Solved Exercises

Optimization and Operations Research

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1 Non-linear optimization

1.1 Karush-Kuhn-Tucker theorem

Exercise 1 — Maximize $f(x,y) = xy$ subject to $100 \geq x + y$ and $x \leq 40$

Solution (Exercise 1) — The Karush Kuhn Tucker (KKT) conditions for optimality are a set of necessary conditions for a solution to be optimal in a mathematical optimization problem. They are necessary and sufficient conditions for a local minimum in nonlinear programming problems. The KKT conditions consist of the following elements:

For an optimization problem in its standard form:

$$\max f(x)$$
$$\text{s.t.} \quad \begin{array}{rcll} g_i(x) - b_i & \leq & 0 & i = 1, \ldots, k \\ g_i(x) - b_i & = & 0 & i = k+1, \ldots, m \end{array}$$

There are 4 KKT conditions for optimal primal ($x4$) and dual (λ) variables. If x^* denotes optimal values:

(A) Primal feasibility: all constraints must be satisfied: $g_i(x^*) - b_i$ is feasible. Applies to both equality and non-equality constraints.

(B) Gradient condition or No feasible descent: No possible improvement at the solution:
$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0$$

(C) Complementariety slackness:
$$\lambda_i^*(g_i(x^*) - b_i) = 0$$

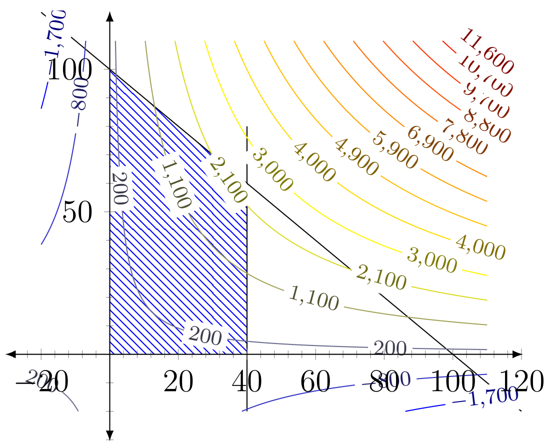
(D) Dual feasibility: $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active (=0) and a zero Lagrange multiplier when the constraint is inactive (>0).

To solve our problem, first we will put it in its standard form:

$$\max f(x,y) = xy$$

$$\text{s.t.} \quad \begin{array}{rcl} g_1(x,y) = x + y - 100 & \leq & 0 \\ g_2(x,y) = x - 40 & \leq & 0 \\ g_3(x,y) = -x & \leq & 0 \\ g_4(x,y) = -y & \leq & 0 \end{array}$$



We will go through the different conditions:

- on the gradient:

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} - \lambda_1 \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_1}{\partial y} \end{pmatrix} - \lambda_2 \begin{pmatrix} \frac{\partial g_2}{\partial x} \\ \frac{\partial g_2}{\partial y} \end{pmatrix} - \lambda_3 \begin{pmatrix} \frac{\partial g_3}{\partial x} \\ \frac{\partial g_3}{\partial y} \end{pmatrix} - \lambda_4 \begin{pmatrix} \frac{\partial g_4}{\partial x} \\ \frac{\partial g_4}{\partial y} \end{pmatrix} = 0$$

  which, in this example, resolves into:

$$y - (\lambda_1 + \lambda_2 - \lambda_3) = 0 \tag{1}$$

$$x - (\lambda_1 - \lambda_4) = 0 \tag{2}$$

- on the complementary slackness:

$$\lambda_1(x + y - 100) = 0 \tag{3}$$

$$\lambda_2(x - 40) = 0 \tag{4}$$

$$\lambda_3 x = 0 \tag{5}$$

$$\lambda_4 y = 0 \tag{6}$$

- on the constraints:

$$x + y \leq 100 \tag{7}$$

$$x \leq 40 \tag{8}$$

$$-x \leq 0 \tag{9}$$

$$-y \leq 0 \tag{10}$$

plus $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$.

We will start by checking Eq. 3:

–Let us seeg what occurs if $\lambda_1 = 0$. Then, from Eq. 2, $x + \lambda_4 = 0$ which implies that $x = \lambda_4 = 0$[1]. But, then, from Eq. 4 we obtain that $\lambda_2 = 0$ which, using Eq. 1 gives $y + \lambda_3 = 0 \Rightarrow y = \lambda_3 = 0$. Indeed, the KKT conditions are satisfied when all variables and multipliers are zero, but it is not a maximum of the function (see figure above).

–So, let us see what happens if $x + y - 100 = 0$ and consider the two possibilities for x:

Case $x = 0$**:**Then, $y = 100$, which would lead (Eq. 6) to $\lambda_4 = 0$ and (Eq. 2) to $x = \lambda_1 = 0$, that was discussed in the previous item. So, we need top explore the other possibility for x.

Case $x > 0$**:**From Eq. 5 $\lambda_3 = 0$ and, from Eqs. 1 and 2:

$$\begin{cases} \quad y = \lambda_1 + \lambda_2 \\ x \quad = \lambda_1 + \lambda_4 \end{cases}$$

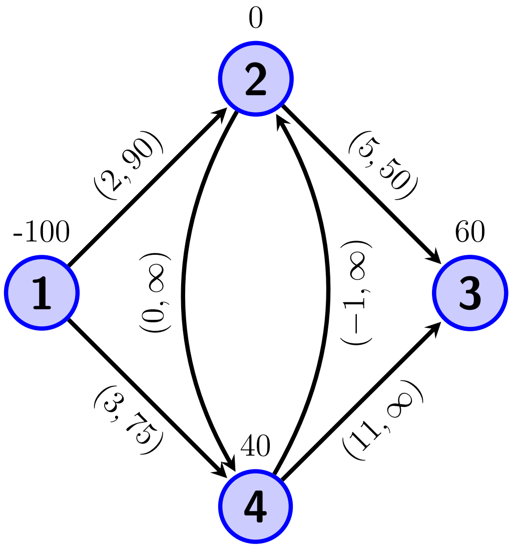
let us try what happens if, e.g., $\lambda_2 \neq 0$ (or, said in other words, if constraint 8 is active): $x = 40$. As we know we do not want $\lambda_1 = 0$, from Eq. 3 we obtain $x + y - 100 = 0 \Rightarrow y = 60$.

The point $(x, y) = (40, 60)$ fullfills the KKT conditions and is a maximum in the constrained maximization problem.

2 Network analysis

Exercise 2 — The figure shows a network on four nodes, including net demands on the vertex, b_k, and cost an capacity on the edges, $(c_{i,j}, u_{i,j})$.(Adapted from [?])

[1]Recall that both variables and multiplieras must be positive or zero, so, the only possibility for the equation to fullfill is that both are zero.



(A)Formulate the corresponding minimum cost network flow model

(B)Classify the nodes as *source*, *sink* or *transshipment*

Solution (Exercise 2) — In this problem, vertex and edges are:

$$\begin{aligned} V &= \{1,2,3,4\} \\ A &= \{(1,2),(1,4),(2,3),(2,4),(4,2),(4,3)\} \end{aligned}$$

we can use the variables $x_{i,j}$ to represent the flows in the different members of set A. Thus, the formulation of the problem is:

$$\min \quad 2x_{1,2} + 3x_{1,4} + 5x_{2,3} - x_{4,2} + 11x_{4,3}$$

$$\text{subject to} \quad \left\{ \begin{array}{ccccccccccccc} -x_{1,2} & - & x_{1,4} & & & & & & & & & = & -100 \\ x_{1,2} & & & - & x_{2,3} & - & x_{2,4} & & & + & x_{4,2} & = & 0 \\ & & & & x_{2,3} & & & + & x_{4,3} & & & = & 60 \\ & & x_{1,4} & & & + & x_{2,4} & - & x_{4,3} & - & x_{4,2} & = & 40 \\ x_{1,2} & & & & & & & & & & & \leq & 90 \\ & & x_{1,4} & & & & & & & & & \leq & 75 \\ & & & & x_{2,3} & & & & & & & \leq & 50 \end{array} \right.$$

and $x_{i,j} \geq 0$.

There are 4 KKT conditions for optimal primal ($x4$) and dual (λ) variables. If x^* denotes optimal values:

(A) Primal feasibility: all constraints must be satisfied: $g_i(x^*) - b_i$ is feasible. Applies to both equality and non-equality constraints.

(B) Gradient condition or No feasible descent: No possible improvement at the solution:

$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0$$

(C) Complementariety slackness:

$$\lambda_i^*(g_i(x^*) - b_i) = 0$$

(D) Dual feasibility: $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active (=0) and a zero Lagrange multiplier when the constraint is inactive (>0).

to solve our problem, first we will put it in its standard form:

$$\begin{aligned} \min\ f(x,y) = -xy \\ \text{subject to} \quad \begin{array}{rcl} -x - y + 100 & \geq & 0 \\ -x - 40 & \geq & 0 \end{array} \end{aligned}$$

We will go through the different conditions:

(A) Primal feasibility: $g_i(x^*) - b_i$ is feasible.

$$-x^* - y^* + 100 = 0$$

$$-x^* - 40 = 0$$

(B) Gradient condition or No feasible descent:

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} - \lambda_1 \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_1}{\partial y} \end{pmatrix} - \lambda_2 \begin{pmatrix} \frac{\partial g_2}{\partial x} \\ \frac{\partial g_2}{\partial y} \end{pmatrix} = 0$$

which, in this example, resolves into:

$$\begin{cases} \quad\ - y + \lambda_1 + \lambda_2 = 0 \\ -x \quad\ - \lambda_1 \qquad\ = 0 \end{cases}$$

(C) Complementariety slackness:

$$\lambda_1^*(-x^* - y^* + 100) = 0$$

$$\lambda_2^*(-x^* - 40) = 0$$

(D)Dual feasibility: $\lambda_1, \lambda_2 \geq 0$

We can put the resulting 5 expressions for conditions 1 and 2 into matrix form:

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -100 \\ 40 \\ 0 \\ 0 \end{pmatrix}$$