## SMALL VOLUME BODIES OF CONSTANT WIDTH

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ABSTRACT. For every large enough n, we explicitly construct a body of constant width 2 that has volume less than  $0.9^n \text{Vol}(\mathbb{B}^n)$ , where  $\mathbb{B}^n$  is the unit ball in  $\mathbb{R}^n$ . This answers a question of O. Schramm.

### 1. Introduction

A convex body K in the n-dimensional Euclidean space  $\mathbb{R}^n$  has constant width w if the length of the projection of K to any line is equal to w.

A body K is said to have effective radius r if  $Vol(K) = Vol(r\mathbb{B}^n)$ , where  $\mathbb{B}^n$  is the unit ball in  $\mathbb{R}^n$ . By the Urysohn's inequality (see, e.g., [2, (7.21), p. 382]), any body of constant width 2 has effective radius at most 1.

Let  $r_n$  denote the smallest effective radius of a body of constant width 2 in  $\mathbb{R}^n$ . Schramm [3] established a non-trivial lower bound  $r_n \geq \sqrt{3 + \frac{2}{n+1}} - 1$  and asked (see also the survey of Kalai [1, Problem 3.4]) if there exists  $\varepsilon > 0$  such that  $r_n \leq 1 - \varepsilon$  for all  $n \geq 2$ . We answer this question in the affirmative.

**Theorem 1.**  $r_n < 0.9$  for all sufficiently large n.

We prove the theorem by providing an explicit construction of a body M of constant width 2 with  $Vol(M) < 0.9^n Vol(\mathbb{B}^n)$ .

## 2. Preliminaries and the construction of M

For every positive integer n, let  $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$  be the unit sphere in  $\mathbb{R}^n$ . Let  $\omega_n$  and  $\Omega_n = \frac{\omega_n}{n}$  be respectively the surface area of  $\mathbb{S}^{n-1}$  and the volume of  $\mathbb{B}^n$ . Define  $\mathbb{R}^n_+$  to be the positive orthant, i.e.,

$$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \ge 0 \text{ for all } i \in \{1, \dots, n\}\}.$$

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Define  $S = \mathbb{S}^{n-1} \cap \mathbb{R}^n_+$  and let

$$L := \left(\sqrt{2}S\right) \cup \left(\left(\sqrt{2} - 2\right)S\right)$$
.

Now we define

(1) 
$$M := \bigcap_{x \in L} (x + 2\mathbb{B}^n).$$

Before we prove that M is a body of constant width 2 (Claim 2), we prove an auxiliary Claim 1. Note that any vector  $v \in \mathbb{R}^n$  has a unique representation as  $v = v_+ - v_-$  with  $v_{\pm} \in \mathbb{R}^n_+$  and  $v_+ \cdot v_- = 0$ . More precisely, for a vector  $v = (v_1, \ldots, v_n)$ , the *i*-th coordinates of the vectors  $v_+$  and  $v_-$  are  $\max\{v_i, 0\}$  and  $\max\{-v_i, 0\}$  respectively.

Define a set

$$A := \{(a, b) \in \mathbb{R}^2_+ : a^2 + (b + \sqrt{2})^2 \le 2^2\}.$$

Claim 1. If  $v \in M$ , then  $(|v_+|, |v_-|) \in A$ .

*Proof.* Define for every nonzero  $x \in \mathbb{R}^n_+$  the vector  $\widehat{x} = \frac{1}{|x|}x \in S$ . Claim 1 holds trivially for v = 0, so assume that  $v \in \mathbb{R}^n$  is a nonzero vector.

If  $v \in \mathbb{R}^n_+$ , then  $v = v_+$  and the distance from v to  $(\sqrt{2} - 2)\widehat{v} \in L$  is  $|v| + 2 - \sqrt{2}$ , and so  $|v| \le \sqrt{2}$  and  $(|v_+|, |v_-|) = (|v|, 0) \in A$ . Otherwise,  $\widehat{v}_-$  is well-defined, and the squared distance from v to  $\sqrt{2}\widehat{v}_- \in L$  is  $|v_+|^2 + (|v_-| + \sqrt{2})^2$ , so  $(|v_+|, |v_-|) \in A$ .

Claim 2. The set M, as defined in (1), is a body of constant width 2. Moreover  $M = \widetilde{M}$ , where

$$\widetilde{M} = \{ v - w : v, w \in \mathbb{R}^n_+, (|v|, |w|) \in A \}.$$

*Proof.* First we show that  $\widetilde{M}$  has diameter at most 2. Indeed, let  $v_1-w_1,v_2-w_2\in\widetilde{M}$ . Then

$$|(v_1 - w_1) - (v_2 - w_2)|^2 = |(v_1 + w_2) - (v_2 + w_1)|^2 \le |v_1 + w_2|^2 + |v_2 + w_1|^2$$
  
$$\le (|v_1| + |w_2|)^2 + (|v_2| + |w_1|)^2 =: d^2.$$

Now, let A' denote the set symmetric to A with respect to the line y = -x. Then d is the distance between  $(|v_1|, |w_1|) \in A$  and  $(-|w_2|, -|v_2|) \in A'$ . Both A and A' are contained in a Reuleaux triangle  $\mathcal{R}$  with vertices  $(\sqrt{2}, 0), (0, -\sqrt{2})$ , as depicted in Figure 1. So  $(|v_1|, |w_1|), (-|w_2|, -|v_2|) \in \mathcal{R}$ , and, since  $\mathcal{R}$  has diameter 2, we conclude that  $d^2 \leq 4$ . Hence the diameter of  $\widetilde{M}$  is at most 2.

By Claim 1,  $M \subseteq \widetilde{M}$ . On the other hand, since  $L \subseteq \widetilde{M}$  and  $\widetilde{M}$  has diameter at most 2, we have

$$\widetilde{M} \subseteq \bigcap_{x \in \widetilde{M}} (x + 2\mathbb{B}^n) \subseteq \bigcap_{x \in L} (x + 2\mathbb{B}^n) = M.$$

Thus  $\widetilde{M} = M$ .

Now we show that the width of  $\widetilde{M}$  in any direction  $\theta \in \mathbb{S}^{n-1}$  is at least 2. Write  $\theta = \theta_+ - \theta_-$ . Then  $|\theta_+|^2 + |\theta_-|^2 = 1$ . Since the width in the direction  $\theta$  is the same as in the direction

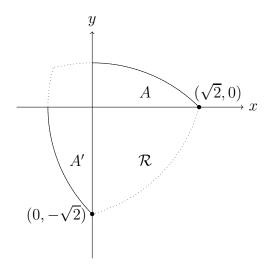


FIGURE 1. Sets A, A', and a Reuleaux triange  $\mathcal{R}$  that contains both A and A'

 $-\theta$ , we can assume without loss of generality that  $|\theta_-| \ge |\theta_+|$ , and so  $|\theta_-| \ge \frac{1}{\sqrt{2}}$ . Now, both vectors  $2\theta_+ - (2|\theta_-| - \sqrt{2})\widehat{\theta}_-$  and  $\sqrt{2}\widehat{\theta}_-$  are in  $\widetilde{M}$ , and the difference between the two vectors is  $2(\theta_+ - \theta_-) = 2\theta$ . Hence the width of  $\widetilde{M}$  in direction  $\theta$  is at least 2.

So 
$$\widetilde{M} = M$$
 is a convex body of constant width 2.

# 3. Estimate for Vol(M)

In order to prove the Theorem, we will prove that there is a positive  $\sigma < 0.9$  such that

(2) 
$$\operatorname{Vol}(M) \le (n+1)\sigma^n \Omega_n.$$

We will estimate the volume of M in each of the  $2^n$  coordinate orthants. We say that an orthant Q is a (k, n - k) orthant if Q consists of the points with exactly k positive and n - k negative coordinates.

If Q is the (n,0) orthant, we have  $\operatorname{Vol}(M \cap Q) = \frac{1}{2^n} \left(\sqrt{2}\right)^n \Omega_n$ , as  $M \cap Q = \sqrt{2}\mathbb{B}^n \cap \mathbb{R}^n_+$ . Similarly, if Q is the (0,n) orthant, we have  $\operatorname{Vol}(M \cap Q) = \frac{1}{2^n} \left(2 - \sqrt{2}\right)^n \Omega_n$ .

Let now Q be a (k, n) orthant with  $1 \le k \le n - 1$ . Then, by Claim 1,

$$Vol(M \cap Q) \le Vol(\{v \in Q : (|v_+|, |v_-|) \in A\}).$$

Let  $Q_+$  and  $Q_-$  be the positive orthants in  $\mathbb{R}^k$  and  $\mathbb{R}^{n-k}$  corresponding to the positive and the negative coordinates of Q, respectively. If we consider an infinitesimal rectangle  $R = [a, a + da] \times [b, b + db]$  with  $R \subset A$ , the set  $\{v \in Q : (|v_+|, |v_-|) \in R\}$  corresponds to the Cartesian product of the portions of the spherical shells  $\{v' \in Q_+ : |v'| \in [a, a + da]\} \times \{v'' \in Q_- : |v''| \in [b, b + db]\}$ , and the corresponding shell volumes in these orthants are  $\frac{\omega_k}{2^k} a^{k-1} da$  and  $\frac{\omega_{n-k}}{2^{n-k}} b^{n-k-1} db$ . Therefore,

$$Vol(M \cap Q) \le \iint_A \frac{\omega_k}{2^k} \frac{\omega_{n-k}}{2^{n-k}} a^{k-1} b^{n-k-1} \ da \ db = \frac{1}{2^n} k(n-k) \ \Omega_k \Omega_{n-k} \iint_A a^{k-1} b^{n-k-1} \ da \ db.$$

Since the number of (k, n-k) orthants is exactly  $\binom{n}{k}$ , we obtain the estimate

(3) 
$$\operatorname{Vol}(M) \le \frac{\Omega_n}{2^n} \left( (\sqrt{2})^n + (2 - \sqrt{2})^n + \sum_{k=1}^{n-1} k(n-k) \binom{n}{k} \frac{\Omega_k \Omega_{n-k}}{\Omega_n} \iint_A a^{k-1} b^{n-k-1} da db \right).$$

Now, let  $\alpha, \beta > 0$  be some real numbers such that the triangle  $T_{\alpha,\beta} := \{a, b \geq 0 : \frac{a}{\alpha} + \frac{b}{\beta} \leq 1\}$  contains the set A (see Figure 2). Note that the condition  $A \subseteq T_{\alpha,\beta}$  is equivalent to the statement that the distance from the line  $\frac{a}{\alpha} + \frac{b}{\beta} = 1$  to  $(0, -\sqrt{2})$  is at least 2, i.e.,  $\alpha(\beta + \sqrt{2}) \geq 2\sqrt{\alpha^2 + \beta^2}$ .

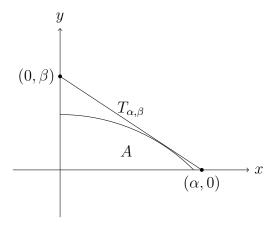


FIGURE 2. Triangle  $T_{\alpha,\beta}$  that contains A.

Then for all  $k \in \{1, \ldots, n-1\}$ , we have

$$\iint\limits_A a^{k-1}b^{n-k-1} \ da \ db \le \iint\limits_{T_{\alpha,\beta}} a^{k-1}b^{n-k-1} \ da \ db = \alpha^k\beta^{n-k} \iint\limits_{T_{1,1}} a^{k-1}b^{n-k-1} \ da \ db.$$

Now, note that  $k(n-k) \iint_{T_{1,1}} a^{k-1}b^{n-k-1} da db = \binom{n}{k}^{-1}$ , see Claim 3 in the Appendix for the proof, and so

$$k(n-k)\binom{n}{k}\frac{\Omega_k\Omega_{n-k}}{\Omega_n}\iint\limits_A a^{k-1}b^{n-k-1}\ da\ db \le \frac{\alpha^k\Omega_k\cdot\beta^{n-k}\Omega_{n-k}}{\Omega_n}.$$

Since  $\alpha \geq \sqrt{2}$  and  $\beta \geq 2 - \sqrt{2}$ , (3) implies

(4) 
$$\operatorname{Vol}(M) \leq \frac{\Omega_n}{2^n} \left( \sum_{k=0}^n \frac{\alpha^k \Omega_k \cdot \beta^{n-k} \Omega_{n-k}}{\Omega_n} \right).$$

Finally,  $\alpha \mathbb{B}^k \times \beta \mathbb{B}^{n-k} \subseteq \sqrt{\alpha^2 + \beta^2} \mathbb{B}^n$ , which implies  $\frac{\alpha^k \Omega_k \cdot \beta^{n-k} \Omega_{n-k}}{\Omega_n} \leq \left(\sqrt{\alpha^2 + \beta^2}\right)^n$ , and so for every  $(\alpha, \beta)$  with  $A \subseteq T_{\alpha, \beta}$ , we have

(5) 
$$r_n \le \frac{1}{2}(n+1)^{1/n} \sqrt{\alpha^2 + \beta^2}.$$

Let

$$s = \min\{\sqrt{\alpha^2 + \beta^2} : \alpha, \beta > 0, \ \alpha(\beta + \sqrt{2}) \ge 2\sqrt{\alpha^2 + \beta^2}\}.$$

Then

(6) 
$$r_n \le \frac{1}{2}(n+1)^{1/n}s.$$

A human verifiable proof of s < 1.8 is obtained by choosing  $\alpha = 1.5$  and  $\beta = 0.7\sqrt{2}$ . For such values we have  $\alpha^2 + \beta^2 = 3.23 < 3.24 = 1.8^2$ . On other hand,  $\alpha(\beta + \sqrt{2}) > 1.5 \cdot 1.7 \cdot \sqrt{2} > 3.6 \ge 2\sqrt{\alpha^2 + \beta^2}$ . The middle inequality is equivalent to  $1.7\sqrt{2} > 2.4$ , i.e.,  $2.89 \cdot 2 > 5.76$ .

So s < 1.8 and inequality (6) finishes the proof of the Theorem.

Remark 1. We will verify in the Appendix that  $\frac{1}{2}s$  is the least positive root of the equation  $8x^6 - 76x^4 + 54x^2 + 1 = 0$ , whose numerical value is 0.89071..., see Claim 4.

Remark 2. Theorem 1 implies that there exists  $\varepsilon > 0$  such that  $r_n \leq 1 - \varepsilon$  for all  $n \geq 2$ . Indeed, there is  $n_0$  such that for  $n > n_0$  we have  $r_n < 0.9$ . For every  $n \in \{2, \ldots, n_0\}$ , the body  $M_n$  that we constructed is a body of constant width 2 in  $\mathbb{R}^n$  different from  $\mathbb{B}^n$ . So by the equality part of Urysohn's inequality (see, e.g., [2, (7.21), p. 382]),  $\varepsilon_n := 1 - \left(\frac{\text{Vol}(M_n)}{\text{Vol}(\mathbb{B}^n)}\right)^{1/n} > 0$ . Therefore we can take  $\varepsilon = \min\{0.1, \varepsilon_2, \ldots, \varepsilon_{n_0}\} > 0$ .

## References

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#### Appendix

Claim 3. Let  $T_{1,1} = \{(x,y) \in \mathbb{R}^2_+ : x+y \leq 1\}$ . Then for all integers n and  $k \in \{1,\ldots,n-1\}$ , we have  $k(n-k) \iint_{T_{1,1}} a^{k-1}b^{n-k-1} da db = \binom{n}{k}^{-1}$ .

*Proof.* First,

$$\iint_{T_{1,1}} a^{k-1}b^{n-k-1} \ da \ db = \int_0^1 \int_0^{1-a} a^{k-1}b^{n-k-1} \ db \ da = \frac{1}{n-k} \int_0^1 a^{k-1}(1-a)^{n-k} \ da.$$

Now, for a variable x consider

$$P(x) = \sum_{k=1}^{n} {n-1 \choose k-1} \left( \int_{0}^{1} a^{k-1} (1-a)^{n-k} da \right) x^{k-1} = \int_{0}^{1} (1-a+ax)^{n-1} da$$
$$= \frac{x^{n}-1}{(x-1)n} = \sum_{k=1}^{n} \frac{1}{n} x^{k-1}.$$

Comparing the coefficients at  $x^{k-1}$  in P(x), we get  $\int_0^1 a^{k-1} (1-a)^{n-k} da = \frac{1}{n\binom{n-1}{k-1}}$ , and so

$$k(n-k) \iint_{T_{1,1}} a^{k-1}b^{n-k-1} da db = \frac{k}{n\binom{n-1}{k-1}} = \binom{n}{k}^{-1}.$$

Claim 4. Let  $s = \min\{\sqrt{\alpha^2 + \beta^2} : \alpha, \beta > 0, \alpha(\beta + \sqrt{2}) \ge 2\sqrt{\alpha^2 + \beta^2}\}$ . Then  $x = \frac{1}{2}s$  is the least positive root of  $P(x) := 8x^6 - 76x^4 + 54x^2 + 1 = 0$ , which is also the only root of P(x) on (0,1).

*Proof.* It follows from the definition of s that x is the least positive number for which there exist  $\alpha, \beta > 0$  such that  $\alpha^2 + \beta^2 = 4x^2$  and  $\alpha(\beta + \sqrt{2}) \ge 4x$ , or equivalently for which

(7) 
$$\max\{\alpha(\beta + \sqrt{2}) : \alpha, \beta > 0, \sqrt{\alpha^2 + \beta^2} = 4x^2\} \ge 4x.$$

Now, the function  $f(\beta) = \sqrt{4x^2 - \beta^2}(\beta + \sqrt{2})$  satisfies f'(0) = 2x > 0, f(2x) = 0, and is positive on [0, 2x). So the maximum of f is attained at a critical point on (0, 2x). The critical point satisfies the quadratic equation  $2\beta^2 + \sqrt{2}\beta - 4x^2 = 0$ , whose only positive root is

$$b = -\frac{1}{2\sqrt{2}} + \sqrt{\frac{1}{8} + 2x^2}.$$

The condition (7) is now equivalent to the inequality  $f(b) \ge 4x$ , or  $f^2(b) \ge 16x^2$ , or

$$(4x^2 - b^2)(b^2 + 2\sqrt{2}b + 2) \ge 16x^2.$$

Since  $b^2 = 2x^2 - \frac{1}{\sqrt{2}}b$ , we can rewrite this as

$$(2x^2 + \frac{1}{\sqrt{2}}b)(2x^2 + \frac{3}{\sqrt{2}}b + 2) \ge 16x^2,$$

or, equivalently,

$$4x^4 + 4x^2 + \left(4\sqrt{2}x^2 + \sqrt{2}\right)b + \frac{3}{2}b^2 \ge 16x^2.$$

Replacing  $b^2$  with  $2x^2 - \frac{1}{\sqrt{2}}b$  once again, we get

$$4x^{4} + \left(4\sqrt{2}x^{2} + \frac{1}{2\sqrt{2}}\right)b \ge 9x^{2},$$

$$4x^{4} + \left(4\sqrt{2}x^{2} + \frac{1}{2\sqrt{2}}\right)\left(\frac{-1}{2\sqrt{2}} + \sqrt{\frac{1}{8} + 2x^{2}}\right) \ge 9x^{2},$$

$$4x^{4} - 2x^{2} - \frac{1}{8} + \sqrt{\frac{1}{8} + 2x^{2}}\left(4\sqrt{2}x^{2} + \frac{1}{2\sqrt{2}}\right) \ge 9x^{2}.$$

The last inequality can be rewritten as

$$\sqrt{\frac{1}{8} + 2x^2} \left( 4\sqrt{2}x^2 + \frac{1}{2\sqrt{2}} \right) \ge \frac{1}{8} + 11x^2 - 4x^4.$$

Note that the right hand side is positive for  $x \in (0,1)$  and the argument after (6) implies x < 0.9. So the last inequality can be squared, and we are looking for the least positive x such that

$$\left(\frac{1}{8} + 2x^2\right) \left(4\sqrt{2}x^2 + \frac{1}{2\sqrt{2}}\right)^2 \ge \left(\frac{1}{8} + 11x^2 - 4x^4\right)^2, \text{ i.e.,}$$

$$\left(\frac{1}{8} + 2x^2\right) \left(32x^4 + 4x^2 + \frac{1}{8}\right) \ge 16x^8 + 121x^4 + \frac{1}{64} - 88x^6 - x^4 + \frac{11}{4}x^2, \text{ or }$$

$$64x^6 + 12x^4 + \frac{3}{4}x^2 + \frac{1}{64} \ge 16x^8 - 88x^6 + 120x^4 + \frac{11}{4}x^2 + \frac{1}{64},$$

which after canceling  $\frac{1}{64}$ , moving all terms to one side and dividing by  $2x^2$  turns into

$$P(x) = 8x^6 - 76x^4 + 54x^2 + 1 \le 0.$$

Note that P(0) = 1 > 0, P(1) = -13 < 0 and  $P(+\infty) = +\infty$ . So P has at least one root on (0,1) and at least one root on  $(1,+\infty)$ . Combining this with the Descartes' rule of signs, we see that P(x) has exactly two roots on  $(0,\infty)$ . Thus the least positive x with  $P(x) \le 0$  is the least positive root of P, which is also the only root of P on (0,1).

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