Solved Exercises

Optimization and Operations Research

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1 Non-linear optimization

1.1 Karush-Kuhn-Tucker theorem

Exercise 1 — Maximize $f(x,y) = xy$ subject to $100 \geq x + y$ and $x \leq 40$

Solution (Exercise 1) — The Karush Kuhn Tucker (KKT) conditions for optimality are a set of necessary conditions for a solution to be optimal in a mathematical optimization problem. They are necessary and sufficient conditions for a local minimum in nonlinear programming problems. The KKT conditions consist of the following elements:

For an optimization problem in its standard form:

$$\min f(x)$$
$$\text{subject to} \quad \begin{array}{rcll} g_i(x) - b_i & \geq & 0 & i = 1, \dots, k \\ g_i(x) - b_i & = & 0 & i = k+1, \dots, m \end{array}$$

There are 4 KKT conditions for optimal primal ($x4$) and dual (λ) variables. If x^* denotes optimal values:

(A) Primal feasibility: all constraints must be satisfied: $g_i(x^*) - b_i$ is feasible. Applies to both equality and non-equality constraints.

(B) Gradient condition or No feasible descent: No possible improvement at the solution:
$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0$$

(C) Complementariety slackness:
$$\lambda_i^*(g_i(x^*) - b_i) = 0$$

(D) Dual feasibility: $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active (=0) and a zero Lagrange multiplier when the constraint is inactive (>0).

to solve our problem, first we will put it in its standard form:

$$\min\ f(x,y) = -xy$$
$$\text{subject to} \quad \begin{array}{rcl} -x-y+100 & \geq & 0 \\ -x-40 & \geq & 0 \end{array}$$

We will go through the different conditions:

(A)Primal feasibility: $g_i(x^*) - b_i$ is feasible.

$$-x^* - y^* + 100 = 0$$

$$-x^* - 40 = 0$$

(B)Gradient condition or No feasible descent:

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} - \lambda_1 \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_1}{\partial y} \end{pmatrix} - \lambda_2 \begin{pmatrix} \frac{\partial g_2}{\partial x} \\ \frac{\partial g_2}{\partial y} \end{pmatrix} = 0$$

which, in this example, resolves into:

$$\begin{cases} \quad\; - y + \lambda_1 + \lambda_2 = 0 \\ -x \quad\;\; - \lambda_1 \qquad\;\; = 0 \end{cases}$$

(C)Complementariety slackness:

$$\lambda_1^*(-x^* - y^* + 100) = 0$$

$$\lambda_2^*(-x^* - 40) = 0$$

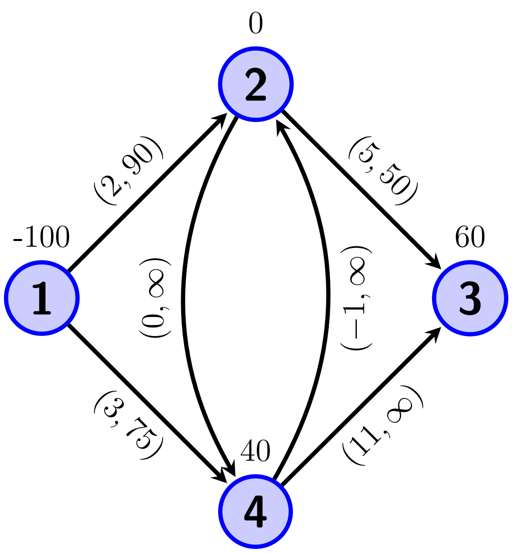
(D)Dual feasibility: $\lambda_1, \lambda_2 \geq 0$

We can put the resulting 5 expressions for conditions 1 and 2 into matrix form:

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -100 \\ 40 \\ 0 \\ 0 \end{pmatrix}$$

2 Network analysis

Exercise 2 — The figure shows a network on four nodes, including net demands on the vertex, b_k, and cost an capacity on the edges, $(c_{i,j}, u_{i,j})$.(Adapted from [1])



(A)Formulate the corresponding minimum cost network flow model

(B)Classify the nodes as *source*, *sink* or *transshipment*

Solution (Exercise 2) — In this problem, vertex and edges are:

$$\begin{aligned} V &= \{1, 2, 3, 4\} \\ A &= \{(1,2), (1,4), (2,3), (2,4), (4,2), (4,3)\} \end{aligned}$$

we can use the variables $x_{i,j}$ to represent the flows in the different members of set A.

Thus, the formulation of the problem is:

$$\min \quad 2x_{1,2} + 3x_{1,4} + 5x_{2,3} - x_{4,2} + 11x_{4,3}$$

$$\text{subject to} \quad \begin{cases} -x_{1,2} - x_{1,4} = -100 \\ x_{1,2} - x_{2,3} - x_{2,4} + x_{4,2} = 0 \\ x_{2,3} + x_{4,3} = 60 \\ x_{1,4} + x_{2,4} - x_{4,3} - x_{4,2} = 40 \\ x_{1,2} \leq 90 \\ x_{1,4} \leq 75 \\ x_{2,3} \leq 50 \end{cases}$$

and $x_{i,j} \geq 0$.

There are 4 KKT conditions for optimal primal ($x4$) and dual (λ) variables. If x^* denotes optimal values:

(A) Primal feasibility: all constraints must be satisfied: $g_i(x^*) - b_i$ is feasible. Applies to both equality and non-equality constraints.

(B) Gradient condition or No feasible descent: No possible improvement at the solution:

$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0$$

(C) Complementariety slackness:

$$\lambda_i^* (g_i(x^*) - b_i) = 0$$

(D) Dual feasibility: $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active (=0) and a zero Lagrange multiplier when the constraint is inactive (>0).

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We can put the resulting 5 expressions for conditions 1 and 2 into matrix form:

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -100 \\ 40 \\ 0 \\ 0 \end{pmatrix}$$

References

[1] Ronald L. Rardin. *Optimization in Operations Research.* Pearson, Boston, second edition edition, 2017.