Unit 2. Linear programming. The Simplex Method

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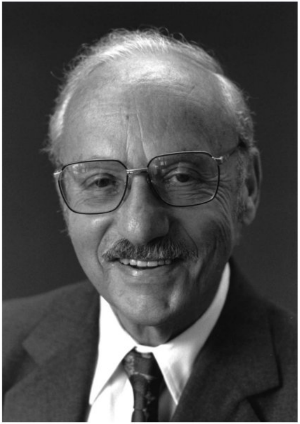
Preliminary

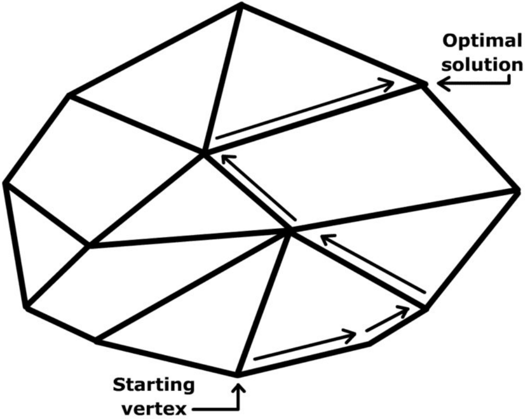
This course is strongly based on the monography on Operations Research by Carter, Price and Rabadi [**?**], and in material obtained from different sources (quoted when needed through the slides).

Learning outcomes

- Understanding the rational behind the Simplex method for LP.
- Understanding and practicing the algorithm in 2 variables.
- Recognizing the different types of results one can achieve in LP from the Simplex Method algorithm.
- Getting familiar with slack, surplus and artificial variables.

Summary

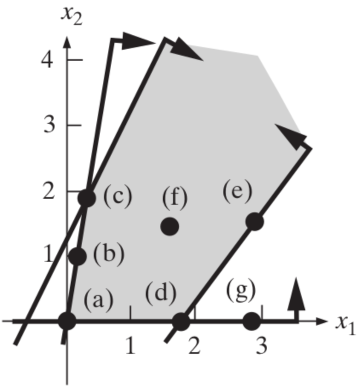




Left: George Dantzig (1914-2005), the American mathematical scientist who devised linear programming and the simplex algorithm. Right: The simplex algorithm moves along the edges of the polytope until it reaches the optimum solution [Image Wikimedia Commons].

Summary

Classification of solutions in an LP problem



(a), (c), and (d) are both extreme points and boundary points.
(b) and (e) are boundary points that are not extreme.
(f) is interior because no constraint is active.
(g) is neither interior nor boundary (nor extreme) because it is infeasible.[**?**]

The standard form

Standard form

For a LP with n variables and m constraints, the standard form is given by:

$$
\begin{array}{ll}
\text{maximize} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} & \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & = & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}
\end{array}
$$

where $x_1, \ldots, x_n \geq 0$ and $b_1, \ldots, b_m \geq 0$

Thus:

$$
\begin{array}{ll}
\text{maximize} & z = cx \\
\text{subject to} & \begin{array}{c} Ax = b \\ x \geq 0 \\ b \geq 0 \end{array}
\end{array}
$$

Of course, not always the system is proposed in this standard form, so[**?**]:

- We need to leave a maximization problem. If the problem is to minimize the objective function, we simply multiply it by (-1).
- In order to change inequality constraints into equality constraints we use *slack* variables for $\leq$ inequalities:

$$3x_1 + 4x_2 \leq 7 \rightarrow 3x_1 + 4x_2 + s_1 = 7$$

  or *surplus* variables for $\geq$ inequalities:

$$x_1 + 3x_2 \geq 10 \rightarrow x_1 + 3x_2 - s_2 = 10$$

- All variables should be non-negative. If some variable x_1 needs to be considered as unrestricted in sign, we will replace it by $x_1' - x_1''$, with $x_1', x_1'' \geq 0$

Summary

- The system of linear equations, $Ax = b$, consists of m equations and n unknowns (the original decision variables plus the rest introduced to get a standard form). If the system has any solution, then $m \leq n$.
- If $m = n$ (and if $rank(A) = m$ and A is nonsingular), then $x = A^{-1}b$. No optimization needed!
- If $m < n$, there are infinitely many solutions and $n - m$ degrees of freedom.
- A **basic solution** is obtained by setting $n - m$ of the variables to zero (**non-basic variables**), and solving for the remaining (**basic**) m variables. The number of basic solutions is $\binom{n}{n-m} = \binom{n}{m} = \frac{n!}{m!(n-m)!}$
- Basic solutions that do not satisfy all problem constraints and non-negativity constraints are **infeasible solutions**.
- An **optimal basic feasible solution** is a basic feasible solution that optimizes the objective function.

Summary

We define two extreme points of the feasible region (or two basic feasible solutions) as being adjacent if all but one of their basic variables are the same. Thus, *a transition from one basic feasible solution to an adjacent basic feasible solution can be thought of as exchanging the roles of one basic variable and one non-basic variable*. The Simplex method performs a sequence of such transitions and thereby examines a succession of adjacent extreme points. A transition to an adjacent extreme point will be made only if by doing so the objective function is improved (or stays the same). It is a property of linear programming problems that this type of search will lead us to the discovery of an optimal solution (if one exists). The Simplex method is not only successful in this sense, but it is remarkably efficient because it succeeds after examining only a fraction of the basic feasible solutions.[**?**]

The algorithm consists in:

1. obtaining an initial feasible solution,
2. make transitions to a better basic feasible solution,
3. recognizing an optimal solution.

From any basic feasible solution, we have the assurance that, if a better solution exists at all, then there is an adjacent solution that is better than the current one.[**?**]

The algorithm

1. Convert the system of inequalities to equations (using slack or surplus variables).
2. Set the objective function to zero.
3. Create the Simplex tableau and label active and basic variables.
4. Select the pivot column (the one with the most negative coefficient in the zeroed objective function). Linked to the **entering vaiable**.
5. Select the pivot row (once divided the entry in the constant column by the ceffient in that row in the pivot column, we choose the smallest ratio). Linked to the **leaving variable**.
6. The pivot is the intersection between the pivot row and pivot column.
7. Use the pivot value to make zeros in the rest of elements in the column.
8. Repeat the process from step 4, until the last row is all non-negative.

Example. Standard form

$$
\begin{array}{ll}
\text{maximize} & z = 8x_1 + 5x_2 \\
\text{subject to} & \begin{array}{rcl} x_1 & \leq & 150 \\ x_2 & \leq & 250 \\ 2x_1 + x_2 & \leq & 500 \\ x_1, x_2 & \geq & 0 \end{array}
\end{array}
$$

We build first the standard form:

$$
\begin{array}{rcl}
-8x_1 - 5x_2 - 0s - 0t - 0u + z & = & 0 \\
x_1 + s & = & 150 \\
x_2 + t & = & 250 \\
2x_1 + x_2 + u & = & 500
\end{array}
$$

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Example. The Simplex Tableau

We write the coefficients matrix and we identify the basic ($m = 3$) and non-basic ($n - m = 5 - 3 = 2$ degrees of freedom) variables:

		z	x_1	x_2	s	t	u	b
	z	1	-8	-5	0	0	0	0
$\rightarrow$	s	0	1	0	1	0	0	150
	t	0	0	1	0	1	0	250
	u	0	2	1	0	0	1	500
basic			$\uparrow$					

(1)

Note that the basic variables are those for which each column is a collection of 1 and zero. We will arbitrarily assign zeros to the non-basic variables. So a possible solution is $\boxed{P_A = (x_1 = 0, x_2 = 0) \Rightarrow z = 0}$. The arrow marks the pivot column. We will take the pivot row by considering which is the lowest value among $150/1$ and $500/2$. So, the pivot row corresponds to the s basic variable.

Example. Entering/leaving variables

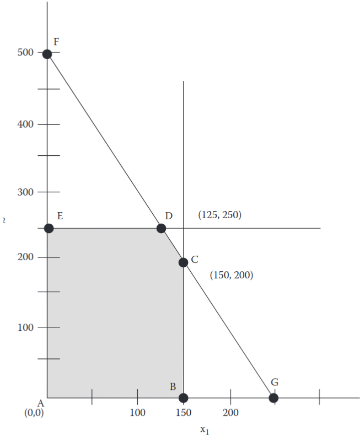
		z	x_1	x_2	s	t	u	b
$R_1 + 8R_2$	z	1	0	-5	8	0	0	1200
	x_1	0	1	0	1	0	0	150
	t	0	0	1	0	1	0	250
$\rightarrow R_4 - 2R_2$	u	0	0	1	-2	0	1	200
	basic			$\uparrow$				

$P_B = (150, 0)$ and $z(P_B) = 1200$ with $t = 250,\ u = 200,\ s = 0.$

		z	x_1	x_2	s	t	u	b
$R_1 + 5R_4$	z	1	0	0	-2	0	5	2200
	x_1	0	1	0	1	0	-1	150
$\to R_3 - R_4$	t	0	0	0	2	0	1	50
	x_2	0	0	1	-2	0	1	200
	basic				$\uparrow$			

		z	x_1	x_2	s	t	u	b
$R_1 + 5R_4$	z	1	0	0	0	1	4	2250
	x_1	0	1	0	0	$-1/2$	$1/2$	125
$\to R_3 - R_4$	s	0	0	0	1	$1/2$	$-1/2$	25
	x_2	0	1	0	1	0	0	250
	basic				$\uparrow$			

$\boxed{P_D = (125, 250)}$ and $\boxed{z(P_D) = 2250}$ with $t, u = 0$ (binding constraints), $s = 25$ (non-binding constraint).



Simplex steps for the given example (taken from [?]).

Exercises

Solve, using the Simplex method, the Exercises 3 through 6 of the previous session.

Summary

Artificial variables

When adding slack variables, we end up finding an initial feasible set of basic variables. But what happens when not all constraints are of the form "$\leq$"?[**?**]

1. Make all right hand sides b_i of the constraint (in)equalities positive. If they are not, multiply the expression by (-1).
2. If we end up just with slack variables, they can be directly used as initial basic variables. If the expressions are of the type $\geq$ the selection of initial basic variables is not trivial.
3. We then create **artificial variables**, as a trick to create a starting basic solution

$$\begin{array}{ll} \text{maximize} & z = x_1 + 3x_2 \\ \text{subject to} & \begin{array}{rcl} 2x_1 - x_2 & \leq & -1 \\ x_1 + x_2 & = & 3 \\ x_1, x_2 & \geq & 0 \end{array} \end{array}$$

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By leaving the right hand side of the constraints positive and adding a surplus variable, we end up having

$$\text{subject to} \quad \begin{array}{rcl} -2x_1 + x_2 - s_1 & = & 1 \\ x_1 + x_2 & = & 3 \end{array}$$

and now we introduce the artificial variables to create an initial solution:

$$\text{subject to} \quad \begin{array}{rcl} -2x_1 + x_2 - s_1 + R_1 & = & 1 \\ x_1 + x_2 + R_2 & = & 3 \end{array}$$

with all variables non-negative. Recall that, in fact, $R_1 = R_2 = 0$, so they will be just used temporarily.

The Two-Phase method I

1. First we will use the Simplex method to minimize the values of R_1 and R_2.
2. If they reach the value of zero, there is a solution for the original problem. If not, there is no feasible solution to the original problem.

In the above example, we want to minimize the function $z_R = R_1 + R_2$ or, what is the same, we want to maximize the function $z_R = -R_1 - R_2$

	x_1	x_2	s_1	R_1	R_2	*Solution*
z_R	0	0	0	1	1	0
R_1	-2	1	-1	1	0	1
R_2	1	1	0	0	1	3

The Two-Phase method II

Next, we transform the problem in order to get basic variables (columns of one 1 and the rest zeros). For example, by using the second and third row to make appear 0 in the first row for the artificial variables:

	x_1	x_2	s_1	R_1	R_2	*Solution*
z_R	1	-2	1	0	0	-4
R_1	-2	1	-1	1	0	1
R_2	1	1	0	0	1	3

We can solve this problem with the Simplex method, obtaining a tableau:

	x_1	x_2	s_1	R_1	R_2	*Solution*
z_R	0	0	0	1	1	0
x_2	0	1	$-1/3$	$1/3$	$2/3$	$7/3$
x_1	1	0	$1/3$	$-1/3$	$1/3$	$2/3$

The Two-Phase method III

which provides the optimal solution for Phase 1. This means that the original problem has solution (as $R_1 = R_2 = 0$). In Phase 2, we will replace the first row by the original maximization problem and leave out the artificial variables, solving the now standard Simplex problem:

	x_1	x_2	s_1	*Solution*
z	-1	-3	0	0
x_2	0	1	$-1/3$	$7/3$
x_1	1	0	$1/3$	$2/3$

that leads to

	x_1	x_2	s_1	*Solution*
z	2	0	0	9
x_2	1	1	0	3
s_1	3	0	1	2

Draw the graphical solution of this problem and interpret the results.

Information in the tableau

Some special cases:

Multiple optimal solutions If a zero appears in the objective function row corresponding to a non-basic variable, then that non-basic variable can enter the basis without changing the value of the objective function. So, two adjacent points have the same objective function value.

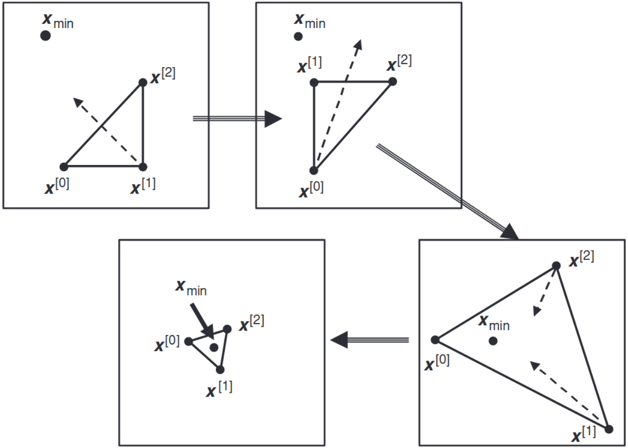
Unbounded solution If in any tableau the constraint coefficients corresponding to a non-basic variable are all either negative or zero, then that non-basic variable can be increased arbitrarily without violating any constraint. Thus, the feasible region is unbounded in the direction of that variable.

Degenerate solutions A solution to a linear programming problem is said to be degenerate if one or more of the basic variables has a value of zero. This shows redundancy in the constraints.

Summary

Algorithm (see example in the next page):

1. In $\mathbf{R}^2$, we form a triangle with vertices at $x^{[0]}$ and the two points $x^{[1]} = x^{[0]} + l_1 e^{[1]}$ and $x^{[2]} = x^{[0]} + l_2 e^{[2]}$. In $\mathbf{R}^N$ the triangle becomes a simplex with $N + 1$ vertices.
2. We aim at moving $x^{[0]}, x^{[1]}, x^{[2]}, \ldots$ so that the triangle comes to enclose a local minimum $x_{\min}$ while shrinking in size. First, the vertex of highest cost function value is moved in the direction of the **triangle's center** towards a region where we expect the cost function values to be lower. After a sequence of such moves, the triangle eventually contains in its interior a local minimum.
3. The size of the triangle is progressively reduced until it tightly bounds the local minimum, as the vertex moves are most likely to succeed when the triangle is small enough that $F(x)$ varies linearly over the triangle region.



The simplex method moves the vertices of the simplex (in two dimensions, a triangle) until it contains the local minimum, and shrinks the simplex to find the local minimum (taken from [?]).

Summary

References

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