Solved Exercises

Optimization and Operations Research

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Contents

1 Non-linear optimization

Ex. 1 — A problem of non-linear optimization: A company wants to maximize its profit function given by:

$$f(x, y) = -x^2 + 4xy - 2y^2$$

where x and y represent the amount of resources allocated to two different projects.
The problem is subject to the following constraints:

1. The total resources used must not exceed 30 units.
2. The product of the resources allocated must be at least 50 units.
3. The relationship between the resource allocations must satisfy the following nonlinear constraint:

$$y \leq \frac{3x^2}{100} + 5$$

The company wants to find the values of x and y that maximize the profit $f(x, y)$ under these constraints.
Can you help the company drawing the problem in a graph? Can you identify the feasible region? Is the region convex?

Answer (Ex. 1) — A company wants to maximize its profit function given by:

$$f(x, y) = -x^2 + 4xy - 2y^2$$

where x and y represent the amount of resources allocated to two different projects.
The problem is subject to the following constraints:

1. The total resources used must not exceed 30 units:

$$x + 2y \leq 30$$

2. The product of the resources allocated must be at least 50 units:

$$xy \geq 50$$

3. The relationship between the resource allocations must satisfy the following nonlinear constraint:

$$y \leq \frac{3x^2}{100} + 5$$

Find the values of x and y that maximize the profit $f(x, y)$ under these constraints. Before solving it, we can draw the problem in a graph to identify the feasible region and the optimal solution. The problem is shown in Figure 1.

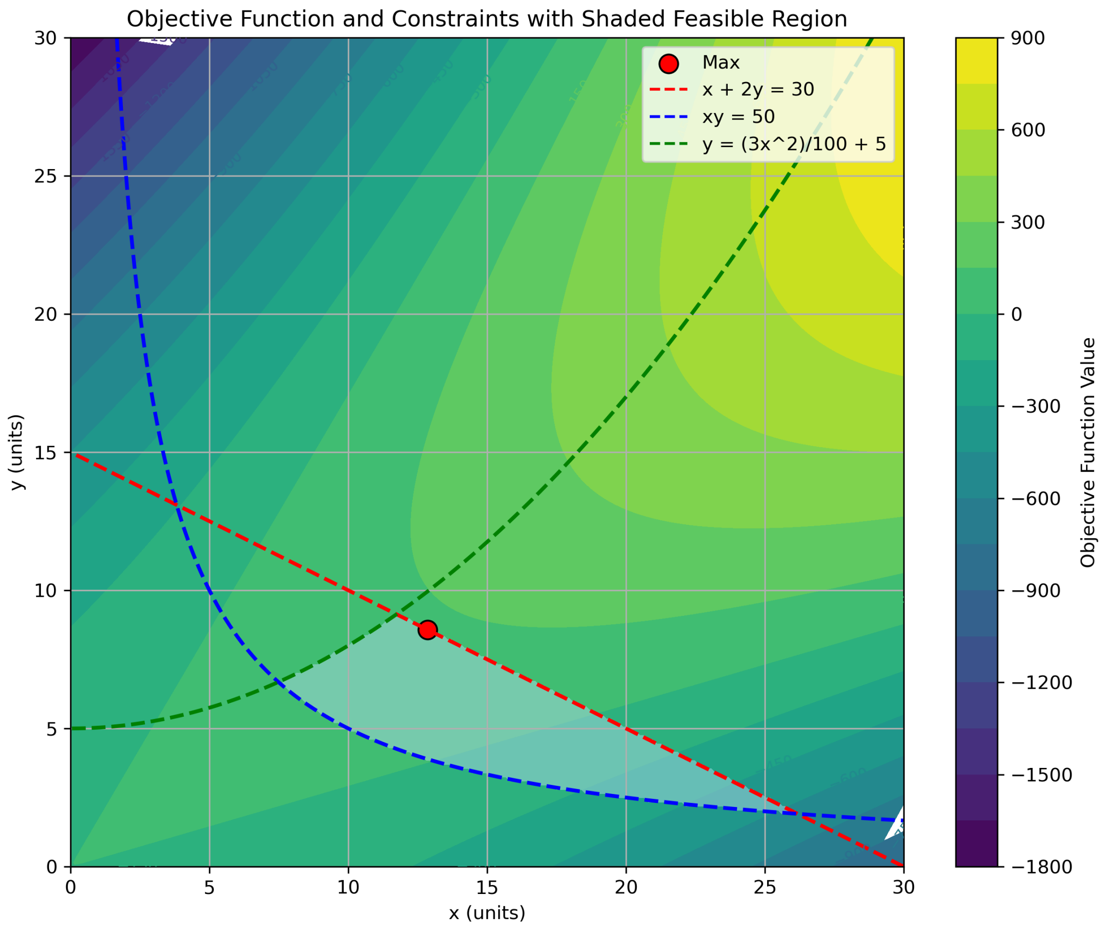


Figure 1: Contour plot of the objective function $f(x, y) = -x^2 + 4xy - 2y^2$ with constraints $x + 2y \leq 30$ (red dashed), $xy \geq 50$ (blue dashed), and $y \leq \frac{3x^2}{100} + 5$ (green dashed). The light blue region represents the feasible area where all constraints are satisfied. The solution has been obtained with Code 1

■

1.1 Unconstrained non-linear optimization

Ex. 2 — Quadratic function minimization: Consider the quadratic function $f(x) = 2x^2 - 4x + 1$. Determine the coordinates of the vertex of the parabola and state whether it represents a minimum or maximum. Compute the minimum (or maximum) value of $f(x)$.

Answer (Ex. 2) — The vertex is at $x = 1$, corresponding to a minimum since the coefficient of x^2 is positive. The minimum value is $f(1) = -1$.

Ex. 3 — Cubic function extrema: Find all critical points of the function $f(x) = x^3 - 3x + 1$ and determine which of them correspond to local minima and maxima.

Answer (Ex. 3) — $f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$. $f''(x) = 6x \Rightarrow f''(1) > 0 \to$ local minimum at $x = 1$; $f''(-1) < 0 \to$ local maximum at $x = -1$. Minimum value: $f(1) = -1$.

Ex. 4 — Stationary point of a quadratic function: Determine the stationary point of $f(x, y) = 3x^2 + 2y^2 - 6x - 8y$ and classify it.

Answer (Ex. 4) — Setting partial derivatives to zero: $6x - 6 = 0 \Rightarrow x = 1$, $4y - 8 = 0 \Rightarrow y = 2$. The point $(1, 2)$ is a minimum.

Ex. 5 — Quadratic maximization with bounds: Maximize $f(x) = -x^2 + 4x$ for $0 \leq x \leq 3$. Determine where the maximum occurs and its value.

Answer (Ex. 5) — $f'(x) = -2x + 4 = 0 \Rightarrow x = 2$. $f(2) = 4$ is the maximum value, since $f(0) = 0$ and $f(3) = 3$.

Ex. 6 — Hessian at a minimum: Suppose $f(x)$ is a twice continuously differentiable function with a global minimum at $x = x^*$. What can be said about the Hessian matrix $H(x^*)$?

Answer (Ex. 6) — At a global minimum, the Hessian must be **positive definite**. All eigenvalues are positive, indicating the surface curves upward in all directions. If the Hessian were negative definite, the point would be a local maximum; if indefinite, a saddle. ■

Ex. 7 — **Hessian concavity criteria:** Given the function $f(x, y) = x^3 + 2y^3 - xy$, determine whether the function at $(0, 0)$ is convex, concave, or neither.

Answer (Ex. 7) — The Hessian is

$$H = \begin{pmatrix} 6x & -1 \\ -1 & 12y \end{pmatrix}.$$

At $(0, 0)$:

$$H(0, 0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Its eigenvalues are $\lambda = 1$ and $\lambda = -1$, so H is **indefinite**. Hence, the function is **neither convex nor concave** at $(0, 0)$ (it has a saddle-type curvature). ■

Ex. 8 — **Golden Section Search:** When applying the Golden Section Search to find the maximum of $f(x)$, the condition $f(x_1) > f(x_2)$ holds for the intermediate points $(x_1 < x_2)$. Explain which region is eliminated from the search interval and why.

Answer (Ex. 8) — Since $f(x_1) > f(x_2)$, the function is decreasing after x_1. Thus, the maximum lies between x_l and x_2, and the interval $[x_2, x_u]$ can be eliminated. The new search region is $[x_l, x_2]$. ■

Ex. 9 — **Newton's Method:** In Newton's method for unconstrained optimization, the search direction is $p_k = -H_k^{-1}\nabla f(x_k)$. Explain under which conditions this direction corresponds to a descent direction.

Answer (Ex. 9) — If the Hessian H_k is **positive definite**, then p_k points in a descent direction (toward a local minimum). If H_k is **negative definite**, the algorithm moves toward a local maximum. If H_k is **indefinite**, the step may not be a descent direction. ■

Ex. 10 — **Gradient Descent:** In the gradient descent method with update $x_{k+1} = x_k - \alpha_k \nabla f(x_k)$, explain the role of the step size α_k.

Answer (Ex. 10) — The step is taken in the **negative gradient direction**, which points toward steepest descent. A small α_k leads to slow convergence, while a large α_k can cause divergence or oscillation. A properly chosen step size ensures steady convergence to a local minimum. ■

Ex. 11 — First- and Second-Order Optimality Conditions: State the first- and second-order conditions that must hold at a local minimum of a differentiable function $f(x)$.

Answer (Ex. 11) — At a local minimum x^*:

$$\nabla f(x^*) = 0 \quad \text{and} \quad H(x^*) \text{ is positive definite.}$$

If $H(x^*)$ is negative definite, the point is a local maximum; if indefinite, it is a saddle point. ■

Ex. 12 — Hessian and Convexity: Explain how the definiteness of the Hessian matrix determines whether a function is convex, concave, or neither.

Answer (Ex. 12) — If $H(x)$ is **positive semi-definite** for all x, the function is convex. If $H(x)$ is **negative semi-definite**, it is concave. If $H(x)$ is **indefinite**, the function is neither convex nor concave. A **singular** Hessian may indicate flat regions or saddle behavior. ■

Ex. 13 — Critical points and curvature classification: Consider the function $f(x, y) = x^3 - 3x + y^2$. Find its critical points and determine their type.

Answer (Ex. 13) —

$$\frac{\partial f}{\partial x} = 3x^2 - 3 = 0 \Rightarrow x = \pm 1, \qquad \frac{\partial f}{\partial y} = 2y = 0 \Rightarrow y = 0.$$

Thus, critical points: $(1, 0)$ and $(-1, 0)$.

$$H = \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix}.$$

At $(1, 0)$: $H = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \to$ positive definite $\to$ local minimum. At $(-1, 0)$: $H = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix} \to$ indefinite $\to$ saddle point. ■

Ex. 14 — Critical points of $\sin(2x) + \cos(y)$: Find and classify the critical points of $f(x, y) = \sin(2x) + \cos(y)$.

Answer (Ex. 14) —

$$\frac{\partial f}{\partial x} = 2\cos(2x), \quad \frac{\partial f}{\partial y} = -\sin(y).$$

Set both to zero: $\cos(2x) = 0$, $\sin(y) = 0$. Thus $x = \pi/4 + k\pi/2$, $y = n\pi$, with $k, n \in \mathbb{Z}$.

$$H = \begin{pmatrix} -4\sin(2x) & 0 \\ 0 & -\cos(y) \end{pmatrix}.$$

If $\sin(2x)$ and $\cos(y)$ have the same sign $\to$ H definite $\to$ local extrema. If signs differ $\to$ H indefinite $\to$ saddle points. ■

Ex. 15 — Critical points of $e^{x\cos y}$: Find and classify the critical points of $f(x, y) = e^{x\cos y}$.

Answer (Ex. 15) —

$$\frac{\partial f}{\partial x} = e^{x\cos y}\cos y, \qquad \frac{\partial f}{\partial y} = -xe^{x\cos y}\sin y.$$

Setting both to zero gives $x = 0$, $\sin y = 0 \Rightarrow y = n\pi$, $n \in \mathbb{Z}$. Thus, critical points are $(0, n\pi)$.

$$H = \begin{pmatrix} \cos y\, e^{x\cos y} & -x\sin y\, e^{x\cos y} \\ -x\sin y\, e^{x\cos y} & -x\cos y\, e^{x\cos y} \end{pmatrix}.$$

At $(0, n\pi)$:

$$H = \begin{pmatrix} (-1)^n & 0 \\ 0 & 0 \end{pmatrix}.$$

One eigenvalue is zero $\to$ Hessian is **singular**. The function is flat in the y-direction at these points. ■

1.2 Optimization with constraints

1.2.1 Lagrange Multipliers

Ex. 16 — Lagrange multipliers: Optimization of $f(x, y) = x^2 - y$ subject to $x^2 + y^2 = 4$

Answer (Ex. 16) — We will use the method of Lagrange multipliers. Define the constraint as $g(x, y) = x^2 + y^2 - 4 = 0$. The Lagrange function is then:

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda \cdot g(x, y) = x^2 - y + \lambda(x^2 + y^2 - 4).$$

Note that we have used $+\lambda \cdot g(x, y)$ in the equation above, but we could also use $-\lambda \cdot g(x, y)$; the results would be equivalent. We now compute the partial derivatives of $\mathcal{L}(x, y, \lambda)$ with respect to x, y, and λ and equal them to zero to obtain the system of equations to solve:

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + \lambda \cdot 2x = 2x(1 + \lambda) = 0, \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -1 + \lambda \cdot 2y = 0 \quad \Rightarrow \quad \lambda = \frac{1}{2y}, \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - 4 = 0. \tag{3}$$

From Eq. 1, we see two possibilities:

- $x = 0$: Substitute $x = 0$ in Eq. 3:

  $$0^2 + y^2 = 4 \quad \Rightarrow \quad y^2 = 4 \quad \Rightarrow \quad y = \pm 2.$$

  For $y = 2$, substitute into $f(x, y) = x^2 - y$:

  $$f(0, 2) = 0^2 - 2 = -2.$$

  For $y = -2$, substitute into $f(x, y)$:

  $$f(0, -2) = 0^2 - (-2) = 2.$$

- $1 + \lambda = 0 \quad \Rightarrow \quad \lambda = -1$. Substitute $\lambda = -1$ into Eq. 2:

  $$-1 = \frac{1}{2y} \quad \Rightarrow \quad y = -\frac{1}{2}.$$

  Now, substitute $y = -\frac{1}{2}$ into the constraint equation $x^2 + y^2 = 4$:

  $$x^2 + \left(-\frac{1}{2}\right)^2 = 4 \quad \Rightarrow \quad x^2 + \frac{1}{4} = 4 \quad \Rightarrow \quad x^2 = \frac{15}{4} \quad \Rightarrow \quad x = \pm\frac{\sqrt{15}}{2}.$$

Now, calculate $f(x,y) = x^2 - y$ for $y = -\frac{1}{2}$ and $x = \pm\frac{\sqrt{15}}{2}$:
For $x = \frac{\sqrt{15}}{2}$:

$$f\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) = \left(\frac{\sqrt{15}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{15}{4} + \frac{1}{2} = \frac{17}{4}.$$

For $x = -\frac{\sqrt{15}}{2}$:

$$f\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) = \left(-\frac{\sqrt{15}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{15}{4} + \frac{1}{2} = \frac{17}{4}.$$

We now compare the function values:

- For $(0,2)$: $f(0,2) = -2$.
- For $(0,-2)$: $f(0,-2) = 2$.
- For $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$: $f = \frac{17}{4}$.

Thus:

- The maximum value is $f = \frac{17}{4}$ at $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$.
- The minimum value is $f = -2$ at $(0,2)$.

For a Python implementation check this file.

Ex. 17 — Lagrange multipliers with linear constraint: Solve the constrained optimization problem:

$$\min f(x,y) = x^2 + y^2 \quad \text{subject to} \quad x + 2y = 4.$$

Find all stationary points and determine which gives the minimum value.

Answer (Ex. 17) — Using the Lagrange multiplier method:

$$\nabla f = (2x, 2y), \quad \nabla g = (1,2) \Rightarrow 2x = \lambda,\ 2y = 2\lambda.$$

From the constraint: $x + 2y = 4$. Solving gives $x = \frac{4}{5}$, $y = \frac{8}{5}$. This is the minimum point.

Ex. 18 — Constrained product minimization: Find the minimum and maximum values of $f(x,y) = xy$ subject to $x^2 + y^2 = 9$.

Answer (Ex. 18) — Using Lagrange multipliers: $y = \lambda 2x$, $x = \lambda 2y \Rightarrow x^2 = y^2 \Rightarrow y = \pm x$. For $y = x$: $2x^2 = 9 \Rightarrow x = \pm\frac{3}{\sqrt{2}} \Rightarrow f = \frac{9}{2}$. For $y = -x$: $f = -\frac{9}{2}$. Minimum value $-\frac{9}{2}$, maximum $\frac{9}{2}$.

Ex. 19 — **Optimization with equality constraint:** Use the Lagrange multiplier method to find the values of x and y that minimize $f(x, y) = x^2 + y^2$ subject to $x + y = 1$.

Answer (Ex. 19) — From the constraint: $y = 1 - x$. Then $f(x) = x^2 + (1-x)^2 = 2x^2 - 2x + 1$. Minimum at $x = \frac{1}{2}$, $y = \frac{1}{2}$.

1.2.2 Karush-Kuhn-Tucker theorem

Ex. 20 — **Complementary slackness computation:** Consider minimizing $f(x) = x_1^2 + x_2^2$ subject to $x_1 + x_2 \geq 2$ and $x_1, x_2 \geq 0$. Use KKT conditions (with multipliers) and illustrate complementary slackness: identify active constraints and multipliers.

Answer (Ex. 20) — Set constraint as $g(x) = 2-(x_1+x_2) \leq 0$. Lagrangian $L = x_1^2 + x_2^2 + \lambda(2 - x_1 - x_2) - \mu_1 x_1 - \mu_2 x_2$.
KKT conditions:

Stationarity:

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda - \mu_1 = 0$$
$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda - \mu_2 = 0$$

Primal feasibility:

$$x_1 + x_2 \geq 2, \quad x_1 \geq 0, \quad x_2 \geq 0$$

Dual feasibility:

$$\lambda \geq 0, \quad \mu_1 \geq 0, \quad \mu_2 \geq 0$$

Complementary slackness:

$$\lambda(2 - x_1 - x_2) = 0$$
$$\mu_1 x_1 = 0$$
$$\mu_2 x_2 = 0$$

Solving: Try symmetric solution $x_1 = x_2 = a$. Then $2a \geq 2 \implies a \geq 1$.
Try $a = 1$ (active constraint).
Plug into stationarity:

$$2x_1 - \lambda - \mu_1 = 0 \implies 2 - \lambda - \mu_1 = 0 2x_2 - \lambda - \mu_2 = 0 \implies 2 - \lambda - \mu_2 = 0$$

By symmetry, $\mu_1 = \mu_2 = \mu$, so $2 - \lambda - \mu = 0 \implies \lambda = 2 - \mu$.
From complementary slackness: $x_1 = x_2 = 1 > 0 \implies \mu_1 = \mu_2 = 0$.
So $\lambda = 2$.
Check complementary slackness for λ: $2 - x_1 - x_2 = 0$, so $\lambda \cdot 0 = 0$.
Summary:

- Active constraint: $x_1 + x_2 = 2$ ($g(x) = 0$), multiplier $\lambda = 2$.
- Inactive constraints: $x_1 = 1 > 0$, $x_2 = 1 > 0$, multipliers $\mu_1 = \mu_2 = 0$.
- All KKT conditions are satisfied.

Interpretation: The only active constraint is $x_1 + x_2 \geq 2$, and its multiplier is positive. The nonnegativity constraints are inactive, so their multipliers are zero. This illustrates complementary slackness: only active constraints can have nonzero multipliers.

Ex. 21 — KKT necessity for optimality: For the constrained problem $\min f(x) = x^2 + y^2$ subject to $x + y = 1$, write the KKT (which reduce to Lagrange) conditions and solve. Explain why KKT are necessary.

Answer (Ex. 21) — Lagrangian $L = x^2 + y^2 + \lambda(x + y - 1)$. Stationarity: $2x + \lambda = 0,\ 2y + \lambda = 0 \to$ x=y. Constraint gives x=y=1/2. KKT (here Lagrange) are necessary to characterize optima under differentiability and constraint qualifications.

Ex. 22 — KKT application domain: Give a nonlinear optimization problem where KKT applies: minimize $f(x) = x_1^2 + 4x_2^2$ subject to $x_1 + x_2 \leq 3,\ x_i \geq 0$. Use KKT to find candidate minimizers.

Answer (Ex. 22) — Formulate KKT with multipliers λ for inequality. Solve stationarity: $2x_1 + \lambda - \mu_1 = 0,\ 8x_2 + \lambda - \mu_2 = 0$. Try interior (mu=0) and inactive constraint: if interior then lambda=0$\to$ x1=0,x2=0 but constraint 0<=3 satisfied; 0 is minimizer.

Ex. 23 — Non-negativity in KKT: Show with a small example how KKT enforces non-negativity constraints. Minimize $f(x) = x_1^2 + x_2^2$ subject to $x_1 \geq 0, x_2 \geq 0$. Find solution and relate to KKT multipliers.

Answer (Ex. 23) — Unconstrained minimum at (0,0) already satisfies non-negativity. KKT multipliers for inequality constraints are zero or positive; stationarity yields $2x_1 - \mu_1 = 0,\ 2x_2 - \mu_2 = 0$. At (0,0) mu>=0 can be zero; complementary slackness holds.

Ex. 24 — Multipliers as shadow prices: Consider minimize cost $f(x) = 2x_1 + 3x_2$ subject to $x_1 + x_2 \geq 5,\ x_i \geq 0$. Using KKT multipliers (dual viewpoint) interpret the multiplier for the constraint as a shadow price: compute multiplier and show sensitivity of optimal cost to RHS change.

Answer (Ex. 24) — Reformulate as equality at optimum: x1+x2=5 minimal cost with nonnegativity -> allocate to cheapest variable x1 (cost 2): x1=5,x2=0, cost=10. Small increase d in RHS increases minimal cost by multiplier equal to dual price; here dual multiplier = 2 (cost of resource) so sensitivity approx 2 per unit. (Interpretation)

Ex. 25 — Linear independence not KKT requirement: Explain with a short counterexample why linear independence (LICQ) is not a component of the KKT equalities themselves: state KKT conditions and show one of them is not 'linear independence'.

Answer (Ex. 25) — KKT conditions include stationarity, primal and dual feasibility, complementary slackness. LICQ is a constraint qualification (assumption) to guarantee KKT necessity/sufficiency. Thus LICQ is not part of KKT equations but an extra assumption—demonstrated by stating KKT and noting LICQ absent.

Ex. 26 — Convexity assumption for sufficiency: Provide a convex constrained example where KKT conditions are sufficient. Minimize $f(x) = x_1^2 + x_2^2$ subject to $x_1 + x_2 \geq 1,\ x_i \geq 0$. Use KKT to find minimizer and argue sufficiency because of convexity.

Answer (Ex. 26) — As earlier, constraint active yields x1=x2=0.5; KKT solved gives multipliers and unique minimizer. Because f and feasible set are convex, KKT conditions ensure global optimality.

Ex. 27 — KKT necessity vs sufficiency: Give an example showing that satisfying KKT is not always sufficient for optimality when convexity fails. Provide a nonconvex problem and describe a KKT point that is not global optimum.

Answer (Ex. 27) — Consider $f(x) = -x^2$ subject to $x^2 \leq 1$. The KKT stationary point $x = 0$ satisfies KKT but is not maximal (the endpoints $x = \pm 1$ give greater objective for maximization). This illustrates KKT is not sufficient without convexity.

Ex. 28 — KKT generalize Lagrange: Show algebraically that KKT conditions generalize Lagrange multipliers to inequalities by writing KKT for a problem with equality and inequality constraints. Use the problem $\min x^2$ s.t. $x \geq 1$ and show correspondence.

Answer (Ex. 28) — For $\min x^2$ s.t. $x \geq 1$, write $g(x) = 1 - x \leq 0$. Lagrangian $L = x^2 + \lambda(1 - x)$. Stationarity: $2x - \lambda = 0$ with $\lambda \geq 0$, complementary slackness: $\lambda(1 - x) = 0$. Solution $x = 1$, $\lambda = 2$ satisfies KKT; reduces to Lagrange when constraint is active as equality.

Ex. 29 — A problem of KKT conditions: Maximize $f(x, y) = xy$ subject to $100 \geq x + y$ and $x \leq 40$ and $(x, y) \geq 0$.

Answer (Ex. 29) — The Karush Kuhn Tucker (KKT) conditions for optimality are a set of necessary conditions for a solution to be optimal in a mathematical optimization problem. They are necessary and sufficient conditions for a local minimum in nonlinear programming problems. The KKT conditions consist of the following elements:
For an optimization problem in its standard form:

$$
\begin{aligned}
&\max f(\mathbf{x}) \\
&\text{s.t.} \quad \begin{array}{rcll} g_i(\mathbf{x}) - b_i & \leq & 0 & i = 1, \ldots, k \\ g_i(\mathbf{x}) - b_i & = & 0 & i = k+1, \ldots, m \end{array}
\end{aligned}
$$

There are 4 KKT conditions for optimal primal ($\mathbf{x}$) and dual (λ) variables. If $\mathbf{x}^*$ denotes optimal values:

1. Primal feasibility: all constraints must be satisfied: $g_i(\mathbf{x}^*) - b_i$ is feasible. Applies to both equality and non-equality constraints.

2. Gradient condition or No feasible descent: No possible improvement at the solution:

$$\nabla f(\mathbf{x}^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(\mathbf{x}^*) = 0$$

3. Complementariety slackness:

$$\lambda_i^* (g_i(\mathbf{x}^*) - b_i) = 0$$

4. Dual feasibility: $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active (=0) and a zero Lagrange multiplier when the constraint is inactive (>0).
To solve our problem, first we will put it in its standard form:

$$\max f(x,y) = xy$$
$$\text{s.t.} \quad \begin{aligned} g_1(x,y) = x + y - 100 &\leq 0 \\ g_2(x,y) = x - 40 &\leq 0 \\ g_3(x,y) = -x &\leq 0 \\ g_4(x,y) = -y &\leq 0 \end{aligned} \tag{4}$$

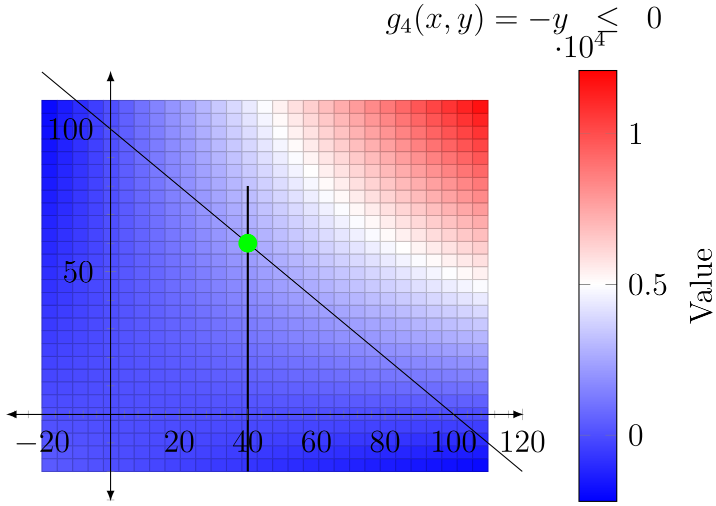


Figure 2: Contour plot of the objective function $f(x,y) = xy$ with constraints in Eq. 4, showing the final optimal value after applying the KKT conditions (green dot).
We will go through the different conditions:

- on the gradient:

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} - \lambda_1 \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_1}{\partial y} \end{pmatrix} - \lambda_2 \begin{pmatrix} \frac{\partial g_2}{\partial x} \\ \frac{\partial g_2}{\partial y} \end{pmatrix} - \lambda_3 \begin{pmatrix} \frac{\partial g_3}{\partial x} \\ \frac{\partial g_3}{\partial y} \end{pmatrix} - \lambda_4 \begin{pmatrix} \frac{\partial g_4}{\partial x} \\ \frac{\partial g_4}{\partial y} \end{pmatrix} = 0$$

which, in this example, resolves into:

$$y - (\lambda_1 + \lambda_2 - \lambda_3) = 0 \tag{5}$$

$$x - (\lambda_1 - \lambda_4) = 0 \tag{6}$$

- on the complementary slackness:

$$\lambda_1(x + y - 100) = 0 \tag{7}$$

$$\lambda_2(x - 40) = 0 \tag{8}$$

$$\lambda_3 x = 0 \tag{9}$$

$$\lambda_4 y = 0 \tag{10}$$

- on the constraints:

$$x + y \leq 100 \tag{11}$$

$$x \leq 40 \tag{12}$$

$$-x \leq 0 \tag{13}$$

$$-y \leq 0 \tag{14}$$

plus $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$.

We will start by checking Eq. 7:

–Let us see what occurs if $\lambda_1 = 0$. Then, from Eq. 6, $x + \lambda_4 = 0$ which implies that $x = \lambda_4 = 0$[1]. But, then, from Eq. 8 we obtain that $\lambda_2 = 0$ which, using Eq. 5 gives $y + \lambda_3 = 0 \Rightarrow y = \lambda_3 = 0$. Indeed, the KKT conditions are satisfied when all variables and multipliers are zero, but it is not a maximum of the function (see figure above).

–So, let us see what happens if $x + y - 100 = 0$ and consider the two possibilities for x:

[1] Recall that both variables and multiplieras must be positive or zero, so, the only possibility for the equation to fullfill is that both are zero.

Case $x = 0$**:**Then, $y = 100$, which would lead (Eq. 10) to $\lambda_4 = 0$ and (Eq. 6) to $x = \lambda_1 = 0$, that was discussed in the previous item. So, we need top explore the other possibility for x.

Case $x > 0$**:**From Eq. 9 $\lambda_3 = 0$ and, from Eqs. 5 and 6:

$$\begin{cases} \quad y = \lambda_1 + \lambda_2 \\ x \quad = \lambda_1 + \lambda_4 \end{cases}$$

let us try what happens if, e.g., $\lambda_2 \neq 0$ (or, said in other words, if constraint 12 is active): $x = 40$. As we know we do not want $\lambda_1 = 0$, from Eq. 7 we obtain $x + y - 100 = 0 \Rightarrow y = 60$.

The point $(x, y) = (40, 60)$ fullfills the KKT conditions and is a maximum in the constrained maximization problem (as can be seen in Figure 2).

Ex. 30 — **KKT conditions - Complete problem:** Solve the following optimization problem:[1]

$$\text{Optimize } f(x,y,z) = x + y + z,$$

$$\text{subject to: } \begin{cases} h_1(x,y,z) = (y-1)^2 + z^2 \leq 1, \\ h_2(x,y,z) = x^2 + (y-1)^2 + z^2 \leq 3. \end{cases}$$

Answer (Ex. 30) — By Weierstrass' theorem, there exist a maximum and a minimum for the problem. The Karush-Kuhn-Tucker (KKT) conditions are:

- Stationarity conditions:

$$\begin{cases} 1 + 2\mu_2 x = 0, \\ 1 + 2\mu_1(y-1) + 2\mu_2(y-1) = 0, \\ 1 + 2\mu_1 z + 2\mu_2 z = 0. \end{cases}$$

- Slackness conditions:

$$\begin{cases} \mu_1\left[(y-1)^2 + z^2 - 1\right] = 0, \\ \mu_2\left[x^2 + (y-1)^2 + z^2 - 3\right] = 0. \end{cases}$$

- Feasibility conditions:

$$(y-1)^2 + z^2 \le 1, \qquad x^2 + (y-1)^2 + z^2 \le 3.$$

- Sign conditions:

$$\mu_1, \mu_2 \ge 0 \Rightarrow \text{local minimum}, \qquad \mu_1, \mu_2 \le 0 \Rightarrow \text{local maximum}.$$

From the slackness conditions we distinguish four cases

$$\begin{cases} \mu_1 = 0 \implies \begin{cases} \mu_2 = 0 & \text{(Case I)} \\ x^2 + (y-1)^2 + z^2 - 3 = 0 & \text{(Case II)} \end{cases} \\ (y-1)^2 + z^2 - 1 = 0 \implies \begin{cases} \mu_2 = 0 & \text{(Case III)} \\ x^2 + (y-1)^2 + z^2 - 3 = 0 & \text{(Case IV)} \end{cases} \end{cases}$$

but the first stationarity equation implies that $\mu_2 \neq 0$, so only cases II and IV must be checked.

- Case II: $\mu_1 = 0$ and $x^2 + (y-1)^2 + z^2 - 3 = 0$.

$$\begin{aligned} P_1 &= (1, 2, 1), & \mu &= (0, -\tfrac{1}{2}), \\ P_2 &= (-1, 0, -1), & \mu &= (0, \tfrac{1}{2}). \end{aligned}$$

- Case IV: $(y-1)^2 + z^2 - 1 = 0$ and $x^2 + (y-1)^2 + z^2 - 3 = 0$.

$$\begin{aligned} P_3 &= \left(\sqrt{2}, 1 + \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), & \mu &= \left(-\tfrac{1}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}\right), \\ P_4 &= \left(\sqrt{2}, 1 - \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right), & \mu &= \left(\tfrac{3}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}\right), \\ P_5 &= \left(-\sqrt{2}, 1 + \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), & \mu &= \left(-\tfrac{3}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}\right), \\ P_6 &= \left(-\sqrt{2}, 1 - \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right), & \mu &= \left(\tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}\right). \end{aligned}$$

After applying feasibility and sign conditions, the results are summarized as follows:

P	μ	Feasibility	Sign	Conclusion
P_1	$(0, -\frac{1}{2})$	NO	–	–
P_2	$(0, \frac{1}{2})$	NO	–	–
P_3	$(-\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}})$	YES	$Negative$	Conditional Maximum
P_4	$(\frac{3}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}})$	YES	NO	–
P_5	$(-\frac{3}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$	YES	NO	–
P_6	$(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$	YES	$Positive$	Conditional Minimum

Therefore:

$$\boxed{\begin{aligned} \text{Conditional Maximum:} \quad & P_3 = \left(\sqrt{2}, 1 + \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \\ \text{Conditional Minimum:} \quad & P_6 = \left(-\sqrt{2}, 1 - \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right). \end{aligned}}$$

Here you have a MATLAB script that implements the KKT conditions to solve this problem:

```matlab
syms x y z mu1 mu2 real

% Objective function
f = x + y + z;

% Constraints
h1 = (y - 1)^2 + z^2 - 1;
h2 = x^2 + (y - 1)^2 + z^2 - 3;

% Stationary conditions
eq1 = 1 + 2*mu2*x == 0;
eq2 = 1 + 2*mu1*(y - 1) + 2*mu2*(y - 1) == 0;
eq3 = 1 + 2*mu1*z + 2*mu2*z == 0;

% Complementary slackness conditions
eq4 = mu1*h1 == 0;
eq5 = mu2*h2 == 0;

% Symbolic resolution (considering cases)
sol = solve([eq1, eq2, eq3, eq4, eq5], [x, y, z, mu1, mu2], 'Real', true);

disp(struct2table(sol))
```

2 Linear Optimization

Ex. 31 — Formulation of an optimization problem: Consider a manufacturing company that produces two products, P_1 and P_2. The company aims to maximize its profit. The profit from each product is given as follows:
- P_1: \$40 per unit - P_2: \$30 per unit
The company has the following constraints based on its resources:
1. The availability of raw material limits production to a maximum of 100 units of P_1 and 80 units of P_2. 2. The total production time available is 160 hours, where producing one unit of P_1 takes 2 hours and one unit of P_2 takes 1 hour.
Can you formulate the operations research problem?

Answer (Ex. 31) — The problem can be stated as:

$$\begin{aligned}
\text{Objective Function:} \quad & \text{Maximize } Z = 40x_1 + 30x_2 \\
\text{Subject to:} \quad & \\
& 2x_1 + x_2 \leq 160 \quad \text{(Total production time constraint)} \\
& x_1 \leq 100 \quad \text{(Raw material constraint for } P_1\text{)} \\
& x_2 \leq 80 \quad \text{(Raw material constraint for } P_2\text{)} \\
& x_1, x_2 \geq 0 \quad \text{(Non-negativity constraints)}
\end{aligned}$$

■

2.1 The Simplex method

Ex. 32 — Simplex optimization: — Use Simplex to solve this LP problem:

$$\begin{aligned}
\max \quad & Z = 1.0x_1 + 2.0x_2 \\
\text{s.t.} \quad & \begin{cases} -1.0x_1 + 1.0x_2 \leq 2.0 \\ 1.0x_1 + 2.0x_2 \leq 8.0 \\ 1.0x_1 + 0.0x_2 \leq 6.0 \end{cases}
\end{aligned}$$

Answer (Ex. 32) — Canonical Form:

$$Z - 1.0x_1 - 2.0x_2 - 0s_1 - 0s_2 = 0$$

$$\begin{cases} s_1 \quad\quad\quad - 1.0x_1 + 1.0x_2 = 2.0 \\ \quad s_2 \quad\quad + 1.0x_1 + 2.0x_2 = 8.0 \\ \quad\quad s_3 + 1.0x_1 + 0.0x_2 = 6.0 \end{cases}$$

Simplex tableau

Iteration 0 Entering and leaving variables:

- Entering variable: x_2
- Leaving variable: Basic variable from row 1

Table 1: Simplex Tableau after iteration 0

Basic	x_1	x_2	s_1	s_2	s_3	RHS
s_1	-1.00	**1.00**	1.00	0.00	0.00	2.00
s_2	1.00	2.00	0.00	1.00	0.00	8.00
s_3	1.00	0.00	0.00	0.00	1.00	6.00
Z	-1.00	**-2.00**	0.00	0.00	0.00	0.00

Iteration 1 Entering and leaving variables:

- Entering variable: x_1
- Leaving variable: Basic variable from row 2

Table 2: Simplex Tableau after iteration 1

Basic	x_1	x_2	s_1	s_2	s_3	RHS
x_2	-1.00	1.00	1.00	0.00	0.00	2.00
s_2	**3.00**	0.00	-2.00	1.00	0.00	4.00
s_3	1.00	0.00	0.00	0.00	1.00	6.00
Z	**-3.00**	0.00	2.00	0.00	0.00	4.00

Iteration 2 **Optimal solution reached**.

Table 3: Simplex Tableau after iteration 2

Basic	x_1	x_2	s_1	s_2	s_3	RHS
x_2	0.00	1.00	0.33	0.33	0.00	3.33
x_1	1.00	0.00	-0.67	0.33	0.00	1.33
s_3	0.00	0.00	0.67	-0.33	1.00	4.67
Z	0.00	0.00	0.00	1.00	0.00	8.00

Optimal solution: $x_1 = 1.33$, $x_2 = 3.33$

Optimal value (Simplex): 8.00
Optimal value (ORTools): 8.00

■

Ex. 33 — Simplex optimization: — Use Simplex to solve this LP problem:

$$\begin{array}{ll} \max & Z = 3.0x_1 + 1.0x_2 \\ \text{s.t.} & \begin{cases} 1.0x_1 + -1.0x_2 \leq -4.0 \\ -1.0x_1 + 2.0x_2 \leq -4.0 \end{cases} \end{array}$$

Answer (Ex. 33) — Canonical Form:

$$Z - 3.0x_1 - 1.0x_2 - 0s_1 - 0s_2 = 0$$

$$\begin{cases} s_1 \quad + 1.0x_1 - 1.0x_2 + \; = -4.0 \\ \quad s_2 - 1.0x_1 + 2.0x_2 \quad = -4.0 \end{cases}$$

Simplex tableau

Iteration 0 System unbound.

Table 4: Simplex Tableau after iteration 0

Basic	x_1	x_2	s_1	s_2	RHS
s_1	1.00	-1.00	1.00	0.00	-4.00
s_2	-1.00	2.00	0.00	1.00	-4.00
Z	-3.00	-1.00	0.00	0.00	0.00

No solution found (Simplex)

The ORTools solver did not find an optimal solution.

■

Ex. 34 — Feasible solution: Give a small linear program (two variables) and determine whether the point $(1, 2)$ is a feasible solution. The LP is:

$$\begin{array}{ll} \text{maximize} & z = 3x_1 + 2x_2 \\ \text{subject to} & x_1 + x_2 \leq 4, \\ & 2x_1 + x_2 \leq 6, \\ & x_1, x_2 \geq 0. \end{array}$$

Decide if $(1,2)$ satisfies all constraints and compute the objective value at that point.

Answer (Ex. 34) — Check constraints: $1+2=3\le 4$ and $2\cdot1+2=4\le 6$, nonnegativity holds. So $(1,2)$ is feasible. Objective value: $z=3\cdot1+2\cdot2=7$.

Ex. 35 — **Feasible region convexity:** Consider a linear program with two variables and linear constraints. Show by a short computation that the feasible region is convex. Provide an explicit example: two feasible points and their convex combination stays feasible.

Answer (Ex. 35) — Take the LP: $x_1+x_2\le 4,\ x_1,x_2\ge 0$. Points $A=(1,1)$ and $B=(2,0)$ are feasible. For $\theta=0.5$, the combination $C=0.5A+0.5B=(1.5,0.5)$. Check constraint: $1.5+0.5=2\le 4$ and nonnegativity holds. Hence convexity illustrated.

Ex. 36 — **Objective function linearity:** Explain and demonstrate with calculation why the LP objective must be linear for standard linear programming. Given $z=ax_1+bx_2$, evaluate z at two points and show linearity property $z(\alpha x+\beta y)=\alpha z(x)+\beta z(y)$ for $\alpha+\beta=1$.

Answer (Ex. 36) — Let $x=(x_1,x_2)=(1,0),\ y=(0,2),\ a=3,b=2$. Then $z(x)=3,\ z(y)=4$. For $\alpha=0.3,\beta=0.7$ the combination is $(0.3,1.4)$. $z(\alpha x+\beta y)=3\cdot0.3+2\cdot1.4=0.9+2.8=3.7=0.3\cdot3+0.7\cdot4$. Linearity verified.

Ex. 37 — **Optimum at vertex:** Consider maximizing $z=4x_1+5x_2$ subject to $x_1+x_2\le 6,\ x_1,x_2\ge 0$. Show that any optimum occurs at a vertex. Compute the candidate vertices and find the maximum.

Answer (Ex. 37) — Vertices: (0,0),(6,0),(0,6). Evaluate: z(0,0)=0, z(6,0)=24, z(0,6)=30. Maximum at vertex (0,6) with z=30. This shows optimum at a vertex.

Ex. 38 — **Graphical method variable limit:** Use the graphical method to solve the LP: maximize $z=x_1+2x_2$ subject to $x_1+x_2\le 5,\ x_1,x_2\ge 0$. Explain why this graphical approach is limited to two variables and solve the LP.

Answer (Ex. 38) — Graphical method needs a 2D plot; for more variables visualization fails. Solve: vertices (0,0),(5,0),(0,5). Evaluate z: 0,5,10. Optimal at (0,5) with z=10.

Ex. 39 — Feasible region shape: Prove that the feasible region of a linear program with linear inequalities is a convex polyhedron. Give an example and sketch reasoning (algebraic).

Answer (Ex. 39) — Each linear inequality defines a half-space; intersection of finitely many half-spaces equals a convex polyhedron. Example: $x_1 + x_2 \leq 4,\ x_1 \geq 0, x_2 \geq 0$ yields a polygon in $\mathbb{R}^2$, hence a convex polyhedron.

Ex. 40 — Uniqueness of LP solutions: Provide a concrete LP example that has multiple optimal solutions (not unique). Solve it and show the set of optima.

Answer (Ex. 40) — LP: maximize $z = x_1 + x_2$ subject to $x_1 + x_2 \leq 2,\ x_1, x_2 \geq 0$. All points on the segment from (0,2) to (2,0) that satisfy $x_1 + x_2 = 2$ are optimal with z=2. Hence not unique.

Ex. 41 — Redundant constraint: Show by example that a redundant constraint does not affect the feasible region. Consider constraints $x_1 + x_2 \leq 4,\ 2x_1 + 2x_2 \leq 8$ and show the second is redundant.

Answer (Ex. 41) — Second inequality is exactly 2 times the first; any point satisfying $x_1 + x_2 \leq 4$ automatically satisfies $2x_1 + 2x_2 \leq 8$. Thus redundant.

Ex. 42 — Max vs min in LP: Demonstrate that linear programming can handle both maximization and minimization. Convert a maximization problem into a minimization equivalent and solve numerically: minimize $z' = -x_1 - 2x_2$ with $x_1 + x_2 \leq 3,\ x_i \geq 0$.

Answer (Ex. 42) — Minimizing $z' = -z$ is equivalent to maximizing z. Solve feasible vertices (0,0),(3,0),(0,3): z' values 0,-3,-6 so minimum at (0,3) which corresponds to maximum of z at (0,3).

Ex. 43 — Feasibility analysis in linear programming: Determine whether the linear programming problem

$$\max z = 2x_1 + 5x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 3,\ x_1 + x_2 \geq 5,\ x_1, x_2 \geq 0$$

has a feasible solution. Justify your answer.

Answer (Ex. 43) — The constraints $x_1 + x_2 \leq 3$ and $x_1 + x_2 \geq 5$ are incompatible. Hence, the problem is infeasible — no feasible region exists.

Ex. 44 — **Multiple optimal solutions in LP:** Analyze the linear program:

$$\max z = x_1 + 2x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 5,\ x_1 + 3x_2 \leq 9,\ x_1, x_2 \geq 0.$$

Determine whether there are multiple optimal solutions.

Answer (Ex. 44) — The objective function is parallel to one of the constraints at optimality, resulting in multiple optimal solutions along a boundary segment.

Ex. 45 — **Unbounded linear programming problem:** Consider the linear program:

$$\max z = 5x_1 + 4x_2 \quad \text{s.t.} \quad x_1 + 2x_2 \geq 4,\ x_1 + x_2 \leq 10,\ x_1, x_2 \geq 0.$$

Determine the nature of its solution.

Answer (Ex. 45) — The feasible region is unbounded in the direction of increasing z; therefore, the problem is unbounded.

Ex. 46 — **Infeasible LP example:** Consider:

$$\max z = x_1 + x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 2,\ x_1 + x_2 \geq 5,\ x_1, x_2 \geq 0.$$

Discuss the feasibility of this system.

Answer (Ex. 46) — The constraints contradict each other; no point satisfies both. The LP is infeasible.

Ex. 47 — **Optimal solution in linear programming:** Solve the linear program:

$$\max z = 3x_1 + 2x_2, \quad \text{s.t.} \quad x_1 + x_2 \leq 4,\ 2x_1 + x_2 \leq 6,\ x_1, x_2 \geq 0.$$

Find the optimal point.

Answer (Ex. 47) — Corner points: $(0,0),(0,4),(2,2),(3,0)$. z is maximized at $(2,2)$ with $z=10$.

Ex. 48 — **Minimization with inequalities:** Find the optimal solution to:

$$\min z = x_1 + 4x_2 \quad \text{s.t.} \quad x_1 + 2x_2 \geq 3,\ -x_1 + x_2 \geq 1,\ x_1, x_2 \geq 0.$$

Answer (Ex. 48) — By graphical or algebraic methods, the minimum occurs at $(x_1, x_2) = (3,0)$.

Ex. 49 — **Maximization in LP:** Maximize $z = 4x_1 + 3x_2$ subject to:

$$\begin{cases} x_1 + 2x_2 \leq 8, \\ x_1 + x_2 \leq 6, \\ x_1, x_2 \geq 0. \end{cases}$$

Find the optimal point and value.

Answer (Ex. 49) — The feasible vertices yield maximum z at $(x_1, x_2) = (4,2)$ with $z = 22$.

Ex. 50 — **Minimization in LP with inequalities:** Minimize $z = 2x_1 + 3x_2$ subject to:

$$\begin{cases} x_1 + x_2 \geq 5, \\ 3x_1 + x_2 \geq 9, \\ x_1, x_2 \geq 0. \end{cases}$$

Find the optimal point.

Answer (Ex. 50) — Optimal solution at $(x_1, x_2) = (3,2)$, yielding $z = 12$.

Ex. 51 — **Optimal vertex in bounded LP:** Solve the LP:

$$\max z = 6x_1 + 7x_2,\ \text{s.t.}\ 2x_1 + x_2 \leq 14,\ x_1 + x_2 \leq 10,\ x_1, x_2 \geq 0.$$

Find the optimal vertex.

Answer (Ex. 51) — Intersection of constraints gives $(x_1, x_2) = (5,5)$. Maximum value: $z = 65$.

Ex. 52 — Profit maximization problem: A company produces two products, A and B, with profits per unit of 10 and 15, respectively. Each unit of A requires 2 units of resource 1 and 1 of resource 2; each unit of B requires 3 and 2 units, respectively. If at most 18 units of resource 1 and 10 of resource 2 are available, determine the optimal production plan.

Answer (Ex. 52) — Formulate:

$$\max z = 10x_1 + 15x_2,\ 2x_1 + 3x_2 \leq 18,\ x_1 + 2x_2 \leq 10.$$

The optimal solution is $(x_1, x_2) = (4, 3)$, with $z = 85$.

2.2 Duality

Ex. 53 — A problem of linear programming: Consider the linear programming problem:

$$\max x_1 + 4x_2 + 2x_3$$

$$\text{s.t.} \quad \begin{array}{rcl} 5x_1 + 2x_2 + 2x_3 & \leq & 145 \\ 4x_1 + 8x_2 - 8x_3 & \leq & 260 \\ x_1 + x_2 + 4x_3 & \leq & 190 \\ x_1, x_2 x_3 & \geq & 0 \end{array}$$

Find x_1, x_2 and x_3 to solve it.

Answer (Ex. 53) — material: complementary slackness.pdf

Ex. 54 — Complementary slackness: Consider the linear programming problem:

$$\max 2x_1 + 16x_2 + 2x_3$$

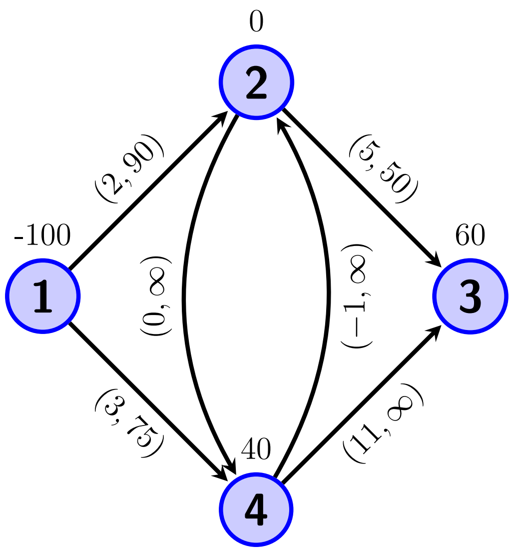
$$\text{s.t.} \quad \begin{array}{rcl} 2x_1 + x_2 - x_3 & \leq & -3 \\ -3x_1 + x_2 + 2x_3 & \leq & 12 \\ x_1, x_2 x_3 & \geq & 0 \end{array}$$

Check whether each of the following is an optimal solution, using complementary slackness:

Answer (Ex. 54) — material: compslack.pdf

2.3 Network Analysis

Ex. 55 — Minimum Cost: The figure shows a network on four nodes, including net demands on the vertex, b_k, and cost an capacity on the edges, $(c_{i,j}, u_{i,j})$.(Adapted from [**?**])



1.Formulate the corresponding minimum cost network flow model

2.Classify the nodes as *source*, *sink* or *transshipment*

Answer (Ex. 55) — In this problem, vertex and edges are:

$$
\begin{aligned}
V &= \{1,2,3,4\} \\
A &= \{(1,2),(1,4),(2,3),(2,4),(4,2),(4,3)\}
\end{aligned}
$$

we can use the variables $x_{i,j}$ to represent the flows in the different members of set A. Thus, the formulation of the problem is:

$$\min \quad 2x_{1,2} + 3x_{1,4} + 5x_{2,3} - x_{4,2} + 11x_{4,3}$$
$$\text{subject to} \quad \begin{cases} -A - B &= -100 \\ A + F - C - D &= 0 \\ C + E &= 60 \\ B + D - F - E &= 40 \\ A &\leq 90 \\ B &\leq 75 \\ C &\leq 50 \end{cases}$$

and $x_{i,j} \geq 0$.
There are 4 KKT conditions for optimal primal ($x4$) and dual (λ) variables. If x^* denotes optimal values:

1. Primal feasibility: all constraints must be satisfied: $g_i(x^*) - b_i$ is feasible. Applies to both equality and non-equality constraints.
2. Gradient condition or No feasible descent: No possible improvement at the solution:
$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0$$
3. Complementariety slackness:
$$\lambda_i^*(g_i(x^*) - b_i) = 0$$
4. Dual feasibility: $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active (=0) and a zero Lagrange multiplier when the constraint is inactive (>0).
to solve our problem, first we will put it in its standard form:

$$\min \ f(x, y) = -xy$$
$$\text{subject to} \quad \begin{array}{rcl} -x - y + 100 &\geq& 0 \\ -x - 40 &\geq& 0 \end{array}$$

We will go through the different conditions:

1.Primal feasibility: $g_i(x^*) - b_i$ is feasible.

$$-x^* - y^* + 100 = 0$$

$$-x^* - 40 = 0$$

2.Gradient condition or No feasible descent:

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} - \lambda_1 \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_1}{\partial y} \end{pmatrix} - \lambda_2 \begin{pmatrix} \frac{\partial g_2}{\partial x} \\ \frac{\partial g_2}{\partial y} \end{pmatrix} = 0$$

which, in this example, resolves into:

$$\begin{cases} \quad\;\; - y + \lambda_1 + \lambda_2 = 0 \\ -x \quad - \quad\;\; \lambda_1 = 0 \end{cases}$$

3.Complementariety slackness:

$$\lambda_1^*(-x^* - y^* + 100) = 0$$

$$\lambda_2^*(-x^* - 40) = 0$$

4.Dual feasibility: $\lambda_1, \lambda_2 \geq 0$

We can put the resulting 5 expressions for conditions 1 and 2 into matrix form:

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -100 \\ 40 \\ 0 \\ 0 \end{pmatrix}$$

3 Appendices

3.1 Python codes

Codes

3.1.1 NLO with constraints

Code 1: Non-linear optimization with constraints

```
from scipy.optimize import minimize

# Define the objective function
def objective(xy):
    x, y = xy
    return -(-x**2 + 4*x*y - 2*y**2)  # Minimization function, so retu

# Define the constraints
def constraint1(xy):
    x, y = xy
    return 30 - (x + 2*y)  # x + 2y <= 30

def constraint2(xy):
    x, y = xy
    return (x * y) - 50  # xy >= 50

def constraint3(xy):
    x, y = xy
    return (3 * x**2 / 100) + 5 - y  # y <= (3x^2)/100 + 5

# Initial guess
initial_guess = [15, 1]

# Define constraints
constraints = [{'type': 'ineq', 'fun': constraint1},
               {'type': 'ineq', 'fun': constraint2},
               {'type': 'ineq', 'fun': constraint3}]

# Perform the optimization
result = minimize(objective, initial_guess, constraints=constraints)

# Display the results
optimal_x, optimal_y = result.x
print(f"Optimal values: x = {optimal_x:.2f}, y = {optimal_y:.2f}")
```

```python
print(f"Maximum profit: {-result.fun:.2f}")
```

References

[1] Fco. Javier Martínez Sánchez. El teorema de karush-kuhn-tucker, una generalización del teorema de los multiplicadores de lagrange, y programación convexa. *TEMat*, 3:33–44, 2019.