Solved Exercises

Optimization and Operations Research

FACULTAT
DE CIÈNCIES, TECNOLOGIA I ENGINYERIES
UVIC | UVIC·UCC

Jordi Villà-Freixa

jordi.villa@uvic.cat

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1 Operations Research

Ex. 1 — Formulation of an optimization problem: Consider a manufacturing company that produces two products, P_1 and P_2. The company aims to maximize its profit. The profit from each product is given as follows:
- P_1: \$40 per unit - P_2: \$30 per unit
The company has the following constraints based on its resources:
1. The availability of raw material limits production to a maximum of 100 units of P_1 and 80 units of P_2. 2. The total production time available is 160 hours, where producing one unit of P_1 takes 2 hours and one unit of P_2 takes 1 hour.
Can you formulate the operations research problem?

Answer (Ex. 1) — The problem can be stated as:

$$\begin{aligned}
\text{Objective Function:} \quad & \text{Maximize } Z = 40x_1 + 30x_2 \\
\text{Subject to:} \quad & \\
& 2x_1 + x_2 \leq 160 \quad \text{(Total production time constraint)} \\
& x_1 \leq 100 \quad \text{(Raw material constraint for } P_1\text{)} \\
& x_2 \leq 80 \quad \text{(Raw material constraint for } P_2\text{)} \\
& x_1, x_2 \geq 0 \quad \text{(Non-negativity constraints)}
\end{aligned}$$

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1.1 Non-linear optimization

Ex. 2 — A problem of non-linear optimization: A company wants to maximize its profit function given by:

$$f(x, y) = -x^2 + 4xy - 2y^2$$

where x and y represent the amount of resources allocated to two different projects.
The problem is subject to the following constraints:

1.The total resources used must not exceed 30 units.

2. The product of the resources allocated must be at least 50 units.
3. The relationship between the resource allocations must satisfy the following nonlinear constraint:

$$y \leq \frac{3x^2}{100} + 5$$

The company wants to find the values of x and y that maximize the profit $f(x, y)$ under these constraints.
Can you help the company drawing the problem in a graph? Can you identify the feasible region? Is the region convex?

Answer (Ex. 2) — A company wants to maximize its profit function given by:

$$f(x, y) = -x^2 + 4xy - 2y^2$$

where x and y represent the amount of resources allocated to two different projects.
The problem is subject to the following constraints:

1. The total resources used must not exceed 30 units:

$$x + 2y \leq 30$$

2. The product of the resources allocated must be at least 50 units:

$$xy \geq 50$$

3. The relationship between the resource allocations must satisfy the following nonlinear constraint:

$$y \leq \frac{3x^2}{100} + 5$$

Find the values of x and y that maximize the profit $f(x, y)$ under these constraints. Before solving it, we can draw the problem in a graph to identify the feasible region and the optimal solution. The problem is shown in Figure 1.

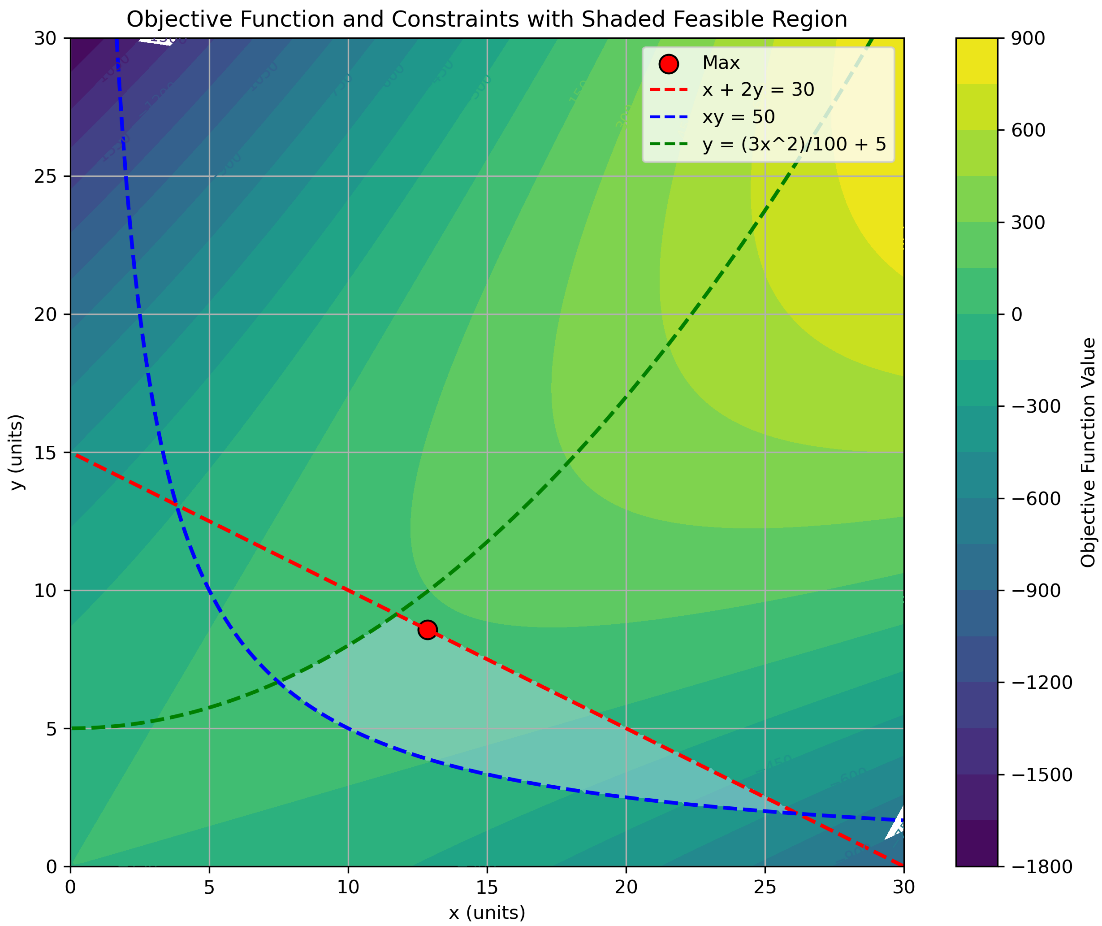

Figure 1: Contour plot of the objective function $f(x, y) = -x^2 + 4xy - 2y^2$ with constraints $x + 2y \leq 30$ (red dashed), $xy \geq 50$ (blue dashed), and $y \leq \frac{3x^2}{100} + 5$ (green dashed). The light blue region represents the feasible area where all constraints are satisfied. The solution has been obtained with Code 1

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1.1.1 Lagrange Multipliers

Ex. 3 — Lagrange multipliers: Optimization of $f(x,y) = x^2 - y$ subject to $x^2 + y^2 = 4$

Answer (Ex. 3) — We will use the method of Lagrange multipliers.
Step 1: Define the Lagrange Function Define the constraint as $g(x,y) = x^2 + y^2 - 4 = 0$. The Lagrange function is then:

$$\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda \cdot g(x,y) = x^2 - y + \lambda(x^2 + y^2 - 4).$$

Step 2: Compute the Partial Derivatives We now compute the partial derivatives of $\mathcal{L}(x,y,\lambda)$ with respect to x, y, and λ:

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + \lambda \cdot 2x = 2x(1+\lambda) = 0,$$
$$\frac{\partial \mathcal{L}}{\partial y} = -1 + \lambda \cdot 2y = 0 \quad \Rightarrow \quad \lambda = \frac{1}{2y},$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - 4 = 0.$$

Step 3: Solve the Equations

From $\frac{\partial \mathcal{L}}{\partial x} = 0$:

$$2x(1+\lambda) = 0.$$

This gives two possibilities:

- $x = 0$, or
- $1 + \lambda = 0 \quad \Rightarrow \quad \lambda = -1.$

Case 1: $x = 0$ Substitute $x = 0$ into the constraint equation $x^2 + y^2 = 4$:

$$0^2 + y^2 = 4 \quad \Rightarrow \quad y^2 = 4 \quad \Rightarrow \quad y = \pm 2.$$

For $y = 2$, substitute into $f(x,y) = x^2 - y$:

$$f(0,2) = 0^2 - 2 = -2.$$

For $y = -2$, substitute into $f(x,y)$:

$$f(0,-2) = 0^2 - (-2) = 2.$$

Case 2: $\lambda = -1$ Substitute $\lambda = -1$ into $\lambda = \frac{1}{2y}$:

$$-1 = \frac{1}{2y} \quad \Rightarrow \quad y = -\frac{1}{2}.$$

Now, substitute $y = -\frac{1}{2}$ into the constraint equation $x^2 + y^2 = 4$:

$$x^2 + \left(-\frac{1}{2}\right)^2 = 4 \quad \Rightarrow \quad x^2 + \frac{1}{4} = 4 \quad \Rightarrow \quad x^2 = \frac{15}{4} \quad \Rightarrow \quad x = \pm\frac{\sqrt{15}}{2}.$$

Now, calculate $f(x, y) = x^2 - y$ for $y = -\frac{1}{2}$ and $x = \pm\frac{\sqrt{15}}{2}$:
For $x = \frac{\sqrt{15}}{2}$:

$$f\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) = \left(\frac{\sqrt{15}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{15}{4} + \frac{1}{2} = \frac{17}{4}.$$

For $x = -\frac{\sqrt{15}}{2}$:

$$f\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) = \left(-\frac{\sqrt{15}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{15}{4} + \frac{1}{2} = \frac{17}{4}.$$

Step 4: Compare the Results We now compare the function values:

- For $(0, 2)$: $f(0, 2) = -2$.
- For $(0, -2)$: $f(0, -2) = 2$.
- For $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$: $f = \frac{17}{4} \approx 4.25$.

Conclusion

- The maximum value is $f = \frac{17}{4} \approx 4.25$ at $\left(\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{15}}{2}, -\frac{1}{2}\right)$.
- The minimum value is $f = -2$ at $(0, 2)$.

For a Python implementation check this file.

1.1.2 Karush-Kuhn-Tucker theorem

Ex. 4 — A problem of KKT conditions: Maximize $f(x, y) = xy$ subject to $100 \geq x + y$ and $x \leq 40$ and $(x, y) \geq 0$.

Answer (Ex. 4) — The Karush Kuhn Tucker (KKT) conditions for optimality are a set of necessary conditions for a solution to be optimal in a mathematical optimization problem. They are necessary and sufficient conditions for a local minimum in nonlinear programming problems. The KKT conditions consist of the following elements:
For an optimization problem in its standard form:

$$\max f(\mathbf{x})$$
$$\text{s.t.} \quad \begin{array}{rcll} g_i(\mathbf{x}) - b_i & \leq & 0 & i = 1, \dots, k \\ g_i(\mathbf{x}) - b_i & = & 0 & i = k+1, \dots, m \end{array}$$

There are 4 KKT conditions for optimal primal ($\mathbf{x}$) and dual (λ) variables. If $\mathbf{x}^*$ denotes optimal values:

1. Primal feasibility: all constraints must be satisfied: $g_i(\mathbf{x}^*) - b_i$ is feasible. Applies to both equality and non-equality constraints.
2. Gradient condition or No feasible descent: No possible improvement at the solution:
$$\nabla f(\mathbf{x}^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(\mathbf{x}^*) = 0$$
3. Complementariety slackness:
$$\lambda_i^* (g_i(\mathbf{x}^*) - b_i) = 0$$
4. Dual feasibility: $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active (=0) and a zero Lagrange multiplier when the constraint is inactive (>0).
To solve our problem, first we will put it in its standard form:

$$\max f(x,y) = xy$$
$$\text{s.t.} \quad \begin{array}{rcl} g_1(x,y) = x + y - 100 & \leq & 0 \\ g_2(x,y) = x - 40 & \leq & 0 \\ g_3(x,y) = -x & \leq & 0 \\ g_4(x,y) = -y & \leq & 0 \end{array} \tag{1}$$

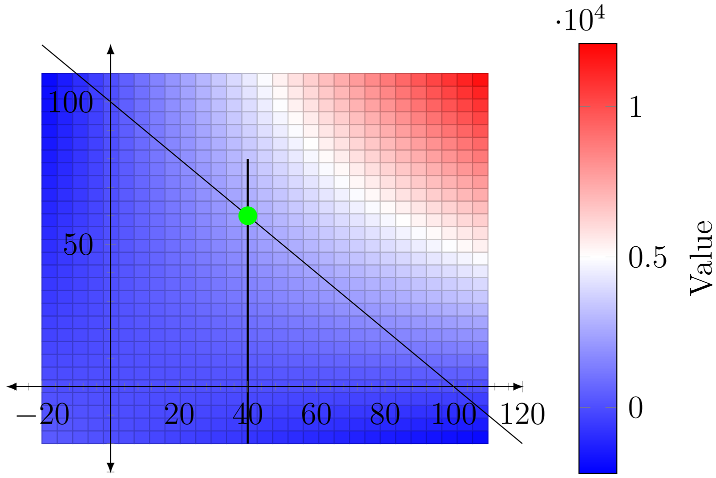

Figure 2: Contour plot of the objective function $f(x, y) = xy$ with constraints in Eq. 1, showing the final optimal value after applying the KKT conditions (green dot).

We will go through the different conditions:

- on the gradient:

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} - \lambda_1 \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_1}{\partial y} \end{pmatrix} - \lambda_2 \begin{pmatrix} \frac{\partial g_2}{\partial x} \\ \frac{\partial g_2}{\partial y} \end{pmatrix} - \lambda_3 \begin{pmatrix} \frac{\partial g_3}{\partial x} \\ \frac{\partial g_3}{\partial y} \end{pmatrix} - \lambda_4 \begin{pmatrix} \frac{\partial g_4}{\partial x} \\ \frac{\partial g_4}{\partial y} \end{pmatrix} = 0$$

  which, in this example, resolves into:

$$y - (\lambda_1 + \lambda_2 - \lambda_3) = 0 \tag{2}$$

$$x - (\lambda_1 - \lambda_4) = 0 \tag{3}$$

- on the complementary slackness:

$$\lambda_1(x + y - 100) = 0 \tag{4}$$

$$\lambda_2(x - 40) = 0 \tag{5}$$

$$\lambda_3 x = 0 \tag{6}$$

$$\lambda_4 y = 0 \tag{7}$$

- on the constraints:

$$x + y \leq 100 \tag{8}$$

$$x \leq 40 \tag{9}$$

$$-x \leq 0 \tag{10}$$

$$-y \leq 0 \tag{11}$$

plus $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$.

We will start by checking Eq. 4:

–Let us see what occurs if $\lambda_1 = 0$. Then, from Eq. 3, $x + \lambda_4 = 0$ which implies that $x = \lambda_4 = 0$[1]. But, then, from Eq. 5 we obtain that $\lambda_2 = 0$ which, using Eq. 2 gives $y + \lambda_3 = 0 \Rightarrow y = \lambda_3 = 0$. Indeed, the KKT conditions are satisfied when all variables and multipliers are zero, but it is not a maximum of the function (see figure above).

–So, let us see what happens if $x + y - 100 = 0$ and consider the two possibilities for x:

Case $x = 0$**:**Then, $y = 100$, which would lead (Eq. 7) to $\lambda_4 = 0$ and (Eq. 3) to $x = \lambda_1 = 0$, that was discussed in the previous item. So, we need top explore the other possibility for x.

Case $x > 0$**:**From Eq. 6 $\lambda_3 = 0$ and, from Eqs. 2 and 3:

$$\begin{cases} \quad y = \lambda_1 + \lambda_2 \\ x \quad = \lambda_1 + \lambda_4 \end{cases}$$

let us try what happens if, e.g., $\lambda_2 \neq 0$ (or, said in other words, if constraint 9 is active): $x = 40$. As we know we do not want $\lambda_1 = 0$, from Eq. 4 we obtain $x + y - 100 = 0 \Rightarrow y = 60$.

The point $(x, y) = (40, 60)$ fullfills the KKT conditions and is a maximum in the constrained maximization problem (as can be seen in Figure 2).

1.2 Simplex

Ex. 5 — Simplex optimization: — Use Simplex to solve this LP problem:

[1]Recall that both variables and multiplieras must be positive or zero, so, the only possibility for the equation to fullfill is that both are zero.

$$\max \quad Z = 1.0x_1 + 2.0x_2$$
$$\text{s.t.} \quad \begin{cases} -1.0x_1 + 1.0x_2 \leq 2.0 \\ 1.0x_1 + 2.0x_2 \leq 8.0 \\ 1.0x_1 + 0.0x_2 \leq 6.0 \end{cases}$$

Answer (Ex. 5) — Canonical Form:

$$Z - 1.0x_1 - 2.0x_2 - 0s_1 - 0s_2 = 0$$

$$\begin{cases} s_1 \qquad\qquad - 1.0x_1 + 1.0x_2 = 2.0 \\ \qquad s_2 \qquad + 1.0x_1 + 2.0x_2 = 8.0 \\ \qquad\qquad s_3 + 1.0x_1 + 0.0x_2 = 6.0 \end{cases}$$

Simplex tableau

Iteration 0 Entering and leaving variables:

- Entering variable: x_2
- Leaving variable: Basic variable from row 1

Table 1: Simplex Tableau after iteration 0

Basic	x_1	x_2	s_1	s_2	s_3	RHS
s_1	-1.00	**1.00**	1.00	0.00	0.00	2.00
s_2	1.00	2.00	0.00	1.00	0.00	8.00
s_3	1.00	0.00	0.00	0.00	1.00	6.00
Z	-1.00	**-2.00**	0.00	0.00	0.00	0.00

Iteration 1 Entering and leaving variables:

- Entering variable: x_1
- Leaving variable: Basic variable from row 2

Table 2: Simplex Tableau after iteration 1

Basic	x_1	x_2	s_1	s_2	s_3	RHS
x_2	-1.00	1.00	1.00	0.00	0.00	2.00
s_2	**3.00**	0.00	-2.00	1.00	0.00	4.00
s_3	1.00	0.00	0.00	0.00	1.00	6.00
Z	**-3.00**	0.00	2.00	0.00	0.00	4.00

Iteration 2 Optimal solution reached.

Table 3: Simplex Tableau after iteration 2

Basic	x_1	x_2	s_1	s_2	s_3	RHS
x_2	0.00	1.00	0.33	0.33	0.00	3.33
x_1	1.00	0.00	-0.67	0.33	0.00	1.33
s_3	0.00	0.00	0.67	-0.33	1.00	4.67
Z	0.00	0.00	0.00	1.00	0.00	8.00

Optimal solution: $x_1 = 1.33$, $x_2 = 3.33$

Optimal value (Simplex): 8.00

Optimal value (ORTools): 8.00

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Ex. 6 — Simplex optimization: — Use Simplex to solve this LP problem:

$$\max \quad Z = 3.0x_1 + 1.0x_2$$
$$\text{s.t.} \quad \begin{cases} 1.0x_1 + -1.0x_2 \leq -4.0 \\ -1.0x_1 + 2.0x_2 \leq -4.0 \end{cases}$$

Answer (Ex. 6) — Canonical Form:

$$Z - 3.0x_1 - 1.0x_2 - 0s_1 - 0s_2 = 0$$
$$\begin{cases} s_1 \quad + 1.0x_1 - 1.0x_2 + \; = -4.0 \\ \quad s_2 - 1.0x_1 + 2.0x_2 \quad = -4.0 \end{cases}$$

Simplex tableau

Iteration 0 System unbound.

Table 4: Simplex Tableau after iteration 0

Basic	x_1	x_2	s_1	s_2	RHS
s_1	1.00	-1.00	1.00	0.00	-4.00
s_2	-1.00	2.00	0.00	1.00	-4.00
Z	-3.00	-1.00	0.00	0.00	0.00

No solution found (Simplex)

The ORTools solver did not find an optimal solution.

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1.3 Duality

Ex. 7 — A problem of linear programming: Consider the linear programming problem:

$$\max x_1 + 4x_2 + 2x_3$$
$$\text{s.t.} \quad \begin{array}{rcl} 5x_1 + 2x_2 + 2x_3 & \leq & 145 \\ 4x_1 + 8x_2 - 8x_3 & \leq & 260 \\ x_1 + x_2 + 4x_3 & \leq & 190 \\ x_1, x_2 x_3 & \geq & 0 \end{array}$$

Find x_1, x_2 and x_3 to solve it.

Answer (Ex. 7) — material: complementary slackness.pdf

Ex. 8 — Complementary slackness: Consider the linear programming problem:

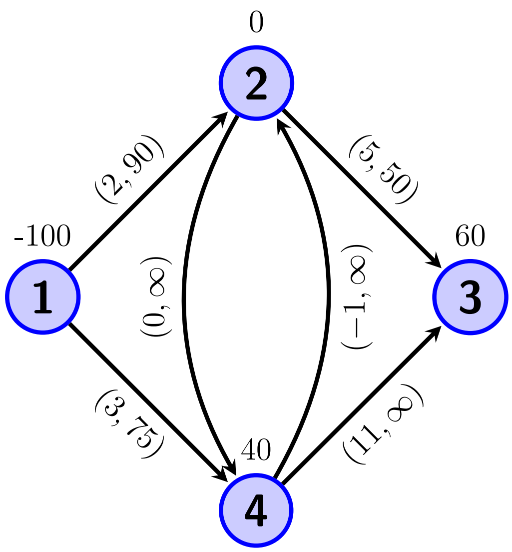
$$\max 2x_1 + 16x_2 + 2x_3$$
$$\text{s.t.} \quad \begin{array}{rcl} 2x_1 + x_2 - x_3 & \leq & -3 \\ -3x_1 + x_2 + 2x_3 & \leq & 12 \\ x_1, x_2 x_3 & \geq & 0 \end{array}$$

Check whether each of the following is an optimal solution, using complementary slackness:

Answer (Ex. 8) — material: compslack.pdf

1.4 Network Analysis

Ex. 9 — Minimum Cost: The figure shows a network on four nodes, including net demands on the vertex, b_k, and cost an capacity on the edges, $(c_{i,j}, u_{i,j})$.(Adapted from [1])



1.Formulate the corresponding minimum cost network flow model

2.Classify the nodes as *source*, *sink* or *transsshipment*

Answer (Ex. 9) — In this problem, vertex and edges are:

$$\begin{aligned} V &= \{1,2,3,4\} \\ A &= \{(1,2),(1,4),(2,3),(2,4),(4,2),(4,3)\} \end{aligned}$$

we can use the variables $x_{i,j}$ to represent the flows in the different members of set A. Thus, the formulation of the problem is:

$$\min \quad 2x_{1,2} + 3x_{1,4} + 5x_{2,3} - x_{4,2} + 11x_{4,3}$$

$$\text{subject to} \quad \left\{ \begin{array}{rcl} -A - B &=& -100 \\ A + F - C - D &=& 0 \\ C + E &=& 60 \\ B + D - F - E &=& 40 \\ A &\leq& 90 \\ B &\leq& 75 \\ C &\leq& 50 \end{array} \right.$$

and $x_{i,j} \geq 0$.
There are 4 KKT conditions for optimal primal ($x4$) and dual (λ) variables. If x^* denotes optimal values:

1.Primal feasibility: all constraints must be satisfied: $g_i(x^*)-b_i$ is feasible. Applies to both equality and non-equality constraints.

2.Gradient condition or No feasible descent: No possible improvement at the solution:

$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0$$

3.Complementariety slackness:

$$\lambda_i^*(g_i(x^*) - b_i) = 0$$

4.Dual feasibility: $\lambda_i^* \geq 0$

The last two conditions (3 and 4) are only required with inequality constraints and enforce a positive Lagrange multiplier when the constraint is active (=0) and a zero Lagrange multiplier when the constraint is inactive (>0).
to solve our problem, first we will put it in its standard form:

$$\min\ f(x,y) = -xy$$

$$\text{subject to} \quad \begin{array}{rcl} -x-y+100 & \geq & 0 \\ -x-40 & \geq & 0 \end{array}$$

We will go through the different conditions:

1.Primal feasibility: $g_i(x^*) - b_i$ is feasible.

$$-x^* - y^* + 100 = 0$$

$$-x^* - 40 = 0$$

2.Gradient condition or No feasible descent:

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} - \lambda_1 \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \frac{\partial g_1}{\partial y} \end{pmatrix} - \lambda_2 \begin{pmatrix} \frac{\partial g_2}{\partial x} \\ \frac{\partial g_2}{\partial y} \end{pmatrix} = 0$$

which, in this example, resolves into:

$$\begin{cases} - y + \lambda_1 + \lambda_2 = 0 \\ -x \quad - \quad\quad \lambda_1 = 0 \end{cases}$$

3.Complementariety slackness:

$$\lambda_1^*(-x^* - y^* + 100) = 0$$

$$\lambda_2^*(-x^* - 40) = 0$$

4.Dual feasibility: $\lambda_1, \lambda_2 \geq 0$

We can put the resulting 5 expressions for conditions 1 and 2 into matrix form:

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -100 \\ 40 \\ 0 \\ 0 \end{pmatrix}$$

2 Appendices

2.1 Python codes

Codes

2.1.1 NLO with constraints

Code 1: Non-linear optimization with constraints

```
from scipy.optimize import minimize

# Define the objective function
def objective(xy):
    x, y = xy
    return -(-x**2 + 4*x*y - 2*y**2)  # Minimization function, so retu

# Define the constraints
def constraint1(xy):
    x, y = xy
    return 30 - (x + 2*y)  # x + 2y <= 30
```

```python
def constraint2(xy):
    x, y = xy
    return (x * y) - 50  # xy >= 50

def constraint3(xy):
    x, y = xy
    return (3 * x**2 / 100) + 5 - y  # y <= (3x^2)/100 + 5

# Initial guess
initial_guess = [15, 1]

# Define constraints
constraints = [{'type': 'ineq', 'fun': constraint1},
               {'type': 'ineq', 'fun': constraint2},
               {'type': 'ineq', 'fun': constraint3}]

# Perform the optimization
result = minimize(objective, initial_guess, constraints=constraints)

# Display the results
optimal_x, optimal_y = result.x
print(f"Optimal values: x = {optimal_x:.2f}, y = {optimal_y:.2f}")
print(f"Maximum profit: {-result.fun:.2f}")
```

References

[1] Ronald L. Rardin. *Optimization in Operations Research.* Pearson, Boston, second edition edition, 2017.