Biomedical Engineering Degree

2. Estimation

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References

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2. B. Caffo. *Statistical Inference for Data Science*. Leanpub. Chapter 7
3. D. Díez, M Cetinkaya-Rundel and CD Barr. *OpenIntro Statistics*. Chapter 5.

Outline

Example

We want to measure the average height of the university students **population** in Spain. Who would you do it?

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2. You measure the height of a **sample** of university student in Spain and then average the results.
   - How to choose this sample? How many samples would you need?
   - How close would our **estimation** be to the real value?
   - How likely would our **estimate** be within a certain range of values?

Example

We want to measure the average height of the university students **population** in Spain. Who would you do it?

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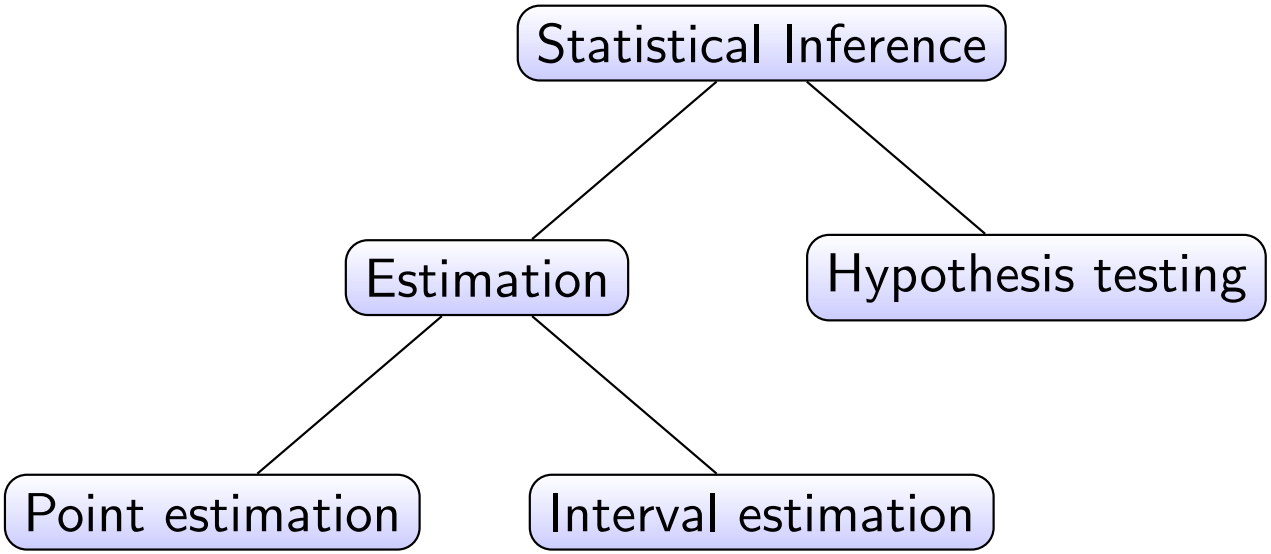
2. You measure the height of a **sample** of university student in Spain and then average the results.
   - How to choose this sample? How many samples would you need?
   - How close would our **estimation** be to the real value?
   - How likely would our **estimate** be within a certain range of values?
3. You assume that the height of university student in Spain follows a Normal distribution with mean value μ and variance σ^2
   - Does this assumption help? Is this a valid assumption?
   - How can we estimate μ? and σ^2?
   - Under this assumption, can we **compare** the height of students from Valencia versus students from Bilbao?

Mind map



- Statistical inference: is the process and result of drawing conclusions about a population from **one or more samples**
- Point estimation: estimating the values of specific population parameters
- Interval estimation: specify a range within which the parameter values are likely to fall
- Hypothesis testing: is concerned with testing whether the value of a population parameter is equal to some specific value.

Random sample vs population

- **Population**, reference, or target refer the group we want to study.
- From the population, a sample is drawn at random (**random sample**) to select some members of the population such that **each member is independently chosen**.
- If we can take action on the sampling process, we must consider:
  1. Building a sample big enough to have reliable data
  2. Building a representative sample of the population
     - ⋆ Example: randomized clinical trials

Outline

Point estimation

- We will study two estimators for different conditions and distributions:
  1. Estimation of the mean
  2. Estimation of the variance

Given a specific random sample $x_1, x_2, \ldots, x_n$, how can we estimate μ and σ^2?

- We will not study how to mathematically derived (robust) estimators using different criteria like
  1. Maximum likelihood, maximum a posteriori
  2. Method of moments
  3. Least squares

Estimation of the mean

Given a specific random sample $x_1, x_2, \ldots, x_n$, how can we estimate μ?

- Answer: use the **sample mean**

$$\hat{\mu} = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

- *But, why?* Let's examine its properties ...
- *... OK, but, how can I do it?* Use the **sampling distribution**

Sampling distribution

We must forget about our particular sample for the moment and consider the set of all possible samples of size n that could have been selected from the population

Sampling distributions are never observed, but we keep them in mind

Example

- Sorry, but I do not believe you, my estimator is better than yours:
  a. Mine: $\hat{\mu}_1 = \frac{1}{n} \sum_{n=i}^{n} x_i$
  b. Yours: $\hat{\mu}_2 = x_1$

Exercise 1: Let's run some simulations

- Represent the sampling distribution of both estimators. To do so, consider:
  1. The population follows a Normal distribution with $\mu = 2$ and $\sigma^2 = 2$
  2. Use $n = 10$

Exercise 1 (cont.): Let's do some thinking (it is free!)

- Which is the best estimator? and why?
- What if we increase/decrease n, how does it affect to our results?

Properties of an estimator

Take-home message

The estimator $\hat{\theta}$ of a distribution parameter θ is always a random variable

- Thus, properties of an estimator have to be assessed statistically:
  - Analytically, through its pdf
  - Computationally, through computer simulations (Monte Carlo methods)

Bias

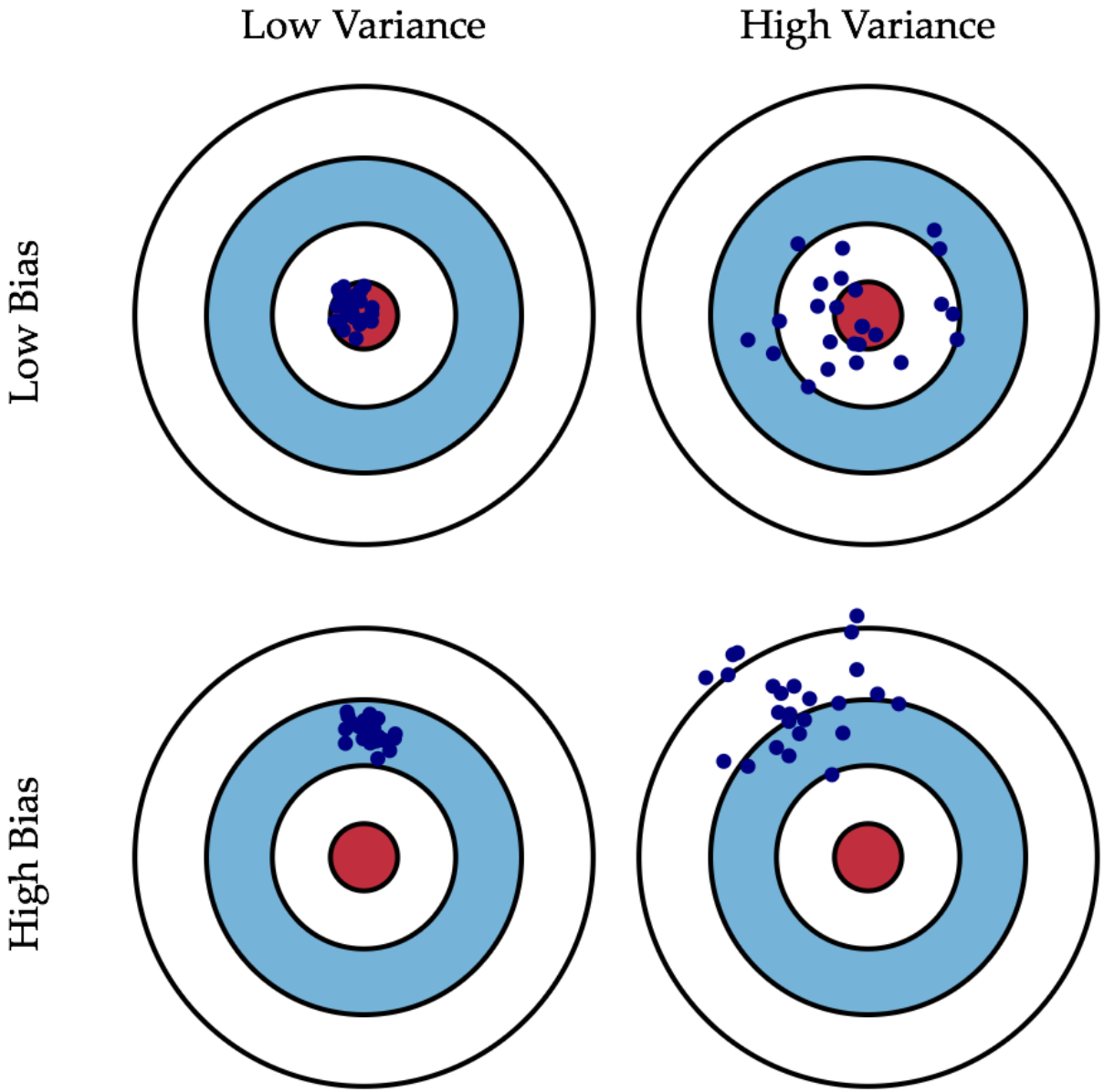
$$b = E[\hat{\theta}] - \theta$$

where b is the **bias**. If $b = 0$ we say that $\hat{\theta}$ is **unbiased**

Variance

$$\mathrm{Var}(\hat{\theta}) = E\left[\left(\hat{\theta} - E[\hat{\theta}]\right)^2\right]$$

Bias vs Variance



Example

Calculate the bias and variance of our estimators

a. Mine: $\hat{\mu}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$

b. Yours: $\hat{\mu}_2 = x_1$

Bias (example solution)

- As for $\hat{\mu}_1$

$$E[\hat{\mu}_1] = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[x_i] = \frac{1}{n}\sum_{n=1}^{n} \mu = \mu$$

  Thus, $\hat{\mu}_1$ is **unbiased**.

- The estimator $\hat{\mu}_2$

$$E[\hat{\mu}_2] = E[x_1] = \mu$$

  is also **unbiased**

- In terms of bias, both estimators are equally good.
- If both are unbiased, which one should I choose?

Variance (example solution)

- Variance for $\hat{\mu}_1$ is

$$\mathrm{Var}(\hat{\mu}_1) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(x_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{\sigma^2}{n}$$

- And for $\hat{\mu}_2$

$$\mathrm{Var}(\hat{\mu}_2) = \mathrm{Var}(x_1) = \sigma^2$$

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Standard error (se) of the mean

$$\mathrm{se} = \frac{\sigma}{\sqrt{n}}$$

Exercise 2

Let X be a r.v. that follows a $\mathcal{N}(\mu, \sigma)$ with $\mu = 100$ and $\sigma = 15$. What's the sampling distribution of $\bar{X}$ for different values of n? Check your analytical solution with computer simulations.

Exercise 3

Let X be a r.v. that follows a uniform $\mathcal{U}(a, b)$ with $a = 150$ and $b = 190$. What's the sampling distribution of $\bar{X}$ for different values of n? Check your analytical solution with computer simulations.

Central-Limit Theorem

- Let $X_1, X_2, \ldots, X_n$ be a random sample from some population with mean μ and variance σ^2.

For large n ($n > 30$), $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n)$ even if the underlying distribution of individual observations in the population is not normal.

- If we standardized the sampling distribution then

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

follows a $\mathcal{N}(0, 1)$

Estimation of the variance

Given a specific random sample $x_1, x_2, \ldots, x_n$, how can we estimate σ^2?

- Answer: use the (corrected) **sample variance**

$$\hat{\sigma}^2 = s_*^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ is the sample mean

- This constitutes an **unbiased** estimator of the variance. Proofs here and here.
- In Python, we can use `np.std(x,ddof=1)` for calculating the (corrected) sample standard deviation s_*

Outline

Interval Estimation

- Interval estimation: specify a range within which the parameter values are likely to fall

Interval estimation of the mean

- From our previous discussion we know that

$$\bar{X} \sim \mathcal{N}(\mu, \sigma^2/n)$$

- In the standardized form,

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$$

- Hence, $95\,\%$ of the z values from repeated samples of size n will fall within the interval $[-1.96, +1.96]$

$$P(z_{0.025} < Z < z_{0.975}) = P(-1.96 < Z < 1.96) = 0.95$$

- We would then have a $95\,\%$ certainty that $\bar{X}$ would fall in the interval

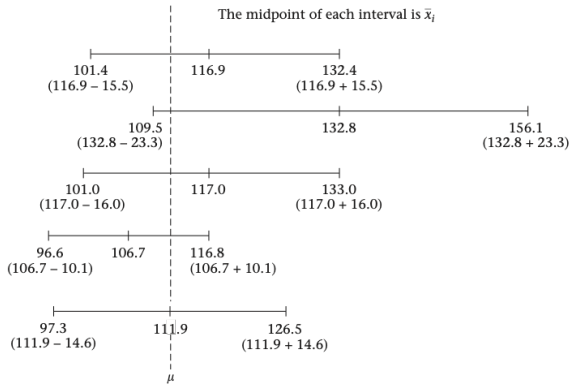
$$[\mu - 1.96 \cdot \sigma/\sqrt{n}, \mu + 1.96 \cdot \sigma/\sqrt{n}]$$

Confidence interval (CI) of the mean

- Or equivalently, the probability that these limits contain μ is $95\,\%$
- The quantity:

$$\boxed{\overline{X} \pm 1.96 \cdot \sigma/\sqrt{n}}$$

  is called a $95\,\%$ **interval** for μ
- Notice that the **interval is different for each sample**



Over the collection of all $95\,\%$ CIs that could be constructed from repeated random samples of size n, $95\,\%$ will contain the parameter μ

Confidence interval (CI) of the mean

- More generally, we can write $95\,\% = 100\,\%(1-\alpha)$, so that

$$P(z_{\alpha/2} < Z < z_{1-\alpha/2}) = 1-\alpha$$

  - $1-\alpha$ is called the **confidence level**
  - α is called the **significance level**

- And then[1]

$$\boxed{\overline{X} \pm z_{1-\alpha/2} \cdot \sigma/\sqrt{n}}$$

  is the confidence interval of μ with a confidence level $100\,\%(1-\alpha)$

[1] Using that $z_{1-\alpha/2} = -z_{\alpha/2}$

$t-$distribution

- In practice σ **is rarely known**.
- Thus, it is reasonable to estimate σ by the sample standard deviation s_*
- However, the quantity

$$t = \frac{\bar{X} - \mu}{s_*/\sqrt{n}}$$

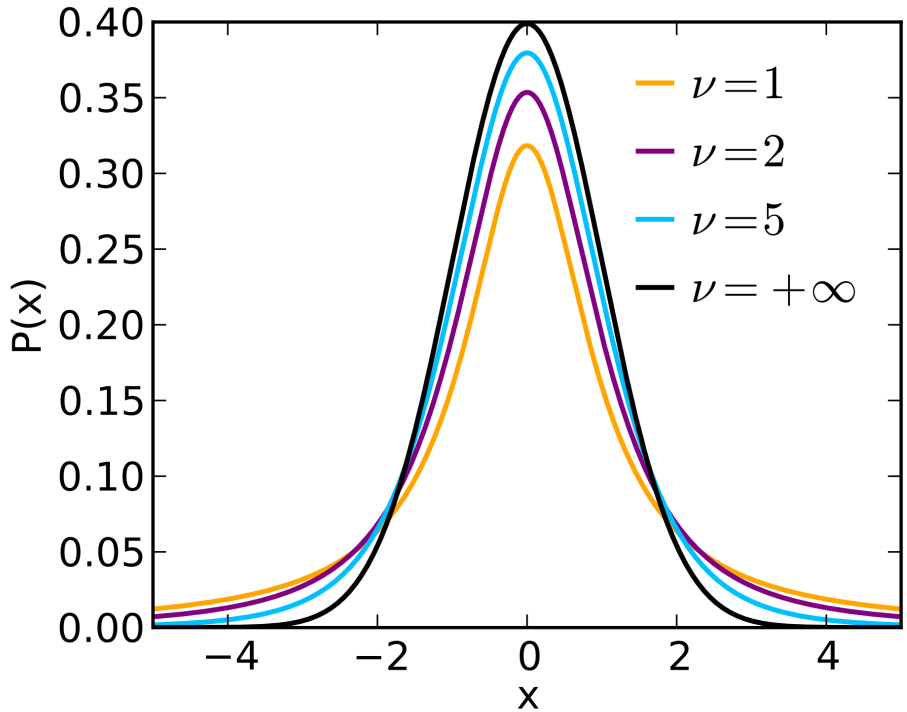
is **no longer normally distributed**. It distributes as a **Student's t-distribution**[2]

  - The shape of this distribution depends on the sample size n.
  - Thus, the t-distribution is not a unique distribution but is instead a **family of distributions** indexed by a parameter referred to as the **degrees of freedom (df)** of the distribution

[2] This problem was first solved in 1908 by a statistician named William Gossett. For his entire professional life, Gossett worked for the Guinness Brewery in Ireland. He chose to identify himself by the pseudonym "Student"

$t-$distribution

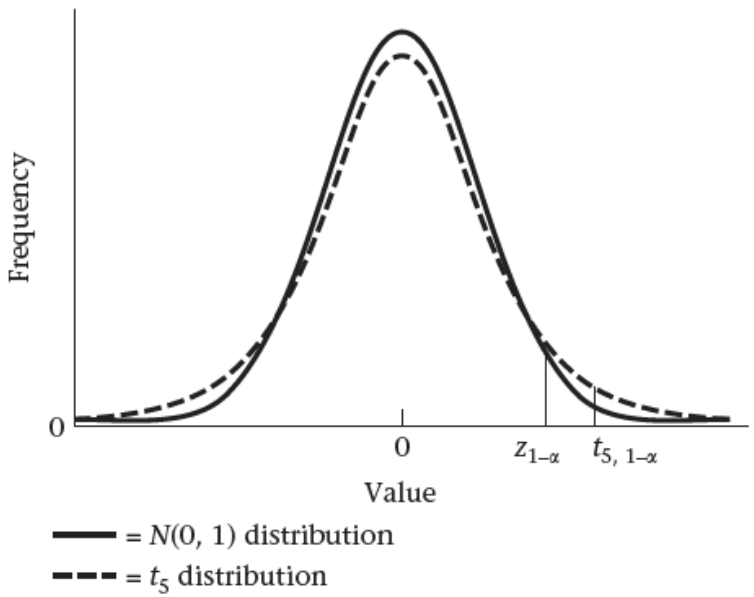
If $x_1, \ldots, x_n \sim \mathcal{N}(\mu, \sigma^2)$ and are independent, then $\frac{\bar{X}-\mu}{s_*/\sqrt{n}}$ is distributed as a $t-$distribution with $d = (n-1)$ df, which is sometimes referred to as the t_d-distribution.



$t-$distribution percentiles

- We denote the $u-$th percentile of the t_d distribution (d degrees of freedom) as $t_{d,u}$ so that

$$P(t_d < t_{d,u}) = u$$



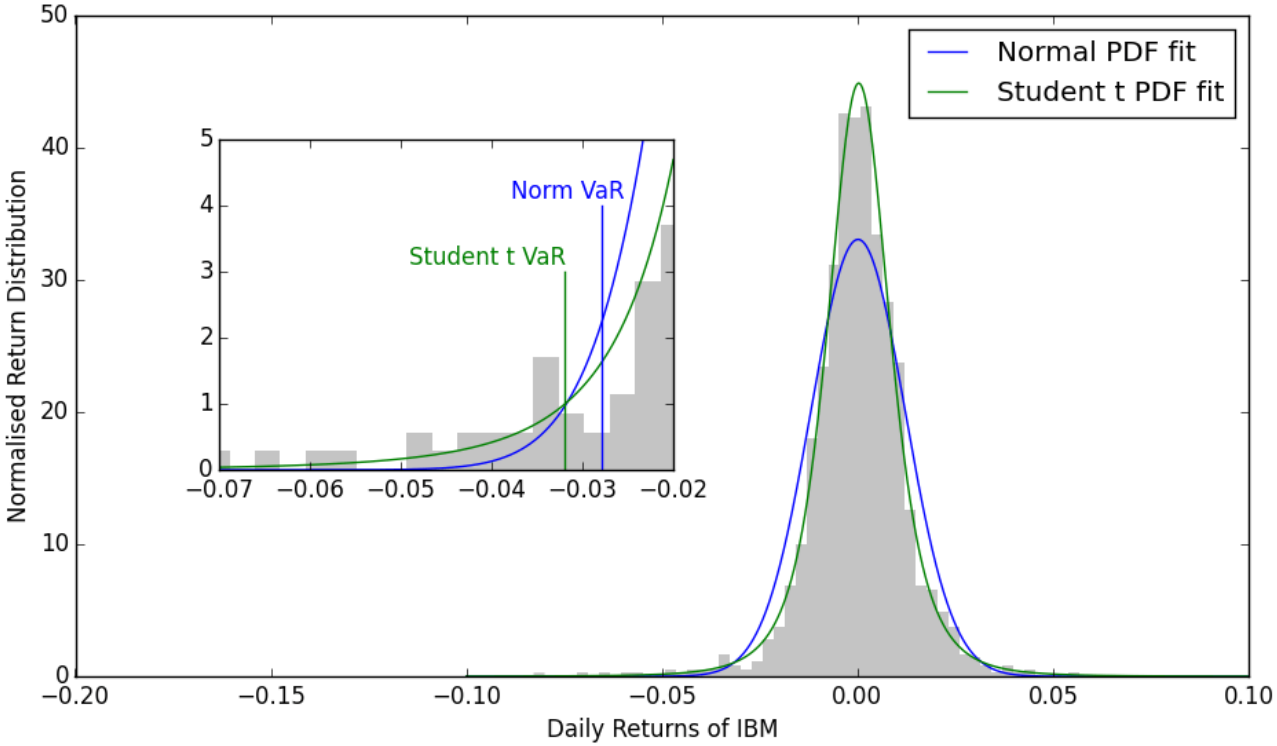
Python code

```
from scipy.stats import norm, t
print(norm.ppf(0.95))
print(t(df=5).ppf(0.95))
```

- >> 1.6448536269514722
- >> 2.015048372669157

$t-$distribution example in the real world

- Asset portfolio optimization Value at Risk metric calculation[3]



[3]This figure was extracted from here

CI of the mean (unknown variance)

- Remember that the confidence interval when the variance is know, can be calculated as

$$\overline{X} \pm z_{1-\alpha/2} \cdot \sigma/\sqrt{n}$$

- And now, if the **variance is unknown**, we have

$$\boxed{\overline{X} \pm t_{n-1,1-\alpha/2} \cdot s_*/\sqrt{n}}$$

  which is the confidence interval of μ with a confidence level $100\,\%(1-\alpha)$

  - if $n > 200$ then $t_{n-1} \sim \mathcal{N}(0,1)$ and in this case

$$\overline{X} \pm z_{1-\alpha/2} \cdot s_*/\sqrt{n}$$

Factors affecting the length of a CI

$$\boxed{\overline{X} \pm t_{n-1,1-\alpha/2} \cdot s_*/\sqrt{n}}$$

- n: if $n \uparrow \Rightarrow$ length of CI $\downarrow$
- s_*: if $s_* \uparrow \Rightarrow$ length of CI $\uparrow$
- α: if $\alpha \uparrow \Rightarrow$ length of CI $\downarrow$

Interval estimation of the variance

- To obtain an interval estimate for σ^2, we need a **new family of distributions** called chi-square (χ^2) distributions

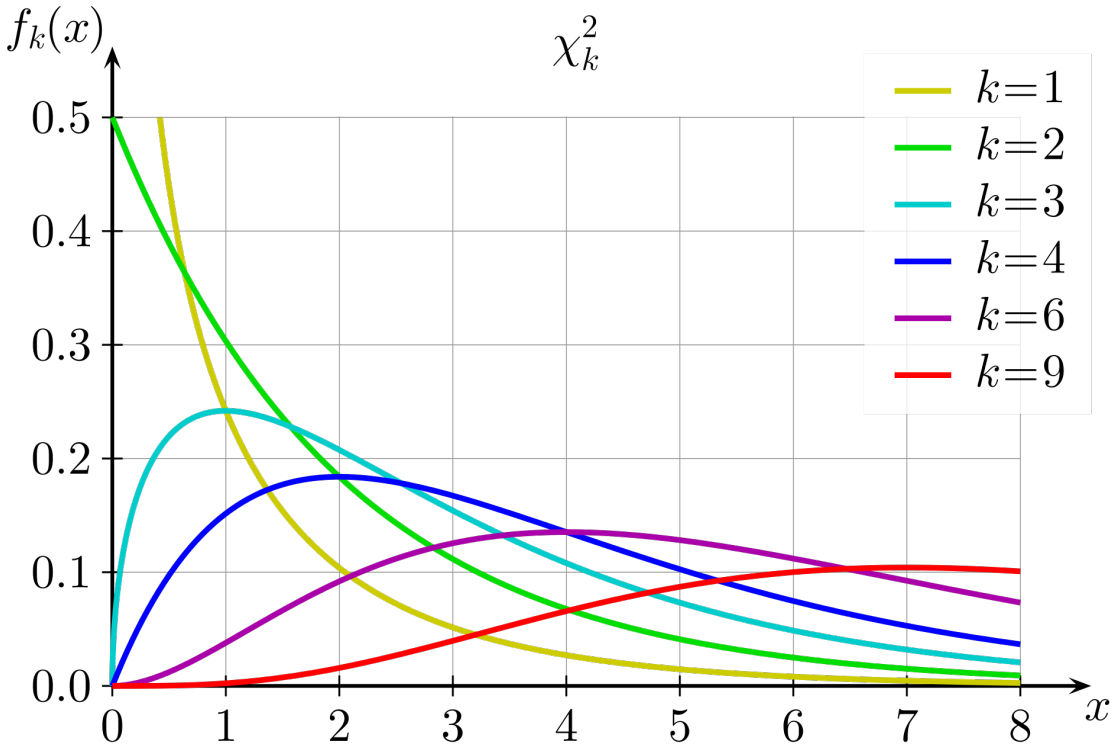
Let $X_1, X_2, \ldots, X_n$ be **independent** r.v.'s following a $\mathcal{N}(0,1)$ distribution. Then,

$$G = \sum_{i=1}^{n} X_i^2$$

is said to follow a chi-square distribution with n degrees of freedom (df), which is denoted by $\boldsymbol{\chi^2_n}$

- Distribution parameters:
  - $E[\chi^2_n] = n$
  - $\mathrm{Var}[\chi^2_n] = 2n$

Chi-square distribution

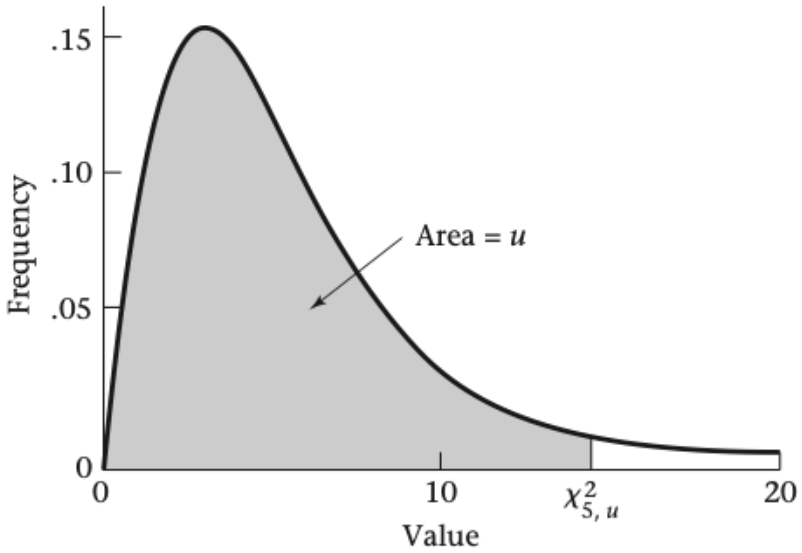


- Only takes on positive values and is always skewed to the right
- The skewness diminishes as n increases

Chi-square percentiles

- We denote the $u-$th percentile of the χ^2_d distribution (d degrees of freedom) as $\chi^2_{d,u}$ so that

$$P(\chi^2_d < \chi^2_{d,u}) = u$$



Python code

```
from scipy.stats import chi2
print(chi2(df=5).ppf(0.95))
```

- >> 11.070497693516351

Interval estimation

- Let Z_i be a standard normal. Then, by definition

$$\sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$$

- Since $Z_i = \frac{X_i - \mu}{\sigma}$, we can write

$$\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi_n^2 \Rightarrow \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 \sim \chi_n^2$$

- If we estimate μ by $\overline{X}$ (we usually don't know μ), then we lose 1 df[4].

$$\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \overline{X})^2 \sim \chi_{n-1}^2$$

[4] You can find the proof here.

Interval estimation

- Then, by using the relationship $s_*^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ it results in

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \frac{(n-1)s_*^2}{\sigma^2} \sim \chi^2_{n-1}$$

- So we can obtain that

$$s_*^2 \sim \frac{\sigma^2}{n-1}\chi^2_{n-1}$$

- And then

$$P\left(\frac{\sigma^2}{n-1}\chi^2_{n-1,\alpha/2} < s_*^2 < \frac{\sigma^2}{n-1}\chi^2_{n-1,1-\alpha/2}\right) = 1-\alpha$$

Confidence interval of the variance

- Thus, the interval

$$\boxed{\left[\frac{(n-1)s_*^2}{\chi^2_{n-1,1-\alpha/2}}, \frac{(n-1)s_*^2}{\chi^2_{n-1,\alpha/2}}\right]}$$

  is a $100\,\%(1-\alpha)$ CI for σ^2

Outline

Estimation for the Binomial distribution

- Recall that if $X \sim \text{Binomial}(n, p)$, then

$$f_X(x) = \left(\begin{array}{c} n \\ x \end{array} \right) p^x (1-p)^{n-x}$$

where p is the proportion of *success*, $E[X] = np$ and $\text{Var}[X] = np(1-p)$

Point Estimation of p

Let $X \sim \text{Binomial}(n, p)$. An unbiased estimator of p is given by the sample proportion of events $\hat{p}$. Its standard error is given exactly by $\sqrt{pq/n}$ and is estimated by $\sqrt{\hat{p}\hat{q}/n}$.

Estimation for the Binomial distribution

Normal approximation

if $n\hat{p}\hat{q} \geq 5$ then an approximate $100\,\% \times (1-\alpha)$ CI for the binomial parameter p based on the normal approximation to the binomial distribution is given by

$$\hat{p} \pm z_{1-\alpha/2}\sqrt{\hat{p}\hat{q}/n}$$

Exact method

if $n\hat{p}\hat{q} < 5$ then an exact $100\,\% \times (1-\alpha)$ CI for the binomial parameter p that is always valid is given by (p_1, p_2), where

$$P(X \geq x | p = p_1) = \frac{\alpha}{2}$$
$$P(X \leq x | p = p_2) = \frac{\alpha}{2}$$

Estimation for the Poisson distribution

- Remember that if $X \sim Poisson(\lambda t)$ then $\lambda = E[X/t]$ is the **expected count per unit of time** and t is the total monitoring time

$$\hat{\lambda} = \frac{X}{t}$$

$$\hat{\lambda} \pm z_{1-\alpha/2}\sqrt{\frac{\hat{\lambda}}{t}}$$