

# The Fourier Transform

## DD2423 Image Analysis and Computer Vision

Mårten Björkman

Robotics, Perception and Learning Division  
School of Electrical Engineering and Computer Science

November 8, 2021

- Continuous

$$\hat{f}(\omega) = \int_{x \in \mathbb{R}^n} f(x) e^{-i\omega^T x} dx$$

- Discrete

$$\hat{f}(u) = \frac{1}{\sqrt{M}^n} \sum_{x \in I^n} f(x) e^{-\frac{2\pi i u^T x}{M}}$$

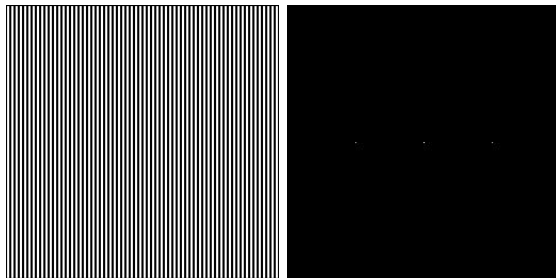
- Frequency variables are related (in 1D) by

$$\omega = \frac{2\pi u}{M}$$

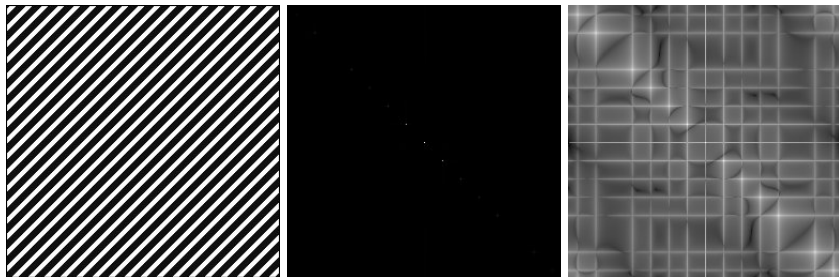
- Note:  $u$  assumes values  $0 \dots M-1 \Rightarrow \omega \in [0, 2\pi)$ .
- By periodic extension, we can map this integral to  $[-\pi, \pi)$ .

# More insight (2 pixel wide stripes)

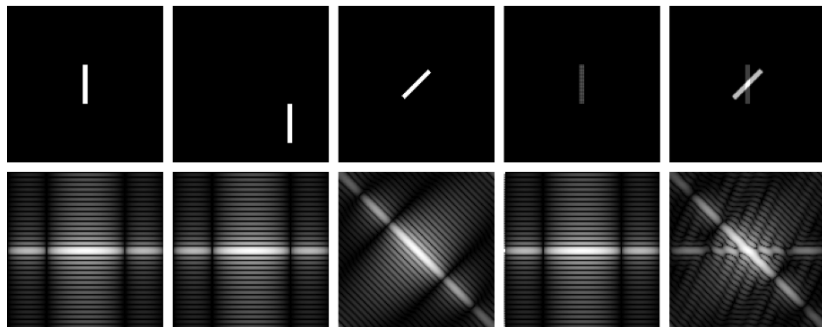
- The maximum frequency which can be represented in the spatial domain are one pixel wide stripes (period=2):  $\omega_{max} = 2\pi\frac{1}{2} = \pi$
- So, 2 pixel wide stripes (period=4) give  $\omega = 2\pi\frac{1}{4} = \frac{1}{2}\omega_{max}$
- Plotting magnitude of the Fourier transform:
  - Two points are halfway between the center and the edge of the image, i.e. the represented frequency is half of the maximum.
  - One point in the middle shows the DC-value (image mean).



- If you take the logarithm of the Fourier transform, you see many minor frequencies. Since an image can only be represented by square pixels, discretization noise appear along diagonals.



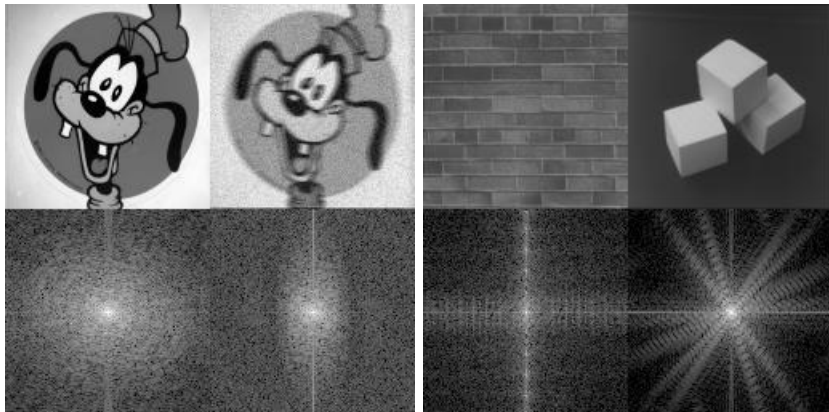
# Example images and Fourier transforms



## Observations:

- Translation only affects phase (not shown), not magnitude (shown)
- A rotation in one domain becomes a rotation in the other domain
- The Fourier transform is a linear operation

# Example images and Fourier transforms



Given a simple 4-pixel “image”  $f(m) = [3, 2, 2, 1]$ , what is its Fourier Transform  $\hat{f}(u)$ ?

Given a simple 4-pixel “image”  $f(m) = [3, 2, 2, 1]$ , what is its Fourier Transform  $\hat{f}(u)$ ?

$$\hat{f}(u) = \frac{1}{\sqrt{M}} \sum_m f(m) e^{-\frac{2\pi i u m}{M}}$$

$$\hat{f}(0) = \frac{1}{2} \sum_m f(m) (1)^m = \frac{1}{2} (3 + 2 + 2 + 1) = 4$$

$$\hat{f}(1) = \frac{1}{2} \sum_m f(m) (-i)^m = \frac{1}{2} (3 + 2(-i) - 2 + 1(i)) = \frac{1}{2} (1 - i)$$

$$\hat{f}(2) = \frac{1}{2} \sum_m f(m) (-1)^m = \frac{1}{2} (3 - 2 + 2 - 1) = 1$$

$$\hat{f}(3) = \frac{1}{2} \sum_m f(m) (i)^m = \frac{1}{2} (3 + 2(i) - 2 + 1(-i)) = \frac{1}{2} (1 + i)$$



$$\hat{f}(u, v) = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) e^{-2\pi i (\frac{mu}{M} + \frac{nv}{N})} \quad (1)$$

$$\hat{f}(u, v) = \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} \left( \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(m, n) e^{-2\pi i \frac{nv}{N}} \right) e^{-2\pi i \frac{mu}{M}}$$

$$f(m, n) = \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} \left( \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \hat{f}(u, v) e^{2\pi i \frac{nv}{N}} \right) e^{2\pi i \frac{mu}{M}}$$

2D DFT can be implemented as a series of 1D DFTs along each column, followed by 1D DFTs along each row.

$$\mathcal{F}[a f_1(m, n) + b f_2(m, n)] = a \hat{f}_1(u, v) + b \hat{f}_2(u, v)$$
$$a f_1(m, n) + b f_2(m, n) = \mathcal{F}^{-1} \left[ a \hat{f}_1(u, v) + b \hat{f}_2(u, v) \right]$$

You can add two functions (images) or rescale a function, either before or after computing the Fourier transform. It leads to the same result.

- If the original function is multiplied with an exponential like the one below and transformed, it will result in a shift of the origin of the frequency plane to point  $(u_0, v_0)$ .

$$\mathcal{F} \left[ f(m, n) e^{2\pi i \left( \frac{mu_0}{M} + \frac{nv_0}{N} \right)} \right] = \hat{f}(u - u_0, v - v_0)$$

- For  $(u_0, v_0) = (M/2, N/2)$

$$e^{2\pi i \left( \frac{mu_0}{M} + \frac{nv_0}{N} \right)} = e^{\pi i(m+n)} = (-1)^{m+n}$$

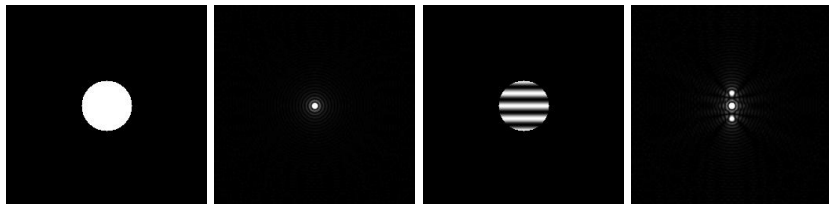
and

$$f(m, n)(-1)^{m+n} \iff \hat{f}(u - M/2, v - N/2)$$

- Conclusion: the origin of the Fourier transform can be moved to the center by multiplying the original function by  $(-1)^{m+n}$ .

# Property III - Modulation/Frequency translation

From left: Original image, magnitude of the Fourier spectrum, original multiplied by  $1 + \cos \omega y$  at a relative frequency of 16, magnitude of the Fourier spectrum.



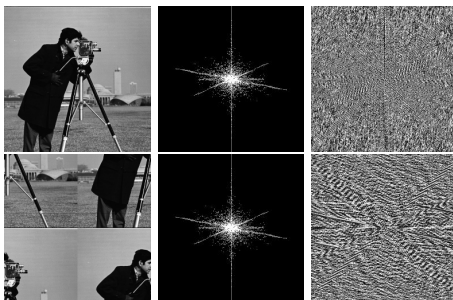
$$\text{Note: } 1 + \cos \omega y = 1 + \frac{1}{2} e^{i\omega y} + \frac{1}{2} e^{-i\omega y}.$$

# Property IV - Translation

- If the image is moved, the Fourier spectrum undergoes a phase shift, but magnitude of the spectrum remains the same.

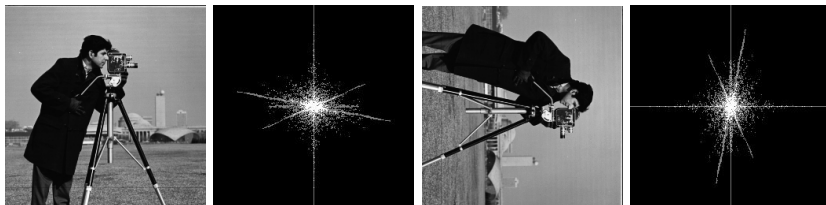
$$\mathcal{F} [f(m - m_0, n - n_0)] = \hat{f}(u, v) e^{-2\pi i (\frac{m_0 u}{M} + \frac{n_0 v}{N})}$$

$$|\hat{f}(u, v) e^{-2\pi i (\frac{m_0 u}{M} + \frac{n_0 v}{N})}| = |\hat{f}(u, v)|$$



# Property V - Rotation

Rotation of the original image rotates  $\hat{f}$  by the same angle.



Exercise: Introduce polar coordinates and perform direct substitution.

$$A = \begin{pmatrix} S_1 & & \\ & \ddots & \\ & & S_n \end{pmatrix} \text{ (diagonal)}$$

$$g(x) = f(S_1 x_1, \dots, S_n x_n)$$

$$\hat{g}(\omega) = \frac{1}{|S_1 \dots S_n|} \hat{f}\left(\frac{\omega_1}{S_1}, \dots, \frac{\omega_n}{S_n}\right)$$

Conclusion: compression in spatial domain is same as expansion in Fourier domain (and vice versa).

- The DFT and its inverse are periodic with period  $N$ , for an  $N \times N$  image. This means:

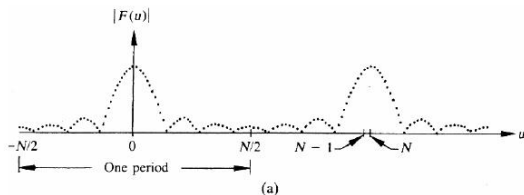
$$\hat{f}(u, v) = \hat{f}(u + N, v) = \hat{f}(u, v + N) = \hat{f}(u + N, v + N)$$

- This property defines the implied symmetry in the Fourier spectrum as well as that  $\hat{f}$  repeats itself infinitely.
- However, only one period is enough to reconstruct the original function  $f$ .



# Property VIII - Conjugate Symmetry

- The Fourier transform satisfies  $\hat{f}(u, v) = \hat{f}^*(-u, -v)$  and  $|\hat{f}(u, v)| = |\hat{f}(-u, -v)|$ .
- With periodicity and above, we have that  $\hat{f}$  has period  $N$  and is (conjugate) symmetric around the origin.

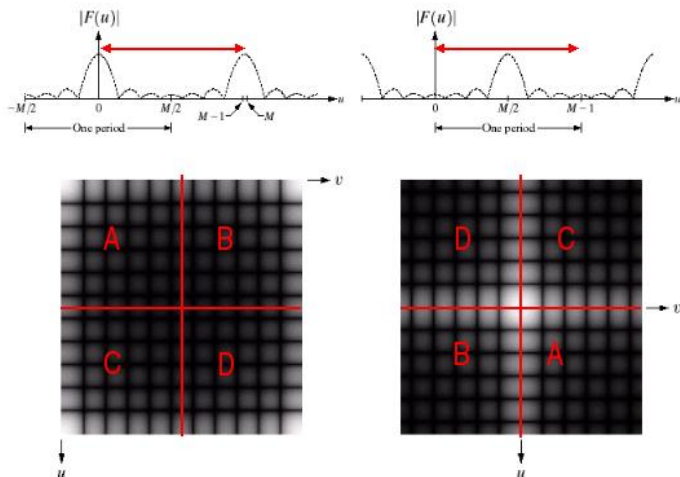


- Thus we don't need  $2N^2$  ( $N^2$  real and  $N^2$  imaginary) values to represent a  $N \times N$  image in Fourier domain, but  $N^2$  if we exploit the symmetry.

a b  
c d

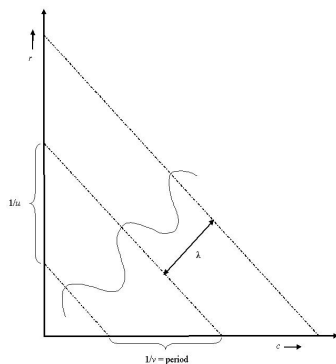
**FIGURE 4.34**

(a) Fourier spectrum showing back-to-back half periods in the interval  $[0, M - 1]$ .  
 (b) Shifted spectrum showing a full period in the same interval.  
 (c) Fourier spectrum of an image, showing the same back-to-back properties as (a), but in two dimensions.  
 (d) Centered Fourier spectrum.



# Wavelength in two dimensions

- The wavelength of the sinusoid is:  $\lambda = \frac{1}{\sqrt{u^2 + v^2}}$ , where  $(u, v)$  are the frequencies along  $(r, c)$  and the periods are  $1/u$  and  $1/v$ .



Multiplication in the spatial domain is same as convolution in the Fourier domain.

$$\mathcal{F}(hf) = \mathcal{F}(h) * \mathcal{F}(f)$$

Exercise: Prove it!

- A linear, shift invariant system (such as a filter) is completely specified by it's response to an impulse, which is called the **impulse response**.
- The **transfer function**  $H$  is the Fourier transform of the impulse response.
- Using the convolution theorem to describe the effects of the system:

$$g(x, y) = h(x, y) * f(x, y)$$

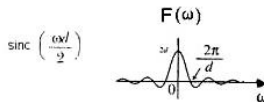
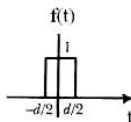
$$G(u, v) = H(u, v) \cdot F(u, v)$$

- Convolution with an impulse function

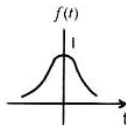
$$h(x, y) * \delta(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x_0, y_0) \delta(x - x_0, y - y_0) dx_0 dy_0 = h(x, y)$$

results in a “copy” of  $h(x, y)$  to the location of the impulse

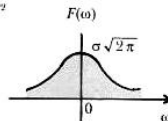
# Transfer function examples



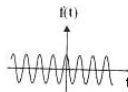
$$e^{-t^2/2\sigma^2}$$



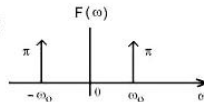
$$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$$



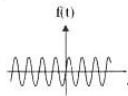
$$\cos \omega_0 t$$



$$\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

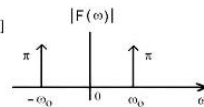


$$\sin \omega_0 t$$



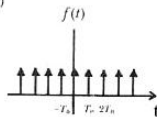
$$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

$$\theta(\omega) = -\pi/2$$

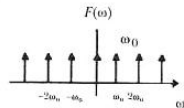


# Transfer function examples

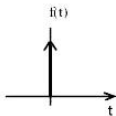
$$\delta T_s(t)$$



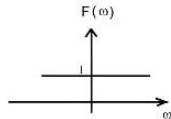
$$\omega_0 \delta \omega_0(\omega)$$



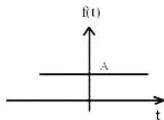
$$\delta(t)$$



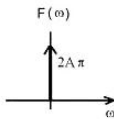
$$1$$



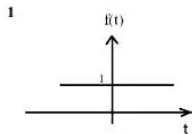
$$A$$



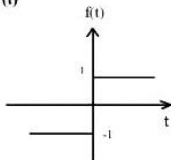
$$2A\pi\delta(\omega)$$



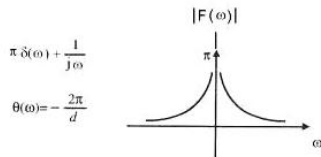
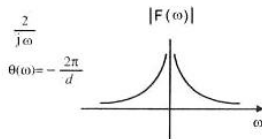
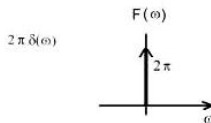
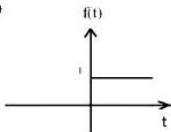
# Transfer function examples



**Sgn (t)**



**u(t)**





# Example application: Image registration

Assume you have two noisy satellite (oh...) images that are shifted with respect to each other, can you determine how large the shift is?



A shift in spatial domain is a modulation in frequency domain:

$$f_2(x, y) = f_1(x - dx, y - dy) \Rightarrow F_2(\omega_x, \omega_y) = e^{i(\omega_x dx + \omega_y dy)} F_1(\omega_x, \omega_y)$$

We can get the modulation

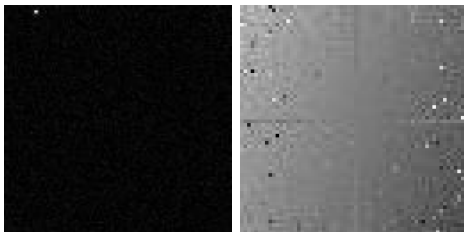
$$R(\omega_x, \omega_y) = \frac{F_2^*(\omega_x, \omega_y) F_1(\omega_x, \omega_y)}{|F_2^*(\omega_x, \omega_y) F_1(\omega_x, \omega_y)|} = e^{-i(\omega_x dx + \omega_y dy)}$$

In spatial domain this just becomes a Dirac-function:

$$\mathcal{F}^{-1}(R(\omega_x, \omega_y)) = \delta(dx, dy)$$

# Example application: Image registration

In this case, we can detect the peak of  $\mathcal{F}^{-1}(R(\omega_x, \omega_y))$  at location  $(\hat{dx}, \hat{dy}) = (17, 3)$  (left image). However, we want sub-pixel accuracy.

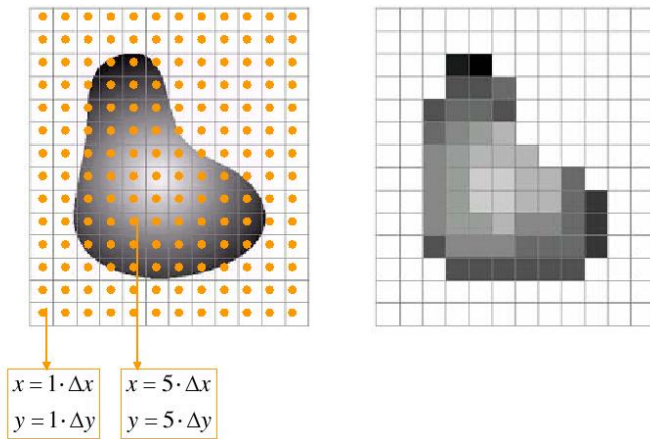


- We can take the estimate  $(\hat{dx}, \hat{dy})$  and shift  $f_2(x, y)$  towards  $f_1(x, y)$ , and then repeat the process.
- The phase angle of  $R(\omega_x, \omega_y)$  is  $\angle R(\omega_x, \omega_y) = -(\omega_x dx + \omega_y dy)$ , which is just a linear function (right image, excluding the highest frequencies).
- If we use least-squares minimization we can find this function and the updated parameters  $(\hat{dx}, \hat{dy}) = (17.49, 3.24)$ . Ground truth:  $(17.50, 3.25)$

Our signals (images) are not in a continuous domain, but in a discrete.

- A continuous function  $f(x, y)$  (an image) can be sampled using a discrete grid of sampling points.
- The image is sampled at points  $(j\Delta x, k\Delta y)$ , with  $j = 1, \dots, M$  and  $k = 1, \dots, N$ , where  $(M, N)$  is the size of the image in pixels.
- Here  $\Delta x$  and  $\Delta y$  are called the sampling interval.

# Sampling

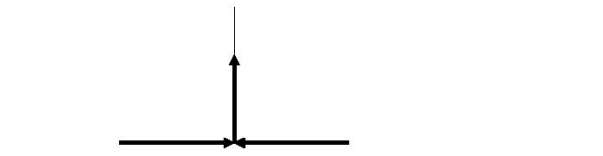


Dirac (continuous domain) and Kronecker (discrete) delta functions.

*Ideal impulse defined using **Dirac distribution***

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y) dx dy = 1$$

*and  $\delta(x, y) = 0$  for all  $x, y \neq 0$*



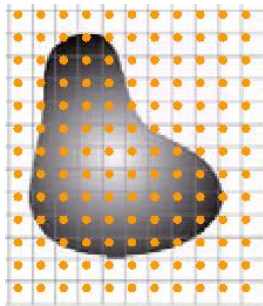
- The 'sifting property' of the dirac function provides a value of the function  $f(x, y)$  at point  $(a, b)$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x - a, y - b) dx dy = f(a, b)$$

- The sifting property can be used to describe the sampling process of a continuous function  $f(x, y)$ .

- The ideal sampling  $s(x, y)$  in the regular grid can be represented using a collection of Dirac functions  $\delta$ .

$$s(x, y) = \sum_{j=1}^M \sum_{k=1}^N \delta(x - j\Delta x, y - k\Delta y)$$

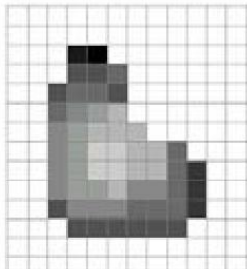


# Sampling

- The sampled image  $f_s(x, y)$  is the product of the continuous image  $f(x, y)$  and the sampling function  $s(x, y)$ .

$$\begin{aligned} f_s(x, y) &= f(x, y)s(x, y) = f(x, y) \sum_{j=1}^M \sum_{k=1}^N \delta(x - j\Delta x, y - k\Delta y) = \\ &= \sum_{j=1}^M \sum_{k=1}^N f(j\Delta x, k\Delta y) \delta(x - j\Delta x, y - k\Delta y) \end{aligned}$$

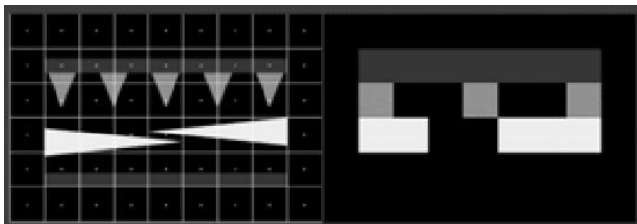
- Note: Sampling is not a convolution, but a product  $f(x, y)s(x, y)$ .



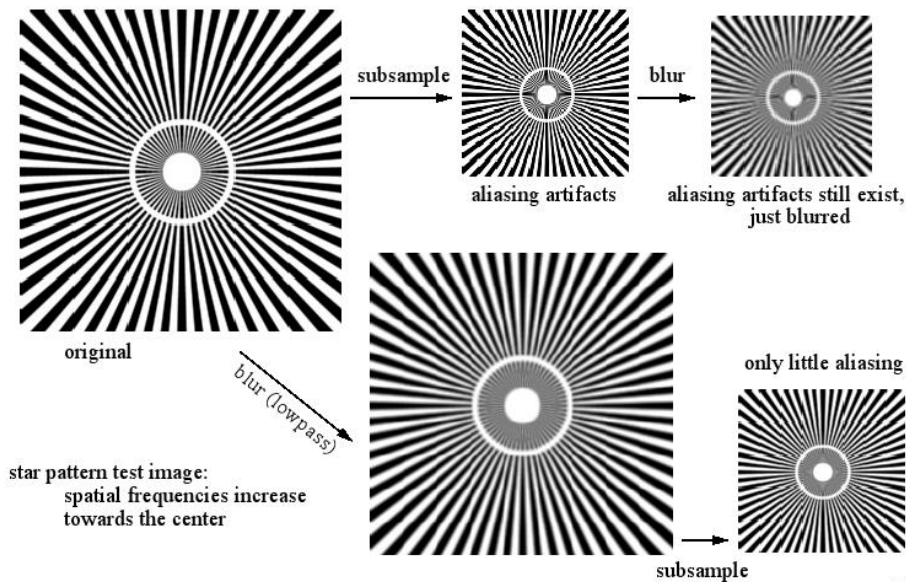


# Aliasing and anti-aliasing

- Sources of error during sampling:
  - Intensity quantization (not enough intensity resolution).
  - Spatial aliasing (not enough spatial resolution).
  - Temporal aliasing (not enough temporal resolution).
- Artifacts produced by under-sampling or poor reconstruction.  
Fine structures disappear and distort coarser structure.
- Anti-aliasing: sample at higher rate or prefiltering.  
Tools: Fourier transform, convolution and the Sampling Theorem.



# Example: Aliasing



## Low pass filtering (blurring) important!

### Nine survivors, 1 body removed from Cuban plane in Gulf of Mexico

Nine survivors and one body have been pulled from the wreckage of a Cuban airplane by a merchant ship in the Gulf of Mexico, about 60 miles (96 kilometers) off the western tip of Cuba, the U.S. Coast Guard said. The rescue at 1:45 p.m. Tuesday came a few hours after officials in Havana, Cuba, reported the plane hijacked.

#### FULL STORY

- **Play related video:** [The sequence of events leading to the rescue](#)
- [Injured Cuban flown to Florida will be allowed to seek asylum](#)
- Major features of Antonov An-2 planes
- History: [Leaving Cuba by air](#)
- Message Board: [U.S./Cuba relations](#)
- Message Board: [Air safety](#)

original

subsample

### Nine survivors, 1 body removed from Cuban plane in Gulf of Mexico

Nine survivors and one body have been pulled from the wreckage of a Cuban airplane by a merchant ship in the Gulf of Mexico, about 60 miles (96 kilometers) off the western tip of Cuba, the U.S. Coast Guard said. The rescue at 1:45 p.m. Tuesday came a few hours after officials in Havana, Cuba, reported the plane hijacked.

#### FULL STORY

- **Play related video:** [The sequence of events leading to the rescue](#)
- [Injured Cuban flown to Florida will be allowed to seek asylum](#)
- Major features of Antonov An-2 planes
- History: [Leaving Cuba by air](#)
- Message Board: [U.S./Cuba relations](#)
- Message Board: [Air safety](#)

aliased text

blur

### Nine survivors, 1 body removed from Cuban plane in Gulf of Mexico

Nine survivors and one body have been pulled from the wreckage of a Cuban airplane by a merchant ship in the Gulf of Mexico, about 60 miles (96 kilometers) off the western tip of Cuba, the U.S. Coast Guard said. The rescue at 1:45 p.m. Tuesday came a few hours after officials in Havana, Cuba, reported the plane hijacked.

#### FULL STORY

- **Play related video:** [The sequence of events leading to the rescue](#)
- [Injured Cuban flown to Florida will be allowed to seek asylum](#)
- Major features of Antonov An-2 planes
- History: [Leaving Cuba by air](#)
- Message Board: [U.S./Cuba relations](#)
- Message Board: [Air safety](#)

blurred, aliased text

blur, then subsample

### Nine survivors, 1 body removed from Cuban plane in Gulf of Mexico

Nine survivors and one body have been pulled from the wreckage of a Cuban airplane by a merchant ship in the Gulf of Mexico, about 60 miles (96 kilometers) off the western tip of Cuba, the U.S. Coast Guard said. The rescue at 1:45 p.m. Tuesday came a few hours after officials in Havana, Cuba, reported the plane hijacked.

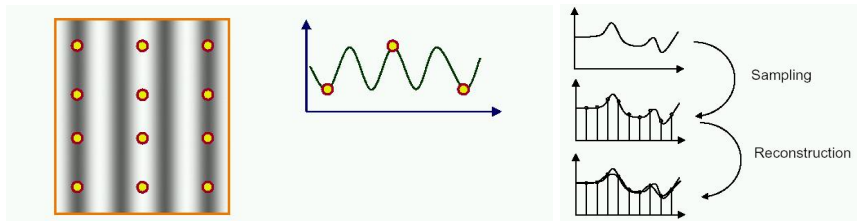
#### FULL STORY

- **Play related video:** [The sequence of events leading to the rescue](#)
- [Injured Cuban flown to Florida will be allowed to seek asylum](#)
- Major features of Antonov An-2 planes
- History: [Leaving Cuba by air](#)
- Message Board: [U.S./Cuba relations](#)
- Message Board: [Air safety](#)

looks more pleasing

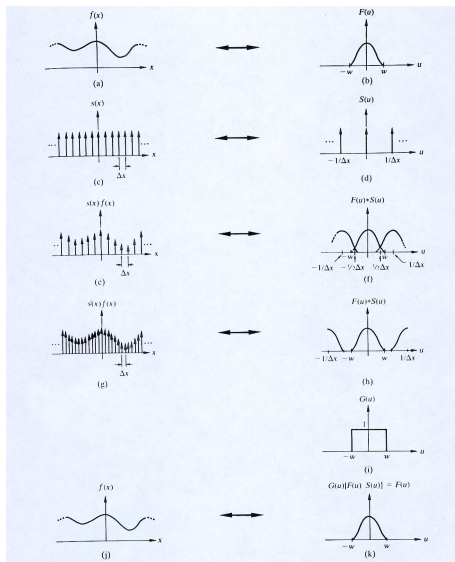
# Sampling Theorem

- How many samples are required to describe the given signal without loss of information?
- What signal can be reconstructed given the current sampling rate?



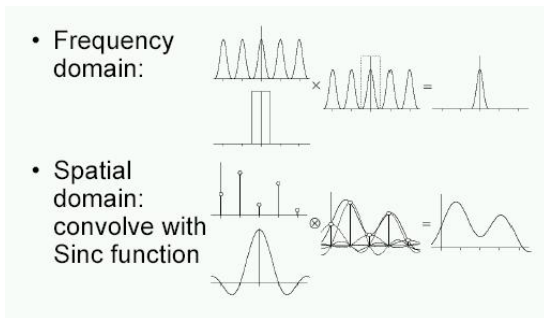
- A signal is band limited if its highest frequency is bounded. This frequency is called the **bandwidth**.
- The sine/cosine component of the highest frequency determines the highest “frequency content” of the signal.
- If the signal is sampled at a rate equal or greater to than twice its highest frequency, the original signal can be completely recovered from its samples (Shannon).
- The minimum sampling rate for a band limited function is called **Nyquist rate**.

# Reconstruction



# Reconstruction

- For reconstruction, we need to convolve with a sinc function.
  - It is the Fourier transform of the box function.
  - It has infinite support.
- May be approximated by a Gaussian, cubic or even “tent” function.



- If the signal is undersampled, aliasing occurs.
- Prevent aliasing by:
  - increasing sampling rate, or [infeasible]
  - decreasing highest frequency before sampling. [blurring]



# Summary of good questions

- How do you interpret a point in the Fourier domain in the spatial domain?
- How do you apply a discrete Fourier transform?
- What happens to the Fourier transform, if you translate an image?
- What happens to the Fourier transform, if you rotate an image?
- In what sense is the Fourier transform symmetric?
- How can a Dirac function be used to model sampling?
- What does the Sampling Theorem mean in practice?
- What can you do to get rid of aliasing in the sampling process?

- Gonzalez and Woods: Chapter 4.6-4.7
- Szeliski: Chapter 3.4