Unit 2. Linear programming. The Simplex Method

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Preliminary

This course is strongly based on the monography on Operations Research by Carter, Price and Rabadi [**?**], and in material obtained from different sources (quoted when needed through the slides).

Learning outcomes

- Understanding the rational behind the Simplex method for LP.
- Understanding and practicing the algorithm in 2 variables.
- Recognizing the different types of results one can achieve in LP from the Simplex Method algorithm.
- Getting familiar with slack, surplus and artificial variables.

Summary

The standard form

Standard form

For a LP with n variables and m constraints, the standard form is given by:

$$
\begin{array}{ll}
\text{maximize} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} & \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & = & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}
\end{array}
$$

where $x_1, \ldots, x_n \geq 0$ and $b_1, \ldots, b_m \geq 0$

Thus:

$$
\begin{array}{ll}
\text{maximize} & z = cx \\
\text{subject to} & \begin{array}{c} Ax = b \\ x \geq 0 \\ b \geq 0 \end{array}
\end{array}
$$

Of course, not always the system is proposed in this standard form, so:

- We need to leave a maximization problem. If the problem is to minimize the objective function, we simply multiply it by (-1).
- In order to change inequality constraints into equality constraints we use *slack* variables for $\leq$ inequalities:

  $$3x_1 + 4x_2 \leq 7 \rightarrow 3x_1 + 4x_2 + s_1 = 7$$

  or *surplus* variables for $\geq$ inequalities:

  $$x_1 + 3x_2 \geq 10 \rightarrow x_1 + 3x_2 - s_2 = 10$$

- All variables should be non-negative.

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The algorithm

1. Convert the system of inequalities to equations (using slack or surplus variables).
2. Set the objective function to zero.
3. Create the Simplex tableau and label active and basic variables.
4. Select the pivot column (the one with the most negative coefficient in the zeroed objective function). This will be linked to the **entering vaiable**.
5. Select the pivot row (once divided the entry in the constant column by the ceffient in that row in the pivot column, we choose the smallest ratio). This will be linked to the **leaving variable**.
6. The pivot is the intersection between the pivot row and pivot column.
7. Use the pivot value to make zeros in the rest of elements in the pivot column.
8. Repeat the process from step 4, until the last row is all non-negative.

Example. Standard form

$$
\begin{array}{ll}
\text{maximize} & z = 8x_1 + 5x_2 \\
\text{subject to} & \begin{array}{rcl} x_1 & \leq & 150 \\ x_2 & \leq & 250 \\ 2x_1 + x_2 & \leq & 500 \\ x_1, x_2 & \geq & 0 \end{array}
\end{array}
$$

We build first the standard form:

$$
\begin{array}{rcl}
-8x_1 - 5x_2 + z & = & 0 \\
x_1 + s & = & 150 \\
x_2 + t & = & 250 \\
2x_1 + x_2 + u & = & 500
\end{array}
$$

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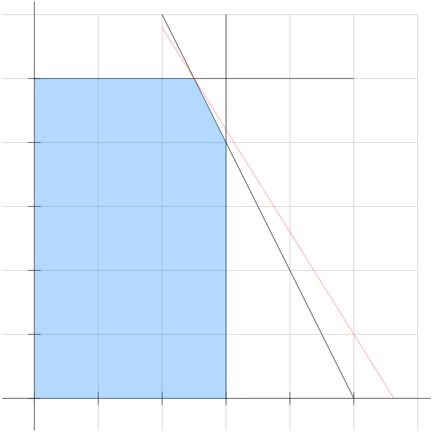
Example. The Simplex Tableau

We write the coefficients matrix and we identify the basic ($m = 3$) and non-basic ($n - m = 5 - 3 = 2$) variables:

$$\begin{array}{cc|c|ccccc|c}
 & & z & x_1 & x_2 & s & t & u & b \\
\hline
 & z & 1 & -8 & -5 & 0 & 0 & 0 & 0 \\
\hline
\rightarrow & s & 0 & 1 & 0 & 1 & 0 & 0 & 150 \\
 & t & 0 & 0 & 1 & 0 & 1 & 0 & 250 \\
 & u & 0 & 2 & 1 & 0 & 0 & 1 & 500 \\
\text{basic} & & & \uparrow & & & & &
\end{array} \tag{1}$$

Note that the basic variables are those for which each column is a collection of 1 and zero. We will arbitrarily assign zeros to the non-basic variables. So a possible solution is $\boxed{P_A = (x_1 = 0, x_2 = 0) \Rightarrow z = 0}$. The arrow marks the pivot column. We will take the pivot row by considering which is the lowest value among $150/1$ and $500/2$. So, the pivot row corresponds to the s basic variable.

Example. Graphic solution



Example. Entering/leaving variables

		z	x_1	x_2	s	t	u	b
$R_1 + 8R_2$	z	1	0	-5	8	0	0	1200
	x_1	0	1	0	1	0	0	150
	t	0	0	1	0	1	0	250
$\rightarrow R_4 - 2R_2$	u	0	0	1	-2	0	1	200
	basic			$\uparrow$				

$P_B = (150, 0)$ and $z(P_B) = 1200$ with $t = 250,\ u = 200,\ s = 0.$

		z	x_1	x_2	s	t	u	b
$R_1 + 5R_4$	z	1	0	0	-2	0	5	2200
	x_1	0	1	0	1	0	-1	150
$\to R_3 - R_4$	t	0	0	0	2	0	1	50
	x_2	0	0	1	-2	0	1	200
	basic				$\uparrow$			

		z	x_1	x_2	s	t	u	b
$R_1 + 5R_4$	z	1	0	0	0	1	4	2250
	x_1	0	1	0	0	$-1/2$	$1/2$	125
$\to R_3 - R_4$	s	0	0	0	1	$1/2$	$-1/2$	25
	x_2	0	1	0	1	0	0	250
	basic				$\uparrow$			

$\boxed{P_D = (125, 250)}$ and $\boxed{z(P_D) = 2250}$ with $t, u = 0$ (binding constraints), $s = 25$ (non-binding constraint).

Exercises

Solve, using the Simplex method, the Exercises 3 (p17) and 6 (p23) of the previous session

References

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