FACULTAT
DE CIÈNCIES, TECNOLOGIA
I ENGINYERIES
UVIC | UVIC·UCC

Solved Exercises

Optimization and Operations Research [*]

Jordi Villà-Freixa

Last update: December 30, 2025

Contents

[*]If you detect an error or would like to make a suggestion, please contact me: jordi.villa@uvic.cat

Scripts

1 Non-linear optimization

Ex. 1 — A problem of non-linear optimization: A company wants to maximize its profit function given by:

$$f(x, y) = -x^2 + 4xy - 2y^2$$

where x and y represent the amount of resources allocated to two different projects.
The problem is subject to the following constraints:

1. The total resources used must not exceed 30 units.
2. The product of the resources allocated must be at least 50 units.
3. The relationship between the resource allocations must satisfy the following nonlinear constraint:

$$y \leq \frac{3x^2}{100} + 5$$

The company wants to find the values of x and y that maximize the profit $f(x, y)$ under these constraints.
Can you help the company drawing the problem in a graph? Can you identify the feasible region? Is the region convex?

Answer (Ex. 1) — A company wants to maximize its profit function given by:

$$f(x, y) = -x^2 + 4xy - 2y^2$$

where x and y represent the amount of resources allocated to two different projects.
The problem is subject to the following constraints:

1. The total resources used must not exceed 30 units:

$$x + 2y \leq 30$$

2. The product of the resources allocated must be at least 50 units:

$$xy \geq 50$$

3.The relationship between the resource allocations must satisfy the following nonlinear constraint:

$$y \leq \frac{3x^2}{100} + 5$$

Find the values of x and y that maximize the profit $f(x, y)$ under these constraints. Before solving it, we can draw the problem in a graph to identify the feasible region and the optimal solution. The problem is shown in Figure 1.

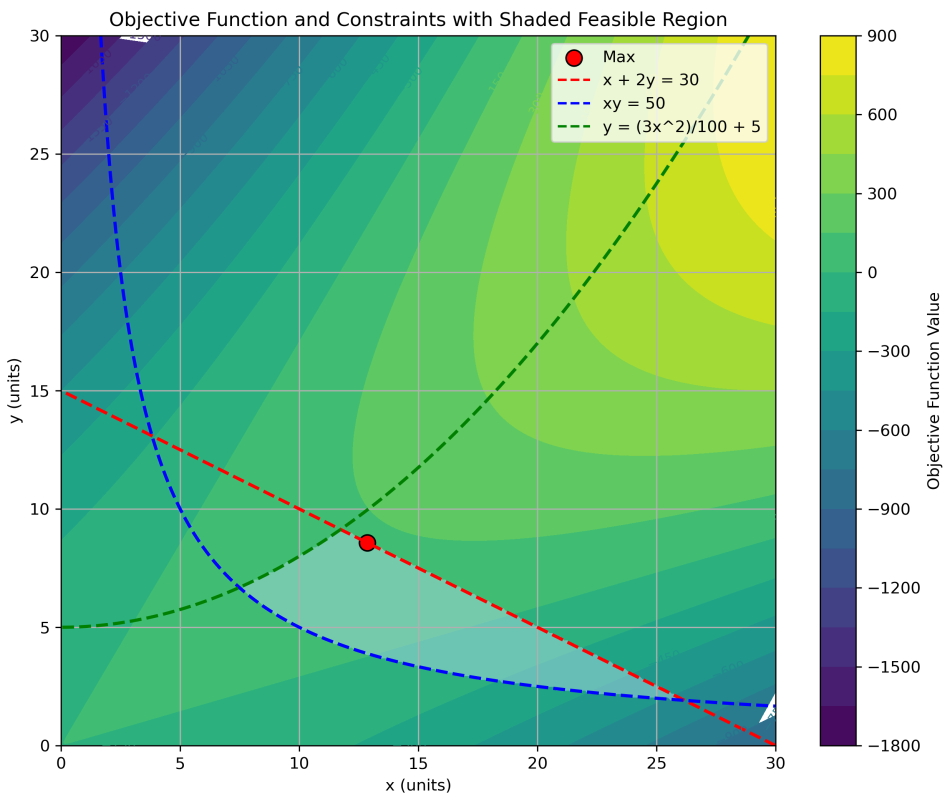


Figure 1: Contour plot of the objective function $f(x, y) = -x^2 + 4xy - 2y^2$ with constraints $x + 2y \leq 30$ (red dashed), $xy \geq 50$ (blue dashed), and $y \leq \frac{3x^2}{100} + 5$ (green dashed). The light blue region represents the feasible area where all constraints are satisfied. The solution has been obtained with Script 1

Script 1: Non-linear optimization with constraints

```python
from scipy.optimize import minimize

# Define the objective function
# Minimization function, so return negative
def objective(xy):
    x, y = xy
    return -(-x**2 + 4*x*y - 2*y**2)

# Define the constraints
def constraint1(xy):
    x, y = xy
    return 30 - (x + 2*y)  # x + 2y <= 30

def constraint2(xy):
    x, y = xy
    return (x * y) - 50  # xy >= 50

def constraint3(xy):
    x, y = xy
    return (3 * x**2 / 100) + 5 - y  # y <= (3x^2)/100 + 5

# Initial guess
initial_guess = [15, 1]

# Define constraints
constraints = [{'type': 'ineq', 'fun': constraint1},
               {'type': 'ineq', 'fun': constraint2},
               {'type': 'ineq', 'fun': constraint3}]

# Perform the optimization
result = minimize(objective, initial_guess, constraints=constraints)

# Display the results
optimal_x, optimal_y = result.x
print(f"Optimal values: x = {optimal_x:.2f}, y = {optimal_y:.2f}")
print(f"Maximum profit: {-result.fun:.2f}")
```

■

1.1 Unconstrained non-linear optimization

Ex. 2 — **Quadratic function minimization:** Consider the quadratic function $f(x) = 2x^2 - 4x + 1$. Determine the coordinates of the vertex of the parabola and state whether it represents a minimum or maximum. Compute the minimum (or maximum) value of $f(x)$.

Answer (Ex. 2) — The vertex is at $x = 1$, corresponding to a minimum since the coefficient of x^2 is positive. The minimum value is $f(1) = -1$.

Ex. 3 — **Cubic function extrema:** Find all critical points of the function $f(x) = x^3 - 3x + 1$ and determine which of them correspond to local minima and maxima.

Answer (Ex. 3) — $f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$. $f''(x) = 6x \Rightarrow f''(1) > 0 \to$ local minimum at $x = 1$; $f''(-1) < 0 \to$ local maximum at $x = -1$. Minimum value: $f(1) = -1$.

Ex. 4 — **Stationary point of a quadratic function:** Determine the stationary point of $f(x, y) = 3x^2 + 2y^2 - 6x - 8y$ and classify it.

Answer (Ex. 4) — Setting partial derivatives to zero: $6x - 6 = 0 \Rightarrow x = 1$, $4y - 8 = 0 \Rightarrow y = 2$. The point $(1, 2)$ is a minimum.

Ex. 5 — **Quadratic maximization with bounds:** Maximize $f(x) = -x^2 + 4x$ for $0 \leq x \leq 3$. Determine where the maximum occurs and its value.

Answer (Ex. 5) — $f'(x) = -2x + 4 = 0 \Rightarrow x = 2$. $f(2) = 4$ is the maximum value, since $f(0) = 0$ and $f(3) = 3$.

Ex. 6 — **Hessian at a minimum:** Suppose $f(x)$ is a twice continuously differentiable function with a global minimum at $x = x^*$. What can be said about the Hessian matrix $H(x^*)$?

Answer (Ex. 6) — At a global minimum, the Hessian must be **positive definite**. All eigenvalues are positive, indicating the surface curves upward in all directions. If the Hessian were negative definite, the point would be a local maximum; if indefinite, a saddle. ■

Ex. 7 — Hessian concavity criteria: Given the function $f(x, y) = x^3 + 2y^3 - xy$, determine whether the function at $(0, 0)$ is convex, concave, or neither.

Answer (Ex. 7) — The Hessian is

$$H = \begin{pmatrix} 6x & -1 \\ -1 & 12y \end{pmatrix}.$$

At $(0, 0)$:

$$H(0, 0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Its eigenvalues are $\lambda = 1$ and $\lambda = -1$, so H is **indefinite**. Hence, the function is **neither convex nor concave** at $(0, 0)$ (it has a saddle-type curvature). ■

Ex. 8 — Golden Section Search: When applying the Golden Section Search to find the maximum of $f(x)$, the condition $f(x_1) > f(x_2)$ holds for the intermediate points ($x_1 < x_2$). Explain which region is eliminated from the search interval and why.

Answer (Ex. 8) — Since $f(x_1) > f(x_2)$, the function is decreasing after x_1. Thus, the maximum lies between x_l and x_2, and the interval $[x_2, x_u]$ can be eliminated. The new search region is $[x_l, x_2]$. ■

Ex. 9 — Newton's Method: In Newton's method for unconstrained optimization, the search direction is $p_k = -H_k^{-1}\nabla f(x_k)$. Explain under which conditions this direction corresponds to a descent direction.

Answer (Ex. 9) — If the Hessian H_k is **positive definite**, then p_k points in a descent direction (toward a local minimum). If H_k is **negative definite**, the algorithm moves toward a local maximum. If H_k is **indefinite**, the step may not be a descent direction. ■

Ex. 10 — Gradient Descent: In the gradient descent method with update $x_{k+1} = x_k - \alpha_k \nabla f(x_k)$, explain the role of the step size α_k.

Answer (Ex. 10) — The step is taken in the **negative gradient direction**, which points toward steepest descent. A small α_k leads to slow convergence, while a large α_k can cause divergence or oscillation. A properly chosen step size ensures steady convergence to a local minimum. ■

Ex. 11 — **First- and Second-Order Optimality Conditions:** State the first- and second-order conditions that must hold at a local minimum of a differentiable function $f(x)$.

Answer (Ex. 11) — At a local minimum x^*:

$$\nabla f(x^*) = 0 \quad \text{and} \quad H(x^*) \text{ is positive definite.}$$

If $H(x^*)$ is negative definite, the point is a local maximum; if indefinite, it is a saddle point. ■

Ex. 12 — **Hessian and Convexity:** Explain how the definiteness of the Hessian matrix determines whether a function is convex, concave, or neither.

Answer (Ex. 12) — If $H(x)$ is **positive semi-definite** for all x, the function is convex. If $H(x)$ is **negative semi-definite**, it is concave. If $H(x)$ is **indefinite**, the function is neither convex nor concave. A **singular** Hessian may indicate flat regions or saddle behavior. ■

Ex. 13 — **Critical points and curvature classification:** Consider the function $f(x, y) = x^3 - 3x + y^2$. Find its critical points and determine their type.

Answer (Ex. 13) —

$$\frac{\partial f}{\partial x} = 3x^2 - 3 = 0 \Rightarrow x = \pm 1, \qquad \frac{\partial f}{\partial y} = 2y = 0 \Rightarrow y = 0.$$

Thus, critical points: $(1, 0)$ and $(-1, 0)$.

$$H = \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix}.$$

At $(1, 0)$: $H = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$ → positive definite → local minimum. At $(-1, 0)$: $H = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix}$ → indefinite → saddle point. ■

Ex. 14 — Critical points of $\sin(2x) + \cos(y)$: Find and classify the critical points of $f(x, y) = \sin(2x) + \cos(y)$.

Answer (Ex. 14) —

$$\frac{\partial f}{\partial x} = 2\cos(2x), \quad \frac{\partial f}{\partial y} = -\sin(y).$$

Set both to zero: $\cos(2x) = 0$, $\sin(y) = 0$. Thus $x = \pi/4 + k\pi/2$, $y = n\pi$, with $k, n \in \mathbb{Z}$.

$$H = \begin{pmatrix} -4\sin(2x) & 0 \\ 0 & -\cos(y) \end{pmatrix}.$$

If $\sin(2x)$ and $\cos(y)$ have the same sign $\to$ H definite $\to$ local extrema. If signs differ $\to$ H indefinite $\to$ saddle points. ■

Ex. 15 — Critical points of $e^{x\cos y}$: Find and classify the critical points of $f(x, y) = e^{x\cos y}$.

Answer (Ex. 15) —

$$\frac{\partial f}{\partial x} = e^{x\cos y}\cos y, \qquad \frac{\partial f}{\partial y} = -xe^{x\cos y}\sin y.$$

Setting both to zero gives $x = 0$, $\sin y = 0 \Rightarrow y = n\pi$, $n \in \mathbb{Z}$. Thus, critical points are $(0, n\pi)$.

$$H = \begin{pmatrix} \cos y\, e^{x\cos y} & -x\sin y\, e^{x\cos y} \\ -x\sin y\, e^{x\cos y} & -x\cos y\, e^{x\cos y} \end{pmatrix}.$$

At $(0, n\pi)$:

$$H = \begin{pmatrix} (-1)^n & 0 \\ 0 & 0 \end{pmatrix}.$$

One eigenvalue is zero $\to$ Hessian is **singular**. The function is flat in the y-direction at these points. ■

Ex. 16 — Critical points of $f(x, y) = \sin\frac{x}{2} + \cos y$: *(JVF,1st25)* Find the critical points of the function $f(x, y) = \sin\frac{x}{2} + \cos y$, and determine their type.

Answer (Ex. 16) — The function is defined on the entire plane (x, y). To find the critical points, we solve the system generated by:

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = 0 = \begin{pmatrix} \frac{1}{2}\cos\frac{x}{2} \\ -\sin y \end{pmatrix}.$$

Hence we obtain

$$\begin{aligned} \cos\frac{x}{2} &= 0, \\ \sin y &= 0. \end{aligned}$$

The first equation is solved for all values of x satisfying

$$\frac{x}{2} = \frac{\pi}{2} + n\pi \Rightarrow x = \pi + 2n\pi = (2n+1)\pi,$$

and the second for all y satisfying

$$y = m\pi,$$

for any integer number m.
We construct the Hessian matrix by taking the second derivatives of the function:

$$\mathbf{H}_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} A & C \\ C & B \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}\sin\frac{x}{2} & 0 \\ 0 & -\cos y \end{pmatrix}.$$

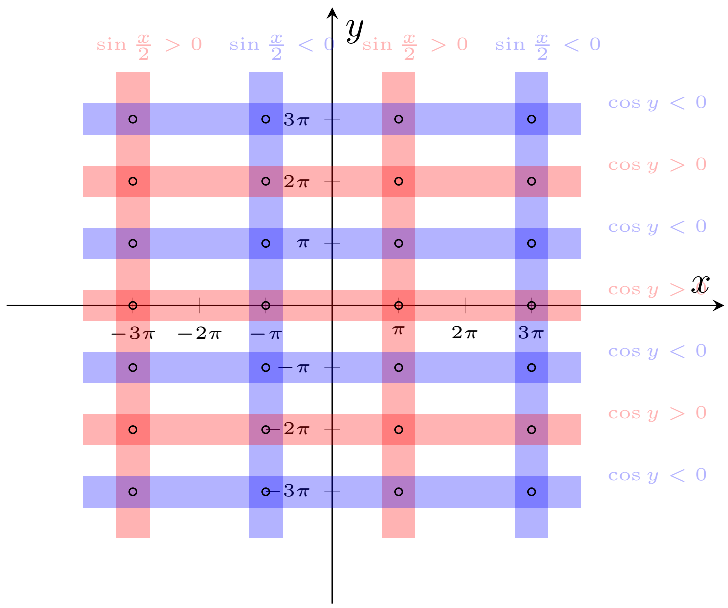
We have used that the first derivatives are continuous on all $\mathbb{R}^2$, so we can apply Schwarz's theorem:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = C.$$

The Hessian determinant is therefore:

$$H_f = \det(\mathbf{H}_f(x,y)) = \frac{1}{4}\sin\frac{x}{2}\cos y.$$

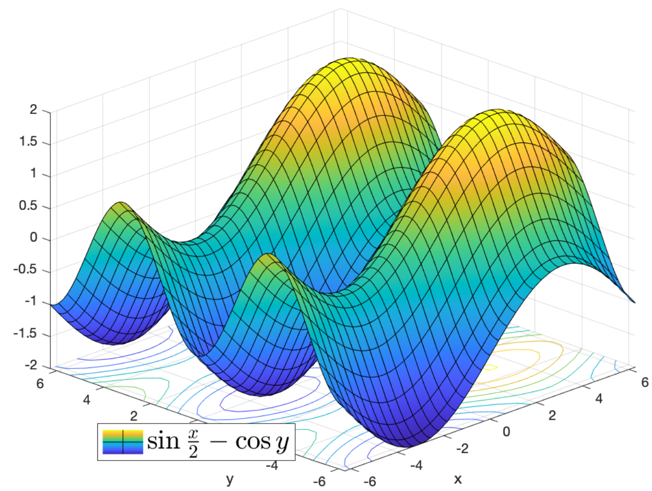
The following figure represents the value of the Hessian at the critical points we found: $(x,y) = ([2n-1]\pi, m\pi)$. The red bands represent the values of x and y for which either $\sin\frac{x}{2} > 0$ or $\cos y > 0$. The blue bands represent the values of x and y for which these same functions are negative. Thus, $H_f < 0$ where the blue and red bands intersect, and $H_f > 0$ when bands of the same color intersect.



In summary, considering that $A = -\frac{1}{4}\sin\frac{x}{2}$:

y \ x	-3π	$-\pi$	π	3π	$\cdots$
3π	$H_f = -\frac{1}{4}$ Saddle point	$H_f = \frac{1}{4}$ $A = \frac{1}{4} > 0$ Minimum	$H_f = -\frac{1}{4}$ Saddle point	$H_f = \frac{1}{4}$ $A = \frac{1}{4} > 0$ Minimum	$\cdots$
2π	$H_f = \frac{1}{4}$ $A = -\frac{1}{4} < 0$ Maximum	$H_f = -\frac{1}{4}$ Saddle point	$H_f = \frac{1}{4}$ $A = -\frac{1}{4} < 0$ Maximum	$H_f = -\frac{1}{4}$ Saddle point	$\cdots$
π	$H_f = -\frac{1}{4}$ Saddle point	$H_f = \frac{1}{4}$ $A = \frac{1}{4} > 0$ Minimum	$H_f = -\frac{1}{4}$ Saddle point	$H_f = \frac{1}{4}$ $A = \frac{1}{4} > 0$ Minimum	$\cdots$
0	$H_f = \frac{1}{4}$ $A = -\frac{1}{4} < 0$ Maximum	$H_f = -\frac{1}{4}$ Saddle point	$H_f = \frac{1}{4}$ $A = -\frac{1}{4} < 0$ Maximum	$H_f = -\frac{1}{4}$ Saddle point	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The graph of the function shows that, along a given X or Y direction, maxima and minima alternate with saddle points.



■

1.2 Optimization with constraints

1.2.1 Lagrange Multipliers

Ex. 17 — Lagrange multipliers: Optimization of $f(x, y) = x^2 - y$ subject to $x^2 + y^2 = 4$

Answer (Ex. 17) — We will use the method of Lagrange multipliers. Define the constraint as $g(x, y) = x^2 + y^2 - 4 = 0$. The Lagrange function is then:

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda \cdot g(x, y) = x^2 - y + \lambda(x^2 + y^2 - 4).$$

Note that we have used $+\lambda \cdot g(x, y)$ in the equation above, but we could also use $-\lambda \cdot g(x, y)$; the results would be equivalent. We now compute the partial derivatives of $\mathcal{L}(x, y, \lambda)$ with respect to x, y, and λ and equal them to zero to obtain the system of equations to solve:

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + \lambda \cdot 2x = 2x(1 + \lambda) = 0, \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -1 + \lambda \cdot 2y = 0 \quad \Rightarrow \quad \lambda = \frac{1}{2y}, \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - 4 = 0. \tag{3}$$

From Eq. 1, we see two possibilities:

- $x = 0$: Substitute $x = 0$ in Eq. 3:

  $$0^2 + y^2 = 4 \quad \Rightarrow \quad y^2 = 4 \quad \Rightarrow \quad y = \pm 2.$$

  For $y = 2$, substitute into $f(x, y) = x^2 - y$:

  $$f(0, 2) = 0^2 - 2 = -2.$$

  For $y = -2$, substitute into $f(x, y)$:

  $$f(0, -2) = 0^2 - (-2) = 2.$$

- $1 + \lambda = 0 \quad \Rightarrow \quad \lambda = -1$. Substitute $\lambda = -1$ into Eq. 2:

  $$-1 = \frac{1}{2y} \quad \Rightarrow \quad y = -\frac{1}{2}.$$

  Now, substitute $y = -\frac{1}{2}$ into the constraint equation $x^2 + y^2 = 4$:

  $$x^2 + \left(-\frac{1}{2}\right)^2 = 4 \quad \Rightarrow \quad x^2 + \frac{1}{4} = 4 \quad \Rightarrow \quad x^2 = \frac{15}{4} \quad \Rightarrow \quad x = \pm\frac{\sqrt{15}}{2}.$$

  Now, calculate $f(x, y) = x^2 - y$ for $y = -\frac{1}{2}$ and $x = \pm\frac{\sqrt{15}}{2}$:
  For $x = \frac{\sqrt{15}}{2}$:

  $$f\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) = \left(\frac{\sqrt{15}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{15}{4} + \frac{1}{2} = \frac{17}{4}.$$

  For $x = -\frac{\sqrt{15}}{2}$:

  $$f\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) = \left(-\frac{\sqrt{15}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{15}{4} + \frac{1}{2} = \frac{17}{4}.$$

We now compare the function values:

- For $(0, 2)$: $f(0, 2) = -2$.
- For $(0, -2)$: $f(0, -2) = 2$.
- For $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$: $f = \frac{17}{4}$.

Thus:

- The maximum value is $f = \frac{17}{4}$ at $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$.
- The minimum value is $f = -2$ at $(0, 2)$.

For a Python implementation check this file.

Ex. 18 — Lagrange multipliers with linear constraint: Solve the constrained optimization problem:

$$\min f(x, y) = x^2 + y^2 \quad \text{subject to} \quad x + 2y = 4.$$

Find all stationary points and determine which gives the minimum value.

Answer (Ex. 18) — Using the Lagrange multiplier method:

$$\nabla f = (2x, 2y), \quad \nabla g = (1, 2) \Rightarrow 2x = \lambda,\ 2y = 2\lambda.$$

From the constraint: $x + 2y = 4$. Solving gives $x = \frac{4}{5}$, $y = \frac{8}{5}$. This is the minimum point.

Ex. 19 — Constrained product minimization: Find the minimum and maximum values of $f(x, y) = xy$ subject to $x^2 + y^2 = 9$.

Answer (Ex. 19) — Using Lagrange multipliers: $y = \lambda 2x$, $x = \lambda 2y \Rightarrow x^2 = y^2 \Rightarrow y = \pm x$. For $y = x$: $2x^2 = 9 \Rightarrow x = \pm\frac{3}{\sqrt{2}} \Rightarrow f = \frac{9}{2}$. For $y = -x$: $f = -\frac{9}{2}$. Minimum value $-\frac{9}{2}$, maximum $\frac{9}{2}$.

Ex. 20 — Optimization with equality constraint: Use the Lagrange multiplier method to find the values of x and y that minimize $f(x, y) = x^2 + y^2$ subject to $x + y = 1$.

Answer (Ex. 20) — From the constraint: $y = 1 - x$. Then $f(x) = x^2 + (1 - x)^2 = 2x^2 - 2x + 1$. Minimum at $x = \frac{1}{2}$, $y = \frac{1}{2}$.

1.2.2 Karush-Kuhn-Tucker theorem

Ex. 21 — Complementary slackness computation: Consider minimizing $f(x) = x_1^2 + x_2^2$ subject to $x_1 + x_2 \geq 2$ and $x_1, x_2 \geq 0$. Use KKT conditions (with multipliers) and illustrate complementary slackness: identify active constraints and multipliers.

Answer (Ex. 21) — Set constraint as $g(x) = 2-(x_1+x_2) \leq 0$. Lagrangian $L = x_1^2 + x_2^2 + \lambda(2 - x_1 - x_2) - \mu_1 x_1 - \mu_2 x_2$.
KKT conditions:

Stationarity:

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda - \mu_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda - \mu_2 = 0$$

Primal feasibility:

$$x_1 + x_2 \geq 2, \quad x_1 \geq 0, \quad x_2 \geq 0$$

Dual feasibility:

$$\lambda \geq 0, \quad \mu_1 \geq 0, \quad \mu_2 \geq 0$$

Complementary slackness:

$$\lambda(2 - x_1 - x_2) = 0$$

$$\mu_1 x_1 = 0$$

$$\mu_2 x_2 = 0$$

Solving: Try symmetric solution $x_1 = x_2 = a$. Then $2a \geq 2 \implies a \geq 1$. Try $a = 1$ (active constraint).
Plug into stationarity:

$$2x_1 - \lambda - \mu_1 = 0 \implies 2 - \lambda - \mu_1 = 02x_2 - \lambda - \mu_2 = 0 \implies 2 - \lambda - \mu_2 = 0$$

By symmetry, $\mu_1 = \mu_2 = \mu$, so $2 - \lambda - \mu = 0 \implies \lambda = 2 - \mu$.
From complementary slackness: $x_1 = x_2 = 1 > 0 \implies \mu_1 = \mu_2 = 0$.
So $\lambda = 2$.
Check complementary slackness for λ: $2 - x_1 - x_2 = 0$, so $\lambda \cdot 0 = 0$.
Summary:

- Active constraint: $x_1 + x_2 = 2$ ($g(x) = 0$), multiplier $\lambda = 2$.
- Inactive constraints: $x_1 = 1 > 0$, $x_2 = 1 > 0$, multipliers $\mu_1 = \mu_2 = 0$.
- All KKT conditions are satisfied.

Interpretation: The only active constraint is $x_1+x_2 \geq 2$, and its multiplier is positive. The nonnegativity constraints are inactive, so their multipliers are zero. This illustrates complementary slackness: only active constraints can have nonzero multipliers.

Ex. 22 — KKT necessity for optimality: For the constrained problem $\min f(x) = x^2 + y^2$ subject to $x + y = 1$, write the KKT (which reduce to Lagrange) conditions and solve. Explain why KKT are necessary.

Answer (Ex. 22) — Lagrangian $L = x^2 + y^2 + \lambda(x + y - 1)$. Stationarity: $2x + \lambda = 0,\ 2y + \lambda = 0 \to$ x=y. Constraint gives x=y=1/2. KKT (here Lagrange) are necessary to characterize optima under differentiability and constraint qualifications.

Ex. 23 — KKT application domain: Give a nonlinear optimization problem where KKT applies: minimize $f(x) = x_1^2 + 4x_2^2$ subject to $x_1 + x_2 \leq 3,\ x_i \geq 0$. Use KKT to find candidate minimizers.

Answer (Ex. 23) — Formulate KKT with multipliers λ for inequality. Solve stationarity: $2x_1 + \lambda - \mu_1 = 0,\ 8x_2 + \lambda - \mu_2 = 0$. Try interior (mu=0) and inactive constraint: if interior then lambda=0→ x1=0,x2=0 but constraint 0<=3 satisfied; 0 is minimizer.

Ex. 24 — Non-negativity in KKT: Show with a small example how KKT enforces non-negativity constraints. Minimize $f(x) = x_1^2 + x_2^2$ subject to $x_1 \geq 0, x_2 \geq 0$. Find solution and relate to KKT multipliers.

Answer (Ex. 24) — Unconstrained minimum at (0,0) already satisfies non-negativity. KKT multipliers for inequality constraints are zero or positive; stationarity yields $2x_1 - \mu_1 = 0,\ 2x_2 - \mu_2 = 0$. At (0,0) mu>=0 can be zero; complementary slackness holds.

Ex. 25 — Multipliers as shadow prices: Consider minimize cost $f(x) = 2x_1 + 3x_2$ subject to $x_1 + x_2 \geq 5,\ x_i \geq 0$. Using KKT multipliers (dual viewpoint) interpret the multiplier for the constraint as a shadow price:

compute multiplier and show sensitivity of optimal cost to RHS change.

Answer (Ex. 25) — Reformulate as equality at optimum: x1+x2=5 minimal cost with nonnegativity -> allocate to cheapest variable x1 (cost 2): x1=5,x2=0, cost=10. Small increase d in RHS increases minimal cost by multiplier equal to dual price; here dual multiplier = 2 (cost of resource) so sensitivity approx 2 per unit. (Interpretation)

Ex. 26 — Linear independence not KKT requirement: Explain with a short counterexample why linear independence (LICQ) is not a component of the KKT equalities themselves: state KKT conditions and show one of them is not 'linear independence'.

Answer (Ex. 26) — KKT conditions include stationarity, primal and dual feasibility, complementary slackness. LICQ is a constraint qualification (assumption) to guarantee KKT necessity/sufficiency. Thus LICQ is not part of KKT equations but an extra assumption—demonstrated by stating KKT and noting LICQ absent.

Ex. 27 — Convexity assumption for sufficiency: Provide a convex constrained example where KKT conditions are sufficient. Minimize $f(x) = x_1^2 + x_2^2$ subject to $x_1 + x_2 \geq 1,\ x_i \geq 0$. Use KKT to find minimizer and argue sufficiency because of convexity.

Answer (Ex. 27) — As earlier, constraint active yields x1=x2=0.5; KKT solved gives multipliers and unique minimizer. Because f and feasible set are convex, KKT conditions ensure global optimality.

Ex. 28 — KKT necessity vs sufficiency: Give an example showing that satisfying KKT is not always sufficient for optimality when convexity fails. Provide a nonconvex problem and describe a KKT point that is not global optimum.

Answer (Ex. 28) — Consider $f(x) = -x^2$ subject to $x^2 \leq 1$. The KKT stationary point $x = 0$ satisfies KKT but is not maximal (the endpoints $x = \pm 1$ give greater objective for maximization). This illustrates KKT is not sufficient without convexity.

Ex. 29 — KKT generalize Lagrange: Show algebraically that

KKT conditions generalize Lagrange multipliers to inequalities by writing KKT for a problem with equality and inequality constraints. Use the problem $\min x^2$ s.t. $x \geq 1$ and show correspondence.

Answer (Ex. 29) — For $\min x^2$ s.t. $x \geq 1$, write $g(x) = 1 - x \leq 0$. Lagrangian $L = x^2 + \lambda(1 - x)$. Stationarity: $2x - \lambda = 0$ with $\lambda \geq 0$, complementary slackness: $\lambda(1 - x) = 0$. Solution $x = 1$, $\lambda = 2$ satisfies KKT; reduces to Lagrange when constraint is active as equality.

Ex. 30 — **A problem of KKT conditions:** Maximize $f(x, y) = xy$ subject to $100 \geq x + y$ and $x \leq 40$ and $(x, y) \geq 0$.

Answer (Ex. 30) — The Karush Kuhn Tucker (KKT) conditions for optimality are a set of necessary conditions for a solution to be optimal in a mathematical optimization problem. They are necessary and sufficient conditions for a local minimum in nonlinear programming problems. The KKT conditions consist of the following elements:
For an optimization problem in its standard form:

$$\max f(\mathbf{x})$$
$$\text{s.t.} \quad \begin{array}{rcll} g_i(\mathbf{x}) - b_i & \leq & 0 & i = 1, \ldots, k \\ g_i(\mathbf{x}) - b_i & = & 0 & i = k+1, \ldots, m \end{array}$$

There are 4 KKT conditions for optimal primal ($\mathbf{x}$) and dual (λ) variables. If $\mathbf{x}^*$ denotes optimal values:

1. Primal feasibility: all constraints must be satisfied: $g_i(\mathbf{x}^*) - b_i$ is feasible. Applies to both equality and non-equality constraints.
2. Gradient condition or No feasible descent: No possible improvement at the solution:
$$\nabla f(\mathbf{x}^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(\mathbf{x}^*) = 0$$
3. Complementariety slackness:
$$\lambda_i^*(g_i(\mathbf{x}^*) - b_i) = 0$$
4. Dual feasibility: $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active (=0) and a zero Lagrange multiplier when the constraint is inactive (>0).
To solve our problem, first we will put it in its standard form:

$$\max f(x,y) = xy$$
$$\text{s.t.} \quad \begin{array}{rcl} g_1(x,y) = x + y - 100 & \leq & 0 \\ g_2(x,y) = x - 40 & \leq & 0 \\ g_3(x,y) = -x & \leq & 0 \\ g_4(x,y) = -y & \leq & 0 \end{array} \tag{4}$$

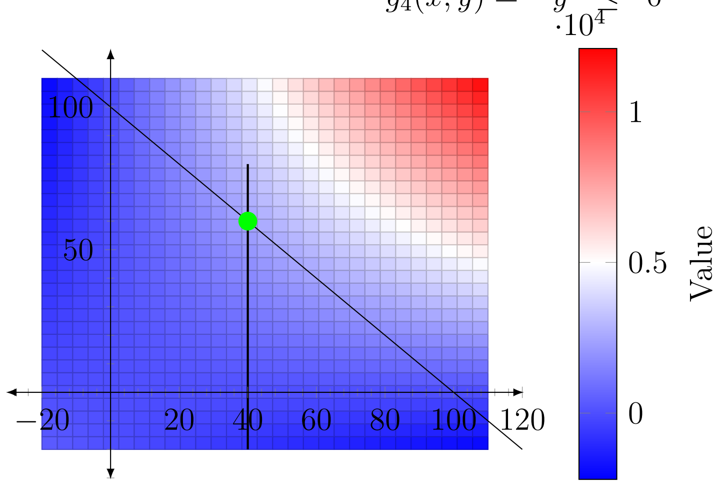


Figure 2: Contour plot of the objective function $f(x,y) = xy$ with constraints in Eq. 4, showing the final optimal value after applying the KKT conditions (green dot).
We will go through the different conditions:

- on the gradient:

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} - \lambda_1 \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_1}{\partial y} \end{pmatrix} - \lambda_2 \begin{pmatrix} \frac{\partial g_2}{\partial x} \\ \frac{\partial g_2}{\partial y} \end{pmatrix} - \lambda_3 \begin{pmatrix} \frac{\partial g_3}{\partial x} \\ \frac{\partial g_3}{\partial y} \end{pmatrix} - \lambda_4 \begin{pmatrix} \frac{\partial g_4}{\partial x} \\ \frac{\partial g_4}{\partial y} \end{pmatrix} = 0$$

  which, in this example, resolves into:

$$y - (\lambda_1 + \lambda_2 - \lambda_3) = 0 \tag{5}$$

$$x - (\lambda_1 - \lambda_4) = 0 \tag{6}$$

- on the complementary slackness:

$$\lambda_1(x+y-100)=0 \tag{7}$$

$$\lambda_2(x-40)=0 \tag{8}$$

$$\lambda_3 x=0 \tag{9}$$

$$\lambda_4 y=0 \tag{10}$$

- on the constraints:

$$x+y\leq 100 \tag{11}$$

$$x\leq 40 \tag{12}$$

$$-x\leq 0 \tag{13}$$

$$-y\leq 0 \tag{14}$$

plus $\lambda_1,\lambda_2,\lambda_3,\lambda_4\geq 0$.

We will start by checking Eq. 7:

–Let us see what occurs if $\lambda_1=0$. Then, from Eq. 6, $x+\lambda_4=0$ which implies that $x=\lambda_4=0$[1]. But, then, from Eq. 8 we obtain that $\lambda_2=0$ which, using Eq. 5 gives $y+\lambda_3=0\Rightarrow y=\lambda_3=0$. Indeed, the KKT conditions are satisfied when all variables and multipliers are zero, but it is not a maximum of the function (see figure above).

–So, let us see what happens if $x+y-100=0$ and consider the two possibilities for x:

Case $x=0$**:**Then, $y=100$, which would lead (Eq. 10) to $\lambda_4=0$ and (Eq. 6) to $x=\lambda_1=0$, that was discussed in the previous item. So, we need top explore the other possibility for x.

Case $x>0$**:**From Eq. 9 $\lambda_3=0$ and, from Eqs. 5 and 6:

$$\begin{cases} \quad y=\lambda_1+\lambda_2 \\ x \quad =\lambda_1+\lambda_4 \end{cases}$$

let us try what happens if, e.g., $\lambda_2\neq 0$ (or, said in other words, if constraint 12 is active): $x=40$. As we know we do not want $\lambda_1=0$, from Eq. 7 we obtain $x+y-100=0\Rightarrow y=60$.

[1]Recall that both variables and multiplieras must be positive or zero, so, the only possibility for the equation to fullfill is that both are zero.

The point $(x, y) = (40, 60)$ fullfills the KKT conditions and is a maximum in the constrained maximization problem (as can be seen in Figure 2).

Ex. 31 — KKT conditions - Complete problem: Solve the following optimization problem:[1]

$$\text{Optimize } f(x,y,z) = x + y + z,$$

$$\text{subject to: } \begin{cases} h_1(x,y,z) = (y-1)^2 + z^2 \leq 1, \\ h_2(x,y,z) = x^2 + (y-1)^2 + z^2 \leq 3. \end{cases}$$

Answer (Ex. 31) — By Weierstrass' theorem, there exist a maximum and a minimum for the problem. The Karush-Kuhn-Tucker (KKT) conditions are:

- Stationarity conditions:

$$\begin{cases} 1 + 2\mu_2 x = 0, \\ 1 + 2\mu_1(y-1) + 2\mu_2(y-1) = 0, \\ 1 + 2\mu_1 z + 2\mu_2 z = 0. \end{cases}$$

- Slackness conditions:

$$\begin{cases} \mu_1\left[(y-1)^2 + z^2 - 1\right] = 0, \\ \mu_2\left[x^2 + (y-1)^2 + z^2 - 3\right] = 0. \end{cases}$$

- Feasibility conditions:

$$(y-1)^2 + z^2 \leq 1, \qquad x^2 + (y-1)^2 + z^2 \leq 3.$$

- Sign conditions:

$$\mu_1, \mu_2 \geq 0 \Rightarrow \text{local minimum}, \qquad \mu_1, \mu_2 \leq 0 \Rightarrow \text{local maximum}.$$

From the slackness conditions we distinguish four cases

$$\begin{cases} \mu_1 = 0 \implies \begin{cases} \mu_2 = 0 & \text{(Case I)} \\ x^2 + (y-1)^2 + z^2 - 3 = 0 & \text{(Case II)} \end{cases} \\ (y-1)^2 + z^2 - 1 = 0 \implies \begin{cases} \mu_2 = 0 & \text{(Case III)} \\ x^2 + (y-1)^2 + z^2 - 3 = 0 & \text{(Case IV)} \end{cases} \end{cases}$$

but the first stationarity equation implies that $\mu_2 \neq 0$, so only cases II and IV must be checked.

- Case II: $\mu_1 = 0$ and $x^2 + (y-1)^2 + z^2 - 3 = 0$.

$$\begin{aligned} P_1 &= (1,2,1), & \mu &= (0, -\tfrac{1}{2}), \\ P_2 &= (-1,0,-1), & \mu &= (0, \tfrac{1}{2}). \end{aligned}$$

- Case IV: $(y-1)^2 + z^2 - 1 = 0$ and $x^2 + (y-1)^2 + z^2 - 3 = 0$.

$$\begin{aligned} P_3 &= \left(\sqrt{2}, 1 + \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), & \mu &= \left(-\tfrac{1}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}\right), \\ P_4 &= \left(\sqrt{2}, 1 - \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right), & \mu &= \left(\tfrac{3}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}\right), \\ P_5 &= \left(-\sqrt{2}, 1 + \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), & \mu &= \left(-\tfrac{3}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}\right), \\ P_6 &= \left(-\sqrt{2}, 1 - \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right), & \mu &= \left(\tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}\right). \end{aligned}$$

After applying feasibility and sign conditions, the results are summarized as follows:

P	μ	Feasibility	Sign	Conclusion
P_1	$(0, -\frac{1}{2})$	*NO*	–	–
P_2	$(0, \frac{1}{2})$	*NO*	–	–
P_3	$(-\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}})$	*YES*	*Negative*	Conditional Maximum
P_4	$(\frac{3}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}})$	*YES*	*NO*	–
P_5	$(-\frac{3}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$	*YES*	*NO*	–
P_6	$(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$	*YES*	*Positive*	Conditional Minimum

Therefore:

$$\boxed{\begin{aligned} &\text{Conditional Maximum:} \quad P_3 = \left(\sqrt{2}, 1 + \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \\ &\text{Conditional Minimum:} \quad P_6 = \left(-\sqrt{2}, 1 - \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right). \end{aligned}}$$

Here you have a MATLAB script that implements the KKT conditions to solve this problem:

Script 2: KKT conditions for problem 31

```
syms x y z mu1 mu2 real

% Objective function
f = x + y + z;

% Constraints
h1 = (y - 1)^2 + z^2 - 1;
h2 = x^2 + (y - 1)^2 + z^2 - 3;

% Stationary conditions
eq1 = 1 + 2*mu2*x == 0;
eq2 = 1 + 2*mu1*(y - 1) + 2*mu2*(y - 1) == 0;
eq3 = 1 + 2*mu1*z + 2*mu2*z == 0;

% Complementary slackness conditions
eq4 = mu1*h1 == 0;
eq5 = mu2*h2 == 0;

% Symbolic resolution (considering cases)
sol = solve([eq1, eq2, eq3, eq4, eq5], [x, y, z, mu1, mu2], 'Real', true);

disp(struct2table(sol))
```

Ex. 32 — KKT example: maximize $f(x, y) = xy$ subject to $x + y^2 \leq 2$: *(JVF,1st*
Consider the nonlinear programming problem

$$\begin{aligned} \max \quad & f(x, y) = xy \\ \text{s.t.} \quad & x + y^2 \leq 2, \\ & x \geq 0,\ y \geq 0. \end{aligned} \tag{15}$$

1. Is the feasible region bounded? Justify your answer.
2. Write the Karush–Kuhn–Tucker (KKT) necessary conditions for this problem.
3. Find the candidate points that satisfy the KKT conditions.

Answer (Ex. 32) — Starting from Eqs. 15, the detailed solution is as follows:

1. The feasible region is bounded, because the constraint $x + y^2 \leq 2$ describes the area below the parabola $y = \sqrt{2 - x}$, which intersects the

axes at $(2, 0)$ and $(0, \sqrt{2})$. Additionally, the nonnegativity constraints $x \geq 0$ and $y \geq 0$ restrict the feasible region to the first quadrant. Thus, the feasible region is enclosed and bounded.

A continuous function like $f(x, y) = xy$ defined on a closed and bounded region attains its maximum and minimum values (Weierstrass theorem).

2. Formulation of KKT conditions.

The Karush-Kuhn-Tucker (KKT) conditions state that for a non-linear optimization problem of the form:

$$\begin{aligned} \text{maximize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_i(\mathbf{x}) = 0, \quad i = 1, \dots, p \\ & h_j(\mathbf{x}) \leq 0, \quad j = 1, \dots, q \end{aligned}$$

where f, g_i, and h_j are continuously differentiable functions, a point $\mathbf{x}^*$ is a local maximum or local minimum if there exist multipliers λ_i (for equality constraints), and μ_j (for inequality constraints) such that the following conditions are satisfied:

(a) **Stationarity:**

$$\nabla f(\mathbf{x}^*) - \sum_{i=1}^{p} \lambda_i \nabla g_i(\mathbf{x}^*) - \sum_{j=1}^{q} \mu_j \nabla h_j(\mathbf{x}^*) = 0$$

(b) **Primal feasibility:**

$$g_i(\mathbf{x}^*) = 0 \quad \text{for } i = 1, \dots, p$$

$$h_j(\mathbf{x}^*) \leq 0 \quad \text{for } j = 1, \dots, q$$

(c) **Complementary slackness:**

$$\mu_j h_j(\mathbf{x}^*) = 0 \quad \text{for } j = 1, \dots, q$$

This is, either the constraint is active ($h_j(\mathbf{x}^*) = 0$) or its multiplier is zero ($\mu_j = 0$).

(d) **Dual feasibility (sign condition):**

$$\mu_j \geq 0 \quad \text{for all } j = 1, \dots, q$$

We first write the constraints in the standard form

$$h_1(x,y) = x + y^2 - 2 \le 0, \quad h_2(x,y) = -x \le 0, \quad h_3(x,y) = -y \le 0. \tag{16}$$

Introduce Lagrange multipliers $\mu_1, \mu_2, \mu_3 \ge 0$ associated with h_1, h_2, h_3. The KKT *stationarity* conditions are

$$\begin{aligned} \frac{\partial f}{\partial x} - \mu_1 \frac{\partial h_1}{\partial x} - \mu_2 \frac{\partial h_2}{\partial x} - \mu_3 \frac{\partial h_3}{\partial x} = 0 \quad &\Longrightarrow \quad y - \mu_1 + \mu_2 = 0, \\ \frac{\partial f}{\partial y} - \mu_1 \frac{\partial h_1}{\partial y} - \mu_2 \frac{\partial h_2}{\partial y} - \mu_3 \frac{\partial h_3}{\partial y} = 0 \quad &\Longrightarrow \quad x - 2y\mu_1 + \mu_3 = 0. \end{aligned} \tag{17}$$

Complementary slackness and primal feasibility give

$$\mu_1(x + y^2 - 2) = 0, \qquad \mu_2 x = 0, \qquad \mu_3 y = 0, \tag{18}$$

and

$$x + y^2 \le 2, \qquad x, y, \mu_1, \mu_2, \mu_3 \ge 0. \tag{19}$$

3. **Case analysis and candidates.**

We now examine the possible cases that arise from the complementary slackness conditions (18).

- **Case 1:** $\mu_1 = 0$.
  Then from the stationarity equations (17) we obtain

  $$y + \mu_2 = 0, \qquad x + \mu_3 = 0.$$

  Since all variables are nonnegative, this forces

  $$x = 0,\ y = 0,\ \mu_2 = 0,\ \mu_3 = 0.$$

  Thus, $(x,y) = (0,0)$ satisfies the KKT conditions, but it is not a maximum because $f(0,0) = 0$ but $f > 0$ for interior points in the feasible region.
- **Case 2:** $\mu_1 > 0$. By complementary slackness in (18), the first constraint is active:

  $$x + y^2 = 2. \tag{20}$$

  We now consider subcases depending on x.

–**Case 2a:** $x > 0$. Then $\mu_2 = 0$, and from the first equation of (17) we have $\mu_1 = y$. Substituting into the second equation of (17) gives

$$x - 2y\mu_1 + \mu_3 = x - 2y^2 + \mu_3 = 0. \tag{21}$$

Using the active constraint (20), we replace $x = 2 - y^2$:

$$(2 - y^2) - 2y^2 + \mu_3 = 2 - 3y^2 + \mu_3 = 0. \tag{22}$$

Since $\mu_3 \geq 0$, it follows that $2 - 3y^2 \leq 0$. If $\mu_3 > 0$, the complementary condition $\mu_3 y = 0$ would force $y = 0$, contradicting $\mu_1 = y > 0$. Hence $\mu_3 = 0$, and from (22) we obtain

$$2 - 3y^2 = 0 \quad \Longrightarrow \quad y^2 = \tfrac{2}{3}, \quad y = \sqrt{\tfrac{2}{3}} > 0. \tag{23}$$

Then, substituting into (20),

$$x = 2 - y^2 = 2 - \tfrac{2}{3} = \tfrac{4}{3}. \tag{24}$$

Therefore, one KKT candidate point is

$$(x, y) = \left(\tfrac{4}{3}, \sqrt{\tfrac{2}{3}}\right). \tag{25}$$

–**Case 2b:** $x = 0$. From (20), $y^2 = 2$ so $y = \sqrt{2} > 0$. Complementary slackness $\mu_3 y = 0$ gives $\mu_3 = 0$. The second equation of (17) becomes

$$0 - 2y\mu_1 + 0 = 0 \quad \Longrightarrow \quad \mu_1 = 0, \tag{26}$$

which contradicts the assumption $\mu_1 > 0$. Thus, this subcase reduces to Case 1 and yields no new feasible maximizer.

- **Conclusion of case analysis.** The only KKT candidates are

$$(0, 0) \quad \text{and} \quad \left(\tfrac{4}{3}, \sqrt{\tfrac{2}{3}}\right). \tag{27}$$

- **Determination of the global maximum.**

Evaluating the objective function for the candidates in (27):

$$f(0, 0) = 0, \qquad f\left(\tfrac{4}{3}, \sqrt{\tfrac{2}{3}}\right) = \tfrac{4}{3} \cdot \sqrt{\tfrac{2}{3}} = \tfrac{4}{3}\sqrt{\tfrac{2}{3}} > 0. \tag{28}$$

The feasible region of (15) is closed and bounded, so a global maximum exists. Comparing the values in (28) shows that the global maximum is attained at the point in (25):

$$\boxed{(x^*, y^*) = \left(\tfrac{4}{3}, \sqrt{\tfrac{2}{3}}\right), \quad f(x^*, y^*) = \frac{4}{3}\sqrt{\frac{2}{3}}.} \tag{29}$$

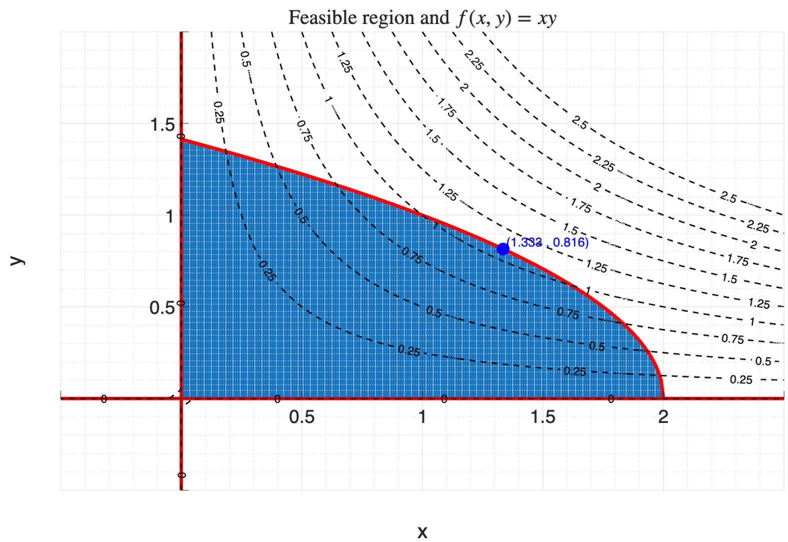


Figure 3: Feasible region, isocontour plot for $f(x, y) = xy$ and optimal point in Exercise 32.

■

2 Linear Optimization

Ex. 33 — Formulation of an optimization problem: Consider a manufacturing company that produces two products, P_1 and P_2. The company aims to maximize its profit. The profit from each product is given as follows:

- P_1: \$40 per unit - P_2: \$30 per unit

The company has the following constraints based on its resources:

1. The availability of raw material limits production to a maximum of 100 units of P_1 and 80 units of P_2. 2. The total production time available is 160

hours, where producing one unit of P_1 takes 2 hours and one unit of P_2 takes 1 hour.
Can you formulate the operations research problem?

Answer (Ex. 33) — The problem can be stated as:

$$\begin{aligned}
\text{Objective Function:}\quad & \text{Maximize } Z = 40x_1 + 30x_2 \\
\text{Subject to:}\quad & \\
& 2x_1 + x_2 \leq 160 \quad \text{(Total production time constraint)} \\
& x_1 \leq 100 \quad \text{(Raw material constraint for } P_1\text{)} \\
& x_2 \leq 80 \quad \text{(Raw material constraint for } P_2\text{)} \\
& x_1, x_2 \geq 0 \quad \text{(Non-negativity constraints)}
\end{aligned}$$

■

2.1 The Simplex method

Ex. 34 — **Unique optimal solution:** Maximize $z = 2x+3y$ subject to:

$$\begin{aligned}
x + y &\leq 6 \\
x &\leq 4 \\
x, y &\geq 0
\end{aligned}$$

Answer (Ex. 34) — Introduce slack variables s_1, s_2 to convert to equalities:

$$\begin{aligned}
x + y + s_1 &= 6, \quad s_1 \geq 0, \\
x + s_2 &= 4, \quad s_2 \geq 0.
\end{aligned}$$

Objective rewritten in standard form:

$$z - 2x - 3y = 0.$$

We form the initial simplex tableau (using the convention where the objective row contains the coefficients of $z -$ (linear form) $= 0$, i.e. the top row for constraints and bottom row for z):

	x	y	s_1	s_2	RHS
s_1	1	1	1	0	6
s_2	1	0	0	1	4
z	-2	-3	0	0	0

Step 1 — choose entering variable. For a maximization problem, we look for negative coefficients in the z-row (since $z - c^T x = 0$); the most negative is -3 corresponding to y. So y will enter.
Step 2 — choose leaving variable (minimum ratio test). Compute RHS / (coefficient of entering variable):
- For row s_1: coefficient of y is 1 so ratio $6/1 = 6$. - For row s_2: coefficient of y is 0 (cannot be pivot row).
Minimum positive ratio is 6 (row of s_1). Thus pivot on the element in row s_1, column y (which is 1). After pivoting, y will replace s_1 as basic variable. Perform the pivot (make pivot = 1 and eliminate y elsewhere).
The pivot row (row s_1) already has a 1 in the y-column, so it stays:

$$\text{New row (basic } y) : \quad y + x + s_1 = 6$$

We will express the pivot row as $y = 6 - x - s_1$.
Eliminate y from the z-row: add $3\times$ (pivot row) to the z-row to remove -3 in the z-row.
Arithmetic for the z-row update:

$$\text{old } z\text{-row } (-2,\ -3,\ 0,\ 0 \mid 0) \quad + \ 3 \times \text{pivot row } (1,\ 1,\ 1,\ 0 \mid 6)$$

gives

$$z\text{-row}_{\text{new}} = (-2+3\cdot1,\ -3+3\cdot1,\ 0+3\cdot1,\ 0+3\cdot0 \mid 0+3\cdot6) = (1,\ 0,\ 3,\ 0 \mid 18).$$

The row s_2 has zero in column y so it is unchanged.
So the tableau after the pivot becomes:

	x	y	s_1	s_2	RHS
y	1	1	1	0	6
s_2	1	0	0	1	4
z	1	0	3	0	18

(Interpretation: the first row represents $y+x+s_1 = 6$, second row $x+s_2 = 4$, and objective equation is $z + x + 3s_1 = 18$.)

Step 3 — check optimality. Using the same sign convention (bottom row is coefficients of $z-(\cdots)=0$), we look for negative coefficients in the z-row to select another entering variable. The current z-row coefficients for nonbasic variables are:
- coefficient of x in the z-row is $+1$, - coefficient of s_1 is $+3$, - coefficient of s_2 is 0.
There are no negative entries in the z-row. Therefore, the current basic feasible solution is optimal.
Read off the basic feasible solution by setting nonbasic variables to zero. From the tableau: - Basic variables are y and s_2 (they correspond to the left labels). Nonbasic variables are x and s_1. Set $x=0$ and $s_1=0$. - From row y: $y=6-x-s_1=6$. - From row s_2: $s_2=4-x=4$.
Thus the primal solution is:

$$(x,y)=(0,6).$$

Compute objective:

$$z=2x+3y=2\cdot 0+3\cdot 6=18.$$

Concluding, the simplex method yields the optimal solution

$$\boxed{(x,y)=(0,6), \qquad z_{\max}=18.}$$

(As a sanity check: the corner points of the feasible polygon are $(0,0),(0,6),(4,0),(4,2)$. Their objective values are $0,18,8,14$ respectively, so indeed $(0,6)$ is best.)

Ex. 35 — Modified problem: maximize $z=x+2y$: Maximize $z=x+2y$ subject to:

$$\begin{aligned} x+2y &\leq 8,\\ 2x+y &\leq 8,\\ x,y &\geq 0. \end{aligned}$$

Answer (Ex. 35) — Introduce slack variables s_1,s_2 to convert inequalities to equalities:

$$x+2y+s_1=8, \qquad 2x+y+s_2=8, \qquad s_1,s_2\geq 0.$$

Objective in standard form:

$$z - x - 2y = 0.$$

Initial simplex tableau (columns: x, y, s_1, s_2)

	x	y	s_1	s_2	RHS
s_1	1	2	1	0	8
s_2	2	1	0	1	8
z	-1	-2	0	0	0

Choose entering variable. For maximization we look for negative reduced costs in the z-row. The most negative entry is -2 under y, so y enters.

Minimum ratio test (RHS / coefficient of y):

$$\begin{aligned} \text{row } s_1 &: \quad 8/2 = 4, \\ \text{row } s_2 &: \quad 8/1 = 8. \end{aligned}$$

Minimum is 4 (row s_1), so pivot on the element in row s_1, column y. Thus y will replace s_1 as a basic variable.

Pivot 1: make pivot = 1 (divide row s_1 by 2).

$$\text{new (row for } y) = \tfrac{1}{2} \cdot (1,\ 2,\ 1,\ 0 \mid 8) = \left(\tfrac{1}{2},\ 1,\ \tfrac{1}{2},\ 0 \mid 4\right).$$

Eliminate y from the other rows:

$$\begin{aligned} \text{new } s_2 &= \text{old } s_2 - 1 \cdot (\text{new } y \text{ row}) \\ &= (2,\ 1,\ 0,\ 1 \mid 8) - (\tfrac{1}{2},\ 1,\ \tfrac{1}{2},\ 0 \mid 4) \\ &= \left(\tfrac{3}{2},\ 0,\ -\tfrac{1}{2},\ 1 \mid 4\right), \end{aligned}$$

$$\begin{aligned} \text{new } z &= \text{old } z + 2 \cdot (\text{new } y \text{ row}) \quad (\text{to eliminate } -2) \\ &= (-1,\ -2,\ 0,\ 0 \mid 0) + 2 \cdot (\tfrac{1}{2},\ 1,\ \tfrac{1}{2},\ 0 \mid 4) \\ &= (0,\ 0,\ 1,\ 0 \mid 8). \end{aligned}$$

Tableau after Pivot 1

	x	y	s_1	s_2	RHS
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	4
s_2	$\frac{3}{2}$	0	$-\frac{1}{2}$	1	4
z	0	0	1	0	8

Optimality check. In the z-row the reduced costs for the original variables x, y are $0, 0$ (non-negative). Thus no negative reduced cost remains and the current basic feasible solution is optimal.

Read the solution from the final tableau. Nonbasic variables are x and s_1 (set them to zero). From the rows:

$$y = 4 - \tfrac{1}{2}x - \tfrac{1}{2}s_1 \Rightarrow y = 4, \qquad s_2 = 4 - \tfrac{3}{2}x + \tfrac{1}{2}s_1 \Rightarrow s_2 = 4.$$

Hence the primal solution is

$$(x, y) = (0, 4),$$

and the objective value is

$$z = x + 2y = 0 + 2 \cdot 4 = 8,$$

which matches the RHS of the z-row.

On multiple optimal solutions. Note the reduced cost of the nonbasic original variable x is 0 in the final tableau. This indicates alternative optima exist: introducing x into the basis would not change the objective value. Indeed, evaluating the objective at the feasible corner $(\frac{8}{3}, \frac{8}{3})$ (the intersection of the two constraint lines) yields

$$z = \tfrac{8}{3} + 2 \cdot \tfrac{8}{3} = \tfrac{24}{3} = 8,$$

so $(\frac{8}{3}, \frac{8}{3})$ is also optimal. Therefore there are infinitely many optimal solutions lying on the segment joining $(0, 4)$ and $(\frac{8}{3}, \frac{8}{3})$, all with $z_{\max} = 8$.

$$\boxed{\text{Optimal value } z_{\max} = 8, \text{ with infinite optima on the segment between } (0,4) \text{ and } (\tfrac{8}{3}, \tfrac{8}{3}).}$$

Ex. 36 — **Infeasible region:** Minimize $z = x + y$ subject to:

$$\begin{aligned} x + y &\geq 10 \\ x + y &\leq 5 \\ x, y &\geq 0 \end{aligned}$$

Answer (Ex. 36) — The constraints $x + y \geq 10$ and $x + y \leq 5$ are contradictory. No feasible region exists, hence the problem is infeasible.

Ex. 37 — Unbounded problem: Maximize $z = 3x + 2y$ subject to:

$$y \geq x + 1,$$
$$x, y \geq 0.$$

Answer (Ex. 37) — We first rewrite the inequality $y \geq x + 1$ in standard form. Move all variables to the left-hand side and introduce a surplus variable $s_1 \geq 0$:

$$-y + x + s_1 = -1.$$

To keep the right-hand side nonnegative (required for a standard initial basic feasible solution), multiply both sides by -1:

$$y - x - s_1 = 1.$$

Thus our constraint system is:

$$y - x - s_1 = 1, \quad x, y, s_1 \geq 0.$$

The objective function in standard form:

$$z - 3x - 2y = 0.$$

Initial simplex tableau (columns: x, y, s_1)

	x	y	s_1	RHS
y	-1	1	-1	1
z	-3	-2	0	0

However, this tableau is not suitable as an initial BFS because the basic variable y has a negative coefficient for x (we cannot express a nonnegative basic variable y with positive coefficients of nonbasic variables). We could attempt to construct an initial feasible point manually.

Feasible region:
From $y \geq x + 1$ and nonnegativity, we see that:

$$\text{Feasible region } = \{(x, y) \mid y \geq x + 1,\ x, y \geq 0\}.$$

Behavior of the objective function:

For any feasible point satisfying $y = x + 1$,

$$z = 3x + 2y = 3x + 2(x + 1) = 5x + 2.$$

As $x \to \infty$, $z \to \infty$.
Hence, there is no finite maximum value of z.

Conclusion:

$$\boxed{\text{The linear program is unbounded.}}$$

Geometrically, the feasible region is the half-plane above the line $y = x + 1$. The objective function lines $3x + 2y = c$ have slope $-\frac{3}{2}$, and by moving this line upward (increasing c) we can always find new feasible points with larger z values, without limit.
Therefore, no optimal solution exists.

Ex. 38 — Unique minimum in Linear Programming: Minimize

$$z = 5x + 2y$$

subject to

$$\begin{aligned} x + y &\geq 4, \\ 2x + y &\geq 5, \\ x, y &\geq 0. \end{aligned}$$

Answer (Ex. 38) — Because the constraints are of type "$\geq$", we introduce surplus and artificial variables:

$$x + y - s_1 + a_1 = 4, \qquad 2x + y - s_2 + a_2 = 5,$$

with

$$s_1, s_2 \geq 0, \qquad a_1, a_2 \geq 0.$$

In Phase I we minimize

$$w = a_1 + a_2.$$

After performing the Phase I simplex procedure (details omitted here for brevity), we obtain the basic feasible solution

$$x = 1, \qquad y = 3,$$

and

$$w = 0,$$

so a feasible solution for the original problem has been found.
We now proceed to Phase II and evaluate the original objective function

$$z = 5x + 2y$$

at the candidate corner points of the feasible region.
The feasible region is defined by

$$x + y \geq 4, \quad 2x + y \geq 5, \quad x, y \geq 0.$$

The corner points of this region are:

$$\begin{aligned} x + y = 4,\ \ 2x + y = 5 \quad &\Rightarrow \quad (x, y) = (1, 3), \\ x + y = 4,\ \ y = 0 \quad &\Rightarrow \quad (x, y) = (4, 0), \\ 2x + y = 5,\ \ x = 0 \quad &\Rightarrow \quad (x, y) = (0, 5). \end{aligned}$$

We evaluate the objective function at these three points:

$$z(1, 3) = 5(1) + 2(3) = 5 + 6 = 11,$$

$$z(4, 0) = 5(4) + 2(0) = 20,$$

$$z(0, 5) = 5(0) + 2(5) = 10.$$

Therefore, the minimum value is

$$\boxed{z_{\min} = 10}$$

and it is attained at

$$\boxed{(x, y) = (0, 5)}.$$

$$\boxed{\text{The unique optimal solution is } (0, 5).}$$

Ex. 39 — Simplex optimization: unique solution: Use Simplex to solve this LP problem:

$$\text{Maximize:} \quad z = 3x_1 + x_2$$

$$\text{subject to:} \quad \begin{aligned} & x_2 \le 5 \\ & x_1 + x_2 \le 10 \\ & -x_1 + x_2 \ge -2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Answer (Ex. 39) — The third constraint can be rewritten as:

$$x_1 - x_2 \le 2$$

We introduce slack variables $s_1, s_2, s_3 \ge 0$:

$$\begin{aligned} x_2 + s_1 &= 5 \\ x_1 + x_2 + s_2 &= 10 \\ x_1 - x_2 + s_3 &= 2 \end{aligned}$$

$$\text{Maximize } z = 3x_1 + x_2$$

Initial Simplex Tableau

Basis	x_1	x_2	s_1	s_2	s_3	RHS
s_1	0	1	1	0	0	5
s_2	1	1	0	1	0	10
s_3	1	-1	0	0	1	2
z	-3	-1	0	0	0	0

Iteration 1 The entering variable is x_1 (most negative coefficient in the z-row).

$$\text{Minimum ratio test: } \frac{5}{0} = \infty, \quad \frac{10}{1} = 10, \quad \frac{2}{1} = 2 \Rightarrow \text{ leaving variable: } s_3$$

Pivot element: row s_3, column x_1.

$$x_1 = 2 + x_2 - s_3$$

Substitute into the other equations to obtain the new tableau:

Basis	x_1	x_2	s_1	s_2	s_3	RHS
s_1	0	1	1	0	0	5
s_2	0	2	0	1	-1	8
x_1	1	-1	0	0	1	2
z	0	-4	0	0	3	6

Iteration 2 The entering variable is x_2 (most negative coefficient: -4).

$$\text{Minimum ratio test: } \frac{5}{1} = 5, \quad \frac{8}{2} = 4, \quad \frac{2}{-1} \text{ invalid} \Rightarrow \text{ leaving variable: } s_2$$

Pivot element: row s_2, column x_2.

$$x_2 = 4 - 0.5s_2 + 0.5s_3$$

Substitute and simplify:

Basis	x_1	x_2	s_1	s_2	s_3	RHS
s_1	0	0	1	-0.5	0.5	1
x_2	0	1	0	0.5	-0.5	4
x_1	1	0	0	0.5	0.5	6
z	0	0	0	2	1	22

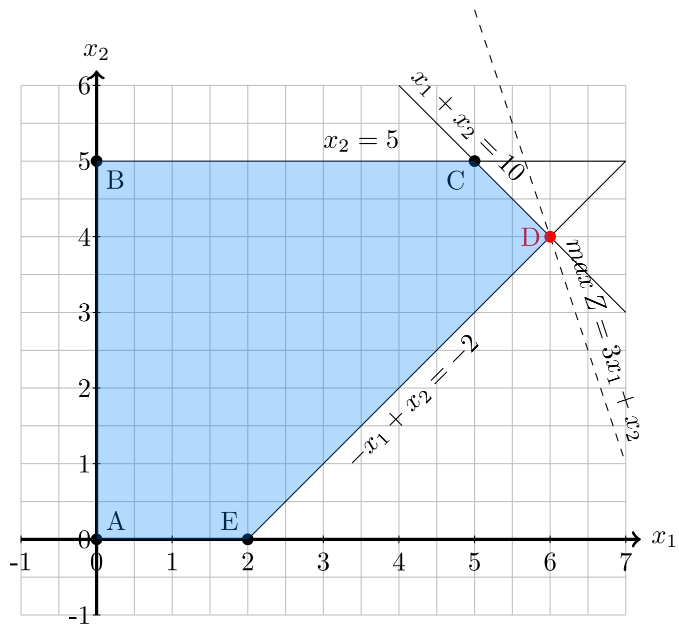


Figure 4: Graphical solution for Exercise 39, showing the unique optimal solution at the corner point.

Optimal Solution Since there are no negative coefficients in the z-row, the solution is optimal:

$$x_1 = 6, \quad x_2 = 4, \quad z_{\max} = 3(6) + 4 = 22$$

$$\boxed{x_1 = 6,\ x_2 = 4,\ z_{\max} = 22}$$

■

Ex. 40 — Simplex optimization: multiple solutions: Use Simplex to solve this LP problem:

$$\begin{aligned} \max \quad & Z = 1.0x_1 + 2.0x_2 \\ \text{s.t.} \quad & \begin{cases} -1.0x_1 + 1.0x_2 \le 2.0 \\ 1.0x_1 + 2.0x_2 \le 8.0 \\ 1.0x_1 + 0.0x_2 \le 6.0 \end{cases} \\ & x_1, x_2 \ge 0 \end{aligned}$$

Canonical Form:

$$Z - 1.0x_1 - 2.0x_2 - 0\,s_1 - 0\,s_2 - 0\,s_3 = 0$$

$$\begin{cases} -1.0x_1 + 1.0x_2 + s_1 = 2.0 \\ 1.0x_1 + 2.0x_2 + s_2 = 8.0 \\ 1.0x_1 + 0.0x_2 + s_3 = 6.0 \end{cases}$$

Simplex Tableau

Answer (Ex. 40) — Iteration 0 Entering and leaving variables:

- Entering variable: x_2
- Leaving variable: s_1

Table 1: Simplex Tableau after iteration 0

Basic	x_1	x_2	s_1	s_2	s_3	RHS
x_2	-1.0	1.0	1.0	0.0	0.0	2.0
s_2	3.0	0.0	-2.0	1.0	0.0	4.0
s_3	1.0	0.0	0.0	0.0	1.0	6.0
Z	-3.0	0.0	2.0	0.0	0.0	4.0

Iteration 1 Entering and leaving variables:

- Entering variable: x_1
- Leaving variable: s_2

Table 2: Simplex Tableau after iteration 1

Basic	x_1	x_2	s_1	s_2	s_3	RHS
x_2	0.0	1.0	0.33	0.33	0.0	3.33
x_1	1.0	0.0	-0.67	0.33	0.0	1.33
s_3	0.0	0.0	0.67	-0.33	1.0	4.67
Z	0.0	0.0	0.0	1.0	0.0	8.0

Observation on Multiple Solutions: In the final tableau, all coefficients in the Z-row are ≥ 0, so the solution is optimal. However, notice that the column corresponding to the non-basic variable s_1 has a coefficient of exactly zero in the Z-row.

- This indicates that we can increase s_1 (relaxing constraint 1) without decreasing Z.
- Geometrically, there is an edge of the feasible polyhedron where Z = 8.
- Therefore, any convex combination of the basic optimal vertices:

$$x^{(A)} = (4/3, 10/3), \quad x^{(B)} = (6, 1)$$

  is also optimal:

$$x(t) = (1-t)x^{(A)} + t\,x^{(B)}, \quad t \in [0, 1].$$

Optimal Solution (example from the final Simplex tableau):

$$x_1 = 1.3333, \quad x_2 = 3.3333, \quad Z = 8$$

Slack Variables:

$$s_1 = 0, \quad s_2 = 0, \quad s_3 = 4.6667$$

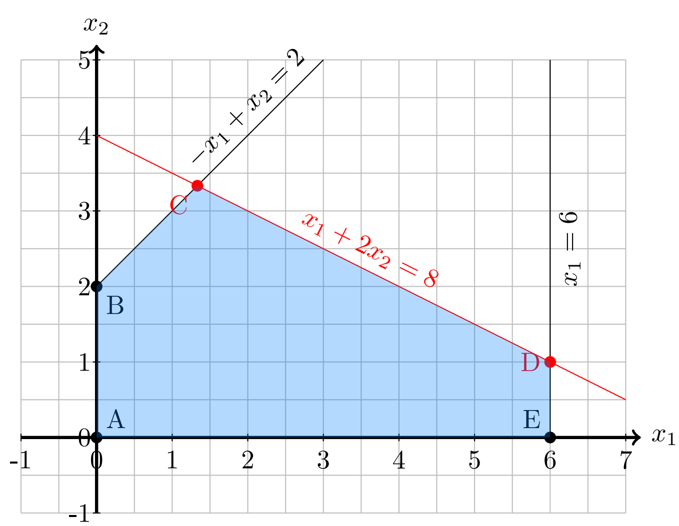


Figure 5: Graphical solution of Exercise 40, showing multiple optimal solutions along the edge between points C and D.

Conclusion: The presence of a zero coefficient in the Z-row for a non-basic variable indicates infinitely many optimal solutions. The solution found is only one of the multiple possible solutions.
■

Ex. 41 — Simplex optimization: unbound solutions: Use the Simplex method to solve the following LP problem:

$$\text{Maximize} \quad z = 3x_1 + x_2,$$
$$\text{subject to}$$
$$\begin{cases} x_1 + x_2 \geq 4, \\ \\ -x_1 + x_2 \leq 4, \\ \\ -x_1 + 2x_2 \geq -4, \\ \\ x_1, x_2 \geq 0. \end{cases}$$

Step 1. Standard form

$$\begin{aligned} (1) &: \ x_1 + x_2 - s_1 + a_1 = 4, \\ (2) &: \ -x_1 + x_2 + s_2 = 4, \\ (3) &: \ x_1 - 2x_2 + s_3 = 4. \end{aligned}$$

All variables nonnegative.

Phase I: minimize $W = a_1$ Initial basis: a_1, s_2, s_3.

Basis	x_1	x_2	s_1	s_2	s_3	RHS
a_1	1	1	−1	0	0	4
s_2	−1	1	0	1	0	4
s_3	1	−2	0	0	1	4
W	−1	−1	+1	0	0	−4

Add row a_1 to the W-row to eliminate a_1 column effect:

Basis	x_1	x_2	s_1	s_2	s_3	RHS
a_1	1	1	-1	0	0	4
s_2	-1	1	0	1	0	4
s_3	1	-2	0	0	1	4
W	0	0	0	0	0	0

No negative coefficients remain, so a_1 can be removed. A feasible basis exists. Choose x_1, s_2, s_3 as the new basis (pivoting x_1 in for a_1). Substitute $x_1 = 4 - x_2 + s_1$ from (1).

Phase II: maximize $z = 3x_1 + x_2$ After substitution:

$$z = 3(4 - x_2 + s_1) + x_2 = 12 - 2x_2 + 3s_1.$$

The canonical tableau becomes:

Basis	x_2	s_1	RHS
x_1	-1	$+1$	4
s_2	-2	$+1$	8
s_3	$+3$	-1	0
z	-2	$+3$	12

The coefficient $+3$ under s_1 indicates z increases if s_1 increases. However, from $s_3 = 3x_2 - s_1 \geq 0$ we have $s_1 \leq 3x_2$. Thus, increasing x_2 allows s_1 to grow indefinitely.
When x_2 increases, all RHS remain feasible (no basic variable becomes negative), hence there is no leaving variable—this is the Simplex signal of unboundedness.

Conclusion The LP is feasible but unbounded. The objective function can increase without limit along the ray:

$$(x_1, x_2) = (4 + t, t), \quad t \geq 0.$$

Hence, no finite maximum exists.

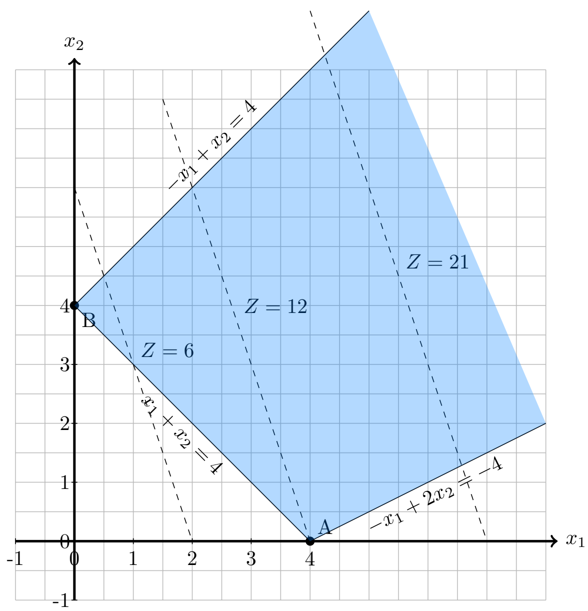


Figure 6: Graphical view of Exercise 41. The feasible region extends infinitely in the direction of increasing z.
■

Answer (Ex. 41) — Ex. 42 — Simplex optimization: unfeasible solutions: — Use Simplex to solve this LP problem:

$$\max \quad Z = 3.0x_1 + 1.0x_2$$
$$\text{s.t.} \quad \begin{cases} 1.0x_1 - 1.0x_2 \leq -4.0 \\ -1.0x_1 + 2.0x_2 \leq -4.0 \end{cases}$$
$$x_1, x_2 \geq 0$$

Canonical Form:

$$Z - 3.0x_1 - 1.0x_2 - 0\,s_1 - 0\,s_2 = 0$$

$$\begin{cases} 1.0x_1 - 1.0x_2 + s_1 = -4.0 \\ -1.0x_1 + 2.0x_2 + s_2 = -4.0 \end{cases}$$

Simplex Tableau

Answer (Ex. 42) — Iteration 0 System unbound / infeasible.

Table 3: Initial Simplex Tableau

Basic	x_1	x_2	s_1	s_2	RHS
s_1	1.0	-1.0	1.0	0.0	-4.0
s_2	-1.0	2.0	0.0	1.0	-4.0
Z	-3.0	-1.0	0.0	0.0	0.0

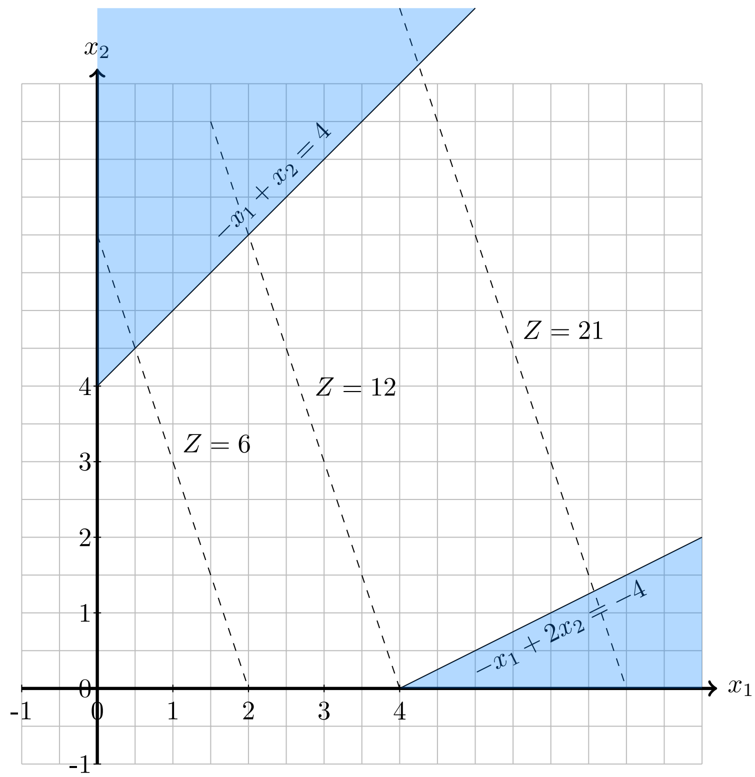


Figure 7: Graphical solution of Exercise 42, showing how the LP problem has no feasible solution.

Feasibility Analysis: - The initial RHS values are negative (-4 and -4). - Since all decision variables must satisfy $x_1, x_2 \geq 0$ and slack variables $s_1, s_2 \geq 0$, there is no combination that satisfies all constraints simultaneously. - Therefore, the system is infeasible, and no optimal solution exists.

Conclusion: This LP has no feasible solution. The initial tableau and Z-row clearly show the inconsistency, as the negative RHS values cannot be compensated by non-negative variables.
■

Ex. 43 — Feasible solution: Give a small linear program (two variables) and determine whether the point $(1, 2)$ is a feasible solution. The LP is:

$$\begin{aligned} \text{maximize} \quad & z = 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 4, \\ & 2x_1 + x_2 \le 6, \\ & x_1, x_2 \ge 0. \end{aligned}$$

Decide if $(1, 2)$ satisfies all constraints and compute the objective value at that point.

Answer (Ex. 43) — Check constraints: $1+2 = 3 \le 4$ and $2\cdot1+2 = 4 \le 6$, nonnegativity holds. So $(1, 2)$ is feasible. Objective value: $z = 3\cdot1+2\cdot2 = 7$.

Ex. 44 — Feasible region convexity: Consider a linear program with two variables and linear constraints. Show by a short computation that the feasible region is convex. Provide an explicit example: two feasible points and their convex combination stays feasible.

Answer (Ex. 44) — Take the LP: $x_1 + x_2 \le 4,\ x_1, x_2 \ge 0$. Points $A = (1, 1)$ and $B = (2, 0)$ are feasible. For $\theta = 0.5$, the combination $C = 0.5A + 0.5B = (1.5, 0.5)$. Check constraint: $1.5 + 0.5 = 2 \le 4$ and nonnegativity holds. Hence convexity illustrated.

Ex. 45 — Objective function linearity: Explain and demonstrate with calculation why the LP objective must be linear for standard linear programming. Given $z = ax_1 + bx_2$, evaluate z at two points and show linearity property $z(\alpha x + \beta y) = \alpha z(x) + \beta z(y)$ for $\alpha + \beta = 1$.

Answer (Ex. 45) — Let $x = (x_1, x_2) = (1, 0)$, $y = (0, 2)$, $a = 3, b = 2$. Then $z(x) = 3$, $z(y) = 4$. For $\alpha = 0.3, \beta = 0.7$ the combination is $(0.3, 1.4)$. $z(\alpha x + \beta y) = 3 \cdot 0.3 + 2 \cdot 1.4 = 0.9 + 2.8 = 3.7 = 0.3 \cdot 3 + 0.7 \cdot 4$. Linearity verified.

Ex. 46 — Optimum at vertex: Consider maximizing $z = 4x_1+$

$5x_2$ subject to $x_1 + x_2 \leq 6,\ x_1, x_2 \geq 0$. Show that any optimum occurs at a vertex. Compute the candidate vertices and find the maximum.

Answer (Ex. 46) — Vertices: (0,0),(6,0),(0,6). Evaluate: z(0,0)=0, z(6,0)=24, z(0,6)=30. Maximum at vertex (0,6) with z=30. This shows optimum at a vertex.

Ex. 47 — Graphical method variable limit: Use the graphical method to solve the LP: maximize $z = x_1 + 2x_2$ subject to $x_1 + x_2 \leq 5,\ x_1, x_2 \geq 0$. Explain why this graphical approach is limited to two variables and solve the LP.

Answer (Ex. 47) — Graphical method needs a 2D plot; for more variables visualization fails. Solve: vertices (0,0),(5,0),(0,5). Evaluate z: 0,5,10. Optimal at (0,5) with z=10.

Ex. 48 — Feasible region shape: Prove that the feasible region of a linear program with linear inequalities is a convex polyhedron. Give an example and sketch reasoning (algebraic).

Answer (Ex. 48) — Each linear inequality defines a half-space; intersection of finitely many half-spaces equals a convex polyhedron. Example: $x_1 + x_2 \leq 4,\ x_1 \geq 0, x_2 \geq 0$ yields a polygon in $\mathbb{R}^2$, hence a convex polyhedron.

Ex. 49 — Uniqueness of LP solutions: Provide a concrete LP example that has multiple optimal solutions (not unique). Solve it and show the set of optima.

Answer (Ex. 49) — LP: maximize $z = x_1 + x_2$ subject to $x_1 + x_2 \leq 2,\ x_1, x_2 \geq 0$. All points on the segment from (0,2) to (2,0) that satisfy $x_1 + x_2 = 2$ are optimal with z=2. Hence not unique.

Ex. 50 — Redundant constraint: Show by example that a redundant constraint does not affect the feasible region. Consider constraints $x_1 + x_2 \leq 4,\ 2x_1 + 2x_2 \leq 8$ and show the second is redundant.

Answer (Ex. 50) — Second inequality is exactly 2 times the first; any point satisfying $x_1 + x_2 \leq 4$ automatically satisfies $2x_1 + 2x_2 \leq 8$. Thus redundant.

Ex. 51 — Max vs min in LP: Demonstrate that linear programming can handle both maximization and minimization. Convert a maximization problem into a minimization equivalent and solve numerically: minimize $z' = -x_1 - 2x_2$ with $x_1 + x_2 \leq 3,\ x_i \geq 0$.

Answer (Ex. 51) — Minimizing $z' = -z$ is equivalent to maximizing z. Solve feasible vertices (0,0),(3,0),(0,3): z' values 0,-3,-6 so minimum at (0,3) which corresponds to maximum of z at (0,3).

Ex. 52 — Feasibility analysis in linear programming: Determine whether the linear programming problem

$$\max z = 2x_1 + 5x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 3,\ x_1 + x_2 \geq 5,\ x_1, x_2 \geq 0$$

has a feasible solution. Justify your answer.

Answer (Ex. 52) — The constraints $x_1 + x_2 \leq 3$ and $x_1 + x_2 \geq 5$ are incompatible. Hence, the problem is infeasible — no feasible region exists.

Ex. 53 — Multiple optimal solutions in LP: Analyze the linear program:

$$\max z = x_1 + 2x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 5,\ x_1 + 3x_2 \leq 9,\ x_1, x_2 \geq 0.$$

Determine whether there are multiple optimal solutions.

Answer (Ex. 53) — The objective function is parallel to one of the constraints at optimality, resulting in multiple optimal solutions along a boundary segment.

Ex. 54 — Unbounded linear programming problem: Consider the linear program:

$$\max z = 5x_1 + 4x_2 \quad \text{s.t.} \quad x_1 + 2x_2 \geq 4,\ x_1 + x_2 \leq 10,\ x_1, x_2 \geq 0.$$

Determine the nature of its solution.

Answer (Ex. 54) — The feasible region is unbounded in the direction of increasing z; therefore, the problem is unbounded.

Ex. 55 — Two–phase simplex detects unboundedness: (*JVF,1st25*)

$$\begin{aligned} \max \quad & Z = 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 - 2x_2 \le 6, \\ & x_1 \le 10, \\ & x_2 \ge 1, \\ & x_1, x_2 \ge 0. \end{aligned}$$

Answer (Ex. 55) — Convert constraints to equalities with slack / surplus / artificial:

$$\begin{aligned} & x_1 - 2x_2 + s_1 = 6, \\ & x_1 + s_2 = 10, \qquad\qquad s_1, s_2, s_3, a_1 \ge 0. \\ & x_2 - s_3 + a_1 = 1, \end{aligned}$$

Initial basis for Phase I: $\{s_1, s_2, a_1\}$.
Columns order in tableau: $[x_1\ x_2\ s_1\ s_2\ s_3\ a_1 \mid RHS]$.

PHASE I — Initial Tableau (minimize $W = a_1$)

Tableau 0:

B	x_1	x_2	s_1	s_2	s_3	a_1	RHS
s_1	1	−2	1	0	0	0	6
s_2	1	0	0	1	0	0	10
a_1	0	1	0	0	−1	1	1
W	0	0	0	0	0	1	0

Pivot: enter x_2, leave a_1. This eliminates the artificial.

Tableau 1:

B	x_1	x_2	s_1	s_2	s_3	a_1	RHS
s_1	1	0	1	0	−2	2	8
s_2	1	0	0	1	0	0	10
x_2	0	1	0	0	−1	1	1
W	0	1	0	0	−1	0	1

Artificial variable = 0. Phase I succeeds. Feasible.

PHASE II — Optimize $Z = 3x_1 + 5x_2$

Current basis is $\{s_1, s_2, x_2\}$. Build reduced costs:

$$c_j - z_j = [3, 0, 0, 0, 5].$$

So the entering variable candidate that improves Z the most is s_3 (5).

Reduced Z row:

	x_1	x_2	s_1	s_2	s_3	Z
$c_j - z_j$	3	0	0	0	5	5

Unboundedness detection (ratio test fails)

Column $s_3 = [-2, 0, -1]^T$. All coefficients ≤ 0. There is no positive entry in that column. Therefore no leaving variable exists.
Since $c_{s_3} - z_{s_3} > 0$ and no pivot is possible:

$$\Rightarrow \textbf{ the problem is unbounded (no upper bound)}.$$

Explicit feasible ray

Let $t \geq 0$. Take $s_3 = t$, $x_1 = 0$.

$$x_2 = 1 + t, \qquad s_1 = 8 + 2t, \qquad s_2 = 10.$$

Then all variables remain feasible for all $t \geq 0$. Objective:

$$Z(t) = 5 + 5t \to +\infty.$$

Conclusion

Two–phase succeeds in Phase I, and Phase II proves the problem is **unbounded**.
■

Ex. 56 — Infeasible LP example: Consider:

$$\max z = x_1 + x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 2,\ x_1 + x_2 \geq 5,\ x_1, x_2 \geq 0.$$

Discuss the feasibility of this system.

Answer (Ex. 56) — The constraints contradict each other; no point satisfies both. The LP is infeasible.

Ex. 57 — **Optimal solution in linear programming:** Solve the linear program:

$$\max z = 3x_1 + 2x_2, \quad \text{s.t.} \quad x_1 + x_2 \leq 4,\ 2x_1 + x_2 \leq 6,\ x_1, x_2 \geq 0.$$

Find the optimal point.

Answer (Ex. 57) — Corner points: $(0,0),(0,4),(2,2),(3,0)$. z is maximized at $(2,2)$ with $z = 10$.

Ex. 58 — **Minimization with inequalities:** Find the optimal solution to:

$$\min z = x_1 + 4x_2 \quad \text{s.t.} \quad x_1 + 2x_2 \geq 3,\ -x_1 + x_2 \geq 1,\ x_1, x_2 \geq 0.$$

Answer (Ex. 58) — By graphical or algebraic methods, the minimum occurs at $(x_1, x_2) = (3, 0)$.

Ex. 59 — **Maximization in LP:** Maximize $z = 4x_1 + 3x_2$ subject to:

$$\begin{cases} x_1 + 2x_2 \leq 8, \\ x_1 + x_2 \leq 6, \\ x_1, x_2 \geq 0. \end{cases}$$

Find the optimal point and value.

Answer (Ex. 59) — The feasible vertices yield maximum z at $(x_1, x_2) = (4, 2)$ with $z = 22$.

Ex. 60 — **Minimization in LP with inequalities:** Minimize $z = 2x_1 + 3x_2$ subject to:

$$\begin{cases} x_1 + x_2 \geq 5, \\ 3x_1 + x_2 \geq 9, \\ x_1, x_2 \geq 0. \end{cases}$$

Find the optimal point.

Answer (Ex. 60) — Optimal solution at $(x_1, x_2) = (3, 2)$, yielding $z = 12$.

Ex. 61 — **Optimal vertex in bounded LP:** Solve the LP:

$$\max z = 6x_1 + 7x_2, \text{ s.t. } 2x_1 + x_2 \leq 14,\ x_1 + x_2 \leq 10,\ x_1, x_2 \geq 0.$$

Find the optimal vertex.

Answer (Ex. 61) — Intersection of constraints gives $(x_1, x_2) = (5, 5)$. Maximum value: $z = 65$.

Ex. 62 — **Profit maximization problem:** A company produces two products, A and B, with profits per unit of 10 and 15, respectively. Each unit of A requires 2 units of resource 1 and 1 of resource 2; each unit of B requires 3 and 2 units, respectively. If at most 18 units of resource 1 and 10 of resource 2 are available, determine the optimal production plan.

Answer (Ex. 62) — Formulate:

$$\max z = 10x_1 + 15x_2,\ 2x_1 + 3x_2 \leq 18,\ x_1 + 2x_2 \leq 10.$$

The optimal solution is $(x_1, x_2) = (4, 3)$, with $z = 85$.

Ex. 63 — **Two–phase simplex detects infeasibility:** (*JVF,1st25*) Given the following linear programming problem:

$$\max Z = -x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 1, \\ 2x_1 + 3x_2 &\geq 6, \\ x_1, x_2 &\geq 0. \end{aligned}$$

1. Write the problem in standard form.
2. Apply the two–phase simplex method to determine whether the problem has a feasible solution. Justify your answer.

Answer (Ex. 63) — 1. **Standard form**

We introduce a slack variable s_1 for the first constraint and a surplus variable s_2 for the second constraint:

$$\begin{aligned} x_1 + x_2 + s_1 &= 1, \\ 2x_1 + 3x_2 - s_2 &= 6, \\ x_1, x_2, s_1, s_2 &\geq 0. \end{aligned}$$

The objective function in standard form is:

$$\max Z = -x_1 + x_2 + 0s_1 + 0s_2.$$

We thus ca apply the two-phase simplex method. Let us start with Phase I.

Since the second constraint is of type "$\geq$", we introduce an artificial variable a_1 to form an initial basic feasible solution (BFS):

$$2x_1 + 3x_2 - s_2 + a_1 = 6.$$

The augmented system becomes:

$$\begin{aligned} x_1 + x_2 + s_1 &= 1, \\ 2x_1 + 3x_2 - s_2 + a_1 &= 6, \\ x_1, x_2, s_1, s_2, a_1 &\geq 0. \end{aligned}$$

The auxiliary objective is to minimize the sum of artificial variables:

$$\min Z^* = a_1.$$

Initial BFS: to build the initial tableau, we choose as basic variables s_1 (from the first constraint) and a_1 (from the second). Setting all nonbasic variables $x_1 = x_2 = s_2 = 0$, we obtain:

$$s_1 = 1, \quad a_1 = 6,$$

which is a valid basic feasible solution since all basic variables are non-negative.

The corresponding value of the auxiliary objective is $Z^* = 6$. Initial Phase I tableau:

Basic Var.	x_1	x_2	s_1	s_2	a_1	RHS
s_1	1	1	1	0	0	1
a_1	2	3	0	-1	1	6
Z^*	-2	-3	0	1	0	-6

The most negative coefficient in the Z^*-row is -3 (column x_2), so x_2 enters the basis.

$$\text{Ratio test: } \frac{1}{1} = 1, \quad \frac{6}{3} = 2 \Rightarrow \text{pivot row: } s_1.$$

The pivot element is the 1 in row s_1, column x_2. We first divide the pivot row by the pivot (which is already 1), then eliminate the x_2 coefficients in the other rows:

$$\begin{aligned}
\text{New } s_1 &\equiv \text{old } s_1, \\
\text{New } a_1 &= \text{old } a_1 - 3(\text{new } s_1), \\
\text{New } Z^* &= \text{old } Z^* + 3(\text{new } s_1).
\end{aligned}$$

This gives:

Basic Var.	x_1	x_2	s_1	s_2	a_1	RHS
x_2	1	1	1	0	0	1
a_1	-1	0	-3	-1	1	3
Z^*	1	0	3	1	0	-3

$$\boxed{\text{Since } a_1 = 3 > 0 \text{ in the final tableau, the original LP is infeasible.}}$$

No need to proceed to Phase II.

■

2.2 Duality

Ex. 64 — Dual of the Problem: Consider the following (primal) minimization problem:

$$\begin{aligned} \text{minimize} \quad & 4x_1 + 4x_2 + x_3 \\ \text{subject to} \quad & \begin{cases} x_1 + x_2 + x_3 \leq 2, \\ 2x_1 + x_2 = 3, \\ 2x_1 + x_2 + 3x_3 \geq 3, \\ x_1, x_2, x_3 \geq 0. \end{cases} \end{aligned}$$

(Trick: normalize the equality as two inequalities.)

Answer (Ex. 64) — First, we rewrite all constraints in the $\geq$ form (to follow a consistent convention):

$$\begin{aligned} -x_1 - x_2 - x_3 &\geq -2, \\ 2x_1 + x_2 &\geq 3, \\ -2x_1 - x_2 &\geq -3, \\ 2x_1 + x_2 + 3x_3 &\geq 3. \end{aligned}$$

Assign dual variables $y_1, y_2, y_3, y_4 \geq 0$ to these four constraints (respectively). The right-hand-side vector is $b = (-2, 3, -3, 3)^T$. The dual problem (in maximization form) is then:

$$\begin{aligned} \text{maximize} \quad & -2y_1 + 3y_2 - 3y_3 + 3y_4 \\ \text{subject to} \quad & \begin{cases} -y_1 + 2y_2 - 2y_3 + 2y_4 \leq 4, \\ -y_1 + y_2 - y_3 + y_4 \leq 4, \\ -y_1 + 3y_4 \leq 1, \\ y_1, y_2, y_3, y_4 \geq 0. \end{cases} \end{aligned}$$

To simplify, note that the original equality constraint $2x_1 + x_2 = 3$ corresponds to a free dual variable. Define $z \in \mathbb{R}$ such that $z = y_2 - y_3$ (with $y_2, y_3 \geq 0$). Keep $y_1, y_4 \geq 0$. This gives the more compact dual formulation:

$$\begin{array}{ll}\text{maximize} & -2y_1 + 3z + 3y_4 \\ \text{subject to} & \begin{cases} -y_1 + 2z + 2y_4 \leq 4, \\ -y_1 + z + y_4 \leq 4, \\ -y_1 + 3y_4 \leq 1, \\ y_1 \geq 0,\ y_4 \geq 0,\ z \in \mathbb{R}. \end{cases}\end{array}$$

■

Ex. 65 — Duality Purpose: In linear programming, every optimization problem (called the *primal*) can be associated with another related problem (the *dual*). Explain the main theoretical and practical purpose of introducing the dual problem in linear programming.

Answer (Ex. 65) — The purpose of duality is to establish a deep connection between two equivalent linear programs: the primal and the dual. This relationship allows us to obtain bounds, check optimality, interpret results economically, and sometimes solve the easier of the two problems. Solving one provides information about the other, and both share the same optimal value under strong duality.■

Ex. 66 — Dual of Maximization Problem: When a primal linear programming problem is formulated as a *maximization* problem with constraints of the form $Ax \leq b$, what can we say about the type of its dual problem? Explain why.

Answer (Ex. 66) — The dual of a primal maximization problem is always a minimization problem. Duality reverses the optimization sense: maximizing profit or output in the primal corresponds to minimizing cost or resources in the dual. This ensures consistency between the two formulations.■

Ex. 67 — Constructing the Dual: Consider the following primal minimization problem:

$$\min\ c^\top x \quad \text{s.t.} \quad Ax \geq b,\ x \geq 0.$$

Construct its dual problem and briefly explain the relationship between both formulations.

Answer (Ex. 67) — The corresponding dual problem is

$$\max b^\top y \quad \text{s.t.} \quad A^\top y \leq c, \ y \geq 0.$$

Here, each primal constraint becomes a variable in the dual, and the inequality directions reverse. The dual gives a complementary view of the same optimization structure, often easier to interpret economically or computationally.■

Ex. 68 — **Economic Interpretation:** In linear programming, dual variables are often given an economic interpretation. Explain what these dual variables represent and how they relate to the primal problem.

Answer (Ex. 68) — Dual variables, also called *shadow prices*, represent the marginal value or worth of each resource in the primal problem. A positive dual variable indicates how much the objective function would improve if the corresponding constraint's right-hand side increased by one unit. They provide valuable economic insight into resource allocation.■

Ex. 69 — **Primal-Dual Relationship:** Describe the fundamental relationship between a primal linear program and its dual. What happens if we take the dual of the dual problem?

Answer (Ex. 69) — The dual of the dual is the primal. This property shows the symmetry and equivalence of the two formulations. It ensures that the transformation between primal and dual is reversible, emphasizing that both contain the same information in different forms.■

Ex. 70 — **Weak Duality Theorem:** State and explain the Weak Duality Theorem in the context of linear programming. What does it imply about the relationship between feasible solutions of the primal and the dual?

Answer (Ex. 70) — The Weak Duality Theorem states that for any feasible primal and dual solutions, the value of the primal objective function is less than or equal to that of the dual. This means the primal provides a lower bound and the dual an upper bound on the optimal value, ensuring consistency between them.■

Ex. 71 — **Unbounded Primal:** Discuss what happens to the dual problem if the primal linear program is feasible but unbounded. Explain the reasoning behind this relationship.

Answer (Ex. 71) — If the primal is unbounded, the dual must be infeasible. If the primal objective can increase without bound, then no dual feasible solution can exist that would impose an upper bound—otherwise, weak duality would be violated.■

Ex. 72 — **Strong Duality Theorem:** State the Strong Duality Theorem in linear programming and explain its significance.

Answer (Ex. 72) — The Strong Duality Theorem asserts that if the primal has an optimal solution, then the dual also has an optimal solution, and both have equal objective values. This powerful result allows us to solve one problem by analyzing the other and to verify optimality by comparing their objective values.■

Ex. 73 — **Equal Objective Values:** If both the primal and the dual have feasible solutions with equal objective values, what conclusion can we draw? Justify your answer.

Answer (Ex. 73) — Both are optimal. Equality of objective values for feasible primal and dual solutions implies that neither can be improved further. This situation occurs only at optimality, according to the Strong Duality Theorem.■

Ex. 74 — **Slack and Surplus Variables:** Why do we introduce slack or surplus variables in linear programming formulations, and what purpose do they serve in solving the problem?

Answer (Ex. 74) — Slack and surplus variables are added to convert inequalities into equalities, enabling the use of algebraic solution methods such as the Simplex algorithm. They also help track unused resources or excess production in the model.■

Ex. 75 — **Non-Binding Constraints:** In an optimal primal solution, what is the relationship between a non-binding constraint and its corresponding dual variable? Explain why.

Answer (Ex. 75) — If a constraint is not binding at the optimum, its corresponding dual variable must be zero. Since relaxing that constraint does not affect the optimal value, its shadow price (dual value) must vanish.■

Ex. 76 — Complementary Slackness: Explain the meaning of the Complementary Slackness Theorem and how it helps determine optimality in linear programming.

Answer (Ex. 76) — Complementary Slackness states that for each constraint and corresponding dual variable, at least one must be zero at optimality. A positive primal variable implies a binding dual constraint, and vice versa. This relationship provides a direct way to test or construct optimal solutions.■

Ex. 77 — Mutual Slackness Prohibition: Can both a primal constraint and its corresponding dual variable be positive at the same time? Explain why or why not.

Answer (Ex. 77) — No. Both cannot be positive simultaneously because this would violate the Complementary Slackness condition. If a constraint has slack (is not tight), the associated dual variable must be zero to maintain optimality.■

Ex. 78 — Simplex Algorithm: Explain how the Simplex algorithm relates the primal and dual problems during the optimization process.

Answer (Ex. 78) — The Simplex algorithm effectively solves the primal and dual simultaneously. As it moves along the edges of the feasible region, it maintains feasibility for one problem and boundedness for the other, converging to a pair of optimal primal and dual solutions.■

Ex. 79 — Feasible but Unbounded Primal: Suppose the primal problem is feasible but unbounded. What can you say about the feasibility of the dual problem?

Answer (Ex. 79) — If the primal is feasible but unbounded, the dual is infeasible. A bounded dual would imply an upper limit to the primal objective, which contradicts unboundedness. Hence, no dual solution satisfies all constraints.■

Ex. 80 — Primal-Dual Relationship Table: Summarize the relationship between the feasibility and boundedness of the primal and dual problems according to the standard primal-dual relationship table.

Answer (Ex. 80) — If the primal is infeasible, the dual may be unbounded. Conversely, if the primal is unbounded, the dual must be infeasible. Only when both are feasible can they have optimal (and equal) objective values.■

Ex. 81 — **Deriving the Dual:** Describe the general process of obtaining the dual problem from a given primal linear program. What changes occur to the constraints, objective function, and variables?

Answer (Ex. 81) — The dual is derived by interchanging the roles of constraints and variables, transposing the coefficient matrix, reversing inequality directions, and switching the optimization sense (min max). Each constraint in the primal corresponds to a variable in the dual.■

Ex. 82 — **Variable-Constraint Correspondence:** Explain the relationship between basic and non-basic variables in the primal and the dual problems.

Answer (Ex. 82) — Basic variables in the primal correspond to non-basic variables in the dual, and vice versa. This structural symmetry ensures that active constraints in one formulation align with zero-valued variables in the other.■

Ex. 83 — **Diet Problem Interpretation:** In the classical diet problem, what does the dual problem represent economically, and how does it relate to the primal formulation?

Answer (Ex. 83) — In the diet problem, the dual represents the nutrient prices that maximize the seller's or supplier's profit. Each dual variable corresponds to the implicit value of a nutrient, showing how the cost-minimization problem of the diet transforms into a value-maximization problem for nutrients.■

Ex. 84 — **Complementarity Condition:** Interpret the condition $(b - Ax^*) \cdot y^* = 0$ in the context of linear programming optimality. What does it express about the relationship between primal and dual solutions?

Answer (Ex. 84) — This condition expresses the Complementarity principle: for each constraint i, either the constraint is binding ($b_i - A_i x^* = 0$) or its dual variable y_i^* is zero. It formalizes the link between feasible and optimal primal-dual pairs.■

Ex. 85 — Complementary Slackness Application: Consider the linear programming problem P:

$$\begin{aligned} \max \quad & x_1 + 4x_2 + 2x_3 \\ \text{s.t.} \quad & 5x_1 + 2x_2 + 2x_3 \leq 145, \\ & 4x_1 + 8x_2 - 8x_3 \leq 260, \\ & x_1 + x_2 + 4x_3 \leq 190, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

(a) Write the dual problem corresponding to P.

(b) Verify that the point $(x_1, x_2, x_3) = (0,\ 52.5,\ 20)$ is feasible for P.

(c) Assuming that $(0, 52.5, 20)$ is an optimal solution to P, use the *Complementary Slackness* conditions to determine a candidate solution for the dual problem. Be explicit about which primal constraints are binding and which are not, and record the implications for the corresponding dual variables.

(d) Check carefully that the candidate dual solution is feasible (including any dual constraints that were not used in the CS system and non-negativity). Then verify equality of primal and dual objective values as the final proof of optimality. Finally, comment on what can go wrong if the primal guess fails (i.e., the "what else?" remarks).

Answer (Ex. 85) — (a) Dual Formulation. Associate dual variables y_1, y_2, y_3 with primal constraints (1),(2),(3). The dual is

$$\begin{aligned} \min \quad & 145y_1 + 260y_2 + 190y_3 \\ \text{s.t.} \quad & 5y_1 + 4y_2 + y_3 \geq 1, \quad \text{(dual for } x_1) \\ & 2y_1 + 8y_2 + y_3 \geq 4, \quad \text{(dual for } x_2) \\ & 2y_1 - 8y_2 + 4y_3 \geq 2, \quad \text{(dual for } x_3) \\ & y_1, y_2, y_3 \geq 0. \end{aligned}$$

(b) Feasibility of the primal candidate. Evaluate each constraint at $x = (0, 52.5, 20)$:

$$\begin{aligned} 5(0) + 2(52.5) + 2(20) &= 145 && \text{(constraint (1): binding)} \\ 4(0) + 8(52.5) - 8(20) &= 260 && \text{(constraint (2): binding)} \\ 0 + 52.5 + 4(20) &= 132.5 < 190 && \text{(constraint (3): non-binding).} \end{aligned}$$

All inequalities hold and $x \geq 0$, so the point is feasible for P.
(c) Complementary Slackness (CS) to construct a dual candidate. CS says: for each primal constraint i, either slack is zero (constraint binding) or $y_i = 0$; and for each primal variable x_j, either $x_j = 0$ or the corresponding dual constraint holds with equality.
From (b) we saw constraint (3) is *non-binding*, hence by CS:

$$y_3^* = 0.$$

Constraints (1) and (2) are binding, so they give no direct information forcing y_1, y_2 to be zero — instead the dual constraints corresponding to positive primal variables must be tight. Look at primal variables:

$$x_1^* = 0, \quad x_2^* = 52.5 > 0, \quad x_3^* = 20 > 0.$$

By CS, dual constraints for x_2 and x_3 must be equalities at optimality:

$$\begin{aligned} 2y_1^* + 8y_2^* + y_3^* &= 4, \\ 2y_1^* - 8y_2^* + 4y_3^* &= 2. \end{aligned}$$

With $y_3^* = 0$ these reduce to:

$$\begin{cases} 2y_1^* + 8y_2^* = 4, \\ 2y_1^* - 8y_2^* = 2. \end{cases}$$

Solve: add the two equations to get $4y_1^* = 6 \Rightarrow y_1^* = 3/2$. Then $2(3/2)+8y_2^* = 4 \Rightarrow 3 + 8y_2^* = 4 \Rightarrow y_2^* = 1/8$. Thus the CS-derived candidate is

$$y^* = \left(\tfrac{3}{2},\ \tfrac{1}{8},\ 0\right).$$

(d) Final feasibility, value check, and “what else?”.
(i) Don't forget: check the dual constraints we didn't use explicitly and non-negativity. We must confirm every dual constraint is satisfied by y^* and $y^* \geq 0$:

$$\begin{aligned} 5y_1^* + 4y_2^* + y_3^* &= 5 \cdot \frac{3}{2} + 4 \cdot \frac{1}{8} + 0 = 7.5 + 0.5 = 8.0 \geq 1, \\ 2y_1^* + 8y_2^* + y_3^* &= 4 \quad \text{(equality used by CS)}, \\ 2y_1^* - 8y_2^* + 4y_3^* &= 2 \quad \text{(equality used by CS)}, \end{aligned}$$

and clearly $y_i^* \geq 0$. So y^* is feasible for the dual.
(ii) Check objective values (primal value = dual value). Primal objective at x^*:

$$z_P = 1 \cdot 0 + 4 \cdot 52.5 + 2 \cdot 20 = 0 + 210 + 40 = 250.$$

Dual objective at y^*:

$$z_D = 145 \cdot \frac{3}{2} + 260 \cdot \frac{1}{8} + 190 \cdot 0 = 217.5 + 32.5 + 0 = 250.$$

Since both are feasible and $z_P = z_D$, by Strong Duality the pair (x^*, y^*) is optimal.
(iii) What else? (Possible failure modes and diagnostic checks.)

- If the primal guess had *failed* feasibility, you would detect it directly in step (b) by violation of some primal constraint — so the approach stops there (the guess is not a candidate solution).
- If the primal guess were feasible but *not* optimal, the CS-derived dual candidate will typically be *infeasible*: one of the dual constraints (often the one we omitted when forming the CS equations) or non-negativity would be violated. That signals the primal guess is not optimal.
- Always perform the three checks in order: (1) primal feasibility of the guess, (2) derive dual candidate via CS, (3) check dual feasibility (including any omitted dual constraints and non-negativity), and (4) compare objective values. Only if all pass can you conclude optimality.

Conclusion. The primal point $x^* = (0, 52.5, 20)$ is feasible, the CS-derived dual $y^* = (3/2, 1/8, 0)$ is feasible, and primal and dual objective values coincide at 250. Therefore both are optimal and all complementary slackness and duality checks are satisfied. ■

Ex. 86 — Complementary Slackness: Consider the problem:

$$\begin{array}{ll} \text{maximize} & 2x_1 + 16x_2 + 2x_3 \\ \text{subject to} & \begin{cases} 2x_1 + x_2 - x_3 \leq -3, \\ -3x_1 + x_2 + 2x_3 \leq 12, \\ x_1, x_2, x_3 \geq 0 \end{cases} \end{array}$$

Check whether each of the following is an optimal solution, using complementary slackness:

$$\begin{aligned} \#1: &\quad x_1 = 6,\ x_2 = 0,\ x_3 = 12 \\ \#2: &\quad x_1 = 0,\ x_2 = 2,\ x_3 = 5 \\ \#3: &\quad x_1 = 0,\ x_2 = 0,\ x_3 = 6 \end{aligned}$$

For what, if any, values of c_2 and c_3 would $x_1 = 0, x_2 = 0, x_3 = 6$ be an optimal solution?

Answer (Ex. 86) — The equations of the dual problem are:

$$\begin{aligned} 2y_1 - 3y_2 - \eta_1 &= 2, \\ y_1 + y_2 - \eta_2 &= 16, \\ -y_1 + 2y_2 - \eta_3 &= 2. \end{aligned}$$

Given values of x_1, x_2, x_3, we first compute the slack variables s_1 and s_2. For optimality, all decision and slack variables must satisfy $x_j, s_i \geq 0$. According to complementary slackness, whenever an x_j or s_i is nonzero, the corresponding η_j or y_i must be zero.

—

Case #1: $x_1 = 6, x_2 = 0, x_3 = 12$

$$s_1 = -3 \quad \Rightarrow \quad \text{not feasible, hence not optimal.}$$

—

Case #2: $x_1 = 0, x_2 = 2, x_3 = 5$

$$s_1 = 0, \quad s_2 = 0.$$

The primal is feasible. From complementary slackness, $\eta_2 = 0$ and $\eta_3 = 0$. The dual equations become:

$$\begin{aligned} 2y_1 - 3y_2 - \eta_1 &= 2, \\ y_1 + y_2 &= 16, \\ -y_1 + 2y_2 &= 2. \end{aligned}$$

From the second and third equations:

$$y_1 = 10, \quad y_2 = 6,$$

and from the first, $\eta_1 = 0$. All variables are nonnegative, so the dual is feasible.

$$\boxed{\text{Conclusion: solution \#2 is optimal.}}$$

—

Case #3: $x_1 = 0, x_2 = 0, x_3 = 6$

$$s_1 = 3, \quad s_2 = 0.$$

The primal is feasible. From complementary slackness, $\eta_3 = 0$ and $y_1 = 0$. The dual becomes:

$$\begin{aligned} -3y_2 - \eta_1 &= 2, \\ y_2 - \eta_2 &= 16, \\ 2y_2 &= 2. \end{aligned}$$

From the third: $y_2 = 1$. Then $\eta_1 = -5$, which is not feasible ($\eta_1 < 0$).

$$\boxed{\text{Conclusion: solution \#3 is not optimal.}}$$

—

Final question: For $x_1 = 0, x_2 = 0, x_3 = 6$, $s_1 = 3, s_2 = 0$, but c_2, c_3 are unspecified ($c_1 = 2$).
With $\eta_3 = 0$ and $y_1 = 0$, the dual becomes:

$$\begin{aligned} -3y_2 - \eta_1 &= 2, \\ y_2 - \eta_2 &= c_2, \\ 2y_2 &= c_3. \end{aligned}$$

If $y_2 \geq 0$ and $\eta_2 \geq 0$, then the left-hand side of the first equation is ≤ 0. Hence, there are no such feasible values of c_2 and c_3.

$$\boxed{\text{No values of } c_2, c_3 \text{ make this solution optimal.}}$$

■

2.3 Sensitivity analysis

Ex. 87 — Sensitivity Analysis in a Paper Mill Production Problem: (*Adapted*
A paper mill converts pulpwood into three grades of newsprint: low, medium, and high. The pulpwood requirements (tons per ton of newsprint), availability of each type of pulpwood, and selling prices are given below:

	Low	Medium	High	Available (tons)
Virginia pine	2	2	1	180
White pine	1	2	3	120
Loblolly pine	1	1	2	160
Price ($/ton)	900	1000	1200	

The associated linear program is:

$$\max\ 900x_1 + 1000x_2 + 1200x_3$$

subject to

$$\begin{aligned} 2x_1 + 2x_2 + x_3 + s_1 &= 180, \\ x_1 + 2x_2 + 3x_3 + s_2 &= 120, \\ x_1 + x_2 + 2x_3 + s_3 &= 160, \\ x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

The optimal tableau is:

	x_1	x_2	x_3	s_1	s_2	s_3	RHS
z	0	200	0	300	300	0	90000
	1	0.8	0	0.6	0.2	0	84
	0	0.4	1	0.2	0.4	0	12
	0	0.6	0	0.2	0.6	1	52

(a) In what range can the price of low-grade paper vary without changing the optimal basis?

(b) What is the new optimal solution if the price of low-grade paper changes to $800?

(c) At what price should medium-grade paper be sold to make it profitable to produce?

(d) In what range can the availability of Virginia pine vary without changing the optimal basis?

(e) If 10 additional tons of Virginia pine are obtained, by how much will the optimal profit increase?

(f) If the pulpwood resources are increased as in part (e), what is the new optimal solution?

(g) What would the plant manager be willing to pay for an additional ton of Loblolly pine?

Answer (Ex. 87) — (a) The variable x_1 (low-grade paper) is a basic variable, since its column forms part of the identity matrix. Let the price of low-grade paper change from 900 to $900 + \varepsilon$.

To restore the correct basis, we add ε times the row corresponding to x_1 to the objective row. This changes the reduced costs of all nonbasic variables. The reduced costs that must remain nonnegative are those of x_2, x_3, s_1, and s_2:

$$\begin{aligned} 200 + 0.8\varepsilon &\geq 0, \\ 0 + 0\varepsilon &\geq 0, \\ 300 + 0.6\varepsilon &\geq 0, \\ 300 + 0.2\varepsilon &\geq 0. \end{aligned}$$

The most restrictive condition is $200 + 0.8\varepsilon \geq 0$, which gives

$$\varepsilon \geq -250.$$

There is no upper bound. Therefore, the price of low-grade paper may decrease to \$650 but may increase without bound without changing the optimal basis.

(b) If the price of low-grade paper is reduced to \$800, this corresponds to $\varepsilon = -100$, which lies within the allowable range found in part (a). Hence, the optimal basis does not change.

From the tableau, the basic variables are:

$$x_1 = 84, \quad x_3 = 12,$$

with $x_2 = 0$. The new optimal profit is

$$800(84) + 1200(12) = 67200 + 14400 = 81600.$$

(c) The reduced cost of x_2 (medium-grade paper) is 200. This means that producing one unit of medium-grade paper would reduce profit by \$200 at current prices. Therefore, its price must increase by at least \$200 to make it profitable. Hence, the minimum price at which medium-grade paper should be sold is

$$1000 + 200 = \$1200.$$

(d) The availability of Virginia pine corresponds to the right-hand side of the first constraint. Let it change from 180 to $180 + \varepsilon$.

The RHS values of the basic variables change according to the column of s_1:

$$x_1 : 84 + 0.6\varepsilon,$$
$$x_3 : 12 + 0.2\varepsilon,$$
$$s_3 : 52 + 0.2\varepsilon.$$

All must remain nonnegative, which gives

$$\varepsilon \geq -60.$$

Thus, the availability of Virginia pine can decrease to 120 tons and can increase without bound without changing the optimal basis.

(e) The shadow price of the Virginia pine constraint is the reduced cost of s_1, which is 300. Therefore, obtaining one additional ton of Virginia pine increases optimal profit by \$300.

If 10 additional tons are obtained, the profit increases by

$$10 \times 300 = 3000.$$

(f) With 10 additional tons of Virginia pine ($\varepsilon = 10$), the new values of the basic variables are:

$$x_1 = 84 + 0.6(10) = 90,$$
$$x_3 = 12 + 0.2(10) = 14.$$

Thus, the new optimal production plan is 90 tons of low-grade paper, 14 tons of high-grade paper, and none of medium-grade paper.

(g) The shadow price of the Loblolly pine constraint is given by the reduced cost of s_3, which is 0. Therefore, an additional ton of Loblolly pine does not increase the optimal profit. The plant manager would be willing to pay \$0 for an extra ton of Loblolly pine.

Ex. 88 — Sensitivity Analysis in a Snack Mix Production Problem: (*Adapted*

Snacks 'R Us is deciding what to produce in the upcoming month. Due to past purchases, they have 500 lb of walnuts, 1000 lb of peanuts, and 500 lb of chocolate. They currently sell three mixes:

- **Trail Mix** (x_1): consists of 1 lb of each material (walnuts, peanuts, chocolate).

- **Nutty Crunch** (x_2): consists of 2 lb peanuts and 1 lb walnuts.
- **Choc-o-plenty** (x_3): consists of 2 lb chocolate and 1 lb peanuts.

They can sell an unlimited amount of these mixes, except they can sell no more than 100 units of Choc-o-plenty. The income on the three mixes is \$2, \$3, and \$4, respectively.
The linear program is:

$$\max\ 2x_1 + 3x_2 + 4x_3$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 500 \quad \text{(walnuts)} \\ x_1 + 2x_2 + x_3 &\leq 1000 \quad \text{(peanuts)} \\ x_1 + 2x_3 &\leq 500 \quad \text{(chocolate)} \\ x_3 &\leq 100 \quad \text{(sales limit)} \\ x_i &\geq 0 \quad \text{for all } i. \end{aligned}$$

After adding slacks s_1, s_2, s_3, s_4, the optimal tableau is:

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
z	0	0	0	1	1	0	3	1800
	1	0	0	2	1	0	1	100
	0	1	0	1	1	0	1	400
	0	0	0	2	1	1	3	200
	0	0	1	0	0	0	1	100

Using the tableau, answer:

(a) What solution is represented (production quantities and total income)? How do we know it is optimal?

(b) If the income for Trail Mix (x_1) is uncertain, for what range of values does the same basis remain optimal? What is the solution if the income is \$2.75?

(c) How much should Snacks 'R Us be willing to spend for one extra pound of peanuts? walnuts? chocolate?

(d) For what range of peanut-stock values does the same basis remain optimal? What is the solution if there are only 900 lb of peanuts?

(e) A new product, Extra-Walnutty Trail Booster, uses 2 lb walnuts, 1 lb peanuts, and 1 lb chocolate per unit. What income is required to consider producing it?

Answer (Ex. 88) — (a)Identify basic variables by columns that form an identity matrix in the tableau. Here, the basic variables are x_1 (row 1), x_2 (row 2), s_3 (row 3), and x_3 (row 4). Therefore,

$$x_1 = 100, \qquad x_2 = 400, \qquad x_3 = 100,$$

and the total income is the RHS of the objective row:

$$z = 1800.$$

Why is it optimal? For a maximization problem in an optimal tableau, all reduced costs (entries in the z-row for nonbasic variables) are nonnegative. In the z-row, the reduced costs for the slack variables are $s_1 : 1$, $s_2 : 1$, $s_3 : 0$, $s_4 : 3$, all ≥ 0, and the reduced costs for x_1, x_2, x_3 are 0 because they are basic. Hence the tableau is optimal.

(b)Let the income coefficient of x_1 change from 2 to $2 + \varepsilon$. Because x_1 is *basic*, we keep the same basis by adding ε times the x_1-row to the z-row. Only reduced costs of *nonbasic* variables must remain ≥ 0; in this basis, the relevant nonbasic variables include s_1, s_2, s_4 (and s_3 is basic).

From the x_1 row, the coefficients under (s_1, s_2, s_4) are $(2, 1, 1)$, while the current reduced costs in the z-row under (s_1, s_2, s_4) are $(1, 1, 3)$. After the change:

$$\begin{aligned}
\bar{c}_{s_1} &= 1 + 2\varepsilon \geq 0 \quad \Rightarrow \quad \varepsilon \geq -\tfrac{1}{2}, \\
\bar{c}_{s_2} &= 1 + \varepsilon \geq 0 \quad \Rightarrow \quad \varepsilon \geq -1, \\
\bar{c}_{s_4} &= 3 + \varepsilon \geq 0 \quad \Rightarrow \quad \varepsilon \geq -3.
\end{aligned}$$

Thus the binding condition is $\varepsilon \geq -\frac{1}{2}$. There is no upper bound, so the coefficient of x_1 can decrease down to $2 - 0.5 = 1.5$ and can increase without limit without changing the optimal basis.

If the income for x_1 is \$2.75, then $\varepsilon = 0.75$, which is feasible. So the production plan stays

$$(x_1, x_2, x_3) = (100, 400, 100).$$

The objective increases by $\varepsilon x_1 = 0.75 \cdot 100 = 75$, hence the new total income is

$$z = 1800 + 75 = 1875.$$

(c) Shadow prices are read from the z-row in the columns of the slack variables corresponding to each resource constraint. Thus:

$$\lambda_{\text{walnuts}} = 1, \qquad \lambda_{\text{peanuts}} = 1, \qquad \lambda_{\text{chocolate}} = 0.$$

Interpretation: one extra pound of walnuts (resp. peanuts) increases optimal income by \$1 (within the allowable range), while extra chocolate has marginal value \$0 (because the chocolate constraint is not binding at optimum).

(d) Let peanut availability change from 1000 to $1000+\varepsilon$. Keeping the same basis, the basic variables change as

$$x_B(\varepsilon) = x_B(0) + B^{-1}(\Delta b),$$

and from the implied sensitivity relationships for this basis we obtain:

$$x_1 = 100 - \varepsilon, \qquad x_2 = 400 + \varepsilon, \qquad s_3 = 200 + \varepsilon, \qquad x_3 = 100.$$

To keep the basis feasible, all basic variables must stay nonnegative:

$$100 - \varepsilon \geq 0 \Rightarrow \varepsilon \leq 100, \qquad 200 + \varepsilon \geq 0 \Rightarrow \varepsilon \geq -200.$$

(And $400 + \varepsilon \geq 0$ is automatically satisfied over $[-200, 100]$.)
Therefore, the peanut stock can vary in the range

$$1000 + \varepsilon \in [800, 1100]$$

without changing the optimal basis.
If there are only 900 lb of peanuts, then $\varepsilon = -100$, and thus

$$x_1 = 100 - (-100) = 200, \quad x_2 = 400 + (-100) = 300, \quad x_3 = 100.$$

The new income is

$$z = 2(200) + 3(300) + 4(100) = 400 + 900 + 400 = 1700.$$

(Equivalently, the objective decreases by the shadow price of peanuts times the change: $1 \cdot 100 = 100$.)

(e) For the new product x_4 (Extra-Walnutty Trail Booster), each unit uses: 2 lb walnuts, 1 lb peanuts, and 1 lb chocolate, and it does not affect the $x_3 \leq 100$ constraint.

The opportunity cost implied by shadow prices is:

$$\lambda_{\text{walnuts}}(2) + \lambda_{\text{peanuts}}(1) + \lambda_{\text{chocolate}}(1) = 1 \cdot 2 + 1 \cdot 1 + 0 \cdot 1 = 3.$$

So the income must be at least \$3 per unit to be worth producing (at exactly \$3 it would be an alternative optimum).

■

2.4 Network optimization

Ex. 89 — Cash Management over Six Months: A company must meet the following cash demands at the *beginning* of each of the next six months:

Month t	1	2	3	4	5	6
Need (\$)	200	100	50	80	160	140

At the beginning of month 1, the company has \$150 in cash and bonds worth \$200 (Bond 1), \$100 (Bond 2), and \$400 (Bond 3). To obtain cash, the company may sell any fraction of each bond in any month. A penalty is charged for selling \$1 worth of each bond in each month:

	1	2	3	4	5	6
Bond 1 penalty	0.07	0.06	0.06	0.04	0.03	0.03
Bond 2 penalty	0.17	0.17	0.17	0.11	0	0
Bond 3 penalty	0.33	0.33	0.33	0.33	0.33	0

(a) Assuming all bills must be paid on time, formulate a *balanced transportation* model to minimize total penalty cost. Also give an optimal plan.

(b) Now allow bills to be paid after they are due. A penalty of \$0.02 per month is assessed for each dollar of demand postponed by one month. All bills must be paid by the end of month 6. Formulate and solve a *minimum-cost network flow / transshipment* model.

Answer (Ex. 89) — (a)Balanced transportation model (on-time payment) Because all penalty rates are nonincreasing over time for each bond, it is never advantageous to sell a bond earlier than the month in which its cash is used. Therefore we may treat "shipping" of dollars from each initial asset directly to each month's demand.

- **Nodes.**
  - Supply nodes: Cash (150), Bond 1 (200), Bond 2 (100), Bond 3 (400).
  - Demand nodes: Month 1–6 needs (200, 100, 50, 80, 160, 140).
  - Since total supply is \$850 and total demand is \$730, add a dummy demand node "Unused" with demand \$120.
- **Arcs and costs.** Let $x_{i,t}$ be dollars sent from asset i to month-t demand. Costs are:
  - Cash $\to$ Month t: cost 0.
  - Bond k $\to$ Month t: cost = (penalty for Bond k sold in month t) per dollar.
  - Any asset $\to$ Unused: cost 0 (unused cash/bonds are simply not sold/used).

  Capacities are the asset amounts (150, 200, 100, 400). Demands are the monthly needs and 120 for Unused. This is a balanced minimum-cost transportation problem, hence also a minimum-cost network flow.

One optimal plan (and minimum penalty). Use cash and the low-penalty bond dollars as early as possible, and postpone selling Bond 2 until month 5 (penalty 0), and Bond 3 until month 6 (penalty 0), except for what is strictly needed earlier. An optimal on-time plan is:

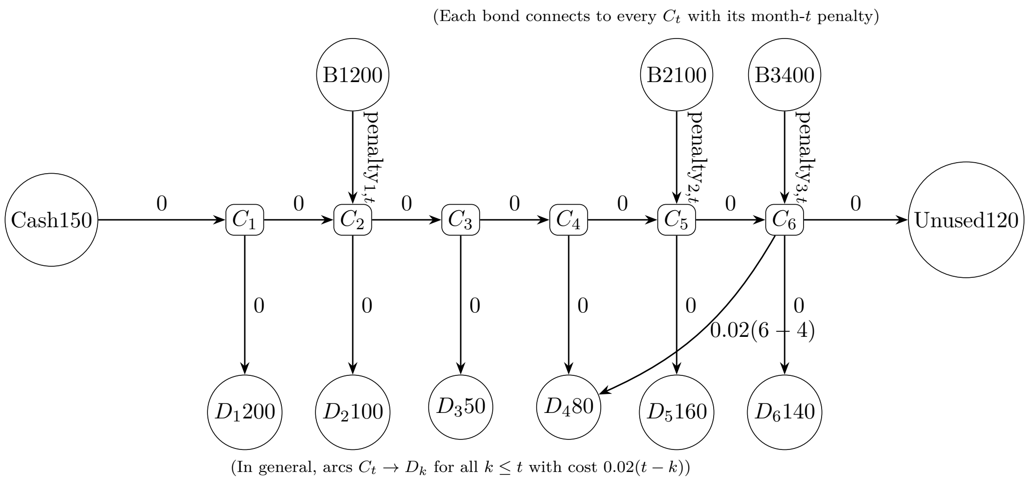
	M1	M2	M3	M4	M5	M6
Cash	150	0	0	0	0	0
Bond 1	50	100	50	0	0	0
Bond 2	0	0	0	0	100	0
Bond 3	0	0	0	80	60	140

with \$120 of Bond 3 left unused (not sold).

Total penalty cost:

$$50(0.07) + 100(0.06) + 50(0.06) + 80(0.33) + 60(0.33) = \$58.70.$$

(b)Transshipment / minimum-cost flow model (late payment allowed)



Now we build a *time-expanded minimum-cost flow network*.

- **Transshipment nodes.** For $t = 1, \ldots, 6$, let C_t denote cash available at the beginning of month t *after selling bonds for month t, but before paying any (possibly overdue) bills.* Let D_t denote the demand node for month t.
- **Supplies/demands.**
  - –Supply nodes: Cash0 with supply 150; Bond 1 with supply 200; Bond 2 with supply 100; Bond 3 with supply 400.
  - –Demand nodes: D_t with demand equal to the required cash in month t.
  - –Dummy sink with demand 120 (unused funds).
- **Arcs (all are directed) and costs.**
  - –Selling bonds: Bond $k \to C_t$ with cost = bond-k penalty in month t, capacity = bond-k amount.
  - –Initial cash: Cash0 $\to C_1$ with cost 0, capacity 150.
  - –Carrying cash forward: $C_t \to C_{t+1}$ with cost 0 (no interest), capacity $+\infty$.
  - –Paying bills: for any $k \leq t$, $C_t \to D_k$ with cost $0.02(t-k)$ per dollar (delay penalty), capacity $+\infty$.
  - –Unused: $C_6 \to$ Dummy with cost 0, capacity $+\infty$.
- **Optimal solution (one optimal flow).** Solving the minimum-cost flow yields the following policy:

–Sell Bond 1: \$50 in month 1 and \$150 in month 2 (penalties 0.07 and 0.06).

–Sell Bond 2: \$100 in month 5 (penalty 0).

–Sell Bond 3: \$400 in month 6 (penalty 0).

Bill payments:

Paid in month	Which demand is paid
1	D_1 (on time)
2	D_2 (on time)
3	D_3 (on time)
5	\$100 of D_5 (on time)
6	D_4 (2 months late), \$60 of D_5 (1 month late), D_6 (on time)

Costs:

$$\underbrace{50(0.07) + 150(0.06)}_{\text{bond penalties } =12.50} + \underbrace{80(0.04) + 60(0.02)}_{\text{late penalties } =4.40} = \$16.90.$$

This is strictly cheaper than paying everything on time because it avoids selling Bond 3 before month 6 (which costs 0.33 per dollar), replacing that with the much smaller delay penalties.

■

Ex. 90 — Production and Inventory Planning over Two Months: A company produces a single product at two plants, A and B. Demand is 12 units this month and 20 units next month. Plant A can produce 7 items per month; Plant B can produce 14 items per month. Production costs are:

	This month	Next month
A	12	18
B	14	20

Production from this month may be held in inventory to satisfy next month's demand at a cost of \$3 per unit.

(a) Formulate as a balanced transportation problem and solve for the least-cost plan.

(b) Suppose no more than 6 items can be held in inventory between this month and next month. Formulate the resulting problem as a minimum-cost network flow and solve it.

Answer (Ex. 90) — We formulate both parts as minimum-cost network flow problems.

1. Production with unlimited inventory: balanced transportation

- Supplies (production capacity). Create four supply nodes: A_1 (7), B_1 (14) for this month, and A_2 (7), B_2 (14) for next month. Total supply is $7 + 14 + 7 + 14 = 42$.
- Demands. Two demand nodes: D_1 (12 units, this month) and D_2 (20 units, next month). Total demand is 32, so add a dummy demand "Unused" of 10 units.
- Arcs and costs. Let $x_{s,d}$ be units shipped from supply node s to demand node d.
  - This-month production to this-month demand: cost = production cost (A:12, B:14).
  - This-month production to next-month demand (inventory): cost = production cost + 3 (A:15, B:17).
  - Next-month production to next-month demand: cost = production cost (A:18, B:20).
  - Any supply to Unused: cost 0.
  - Next-month production to this-month demand is disallowed (no backordering), so that arc is omitted.

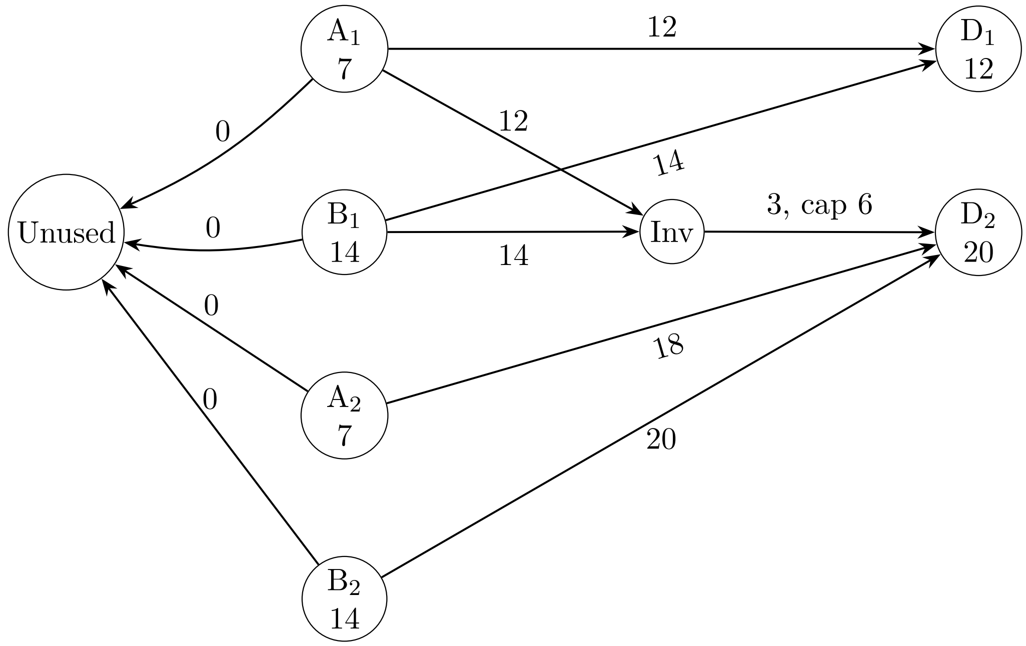
  This is a minimum-cost transportation problem, hence a minimum-cost flow.
- Optimal plan. An optimal shipping plan is:

	$D_1(12)$	$D_2(20)$
$A_1(7)$	0	7
$B_1(14)$	12	2
$A_2(7)$	–	7
$B_2(14)$	–	4

  (remaining 10 units of capacity go to Unused).
  Interpretation: produce 14 at B this month (12 used now, 2 stored), produce 7 at A this month (all 7 stored), and next month produce 7 at A and 4 at B.
  Total cost:

$$7(15) + 12(14) + 2(17) + 7(18) + 4(20) = 513.$$

2.Inventory limited to 6 units: minimum-cost flow.



A single capacity limit on *total inventory* is most naturally modeled with a network arc capacity.

- Network. Use supply nodes A_1, B_1, A_2, B_2 as before, demand nodes D_1, D_2, and add an intermediate transshipment node Inv representing "items carried from this month to next month".
- Arcs, costs, capacities.
  - $A_1 \to D_1$ cost 12, $B_1 \to D_1$ cost 14 (items used immediately).
  - $A_1 \to$Inv cost 12, $B_1 \to$Inv cost 14 (produce now, enter inventory pool).
  - Inv$\to D_2$ cost 3 with **capacity 6** (inventory holding cost and inventory limit).
  - $A_2 \to D_2$ cost 18, $B_2 \to D_2$ cost 20 (produce next month).
  - Disposal arcs to a dummy sink (Unused) with cost 0 to balance total supply and demand.
- Optimal plan with inventory cap 6. An optimal solution is:

$$\text{This month: } A_1 \to D_1 : 1,\ A_1 \to Inv : 6,\ B_1 \to D_1 : 11,$$
$$\text{Next month: } Inv \to D_2 : 6,\ A_2 \to D_2 : 7,\ B_2 \to D_2 : 7.$$

So exactly 6 units are carried in inventory, and next-month production increases accordingly.
Total cost:

$$1(12) + 6(12) + 6(3) + 11(14) + 7(18) + 7(20) = 522.$$

■

2.5 Integer linear programming

Ex. 91 — Lab equipment: A small university computer laboratory has a budget of \$10,800 that can be used to purchase any or all of the four items described below. Each item's value has been assessed by the lab director, and is based on the projected utilization of the item. Use a branch-and-bound technique to determine the optimal selection of items to purchase to enhance the computing laboratory facilities. Show your branch-and-bound tree and give the total cost and total value of the items chosen for purchase.

Item	Cost (\$)	Value
NanoRobot	4,500	8
WinDoze simulator	3,000	3
Network pods	2,500	12
BioPrinter	4,000	10

Budget: \$10,800

Answer (Ex. 91) — This is a 0–1 knapsack problem: choose items (0 or 1 of each) to maximize total value subject to the budget. We solve it with branch-and-bound using the *fractional knapsack relaxation* to compute an upper bound at each node.

It is convenient to order the items by decreasing ratio value/cost (most "profitable" first), and branch in that order. Compute:

$$\frac{12}{2500} = 0.0048, \qquad \frac{10}{4000} = 0.0025, \qquad \frac{8}{4500} \approx 0.00178, \qquad \frac{3}{3000} = 0.001.$$

Hence the branching order is:

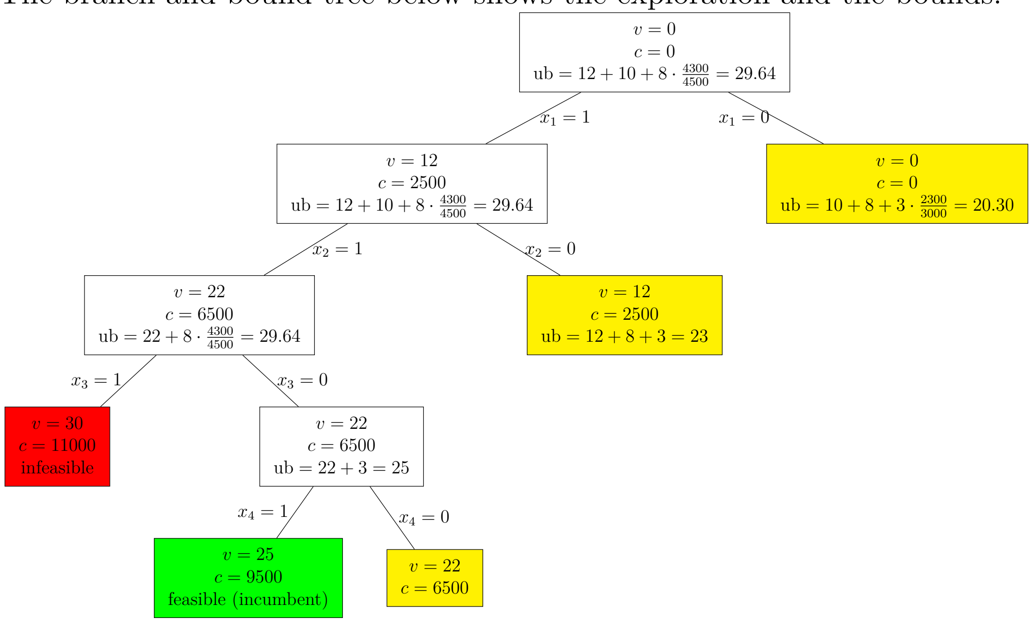
$$x_1 = \text{Network pods}, \quad x_2 = \text{BioPrinter}, \quad x_3 = \text{NanoRobot}, \quad x_4 = \text{WinDoze simulator}.$$

- Model.

$$\max\ Z = 12x_1+10x_2+8x_3+3x_4 \quad \text{s.t.} \quad \begin{cases} 2500x_1 + 4000x_2 + 4500x_3 + 3000x_4 \le 10800, \\ x_i \in \{0,1\} \end{cases}$$

- Upper bound (LP / fractional knapsack). At any partial assignment, the upper bound ub is obtained by filling the remaining budget greedily in the above ratio order, allowing the next item to be taken fractionally. We record at each node: current value v, current cost c, and the bound ub.

The branch-and-bound tree below shows the exploration and the bounds.



- Starting from the root, the fractional relaxation gives ub $= 29.64$, so the node is worth exploring.
- Along the branch $x_1 = 1$, $x_2 = 1$ we have $(v, c) = (22, 6500)$ and still ub $= 29.64$.
- If we then set $x_3 = 1$, the cost becomes $6500 + 4500 = 11000 > 10800$, so that node is infeasible and is pruned.
- Setting instead $x_3 = 0$, the remaining item is x_4 and the bound becomes ub $= 22 + 3 = 25$. Taking $x_4 = 1$ yields a feasible solution with $(v, c) = (25, 9500)$, which becomes the incumbent.

- Any other open node has upper bound at most 23 (or 20.30), which is strictly less than the incumbent value 25, so all remaining nodes are pruned by bound. Hence the incumbent is optimal.

Therefore, the optimal selection is:

$$x_1 = 1, \quad x_2 = 1, \quad x_3 = 0, \quad x_4 = 1,$$

i.e., buy Network pods, BioPrinter, and WinDoze simulator.
Total cost and value:

$$\text{Cost} = 2500 + 4000 + 3000 = 9500, \qquad \text{Value} = 12 + 10 + 3 = 25.$$

■

Ex. 92 — Branch-and-Bound for a 0–1 Integer Programming Problem: Solve the following 0–1 integer programming problem:

$$\begin{aligned} \max \quad & -10x_1 + 3x_2 - 4x_3 \\ \text{s.t.} \quad & -5x_1 + x_2 + x_3 \leq 3, \\ & 3x_1 - 2x_2 - 2x_3 \leq -3, \\ & -2x_1 + 2x_2 + x_3 \leq 3, \\ & x_1, x_2, x_3 \in \{0, 1\}. \end{aligned}$$

Answer (Ex. 92) — We first explore a non-integer relaxation of the problem, then solve it using branch-and-bound.

1. LP relaxation

   - Replace the integrality constraints by bounds:

   $$0 \leq x_1, x_2, x_3 \leq 1.$$

   - The LP relaxation is

   $$\begin{aligned} \max \quad & -10x_1 + 3x_2 - 4x_3 \\ \text{s.t.} \quad & -5x_1 + x_2 + x_3 \leq 3, \\ & 3x_1 - 2x_2 - 2x_3 \leq -3, \\ & -2x_1 + 2x_2 + x_3 \leq 3 \\ & 0 \leq x_1, x_2, x_3 \leq 1. \end{aligned}$$

2. Analysis of the feasible region

- Rewrite the second constraint as

$$2x_2 + 2x_3 - 3x_1 \geq 3. \tag{30}$$

- Since $0 \leq x_2, x_3 \leq 1$, we have $2x_2 + 2x_3 \leq 4$. Therefore, to satisfy the inequality in Eq. 30, both integer values x_2 and x_3 must be equal to 1. Hence

$$x_2 = 1, \qquad x_3 = 1.$$

- Substituting $x_2 = x_3 = 1$ into the second constraint gives

$$3x_1 - 2 - 2 \leq -3 \quad \Longleftrightarrow \quad 3x_1 \leq 1 \quad \Longleftrightarrow \quad x_1 \leq \tfrac{1}{3}.$$

- The remaining constraints are then automatically satisfied for all $0 \leq x_1 \leq \frac{1}{3}$:

$$-5x_1 + 2 \leq 3, \qquad -2x_1 + 3 \leq 3.$$

- Thus, the feasible set of the LP relaxation is the line segment

$$x_2 = 1, \quad x_3 = 1, \quad 0 \leq x_1 \leq \tfrac{1}{3}.$$

3. Root LP optimum

- On the feasible set, the objective function becomes

$$z = -10x_1 + 3(1) - 4(1) = -10x_1 - 1.$$

- Since $-10x_1 - 1$ is strictly decreasing in x_1, the maximum is attained at the smallest feasible value of x_1, namely $x_1 = 0$.
- Therefore, the LP relaxation optimum is

$$(x_1, x_2, x_3) = (0, 1, 1), \qquad z_{\text{LP}} = -1.$$

4. Branch-and-bound conclusion

- The LP relaxation optimum is already integral.
- Hence it is feasible for the original 0–1 problem and provides an upper bound that cannot be improved.
- No branching is required.
- The optimal 0–1 solution is

$$(x_1, x_2, x_3) = (0, 1, 1), \qquad z^* = -1.$$

5.Simplex setup (two-phase formulation, for completeness)

- Multiply the second constraint by -1 to obtain a positive right-hand side:
$$-3x_1 + 2x_2 + 2x_3 \geq 3.$$
- Introduce slack variables for the $\leq$ constraints and a surplus plus artificial variable for the $\geq$ constraint:
$$\begin{aligned} -5x_1 + x_2 + x_3 + s_1 &= 3, \\ -3x_1 + 2x_2 + 2x_3 - s_2 + a_2 &= 3, \\ -2x_1 + 2x_2 + x_3 + s_3 &= 3, \\ x_1 + u_1 &= 1, \\ x_2 + u_2 &= 1, \\ x_3 + u_3 &= 1, \\ x_1, x_2, x_3, s_1, s_2, s_3, a_2, u_1, u_2, u_3 &\geq 0. \end{aligned}$$
- Phase I minimizes $w = a_2$ to obtain feasibility. Phase II restores the objective $\max z = -10x_1 + 3x_2 - 4x_3$.
- The feasible region analysis above already shows that the simplex method must terminate at $(x_1, x_2, x_3) = (0, 1, 1)$, which is therefore the LP (and integer) optimum.

■

Ex. 93 — Lab projects assignment using Branch-and-Bound: An applied research lab has a collection of knowledge transfer projects to choose from and a limited amount of researcher days. The principal investigator wants to choose which projects to take ($x_i = 1$) and which to discard ($x_i = 0$).
The following table shows the revenues and the number of researcher days required:

Project i	1	2	3	4	5	6
Revenue r_i	15	20	5	25	22	17
Days d_i	51	60	35	60	53	10

Assume the lab has at most $D = 120$ researcher days available. Formulate and solve the problem using branch-and-bound, and include a graphical representation of the process.

Answer (Ex. 93) — We formulate and solve the 0–1 integer programming problem using branch-and-bound.

1.Model

- Decision variables: $x_i \in \{0, 1\}$ indicates whether project i is selected.
- Objective:

$$\max\ 15x_1 + 20x_2 + 5x_3 + 25x_4 + 22x_5 + 17x_6.$$

- Capacity (researcher days):

$$51x_1 + 60x_2 + 35x_3 + 60x_4 + 53x_5 + 10x_6 \leq 120.$$

2.LP relaxation (root bound)

- Relax integrality to $0 \leq x_i \leq 1$.
- Use the fractional knapsack bound by sorting projects by revenue per day:

$$\frac{r_6}{d_6} = 1.7,\ \frac{r_4}{d_4} \approx 0.4167,\ \frac{r_5}{d_5} \approx 0.4151,\ \frac{r_2}{d_2} \approx 0.3333,\ \frac{r_1}{d_1} \approx 0.2941,\ \frac{r_3}{d_3} \approx 0.1429.$$

- Fill the 120 days in this order. Take project 6: uses 10 days, revenue 17, remaining 110 days. Take project 4 fully: uses 60 days, revenue 25, remaining 50 days. Next is project 5 (53 days), so take a fraction 50/53:

$$x_5 = \frac{50}{53}.$$

- Root LP solution (one optimal fractional solution):

$$x_6 = 1,\ x_4 = 1,\ x_5 = \frac{50}{53},\ \text{others } 0.$$

- Root upper bound:

$$\mathrm{UB}_{\mathrm{root}} = 17 + 25 + 22 \cdot \frac{50}{53} = 42 + \frac{1100}{53} \approx 62.7547.$$

- The solution is fractional (because $x_5 \notin \{0, 1\}$), so we branch on x_5.

3.A good initial incumbent

- Take the rounded-down integer choice suggested by the LP: $x_6 = 1$, $x_4 = 1$, $x_5 = 0$. This uses $10 + 60 = 70$ days and yields revenue $17 + 25 = 42$.
- Improve the incumbent by adding the best remaining feasible project(s). Remaining capacity is $120 - 70 = 50$ days. Project 3 (35 days, revenue 5) fits; project 1 (51 days) does not; project 2 (60) does not; project 5 (53) does not. So we can add project 3:

$$x_6 = 1,\ x_4 = 1,\ x_3 = 1,\ \text{others } 0,$$

  days $= 10 + 60 + 35 = 105 \leq 120$ and revenue $= 17 + 25 + 5 = 47$.
- Incumbent: $z^* = 47$.

4.Branch on x_5

- Node A: $x_5 = 1$.
- Node B: $x_5 = 0$.

5.Node A: $x_5 = 1$

- Fix $x_5 = 1$ uses 53 days, remaining capacity is $120 - 53 = 67$.
- Compute an LP bound by filling remaining capacity with the best ratios excluding project 5. Take project 6: 10 days, revenue 17, remaining 57. Take project 4: 60 days does not fit, so take a fraction 57/60 of project 4:

$$x_4 = \frac{57}{60}.$$

- Node A upper bound:

$$\mathrm{UB}_A = 22 + 17 + 25 \cdot \frac{57}{60} = 39 + 23.75 = 62.75.$$

- The bound is above the incumbent 47, so we must continue branching. The fractional variable here is x_4, so branch on x_4.

6.Node A1: $x_5 = 1,\ x_4 = 1$

- Days used: $53 + 60 = 113$, remaining capacity is 7.
- Only project 6 needs 10 days, so nothing else fits.
- Best feasible integer solution in this node is $x_4 = x_5 = 1$ with revenue $25 + 22 = 47$.

- This ties the incumbent (47). Fathom this node.

7. Node A2: $x_5 = 1,\ x_4 = 0$
    - Remaining capacity is 67.
    - Best ratio items now: project 6 first (10 days, revenue 17), remaining 57.
    - Next is project 2 or 1 (but 2 needs 60, 1 needs 51). Take project 1 fully (51 days, revenue 15), remaining 6.
    - No other project fits in 6 days.
    - Integer solution: $x_5 = 1,\ x_6 = 1,\ x_1 = 1$ gives days $53 + 10 + 51 = 114$ and revenue $22 + 17 + 15 = 54$.
    - Update incumbent: $z^* = 54$.

8. Node B: $x_5 = 0$
    - This is the branch suggested by the root LP rounding. We already found an integer solution with revenue 47 in this branch ($x_6 = x_4 = x_3 = 1$).
    - Compute an LP upper bound for this node. Fill capacity with best ratios excluding project 5: take 6 (10 days, 17), remaining 110; take 4 (60 days, 25), remaining 50; next 2 (60) does not fit, next 1 (51) does not fit, take 3 fractionally: $50/35 > 1$ so take 3 fully (35 days, 5), remaining 15. Nothing else fits in 15 days.
    - So the best LP bound in this node is actually integer:

      $$x_6 = 1,\ x_4 = 1,\ x_3 = 1,\ \text{others } 0,$$

      with value $17 + 25 + 5 = 47$.
    - Since $\text{UB}_B = 47 < z^* = 54$, prune Node B by bound.

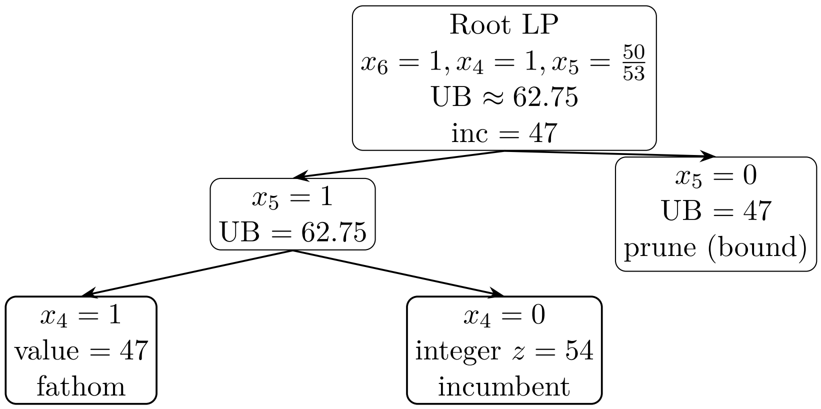
Conclusion:

- The best solution found is in Node A2:

  $$(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 0, 0, 0, 1, 1),$$

  with days $51 + 53 + 10 = 114 \leq 120$ and revenue $15 + 22 + 17 = 54$.
- Therefore the optimal strategy is to take projects 1, 5, and 6, yielding maximum revenue $z^* = 54$.

Branch-and-bound tree:



■

3 Origin of the exercises

The exercises in this compilation come from different sources. Some of them are identified in the statement by acronyms indicating:

Authors:

MC Montserrat Corbera.

JA Josep Ayats.

JVF Jordi Villà-Freixa.

CGPT Adapted from Chat GPT[**?**].

Exams:

1st25 1st Midterm exam, academic year 2025-2026.

If no explicit origin is indicated, it means it has been lost. If you detect any mistake or inaccurate information, please contact the author.

References

[1] Fco. Javier Martínez Sánchez. El teorema de karush-kuhn-tucker, una generalización del teorema de los multiplicadores de lagrange, y programación convexa. *TEMat*, 3:33–44, 2019.