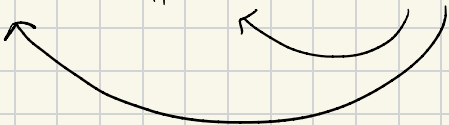




x and $f(x)$ are column vectors

$$1. (a) \frac{\partial}{\partial x_k} (x^T A x) = \sum_{j \neq k} a_{kj} x_j + \sum_{i \neq k} a_{ik} x_i + 2a_{kk} x_k$$


$$\Rightarrow \nabla (x^T A x) = A x + A^T x \\ = 2A x \quad (\text{since } A = A^T)$$

$$\Rightarrow \nabla f(x) = A x + b$$

$$(b) \frac{\partial f}{\partial x_i} = g'(h(x)) \frac{\partial h}{\partial x_i} \Rightarrow \nabla f = g'(h(x)) \cdot \nabla h$$

$$(c) (\nabla^2 (x^T A x))_{kl} = a_{kl} + a_{lk}$$

$$\Rightarrow \nabla^2 (x^T A x) = A + A^T \Rightarrow \nabla^2 (f(x)) = A$$

$$(d) \nabla f = g'(a^T x) a \quad \text{from (c)}$$

$$(\nabla^2 f)_{ij} = g''(a^T x) a_i a_j \Rightarrow \nabla^2 f = g''(a^T x) a a^T$$

$$2 \text{ (a) } x^T A x = x^T z z^T x = (z^T x)^2 \geq 0$$

$$\text{cb) } Ax = 0 \Leftrightarrow z z^T x = 0 \Leftrightarrow z^T x = 0$$

$$\text{Nul}(A) = \{z\}^\perp$$

$$\text{rk}(A) = 1$$

$$\text{(c) YES} \quad x^T B A B^T x = (B^T x)^T A (B^T x) \geq 0$$

$$3. \text{ (a) } A = T \Lambda T^{-1} \Rightarrow AT = T \Lambda$$

$$\Rightarrow A \begin{bmatrix} t^{(1)} & \dots & t^{(n)} \end{bmatrix} = \begin{bmatrix} t^{(1)} & \dots & t^{(n)} \end{bmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$\Rightarrow [A t^{(1)} \dots A t^{(n)}] = [\lambda_1 t^{(1)} \dots \lambda_n t^{(n)}]$$

$$\Rightarrow A t^{(i)} = \lambda_i t^{(i)}$$

$$\text{cb) same as (a) since } U^T = U^{-1}$$

$$\text{(c) from cb) } u^{(i)T} A u^{(i)} = \lambda_i \|u^{(i)}\|^2 \geq 0$$

$$\Rightarrow \lambda_i \geq 0$$