

Biomedical Engineering Degree

5. CATEGORICAL DATA

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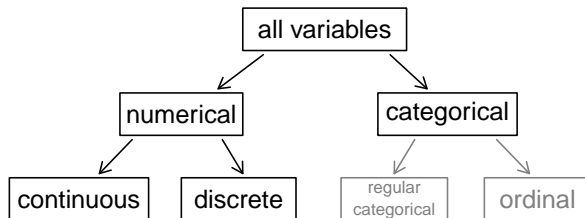
References

- ① R. Bernard. *Fundamentals of Biostatistics*. Ed.: Thompson. Chapter 10
- ② D. Díez, M Cetinkaya-Rundel and CD Barr. *OpenIntro Statistics*. Chapter 6.
- ③ J. Oakley. *MAS113 Introduction to Probability and Statistics (Part 2): Data Science*. Chapters 8, 10.

Outline

- 1 Types of variables
- 2 (Revisiting) single proportion inference
- 3 Difference of two proportions
- 4 Chi-square tests for contingency tables
 - Goodness-of-fit test
 - Independence
- 5 Fisher's exact test

Types of variables



Fuente: [OpenIntro Statistics](#)

- Categorical data: the variable under study is not continuous but instead **may be divided into groups**, so called categories.
 - ▶ Blood type: A, B, AB, O
 - ▶ Gender: M/F
 - ▶ Age group: 18-24, 25-30, 31-35, etc.
 - ▶ Educational level: primary school, high school, college, etc.

Types of variables (cont.)

	gender	sleep	bedtime	countries	rating
1	male	5	12-2	13	3
2	female	7	10-12	7	2
3	female	5.5	12-2	1	4
4	female	7	12-2		2
5	female	3	12-2	1	3
6	female	3	12-2	9	4

- gender: categorical
- sleep: numerical, continuous
- bedtime categorical, ordinal
- countries: numerical, discrete
- rating: categorical, ordinal - could also be used as numerical

Two-way table

- Categorical data: they are normally represented in a **two-way table**¹ that counts the number of observations that fall into each group for two variables

	Survived	Died	Total
Control	11	39	50
Treatment	14	26	40
Total	25	65	90

¹Also known as *contingency* table

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Sampling distribution of \hat{p}

- Recall that we estimate a population proportion p as the sample proportion

$$\hat{p} = \frac{x}{n}$$

where x is the total number of successes and n is the sample size.

The sampling distribution for \hat{p} based on a sample of size n from a population with a true proportion p is nearly normal

$$\hat{p} \sim \mathcal{N} \left(p, \underbrace{\sqrt{\frac{p(1-p)}{n}}}_{\text{SE}^a} \right)$$

- 1 The sample's observations are independent, e.g. are from a simple random sample.
- 2 $np(1-p) \geq 5$

^aIf p is unknown (most cases), we use \hat{p} in the calculation of the standard error

Confidence interval for a proportion

- When \hat{p} can be modeled using a normal distribution, the **confidence interval** for p takes the form

$$\hat{p} \pm z_{1-\alpha/2} \times \text{SE} = \hat{p} \pm z_{1-\alpha/2} \sqrt{\frac{p(1-p)}{n}}$$

Example

We are given that $n = 670$, $\hat{p} = 0.85$. Which of the below is the correct calculation of the 95 % confidence interval?

- (a) $0.85 \pm 1.96 \times \sqrt{\frac{0.85 \times 0.15}{670}}$
- (b) $0.85 \pm 1.65 \times \sqrt{\frac{0.85 \times 0.15}{670}}$
- (c) $0.85 \pm 1.96 \times \frac{0.85 \times 0.15}{\sqrt{670}}$
- (d) $571 \pm 1.96 \times \sqrt{\frac{571 \times 99}{670}}$

Choosing a sample size

Example

We are given that $n = 670$, $\hat{p} = 0.85$. How big a sample is required to ensure the margin of error is smaller than 0.01 using a 95 % confidence level?

$$\begin{aligned}1.96 \times \sqrt{\frac{0.85 \times 0.15}{n}} &\leq 0.01 \\1.96^2 \times \frac{0.85 \times 0.15}{n} &\leq 0.01^2 \\ \frac{1.96^2 \times 0.85 \times 0.15}{0.01^2} &\leq n \\ n &\geq 4898.04\end{aligned}$$

We need at least 4899 participants

Choosing a sample size

Example

A university newspaper is conducting a survey to determine what fraction of students support a \$200 per year increase in fees to pay for a new football stadium. How big a sample is required to ensure the margin of error is smaller than 0.04 using a 95 % confidence level?

- Use $\hat{p} = 0.5$ the most conservative estimate (worst case scenario), yielding the highest possible sample size

$$\begin{aligned} 1.96 \times \sqrt{\frac{0.5 \times (1 - 0.5)}{n}} &< 0.04 \\ 1.96^2 \times \frac{0.5 \times 0.5}{n} &< 0.04^2 \\ \frac{1.96^2 \times 0.5 \times 0.5}{0.04^2} &< n \\ 600.25 &< n \end{aligned}$$

We need 601 participants or more

Hypothesis testing for proportions

- To test the hypothesis $H_0 : p = p_0$ vs $H_1 : p \neq p_0$ with a significance level of α

Test ($p \neq p_0$)

Compute

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}} \sim \mathcal{N}(0, 1)$$

- if $|z| > z_{1-\alpha/2}$, then we reject H_0

Confidence interval

$$\hat{p} \pm z_{1-\alpha/2} \sqrt{p_0(1 - p_0)/n}$$

- if $100\%(1 - \alpha)$ CI **does not contain** p_0 , then we reject H_0

Hypothesis testing for proportions

Homework

Suppose that 8 % of college students are vegetarians. Determine if the following statements are true or false, and explain your reasoning.

- (a) A random sample of 125 college students where 12 % are vegetarians would be considered unusual.
- (b) A random sample of 250 college students where 12 % are vegetarians would be considered unusual.

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Difference of two proportions

Example

Consider an experiment for patients who underwent cardiopulmonary resuscitation (CPR) for a heart attack and were subsequently admitted to a hospital. These patients were randomly divided into a treatment group where they received a blood thinner (anticoagulant) or the control group where they did not receive a blood thinner. The outcome variable of interest was whether the patients survived for at least 24 hours.

	Survived	Died	Total
Control	11	39	50
Treatment	14	26	40
Total	25	65	90

Point estimation of the difference of two proportions

- We estimate the **difference** between two population proportion $p_1 - p_2$ using the sample proportions

$$\hat{p}_1 - \hat{p}_2$$

based on sample sizes n_1, n_2

Sampling distribution of the difference of two proportions

- The sampling distribution for $\hat{p}_1 - \hat{p}_2$ is nearly normal

$$\hat{p}_1 - \hat{p}_2 \sim \mathcal{N} \left(p_1 - p_2, \underbrace{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}}_{SE^2} \right)$$

when

- 1 **Independence:** within groups and between groups (satisfied if the data come from two independent random samples or if the data come from a randomized experiment)
- 2 **Success-failure:** At least 10 observed successes and 10 observed failures in the two groups

²If p_1, p_2 are unknown (most cases), we use \hat{p}_1, \hat{p}_2 in the calculation of the standard error

Confidence interval for $p_1 - p_2$

- When $\hat{p}_1 - \hat{p}_2$ can be modeled using a normal distribution, the **confidence interval** for $p_1 - p_2$ takes the form

$$\hat{p}_1 - \hat{p}_2 \pm z_{1-\alpha/2} \times \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

Example

Calculate a 90 % confidence interval of the difference for the survival rates in the CPR study.

Example solution

- We first calculate the sample proportion difference

$$\hat{p}_1 - \hat{p}_2 = \frac{14}{40} - \frac{11}{50} = 0.35 - 0.22 = 0.13$$

- Then we calculate the standard error

$$SE \approx \sqrt{\frac{0.35(1 - 0.35)}{40} + \frac{0.22(1 - 0.22)}{50}} = 0.095$$

- For a 90 % confidence interval we use $z_{1-\alpha/2} = 1.65$, therefore

$$CI_{90\%} = 0.13 \pm 1.65 \times 0.095 \rightarrow (-0.027, 0.287)$$

Hypothesis testing for the difference of two proportions

- We would like to test the hypothesis

$$H_0 : p_1 = p_2 \quad \text{vs} \quad H_1 : p_1 \neq p_2$$

with a significance level of α

- Or equivalently,

$$H_0 : p_1 - p_2 = 0 \quad \text{vs} \quad H_1 : p_1 - p_2 \neq 0$$

with a significance level of α

Hypothesis testing for the difference of two proportions

- So we define the statistic

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\frac{\hat{p}_{\text{pooled}}(1-\hat{p}_{\text{pooled}})}{n_1} + \frac{\hat{p}_{\text{pooled}}(1-\hat{p}_{\text{pooled}})}{n_2}}} \sim \mathcal{N}(0, 1)$$

where \hat{p}_{pooled} is the **expected number of successes and failures across the entire study**, which is calculated as

$$\hat{p}_{\text{pooled}} = \frac{\# \text{successes}_1 + \# \text{successes}_2}{n_1 + n_2} = \frac{\hat{p}_1 n_1 + \hat{p}_2 n_2}{n_1 + n_2}$$

Example

Calculate \hat{p}_{pooled} in the CPR study.

Example solution

	Survived	Died	Total	\hat{p}
Control	11	39	50	0.220
Treatment	14	26	40	0.350
Total	25	65	90	0.278

- In this case

$$\hat{p}_{\text{pooled}} = \frac{11 + 14}{50 + 40} = \frac{25}{90} = 0.278$$

Hypothesis testing for the difference of two proportions

- To test the hypothesis

$$H_0 : p_1 = p_2 \quad \text{vs} \quad H_1 : p_1 \neq p_2$$

with a significance level of α

Test

Compute the statistic

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\frac{\hat{p}_{\text{pooled}}(1-\hat{p}_{\text{pooled}})}{n_1} + \frac{\hat{p}_{\text{pooled}}(1-\hat{p}_{\text{pooled}})}{n_2}}} \sim \mathcal{N}(0, 1)$$

- if $|z| > z_{1-\alpha/2}$, then we reject H_0
- if $|z| \leq z_{1-\alpha/2}$, then we fail to reject H_0

Hypothesis testing for the difference of two proportions

Example

Consider an experiment for patients who underwent cardiopulmonary resuscitation (CPR) for a heart attack and were subsequently admitted to a hospital. These patients were randomly divided into a treatment group where they received a blood thinner (anticoagulant) or the control group where they did not receive a blood thinner. The outcome variable of interest was whether the patients survived for at least 24 hours.

	Survived	Died	Total
Control	11	39	50
Treatment	14	26	40
Total	25	65	90

Is the blood thinner useful for a 5 % significance level?

Example solution

- We know $\hat{p}_1 = 0.35$, $\hat{p}_2 = 0.22$, $\hat{p}_{\text{pooled}} = 0.278$, $n_1 = 40$, $n_2 = 50$, so

$$z = \frac{0.35 - 0.22}{\sqrt{\frac{0.278(1-0.278)}{40} + \frac{0.278(1-0.278)}{50}}} = 1.367$$

- For a $\alpha = 0.05$ significance level, we have $z_{1-\alpha/2} = 1.96$
- Therefore

We fail to reject the H_0 at a significance level of 5 %

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One-way vs Two-way tables

- A one-way table describes counts for each outcome in a single variable.
 - ▶ Test: **goodness-of-fit**. *"Does the data fit a particular distribution?", "is there any inconsistency between the observed and the expected counts?"*

Outcome	Observed	Expected
1	53,222	52,612
2	52,118	52,612
3	52,465	52,612
4	52,338	52,612
5	52,244	52,612
6	53,285	52,612
Total	315,672	315,672

Zacariah Labby [experiment](#), rolling 12 dice 26,306 times

One-way vs Two-way tables

- A two-way table describes counts for combinations of outcomes for two variables.
 - ▶ Test: **independence**
 - ★ Row homogeneity: *"are proportions the same for every row at the different columns?"*

	Excellent	Very Good	Average	Poor	Terrible	Total
Restaurant <i>A</i>	146	70	33	24	25	298
Restaurant <i>B</i>	419	277	102	66	52	916

Ratings for two restaurants on Tripadvisor. *A* was ranked 187/1257, and *B* was ranked 116/1257

One-way vs Two-way tables

- A two-way table describes counts for combinations of outcomes for two variables.
 - ▶ Test: **independence**
 - ★ Independence: *"are two measurements somehow related?"*

Smoking status	exercise: regular	exercise: some/none	Total
Never	87	102	189
Occasional	12	7	19
Regular	9	8	17
Heavy	7	4	11
Total	115	121	236

Smoking habits vs exercise level. Is smoking status independent of exercise level?

Goodness-of-fit test

H_0 : The observed counts follow the same distribution as the expected counts

H_1 : The observed counts **do not** follow the same distribution as the expected counts

- Quantify how different the observed counts are from the expected counts

Outcome	Observed	Expected
1	53,222	52,612
2	52,118	52,612
3	52,465	52,612
4	52,338	52,612
5	52,244	52,612
6	53,285	52,612
Total	315,672	315,672

Goodness-of-fit test: chi-square statistic

- Quantify how different the observed counts are from the expected counts we will use the **chi-square statistic**

$$\chi^2 = \sum_{i=1}^k \frac{(O_i - E_i)^2}{E_i} \sim \chi_{k-1}^2$$

where

- ▶ O_i is the observed count in cell i
 - ▶ E_i is the expected count in cell i
 - ▶ k is the total number of cells
- Notice that this statistic can be written as $\chi^2 = \sum_{i=1}^k Z_i^2$, where

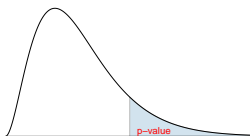
$$Z_i = \frac{O_i - E_i}{\sqrt{E_i}} = \frac{\text{point estimate} - \text{null value}}{\text{SE of point estimate}}$$

Calculating the chi-square statistic

Outcome	Observed	Expected	$\frac{(O_i - E_i)^2}{E_i}$
1	53,222	52,612	$\frac{(53,222 - 52,612)^2}{52,612} = 7.07$
2	52,118	52,612	$\frac{(52,118 - 52,612)^2}{52,612} = 4.64$
3	52,465	52,612	$\frac{(52,465 - 52,612)^2}{52,612} = 0.41$
4	52,338	52,612	$\frac{(52,338 - 52,612)^2}{52,612} = 1.43$
5	52,244	52,612	$\frac{(52,244 - 52,612)^2}{52,612} = 2.57$
6	53,285	52,612	$\frac{(53,285 - 52,612)^2}{52,612} = 8.61$
Total	315,672	315,672	24.73

Finding a p-value for a chi-square test

- We have calculated a test statistic of $\chi^2 = 24.67$
- We have $df = k - 1 = 6 - 1 = 5$ degrees of freedom
- Then we compute the p -value as



$$\begin{aligned} p &= P(\chi_5^2 > 24.67) \\ &= 1 - \text{chi2}(\text{df}=5) . \text{cdf}(24.67) \\ &= 0.00016 \end{aligned}$$

- Therefore, at a significance level of 5%

We **reject** H_0 , the data provide convincing evidence that the dice are biased!

- It turns out that the 1-6 axis is consistently shorter than the other two (2-5 and 3-4), thereby supporting the hypothesis that the faces with one and six pips are larger than the other faces.

Conditions for the chi-square goodness-of-fit test

- 1 **Independence:** each case that contributes a count to the table must be independent of all the other cases in the table.
- 2 **Sample size:** each particular scenario (i.e. cell) must have at least 5 **expected** cases.
- 3 **$df > 1$:** degrees of freedom must be greater than 1.

Failing to check conditions may unintentionally affect the test's error rates.

Independence test

	Excellent	Very Good	Average	Poor	Terrible	Total
Restaurant <i>A</i>	146	70	33	24	25	298
Restaurant <i>B</i>	419	277	102	66	52	916
Total	565	347	135	90	77	1214

Ratings for two restaurants on Tripadvisor. *A* was ranked 187/1257, and *B* was ranked 116/1257

H_0 : The probability of a particular rating is the same for either restaurant (ratings are independent of the restaurant)

H_1 : The probability of a particular rating **is not** the same for either restaurant (ratings depend on the restaurant)

- ▶ We might wonder whether the customer ratings are significantly different; if they are not, one could argue that the rankings are not meaningful.

Chi-square test of independence

- The test statistic is calculated as

$$\chi^2 = \sum_{i=1}^R \sum_{j=1}^C \frac{(O_{i,j} - E_{i,j})^2}{E_{i,j}} \sim \chi^2_{(R-1) \times (C-1)}$$

where

- ▶ $O_{i,j}$ is the observed count in row i , column j
- ▶ $E_{i,j}$ is the expected count in row i , column j
- ▶ R is the number of rows
- ▶ C is the number of columns
- ▶ $df = (R - 1) \times (C - 1)$

- The expected counts can be computed as

$$E_{i,j} = \frac{(\text{total in row } i) \times (\text{total in column } j)}{\text{table total}}$$

Expected counts

	Excellent	Very Good	Average	Poor	Terrible	Total
Restaurant <i>A</i>	146	70	33	24	25	298
Restaurant <i>B</i>	419	277	102	66	52	916
Total	565	347	135	90	77	1214

- The expected counts can be computed as

$$E_{i,j} = \frac{(\text{total in row } i) \times (\text{total in column } j)}{\text{table total}}$$

	Excellent	Very Good	Average	Poor	Terrible	Total
Restaurant <i>A</i>	$\frac{298 \times 565}{1214}$	$\frac{298 \times 347}{1214}$	$\frac{298 \times 135}{1214}$	$\frac{298 \times 90}{1214}$	$\frac{298 \times 77}{1214}$	298
Restaurant <i>B</i>	$\frac{916 \times 565}{1214}$	$\frac{916 \times 347}{1214}$	$\frac{916 \times 135}{1214}$	$\frac{916 \times 90}{1214}$	$\frac{916 \times 77}{1214}$	916
Total	565	347	135	90	77	1214

Calculating the test statistic

- Expected counts are shown in blue next to the observed counts

	Excellent	Very Good	Average	Poor	Terrible	Total
Restaurant A	146 138.7	70 85.2	33 33.1	24 22.1	25 18.9	298
Restaurant B	419 426.3	277 261.8	102 101.9	66 67.9	52 58.1	916
Total	565	347	135	90	77	1214

- We can now compute our observed test statistic:

$$\chi^2 = \frac{(146 - 138.7)^2}{138.7} + \frac{(70 - 85.2)^2}{85.2} + \dots + \frac{(52 - 58.1)^2}{58.1} = 6.92$$

and the degrees of freedom

$$\text{df} = (R - 1) \times (C - 1) = (2 - 1) \times (5 - 1) = 4$$

Calculating the p -value

- Having $\chi^2 = 6.92$ and $df = 4$, then

$$p = P(\chi_4^2 > 6.92) = 1 - \text{chi2}(\text{df}=4).cdf(6.92) = 0.14$$

- Therefore, at a significance level of 5 %

We fail to reject H_0

- Thus, there is no evidence to say that a customer is more likely to rate one restaurant higher than the other. This would suggest that the difference in rankings between the two restaurants (116 and 187) is not particularly meaningful

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Fisher's exact test

- **Nonparametric** test for testing independence
- Typically used only for 2×2 contingency table
- An alternative to Pearson's chi-squared test for independence
 - ✓ Fisher's exact test yields **exact p-values**
 - ✓ Typically used for tables with **small expected values**, where chi-squared test assumptions are not valid
 - ✗ Fisher's exact test may **not work for large sample sizes**, due to computational reasons

Fisher's exact test

	N	Y	total
A	18	2	20
B	11	9	20
total	29	11	40

- Assume the null hypothesis (independence) is true
 - Constrain the marginal counts to be as observed
 - What's the chance of getting this exact table?
- Fisher showed that the probability of obtaining this exact table was given by the *hypergeometric* distribution:

$$p = \frac{\binom{a+b}{a} \binom{c+d}{c}}{\binom{n}{a+c}} = \frac{\binom{a+b}{b} \binom{c+d}{d}}{\binom{n}{b+d}} = \frac{(a+b)! (c+d)! (a+c)! (b+d)!}{a! b! c! d! n!}$$

Fisher's exact test

	N	Y	total
A	18	2	20
B	11	9	20
total	29	11	40

- In Python, probability of obtaining this exact table can be calculated as

Option 1

```
from scipy.special import comb

a = 18; b = 2; c = 11; d = 9
n = a+b+c+d
p = comb(a+b,a) * comb(c+d,c) / comb(n,a+c)

>> 0.013804
```

Option 2

```
from scipy.stats import hypergeom

rv = hypergeom(M=a+b+c+d, n=a+c, N=a+b)
p = rv.pmf(18)

>> 0.013804
```

Fisher's exact test: p -value calculation

- 1 For all possible tables (with the observed marginal counts), calculate the relevant hypergeometric probability

20	0
9	11

 $\rightarrow 0.00007$

19	1
10	10

 $\rightarrow 0.00160$

18	2
11	9

 $\rightarrow \mathbf{0.01380}$

17	3
12	8

 $\rightarrow 0.06212$

16	4
13	7

 $\rightarrow 0.16246$

15	5
14	6

 $\rightarrow 0.25994$

14	6
15	5

 $\rightarrow 0.25994$

13	7
16	4

 $\rightarrow 0.16246$

12	8
17	3

 $\rightarrow 0.06212$

11	9
18	2

 $\rightarrow 0.01380$

10	10
19	1

 $\rightarrow 0.00160$

9	11
20	0

 $\rightarrow 0.00007$

Fisher's exact test: p -value calculation

- 2 Calculate the p -value as the sum of the probabilities for all tables having a probability equal to or smaller than that observed.

20	0
9	11

 → 0.00007

19	1
10	10

 → 0.00160

18	2
11	9

 → 0.01380

17	3
12	8

 → 0.06212

16	4
13	7

 → 0.16246

15	5
14	6

 → 0.25994

14	6
15	5

 → 0.25994

13	7
16	4

 → 0.16246

12	8
17	3

 → 0.06212

11	9
18	2

 → 0.01380

10	10
19	1

 → 0.00160

9	11
20	0

 → 0.00007

$$p = 2*(rv.pmf(20) + rv.pmf(19) + rv.pmf(18)) = 0.03095$$

Fisher's exact test: p -value calculation

	N	Y	total
A	18	2	20
B	11	9	20
total	29	11	40

- In Python, the p -value for the Fisher's exact test can be directly calculated as

```
_, p_value = fisher_exact( [ [18, 2] , [11,9] ] ) = 0.03095
```

Chi-square test vs Fisher's exact test

For the above table, calculate the chi-square test and compare its p -value with respect to the Fisher's exact test.

```
SOL: chi2_contingency( [ [18, 2] , [11,9] ] ) = 0.03361
```