Q1 (a) MM^T is of size $p \times p$

entries of MM^T are real.

$(MM^T)^T = MM^T \Rightarrow MM^T$ is symmetric

Similarly, one can show that all properties hold for M^TM.

(b) Let v be an eigenvector of MM^T with eigenvalue $\lambda \neq 0$

$MM^Tv = \lambda v$ of course $M^Tv \neq \vec{0}$

then $M^TMM^Tv = \lambda M^Tv$

$\Rightarrow$ M^Tv is an eigenvector of M^TM with eigenvalue λ.

$\Rightarrow$ M^TM and MM^T share same non-zero eigenvalues.

But eigenvectors are not necessarily the same from the proof above.

(c) $M^TM = Q\Lambda Q^T$

(d) $M^T M = V \Sigma U^T U \Sigma V^T = V \Sigma^2 V^T$

(e) U Σ V^T can be found in codes

Evals and Evecs can also be found in codes

vectors in Evecs $= \pm$ columns of V.

eigenvalues of $M^T M$ = (singular values of M)^2

Q2 See the codes

Q3 (a) $E = \left(\sum_{(i,u) \in \text{ratings}} (R_{iu} - q_i \cdot P_u^T)^2 \right) + \lambda \left[\sum_u \|P_u\|_2^2 + \sum_i \|q_i\|_2^2 \right]$

$$\varepsilon_{iu} = \frac{\partial E}{\partial R_{iu}} = 2(R_{iu} - q_i \cdot P_u^T)$$

$$\frac{\partial E}{\partial q_i} = -\varepsilon_{iu} P_u + 2\lambda q_i$$

$$\frac{\partial E}{\partial P_u} = -\varepsilon_{iu} q_i + 2\lambda P_u$$

$$q_i := q_i + \eta(\varepsilon_{iu} P_u - 2\lambda q_i)$$

$$P_u := P_u + \eta(\varepsilon_{iu} q_i - 2\lambda P_u)$$

(b) See the codes

Q4 (a) $T = R \cdot R^T$

T_{ii} = degree of node i

T_{ij} = number of paths between node i and node j ($i \neq j$)

(b) $(R^T R)_{ij} = (\text{item } i) \cdot (\text{item } j)$

$$\left(Q^{-\frac{1}{2}} R^T R Q^{-\frac{1}{2}}\right)_{ij} = \frac{(\text{item } i) \cdot (\text{item } j)}{\|\text{item } i\| \cdot \|\text{item } j\|} = (S_I)_{ij}$$

$\Rightarrow S_I = Q^{-\frac{1}{2}} R^T R Q^{-\frac{1}{2}}$

$S_U = P^{-\frac{1}{2}} R R^T P^{-\frac{1}{2}}$ and the proof is similar

(c) From the definition,

item-item: $\Gamma = R \cdot S_I = R Q^{-\frac{1}{2}} R^T R Q^{-\frac{1}{2}}$

user-user: $\Gamma = S_U \cdot R = P^{-\frac{1}{2}} R R^T P^{-\frac{1}{2}} R$

(d) See the codes