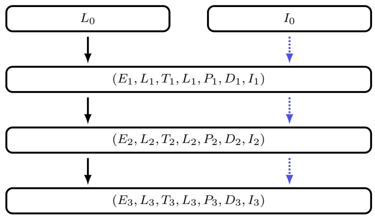
1 Session 2. Nonlinear Optimization

Example Production cost:

A company aims at minimizing the production cost during a series of production periods of time T_i, characterised by the demand $D_i i$, equipment capacities and material limitations E_i, labor force L_i (which cost depends quadratically on the difference of labor force between two periods $C_L(L_i - L_{i-1})^2$), productivity of each worker P_i, the number of units of inventory at the end of each production period I_i and the cost to bring them to the next production period C_I.



The problem, as stated, aims at minimizing the function:

$$f(\vec{L}, \vec{I}) = \sum_{i=1}^{T} C_L(L_i - L_{i-1})^2 + C_I I_i$$

subject to:

$$\begin{cases} L_i P_i \leq E_i \\ I_{i-1} + L_i P_i \geq D_i \\ I_i = I_{i-1} + L_i P_i - D_i \\ L_i, I_i \geq 0, \ \forall i = 1, \ldots, T \end{cases}$$

Quadratic objective function with linear constraints.

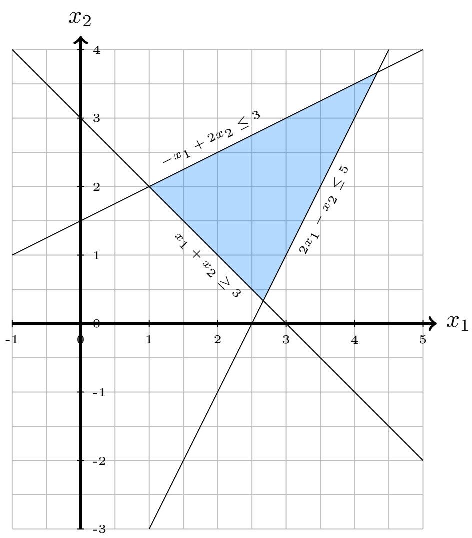
- We want to obtain the best solution to a mathematical programming problem in which both objective function and constraints have general non-linear forms.
- Most of the problems are non-linear indeed!
  - Unconstrained Problems (often dealt with differential calculus)
  - Constrained Problems (may include systems of equations to be solved)
- Classical underlying mathematical theories do not necessarily provide practical methods suitable for efficient numerical computation.
- Points of optimality can be anywhere inside the problem boundaries
- No methods applicable to all non-linear problems

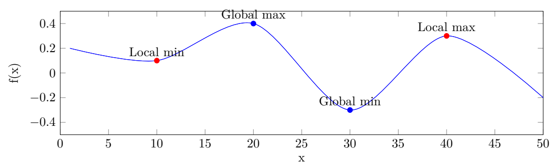
1.1 Some theoretical background

material: convex and non linear problem.pdf

1.2 Feasible region

Set of points satisfying all the constraints (the area between constraint boundaries).





A local maximum of the function $f(x)$ exists in x^* if there is a small positive number ϵ such that

$$f(x^*) > f(x),\ \forall x \in \mathbf{R} : \|x - x^*\| < \epsilon$$

A global maximum of $f(x)$ exists in x^* if

$$f(x^*) > f(x),\ \forall x \in \mathbf{R}$$

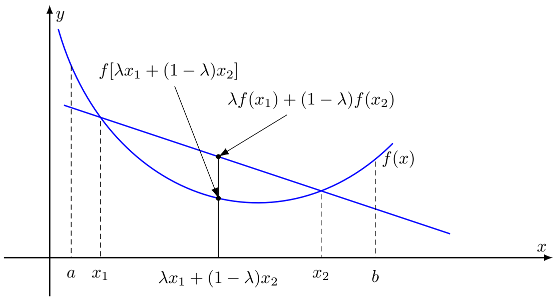
(analogous definitions for local/global minimum)

1.3 Concavity/convexity

For a continuous convex function, given any two points x_1 and x_2:

$$f[\lambda x_1 + (1-\lambda)x_2] \leq \lambda f(x_1) + (1-\lambda)f(x_2),\ 0 \leq \lambda \leq 1 \tag{1}$$

(analogous situation for a continuous concave function)



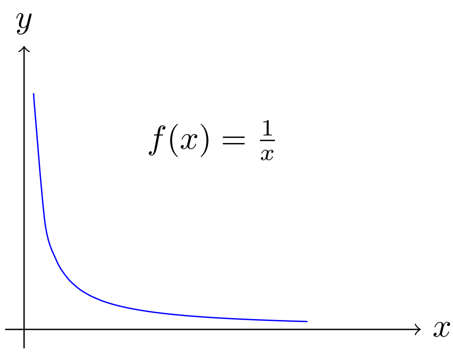
The term "convex" can be applied both to sets and functions. A set $S \in \mathbf{R}^n$ is a *convex set* if the straight line segment connecting any two points in S lies entirely inside S.

Note that $f(x)$ is a convex function if its domain S is a convex set and if for any two points x_1 and x_2 in S Eq. 1 holds.

Exercise 1

Can you draw a convex function that depends on two variables, $f(x, y)$? Can you generalize Eq. 1?

- If a convex nonlinear function is to be optimized without constraints, a global minimum may occur when $f'(x) = 0$. Not always this is the case:



- If the feasible region for a nonlinear programming problem is convex, each of the constraint functions is convex and are of the form $g_i(x) \leq b_i$
- A local minimum is guaranteed to be a global minimum for a convex objective function in a convex feasible region, and
- a local maximum is guaranteed to be a global maximum for a concave objective function in a convex feasible region.

Many functions in nonlinear programming problems are neither concave nor convex!

Exercise 2

Is the function $f(x) = x^2$ convex? Are the regions defined by $x^2 = 4$ or $x^2 \geq 9$ convex?

Exercise 3

Are the functions $f(x,y) = 3x^2 - 2xy + y^2 + 3e^{-x}$ and $g(x,y) = x^4 - 8x^3 + 24x^2 - 32x + 16$ convex?

Exercise 4

Consider these two nonlinear problems[1]:

$$\begin{array}{ll|ll} \text{minimize} & f(x,y) = x - 2xy + 2y & \text{minimize} & f(x,y) = x - 2xy + 2y \\ \text{subject to} & \begin{cases} x^2 + 3y^2 \le 10 \\ 3x + 2y \ge 1 \\ x, y \ge 0 \end{cases} & \text{subject to} & \begin{cases} x^3 - 12x - y \ge 0 \\ x \ge 1 \end{cases} \end{array}$$

Are their feasible regions convex?

$$(\text{Hess } f)_{ij} \equiv \frac{\partial^2 f}{\partial x_i \partial x_j} \tag{2}$$

$$\text{Hess}\left(\frac{x^2}{y}\right) = \begin{bmatrix} \frac{2}{y} & -\frac{2x}{y^2} \\ -\frac{2x}{y^2} & \frac{2x^2}{y^3} \end{bmatrix} \tag{3}$$

References

[1] Michael Carter, Camille C Price, and Ghaith Rabadi. *Operations Research. A Practical Introduction. Second Edition.* CRC Press, 2019.