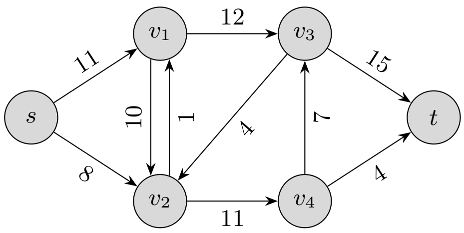
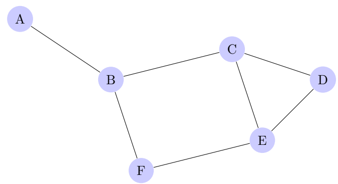
1 Session 13: Network Analysis

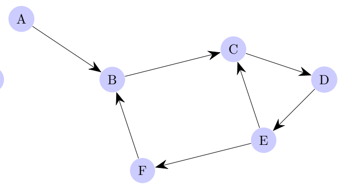
- Getting familiar with the use of network analysis in OR
- Understanding network flow in graphs
- Understanding minimum cost network flow problems
- Applying network connectivity to LP problems
- Solving shortest path problems
- Understanding and applying dynamic programming
- Some of these problems correspond to a physical or geographical network of elements within a system, while others correspond more abstractly to a graphical approach to planning or grouping or arranging the elements of a system.



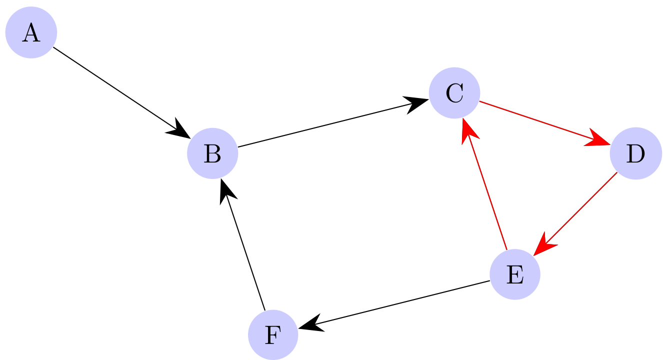
- Systems of highways, railroads, shipping lanes, or aviation patterns, where some supply of a commodity is transported or distributed to satisfy a demand;
- pipeline systems or utility grids can be viewed as fluid flow or power flow networks;
- computer communication networks represent the flow of information;
- an economic system may represent the flow of wealth;
- routing a vehicle or a commodity between certain specified points in the network;
- assigning jobs to machines, or matching workers with jobs for maximum efficiency;
- project planning and project management, where various activities must be scheduled in order to minimize the duration of a project or to meet specified completion dates, subject to the availability of resources;
- and many, many others.

1.1 Definitions

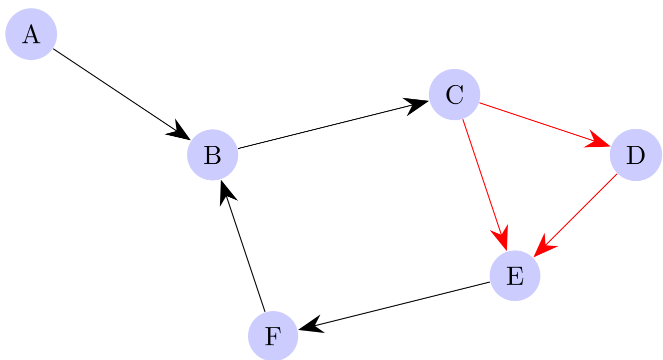




- A graph consists of a set of nodes V (vertices, points, or junctions) and a set of connections called arcs A (edges, links, or branches).
- Each connection is associated with a pair of nodes and is usually drawn as a line joining two points. The graph can be defined as *directed* or *undirected*.
- The **degree of a node** is the number of arcs attached to it. An isolated node is of degree zero.
- In a directed graph, or digraph, the arc is often designated by the ordered pair (A, B). In digraphs, the direction of the flow matters.

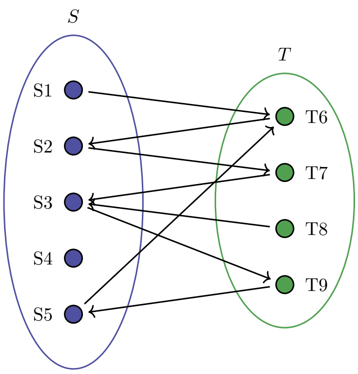


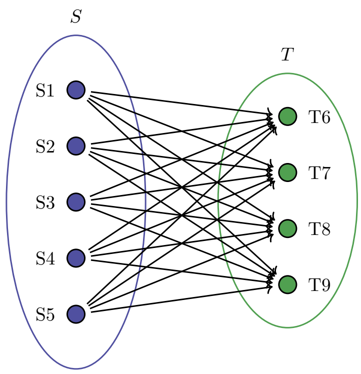
$$A, (A, B), B, (B, C), \underbrace{C, (C, D), D, (D, E), E, (E, C), C}_{\text{cyclic path and chain}}$$



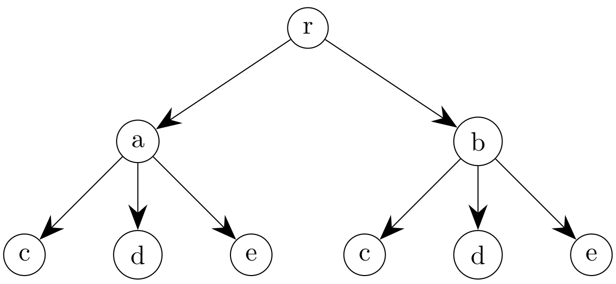
$$A, (A, B), B, (B, C), \underbrace{C, (C, D), D, (D, E), E, (C, E), C}_{\text{cyclic path, non cyclic chain}}$$

- If all the arcs in a path are forward arcs, the path is called **directed chain** or simply **chain**.
- **path** and **chain** are synonimous if the graph is undirected.
- In the second example above we saw a cyclic path but not a cyclic chain, as it included the backward arc (C, E).
- A **connected graph** has at least one path connecting every pair of nodes.
- In a **bipartite graph** the nodes can be partitioned into two subsets S and T, such that each node is in exactly one of the subsets, and every arc in the graph connects a node in set S with a node in set T.
- Such a graph is **complete bipartite** if each node in S is connected to every node in T.





- A **tree** is a directed connected graph in which each node has at most on predecessor, and one node (the root node) has none. In an undirected graph, we have a tree if the graph is connected and contains no cycles.

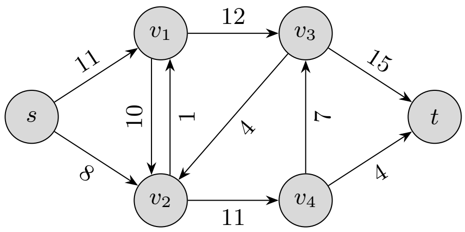


- A **network** is a directed connected graph that is used to model/represent a system/process. The arcs are typically assigned weights representing cost, value or capacity corresponding to each link.
- Nodes in networks can be designated as **sources**, **sinks** or **transshipments**. A **cut set** is any set of arcs which, if removed from the network, would disconnet the sources(s) from the sink(s).
- **Flow** can be thought of as the total amount of an entity that originates at the source, makes it through the different nodes and reaches the sink.

1.2 Maximum flow

Determine the maximum possible flow that can be routed through the various network links, from source (s) to sink (t), without violating the capacity constraints.

Important! the commodity is only generated at the source and consumed at the sink.



The **maximum flow problem** can be stated as a LP formulation.

$$
\begin{array}{lrll}
\text{maximize} & z = f & & \\
\text{subject to} & \sum_{i=2}^{n} x_{1i} = f & & \\
 & \sum_{i=1}^{n-1} x_{in} = f & & \\
 & \sum_{i=1}^{n} x_{ij} = \sum_{k=1}^{n} x_{jk}, & \text{for} & j = 2, 3, \ldots, n-1 \\
 & x_{ij} \leq u_{ij}, & \text{for all} & i, j = 1, 2, \ldots, n
\end{array}
$$

[1]

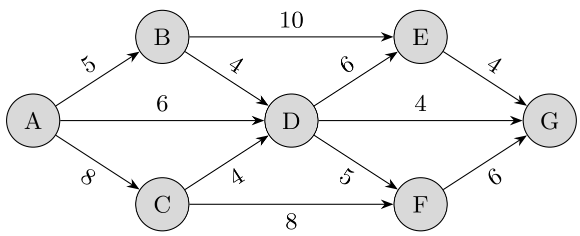
All network problems here can be solved using the Simplex method, but the network structure can help us solving it more efficiently. In the **Ford-Fulkerson labelling algorithm**:

1. Use a labelling procedure to look for a flow augmenting path. If none can be found, stop; the current flow is optimal;

2. Increase the current flow as much as possible in the flow augmenting path, until reaching capacity of some arc. Come back to step 1.

Exercise 1

Find the maximum flow in this network using the Ford-Fulkerson labelling algorithm:

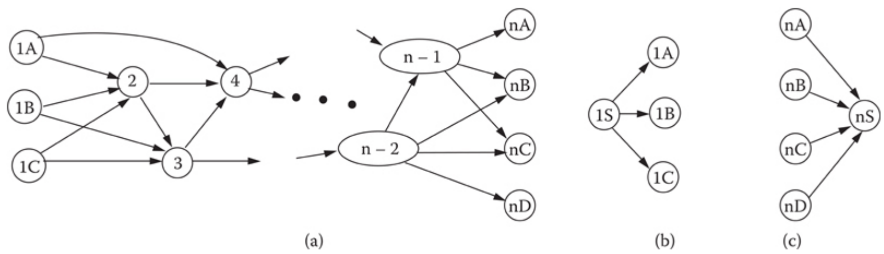


[1] Constraints (1) and (2) state that all the flow is generated at the source and consumed at the sink. Constraint (1) ensures that a flow of f leaves the source, and because of conservation of flow, that flow stops only at the sink. Constraints (3) are the flow conservation equations for all the intermediate nodes; nothing is generated or consumed at these nodes. Constraints (4) enforce arc capacity restrictions. All flow amounts xij must be non-negative. Actually, constraint (2) is redundant[1]

2

- In any network, there is always a bottleneck that in some sense impedes the flow through the network.
- The total capacity of the bottleneck is an upper bound on the total flow in the network.
- Cut sets are, by definition, essential in order for there to be a flow from source to sink, since removal of the cut set links would render the sink unreachable from the source.
- The capacities on the links in any cut set potentially limit the total flow.
- The minimum cut (i.e., the cut set with minimum total capacity) is in fact the bottleneck that precisely determines the maximum possible flow in the network (Max-Flow Min-Cut Theorem): the capacity of the cut is precisely equal to the current flow and this flow is optimal. In other words, a saturated cut defines the maximum flow.

We can generate a supersource or a supersink node with unlimited capacity and repeat the process of optimization as above:[1]



1.3 Minimum Cost Network Flow

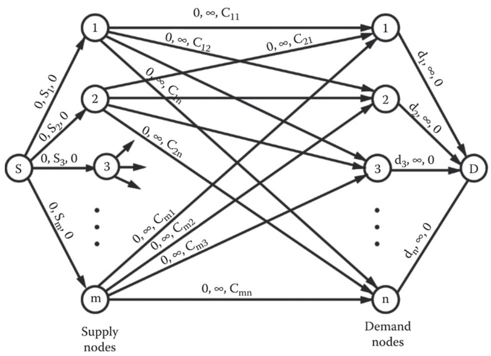
- Useful when there are costs associated with the flow, given a link capacity.
- Let us assume that every node is a source (supply) and a sink (demand). Imagine a distributor with several warehouses and a group of costumers. Serving each customer from a given warehouse has an associated cost.

[2]Let us assume initially that the flow in all arcs is zero, x ij = 0 and f = 0. In the first iteration, we label nodes A, B, C, D, and G, and discover the flow augmenting path (A, D) and (D, G), across which we can increase the flow by 4. So now, x AD = 4, xDG = 4, and f = 4. In the second iteration, we label nodes A, B, C, then nodes E, D, and F, and finally node G. A flow augmenting path consists of links (A, B), (B, D), (D, E), and (E, G) and flow on this path can be increased by 4. Now x AB = 4, xBD = 4, xDE = 4, x EG = 4, and f = 8. In the third iteration, we see that there remains some unused capacity on link (A, B), so we can label nodes A, B, and E, but not G. It appears we cannot use the full capacity of link (A, B). However, we can also label nodes C, D, F, and G, and augment the flow along the links (A, D), (D, F), and (F, G) by 2, the amount of remaining capacity in (A, D). Now x AD = 6, xDF = 2, x FG = 2, and f = 10. In the fourth iteration, we can label nodes A, B, C, D, F, and G. Along the path from A, C, D, F, to G, we can add a flow of 4, the remaining capacity in (F, G). So x AC = 4, xCF = 4, x FG = 6, and f = 14. In the fifth iteration, we can label all nodes except G. Therefore, there is no flow aug- menting path, and the current flow of 14 is optimal[1]

- For m supply nodes, each providing s_i suplly, and n demand nodes, each demanding d_j. Assuming that the total demand eaquals the total supply: $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$ we aim at satisfying the demand using the available supply minimizing cost routes.

$$\text{minimize} \quad z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\begin{array}{lrll} & \sum_{j=1}^{n} x_{ij} = s_i & \text{for} & i = 1, \ldots, m \\ \text{subject to} & \sum_{i=1}^{m} x_{ij} = d_j & \text{for} & j = 1, \ldots, n \\ & x_{ij} \geq 0 & \text{for all} & i, j \end{array}$$



Exercise 2

Find the minimum cost in this trasportation problem:[1]

Sources (Warehouses)	**Sinks (Customers)** 1	2	3	4	5	**Supply**
1	28	7	16	2	30	20
2	18	8	14	4	20	20
3	10	12	13	5	28	25
Demand	12	14	12	18	9	65

NOTE: The Simplex method says that we should first find any basic feasible solution and then look for a simple pivot to improve the solution. repeat until the optimal solution is found.

1. Finding initial solution.
   - Northwest corner rule
   - Minimum cost method
   - Minimum "row" cost method
   - Vogel's method (opportunity cost)

2. Transportation simplex

Once we have any feasible solution, we aim at finding the optimal one. Consider:

Minimum Row Cost Final Solution

Sources	Sinks (Customers)					
(Warehouses)	1	2	3	4	5	Supply
1	$\boxed{28}$	2 $\boxed{7}$	$\boxed{16}$	18 $\boxed{2}$	$\boxed{30}$	20
2	8 $\boxed{18}$	12 $\boxed{8}$	$\boxed{14}$	$\boxed{4}$	$\boxed{20}$	20
3	4 $\boxed{10}$	$\boxed{12}$	12 $\boxed{13}$	$\boxed{5}$	9 $\boxed{28}$	25
Demand	12	14	12	18	9	65

We can reduce the total cost by reducing the individual costs in every row i by u_i and in every column j by v_j:

$$c'_{ij} = c_{ij} - u_i - v_j$$

Check that, now:

- $\sum_i \sum_j x_{ij} c'_{ij} = 0$
- Check how some costs are now negative in non-basic cells.
- If we increase the number of units in those non-basic cells from 0 to some value, reducing at the same time the number of units in the basic cells, we can reduce the overall cost $\sum_i \sum_j x_{ij} c_{ij}$

In practice:

1. We find first an initial feasible solution as explained above
2. Calculate the u_i and v_j, taking into account that $c_{ij} = u_i + v_j$, for all basic variables (used squares in the table). We start by assigning $u_1 = 0$.
3. We calculate the *improvement index* by $I_{ij} = c_{ij} - u_i - v_j$ for all non-used squares in the table.
4. If all I_{ij} are positive, the solution is already optimal and we are done.
5. If some $I_{ij} < 0$, then build a loop with such value in the corner and alternative $\pm$ signs in all vertex.
6. Use the above $\pm$ to increase the number of units in the position that had $I_{ij} < 0$
7. We return to step 2.

[3]

[3]dynamic programming: https://www.youtube.com/watch?v=jqyfhFziDjw
https://www.techiedelight.com/word-break-problem/

References

[1] Michael Carter, Camille C Price, and Ghaith Rabadi. *Operations Research. A Practical Introduction. Second Edition.* CRC Press, 2019.