Farey Sequences

A Farey sequence F_N of degree N (Farey fractions of degree N) is an ordered set of reduced fractions

$$\frac{p_i}{q_i} \qquad \text{with} \qquad p_i \le q_i \le N \qquad \text{and} \qquad 0 \le i < |F_N|$$

and

$$\frac{p_i}{q_i} < \frac{p_j}{q_j} \qquad \forall\, 0 \le i < j < |F_N|.$$

Write a function (using C++)

void Farey(T N)

which calculates the Farey fractions up to degree N and prints the resulting Farey sequences up to degree N on the screen.

Algorithm: The sequences can be computed recursively. The first sequence is given by

$$F_1 = \left(\tfrac{0}{1}, \tfrac{1}{1}\right)$$

For a known sequence F_N one can get F_{N+1} by inserting an additional fraction $\frac{p_i + p_{i+1}}{q_i + q_{i+1}}$ between two consecutive entries $\frac{p_i}{q_i}$ and $\frac{p_{i+1}}{q_{i+1}}$ if $q_i + q_{i+1} = N + 1$ holds for the sum of denominators.

Example: Determining F_7 from F_6 results in the following construction:

$$F_6 = \Bigg(\underbrace{\tfrac{0}{1}, \tfrac{1}{6}}_{\frac{1}{7}}, \tfrac{1}{5}, \underbrace{\tfrac{1}{4}, \tfrac{1}{3}}_{\frac{2}{7}}, \underbrace{\tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}}_{\frac{3}{7} \text{ and } \frac{4}{7}}, \underbrace{\tfrac{2}{3}, \tfrac{3}{4}}_{\frac{5}{7}}, \tfrac{4}{5}, \underbrace{\tfrac{5}{6}, \tfrac{1}{1}}_{\frac{6}{7}} \Bigg)$$

The new elements are:

$$\tfrac{0+1}{1+6} = \tfrac{1}{7} \;;\; \tfrac{1+1}{4+3} = \tfrac{2}{7} \;;\; \tfrac{2+1}{5+2} = \tfrac{3}{7} \;;\; \tfrac{1+3}{2+5} = \tfrac{4}{7} \;;\; \tfrac{2+3}{3+4} = \tfrac{5}{7} \;;\; \tfrac{5+1}{6+1} = \tfrac{6}{7}$$

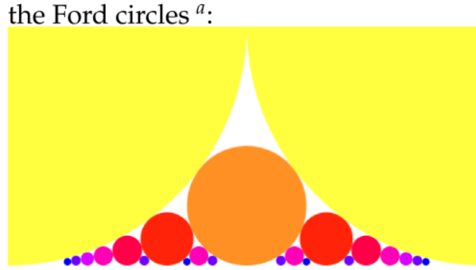
The sorted sequence then is:

$$F_7 = \left(\tfrac{0}{1}, \mathbf{\tfrac{1}{7}}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}, \mathbf{\tfrac{2}{7}}, \tfrac{1}{3}, \tfrac{2}{5}, \mathbf{\tfrac{3}{7}}, \tfrac{1}{2}, \mathbf{\tfrac{4}{7}}, \tfrac{3}{5}, \tfrac{2}{3}, \mathbf{\tfrac{5}{7}} \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \mathbf{\tfrac{6}{7}}, \tfrac{1}{1} \right)$$

For checking:
The Farey sequences up to degree 6

$$\begin{aligned}
F_1 &= \left(\tfrac{0}{1}, \tfrac{1}{1}\right) \\
F_2 &= \left(\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1}\right) \\
F_3 &= \left(\tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{1}\right) \\
F_4 &= \left(\tfrac{0}{1}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{1}\right) \\
F_5 &= \left(\tfrac{0}{1}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{1}{1}\right) \\
F_6 &= \left(\tfrac{0}{1}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \tfrac{1}{1}\right).
\end{aligned}$$

There is a beautiful illustration of these fractions, the Ford circles [a]:



[a] see `http://en.wikipedia.org/wiki/Ford_circle`