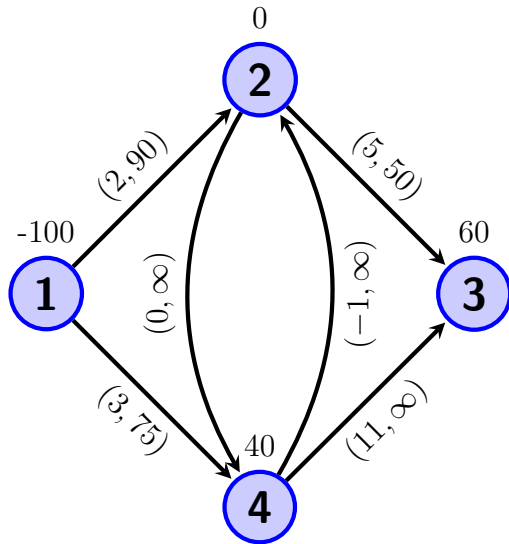
$$\begin{aligned} \max \quad & 2x_1 + 16x_2 + 2x_3 \\ \text{s.t.} \quad & 2x_1 + x_2 - x_3 \leq -3 \\ & -3x_1 + x_2 + 2x_3 \leq 12 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Check whether each of the following is an optimal solution, using complementary slackness:

Solution (Exercise 4) — material: compslack.pdf

3 Network analysis

Exercise 5 — The figure shows a network on four nodes, including net demands on the vertex, b_k , and cost and capacity on the edges, $(c_{i,j}, u_{i,j})$. (Adapted from [1])



(A) Formulate the corresponding minimum cost network flow model

(B) Classify the nodes as *source*, *sink* or *transshipment*

Solution (Exercise 5) — In this problem, vertex and edges are:

$$V = \{1, 2, 3, 4\}$$

$$A = \{(1, 2), (1, 4), (2, 3), (2, 4), (4, 2), (4, 3)\}$$

we can use the variables $x_{i,j}$ to represent the flows in the different members of set A . Thus, the formulation of the problem is:

$$\begin{aligned} \min \quad & 2x_{1,2} + 3x_{1,4} + 5x_{2,3} - x_{4,2} + 11x_{4,3} \\ \text{subject to} \quad & \left\{ \begin{array}{lcl} -x_{1,2} & - & x_{1,4} & = & -100 \\ x_{1,2} & & -x_{2,3} & - & x_{2,4} & + & x_{4,2} & = & 0 \\ & & x_{2,3} & & + & x_{4,3} & = & 60 \\ & & x_{1,4} & + & x_{2,4} & - & x_{4,3} & - & x_{4,2} & = & 40 \\ x_{1,2} & & & & & & \leq & 90 \\ & x_{1,4} & & & & & \leq & 75 \\ & & x_{2,3} & & & & \leq & 50 \end{array} \right. \end{aligned}$$

and $x_{i,j} \geq 0$.

There are 4 KKT conditions for optimal primal (x) and dual (λ) variables. If x^* denotes optimal values:

(A) Primal feasibility: all constraints must be satisfied: $g_i(x^*) - b_i$ is feasible. Applies to both equality and non-equality constraints.

(B) Gradient condition or No feasible descent: No possible improvement at the solution:

$$\nabla f(x^*) - \sum_{i=1}^m \lambda_i^* \nabla g_i(x^*) = 0$$

(C) Complementarity slackness:

$$\lambda_i^* (g_i(x^*) - b_i) = 0$$

(D) Dual feasibility: $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active ($=0$) and a zero Lagrange multiplier when the constraint is inactive (>0).

to solve our problem, first we will put it in its standard form:

$$\begin{aligned} \min f(x, y) &= -xy \\ \text{subject to} \quad & -x - y + 100 \geq 0 \\ & -x - 40 \geq 0 \end{aligned}$$

We will go through the different conditions:

(A) Primal feasibility: $g_i(x^*) - b_i$ is feasible.

$$-x^* - y^* + 100 = 0$$

$$-x^* - 40 = 0$$

(B) Gradient condition or No feasible descent:

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} - \lambda_1 \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_1}{\partial y} \end{pmatrix} - \lambda_2 \begin{pmatrix} \frac{\partial g_2}{\partial x} \\ \frac{\partial g_2}{\partial y} \end{pmatrix} = 0$$

which, in this example, resolves into:

$$\begin{cases} -y + \lambda_1 + \lambda_2 = 0 \\ -x - \lambda_1 = 0 \end{cases}$$

(C) Complementarity slackness:

$$\lambda_1^* (-x^* - y^* + 100) = 0$$

$$\lambda_2^* (-x^* - 40) = 0$$

(D)Dual feasibility: $\lambda_1, \lambda_2 \geq 0$

We can put the resulting 5 expressions for conditions 1 and 2 into matrix form:

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -100 \\ 40 \\ 0 \\ 0 \end{pmatrix}$$

References

- [1] Ronald L. Rardin. *Optimization in Operations Research*. Pearson, Boston, second edition edition, 2017.