1 Initial solutions for General Constraints: Artificial Variables

additional material: artif.pdf

When adding slack variables, we end up finding an initial feasible set of basic variables. But what happens when not all constraints are of the form "$\leq$"?[1]

1. Make all right hand sides b_i of the constraint (in)equalities positive. If they are not, multiply the expression by (-1).
2. If we end up just with slack variables, they can be directly used as initial basic variables. If the expressions are of the type $\geq$ the selection of initial basic variables is not trivial.
3. We then create **artificial variables**, as a trick to create a starting basic solution

$$\begin{array}{ll} \text{maximize} & z = x_1 + 3x_2 \\ \text{subject to} & \begin{array}{rcl} 2x_1 - x_2 & \leq & -1 \\ x_1 + x_2 & = & 3 \\ x_1, x_2 & \geq & 0 \end{array} \end{array}$$

By leaving the right hand side of the constraints positive and adding a surplus variable, we end up having

$$\text{subject to} \quad \begin{array}{rcl} -2x_1 + x_2 - s_1 & = & 1 \\ x_1 + x_2 & = & 3 \end{array}$$

and now we introduce the artificial variables to create an initial solution:

$$\text{subject to} \quad \begin{array}{rcl} -2x_1 + x_2 - s_1 + R_1 & = & 1 \\ x_1 + x_2 + R_2 & = & 3 \end{array}$$

with all variables non-negative. Recall that, in fact, $R_1 = R_2 = 0$, so they will be just used temporarily.

There are two main methods to solve the problem: the two-phase method, explained next, and the big-M method, that we are not explaining here.

1. First we will use the Simplex method to minimize the values of R_1 and R_2.
2. If they reach the value of zero, there is a solution for the original problem. If not, there is no feasible solution to the original problem.

In the above example, we want to minimize the function $z_R = R_1 + R_2$ or, what is the same, we want to maximize the function $z_R = -R_1 - R_2$

	x_1	x_2	s_1	R_1	R_2	*Solution*
z_R	0	0	0	1	1	0
R_1	-2	1	-1	1	0	1
R_2	1	1	0	0	1	3

Next, we transform the problem in order to get basic variables (columns of one 1 and the rest zeros). For example, by using the second and third row to make appear 0 in the first row for the artificial variables:

	x_1	x_2	s_1	R_1	R_2	*Solution*
z_R	1	-2	1	0	0	-4
R_1	-2	1	-1	1	0	1
R_2	1	1	0	0	1	3

We can solve this problem with the Simplex method, obtaining a tableau:

	x_1	x_2	s_1	R_1	R_2	*Solution*
z_R	0	0	0	1	1	0
x_2	0	1	$-1/3$	$1/3$	$2/3$	$7/3$
x_1	1	0	$1/3$	$-1/3$	$1/3$	$2/3$

which provides the optimal solution for Phase 1. This means that the original problem has solution (as $R_1 = R_2 = 0$). In Phase 2, we will replace the first row by the original maximization problem and leave out the artificial variables, solving the now standard Simplex problem:

	x_1	x_2	s_1	*Solution*
z	-1	-3	0	0
x_2	0	1	$-1/3$	$7/3$
x_1	1	0	$1/3$	$2/3$

that leads to

	x_1	x_2	s_1	*Solution*
z	2	0	0	9
x_2	1	1	0	3
s_1	3	0	1	2

Draw the graphical solution of this problem and interpret the results.

References

[1] Michael Carter, Camille C Price, and Ghaith Rabadi. *Operations Research. A Practical Introduction. Second Edition.* CRC Press, 2019.