Approximation Algorithms for the traveling salesman problem”

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Differential approximation results for traveling salesman problem

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Résumé

Nous commençons par démontrer que les versions maximisation et minimisation du problème du voyageur de commerce sont approximables à rapport différentiel $1/2$. Nous présentons ensuite une $3/4$-approximation polynomiale du cas particulier à distances 1 et 2 ; ce résultat nous permet notamment de ramener le rapport standard connu pour la version maxisation de ce sous-problème de $5/7$ à $7/8$. Nous proposons enfin un résultat négatif : approximer le voyageur de commerce, à coût minimum comme maximum, à mieux que $3475/3476 + \epsilon$ est **NP**-difficile pour tout $\epsilon > 0$.

Mots-clé : algorithme d'approximation, rapport d'approximation, problème **NP**-complet, complexité, réduction, voyageur du commerce.

Differential approximation results for traveling salesman problem

Abstract

We prove that both minimum and maximum traveling salesman problems can be approximately solved, in polynomial time within approximation ratio bounded above by $1/2$. We next prove that, when dealing with edge-distances 1 and 2, both versions are approximable within $3/4$. Based upon this result, we then improve the standard approximation ratio known for maximum traveling salesman with distances 1 and 2 from $5/7$ to $7/8$. Finally, we prove that, for any $\epsilon > 0$, it is **NP**-hard to approximate both problems within better than $3475/3476 + \epsilon$.

Keywords: approximation algorithm, approximation ratio, **NP**-complete problem, complexity, reduction, traveling salesman.

1 Introduction

Given a complete graph on n vertices, denoted by K_n, with positive distances on its edges, the minimum traveling salesman problem (min_TSP) consists in minimizing the cost of a Hamiltonian cycle, the cost of such a cycle being the sum of the distances on its edges. The maximum traveling salesman problem (max_TSP) consists in maximizing the cost of a Hamiltonian cycle. Further special but very natural cases of TSP are the ones where edge-distances are defined using the ℓ_2 norm (Euclidean TSP), or where edge-distances verify triangle inequalities (metric TSP); an interesting sub-case of the metric TSP is the one in which edge-distances are only 1 or 2 (TSP12). Both min_ and max_TSP, even in their restricted versions mentioned just mentioned above, are famous **NP**-hard problems.

In general, **NP** optimization (**NPO**) problems are commonly defined as follows.

Definition 1. An **NPO** problem Π is as a four-tuple $(\mathcal{I}, \mathsf{S}, v_I, \text{opt})$ such that:

1. $\mathcal{I}$ is the set of instances of Π and it can be recognized in polynomial time;
2. given $I \in \mathcal{I}$ (let $|I|$ be the size of I), $\mathsf{S}(I)$ denotes the set of feasible solutions of I; moreover, there exists a polynomial P such that, for every $S \in \mathsf{S}(I)$ (let $|S|$ be the size of S), $|S| = O(P(|I|))$; furthermore, given any I and any S with $|S| = O(P(|I|))$, one can decide in polynomial time if $S \in \mathsf{S}(I)$;
3. given $I \in \mathcal{I}$ and $S \in \mathsf{S}(I)$, $v_I(S)$ denotes the value of S; v_I is integer, polynomially computable and is commonly called objective function;
4. $\text{opt} \in \{\max, \min\}$. ∎

Given an instance I of an **NPO** problem Π and a polynomial time approximation algorithm $\mathtt{A}$ feasibly solving Π, we will denote by $\omega(I)$, $\lambda_{\mathtt{A}}(I)$ and $\beta(I)$ the values of the worst solution of I, of the approximated one (provided by $\mathtt{A}$ when running on I), and the optimal one for I, respectively. There exist mainly two thought processes dealing with polynomial approximation. Commonly ([13]), the quality of an approximation algorithm for an **NP**-hard minimization (resp., maximization) problem Π is expressed by the ratio (called standard in what follows) $\rho_{\mathtt{A}}(I) = \lambda(I)/\beta(I)$, and the quantity $\rho_{\mathtt{A}} = \inf\{r : \rho_{\mathtt{A}}(I) < r, I \text{ instance of } \Pi\}$ (resp., $\rho_{\mathtt{A}} = \sup\{r : \rho_{\mathtt{A}}(I) > r, I \text{ instance of } \Pi\}$) constitutes the approximation ratio of $\mathtt{A}$ for Π. Recent works ([9, 8]), strongly inspired by [3] (see also [12, 23]), bring to the fore another approximation measure, as powerful as the traditional one (concerning the type, the diversity and the quantity of the results produced), the ratio (called differential in what follows) $\delta_{\mathtt{A}}(I) = (\omega(I) - \lambda(I))/(\omega(I) - \beta(I))$. The quantity $\delta_{\mathtt{A}} = \sup\{r : \delta_{\mathtt{A}}(I) > r, I \text{ instance of } \Pi\}$ is the differential approximation ratio of $\mathtt{A}$ for Π. In what follows, we use notation ρ when dealing with standard ratio and notation δ when dealing with the differential one. Moreover $\rho(\Pi)$ (resp., $\delta(\Pi)$) will denote the best standard (resp., differential) approximation ratio for Π.

In [3], the term "trivial solution" is used to denote what in [9, 8] and here is called *worst solution*. Moreover, all the examples in [3] carry over **NP**-hard problems for which worst solution can be trivially computed. This is for example the case of maximum independent set where, given a graph, the worst solution is the empty set, or of minimum vertex cover, where the worst solution is the vertex-set of the input-graph, or even of the minimum graph-coloring where one can trivially color the vertices of the input-graph using a distinct color per vertex. On the contrary, for TSP things are very different. Let us take for example min_TSP. Here, given a graph K_n, the worst solution for K_n is a maximum total-distance Hamiltonian cycle, i.e., the optimal solution of max_TSP in K_n. The computation of such a solution is very far from being trivial since max_TSP is **NP**-hard. Obviously, the same holds when one considers max_TSP and tries to compute a worst solution for its instance. In order to remove ambiguities about the concept of the worst solution, the following definition, proposed in [9], will be used here.

Definition 2. Given a typical instance I of an **NPO** problem Π, the worst solution of I is the optimal solution of a new **NPO** problem $\bar{\Pi}$ where items 1 to 3 of definition 1 are identical for both Π and $\bar{\Pi}$, and

$$\mathrm{opt}(\bar{\Pi}) = \begin{cases} \max & \mathrm{opt}(\Pi) = \min \\ \min & \mathrm{opt}(\Pi) = \max \end{cases} \blacksquare$$

One of the features of the differential ratio is to be stable under affine transformation of the objective function of a problem and so it does not create a dissymmetry between minimization and maximization problems. This is very clear in the case of TSP. Dealing with min_TSP it is very well-known that its general version is not approximable in polynomial time within better than $2^{p(n)}$ for a polynomial p. On the other hand, its maximization version, max_TSP, the **NP**-hardness of which is immediately proved if one replaces distance $d(i,j)$ for min_TSP by $M - d(i,j)$ in max_TSP, for an M greater than the largest edge distance in the input graph of min_TSP, can be approximated in polynomial time within 5/7 ([20]).

Let us recall some standard terminology from the theory of the polynomial approximation of the **NP**-hard problems (for the standard approximation framework). Given an **NP** minimization (resp., maximization) problem Π, a *constant-ratio approximation algorithm* for Π is a polynomial time approximation algorithm (PTAA) guaranteeing approximation ratio bounded above (resp., below) by a fixed constant, i.e., by a constant that does not depend on any input-parameter of Π. **APX** is the class of the **NP** optimization problems solved by constant-ratio PTAAs. A *polynomial time approximation schema* (PTAS) for Π is a sequence of PTAAs (receiving as inputs any instance of Π and a fixed constant ϵ) guaranteeing approximation ratio bounded above (resp., below) by $1 + \epsilon$ (resp., $1 - \epsilon$), for every $\epsilon > 0$. If a PTAS is polynomial in both n and $1/\epsilon$, then it is called *fully polynomial time approximation schema* (FPTAS). For the differential approximation, the ratio achieved by polynomial time approximation schemata is $1 - \epsilon$ for both minimization and maximization. Finally, **APX**-complete is the class of problems in **APX**, which, in addition, are complete with respect to the existence of a PTAS solving them, in other words, if any **APX**-complete problem could be solved by a PTAS, then any other **APX**-complete problem could be so.

As it is shown in [9, 8], many problems behave in completely different ways regarding traditional or differential approximation. This is, for example, the case of minimum graph-coloring or, even, of minimum vertex-covering. This paper deals with another example of the diversity in the nature of approximation results achieved within the two frameworks, the TSP. For this problem and its versions mentioned above, a bunch of standard-approximation results (positive or negative) have been obtained until nowadays. The first inapproximability result is the one of [21] (see also [13]) affirming that it is **NP**-hard to approximate min_TSP within any constant factor; with the same proof, one can easily refine the result of [21] to deduce the inaproximability of min_TSP within any ratio of the form $2^{p(n)}$ for any polynomial p. On the other hand, the metric min_TSP is approximable within 3/2 ([5]), the symmetric min_TSP12 within 7/6 ([18]) (recall that the original proof of the **NP**-completeness of the min_TSP is done by reduction to min_TSP12), while the asymmetric version of min_TSP12 is approximable within 17/12 ([22]). Moreover, min_TSP12 is **APX**-complete ([18]), consequently, given the result of [2], it cannot be solved by a PTAS unless **P**=**NP**; in other words, $\exists\epsilon > 0$ for which approximation of min_TSP12 within ratio smaller than $1 + \epsilon$ is **NP**- hard. Furthermore, even in graphs where the density of the subgraph spanned by the edges of length 1 is bounded below by a constant $c \in]0, 1/2[$, min_TSP12 cannot be solved by a polynomial time approximation schema ([11]). The works of [10] and more recently of [4] refine the result of [18] specifying for ϵ. In [4] is proved that for any $\epsilon > 0$, it is **NP**-hard to approximate min_TSP12 within ratio smaller than, or equal to, $3475/3476 - \epsilon$; in other words the result of [10] gives a value – equal to $1/3476 - \epsilon'$, $\forall \epsilon' > 0$ – for the hardness threshold ϵ of min_TSP12 refining so the negative results of [18, 10]. Finally another restrictive version of the metric min_TSP, the Euclidean min_TSP can be solved by a standard PTAS ([1]). A complete list of standard-approximation results for min_TSP is given in [6].

In what follows, we show that, in the differential approximation framework the classical `2_OPT` algorithm, originally devised in [7] and revisited in numerous works (see, for example, [15]), approximately solves min_TSP with edge-distances bounded by a polynomial of n within differential approximation ratio $1/2$. In other words, `2_OPT` provides for these graphs solutions "fairly close" to the optimal and, simultaneously, "fairly far" from the worst one. We also prove that, in the opposite of what happens in the standard framework, metric min_TSP and general min_TSP are equi-approximable in the differential framework. Moreover we prove that min_TSP12 is approximable within $3/4$.

For max_TSP things are much more optimistic for standard approximation, since this problem is in **APX**. By the end of 70s it has been proved in [12] that `2_OPT` guarantees approximation ratio $1/2$ for max_TSP. More recently, in [20] is proved that max_TSP can be solved by a standard PTAA within ratio $5/7$, if the distance-vector is symmetric and within $38/63$, if it is asymmetric. The dissymmetry in the approximability of min_ and max_TSP can be considered as somewhat curious given the structural symmetry existing between them. In fact the transformation $d \mapsto M - d$ mentioned above and revisited in detail in section 6 is affine. Since differential approximation is stable under affine transformation of the objective function, min_TSP and max_TSP are equi-approximable.

In what follows, we will denote by $V = \{v_1, \ldots, v_n\}$ the vertex-set of K_n, by E its edge-set and, for $v_iv_j \in E$, we denote by $d(v_i, v_j)$ (or by $d(i,j)$ when no ambiguity occurs) the distance of the edge $v_iv_j \in E$; we consider that the distance-vector is symmetric and integer. Given a feasible TSP-solution $T(K_n)$ of K_n (both min_ and max_TSP have the same set of feasible solutions), we denote by $d(T(K_n))$ its (objective) value; T will be indexed by min or max depending on whether it deals with min_ or max_TSP. When necessary, the values of the worst case solution, the approximated one and the optimal one for min_TSP (max_TSP) will be denoted by $\omega_{\min}(K_n)$, $\lambda_{\mathtt{A}}^{\min}(K_n)$ and $\beta_{\min}(K_n)$ ($\omega_{\max}(K_n)$, $\lambda_{\mathtt{A}}^{\max}(K_n)$ and $\beta_{\max}(K_n)$), respectively. Given a graph G induced by K_n, we denote by $V(G)$ its vertex-set. Finally, given any set C of edges, we denote by $d(C)$ the total distance of C, i.e., the quantity $\sum_{v_iv_j \in C} d(i,j)$.

2 Preserving differential approximation for several min_TSP versions

This section is a preliminary one containing several results about how differential approximation is preserved between several restrictive versions of min_TSP.

Proposition 1. *Metric min_ TSP and general min_ TSP are differentially equi-approximable.*

Proof. Obviously, metric min_TSP being a special case of the general one, can be solved within the same differential approximation ratio with the latter.

Suppose now that metric min_TSP is approximately solved within differential ratio δ. Given an instance $I = (K_n, \vec{d})$ of general min_TSP (set $d_{\max} = \max\{d(i,j) : v_iv_j \in E\}$), one can transform it into a new one $I' = (K_n, \vec{d'})$ by changing, $\forall v_ivj \in E$, distance $d(i,j)$ of the former to $d'(i,j) = d_{\max} + d(i,j)$. It is easy to see that I' is metric, and that every feasible tour $T(I)$ of $(K_n, \vec{d})$ remains feasible for $(K_n, \vec{d'})$. The cost of such a tour becomes $d(T(I')) = d(T(I)) + nd_{\max}$. Then the δ-PTAA for metric min_TSP will achieve

$$\delta = \frac{\omega(I') - d(T(I'))}{\omega(I') - \beta(I')} = \frac{\omega(I) + nd_{\max} - d(T(I)) - nd_{\max}}{\omega(I) + nd_{\max} - \beta(I) - nd_{\max}} = \frac{\omega(I) - d(T(I))}{\omega(I) - \beta(I)}.$$

Consequently, every PTAA for metric min_TSP can simultaneously solve general min_TSP within the same differential approximation ratio. ∎

Let $d_{\min} = \min\{d(i,j) : v_i v_j \in E\}$. Then, if one transforms every distance $d(i,j)$ into $d(i,j) - d_{\min} + 1$, one obtains a complete graph where $d_{\min} = 1$ and with arguments completely analogous to the ones of proposition 1, the following holds.

Proposition 2. *General min_ TSP and min_ TSP with $d_{\min} = 1$ are differentially equi-approximable.*

We next consider another class of instances, the one where the edge-distances are either a, or b (notorious member of this class of min_TSP-problems, denoted by min_TSPab, is the min_TSP12). Suppose, without loss of generality that $a < b$. Then, by proposition 2, min_TSPab is equi-approximable with min_TSP1b. Consider now an instance of the latter problem. If one sets $b = 2$ for all the b-edges (edges of distance b), then by arguments completely similar to the ones of the proof of proposition 1 (and since for a tour T containing k_b b-edges, $d(T) = n + (b-1)k_b$), the following result holds.

Proposition 3. *min_ TSPab and min_ TSP12 are differentially equi-approximable.*

Note that results analogous to the ones of propositions 1, 2 and 3 do not hold in the standard approximation framework.

3 2_OPT and differential approximation for the general minimum traveling salesman

In what follows, we denote by **D-APX** the analogous of the class **APX**, the class of **NPO** problems solved by a constant-ratio PTAA, for the differential approximation framework.

Theorem 1. *min_ TSP is differentially approximable within approximation ratio 1/2 and this ratio is tight.*

Proof. In what follows, suppose that a tour is listed as the set of its edges and consider the following algorithm of [7].

```
BEGIN /2_OPT/
(1)   start from any feasible tour T;
(2)   REPEAT
(3)          pick a new set {v_i v_j, v_i' v_j'} ⊂ T;
(4)          IF d(i,j) + d(i',j') > d(i,i') + d(j,j') THEN T ← (T \ {v_i v_j, v_i' v_j'}) ∪ {v_i v_i', v_j v_j'} FI
(5)   UNTIL no improvement of d(T) is possible;
(6)   OUTPUT T;
END. /2_OPT/
```

Suppose now that, starting from a vertex denoted by v_1, the rest of the vertices is ordered following the tour T finally computed by 2_OPT (so, given a vertex v_i, $i = 1, \ldots, n-1$, v_{i+1} is its successor with respect to T; $v_{n+1} = v_1$). Let us fix one optimal tour and denote it by T^*. Given a vertex v_i, denote by $v_{s^*(i)}$ its successor in T^* (remark that $v_{s^*(i)+1}$ is the successor of $v_{s^*(i)}$ in T; in other words, edge $v_{s^*(i)} v_{s^*(i)+1} \in T$). Finally let us fix one (of the eventually many) worst-case (maximum total-distance) tour T_ω.

The tour T computed by 2_OPT is a local optimum for the 2-exchange of edges in the sense that every interchange between two non-intersecting edges of T and two non-intersecting edges of $E \setminus T$ will produce a tour of total distance at least equal to $d(T)$. This implies in particular that, $\forall i \in \{1, \ldots, n\}$,

$$d(i, i+1) + d\left(s^*(i), s^*(i)+1\right) \leqslant d\left(i, s^*(i)\right) + d\left(i+1, s^*(i)+1\right);$$

so, writing the expression above for all $i \in \{1, \ldots, n\}$, we get

$$\sum_{i=1}^{n} \left(d(i, i+1) + d\left(s^*(i), s^*(i)+1\right)\right) \leqslant \sum_{i=1}^{n} \left(d\left(i, s^*(i)\right) + d\left(i+1, s^*(i)+1\right)\right) \tag{1}$$

Moreover, it is easy to see that the following holds:

$$\begin{aligned}
\bigcup_{i=1,\ldots,n} \{v_i v_{i+1}\} \quad = \quad \bigcup_{i=1,\ldots,n} \left\{v_{s^*(i)} v_{s^*(i)+1}\right\} \quad &= \quad T && (2)\\
\bigcup_{i=1,\ldots,n} \left\{v_i v_{s^*(i)}\right\} \quad &= \quad T^* && (3)\\
\bigcup_{i=1,\ldots,n} \left\{v_{i+1} v_{s^*(i)+1}\right\} \quad &= \quad \text{some feasible tour } T' && (4)
\end{aligned}$$

Let us show that $T' = \cup_{i=1,\ldots,n}\{v_{i+1}v_{s^*(i)+1}\}$ is feasible. Recall that an acyclic permutation is a bijective function $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that, $\forall i \in \{1, \ldots, n\}$:

$$\begin{cases} f^{(k)}(i) \neq i & k < n \\ f^{(n)}(i) = i \end{cases}$$

Every feasible tour T, oriented as mentioned above, can be seen as an acyclic permutation. Consider now the following mappings

$$\begin{aligned}
s^* &: i \mapsto s^*(i)\\
f &: i \mapsto i+1\\
h &: i \mapsto s^*(i-1)+1.
\end{aligned}$$

It is easy to see that if s^* is an acyclic permutation and f is a permutation, then $h = f \circ s^* \circ f^{-1}$ is an acyclic permutation. Moreover, it is not hard to see that pairs $(i, h(i))$ correspond $(\bmod\, n)$ to the edges of T'.

Combining expression (1) with expressions (2), (3) and (4), one gets:

$$\begin{aligned}
(2) &\implies \sum_{i=1}^{n} d(i, i+1) + \sum_{i=1}^{n} d\left(s^*(i), s^*(i)+1\right) \;=\; 2\lambda_{2_\texttt{OPT}}(K_n)\\
(3) &\implies \sum_{i=1}^{n} d\left(i, s^*(i)\right) \;=\; \beta(K_n)\\
(4) &\implies \sum_{i=1}^{n} d\left(i+1, s^*(i)+1\right) \;=\; d(T') \;\leqslant\; \omega(K_n)
\end{aligned} \tag{5}$$

and expressions (1) and (5) lead to

$$2\lambda_{2\_\texttt{OPT}}\left(K_n\right) \leqslant \beta\left(K_n\right) + \omega\left(K_n\right) \iff \frac{\omega\left(K_n\right) - \lambda_{2\_\texttt{OPT}}\left(K_n\right)}{\omega\left(K_n\right) - \beta\left(K_n\right)} \geqslant \frac{1}{2}.$$

Consequently, $\delta_{2\_\texttt{OPT}} \geqslant 1/2$.

Consider now a K_{4n+8}, $n \geqslant 0$, set $V = \{v_i : i = 1, \ldots, 4n+8\}$, let

$$\begin{aligned}
d(2k+1, 2k+2) = 1 &\quad k = 0, 1, \ldots, 2n+3\\
d(4k+2, 4k+4) = 1 &\quad k = 0, 1, \ldots, n+1\\
d(4k+3, 4k+5) = 1 &\quad k = 0, 1, \ldots, n\\
d(4n+7, 1) = 1 &
\end{aligned}$$

and set the distances of all the remaining edges to 2. Then,

$$\begin{aligned}
T &= \{v_i v_{i+1} : i = 1, \ldots, 4n+7\} \cup \{v_{4n+8} v_1\} \\
T^* &= \{v_{2k+1} v_{2k+2} : k = 0, \ldots, 2n+3\} \cup \{v_{4k+2} v_{4k+4} : k = 0, \ldots, n+1\} \\
&\quad \cup \{v_{4k+3} v_{4k+5} : k = 0, \ldots, n\} \cup \{v_{4n+7} v_1\} \\
T_w &= \{v_{2k+2} v_{2k+3} : k = 0, \ldots, 2n+2\} \cup \{v_{2k+1} v_{2k+4} : k = 0, \ldots, 2n+1\} \\
&\quad \cup \{v_{2k+2} v_{2k+5} : k = 0, \ldots, 2n+1\} \cup \{v_{4n+8} v_1\}.
\end{aligned}$$

In figure 1, T^* and T_w are shown for $n = 1$ ($T = \{1, \ldots, 11, 12, 1\}$). Hence, $\delta_{2\_\mathtt{OPT}}(K_{4n+8}) = 1/2$ and this completes the proof of the theorem. ∎

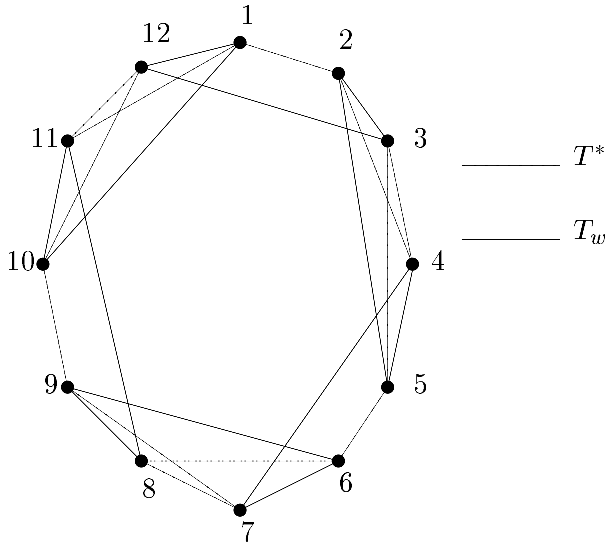

Figure 1. Tightness of the 2_OPT approximation ratio for $n = 1$.

From the proof of the tightness of the ratio of 2_OPT, the following corollary is immediately deduced.

Corollary 1. $\delta_{2\_\mathtt{OPT}} = 1/2$ *is tight even for min_ TSP12.*

A first case of polynomial complexity for algorithm 2_OPT (even if edge-distances of the graph are exponential in n) is for graphs where the number of (feasible) tour-values, denoted by $\sigma(K_n)$, is polynomial in n. Here, since there exists a polynomial number of different min_TSP solution-values, achievement of a locally minimal solution (starting, at worst for the worst-value solution) will need a polynomial number of steps (at most $\sigma(K_n)$) for 2_OPT.

Theorem 1 obviously works in polynomial time when $d_{\max}$ is bounded above by a polynomial of n. However, even when this condition is not satisfied, there exist restrictive cases of min_TSP for which 2_OPT remains polynomial.

Consider now complete graphs with a fixed number $k \in \mathbb{N}$ of distinct edge-distances, $d_1, d_2, \ldots, d_k$. Then, any tour-value can be seen as k-tuple $(n_1, n_2, \ldots n_k)$ with $n_1 + n_2 + \ldots + n_k = n$, where n_1 edges of the tour are of distance d_1, $\ldots$, n_k edges are of distance d_k ($\sum_{i=1}^{k} n_i d_i = d(T)$). Consequently, the consecutive solutions retained by 2_OPT (in line (4)) before attaining a local minimum are, at most, as many as the number of the arrangements with repetitions of k distinct items between n items (in other words, the number of all the distinct k-tuples formed by all the numbers in $\{1, \ldots, n\}$), i.e., bounded above by $O(n^k)$.

Another class of polynomially solved instances is the one where $\beta(K_n) = O(P(n))$ where P is a polynomial of n. Recall that, from proposition 1, general and metric min_TSP are differentially equi-approximable. Consequently, given an instance K_n where $\beta(K_n)$ is polynomial, K_n can be transformed into a graph K'_n as in proposition 1. Then, if one runs the algorithm of [5] in order to obtain an initial feasible tour T (line `(1)` of algorithm `2_OPT`), then its total distance, at most $3/2$ times the optimal one, will be of polynomial value and, consequently, `2_OPT` will need a polynomial number of steps until attaining a local minimum.

Let us note that the first and the fourth of the above cases cannot be decided in polynomial time. However, if one systematically transforms general min_TSP into a metric one (proposition 1) and then uses the algorithm of [5] in line `(1)` of `2_OPT`, then all instances meeting the second item of corollary 2 will be solved in polynomial time even if we cannot recognize them.

Corollary 2. *The following versions of min_TSP are in* ***D-APX*** *(solved by* `2_OPT` *within ratio 1/2):*

- *on graphs where the optimal tour-value is polynomial in n;*
- *on graphs where the number of feasible tour-values is polynomial in n (examples of these graphs are the ones where edge-distances are polynomially bounded, or even the ones where there exists a fixed number of distinct edge-distances).*

4 Approximating min_TSP12

Let us first recall that, given a graph G, a 2-matching is a a set M of edges of G such that if $V(M)$ is the set of the endpoints of M, the vertices of the graph $(V(M), M)$ have degree at most 2, in other words, the graph $(V(M), M)$ is a collection of cycles and simple paths. A 2-matching is optimal if it is the largest over all the 2-matchings of G. As it is shown in [14], *an optimal triangle-free 2-matching can be computed in polynomial time.*

Our min_TSP12 PTAA is based upon a special kind of triangle-free 2-matching in K_n, the cycles of which will be progressively patched in order to produce a Hamiltonian tour. In what follows, we deal with optimal triangle-free 2-matchings, i.e., with *triangle-free collections of cycles.*

Theorem 2. *min_TSP12 is approximable within differential approximation ratio $\delta \geqslant 3/4$. This ratio is tight for the algorithm devised.*

Proof. Let $M = (C_1, C_2, \ldots)$ be any maximal triangle-free 2-matching of K_n. In the sequel, we call by *value of a 2-matching* the sum of the distances of its edges. For any matching M, we will denote its value by $d(M)$. Also, let us call *cycle-patching* (see also [18]) the operation consisting in taking two cycles C_i and C_j of M, in picking edges $v_kv_l \in C_i$, $v_pv_q \in C_j$ and in transforming C_i, and C_j into a unique cycle $C = C_i \cap C_j \setminus \{v_kv_l, v_pv_q\} \cap \{e_{ij}, e'_{ij}\}$, where $\{e_{ij}, e'_{ij}\} = \{v_kv_p, v_lv_q\}$, or $\{e_{ij}, e'_{ij}\} = \{v_kv_q, v_lv_p\}$. This specifies the following procedure, polynomial in n, computing, in addition, the total distance of the cycle resulting from cycle patching.

```
BEGIN /CYCLE_PATCH/
      take edges v_k v_l ∈ C_i and v_p v_q ∈ C_j;
      C^1_ij ← C_i ∪ C_j \ {v_k v_l, v_p v_q} ∪ {v_k v_p, v_l v_q};
      C^2_ij ← C_i ∪ C_j \ {v_k v_l, v_p v_q} ∪ {v_k v_q, v_l v_p};
      OUTPUT C_ij ← argmin{d(C^1_ij), d(C^2_ij)};
END. /CYCLE_PATCH/
```

4.1 Specification of the min_TSP12-algorithm and evaluation of $\lambda(K_n)$

In the sequel, we will first specify a PTAA `min_TSP12` and estimate the value $\lambda_{\texttt{TSP12}}(K_n) = d(T(K_n))$ of the Hamiltonian tour computed. Next, we will compute a lower bound for $\omega(K_n)$. As for theorem 1, we will exhibit a feasible tour of a certain value. Since worst solution's value is larger than the value of every other Hamiltonian tour of K_n, the value of the tour exhibited will be the bound claimed.

Let $\hat{M}$ be an optimal triangle-free 2-matching of K_n (recall that, as we have mentioned, such a matching is maximal, i.e, it does only contains cycles). Starting from $\hat{M}$, one can easily construct an optimal 2-matching M^* where every patching of two cycles strictly increases its value. In what follows, we will call M^* *2-minimal*. Construction of M^* can be done in polynomial time by the following procedure.

```
BEGIN /2_MIN/
      M_p ← ∅;
      REPEAT
             pick a new set {C_i, C_j} ⊆ M̂;
             FOR all v_k v_l ∈ C_i, v_p v_q ∈ C_j DO
                 M_p ← M̂ \ {C_i, C_j} ∪ CYCLE_PATCH(C_i, C_j)
                 IF d(M̂) > d(M_p) THEN M̂ ← M_p FI
             OD
      UNTIL no improvement of d(M̂) is possible;
      OUTPUT M* ← M̂;
END. /2_MIN/
```

Remark 1. In any 2-minimal matching M there exists *at most* one cycle C containing 2-edges (edges of distance 2). In fact, if not, procedure `CYCLE_PATCHING` can always be applied in order to patch two distinct cycles containing 2-edges into one cycle with total distance no longer than the sum of the distances of the two cycles patched. Moreover, if $\mathsf{M} = \mathsf{C}$, then M is an optimal solution for min_TSP (in general, a Hamiltonian cycle being a particular triangle-free 2-matching, $d(\mathsf{M}) \leqslant \beta_{min}(K_n)$). ∎

Fix a 2-minimal triangle-free matching $M^* = (C_1, C_2, \ldots, C_{p+1})$ (recall that M^* is a minimum total-distance triangle-free 2-matching) and suppose, without loss of generality, that $p > 0$ and that cycles $C_1, \ldots, C_p$ contain only 1-edges (edges of distance 1) and that only cycle C_{p+1} contains, eventually, some 2-edges. Finally, recall that it is assumed that $|C_i| \geqslant 4$. The following facts can be concluded regarding M^*.

Fact 1. $\forall (C, C') \in M^* \times M^*$ such that $C \neq C'$, $\forall uv \in C$, $\forall u'v' \in C'$, $\max\{d(u,u'), d(v,v')\} = \max\{d(u,v'), d(v,u')\} = 2$. ∎

Fact 2. If vertex u is adjacent to a 2-edge in C_{p+1}, then, $\forall u' \notin V(C_{p+1})$, $d(u,u') = 2$. ∎

Fact 3. If uu' and vv' are two distinct non-adjacent 2-edges of C_{p+1}, then $d(u,v) = d(u,v') = d(u'v) = d(u'v') = 2$. ∎

Given $M^* = (C_1, \ldots, C_{p+1})$, we first perform the following preprocessing on $C_1, \ldots, C_p$.

```
BEGIN /PREPROCESS/
      PR ← ∅;
      WHILE possible DO
            arbitrarily pick C_i, C_j ∈ M* \ {C_{p+1}} linked by at least one 1-edge;
```

```
        PR ← PR ∪ {Ci, Cj};
        M* ← M* \ {Ci, Cj};
    OD
    OUTPUT PR;
END. /PREPROCESS/
```

Suppose the `WHILE` loop of `PREPROCESS` executed q times and denote by $\{C_1^s, C_2^s\}$ the cycles considered during the sth execution of the loop, $s = 1, \ldots, q$. Then $PR = \cup_{s=1}^{q}\{C_1^s, C_2^s\}$. Set $r = p - 2q$ and denote by D_t, $t = 1, \ldots, r$, the cycles in $\{C_1, \ldots, C_p\} \setminus \cup_{s=1}^{q}\{C_1^s, C_2^s\}$. Under all this,

$$M^* = \left\{\bigcup_{s=1}^{q}\{C_1^s, C_2^s\}\right\} \bigcup \left\{\bigcup_{t=1}^{r} D_t\right\} \bigcup C_{p+1}.$$

The following facts hold and complete the above discussion.

Fact 4. $2q + r \geqslant 1$; if $2q + r = 1$ then $C_{p+1} \neq \emptyset$. ∎

Fact 5. $\forall s \in \{1, \ldots, q\}$, $\forall \ell \in \{1, 2\}$, $\forall e \in C_\ell^s$, $d(e) = 1$. ∎

Fact 6. $\forall t \in \{1, \ldots, r\}$, $\forall e \in D_t$, $d(e) = 1$. ∎

Fact 7. $\forall s \in \{1, \ldots, q\}$, $\exists i^s \in V(C_1^s)$, $\exists I^s \in V(C_2^s)$ such that $d(i^s, I^s) = 1$. ∎

Fact 8. $\forall (t, t') \in \{1, \ldots, r\} \times \{1, \ldots, r\}$, $t \neq t'$, $\forall (u, v) \in V(D_t) \times V(D_{t'})$, $d(u, v) = 2$. ∎

In the sequel, for $s = 1, \ldots, q$, we denote by a^s and b^s (resp., A^s and B^s) the vertices adjacent to i^s (resp., I^s) in C_1^s (resp., C_2^s). We set $c = \sum_{s=1}^{q}(|C_1^s| + |C_2^s|)$, $d = \sum_{t=1}^{r}|D_t|$, $E2 = \{e \in C_{p+1} : d(e) = 2\}$. Following these notations, $n = c + d + |C_{p+1}|$ and, denoting by $|E2|$ cardinality of the set $E2$,

$$d(M^*) = n + |E2| \tag{6}$$

We are well-prepared now to describe the algorithm proposed. Informally, it first patches cycles C_1^s and C_2^s into a single cycle C^s, $s = 1, \ldots, q$. Next, it patches cycle C^1 with C^2 to produce a cycle C which will be patched with C^3, and so on, finally producing a single cycle C. It does so for the cycles D_t, $t = 1, \ldots, r$, producing a single cycle D. Then it patches C and D in order to produce a partial tour $\tilde{T}$ and finally it patches $\tilde{T}$ and C_{p+1} obtaining so the final TSP-tour $T(K_n)$.

4.1.1 Construction and evaluation of C

Construction of C is performed by means of the following procedure.

```
BEGIN /C/
    FOR s ← 1 to q DO using edge i^s I^s  C^s ← CYCLE_PATCH(C^s_1, C^s_2); OD
    C ← C^1;
    FOR s ← 1 TO q − 1 DO
       replacing as many 2-edges as possible C ← CYCLE_PATCH(C, C^{s+1});
    OD
    OUTPUT C;
END. /C/
```

The call of algorithm `CYCLE_PATCH` in the first `FOR`-loop of `C` is a very slightly different variant of the corresponding procedure presented above where one imposes to the 1-edge $i^s I^s$ (fact 7) to be one of the cross-edges entering cycle C^s.

Lemma 1. *The 2-matching $(C^1, C^2, \ldots, C^q)$ produced during the q executions of the first* `FOR`*-loop of algorithm* `C` *has value $d(C^1, C^2, \ldots, C^q) = c + q$.*

Proof of lemma 1. Patching of C_1^s and C_2^s into C^s is done using 1-edge $i^s I^s$ (fact 7), $s = 1, \ldots, q$. Consequently, only one 2-edge has been included in C^s (the one used with $i^s I^s$ to patch C_1^s and C_2^s). Such an edge always exists because of fact 1. So, for $s = 1, \ldots, q$, execution of $\mathtt{CYCLE\_PATCH}(\mathtt{C_1^s}, \mathtt{C_2^s})$ in the first `FOR`-loop of `C` will produce in all exactly q 2-edges replacing and q 1-edges replacing $2q$ 1-edges. Consequently, $d(C^1, C^2, \ldots, C^q) = \sum_{s=1}^{q}(|C_1^s| + |C_2^s|) + q = c + q$ and this completes the proof of lemma 1. ∎

During the executions of `CYCLE_PATCH` in the second `FOR`-loop of `C`, we try that the total distance of the resulting cycle is no longer than the sum of the total distances of the cycles patched. In other words, we try to not produce additional 2-edges in the resulting cycle. Here the following lemma holds.

Lemma 2. *The cycle C produced during the second* `FOR`*-loop of algorithm* `C` *does not increase $d(C^1, C^2, \ldots, C^q)$.*

Proof of lemma 2. The proof is done by induction on q.

4.1.1.1 $q = 1$

The proof of this case is an immediate application of lemma 1 with $q = 1$.

4.1.1.2 $q = k$

Suppose that during the k first executions of the `FOR`-loop, the number of 2-edges is at most k.

4.1.1.3 $q = k + 1$

Suppose now that there exists at least one 2-edge in C (note also that C^{s+1}, since it has been not processed yet, always contains the 2-edge produced by the execution of the first `FOR`-loop). Since the patching with C^{s+1} is done by algorithm `C` using two 2-edges, there is no additional 2-edge created. On the other hand, if no 2-edge exists in C, the patching of C with C^{s+1} will produce at most $2 \leqslant k+1$ new 2-edges and this concludes induction and the proof of lemma 2. ∎

Lemmata 1 and 2 induce

$$d(C) \leqslant c + q \tag{7}$$

4.1.2 Construction and evaluation of D

The following procedure is used to construct D.

```
BEGIN /D/
      D ← D₁;
      FOR t ← 1 TO r − 1 DO
          replacing as many 2-edges as possible D ← CYCLE_PATCH(D, D_{t+1});
      OD
      OUTPUT D;
END. /D/
```

Exactly analogous arguments to the ones of the proof of lemma 2 applied to algorithm `D` and thanks to fact 8 induce

$$d(D) \leqslant d + r \tag{8}$$

Also, let us note that any patching of cycles D_i between them will create an additional cost of r.

4.1.3 Construction and evaluation of $\tilde{T}$

```
BEGIN /T̃/
      replacing as many 2-edges as possible T̃ ← CYCLE_PATCH(C,D);
      OUTPUT T̃;
END. /T̃/
```

With the same arguments as in lemma 2, the following holds for $|\tilde{T}|$:

$$\begin{cases} d(\tilde{T}) & \leqslant & c+d+q+r & 2q+r \geqslant 2 \\ d(\tilde{T}) & = & d & (q,r)=(0,1) \end{cases} \tag{9}$$

4.1.4 Overall specification of the min_TSP12-algorithm, construction and evaluation of T

Once $\tilde{T}$ constructed, call of $\mathtt{CYCLE\_PATCH}(\tilde{T}, C_{p+1})$, changing as many as 2-edges (at most 2) as possible, constructs the final TSP-solution $T(K_n)$ and the whole min_TSP12-PTAA is the following. The 2-matching $\hat{M}$ produced in the first line of the algorithm below is supposed to be optimal and without cycles on less than, or equal to, four edges.

```
BEGIN /TSP12/
      call the algorithm of [14] to produce M̂;
      M̂ = (C_1,...,C_{p+1}) ← 2_MIN(M̂);
      M* ← PREPROCESS(M̂) ∪_{t=1}^{r} {D_t} ∪ C_{p+1};
      C ← C(M*);
      D ← D(M*);
      T̃ ← CYCLE_PATCH(C,D);
      OUTPUTT(K_n) ← CYCLE_PATCH(T̃,C_{p+1});
END. /TSP12/
```

It is easy to see that, since all the algorithms called are polynomial, TSP12 works in polynomial time.

If $(q,r)=(0,1)$ (in this case, by fact 4, $C_{p+1} \neq \emptyset$), then patching of D_1 with C_{p+1} constructs a tour with $d(T(K_n)) = d(D_1)+d(C_{p+1})+1 = d(M^*)+1 = d(M^*)+q+r$.

Suppose $2q+r \geqslant 2$. Then, by expression (9), $d(\tilde{T}) \leqslant c+d+q+r$. If $d(\tilde{T}) < c+d+q+r$, i.e., $d(\tilde{T}) \leqslant c+d+q+r-1$, even if $\tilde{T}$ does not contain any 2-edge, patching of $\tilde{T}$ with C_{p+1} will create only one additional 2-edge so, finally, $d(T(K_n)) \leqslant d(\tilde{T})+d(C_{p+1}) \leqslant c+d+q+r+|C_{p+1}|+|E2|$ and, by expression (6), $d(T(K_n)) \leqslant d(M^*)+q+r$. If $d(\tilde{T}) = c+d+q+r$, we simply exchange two 2-edges and the same expression for $d(T(K_n))$ always holds.

The discussion above leads to the following concluding expression for the quantity $d(T(K_n))$:

$$d\left(T\left(K_n\right)\right) = \lambda_{\mathtt{TSP12}}\left(K_n\right) \leqslant v\left(M^*\right)+q+r \tag{10}$$

4.2 A bound for $\omega(K_n)$

In what follows, we will exhibit a TSP12-solution, the objective value of which will provide us with a lower bound for the value $\omega(K_n)$ of the worst TSP12-solution on K_n. For this we define a set W of disjoint elementary paths (d.e.p.), *any one of them containing only 2-edges*. Obviously, if $W = \{w_1, \ldots, w_{|W|}\}$, one, by properly linking w_i's, can easily construct a tour T' verifying $d(T') \geqslant n + \sum_{w_i \in W} |w_i|$ which is a lower bound for $\omega(K_n)$.

4.2.1 Disjoint elementary paths on $V(C)$

Recall that, for $q \neq 0$, $d(i^s, I^s) = 1$; hence, by fact 1, $d(a^s, A^s) = d(a^s, B^s) = d(b^s, A^s) = d(b^s, B^s) = 2$, $s = 1, \dots, q$. Always by fact 1, either $d(i^s, B^s) = 2$, or $d(I^s, b^s) = 2$. Without loss of generality, we suppose all over the rest of the proof of theorem 2 that $d(i^s, B^s) = 2$.

Consequently, for $s = 1, \dots, q$, set $W_{C^s} = \{b^s A^s, A^s a^s, a^s B^s, B^s i^s\}$ and the set of d.e.p. on the vertices of C is $W_C = \cup_{s=1}^{q} W_{C^s}$ with

$$|W_C| = 4q \tag{11}$$

4.2.2 Disjoint elementary paths on $V(D)$

If $r > 1$, we choose, for $t = 1, \dots, r$, a sequence $\{w_t, x_t, y_t, z_t\} \in V(D_t)$. Then, the set of d.e.p. and its cardinality on $V(D)$ is

$$\begin{aligned} W_D &= \{w_1, w_2, \dots, w_t, w_{t+1}, \dots, w_r, x_1, \dots, x_t, \dots, x_r, y_1, \dots, y_r, z_1, \dots, z_r\} && (12) \\ |W_D| &= 4(r-1) + 3 = 4r - 1 && (13) \end{aligned}$$

If $r \leqslant 1$, then we set $W_D = \emptyset$.

4.2.3 Disjoint elementary paths on $V(\hat{T})$

4.2.3.1 $q > 0$

Suppose first $\boldsymbol{r \neq 1}$. If $\boldsymbol{r = 0}$, then $W_{\hat{T}} = W_C$. Suppose now $r > 1$. Then, by fact 1, there exists vertex $v_1 \in V(D_1)$ such that either $d(v_1, B^1) = 2$, or $d(v_1, I^1) = 2$. Let e be this 2-edge. Without loss of generality, we can suppose $v_1 = w_1$ (see the paragraph just above). Then the set of d.e.p. on $V(\hat{T})$ is $W_{\hat{T}} = W_C \cup W_D \cup \{e\}$ with (see expressions (11) and (13))

$$\left|W_{\hat{T}}\right| = 4q + 4r - 1 + 1 = 4(q + r) \tag{14}$$

Suppose $\boldsymbol{r = 1}$. Consider cycles C_1^1, C_2^1 and D_1 and denote by a'^1 (resp., A'^1) the vertex, distinct from i^1 (resp., I^1) in C_1^1 (resp., C_2^1) adjacent to a^1 (resp., A^1). If for any $u \in V(D_1)$ and for any $v \in \{a^1, b^1, A^1, B^1\}$, $d(u, v) = 2$, then let w, x, y, z be four vertices of $V(D_1)$ and set $W_1 = \{wa^1, a^1x, xb^1, b^1y, yA^1, A^1z, zB^1, B^1i^1\}$.

Suppose now that there exist $x \in v(D_1)$ and $v \in \{a^1, b^1, A^1, B^1\}$ with $d(x, v) = 1$; assume $v = a^1$ (so, $d(xa^1) = 1$). Let w, y, z be three vertices in $V(D_1)$ such that w, x, y, z are subsequent in D_1. Then, by fact 1, $d(w, a'^1) = d(w, i^1) = d(y, a'^1) = d(y, i^1) = 2$.

If $d(y, I^1) = d(x, A^1) = 2$, then $W_1 = \{I^1y^1, ya'^1, a'^1w, wi^1, a^1B^1, B^1b^1, b^1A^1, A^1x\}$. If not, we can suppose (up to renaming of cycles C_1^1, C_2^1 and D_1 in the discussion that follows) $d(y, I^1) = 1$. Then, by fact 1 one of the edges i^1x and a^1y is a 2-edge; let us denote it by e. Set $f = a'^1y$ if $e = i^1x$, or $f = i^1w$ if $e = a^1y$. Then, $W_1 = \{a'^1w, i^1y, a^1A^1, A^1b^1, b^1B^1, B^1z\} \cup \{e, f\}$. Figure 2 illustrates this case. In all the above cases set $W_{\hat{T}} = (W_C \setminus W_{C_1}) \cup W_1$ is a set of d.e.p (remark that the hypothesis $d(i^s, B^S) = 2$ does not intervene in the specification of the set $W_{\hat{T}}$) of cardinality

$$\left|W_{\hat{T}}\right| = 4q + 4 + 8 = 4(q + r) \tag{15}$$

From expressions (14) and (15) we conclude for the case $q > 0$:

$$\left|W_{\hat{T}}\right| = 4(q + r) \tag{16}$$

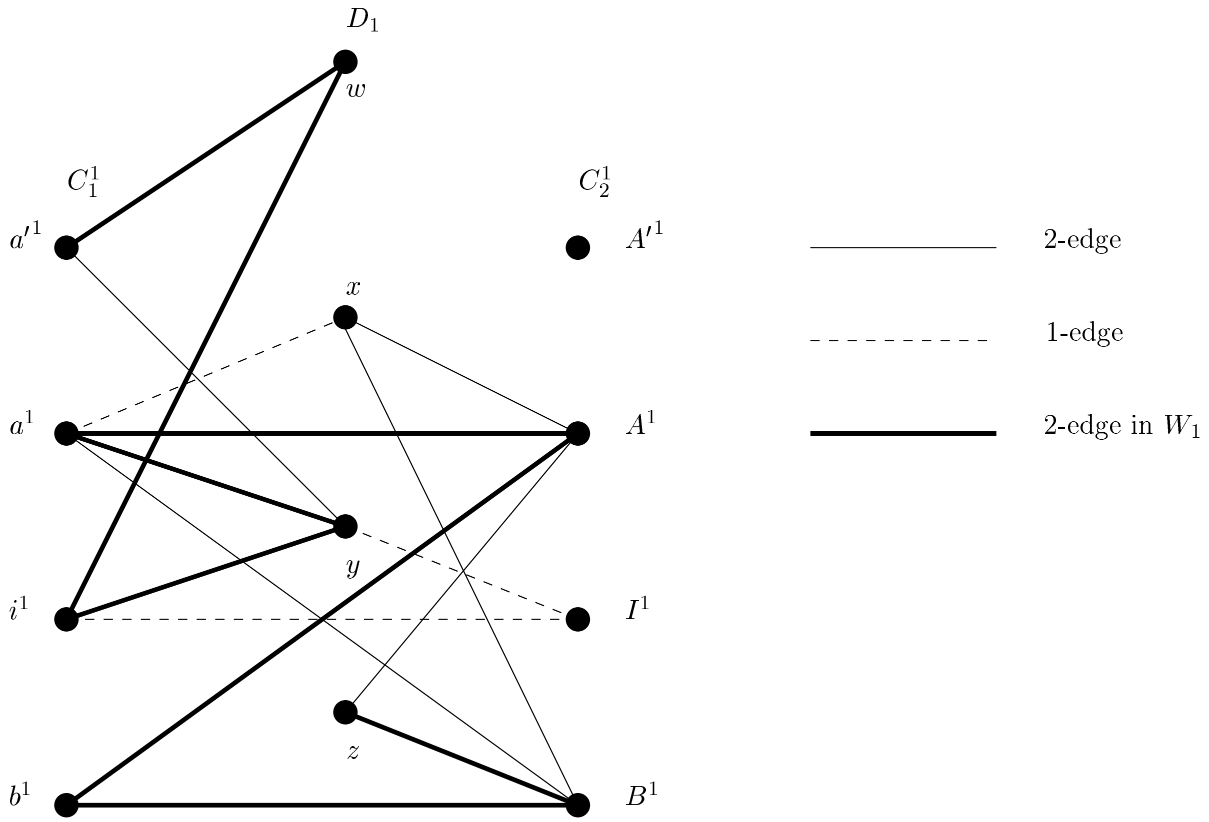

Figure 2. Construction of W_1.

4.2.3.2 $q = 0$

Consider first $|\boldsymbol{E2}| = |\boldsymbol{C_{p+1}}|$. Remark that $\boldsymbol{r} \geqslant \mathbf{1}$ (if not $T(K_n) = C_{p+1}$ is a minimum-distance Hamiltonian tour); note also that C_{p+1} can eventually be empty. From facts 2, 3 and 8, for any cycle D_t, $t = 1, \ldots, r$ the only 1-edges (other than the ones of D_t) incident to vertices of $V(D_t)$ are pairs of $V(D_t) \times V(D_t)$ not included in D_t. However, in any feasible Hamiltonian tour, one cannot use more than $\sum_{t=1}^{t=r}(|D_t| - 1) = d - r$ of them and, consequently, no less than $n - (d - r) = |C_{p+1}| + r$ 2-edges. Hence, $\beta(K_n) \geqslant n + |C_{p+1}| + r = d(M^*) + r = |T(K_n)|$ and the solution computed by algorithm `TSP12` is optimal. For case $|\boldsymbol{E2}| < |\boldsymbol{C_{p+1}}|$, we set $W_{\hat{T}} = W_D$ if $\boldsymbol{r} > \mathbf{1}$, and $W_{\hat{T}} = \emptyset$, if $\boldsymbol{r} = \mathbf{1}$.

4.2.4 Disjoint elementary paths on $\boldsymbol{V(K_n)}$

4.2.4.1 $q > 0, |\boldsymbol{E2}| < |\boldsymbol{C_{p+1}}|$

Consider set $W = W_{\hat{T}} \cup E2$. Using expression (16), we get $|W| = 4(q + r) + |E2|$.

4.2.4.2 $q > 0, |\boldsymbol{E2}| = |\boldsymbol{C_{p+1}}|$

Let $uv \in C_{p+1}$ and u' be a vertex with $|\Gamma_{W_{\hat{T}}}(u')| = 1$, where by $\Gamma_{W_{\hat{T}}}(u')$ we denote the set of neighbors of u' belonging also to $V(W_{\hat{T}})$. Remark that such a vertex u' exists because $W_{\hat{T}}$ is a simple set of paths. Fact 2 ensures $d(u, u') = 2$. We then set $W = W_{\hat{T}} \cup (C_{p+1} \setminus \{uv\}) \cup \{uu'\}$ with (see expression (16)) $|W| = 4(q + r) + |E2|$.

4.2.4.3 $q = 0, |E2| < |C_{p+1}|$

Let us first suppose $r = 1$. Then, let $H = \{e_1, e_2, e_3, e_4\}$ be an elementary path on four edges in C_{p+1} with endpoints u and v and such that $d(e_1) = 1$ and $d(e_2) = 2$; let $M = \{w, x, y, z\}$ be a sequence of four successive vertices in $V(D_1)$ and set $H2 = \{e \in H : d(e) = 2\}$. Then, using facts 1 and 2, we can construct (see figure 3), between paths H and M, a path P containing at least $4 + |H2|$ 2-edges where $|\Gamma_P(v)| \leq 1$. We set $W = P \cup (E2 \setminus H2)$ that constitutes a d.e.p with $|W| = (4 + |H2|) + (|E2| - |H2|) = |E2| + 4$.

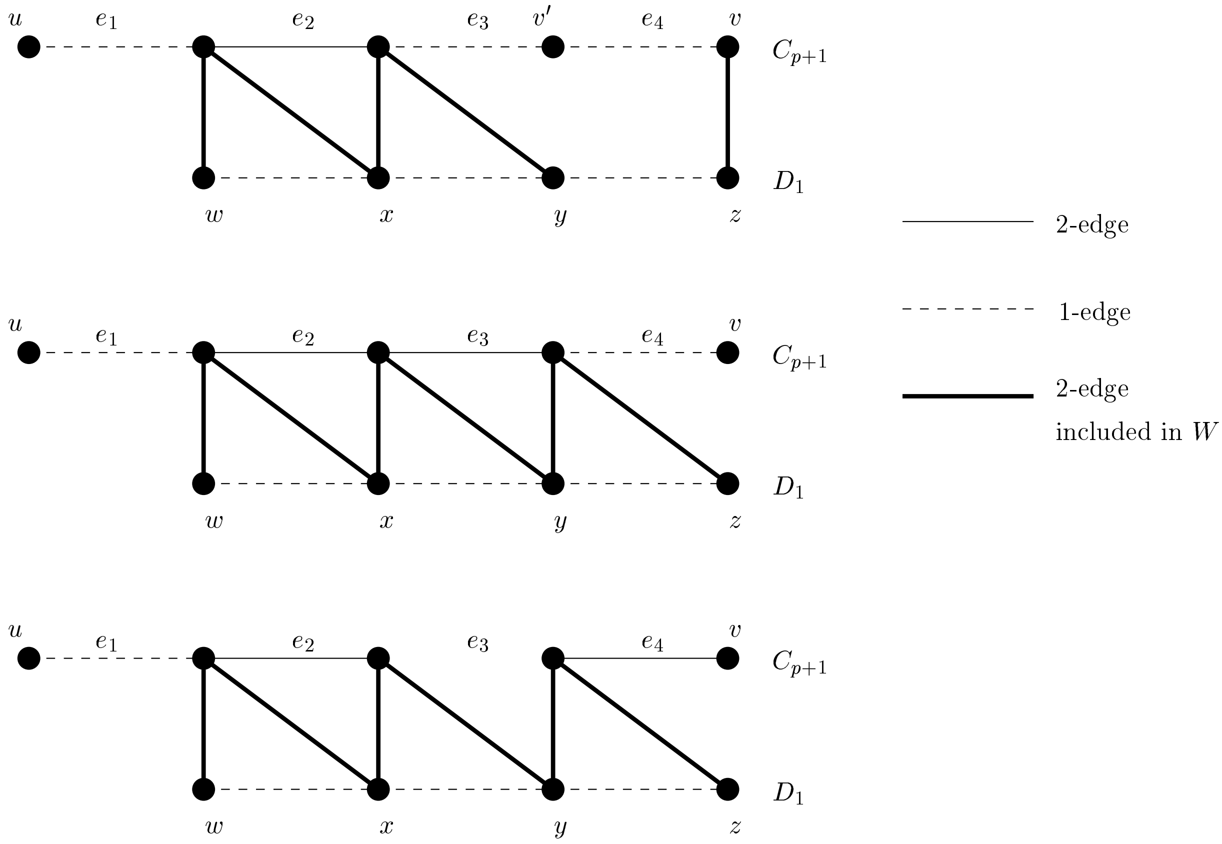

Figure 3. Construction of W supposing $vz = \text{argmax}\{d(v', y), d(v, z)\}$.

Let us now suppose $r > 1$ and let uv be an 1-edge of C_{p+1}. Moreover, from the previous paragraph, for the case we deal with, $W_{\hat{T}} = W_D$, where W_D is given by expression (12). By fact 1 we have that either x_1u, or y_1v is a 2-edge; let us suppose $d(x_1, u) = 2$. Then, the set $W = W_{\hat{T}} \cup E2 \cup \{x_1u\}$ forms a d.e.p. composed of $|W| = (4r - 1) + |E2| + 1 = 4r + |E2| = 4(q + r) + |E2|$ 2-edges.

Consequently, dealing with W, we always have $|W| \geqslant 4(q + r) + |E2|$. One can obtain a tour $T_w(K_n)$ by properly linking d.e.ps by edges (at worst by 1-edges) in order that they form a Hamiltonian cycle on K_n. The so obtained $T_w(K_n)$ has objective value $d(T_w(K_n)) \geqslant n + 4(q + r) + |E2|$; so, using expression (6)

$$\omega(K_n) \geqslant d(T_w(K_n)) \geqslant n + 4(q + r) + |E2| = d(M^*) + 4(q + r). \tag{17}$$

4.3 The differential approximation ratio of TSP12

We have already seen that if $q = 0$ and $|E2| = |C_{p+1}|$, then $\delta(\min\_\mathrm{TSP12}) = 1$. So, for $q > 0$ or $q = 0$ and $|E2| < |C_{p+1}|$ expressions (10), (17) and the fact that $\beta(K_n) \geqslant d(M^*)$, we get

$$\delta_{\texttt{TSP12}}(K_n) = \frac{\omega(K_n) - \lambda_{\texttt{TSP12}}(K_n)}{\omega(K_n) - \beta(K_n)} \geqslant \frac{d(M^*) + 4(q+r) - (d(M^*) + (q+r))}{d(M^*) + 4(q+r) - d(M^*)} = \frac{3(q+r)}{4(q+r)} = \frac{3}{4}.$$

4.4 Ratio 3/4 is tight for TSP12

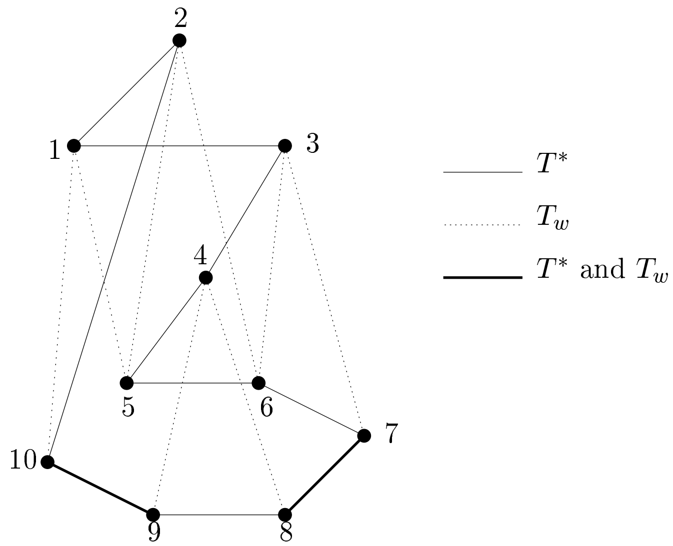

Figure 4. Tightness of the TSP12 approximation ratio.

Consider two cliques and number their vertices by $\{1, \ldots, 4\}$ and by $\{5, 6, \ldots, n+8\}$, respectively. Edges of both cliques have all distance 1. Cross-edges ij, $i = 1, 3$, $j = 5, \ldots, n+8$, are all of distance 2, while every other cross-edge is of distance 1.

Unraveling of TSP12 will produce:

T	$=$	$\{1, 2, 3, 4, 5, 6, \ldots, n+7, n+8, 1\}$	cycle-pathing on edges $(1, 4)$ and $(5, n+8)$
T_w	$=$	$\{1, 5, 2, 6, 3, 7, 4, 8, 9 \ldots, n+7, n+8, 1\}$	using 2-edges $(1, 5), (6, 3), (3, 7)$ and $(n+8, 1)$
T^*	$=$	$\{1, 2, n+8, n+7, ..., 5, 4, 3, 1\}$	using 1-edges $(4, 5), (2, n+8)$

i.e., $\lambda(K_{n+8}) = n+9$, $\beta(K_{n+8}) = n+8$ and $\omega(K_{n+8}) = n+12$ (in figure 4, T^* and T_w are shown for $n = 2$; $T = \{1, \ldots, 10, 1\}$). Consequently, $\delta_{\texttt{TSP12}}(K_{n+8}) = 3/4$ and this completes the proof of theorem 2. ∎

Let us note that the differential approximation ratio of the 7/6-algorithm of [18], when running on K_{n+8}, is also 3/4. The authors of [18] bring also to the fore a family of worst-case instances for their algorithm: one has k cycles of length four arranged around a cycle of length $2k$. We have performed a limited comparative study between their algorithm and the our one, for $k = 3, 4, 5, 6$ (on 24 graphs). The average differential and standard approximation ratios for the two algorithms are presented in table 1.

5 Further results for minimum traveling salesman

5.1 Bridges between differential and standard approximation

Let us consider the following approximation-preserving reduction proposed in [16], strongly inspired by the A-reduction of [17] between pairs (Π, R), where Π is an **NPO** problem and R an

	k	`TSP12`	The algorithm of [18]
Differential ratio	3	0,931100364	0,846702091
	4	0,9000002	0,833333
	5	0,920289696	0,833333
	6	0,9222222	0,833333
Standard ratio	3	0,923350955	0,87013
	4	0,9094018	0,857143
	5	0,92646313	0,857143
	6	0,928178	0,857143

Table 1: A limited comparison between `TSP12` and the algorithm of [18] on some worst-case instances of the latter.

approximation measure. In what follows, we denote by $R[\Pi](I,S)$ the value of the approximation measure R relative to a solution S of an instance I of Π. We suppose that R has values in $[0,1]$ (for the standard approximation, we inverse the approximation ratio in the case of minimization problems).

Definition 3. A G-reduction of the pair (Π_1, R_1) to (Π_2, R_2), denoted by $(\Pi_1, R_1) \leqslant^G (\Pi_2, R_2)$, is a triplet $(\propto, g, c)$ such that:

- $\propto : \mathcal{I}_1 \to \mathcal{I}_2$ polynomially transforms instances of Π_1 into instances of Π_2;
- $g : \mathsf{S}(\propto(I)) \to \mathsf{S}(I)$ polynomially transforms solutions for Π_2 into solutions for Π_1;
- $c : [0,1] \to [0,1]$ $(c^{-1}(0) = \{0\})$ is such that, $\forall \epsilon \in [0,1], \forall I \in \mathcal{I}_1, \forall S \in \mathsf{S}(\propto(I))$,

$$R_2[\Pi_2](\propto(I), S) \geqslant \epsilon \Longrightarrow R_1[\Pi_1](I, g(S)) \geqslant c(\epsilon). \quad \blacksquare$$

The following easy lemma holds.

Lemma 3. *Consider an **NPO** problem $\Pi = (\mathcal{I}, \mathsf{S}, v_I, \text{opt})$. If $\exists t > 0$ such that, $\forall I \in \mathcal{I}$, $|\omega(I) - \beta(I)| \leqslant t \min\{\omega(I), \beta(I)\}$, then $(\Pi, \rho) \leqslant^G (\Pi, \delta)$ with*

$$c_t(\epsilon) = \begin{cases} \frac{t\epsilon+1}{t+1} & \text{opt} = \max \\ \frac{1}{t+1-t\epsilon} & \text{opt} = \min \end{cases}$$

Remark that for min_TSPab we have

$$\omega(K_n) - \beta(K_n) \leqslant bn - an \leqslant (b-a)n \leqslant \frac{b-a}{a}\beta(K_n)$$

and by application of lemma 3 the following theorem holds.

Theorem 3.

$$\begin{aligned} (\text{min\_TSP}ab, \rho) &\leqslant^G (\text{min\_TSP}ab, \delta) \textit{ with } c(\epsilon) = \frac{a}{b-(b-a)\epsilon} \\ (\text{min\_TSP12}, \rho) &\leqslant^G (\text{min\_TSP12}, \delta) \textit{ with } c(\epsilon) = \frac{1}{2-\epsilon}. \end{aligned}$$

Theorem 3 implies $1/\rho_{\texttt{TSP12}} \geqslant 4/5$; in other words, $\rho_{\texttt{TSP12}} \leqslant 5/4$. This ratio is better than the one of [5] for this particular case, but with no operational impact since it is dominated by the result of [18].

Recall that min_TSP12 and min_TSPab are equi-approximable in the differential approximation framework. Consequently, using theorem 3 with $\delta = 3/4$, the following corollary holds.

Corollary 3. *min_ TSPab is approximable within*

$$\rho \leqslant \frac{3}{4} + \frac{1}{4}\frac{b}{a}$$

in the standard framework. This ratio tends to ∞ with b.

Let us now denote by $A_{\max}$ and $A_{\min}$ a maximum and a minimum spanning trees of K_n, respectively, and by $c(A_{\max})$ and $c(A_{\min})$ their respective costs. Then, the following proposition holds.

Proposition 4. *If* $c(A_{\max})/c(A_{\min}) \leqslant \nu$, $\nu > 1$, *then* $(\text{min\_TSP}, \rho) \leqslant^G (\text{min\_TSP}, \delta)$ *with* $c(\epsilon) = 1/(\nu(1-\epsilon)+\epsilon)$.

Proof. Let $T_w(K_n)$ and $T^*(K_n)$ be a worst-value tour and an optimal tour of K_n, respectively. Set $d_w = \min_{v_i v_j \in T_w(K_n)}\{d(i,j)\}$ and $d_\beta = \max_{v_i v_j \in T^*(K_n)}\{d(i,j)\}$. Since $T_w(K_n) \setminus \{\text{argmin}_{v_i v_j \in T_w(K_n)}\{d(i,j)\}\}$ and $T^*(K_n) \setminus \{\text{argmax}_{v_i v_j \in T^*(K_n)}\{d(i,j)\}\}$ are obviously spanning trees of K_n: $c(A_{max}) \geqslant \omega(K_n) - d_w$, $c(A_{min}) \leqslant \beta(K_n) - d_\beta$. Remark also that $d_w \leqslant \omega(K_n)/n$ and $d_\beta \geqslant \beta(K_n)/n$. So, $c(A_{max}) \geqslant \omega(K_n)(1-1/n)$ and $c(A_{min}) \leqslant \beta(K_n)(1-1/n)$. Consequently,

$$\frac{\omega(K_n)}{\beta(K_n)} \leqslant \frac{c(A_{max})\left(1-\frac{1}{n}\right)}{c(A_{min})\left(1-\frac{1}{n}\right)} \leqslant \frac{c(A_{max})}{c(A_{min})} \leq \nu.$$

Hence, $\omega(K_n) - \beta(K_n) \leq (\nu-1)\beta(K_n)$, and using lemma 3 for $t = (\nu - 1)$ we get $c(\epsilon) = (\nu(1-\epsilon)+\epsilon)^{-1}$. ∎

5.2 An inapproximability result

We first note that one can prove very easily (with arguments similar to the ones of theorem 6.13 in [13]) that *min_ TSP cannot be solved by a differential FPTAS unless **P**=**NP***. We now restrict ourselves to min_TSP12 and revisit theorem 3. It is easy to see that it does not only establish links between the approximabilities of min_TSP12 in standard and differential frameworks, but it also establishes limits on its approximability in the two frameworks. Plainly, since approximation of min_TSP12 within $\delta = 1-\epsilon$ implies its approximation within $\rho = 2-(1-\epsilon) = 1+\epsilon$, $0 \leqslant \epsilon \leqslant 1$, if there exists an ϵ_0 such that, under a very likely complexity theory hypothesis, min_TSP12 is inapproximable within $\rho_0 \leqslant 1+\epsilon_0$, then it is inapproximable within $\delta_0 \geqslant 1-\epsilon_0$. In other words, the hardness thresholds for standard and differential frameworks are identical.

Theorem 4. *If under a complexity theory hypothesis min_ TSP12 is inapproximable within $1+\epsilon_0$, then, under the same hypothesis, min_ TSP12 is differentially inapproximable within $1-\epsilon_0$.*

Recall the negative result of [4]: $\forall \epsilon > 0$, no PTAA can guarantee standard approximation ratio less than, or equal to, $3477/3476 - \epsilon$ unless **P**=**NP**. Using theorem 4, $\forall \epsilon > 0$, it is **NP**-hard to approximate min_TSP12 with differential ratio better than $3475/3476 + \epsilon$. Since min_TSP12 is a special case of general min_TSP, the following corollary holds concluding the section.

Corollary 4. *min_ TSP cannot be approximated within differential ratio greater than, or equal to, $3475/3476 + \epsilon$, for every positive ϵ, unless **P**=**NP**.*

Finally, let us note that the inapproximability result of [11] for dense graphs holds also in the differential approximation framework with the same hardness threshold.

6 Differential approximation of maximum traveling salesman

We have also mentioned that in the opposite of min_TSP, max_TSP (certainly less popular than its cousin), although it is **APX**-hard ([20, 18]), can be solved by a PTAA achieving standard approximation ratio $\rho = 5/7$ (this ratio is somewhat worst – 38/63 – when the input-graph is directed).

The purpose of this section is to show that, in the differential approximation framework, the two cousins are equi-approximable establishing so a kind of natural symmetry between the two problems at hand.

Theorem 5. *max_TSP is equi-approximable with min_TSP; consequently it is in **D-APX**.*

Proof. Observe first that, given a graph K_n, there exists a very interesting symmetry between min_ and max_TSP with respect to worst-case and best objective values:

$$\left\{\begin{array}{rcl} \beta_{\min}(K_n) & = & \omega_{\max}(K_n) \\ \beta_{\max}(K_n) & = & \omega_{\min}(K_n) \end{array}\right. \tag{18}$$

Expression (18) confirms what we said in the introduction of the paper that the worst value of a problem can be as hard to compute as the optimal one.

Given a complete graph K_n, let us denote by $\bar{K}_n$ the complete graph on n vertices when one replaces distance $d(i,j)$ by $\bar{d}(i,j) = M - d(i,j)$, $i,j = 1,\dots,n$, for $M = \max_{v_iv_j \in E}\{d(i,j)\} + \min_{v_iv_j \in E}\{d(i,j)\}$. It is easy to see that $\bar{\bar{K}}_n = K_n$. Moreover, any TSP-feasible solution for K_n is TSP-feasible for $\bar{K}_n$.

Given a Hamiltonian cycle T, we use notation $T_{\min}$ (resp., $T_{\max}$) in order to indicate that we deal with a solution of min_TSP (resp., max_TSP). We then have

$$\begin{array}{rcl} |T_{\min}(K_n)| & = & Mn - \left|T_{\max}(\bar{K}_n)\right| \\ |T_{\max}(K_n)| & = & Mn - \left|T_{\min}\left(\bar{K}_n\right)\right| \end{array}$$

and, more particularly,

$$\omega_{\min}(K_n) = Mn - \beta_{\min}\left(\bar{K}_n\right) = Mn - \omega_{\max}\left(\bar{K}_n\right) \tag{19}$$

$$\beta_{\min}(K_n) = Mn - \omega_{\min}\left(\bar{K}_n\right) = Mn - \beta_{\max}\left(\bar{K}_n\right) \tag{20}$$

$$\lambda_{\mathtt{A}}^{\min}(K_n) = Mn - \lambda_{\mathtt{A}}^{\max}\left(\bar{K}_n\right) \tag{21}$$

By the discussion above, one can immediately conclude that for every PTAA A and for every K_n, $\delta_{\mathtt{A}}^{\min}(K_n) = \delta_{\mathtt{A}}^{\max}(\bar{K}_n)$ (where, once again, indices min and max are used to denote min_TSP and max_TSP, respectively). Consequently, $\delta_{\mathtt{A}}^{\min} = \delta_{\mathtt{A}}^{\max}$, $\forall \mathtt{A}$. Since $\delta_{\mathtt{2\_OPT}}^{\min} \geqslant 1/2$, the same holds for $\delta_{\mathtt{2\_OPT}}^{\max}$ and this completes the proof of the theorem. ∎

For $d(i,j) \in \{a,b\}$, $\max_{v_iv_j \in E}\{d(i,j)\} + \min_{v_iv_j \in E}\{d(i,j)\} - d(i,j) \in \{a,b\}$ $\forall v_iv_j \in E$; so, the proof of theorem 5 establishes also equi-approximability between min_TSPab and max_TSPab and the following theorem summarizes differential approximation results for max_TSP.

Theorem 6.

- *max_TSP is approximable within differential approximation ratio 1/2;*
- *max_TSP12 and max_TSPab are approximable within differential approximation ratio 3/4;*
- *for every $\epsilon > 0$, max_TSP cannot be approximated within differential ratio greater than, or equal to, $5379/5380 + \epsilon$, unless **P=NP**.*

7 An improvement of the standard ratio for the maximum traveling salesman with distances 1 and 2

Application of lemma 3 in the case of max_TSPab with $t = (b-a)/a$ gets

$$\rho = c_{\frac{b-a}{a}}(\delta) = \frac{b-a}{b}\delta + \frac{a}{b}$$

and for $\delta = 3/4$ we have

$$\rho = c_{\frac{b-a}{a}}\left(\frac{3}{4}\right) = \frac{3}{4} + \frac{1}{4}\frac{a}{b} \tag{22}$$

The above ratio is always bounded below by 3/4. Here we see another impact of the asymmetry between minimization and maximization versions of TSP in the standard approximation framework. Recall that, as we have seen in section 5.1, the standard approximation ratio for min_TSPab tends to ∞ with b and this obviously holds for every PTAA.

Set now $a = 1$ and $b = 2$ and revisit expression (22). Then, the following theorem immediately holds.

Theorem 7. *max_ TSP12 is polynomially approximable within $\rho^{\max} \geqslant 7/8$.*

Such an improved ratio ($7/8 > 5/7$) for max_TSP12 seems that it cannot be immediately achieved by the interesting work of [20].

Consider now the following algorithm for max_TSP.

```
BEGIN /MTSPALG/
      construct K̄n;
      call the algorithm of [18] to compute a tour Tmin(K̄n);
      OUTPUT Tmax(Kn) ← Tmin(K̄n);
END.
```

Recall that the algorithm called in the first line of the algorithm just above guarantees $\rho^{\min} \leqslant 7/6$. Then, using expressions (20), (21) and some easy algebra, one gets $\rho^{\max}_{\mathtt{MTSPALG}}(K_n) \geqslant 2/3$.

8 Towards stronger differential-inapproximability results for the traveling salesman

Recall that an **NPO** problem Π is called *simple* ([19]) if its restriction Π_k to instances verifying, for every fixed constant $k \in \mathbb{N}$, $\beta(I) \leqslant k$ can be solved in polynomial time. Analogously, we will call Π D-simple if its restriction Π_k to instances verifying, for every fixed constant $k \in \mathbb{N}$, $|\omega(I) - \beta(I)| \leqslant k$ is polynomial. Then the following proposition holds.

Proposition 5. *If Π is not D-simple, then there exists $k_0 \in \mathbb{N}$ such that $\delta(\Pi) < k_0/(k_0+1)$.*

Proof. Suppose Π not D-simple and $\delta(\Pi) \geqslant k/(k+1)$, $\forall k \in \mathbb{N}$. Then, $\forall I \in \mathcal{I}$,

$$\omega(I) - \lambda(I) \geqslant \omega(I) - \beta(I) - \frac{\omega(I) - \beta(I)}{k+1} \tag{23}$$

Consider now an instance $I' \in \mathcal{I}$ such that $\omega(I') - \beta(I') \leqslant k$. Then, since objective function's values are integer (item 3 of definition 1), expression (23) gives $\omega(I') - \lambda(I') = \omega(I') - \beta(I')$. Consequently, it suffices to set $k_0 = \min\{k : \Pi_k \text{ non polynomial}\}$ in order to complete the proof. ∎

In other words, problems which are not D-simple do not admit differential PTAS (this is the differential-equivalent of the result of [19] for the standard approximation). The proposition above allows achievement of stronger hardness thresholds provided that k_0 is a fixed constant. We conjecture that TSP is not D-simple and this for a small k_0. If this was true, the hardness threshold of corollary 4 could be meaningfully improved.

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