Unit 4. Integer Programming

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Preliminary

This course is strongly based on the monography on Operations Research by Carter, Price and Rabadi [1], and in material obtained from different sources (quoted when needed through the slides).

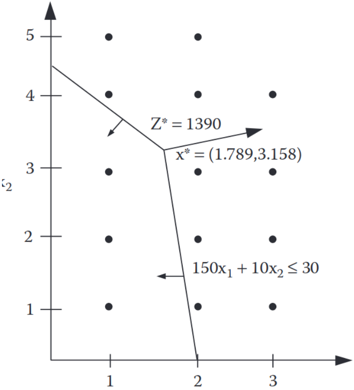
Learning outcomes

- Getting familiar with the use of integer programming
- Solving integer programming problems

The concept

- Problems in which the feasible set is composed of only integer values.
- The feasible set is neither continuous nor feasible.
- NP-hard problems in general.
- Integer problems that have a network structure are easy to solve using the Simplex method (assignment and matching problems, transportation and transshipment problems, and network flow problems always produce integer results, provided that the problem bounds are integers).
- Rounding can be effective in some problems and clearly not in others:
  - not the same tires than aircrafts!
  - values 0/1 for variable: zero-one or binary integer programming (produce or not produce cars in this factory)
  - mixed integer programming problems

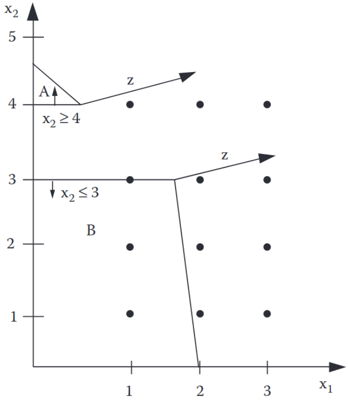
General integer programming problems



Graphical representation of a typical 2 dimensional integer programming[1].

Zero-One (0-1) problems

Airline crew scheduling problem: The airlines first design a flight schedule composed of a large number of flight legs. A flight leg is a specific flight on a specific piece of equipment, such as a 747 from New York to Chicago departing at 6:27 a.m. A flight crew is a complete set of people, including pilots, navigator, and flight attendants who are trained for a specific airplane. A work schedule or rotation is a collection of flight legs that are feasible for a flight crew, and that normally terminate at the point of origin. Variables x_{ij} have value 1 if flight leg i is assigned to crew j. All flight legs should be covered at minimum total cost.



Separation into two subproblems in the Branch-and-Bound method[1].

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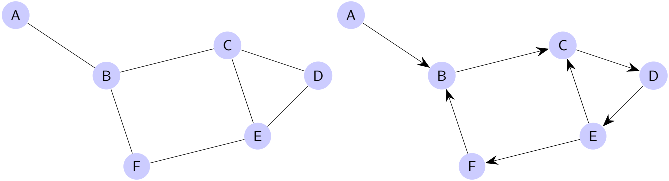
Summary

Graphs and networks: definitions



- A graph is a structure consisting of a set of nodes (vertices, points, or junctions) and a set of connections called arcs (edges, links, or branches).
- Each connection is associated with a pair of nodes and is usually drawn as a line joining two points. The graph can be defined as directed or undirected.
- The degree of a node is the number of arcs attached to it. An isolated node is of degree zero.
- In a directed graph, the arc is often designated by the ordered pair (A, B).

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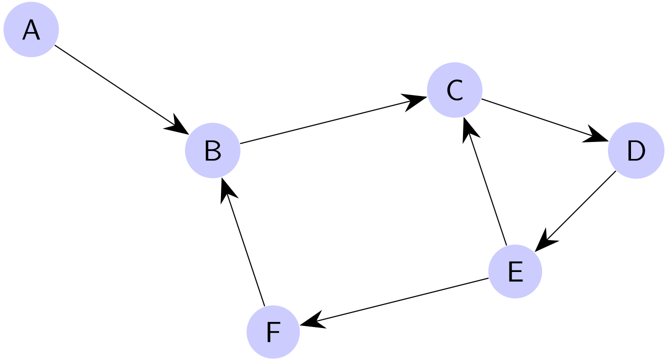
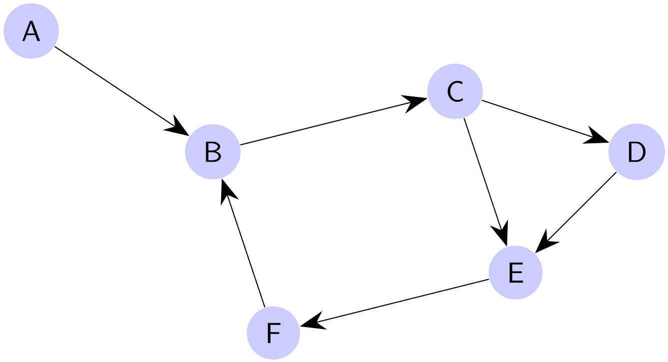
Paths



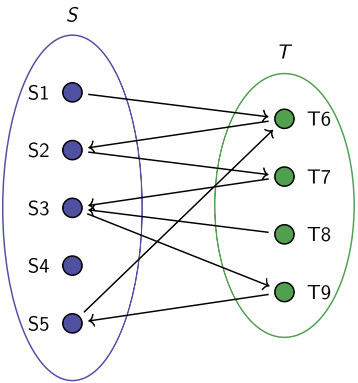
$$A, (A, B), B, (B, C), \underbrace{C, (C, D), D, (D, E), E, (E, C), C}_{\textit{cyclic}}$$

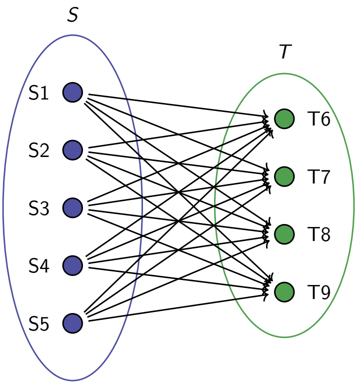


$$A, (A, B), B, (B, C), \underbrace{C, (C, D), D, (D, E), E, (C, E), C}_{\textit{noncyclic}}$$

- If all the arcs in a path are forward arcs, the path is called **directed chain** or simply **chain**.
- **path** and **chain** are synonimous if the graph is undirected.
- In the second example above we saw a cyclic path but not a cyclic chain, as it included the backward arc (C, E).
- A **connected graph** has at least one path connecting every pair of nodes.
- In a **bipartite graph** the nodes can be partitioned into two subsets S and T, such that each node is in exactly one of the subsets, and every arc in the graph connects a node in set S with a node in set T.
- Such a graph is **complete bipartite** if each node in S is connected to every node in T.

Bipartite vs complete bipartite graphs



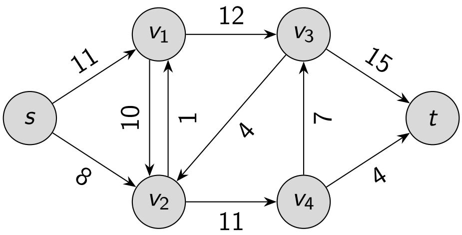


- A **tree** is a directed connected graph in which each node has at most on predecessor, and one node (the root node) has none. In an undirected graph, we have a tree if the graph is connected and contains no cycles.
- A **network** is a directed connected graph that is used to model/represent a system/process. The arcs are typically assigned weights representing cost, value or capacity corresponding to each link.
- Nodes in networks can be designated as **sources** or **sinks**. A **cut set** is any set of arcs which, if removed from the network, would disconnet the sources(s) from the sink(s).
- **Flow** can be thought of as the total amount of an entity that originates at the source, makes it through the different nodes and reaches the sink.

Summary

Maximum flow in networks

Determine the maximum possible flow that can be routed through the various network links, from source to sink, without violating the capacity constraints.
Important!: the commodity is only generated at the source and consumed at the sink.



The **maximum flow problem** can be stated as a LP formulation.

$$
\begin{array}{lrcl}
\text{maximize} & z = f & & \\
\text{subject to} & \sum_{i=2}^{n} x_{1i} = f & & \\
& \sum_{i=1}^{n-1} x_{in} = f & & \\
& \sum_{i=1}^{n} x_{ij} = \sum_{k=1}^{n} x_{jk}, & \text{for} & j = 2, 3, \ldots, n-1 \\
& x_{ij} \leq u_{ij}, & \text{for all} & i, j = 1, 2, \ldots, n
\end{array}
$$

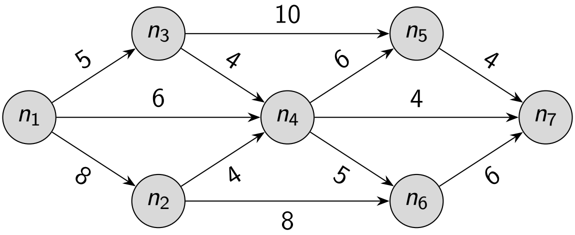
Maximum flow algorithm

All network problems here can be solved using the Simplex method, but the network structure can help us solving it more efficiently. In the **Ford-Fulkerson labelling algorithm**:

1. Use a labelling procedure to look for a flow augmenting path. If none can be found, stop; the current flow is optimal;
2. Increase the current flow as much as possible in the flow augmenting path, until reaching capacity of some arc. Come back to step 1.

Exercise 1

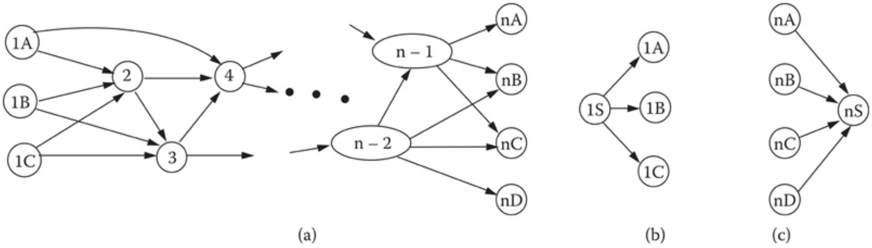
Find the maximum flow in this network using the Ford-Fulkerson labelling algorithm:



- In any network, there is always a bottleneck that in some sense impedes the flow through the network.
- The total capacity of the bottleneck is an upper bound on the total flow in the network.
- Cut sets are, by definition, essential in order for there to be a flow from source to sink, since removal of the cut set links would render the sink unreachable from the source.
- The capacities on the links in any cut set potentially limit the total flow.
- The minimum cut (i.e., the cut set with minimum total capacity) is in fact the bottleneck that precisely determines the maximum possible flow in the network (Max-Flow Min-Cut Theorem): the capacity of the cut is precisely equal to the current flow and this flow is optimal. In other words, a saturated cut defines the maximum flow.

Multiple sinks and sources

We can generate a supersource or a supersink node with unlimited capacity and repeat the process of optimization as above:



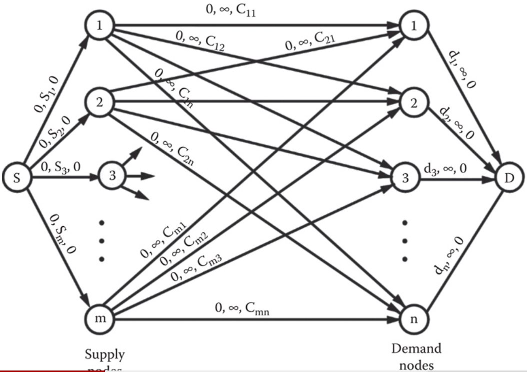
Summary

Transportation problem

- Useful when there are costs associated with the flow, given a link capacity.
- Let us assume that every node is a source (supply) and a sink (demand). Imagine a distributor with several warehouses and a group of costumers. Serving each customer from a given warehouse has an associated cost.
- For m supply nodes, each providing s_i suplly, and n demand nodes, each demanding d_j. Assuming that the total demand eaquals the total supply: $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$ we aim at satisfying the demand using the available supply minimizing cost routes.

$$\text{minimize} \quad z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to} \quad \begin{array}{lll} \sum_{j=1}^{n} x_{ij} = s_i & \text{for} & i = 1, \ldots, m \\ \sum_{i=1}^{m} x_{ij} = d_j & \text{for} & j = 1, \ldots, n \\ x_{ij} \geq 0 & \text{for all} & i, j \end{array}$$



Exercise 2

Find the minimum cost in this trasportation problem:

Sources (Warehouses)	Sinks (Customers) 1	2	3	4	5	Supply
1	28	7	16	2	30	20
2	18	8	14	4	20	20
3	10	12	13	5	28	25
Demand	12	14	12	18	9	65

NOTE: The Simplex method says that we should first find any basic feasible solution and then look for a simple pivot to improve the solution. repeat until the optimal solution is found.

Optimizing

1. Finding initial solution.
    - Northwest corner rule
    - Minimum cost method
    - Minimum ”row” cost method
    - Vogel's method
2. Transportation simplex

Transportation simplex

Once we have any feasible solution, we aim at finding the optimal one. Consider:

Minimum Row Cost Final Solution

Sources (Warehouses)	Sinks (Customers) 1	2	3	4	5	Supply
1	$\boxed{28}$	2 $\boxed{7}$	$\boxed{16}$	18 $\boxed{2}$	$\boxed{30}$	20
2	8 $\boxed{18}$	12 $\boxed{8}$	$\boxed{14}$	$\boxed{4}$	$\boxed{20}$	20
3	4 $\boxed{10}$	$\boxed{12}$	12 $\boxed{13}$	$\boxed{5}$	9 $\boxed{28}$	25
Demand	12	14	12	18	9	65

Transportation simplex

We can reduce the total cost by reducing the individual costs in every row i by u_i and in every column j by v_j:

$$c'_{ij} = c_{ij} - u_i - v_j$$

Check that, now:

- $\sum_i \sum_j x_{ij} c'_{ij} = 0$
- Check how some costs are now negative in non-basic cells.
- If we increase the number of units in those non-basic cells from 0 to some value, reducing at the same time the number of units in the basic cells, we can reduce the overall cost $\sum_i \sum_j x_{ij} c_{ij}$

Transportation simplex

In practice:

1. We find first an initial feasible solution as explained above
2. Calculate the u_i and v_j, taking into account that $c_{ij} = u_i + v_j$, for all basic variables (used squares in the table). We start by assigning $u_1 = 0$.
3. We calculate the *improvement index* by $I_{ij} = c_{ij} - u_i - v_j$ for all non-used squares in the table.
4. If all I_{ij} are positive, the solution is already optimal and we are done.
5. If some $I_{ij} < 0$, then build a loop with such value in the corner and alternative $\pm$ signs in all vertex.
6. Use the above $\pm$ to increase the number of units in the position that had $I_{ij} < 0$
7. We return to step 2.

Summary

Summary

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References

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