x $f(x)$ are column vectors

1\. (a) $\frac{\partial}{\partial x_k}(x^TAx) = \sum_{j\neq k} a_{kj}x_j + \sum_{i\neq k} a_{ik}x_i + 2a_{kk}x_k$

$\Rightarrow \nabla(x^TAx) = Ax + A^Tx$

$= 2Ax$ (since $A = A^T$)

$\Rightarrow \nabla f(x) = Ax + b$

(b) $\frac{\partial f}{\partial x_i} = g'(h(x))\frac{\partial h}{\partial x_i} \Rightarrow \nabla f = g'(h(x)) \cdot \nabla h$

(c) $(\nabla^2(x^TAx))_{kl} = a_{kl} + a_{lk}$

$\Rightarrow \nabla^2(x^TAx) = A + A^T \Rightarrow \nabla^2(f(x)) = A$

(d) $\nabla f = g'(a^Tx)\, a$ from (c)

$(\nabla^2 f)_{ij} = g''(a^Tx)\, a_i a_j \Rightarrow \nabla^2 f = g''(a^Tx) \cdot a \cdot a^T$

2 (a) $x^TAx = x^Tzz^Tx = (z^Tx)^2 \geqslant 0$

(b) $Ax = 0 \Leftrightarrow zz^Tx = 0 \Leftrightarrow z^Tx = 0$

$Nul(A) = \{z\}^\perp$

$rk(A) = 1$

(c) YES $\quad x^TBAB^Tx = (B^Tx)^TA(B^Tx) \geqslant 0$

3. (a) $A = T\Lambda T^{-1} \Rightarrow AT = T\Lambda$

$$\Rightarrow A[t^{(1)} \cdots\cdots t^{(n)}] = [t^{(1)} \cdots\cdots t^{(n)}]\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$\Rightarrow [At^{(1)} \cdots\cdots At^{(n)}] = [\lambda_1 t^{(1)} \cdots\cdots \lambda_n t^{(n)}]$$

$$\Rightarrow At^{(i)} = \lambda_i t^{(i)}$$

(b) Same as (a) since $U^T = U^{-1}$

(c) from (b) $\quad u^{(i)T}Au^{(i)} = \lambda_i \|u^{(i)}\|^2 \geqslant 0$

$$\Rightarrow \lambda_i \geqslant 0$$