Solved Exercises

Optimization and Operations Research

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Contents

1 Non-linear optimization

Ex. 1 — A problem of non-linear optimization: A company wants to maximize its profit function given by:

$$f(x, y) = -x^2 + 4xy - 2y^2$$

where x and y represent the amount of resources allocated to two different projects.
The problem is subject to the following constraints:

1. The total resources used must not exceed 30 units.
2. The product of the resources allocated must be at least 50 units.
3. The relationship between the resource allocations must satisfy the following nonlinear constraint:

$$y \leq \frac{3x^2}{100} + 5$$

The company wants to find the values of x and y that maximize the profit $f(x, y)$ under these constraints.
Can you help the company drawing the problem in a graph? Can you identify the feasible region? Is the region convex?

Answer (Ex. 1) — A company wants to maximize its profit function given by:

$$f(x, y) = -x^2 + 4xy - 2y^2$$

where x and y represent the amount of resources allocated to two different projects.
The problem is subject to the following constraints:

1. The total resources used must not exceed 30 units:

$$x + 2y \leq 30$$

2. The product of the resources allocated must be at least 50 units:

$$xy \geq 50$$

3. The relationship between the resource allocations must satisfy the following nonlinear constraint:

$$y \leq \frac{3x^2}{100} + 5$$

Find the values of x and y that maximize the profit $f(x, y)$ under these constraints. Before solving it, we can draw the problem in a graph to identify the feasible region and the optimal solution. The problem is shown in Figure 1.

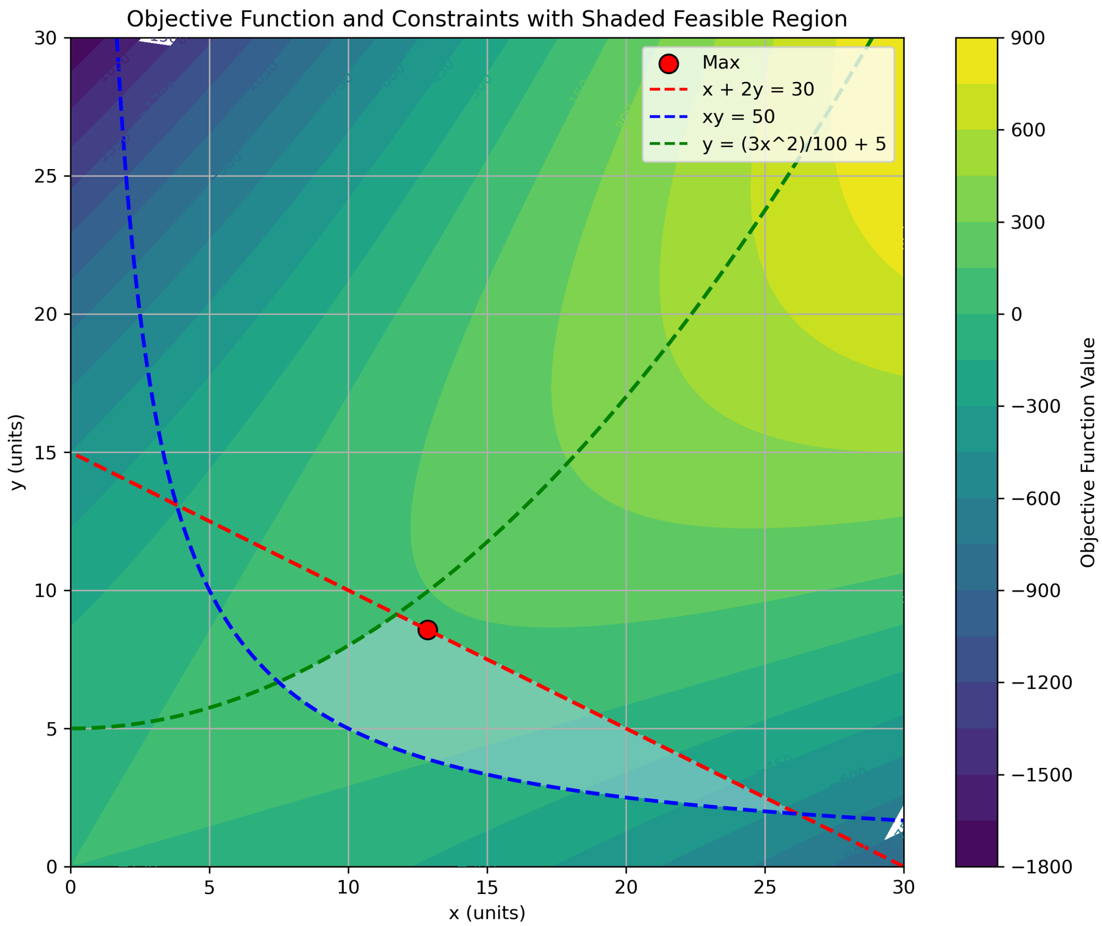


Figure 1: Contour plot of the objective function $f(x, y) = -x^2 + 4xy - 2y^2$ with constraints $x + 2y \leq 30$ (red dashed), $xy \geq 50$ (blue dashed), and $y \leq \frac{3x^2}{100} + 5$ (green dashed). The light blue region represents the feasible area where all constraints are satisfied. The solution has been obtained with Code 1

■

1.1 Unconstrained non-linear optimization

Ex. 2 — Quadratic function minimization: Consider the quadratic function $f(x) = 2x^2 - 4x + 1$. Determine the coordinates of the vertex of the parabola and state whether it represents a minimum or maximum. Compute the minimum (or maximum) value of $f(x)$.

Answer (Ex. 2) — The vertex is at $x = 1$, corresponding to a minimum since the coefficient of x^2 is positive. The minimum value is $f(1) = -1$.

Ex. 3 — Cubic function extrema: Find all critical points of the function $f(x) = x^3 - 3x + 1$ and determine which of them correspond to local minima and maxima.

Answer (Ex. 3) — $f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$. $f''(x) = 6x \Rightarrow f''(1) > 0 \rightarrow$ local minimum at $x = 1$; $f''(-1) < 0 \rightarrow$ local maximum at $x = -1$. Minimum value: $f(1) = -1$.

Ex. 4 — Stationary point of a quadratic function: Determine the stationary point of $f(x, y) = 3x^2 + 2y^2 - 6x - 8y$ and classify it.

Answer (Ex. 4) — Setting partial derivatives to zero: $6x - 6 = 0 \Rightarrow x = 1$, $4y - 8 = 0 \Rightarrow y = 2$. The point $(1, 2)$ is a minimum.

Ex. 5 — Quadratic maximization with bounds: Maximize $f(x) = -x^2 + 4x$ for $0 \leq x \leq 3$. Determine where the maximum occurs and its value.

Answer (Ex. 5) — $f'(x) = -2x + 4 = 0 \Rightarrow x = 2$. $f(2) = 4$ is the maximum value, since $f(0) = 0$ and $f(3) = 3$.

Ex. 6 — Hessian at a minimum: Suppose $f(x)$ is a twice continuously differentiable function with a global minimum at $x = x^*$. What can be said about the Hessian matrix $H(x^*)$?

Answer (Ex. 6) — At a global minimum, the Hessian must be **positive definite**. All eigenvalues are positive, indicating the surface curves upward in all directions. If the Hessian were negative definite, the point would be a local maximum; if indefinite, a saddle. ■

Ex. 7 — Hessian concavity criteria: Given the function $f(x, y) = x^3 + 2y^3 - xy$, determine whether the function at $(0, 0)$ is convex, concave, or neither.

Answer (Ex. 7) — The Hessian is

$$H = \begin{pmatrix} 6x & -1 \\ -1 & 12y \end{pmatrix}.$$

At $(0, 0)$:

$$H(0, 0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Its eigenvalues are $\lambda = 1$ and $\lambda = -1$, so H is **indefinite**. Hence, the function is **neither convex nor concave** at $(0, 0)$ (it has a saddle-type curvature). ■

Ex. 8 — Golden Section Search: When applying the Golden Section Search to find the maximum of $f(x)$, the condition $f(x_1) > f(x_2)$ holds for the intermediate points $(x_1 < x_2)$. Explain which region is eliminated from the search interval and why.

Answer (Ex. 8) — Since $f(x_1) > f(x_2)$, the function is decreasing after x_1. Thus, the maximum lies between x_l and x_2, and the interval $[x_2, x_u]$ can be eliminated. The new search region is $[x_l, x_2]$. ■

Ex. 9 — Newton's Method: In Newton's method for unconstrained optimization, the search direction is $p_k = -H_k^{-1}\nabla f(x_k)$. Explain under which conditions this direction corresponds to a descent direction.

Answer (Ex. 9) — If the Hessian H_k is **positive definite**, then p_k points in a descent direction (toward a local minimum). If H_k is **negative definite**, the algorithm moves toward a local maximum. If H_k is **indefinite**, the step may not be a descent direction. ■

Ex. 10 — Gradient Descent: In the gradient descent method with update $x_{k+1} = x_k - \alpha_k \nabla f(x_k)$, explain the role of the step size α_k.

Answer (Ex. 10) — The step is taken in the **negative gradient direction**, which points toward steepest descent. A small α_k leads to slow convergence, while a large α_k can cause divergence or oscillation. A properly chosen step size ensures steady convergence to a local minimum. ■

Ex. 11 — First- and Second-Order Optimality Conditions: State the first- and second-order conditions that must hold at a local minimum of a differentiable function $f(x)$.

Answer (Ex. 11) — At a local minimum x^*:

$$\nabla f(x^*) = 0 \quad \text{and} \quad H(x^*) \text{ is positive definite.}$$

If $H(x^*)$ is negative definite, the point is a local maximum; if indefinite, it is a saddle point. ■

Ex. 12 — Hessian and Convexity: Explain how the definiteness of the Hessian matrix determines whether a function is convex, concave, or neither.

Answer (Ex. 12) — If $H(x)$ is **positive semi-definite** for all x, the function is convex. If $H(x)$ is **negative semi-definite**, it is concave. If $H(x)$ is **indefinite**, the function is neither convex nor concave. A **singular** Hessian may indicate flat regions or saddle behavior. ■

Ex. 13 — Critical points and curvature classification: Consider the function $f(x, y) = x^3-3x+y^2$. Find its critical points and determine their type.

Answer (Ex. 13) —

$$\frac{\partial f}{\partial x} = 3x^2 - 3 = 0 \Rightarrow x = \pm 1, \qquad \frac{\partial f}{\partial y} = 2y = 0 \Rightarrow y = 0.$$

Thus, critical points: $(1, 0)$ and $(-1, 0)$.

$$H = \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix}.$$

At $(1, 0)$: $H = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \rightarrow$ positive definite $\rightarrow$ local minimum. At $(-1, 0)$: $H = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix} \rightarrow$ indefinite $\rightarrow$ saddle point. ■

Ex. 14 — Critical points of $\sin(2x) + \cos(y)$: Find and classify the critical points of $f(x, y) = \sin(2x) + \cos(y)$.

Answer (Ex. 14) —

$$\frac{\partial f}{\partial x} = 2\cos(2x), \quad \frac{\partial f}{\partial y} = -\sin(y).$$

Set both to zero: $\cos(2x) = 0$, $\sin(y) = 0$. Thus $x = \pi/4 + k\pi/2$, $y = n\pi$, with $k, n \in \mathbb{Z}$.

$$H = \begin{pmatrix} -4\sin(2x) & 0 \\ 0 & -\cos(y) \end{pmatrix}.$$

If $\sin(2x)$ and $\cos(y)$ have the same sign $\to$ H definite $\to$ local extrema. If signs differ $\to$ H indefinite $\to$ saddle points. ■

Ex. 15 — Critical points of $e^{x\cos y}$: Find and classify the critical points of $f(x, y) = e^{x\cos y}$.

Answer (Ex. 15) —

$$\frac{\partial f}{\partial x} = e^{x\cos y}\cos y, \qquad \frac{\partial f}{\partial y} = -xe^{x\cos y}\sin y.$$

Setting both to zero gives $x = 0$, $\sin y = 0 \Rightarrow y = n\pi$, $n \in \mathbb{Z}$. Thus, critical points are $(0, n\pi)$.

$$H = \begin{pmatrix} \cos y\, e^{x\cos y} & -x\sin y\, e^{x\cos y} \\ -x\sin y\, e^{x\cos y} & -x\cos y\, e^{x\cos y} \end{pmatrix}.$$

At $(0, n\pi)$:

$$H = \begin{pmatrix} (-1)^n & 0 \\ 0 & 0 \end{pmatrix}.$$

One eigenvalue is zero $\to$ Hessian is **singular**. The function is flat in the y-direction at these points. ■

1.2 Optimization with constraints

1.2.1 Lagrange Multipliers

Ex. 16 — Lagrange multipliers: Optimization of $f(x, y) = x^2 - y$ subject to $x^2 + y^2 = 4$

Answer (Ex. 16) — We will use the method of Lagrange multipliers. Define the constraint as $g(x, y) = x^2 + y^2 - 4 = 0$. The Lagrange function is then:

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda \cdot g(x, y) = x^2 - y + \lambda(x^2 + y^2 - 4).$$

Note that we have used $+\lambda \cdot g(x, y)$ in the equation above, but we could also use $-\lambda \cdot g(x, y)$; the results would be equivalent. We now compute the partial derivatives of $\mathcal{L}(x, y, \lambda)$ with respect to x, y, and λ and equal them to zero to obtain the system of equations to solve:

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + \lambda \cdot 2x = 2x(1 + \lambda) = 0, \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -1 + \lambda \cdot 2y = 0 \quad \Rightarrow \quad \lambda = \frac{1}{2y}, \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - 4 = 0. \tag{3}$$

From Eq. 1, we see two possibilities:

- $x = 0$: Substitute $x = 0$ in Eq. 3:

$$0^2 + y^2 = 4 \quad \Rightarrow \quad y^2 = 4 \quad \Rightarrow \quad y = \pm 2.$$

  For $y = 2$, substitute into $f(x, y) = x^2 - y$:

$$f(0, 2) = 0^2 - 2 = -2.$$

  For $y = -2$, substitute into $f(x, y)$:

$$f(0, -2) = 0^2 - (-2) = 2.$$

- $1 + \lambda = 0 \quad \Rightarrow \quad \lambda = -1$. Substitute $\lambda = -1$ into Eq. 2:

$$-1 = \frac{1}{2y} \quad \Rightarrow \quad y = -\frac{1}{2}.$$

  Now, substitute $y = -\frac{1}{2}$ into the constraint equation $x^2 + y^2 = 4$:

$$x^2 + \left(-\frac{1}{2}\right)^2 = 4 \quad \Rightarrow \quad x^2 + \frac{1}{4} = 4 \quad \Rightarrow \quad x^2 = \frac{15}{4} \quad \Rightarrow \quad x = \pm\frac{\sqrt{15}}{2}.$$

Now, calculate $f(x, y) = x^2 - y$ for $y = -\frac{1}{2}$ and $x = \pm\frac{\sqrt{15}}{2}$:
For $x = \frac{\sqrt{15}}{2}$:

$$f\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) = \left(\frac{\sqrt{15}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{15}{4} + \frac{1}{2} = \frac{17}{4}.$$

For $x = -\frac{\sqrt{15}}{2}$:

$$f\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) = \left(-\frac{\sqrt{15}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{15}{4} + \frac{1}{2} = \frac{17}{4}.$$

We now compare the function values:

- For $(0, 2)$: $f(0, 2) = -2$.
- For $(0, -2)$: $f(0, -2) = 2$.
- For $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$: $f = \frac{17}{4}$.

Thus:

- The maximum value is $f = \frac{17}{4}$ at $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$.
- The minimum value is $f = -2$ at $(0, 2)$.

For a Python implementation check this file.

Ex. 17 — Lagrange multipliers with linear constraint: Solve the constrained optimization problem:

$$\min f(x, y) = x^2 + y^2 \quad \text{subject to} \quad x + 2y = 4.$$

Find all stationary points and determine which gives the minimum value.

Answer (Ex. 17) — Using the Lagrange multiplier method:

$$\nabla f = (2x, 2y), \quad \nabla g = (1, 2) \Rightarrow 2x = \lambda,\ 2y = 2\lambda.$$

From the constraint: $x + 2y = 4$. Solving gives $x = \frac{4}{5}$, $y = \frac{8}{5}$. This is the minimum point.

Ex. 18 — Constrained product minimization: Find the minimum and maximum values of $f(x, y) = xy$ subject to $x^2 + y^2 = 9$.

Answer (Ex. 18) — Using Lagrange multipliers: $y = \lambda 2x$, $x = \lambda 2y \Rightarrow x^2 = y^2 \Rightarrow y = \pm x$. For $y = x$: $2x^2 = 9 \Rightarrow x = \pm\frac{3}{\sqrt{2}} \Rightarrow f = \frac{9}{2}$. For $y = -x$: $f = -\frac{9}{2}$. Minimum value $-\frac{9}{2}$, maximum $\frac{9}{2}$.

Ex. 19 — **Optimization with equality constraint:** Use the Lagrange multiplier method to find the values of x and y that minimize $f(x, y) = x^2 + y^2$ subject to $x + y = 1$.

Answer (Ex. 19) — From the constraint: $y = 1 - x$. Then $f(x) = x^2 + (1-x)^2 = 2x^2 - 2x + 1$. Minimum at $x = \frac{1}{2}$, $y = \frac{1}{2}$.

1.2.2 Karush-Kuhn-Tucker theorem

Ex. 20 — **Complementary slackness computation:** Consider minimizing $f(x) = x_1^2 + x_2^2$ subject to $x_1 + x_2 \geq 2$ and $x_1, x_2 \geq 0$. Use KKT conditions (with multipliers) and illustrate complementary slackness: identify active constraints and multipliers.

Answer (Ex. 20) — Set constraint as $g(x) = 2-(x_1+x_2) \leq 0$. Lagrangian $L = x_1^2 + x_2^2 + \lambda(2 - x_1 - x_2) - \mu_1 x_1 - \mu_2 x_2$.
KKT conditions:

Stationarity:

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda - \mu_1 = 0$$
$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda - \mu_2 = 0$$

Primal feasibility:

$$x_1 + x_2 \geq 2, \quad x_1 \geq 0, \quad x_2 \geq 0$$

Dual feasibility:

$$\lambda \geq 0, \quad \mu_1 \geq 0, \quad \mu_2 \geq 0$$

Complementary slackness:

$$\lambda(2 - x_1 - x_2) = 0$$
$$\mu_1 x_1 = 0$$
$$\mu_2 x_2 = 0$$

Solving: Try symmetric solution $x_1 = x_2 = a$. Then $2a \geq 2 \implies a \geq 1$.
Try $a = 1$ (active constraint).
Plug into stationarity:

$$2x_1 - \lambda - \mu_1 = 0 \implies 2 - \lambda - \mu_1 = 02x_2 - \lambda - \mu_2 = 0 \implies 2 - \lambda - \mu_2 = 0$$

By symmetry, $\mu_1 = \mu_2 = \mu$, so $2 - \lambda - \mu = 0 \implies \lambda = 2 - \mu$.
From complementary slackness: $x_1 = x_2 = 1 > 0 \implies \mu_1 = \mu_2 = 0$.
So $\lambda = 2$.
Check complementary slackness for λ: $2 - x_1 - x_2 = 0$, so $\lambda \cdot 0 = 0$.
Summary:

- Active constraint: $x_1 + x_2 = 2$ $(g(x) = 0)$, multiplier $\lambda = 2$.
- Inactive constraints: $x_1 = 1 > 0$, $x_2 = 1 > 0$, multipliers $\mu_1 = \mu_2 = 0$.
- All KKT conditions are satisfied.

Interpretation: The only active constraint is $x_1 + x_2 \geq 2$, and its multiplier is positive. The nonnegativity constraints are inactive, so their multipliers are zero. This illustrates complementary slackness: only active constraints can have nonzero multipliers.

Ex. 21 — KKT necessity for optimality: For the constrained problem $\min f(x) = x^2 + y^2$ subject to $x + y = 1$, write the KKT (which reduce to Lagrange) conditions and solve. Explain why KKT are necessary.

Answer (Ex. 21) — Lagrangian $L = x^2 + y^2 + \lambda(x + y - 1)$. Stationarity: $2x + \lambda = 0,\ 2y + \lambda = 0 \rightarrow$ x=y. Constraint gives x=y=1/2. KKT (here Lagrange) are necessary to characterize optima under differentiability and constraint qualifications.

Ex. 22 — KKT application domain: Give a nonlinear optimization problem where KKT applies: minimize $f(x) = x_1^2 + 4x_2^2$ subject to $x_1 + x_2 \leq 3,\ x_i \geq 0$. Use KKT to find candidate minimizers.

Answer (Ex. 22) — Formulate KKT with multipliers λ for inequality. Solve stationarity: $2x_1 + \lambda - \mu_1 = 0,\ 8x_2 + \lambda - \mu_2 = 0$. Try interior (mu=0) and inactive constraint: if interior then lambda=0$\rightarrow$ x1=0,x2=0 but constraint 0<=3 satisfied; 0 is minimizer.

Ex. 23 — Non-negativity in KKT: Show with a small example how KKT enforces non-negativity constraints. Minimize $f(x) = x_1^2 + x_2^2$ subject to $x_1 \geq 0, x_2 \geq 0$. Find solution and relate to KKT multipliers.

Answer (Ex. 23) — Unconstrained minimum at (0,0) already satisfies non-negativity. KKT multipliers for inequality constraints are zero or positive; stationarity yields $2x_1 - \mu_1 = 0,\ 2x_2 - \mu_2 = 0$. At (0,0) mu>=0 can be zero; complementary slackness holds.

Ex. 24 — Multipliers as shadow prices: Consider minimize cost $f(x) = 2x_1 + 3x_2$ subject to $x_1 + x_2 \geq 5,\ x_i \geq 0$. Using KKT multipliers (dual viewpoint) interpret the multiplier for the constraint as a shadow price: compute multiplier and show sensitivity of optimal cost to RHS change.

Answer (Ex. 24) — Reformulate as equality at optimum: x1+x2=5 minimal cost with nonnegativity -> allocate to cheapest variable x1 (cost 2): x1=5,x2=0, cost=10. Small increase d in RHS increases minimal cost by multiplier equal to dual price; here dual multiplier = 2 (cost of resource) so sensitivity approx 2 per unit. (Interpretation)

Ex. 25 — Linear independence not KKT requirement: Explain with a short counterexample why linear independence (LICQ) is not a component of the KKT equalities themselves: state KKT conditions and show one of them is not 'linear independence'.

Answer (Ex. 25) — KKT conditions include stationarity, primal and dual feasibility, complementary slackness. LICQ is a constraint qualification (assumption) to guarantee KKT necessity/sufficiency. Thus LICQ is not part of KKT equations but an extra assumption—demonstrated by stating KKT and noting LICQ absent.

Ex. 26 — Convexity assumption for sufficiency: Provide a convex constrained example where KKT conditions are sufficient. Minimize $f(x) = x_1^2 + x_2^2$ subject to $x_1 + x_2 \geq 1,\ x_i \geq 0$. Use KKT to find minimizer and argue sufficiency because of convexity.

Answer (Ex. 26) — As earlier, constraint active yields x1=x2=0.5; KKT solved gives multipliers and unique minimizer. Because f and feasible set are convex, KKT conditions ensure global optimality.

Ex. 27 — KKT necessity vs sufficiency: Give an example showing that satisfying KKT is not always sufficient for optimality when convexity fails. Provide a nonconvex problem and describe a KKT point that is not global optimum.

Answer (Ex. 27) — Consider $f(x) = -x^2$ subject to $x^2 \leq 1$. The KKT stationary point $x = 0$ satisfies KKT but is not maximal (the endpoints $x = \pm 1$ give greater objective for maximization). This illustrates KKT is not sufficient without convexity.

Ex. 28 — KKT generalize Lagrange: Show algebraically that KKT conditions generalize Lagrange multipliers to inequalities by writing KKT for a problem with equality and inequality constraints. Use the problem $\min x^2$ s.t. $x \geq 1$ and show correspondence.

Answer (Ex. 28) — For $\min x^2$ s.t. $x \geq 1$, write $g(x) = 1 - x \leq 0$. Lagrangian $L = x^2 + \lambda(1 - x)$. Stationarity: $2x - \lambda = 0$ with $\lambda \geq 0$, complementary slackness: $\lambda(1 - x) = 0$. Solution $x = 1$, $\lambda = 2$ satisfies KKT; reduces to Lagrange when constraint is active as equality.

Ex. 29 — A problem of KKT conditions: Maximize $f(x, y) = xy$ subject to $100 \geq x + y$ and $x \leq 40$ and $(x, y) \geq 0$.

Answer (Ex. 29) — The Karush Kuhn Tucker (KKT) conditions for optimality are a set of necessary conditions for a solution to be optimal in a mathematical optimization problem. They are necessary and sufficient conditions for a local minimum in nonlinear programming problems. The KKT conditions consist of the following elements:
For an optimization problem in its standard form:

$$\max f(\mathbf{x})$$
$$\text{s.t.} \quad \begin{array}{rcll} g_i(\mathbf{x}) - b_i & \leq & 0 & i = 1, \ldots, k \\ g_i(\mathbf{x}) - b_i & = & 0 & i = k+1, \ldots, m \end{array}$$

There are 4 KKT conditions for optimal primal ($\mathbf{x}$) and dual (λ) variables. If $\mathbf{x}^*$ denotes optimal values:

1. Primal feasibility: all constraints must be satisfied: $g_i(\mathbf{x}^*) - b_i$ is feasible. Applies to both equality and non-equality constraints.

2.Gradient condition or No feasible descent: No possible improvement at the solution:

$$\nabla f(\mathbf{x}^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(\mathbf{x}^*) = 0$$

3.Complementariety slackness:

$$\lambda_i^*(g_i(\mathbf{x}^*) - b_i) = 0$$

4.Dual feasibility: $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active (=0) and a zero Lagrange multiplier when the constraint is inactive (>0).
To solve our problem, first we will put it in its standard form:

$$\max f(x,y) = xy$$
$$\text{s.t.} \quad \begin{aligned} g_1(x,y) = x + y - 100 &\leq 0 \\ g_2(x,y) = x - 40 &\leq 0 \\ g_3(x,y) = -x &\leq 0 \\ g_4(x,y) = -y &\leq 0 \end{aligned} \tag{4}$$

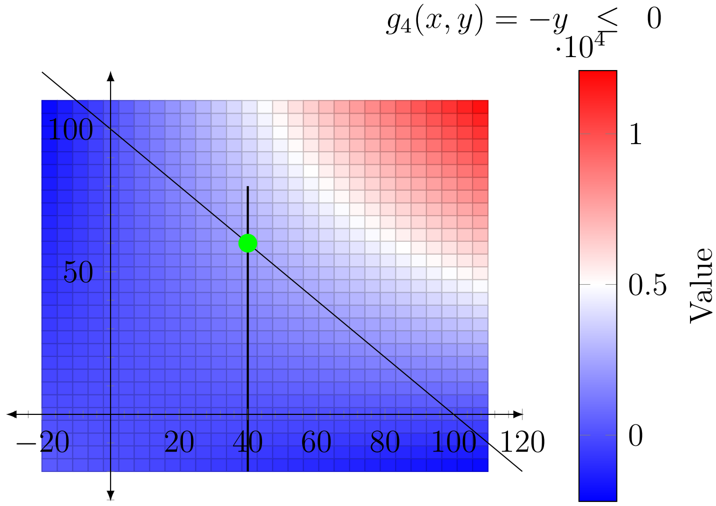


Figure 2: Contour plot of the objective function $f(x, y) = xy$ with constraints in Eq. 4, showing the final optimal value after applying the KKT conditions (green dot).
We will go through the different conditions:

- on the gradient:

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} - \lambda_1 \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_1}{\partial y} \end{pmatrix} - \lambda_2 \begin{pmatrix} \frac{\partial g_2}{\partial x} \\ \frac{\partial g_2}{\partial y} \end{pmatrix} - \lambda_3 \begin{pmatrix} \frac{\partial g_3}{\partial x} \\ \frac{\partial g_3}{\partial y} \end{pmatrix} - \lambda_4 \begin{pmatrix} \frac{\partial g_4}{\partial x} \\ \frac{\partial g_4}{\partial y} \end{pmatrix} = 0$$

which, in this example, resolves into:

$$y - (\lambda_1 + \lambda_2 - \lambda_3) = 0 \tag{5}$$

$$x - (\lambda_1 - \lambda_4) = 0 \tag{6}$$

- on the complementary slackness:

$$\lambda_1(x + y - 100) = 0 \tag{7}$$

$$\lambda_2(x - 40) = 0 \tag{8}$$

$$\lambda_3 x = 0 \tag{9}$$

$$\lambda_4 y = 0 \tag{10}$$

- on the constraints:

$$x + y \leq 100 \tag{11}$$

$$x \leq 40 \tag{12}$$

$$-x \leq 0 \tag{13}$$

$$-y \leq 0 \tag{14}$$

plus $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$.

We will start by checking Eq. 7:

- Let us see what occurs if $\lambda_1 = 0$. Then, from Eq. 6, $x + \lambda_4 = 0$ which implies that $x = \lambda_4 = 0$[1]. But, then, from Eq. 8 we obtain that $\lambda_2 = 0$ which, using Eq. 5 gives $y + \lambda_3 = 0 \Rightarrow y = \lambda_3 = 0$. Indeed, the KKT conditions are satisfied when all variables and multipliers are zero, but it is not a maximum of the function (see figure above).
- So, let us see what happens if $x + y - 100 = 0$ and consider the two possibilities for x:

[1] Recall that both variables and multiplieras must be positive or zero, so, the only possibility for the equation to fullfill is that both are zero.

Case $x = 0$**:**Then, $y = 100$, which would lead (Eq. 10) to $\lambda_4 = 0$ and (Eq. 6) to $x = \lambda_1 = 0$, that was discussed in the previous item. So, we need top explore the other possibility for x.

Case $x > 0$**:**From Eq. 9 $\lambda_3 = 0$ and, from Eqs. 5 and 6:

$$\begin{cases} \quad y = \lambda_1 + \lambda_2 \\ x \quad\ = \lambda_1 + \lambda_4 \end{cases}$$

let us try what happens if, e.g., $\lambda_2 \neq 0$ (or, said in other words, if constraint 12 is active): $x = 40$. As we know we do not want $\lambda_1 = 0$, from Eq. 7 we obtain $x + y - 100 = 0 \Rightarrow y = 60$.

The point $(x, y) = (40, 60)$ fullfills the KKT conditions and is a maximum in the constrained maximization problem (as can be seen in Figure 2).

Ex. 30 — KKT conditions - Complete problem: Solve the following optimization problem:[1]

$$\text{Optimize } f(x, y, z) = x + y + z,$$

$$\text{subject to: } \begin{cases} h_1(x, y, z) = (y-1)^2 + z^2 \leq 1, \\ h_2(x, y, z) = x^2 + (y-1)^2 + z^2 \leq 3. \end{cases}$$

Answer (Ex. 30) — By Weierstrass' theorem, there exist a maximum and a minimum for the problem. The Karush-Kuhn-Tucker (KKT) conditions are:

- Stationarity conditions:

$$\begin{cases} 1 + 2\mu_2 x = 0, \\ 1 + 2\mu_1(y-1) + 2\mu_2(y-1) = 0, \\ 1 + 2\mu_1 z + 2\mu_2 z = 0. \end{cases}$$

- Slackness conditions:

$$\begin{cases} \mu_1 \left[(y-1)^2 + z^2 - 1\right] = 0, \\ \mu_2 \left[x^2 + (y-1)^2 + z^2 - 3\right] = 0. \end{cases}$$

- Feasibility conditions:

$$(y-1)^2 + z^2 \leq 1, \qquad x^2 + (y-1)^2 + z^2 \leq 3.$$

- Sign conditions:

$$\mu_1, \mu_2 \geq 0 \Rightarrow \text{local minimum}, \qquad \mu_1, \mu_2 \leq 0 \Rightarrow \text{local maximum}.$$

From the slackness conditions we distinguish four cases

$$\begin{cases} \mu_1 = 0 \implies \begin{cases} \mu_2 = 0 & \text{(Case I)} \\ x^2 + (y-1)^2 + z^2 - 3 = 0 & \text{(Case II)} \end{cases} \\ (y-1)^2 + z^2 - 1 = 0 \implies \begin{cases} \mu_2 = 0 & \text{(Case III)} \\ x^2 + (y-1)^2 + z^2 - 3 = 0 & \text{(Case IV)} \end{cases} \end{cases}$$

but the first stationarity equation implies that $\mu_2 \neq 0$, so only cases II and IV must be checked.

- Case II: $\mu_1 = 0$ and $x^2 + (y-1)^2 + z^2 - 3 = 0$.

$$\begin{aligned} P_1 &= (1,2,1), & \mu &= (0, -\tfrac{1}{2}), \\ P_2 &= (-1,0,-1), & \mu &= (0, \tfrac{1}{2}). \end{aligned}$$

- Case IV: $(y-1)^2 + z^2 - 1 = 0$ and $x^2 + (y-1)^2 + z^2 - 3 = 0$.

$$\begin{aligned} P_3 &= \left(\sqrt{2}, 1 + \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), & \mu &= \left(-\tfrac{1}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}\right), \\ P_4 &= \left(\sqrt{2}, 1 - \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right), & \mu &= \left(\tfrac{3}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}\right), \\ P_5 &= \left(-\sqrt{2}, 1 + \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), & \mu &= \left(-\tfrac{3}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}\right), \\ P_6 &= \left(-\sqrt{2}, 1 - \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right), & \mu &= \left(\tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}\right). \end{aligned}$$

After applying feasibility and sign conditions, the results are summarized as follows:

P	μ	Feasibility	Sign	Conclusion
P_1	$(0, -\frac{1}{2})$	NO	$-$	$-$
P_2	$(0, \frac{1}{2})$	NO	$-$	$-$
P_3	$(-\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}})$	YES	$Negative$	Conditional Maximum
P_4	$(\frac{3}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}})$	YES	NO	$-$
P_5	$(-\frac{3}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$	YES	NO	$-$
P_6	$(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$	YES	$Positive$	Conditional Minimum

Therefore:

$$\boxed{\begin{array}{rl} \text{Conditional Maximum:} & P_3 = \left(\sqrt{2}, 1 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \\ \text{Conditional Minimum:} & P_6 = \left(-\sqrt{2}, 1 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right). \end{array}}$$

Here you have a MATLAB script that implements the KKT conditions to solve this problem:

```
syms x y z mu1 mu2 real

% Objective function
f = x + y + z;

% Constraints
h1 = (y - 1)^2 + z^2 - 1;
h2 = x^2 + (y - 1)^2 + z^2 - 3;

% Stationary conditions
eq1 = 1 + 2*mu2*x == 0;
eq2 = 1 + 2*mu1*(y - 1) + 2*mu2*(y - 1) == 0;
eq3 = 1 + 2*mu1*z + 2*mu2*z == 0;

% Complementary slackness conditions
eq4 = mu1*h1 == 0;
eq5 = mu2*h2 == 0;

% Symbolic resolution (considering cases)
sol = solve([eq1, eq2, eq3, eq4, eq5], [x, y, z, mu1, mu2], 'Real', true);

disp(struct2table(sol))
```

2 Linear Optimization

Ex. 31 — Formulation of an optimization problem: Consider a manufacturing company that produces two products, P_1 and P_2. The company aims to maximize its profit. The profit from each product is given as follows:
- P_1: \$40 per unit - P_2: \$30 per unit
The company has the following constraints based on its resources:
1. The availability of raw material limits production to a maximum of 100 units of P_1 and 80 units of P_2. 2. The total production time available is 160 hours, where producing one unit of P_1 takes 2 hours and one unit of P_2 takes 1 hour.
Can you formulate the operations research problem?

Answer (Ex. 31) — The problem can be stated as:

$$
\begin{aligned}
\text{Objective Function:}\quad & \text{Maximize } Z = 40x_1 + 30x_2 \\
\text{Subject to:}\quad & \\
& 2x_1 + x_2 \leq 160 \quad \text{(Total production time constraint)} \\
& x_1 \leq 100 \quad \text{(Raw material constraint for } P_1\text{)} \\
& x_2 \leq 80 \quad \text{(Raw material constraint for } P_2\text{)} \\
& x_1, x_2 \geq 0 \quad \text{(Non-negativity constraints)}
\end{aligned}
$$

■

2.1 The Simplex method

Ex. 32 — Unique optimal solution: Maximize $z = 2x+3y$ subject to:

$$
\begin{aligned}
x + y &\leq 6 \\
x &\leq 4 \\
x, y &\geq 0
\end{aligned}
$$

Answer (Ex. 32) — Introduce slack variables s_1, s_2 to convert to equalities:

$$\begin{aligned} x + y + s_1 &= 6, \quad s_1 \geq 0, \\ x + s_2 &= 4, \quad s_2 \geq 0. \end{aligned}$$

Objective rewritten in standard form:

$$z - 2x - 3y = 0.$$

We form the initial simplex tableau (using the convention where the objective row contains the coefficients of $z -$ (linear form) $= 0$, i.e. the top row for constraints and bottom row for z):

	x	y	s_1	s_2	RHS
s_1	1	1	1	0	6
s_2	1	0	0	1	4
z	-2	-3	0	0	0

Step 1 — choose entering variable. For a maximization problem, we look for negative coefficients in the z-row (since $z - c^T x = 0$); the most negative is -3 corresponding to y. So y will enter.
Step 2 — choose leaving variable (minimum ratio test). Compute RHS / (coefficient of entering variable):
- For row s_1: coefficient of y is 1 so ratio $6/1 = 6$. - For row s_2: coefficient of y is 0 (cannot be pivot row).
Minimum positive ratio is 6 (row of s_1). Thus pivot on the element in row s_1, column y (which is 1). After pivoting, y will replace s_1 as basic variable.
Perform the pivot (make pivot = 1 and eliminate y elsewhere).
The pivot row (row s_1) already has a 1 in the y-column, so it stays:

$$\text{New row (basic } y) : \quad y + x + s_1 = 6$$

We will express the pivot row as $y = 6 - x - s_1$.
Eliminate y from the z-row: add $3\times$ (pivot row) to the z-row to remove -3 in the z-row.
Arithmetic for the z-row update:

$$\text{old } z\text{-row } (-2,\ -3,\ 0,\ 0 \mid 0) \quad + \ 3 \times \text{pivot row } (1,\ 1,\ 1,\ 0 \mid 6)$$

gives

$z\text{-row}_{\text{new}} = (-2+3\cdot 1,\ -3+3\cdot 1,\ 0+3\cdot 1,\ 0+3\cdot 0 \mid 0+3\cdot 6) = (1,\ 0,\ 3,\ 0 \mid 18).$

The row s_2 has zero in column y so it is unchanged.
So the tableau after the pivot becomes:

	x	y	s_1	s_2	RHS
y	1	1	1	0	6
s_2	1	0	0	1	4
z	1	0	3	0	18

(Interpretation: the first row represents $y+x+s_1 = 6$, second row $x+s_2 = 4$, and objective equation is $z + x + 3s_1 = 18$.)
Step 3 — check optimality. Using the same sign convention (bottom row is coefficients of $z-(\cdots) = 0$), we look for negative coefficients in the z-row to select another entering variable. The current z-row coefficients for nonbasic variables are:
- coefficient of x in the z-row is $+1$, - coefficient of s_1 is $+3$, - coefficient of s_2 is 0.
There are no negative entries in the z-row. Therefore, the current basic feasible solution is optimal.
Read off the basic feasible solution by setting nonbasic variables to zero. From the tableau: - Basic variables are y and s_2 (they correspond to the left labels). Nonbasic variables are x and s_1. Set $x = 0$ and $s_1 = 0$. - From row y: $y = 6 - x - s_1 = 6$. - From row s_2: $s_2 = 4 - x = 4$.
Thus the primal solution is:

$$(x, y) = (0, 6).$$

Compute objective:

$$z = 2x + 3y = 2 \cdot 0 + 3 \cdot 6 = 18.$$

Concluding, the simplex method yields the optimal solution

$$\boxed{(x, y) = (0, 6), \qquad z_{\max} = 18.}$$

(As a sanity check: the corner points of the feasible polygon are $(0, 0), (0, 6), (4, 0), (4, 2)$. Their objective values are $0, 18, 8, 14$ respectively, so indeed $(0, 6)$ is best.)

Ex. 33 — Modified problem: maximize $z = x + 2y$: Maximize $z = x + 2y$ subject to:

$$\begin{aligned} x + 2y &\leq 8, \\ 2x + y &\leq 8, \\ x, y &\geq 0. \end{aligned}$$

Answer (Ex. 33) — Introduce slack variables s_1, s_2 to convert inequalities to equalities:

$$x + 2y + s_1 = 8, \qquad 2x + y + s_2 = 8, \qquad s_1, s_2 \geq 0.$$

Objective in standard form:

$$z - x - 2y = 0.$$

Initial simplex tableau (columns: x, y, s_1, s_2)

	x	y	s_1	s_2	RHS
s_1	1	2	1	0	8
s_2	2	1	0	1	8
z	-1	-2	0	0	0

Choose entering variable. For maximization we look for negative reduced costs in the z-row. The most negative entry is -2 under y, so y enters.

Minimum ratio test (RHS / coefficient of y):

$$\begin{aligned} \text{row } s_1 &: \quad 8/2 = 4, \\ \text{row } s_2 &: \quad 8/1 = 8. \end{aligned}$$

Minimum is 4 (row s_1), so pivot on the element in row s_1, column y. Thus y will replace s_1 as a basic variable.

Pivot 1: make pivot = 1 (divide row s_1 by 2).

$$\text{new (row for } y) = \tfrac{1}{2} \cdot (1,\ 2,\ 1,\ 0 \mid 8) = \left(\tfrac{1}{2},\ 1,\ \tfrac{1}{2},\ 0 \mid 4\right).$$

Eliminate y from the other rows:

$$\begin{aligned}
\text{new } s_2 &= \text{old } s_2 - 1 \cdot (\text{new } y \text{ row}) \\
&= (2,\ 1,\ 0,\ 1 \mid 8) - (\tfrac{1}{2},\ 1,\ \tfrac{1}{2},\ 0 \mid 4) \\
&= \left(\tfrac{3}{2},\ 0,\ -\tfrac{1}{2},\ 1 \mid 4\right),
\end{aligned}$$

$$\begin{aligned}
\text{new } z &= \text{old } z + 2 \cdot (\text{new } y \text{ row}) \quad (\text{to eliminate } -2) \\
&= (-1,\ -2,\ 0,\ 0 \mid 0) + 2 \cdot (\tfrac{1}{2},\ 1,\ \tfrac{1}{2},\ 0 \mid 4) \\
&= (0,\ 0,\ 1,\ 0 \mid 8).
\end{aligned}$$

Tableau after Pivot 1

	x	y	s_1	s_2	RHS
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	4
s_2	$\frac{3}{2}$	0	$-\frac{1}{2}$	1	4
z	0	0	1	0	8

Optimality check. In the z-row the reduced costs for the original variables x, y are $0, 0$ (non-negative). Thus no negative reduced cost remains and the current basic feasible solution is optimal.

Read the solution from the final tableau. Nonbasic variables are x and s_1 (set them to zero). From the rows:

$$y = 4 - \tfrac{1}{2}x - \tfrac{1}{2}s_1 \Rightarrow y = 4, \qquad s_2 = 4 - \tfrac{3}{2}x + \tfrac{1}{2}s_1 \Rightarrow s_2 = 4.$$

Hence the primal solution is

$$(x, y) = (0, 4),$$

and the objective value is

$$z = x + 2y = 0 + 2 \cdot 4 = 8,$$

which matches the RHS of the z-row.

On multiple optimal solutions. Note the reduced cost of the nonbasic original variable x is 0 in the final tableau. This indicates alternative optima exist: introducing x into the basis would not change the objective value. Indeed, evaluating the objective at the feasible corner $(\frac{8}{3}, \frac{8}{3})$ (the intersection of the two constraint lines) yields

$$z = \tfrac{8}{3} + 2 \cdot \tfrac{8}{3} = \tfrac{24}{3} = 8,$$

so $(\frac{8}{3}, \frac{8}{3})$ is also optimal. Therefore there are infinitely many optimal solutions lying on the segment joining $(0, 4)$ and $(\frac{8}{3}, \frac{8}{3})$, all with $z_{\max} = 8$.

$$\boxed{\text{Optimal value } z_{\max} = 8, \text{ with infinite optima on the segment between } (0,4) \text{ and } (\tfrac{8}{3}, \tfrac{8}{3}).}$$

Ex. 34 — **Infeasible region:** Minimize $z = x + y$ subject to:

$$\begin{aligned} x + y &\geq 10 \\ x + y &\leq 5 \\ x, y &\geq 0 \end{aligned}$$

Answer (Ex. 34) — The constraints $x + y \geq 10$ and $x + y \leq 5$ are contradictory. No feasible region exists, hence the problem is infeasible.

Ex. 35 — **Unbounded problem:** Maximize $z = 3x + 2y$ subject to:

$$\begin{aligned} y &\geq x + 1, \\ x, y &\geq 0. \end{aligned}$$

Answer (Ex. 35) — We first rewrite the inequality $y \geq x + 1$ in standard form. Move all variables to the left-hand side and introduce a surplus variable $s_1 \geq 0$:

$$-y + x + s_1 = -1.$$

To keep the right-hand side nonnegative (required for a standard initial basic feasible solution), multiply both sides by -1:

$$y - x - s_1 = 1.$$

Thus our constraint system is:

$$y - x - s_1 = 1, \quad x, y, s_1 \geq 0.$$

The objective function in standard form:

$$z - 3x - 2y = 0.$$

Initial simplex tableau (columns: x, y, s_1)

	x	y	s_1	RHS
y	-1	1	-1	1
z	-3	-2	0	0

However, this tableau is not suitable as an initial BFS because the basic variable y has a negative coefficient for x (we cannot express a nonnegative basic variable y with positive coefficients of nonbasic variables). We could attempt to construct an initial feasible point manually.

Feasible region:
From $y \geq x+1$ and nonnegativity, we see that:

$$\text{Feasible region } = \{(x,y) \,|\, y \geq x+1,\ x,y \geq 0\}.$$

Behavior of the objective function:
For any feasible point satisfying $y = x+1$,

$$z = 3x + 2y = 3x + 2(x+1) = 5x + 2.$$

As $x \to \infty$, $z \to \infty$.
Hence, there is no finite maximum value of z.

Conclusion:

$$\boxed{\text{The linear program is unbounded.}}$$

Geometrically, the feasible region is the half-plane above the line $y = x+1$. The objective function lines $3x + 2y = c$ have slope $-\frac{3}{2}$, and by moving this line upward (increasing c) we can always find new feasible points with larger z values, without limit.
Therefore, no optimal solution exists.

Ex. 36 — **Unique minimum:** Minimize $z = 5x + 2y$ subject to:

$$\begin{aligned} x + y &\geq 4, \\ 2x + y &\geq 5, \\ x, y &\geq 0. \end{aligned}$$

Answer (Ex. 36) — Because the constraints are "$\geq$"-type we introduce surplus and artificial variables:

$$x + y - s_1 + a_1 = 4, \qquad 2x + y - s_2 + a_2 = 5,$$

with $s_1, s_2 \geq 0$ (surplus) and $a_1, a_2 \geq 0$ (artificials). Phase-I objective: minimize $w = a_1 + a_2$. In tableau form we set $w - a_1 - a_2 = 0$.
We order columns as $x,\ y,\ s_1,\ s_2,\ a_1,\ a_2$ and show RHS on the far right. The initial Phase-I tableau (basic variables a_1, a_2) is:

Basis	x	y	s_1	s_2	a_1	a_2	RHS
a_1	1	1	-1	0	1	0	4
a_2	2	1	0	-1	0	1	5
w	-3	-2	1	1	0	0	-9

(Explanation of the w-row: since $w = a_1 + a_2$ and a_1, a_2 are basic, the reduced costs are $c_j - (1 \cdot a_{1j} + 1 \cdot a_{2j})$. For example for x: $0 - (1+2) = -3$; RHS is $0 - (4+5) = -9$.)

Phase-I — Pivot 1. Select entering variable: look at the w-row negative entries (we want to reduce w), the most negative is -3 under x. So x enters. Minimum ratio test (RHS / coeff of x, only rows with positive coeff):

$$\text{row } a_1: \ 4/1 = 4, \qquad \text{row } a_2: \ 5/2 = 2.$$

Minimum ratio is 2 (row a_2). Pivot on the element in row a_2, column x. Divide row a_2 by 2 to make the pivot 1:

$$\text{New row (basic } x) = \tfrac{1}{2} \cdot (2,\ 1,\ 0,\ -1,\ 0,\ 1 \mid 5) = \big(1,\ \tfrac{1}{2},\ 0,\ -\tfrac{1}{2},\ 0,\ \tfrac{1}{2} \mid \tfrac{5}{2}\big).$$

Eliminate x from the other rows:
- Update row a_1: old a_1 minus $1\times$ new x-row

$$(1, 1, -1, 0, 1, 0 \mid 4) - (1, \tfrac{1}{2}, 0, -\tfrac{1}{2}, 0, \tfrac{1}{2} \mid \tfrac{5}{2}) = \big(0, \tfrac{1}{2}, -1, \tfrac{1}{2}, 1, -\tfrac{1}{2} \mid \tfrac{3}{2}\big).$$

- Update w-row: old w-row + $3\times$ new x-row (to remove -3 under x)

$$(-3, -2, 1, 1, 0, 0 \mid -9) + 3 \cdot (1, \tfrac{1}{2}, 0, -\tfrac{1}{2}, 0, \tfrac{1}{2} \mid \tfrac{5}{2}) = \big(0, -\tfrac{1}{2}, 1, -\tfrac{1}{2}, 0, \tfrac{3}{2} \mid -\tfrac{3}{2}\big).$$

The tableau after Pivot 1:

Basis	x	y	s_1	s_2	a_1	a_2	RHS
a_1	0	$\frac{1}{2}$	-1	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{3}{2}$
x	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{5}{2}$
w	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$	0	$\frac{3}{2}$	$-\frac{3}{2}$

Phase-I — Pivot 2. There is still a negative entry in the w-row: $-\frac{1}{2}$ under y. So y enters. Minimum ratio test for y (only rows with positive coefficient in y column):

$$\text{row } a_1 : \ \frac{3/2}{1/2} = 3, \qquad \text{row } x : \ \frac{5/2}{1/2} = 5.$$

Minimum is 3 (row a_1), so pivot at row a_1, column y. Make pivot 1 by dividing row a_1 by 1/2 (i.e. multiply by 2):

$$\text{New row (basic } y) = 2 \cdot \left(0, \tfrac{1}{2}, -1, \tfrac{1}{2}, 1, -\tfrac{1}{2} \mid \tfrac{3}{2}\right) = \left(0, 1, -2, 1, 2, -1 \mid 3\right).$$

Eliminate y from the other rows:
- Update row x: old row x minus $\frac{1}{2}$ × new y-row

$$(1, \tfrac{1}{2}, 0, -\tfrac{1}{2}, 0, \tfrac{1}{2} \mid \tfrac{5}{2}) - \tfrac{1}{2}(0, 1, -2, 1, 2, -1 \mid 3) = \left(1, 0, 1, -1, -1, 1 \mid 1\right).$$

- Update w-row: old w plus $\frac{1}{2}$ × new y-row (to remove $-\frac{1}{2}$)

$$(0, -\tfrac{1}{2}, 1, -\tfrac{1}{2}, 0, \tfrac{3}{2} \mid -\tfrac{3}{2}) + \tfrac{1}{2}(0, 1, -2, 1, 2, -1 \mid 3) = \left(0, 0, 0, 0, 1, 1 \mid 0\right).$$

The tableau after Pivot 2 is:

Basis	x	y	s_1	s_2	a_1	a_2	RHS
y	0	1	-2	1	2	-1	3
x	1	0	1	-1	-1	1	1
w	0	0	0	0	1	1	0

Observe the Phase-I objective value is $w = 0$ (RHS of w-row is 0). Since $w = 0$ we have a feasible solution for the original problem and may drop the artificial variables a_1, a_2 (their columns can be removed from the Phase-II tableau). The current basic variables are x and y with values shown in the RHS.

Interpretation of the current basic feasible solution: From the two constraint rows

$$\begin{aligned} y - 2s_1 + s_2 &= 3, \\ x + s_1 - s_2 &= 1, \end{aligned}$$

setting the nonbasics $s_1 = s_2 = 0$ gives

$$x = 1, \qquad y = 3.$$

This is a feasible point for the original constraints.

Phase-II (original objective). We now resume with the original minimization objective $z = 5x + 2y$. Using the current basis (x, y) we express the original objective in terms of the nonbasic variables s_1, s_2 by substitution. From the tableau rows (dropping artificial columns), rewrite basic variables:

$$\begin{aligned} x &= 1 - s_1 + s_2, \\ y &= 3 + 2s_1 - s_2. \end{aligned}$$

Substitute into the objective:

$$z = 5x+2y = 5(1-s_1+s_2)+2(3+2s_1-s_2) = 5-5s_1+5s_2+6+4s_1-2s_2 = 11-s_1+3s_2.$$

Rearrange to tableau form (bring all terms to left): $z + s_1 - 3s_2 = 11$. Thus the Phase-II tableau (columns x, y, s_1, s_2, basic vars y, x) is

Basis	x	y	s_1	s_2	RHS
y	0	1	-2	1	3
x	1	0	1	-1	1
z	0	0	1	-3	11

(Interpretation of the z-row: $z - 11 + s_1 - 3s_2 = 0$, equivalently $z = 11 - s_1 + 3s_2$.)

Optimality check for minimization. In Phase-II tableau using the convention above, the coefficients of nonbasic variables in the z-row (the row written as $z - \cdots = \text{RHS}$) indicate how z changes if we increase that nonbasic variable from 0. Here the nonbasic variables are s_1 and s_2; their coefficients (in the form $z - (\cdots) = \text{RHS}$) are:

$$\text{coef of } s_1 : \ +1, \qquad \text{coef of } s_2 : \ -3.$$

A positive coefficient means increasing that nonbasic variable will increase z; a negative coefficient would indicate a possible decrease of z (improvement) if that variable can be increased while preserving feasibility. However, s_2 is a surplus/slack variable that cannot be increased independently without violating the equalities unless a pivot permits it. Checking the ratio tests for potential pivots would show no allowed pivot that decreases z (because the feasible directions corresponding to the negative coefficient are blocked by nonnegativity of current basic variables). Equivalently, we can inspect the feasible corner points of the original region (intersection points) to be certain.

Verification by evaluating corner points. The feasible region is the intersection of two half-planes $x + y \geq 4$ and $2x + y \geq 5$ with $x, y \geq 0$. The corner (intersection) points are:

Intersection of $x + y = 4$ and $2x + y = 5$: $\begin{cases} x + y = 4 \\ 2x + y = 5 \end{cases} \Rightarrow x = 1,\ y = 3.$

Intersection with axes (where constraints active): $(5, 0)$ and $(0, 4)$.

Compute the objective $z = 5x + 2y$ at these candidate points:

$$\begin{aligned} (1,3):&\quad z = 5 \cdot 1 + 2 \cdot 3 = 5 + 6 = 11, \\ (2,2)\ \text{(if considered)}:&\quad z = 5 \cdot 2 + 2 \cdot 2 = 10 + 4 = 14, \\ (5,0):&\quad z = 25, \\ (0,4):&\quad z = 8\ (\text{not feasible for the second inequality } 2x + y \geq 5 \text{ since } 0 + 4 = 4 < \end{aligned}$$

Hence among the feasible corners the minimum value is $z_{\min} = 11$ attained at $(x, y) = (1, 3)$.

Conclusion:

$$\boxed{(x, y) = (1, 3), \qquad z_{\min} = 11.}$$

This completes the two-phase simplex development: Phase-I found a feasible basis $(x, y) = (1, 3)$ with $w = 0$; Phase-II shows the objective value at that feasible basis is 11 and that no feasible move decreases the objective further, so $(1, 3)$ is the unique minimizer.

Ex. 37 — Simplex optimization: unique solution: Use Simplex to solve this LP problem:

$$\begin{aligned} \text{Maximize:}\quad & z = 3x_1 + x_2 \\ \text{subject to:}\quad & x_2 \leq 5 \\ & x_1 + x_2 \leq 10 \\ & -x_1 + x_2 \geq -2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Answer (Ex. 37) — The third constraint can be rewritten as:

$$x_1 - x_2 \leq 2$$

We introduce slack variables $s_1, s_2, s_3 \geq 0$:

$$\begin{aligned} x_2 + s_1 &= 5 \\ x_1 + x_2 + s_2 &= 10 \\ x_1 - x_2 + s_3 &= 2 \end{aligned}$$

$$\text{Maximize } z = 3x_1 + x_2$$

Initial Simplex Tableau

Basis	x_1	x_2	s_1	s_2	s_3	RHS
s_1	0	1	1	0	0	5
s_2	1	1	0	1	0	10
s_3	1	-1	0	0	1	2
z	-3	-1	0	0	0	0

Iteration 1 The entering variable is x_1 (most negative coefficient in the z-row).

$$\text{Minimum ratio test: } \frac{5}{0} = \infty, \quad \frac{10}{1} = 10, \quad \frac{2}{1} = 2 \Rightarrow \text{ leaving variable: } s_3$$

Pivot element: row s_3, column x_1.

$$x_1 = 2 + x_2 - s_3$$

Substitute into the other equations to obtain the new tableau:

Basis	x_1	x_2	s_1	s_2	s_3	RHS
s_1	0	1	1	0	0	5
s_2	0	2	0	1	-1	8
x_1	1	-1	0	0	1	2
z	0	-4	0	0	3	6

Iteration 2 The entering variable is x_2 (most negative coefficient: -4).

$$\text{Minimum ratio test: } \frac{5}{1} = 5, \quad \frac{8}{2} = 4, \quad \frac{2}{-1} \text{ invalid} \Rightarrow \text{ leaving variable: } s_2$$

Pivot element: row s_2, column x_2.

$$x_2 = 4 - 0.5s_2 + 0.5s_3$$

Substitute and simplify:

Basis	x_1	x_2	s_1	s_2	s_3	RHS
s_1	0	0	1	-0.5	0.5	1
x_2	0	1	0	0.5	-0.5	4
x_1	1	0	0	0.5	0.5	6
z	0	0	0	2	1	22

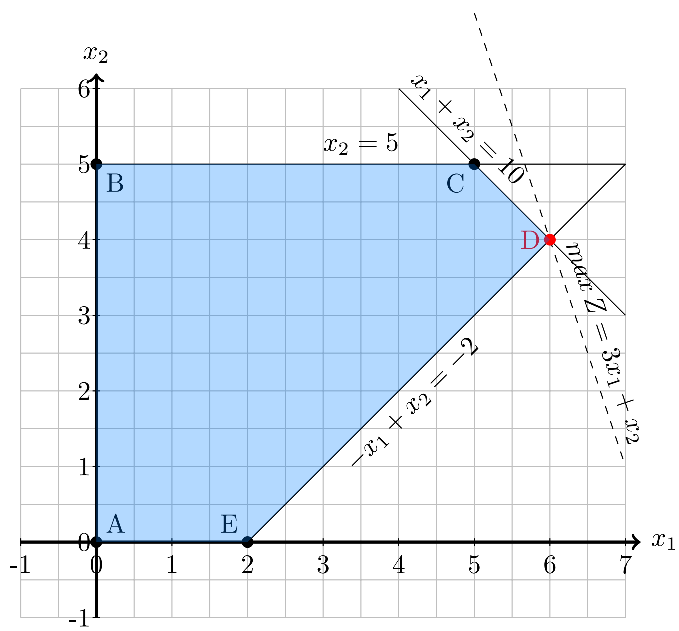


Figure 3: Graphical solution for Exercise 37, showing the unique optimal solution at the corner point.

Optimal Solution Since there are no negative coefficients in the z-row, the solution is optimal:

$$x_1 = 6, \quad x_2 = 4, \quad z_{\max} = 3(6) + 4 = 22$$

$$\boxed{x_1 = 6,\ x_2 = 4,\ z_{\max} = 22}$$

■

Ex. 38 — Simplex optimization: multiple solutions: Use Simplex to solve this LP problem:

$$\begin{aligned} \max \quad & Z = 1.0x_1 + 2.0x_2 \\ \text{s.t.} \quad & \begin{cases} -1.0x_1 + 1.0x_2 \le 2.0 \\ \ \ 1.0x_1 + 2.0x_2 \le 8.0 \\ \ \ 1.0x_1 + 0.0x_2 \le 6.0 \end{cases} \\ & x_1, x_2 \ge 0 \end{aligned}$$

Canonical Form:

$$Z - 1.0x_1 - 2.0x_2 - 0\,s_1 - 0\,s_2 - 0\,s_3 = 0$$

$$\begin{cases} s_1 \qquad\qquad - 1.0x_1 + 1.0x_2 = 2.0 \\ \qquad s_2 \qquad + 1.0x_1 + 2.0x_2 = 8.0 \\ \qquad\qquad s_3 + 1.0x_1 + 0.0x_2 = 6.0 \end{cases}$$

Simplex Tableau

Answer (Ex. 38) — Iteration 0 Entering and leaving variables:

- Entering variable: x_2
- Leaving variable: s_1

Table 1: Simplex Tableau after iteration 0

Basic	x_1	x_2	s_1	s_2	s_3	RHS
x_2	-1.0	1.0	1.0	0.0	0.0	2.0
s_2	3.0	0.0	-2.0	1.0	0.0	4.0
s_3	1.0	0.0	0.0	0.0	1.0	6.0
Z	-3.0	0.0	2.0	0.0	0.0	4.0

Iteration 1 Entering and leaving variables:

- Entering variable: x_1
- Leaving variable: s_2

Table 2: Simplex Tableau after iteration 1

Basic	x_1	x_2	s_1	s_2	s_3	RHS
x_2	0.0	1.0	0.33	0.33	0.0	3.33
x_1	1.0	0.0	-0.67	0.33	0.0	1.33
s_3	0.0	0.0	0.67	-0.33	1.0	4.67
Z	0.0	0.0	0.0	1.0	0.0	8.0

Observation on Multiple Solutions: In the final tableau, all coefficients in the Z-row are ≥ 0, so the solution is optimal. However, notice that the column corresponding to the non-basic variable s_2 has a coefficient of exactly zero in the Z-row.
- This indicates that we can increase s_2 (relaxing constraint 2) without decreasing Z. - Geometrically, there is an edge of the feasible polyhedron where Z = 8. - Therefore, any convex combination of the basic optimal vertices:

$$x^{(A)} = (4/3, 10/3), \quad x^{(B)} = (6, 1)$$

is also optimal:

$$x(t) = (1-t)x^{(A)} + t\,x^{(B)}, \quad t \in [0, 1].$$

Optimal Solution (example from the final Simplex tableau):

$$x_1 = 1.3333, \quad x_2 = 3.3333, \quad Z = 8$$

Slack Variables:

$$s_1 = 0, \quad s_2 = 0, \quad s_3 = 4.6667$$

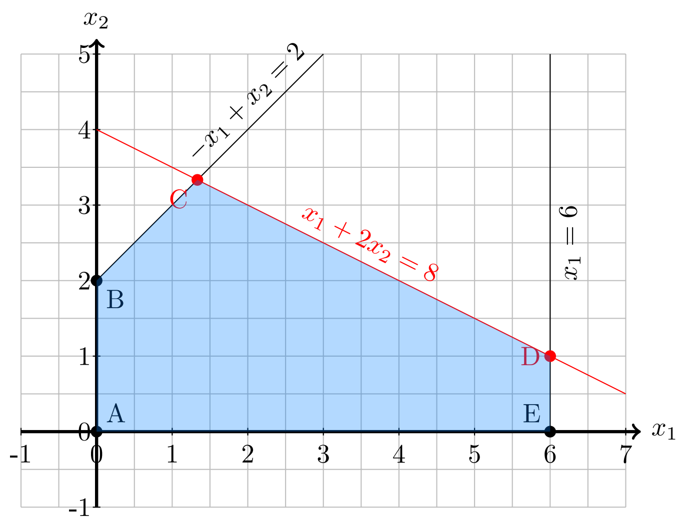


Figure 4: Graphical solution of Exercise 38, showing multiple optimal solutions along the edge between points C and D.

Conclusion: The presence of a zero coefficient in the Z-row for a non-basic variable indicates infinitely many optimal solutions. The solution found is only one of the multiple possible solutions.
■

Ex. 39 — Simplex optimization: unbound solutions: Use Simplex to solve this LP problem:

$$\text{Maximize} \quad z = 3x_1 + x_2,$$
$$\text{subject to} \quad \begin{cases} x_1 + x_2 \geq 4, \\ -x_1 + x_2 \leq 4, \\ -x_1 + 2x_2 \geq -4, \\ x_1,\ x_2 \geq 0. \end{cases}$$

Step 0: Put constraints into equality form Rewrite the "≥" constraints by introducing surplus and (if needed) artificial variables.

(1) $x_1 + x_2 - s_1 + a_1 = 4$ (surplus $s_1 \geq 0$, artificial $a_1 \geq 0$)
(2) $-x_1 + x_2 + s_2 = 4$ (slack $s_2 \geq 0$)
(3) $-x_1 + 2x_2 \geq -4 \implies x_1 - 2x_2 + s_3 = 4$ ($s_3 \geq 0$)

So the equality system is

$$\begin{aligned} x_1 + x_2 - s_1 + a_1 &= 4, \\ -x_1 + x_2 + s_2 &= 4, \\ x_1 - 2x_2 + s_3 &= 4, \end{aligned}$$

with nonnegative variables $x_1, x_2, s_1, s_2, s_3, a_1$.

Phase I: find a feasible basic solution Phase-I objective is to *minimize* the sum of artificials. Here there is one artificial a_1; the Phase-I objective is

$$W = a_1 \quad \text{(minimize)}.$$

Take the initial basis as $(a_1,\ s_2,\ s_3)$ with RHS $(4, 4, 4)$. The initial tableau (only showing the structural part) is:

Basis	x_1	x_2	s_1	s_2	s_3	RHS
a_1	1	1	-1	0	0	4
s_2	-1	1	0	1	0	4
s_3	1	-2	0	0	1	4
W	0	0	0	0	0	0

We eliminate the artificial column from the Phase-I objective row by subtracting the artificial row (this is the usual initialization step). After elimination the Phase-I objective row becomes (the algebraic effect is to set the reduced costs so we aim to drive a_1 to zero). In practice we then attempt to pivot so that a_1 leaves the basis.
Observe the following feasible substitution point:

$$x_1 = 4,\ x_2 = 0 \qquad \Rightarrow \qquad \begin{cases} (1) : 4 + 0 - s_1 + a_1 = 4 \Rightarrow a_1 = s_1, \\ (2) : -4 + 0 + s_2 = 4 \Rightarrow s_2 = 8, \\ (3) : 4 - 0 + s_3 = 4 \Rightarrow s_3 = 0. \end{cases}$$

Choosing $s_1 = 0$ yields the basic feasible point

$$x_1 = 4,\ x_2 = 0,\ s_1 = 0,\ s_2 = 8,\ s_3 = 0,\ a_1 = 0.$$

Hence a_1 can be driven to zero and Phase-I finds a feasible solution. We may therefore remove a_1 and proceed to Phase-II with a feasible basis. One convenient basic set after Phase-I is:

$$\text{Basis: } \{x_1,\ s_2,\ s_3\}, \quad \text{with } (x_1, s_2, s_3) = (4, 8, 0).$$

(If you prefer, you may view this as performing one pivot in the initial Phase-I tableau with x_1 entering and a_1 leaving; algebraic elimination yields the same basis and $a_1 = 0$.)

Canonical form at start of Phase II Use the basis $\{x_1, s_2, s_3\}$. Express the basic variables in terms of the nonbasics x_2 and s_1. Solving the three equalities for the chosen basics gives:

$$\begin{aligned} x_1 &= 4 - x_2 + s_1, \\ s_2 &= 8 - 2x_2 + s_1, \\ s_3 &= 3x_2 - s_1. \end{aligned}$$

(The RHS values above correspond to the basic solution obtained when the nonbasics $x_2 = s_1 = 0$.)
Now substitute $x_1 = 4 - x_2 + s_1$ into the original objective

$$z = 3x_1 + x_2 = 3(4 - x_2 + s_1) + x_2 = 12 - 2x_2 + 3s_1.$$

So, in the Phase-II tableau (canonical form with basis x_1, s_2, s_3 and nonbasics x_2, s_1) the objective row (written as z expressed in nonbasics) is

$$z = 12 - 2x_2 + 3s_1.$$

Simplex interpretation of Phase II To increase z we look at coefficients of nonbasic variables in the expression for z.

Answer (Ex. 39) —
- Coefficient of x_2 is -2 (so increasing x_2 actually *decreases* z if we keep s_1 fixed).

• Coefficient of s_1 is $+3$ (so increasing s_1 *increases* z).
However s_1 cannot be increased arbitrarily while keeping feasibility: from the basic-variable expressions,

$$s_3 = 3x_2 - s_1 \geq 0 \quad \Longrightarrow \quad s_1 \leq 3x_2.$$

Thus s_1 is bounded above unless x_2 grows.

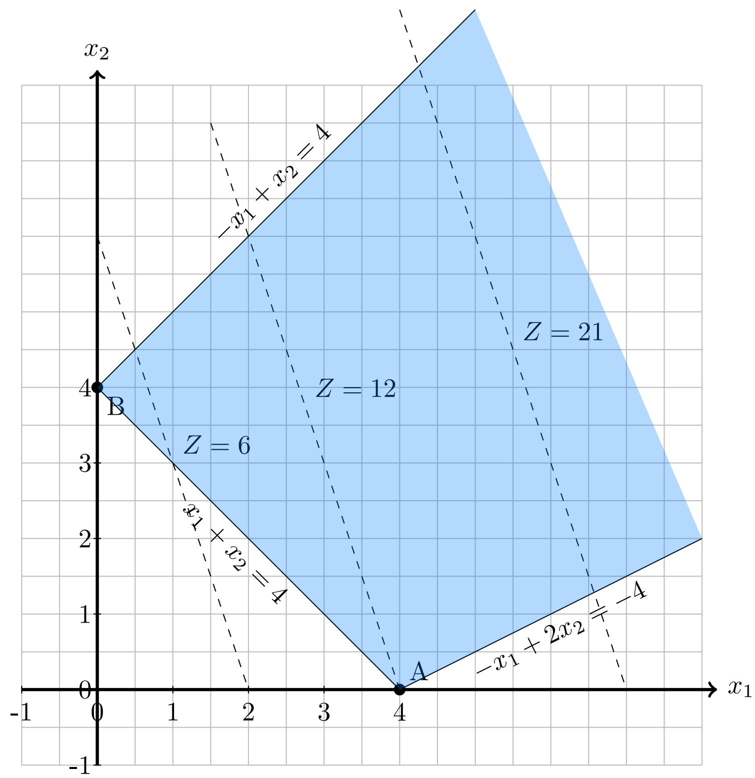


Figure 5: Graphical solution of Exercise 39, showing unbounded feasible region extending infinitely in the direction of increasing objective value.

Conclusion (Phase-II / final) Although Phase-I finds a feasible basis (so the problem is feasible), Phase-II reveals that the objective can be made arbitrarily large along the feasible ray $(t, (t-4)/2)$. In simplex language this corresponds to encountering an entering column that has no valid leaving row (i.e. all entries in that column are nonpositive), which signals *unboundedness.*

Hence the linear program has no finite maximum and, thus, te LP is feasible but unbounded.
■

Ex. 40 — Simplex optimization: unfeasible solutions: — Use Simplex to solve this LP problem:

$$\begin{aligned} \max \quad & Z = 3.0x_1 + 1.0x_2 \\ \text{s.t.} \quad & \begin{cases} 1.0x_1 - 1.0x_2 \leq -4.0 \\ -1.0x_1 + 2.0x_2 \leq -4.0 \end{cases} \\ & x_1, x_2 \geq 0 \end{aligned}$$

Canonical Form:

$$Z - 3.0x_1 - 1.0x_2 - 0\,s_1 - 0\,s_2 = 0$$

$$\begin{cases} s_1 \quad + 1.0x_1 - 1.0x_2 = -4.0 \\ \quad s_2 - 1.0x_1 + 2.0x_2 = -4.0 \end{cases}$$

Simplex Tableau

Answer (Ex. 40) — Iteration 0 System unbound / infeasible.

Table 3: Initial Simplex Tableau

Basic	x_1	x_2	s_1	s_2	RHS
s_1	1.0	-1.0	1.0	0.0	-4.0
s_2	-1.0	2.0	0.0	1.0	-4.0
Z	-3.0	-1.0	0.0	0.0	0.0

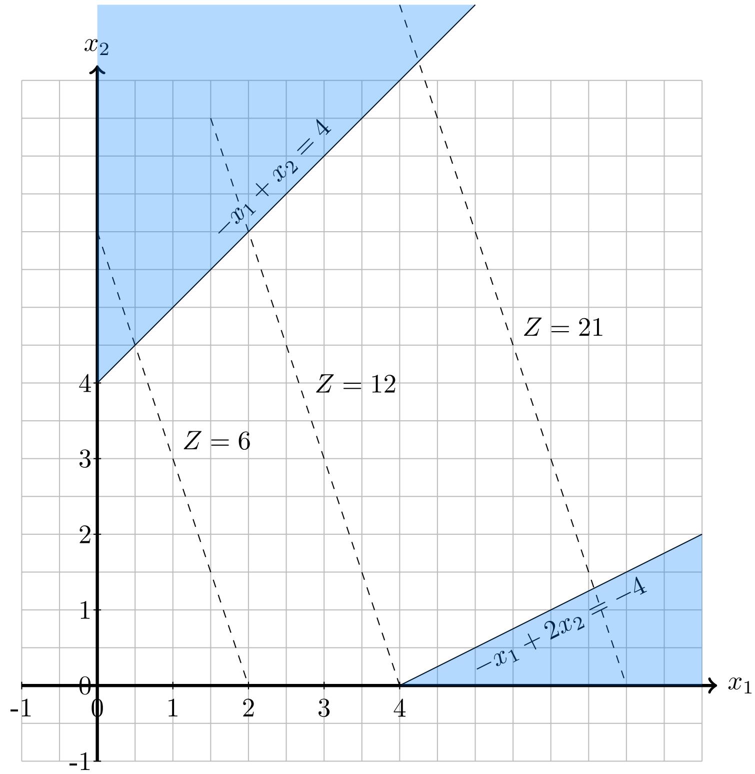


Figure 6: Graphical solution of Exercise 40, showing how the LP problem has no feasible solution.

Feasibility Analysis: - The initial RHS values are negative (-4 and -4). - Since all decision variables must satisfy $x_1, x_2 \geq 0$ and slack variables $s_1, s_2 \geq 0$, there is no combination that satisfies all constraints simultaneously. - Therefore, the system is infeasible, and no optimal solution exists.

Conclusion: This LP has no feasible solution. The initial tableau and Z-row clearly show the inconsistency, as the negative RHS values cannot be compensated by non-negative variables.
■

Ex. 41 — **Feasible solution:** Give a small linear program (two variables) and determine whether the point $(1, 2)$ is a feasible solution. The

LP is:

$$\begin{aligned} \text{maximize} \quad & z = 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 4, \\ & 2x_1 + x_2 \le 6, \\ & x_1, x_2 \ge 0. \end{aligned}$$

Decide if $(1,2)$ satisfies all constraints and compute the objective value at that point.

Answer (Ex. 41) — Check constraints: $1+2 = 3 \le 4$ and $2\cdot 1+2 = 4 \le 6$, nonnegativity holds. So $(1,2)$ is feasible. Objective value: $z = 3\cdot 1+2\cdot 2 = 7$.

Ex. 42 — **Feasible region convexity:** Consider a linear program with two variables and linear constraints. Show by a short computation that the feasible region is convex. Provide an explicit example: two feasible points and their convex combination stays feasible.

Answer (Ex. 42) — Take the LP: $x_1 + x_2 \le 4,\ x_1, x_2 \ge 0$. Points $A = (1,1)$ and $B = (2,0)$ are feasible. For $\theta = 0.5$, the combination $C = 0.5A + 0.5B = (1.5, 0.5)$. Check constraint: $1.5 + 0.5 = 2 \le 4$ and nonnegativity holds. Hence convexity illustrated.

Ex. 43 — **Objective function linearity:** Explain and demonstrate with calculation why the LP objective must be linear for standard linear programming. Given $z = ax_1 + bx_2$, evaluate z at two points and show linearity property $z(\alpha x + \beta y) = \alpha z(x) + \beta z(y)$ for $\alpha + \beta = 1$.

Answer (Ex. 43) — Let $x = (x_1, x_2) = (1, 0),\ y = (0, 2),\ a = 3, b = 2$. Then $z(x) = 3,\ z(y) = 4$. For $\alpha = 0.3, \beta = 0.7$ the combination is $(0.3, 1.4)$. $z(\alpha x + \beta y) = 3 \cdot 0.3 + 2 \cdot 1.4 = 0.9 + 2.8 = 3.7 = 0.3 \cdot 3 + 0.7 \cdot 4$. Linearity verified.

Ex. 44 — **Optimum at vertex:** Consider maximizing $z = 4x_1 + 5x_2$ subject to $x_1 + x_2 \le 6,\ x_1, x_2 \ge 0$. Show that any optimum occurs at a vertex. Compute the candidate vertices and find the maximum.

Answer (Ex. 44) — Vertices: (0,0),(6,0),(0,6). Evaluate: z(0,0)=0, z(6,0)=24, z(0,6)=30. Maximum at vertex (0,6) with z=30. This shows optimum at a vertex.

Ex. 45 — Graphical method variable limit: Use the graphical method to solve the LP: maximize $z = x_1 + 2x_2$ subject to $x_1 + x_2 \leq 5,\ x_1, x_2 \geq 0$. Explain why this graphical approach is limited to two variables and solve the LP.

Answer (Ex. 45) — Graphical method needs a 2D plot; for more variables visualization fails. Solve: vertices (0,0),(5,0),(0,5). Evaluate z: 0,5,10. Optimal at (0,5) with z=10.

Ex. 46 — Feasible region shape: Prove that the feasible region of a linear program with linear inequalities is a convex polyhedron. Give an example and sketch reasoning (algebraic).

Answer (Ex. 46) — Each linear inequality defines a half-space; intersection of finitely many half-spaces equals a convex polyhedron. Example: $x_1 + x_2 \leq 4,\ x_1 \geq 0, x_2 \geq 0$ yields a polygon in $\mathbb{R}^2$, hence a convex polyhedron.

Ex. 47 — Uniqueness of LP solutions: Provide a concrete LP example that has multiple optimal solutions (not unique). Solve it and show the set of optima.

Answer (Ex. 47) — LP: maximize $z = x_1 + x_2$ subject to $x_1 + x_2 \leq 2,\ x_1, x_2 \geq 0$. All points on the segment from (0,2) to (2,0) that satisfy $x_1 + x_2 = 2$ are optimal with z=2. Hence not unique.

Ex. 48 — Redundant constraint: Show by example that a redundant constraint does not affect the feasible region. Consider constraints $x_1 + x_2 \leq 4,\ 2x_1 + 2x_2 \leq 8$ and show the second is redundant.

Answer (Ex. 48) — Second inequality is exactly 2 times the first; any point satisfying $x_1 + x_2 \leq 4$ automatically satisfies $2x_1 + 2x_2 \leq 8$. Thus redundant.

Ex. 49 — Max vs min in LP: Demonstrate that linear programming can handle both maximization and minimization. Convert a maximization problem into a minimization equivalent and solve numerically: minimize $z' = -x_1 - 2x_2$ with $x_1 + x_2 \leq 3,\ x_i \geq 0$.

Answer (Ex. 49) — Minimizing $z' = -z$ is equivalent to maximizing z.

Solve feasible vertices (0,0),(3,0),(0,3): z' values 0,-3,-6 so minimum at (0,3) which corresponds to maximum of z at (0,3).

Ex. 50 — Feasibility analysis in linear programming: Determine whether the linear programming problem

$$\max z = 2x_1 + 5x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 3,\ x_1 + x_2 \geq 5,\ x_1, x_2 \geq 0$$

has a feasible solution. Justify your answer.

Answer (Ex. 50) — The constraints $x_1 + x_2 \leq 3$ and $x_1 + x_2 \geq 5$ are incompatible. Hence, the problem is infeasible — no feasible region exists.

Ex. 51 — Multiple optimal solutions in LP: Analyze the linear program:

$$\max z = x_1 + 2x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 5,\ x_1 + 3x_2 \leq 9,\ x_1, x_2 \geq 0.$$

Determine whether there are multiple optimal solutions.

Answer (Ex. 51) — The objective function is parallel to one of the constraints at optimality, resulting in multiple optimal solutions along a boundary segment.

Ex. 52 — Unbounded linear programming problem: Consider the linear program:

$$\max z = 5x_1 + 4x_2 \quad \text{s.t.} \quad x_1 + 2x_2 \geq 4,\ x_1 + x_2 \leq 10,\ x_1, x_2 \geq 0.$$

Determine the nature of its solution.

Answer (Ex. 52) — The feasible region is unbounded in the direction of increasing z; therefore, the problem is unbounded.

Ex. 53 — Infeasible LP example: Consider:

$$\max z = x_1 + x_2 \quad \text{s.t.} \quad x_1 + x_2 \leq 2,\ x_1 + x_2 \geq 5,\ x_1, x_2 \geq 0.$$

Discuss the feasibility of this system.

Answer (Ex. 53) — The constraints contradict each other; no point satisfies both. The LP is infeasible.

Ex. 54 — Optimal solution in linear programming: Solve the linear program:

$$\max z = 3x_1 + 2x_2, \quad \text{s.t.} \quad x_1 + x_2 \le 4,\ 2x_1 + x_2 \le 6,\ x_1, x_2 \ge 0.$$

Find the optimal point.

Answer (Ex. 54) — Corner points: $(0,0), (0,4), (2,2), (3,0)$. z is maximized at $(2,2)$ with $z = 10$.

Ex. 55 — Minimization with inequalities: Find the optimal solution to:

$$\min z = x_1 + 4x_2 \quad \text{s.t.} \quad x_1 + 2x_2 \ge 3,\ -x_1 + x_2 \ge 1,\ x_1, x_2 \ge 0.$$

Answer (Ex. 55) — By graphical or algebraic methods, the minimum occurs at $(x_1, x_2) = (3, 0)$.

Ex. 56 — Maximization in LP: Maximize $z = 4x_1 + 3x_2$ subject to:

$$\begin{cases} x_1 + 2x_2 \le 8, \\ x_1 + x_2 \le 6, \\ x_1, x_2 \ge 0. \end{cases}$$

Find the optimal point and value.

Answer (Ex. 56) — The feasible vertices yield maximum z at $(x_1, x_2) = (4, 2)$ with $z = 22$.

Ex. 57 — Minimization in LP with inequalities: Minimize $z = 2x_1 + 3x_2$ subject to:

$$\begin{cases} x_1 + x_2 \ge 5, \\ 3x_1 + x_2 \ge 9, \\ x_1, x_2 \ge 0. \end{cases}$$

Find the optimal point.

Answer (Ex. 57) — Optimal solution at $(x_1, x_2) = (3, 2)$, yielding $z = 12$.

Ex. 58 — Optimal vertex in bounded LP: Solve the LP:

$$\max z = 6x_1 + 7x_2, \text{ s.t. } 2x_1 + x_2 \leq 14,\ x_1 + x_2 \leq 10,\ x_1, x_2 \geq 0.$$

Find the optimal vertex.

Answer (Ex. 58) — Intersection of constraints gives $(x_1, x_2) = (5, 5)$. Maximum value: $z = 65$.

Ex. 59 — Profit maximization problem: A company produces two products, A and B, with profits per unit of 10 and 15, respectively. Each unit of A requires 2 units of resource 1 and 1 of resource 2; each unit of B requires 3 and 2 units, respectively. If at most 18 units of resource 1 and 10 of resource 2 are available, determine the optimal production plan.

Answer (Ex. 59) — Formulate:

$$\max z = 10x_1 + 15x_2,\ 2x_1 + 3x_2 \leq 18,\ x_1 + 2x_2 \leq 10.$$

The optimal solution is $(x_1, x_2) = (4, 3)$, with $z = 85$.

2.2 Duality

Ex. 60 — A problem of linear programming: Consider the linear programming problem:

$$\max x_1 + 4x_2 + 2x_3$$
$$\text{s.t.} \quad \begin{array}{rcl} 5x_1 + 2x_2 + 2x_3 & \leq & 145 \\ 4x_1 + 8x_2 - 8x_3 & \leq & 260 \\ x_1 + x_2 + 4x_3 & \leq & 190 \\ x_1, x_2 x_3 & \geq & 0 \end{array}$$

Find x_1, x_2 and x_3 to solve it.

Answer (Ex. 60) — material: complementary slackness.pdf

Ex. 61 — Complementary slackness: Consider the linear programming problem:

$$\max 2x_1 + 16x_2 + 2x_3$$

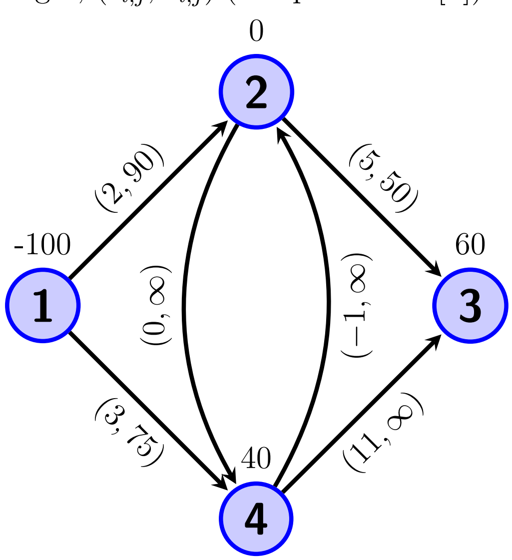
$$\text{s.t.} \quad \begin{array}{rcl} 2x_1 + x_2 - x_3 & \leq & -3 \\ -3x_1 + x_2 + 2x_3 & \leq & 12 \\ x_1, x_2 x_3 & \geq & 0 \end{array}$$

Check whether each of the following is an optimal solution, using complementary slackness:

Answer (Ex. 61) — material: compslack.pdf

2.3 Network Analysis

Ex. 62 — **Minimum Cost:** The figure shows a network on four nodes, including net demands on the vertex, b_k, and cost an capacity on the edges, $(c_{i,j}, u_{i,j})$.(Adapted from [**?**])



1. Formulate the corresponding minimum cost network flow model
2. Classify the nodes as *source*, *sink* or *transshipment*

Answer (Ex. 62) — In this problem, vertex and edges are:

$$V = \{1, 2, 3, 4\}$$

$$A = \{(1,2),(1,4),(2,3),(2,4),(4,2),(4,3)\}$$

we can use the variables $x_{i,j}$ to represent the flows in the different members of set A. Thus, the formulation of the problem is:

$$\min \quad 2x_{1,2} + 3x_{1,4} + 5x_{2,3} - x_{4,2} + 11x_{4,3}$$

$$\text{subject to} \quad \begin{cases} -A - B &= -100 \\ A + F - C - D &= 0 \\ C + E &= 60 \\ B + D - F - E &= 40 \\ A &\leq 90 \\ B &\leq 75 \\ C &\leq 50 \end{cases}$$

and $x_{i,j} \geq 0$.
There are 4 KKT conditions for optimal primal ($x4$) and dual (λ) variables. If x^* denotes optimal values:

1. Primal feasibility: all constraints must be satisfied: $g_i(x^*) - b_i$ is feasible. Applies to both equality and non-equality constraints.
2. Gradient condition or No feasible descent: No possible improvement at the solution:
$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0$$
3. Complementariety slackness:
$$\lambda_i^*(g_i(x^*) - b_i) = 0$$
4. Dual feasibility: $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active (=0) and a zero Lagrange multiplier when the constraint is inactive (>0).
to solve our problem, first we will put it in its standard form:

$$\min \; f(x,y) = -xy$$

$$\text{subject to} \quad \begin{aligned} -x - y + 100 &\geq 0 \\ -x - 40 &\geq 0 \end{aligned}$$

We will go through the different conditions:

1.Primal feasibility: $g_i(x^*) - b_i$ is feasible.

$$-x^* - y^* + 100 = 0$$

$$-x^* - 40 = 0$$

2.Gradient condition or No feasible descent:

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} - \lambda_1 \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_1}{\partial y} \end{pmatrix} - \lambda_2 \begin{pmatrix} \frac{\partial g_2}{\partial x} \\ \frac{\partial g_2}{\partial y} \end{pmatrix} = 0$$

which, in this example, resolves into:

$$\begin{cases} \quad\;\; - y + \lambda_1 + \lambda_2 = 0 \\ -x \quad - \quad\quad \lambda_1 = 0 \end{cases}$$

3.Complementariety slackness:

$$\lambda_1^*(-x^* - y^* + 100) = 0$$

$$\lambda_2^*(-x^* - 40) = 0$$

4.Dual feasibility: $\lambda_1, \lambda_2 \geq 0$

We can put the resulting 5 expressions for conditions 1 and 2 into matrix form:

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -100 \\ 40 \\ 0 \\ 0 \end{pmatrix}$$

3 Appendices

3.1 Python codes

Codes

3.1.1 NLO with constraints

Code 1: Non-linear optimization with constraints

```
from scipy.optimize import minimize

# Define the objective function
def objective(xy):
    x, y = xy
    return -(-x**2 + 4*x*y - 2*y**2)  # Minimization function, so retu

# Define the constraints
def constraint1(xy):
    x, y = xy
    return 30 - (x + 2*y)  # x + 2y <= 30

def constraint2(xy):
    x, y = xy
    return (x * y) - 50  # xy >= 50

def constraint3(xy):
    x, y = xy
    return (3 * x**2 / 100) + 5 - y  # y <= (3x^2)/100 + 5

# Initial guess
initial_guess = [15, 1]

# Define constraints
constraints = [{'type': 'ineq', 'fun': constraint1},
               {'type': 'ineq', 'fun': constraint2},
               {'type': 'ineq', 'fun': constraint3}]

# Perform the optimization
result = minimize(objective, initial_guess, constraints=constraints)

# Display the results
optimal_x, optimal_y = result.x
print(f"Optimal values: x = {optimal_x:.2f}, y = {optimal_y:.2f}")
```

```
print(f"Maximum profit: {-result.fun:.2f}")
```

References

[1] Fco. Javier Martínez Sánchez. El teorema de karush-kuhn-tucker, una generalización del teorema de los multiplicadores de lagrange, y programación convexa. *TEMat*, 3:33–44, 2019.