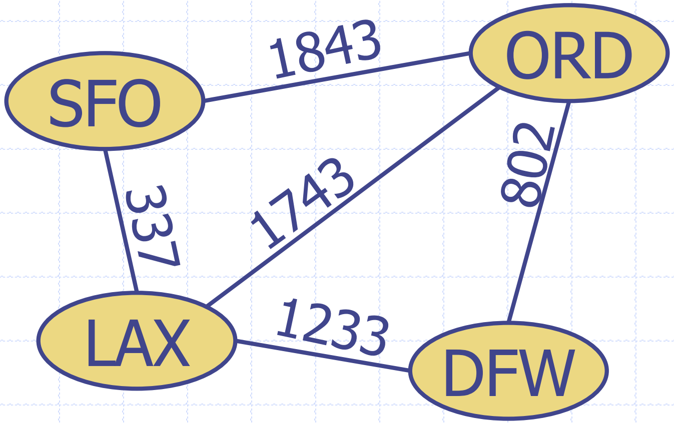
Lecture 16:
BFS Graph Traversal

Principle of Transcending



Wholeness Statement

Graphs have many useful applications in different areas of computer science. However, to be useful we have to be able to traverse them. One of the two primary ways that graphs are systematically explored, is using the breadth-first search algorithm. *Science of Consciousness:* The TM technique provides a simple, effortless way to systematically explore the different levels of the conscious mind until the process of thinking is transcended and unbounded silence is experienced; this is the field of wholeness of individual and cosmic intelligence.

List of Terms

- Graph

- Vertex, vertices
- End vertices
- Adjacent vertices
- Degree of a vertex

- Edges
- Incident edges
- Directed edge, undirected edge
- Directed graph, undirected graph, mixed graph

- Path, simple path
- Cycle, simple cycle

More Terms

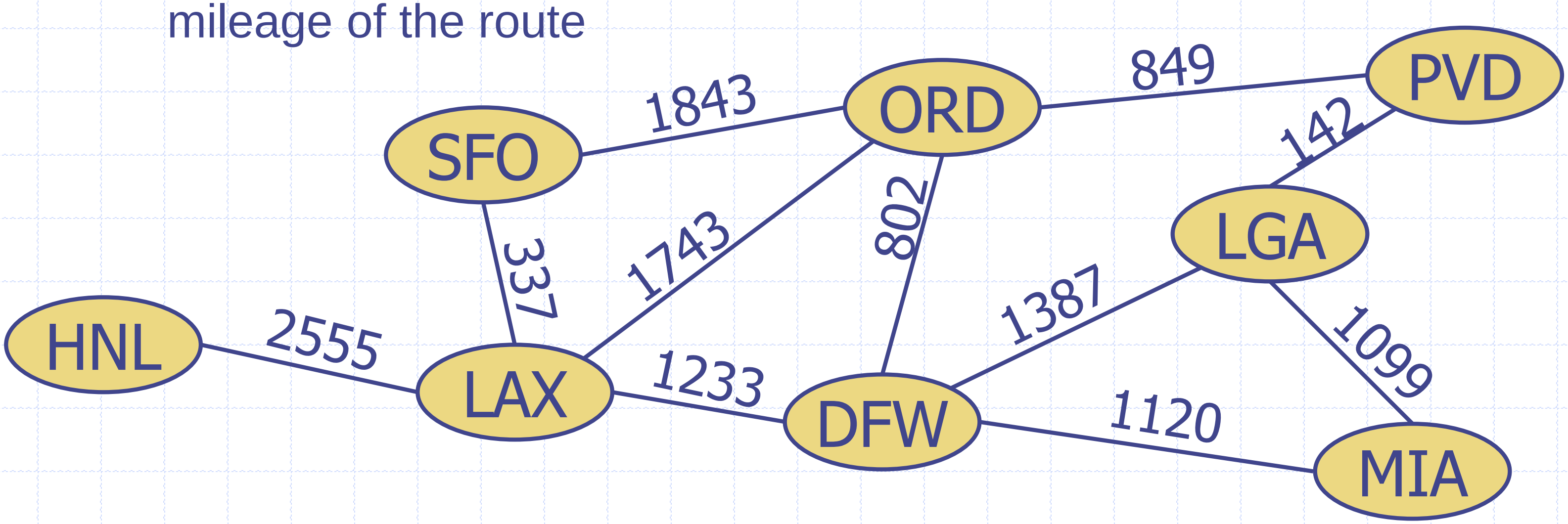
- Subgraph
- Connectivity
  - Connected Vertices (path between them)
  - Connected Graph (all vertices are connected)
  - Connected Component (maximal connected subgraph)
- Tree (connected, no cycles)
- Forest (one or more trees)
- Spanning Tree and Spanning Forest

Breadth-First Search Outline and Reading

- Breadth-first search
  - Algorithm
  - Example
  - Properties
  - Analysis
  - Applications
- DFS vs. BFS
  - Comparison of applications
  - Comparison of edge labels

Graph

- A graph is a pair (V, E), where
  - V is a set of nodes, called vertices
  - E is a collection of pairs of vertices, called edges
  - Vertices and edges are positions and store elements
- Example:
  - A vertex represents an airport and stores the three-letter airport code
  - An edge represents a flight route between two airports and stores the mileage of the route



Properties

Property 1

$\sum_v \deg(v) = 2m$

Proof: each edge is counted twice

Property 2

In an undirected graph with no self-loops and no parallel edges

$m \leq n\,(n-1)/2$

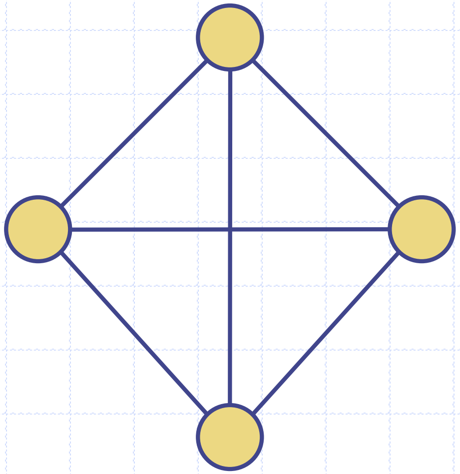
Proof: each vertex has degree at most $(n-1)$

What is the bound for a directed graph?

$m \leq n\ (n\text{-}1)$

Notation

n	number of vertices
m	number of edges
$\deg(v)$	degree of vertex v



Example

- $n = 4$
- $m = 6$
- $\deg(v) = 3$

Main Methods of the Graph ADT

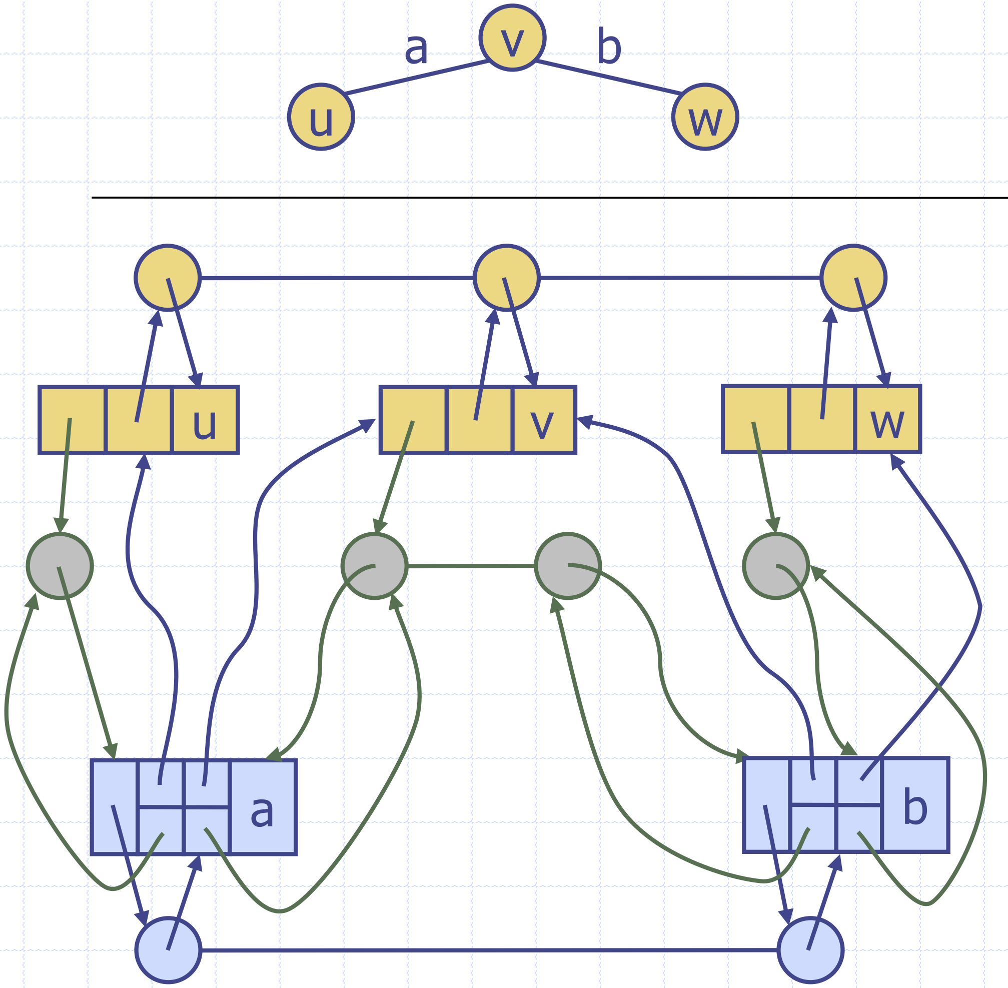
- Vertices and edges
  - are Positions
  - store elements
- Accessor methods
  - aVertex()
  - incidentEdges(v)
  - endVertices(e)
  - opposite(v, e)
  - areAdjacent(v, w)
- Update methods
  - insertVertex(o)
  - insertEdge(v, w, o)
  - removeVertex(v)
  - removeEdge(e)
- Generic methods
  - numVertices()
  - numEdges()
  - vertices()
  - edges()
  - degree(v)

Graph Data Structures

Adjacency list

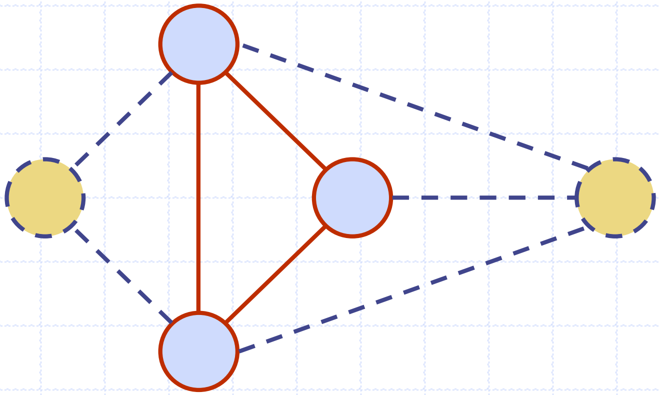
Adjacency List Structure

- Edge list structure
- Incidence sequence for each vertex
  - sequence of references to edge objects of incident edges
- Augmented edge objects
  - references to associated positions in incidence sequences of end vertices

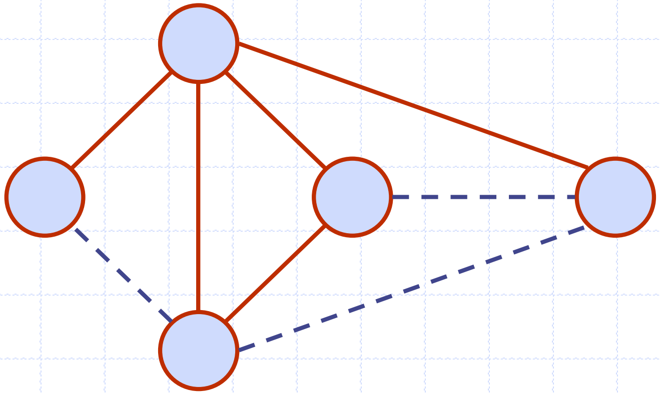


Subgraphs

- A subgraph S of a graph G is a graph such that
  - vertices(S) $\subseteq$ vertices(G)
  - edges(S) $\subseteq$ edges(G)
- A spanning subgraph of G is a subgraph that contains all the vertices of G, i.e., vertices(S) $=$ vertices(G)



Subgraph

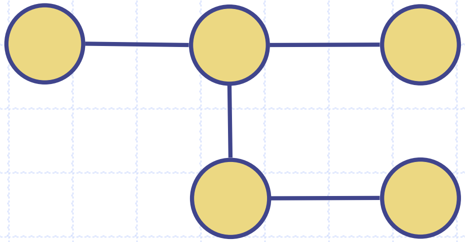


Spanning subgraph

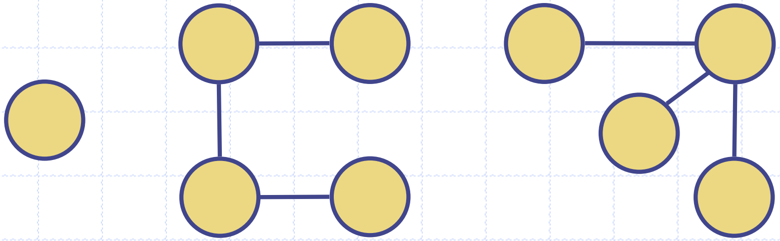
Trees and Forests

- A (free) *tree* is an **undirected** graph T such that
  - T is **connected**
  - T has no **cycles**

  This definition is different from the definition of a rooted tree
- A *forest* is an undirected graph without cycles
- The connected components of a forest are trees



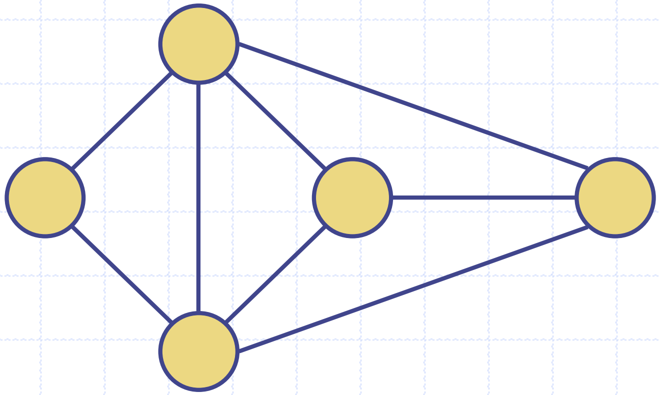
Tree



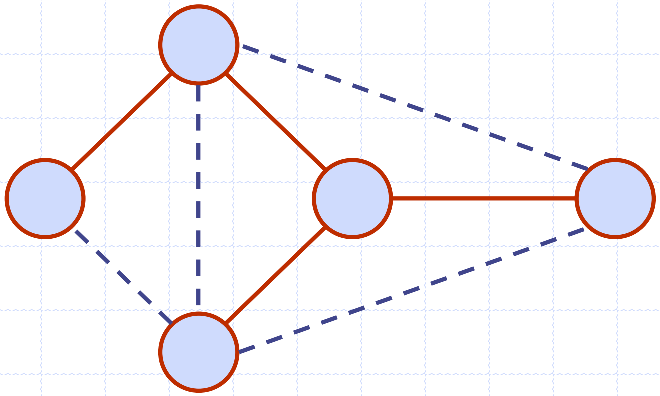
Forest

Spanning Trees and Forests

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest

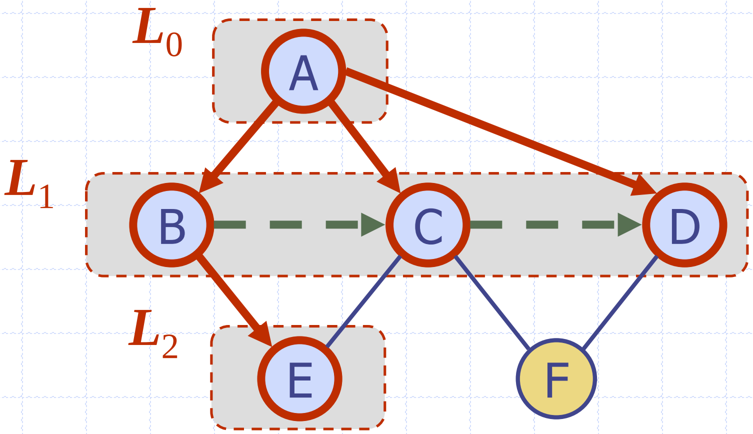


Graph



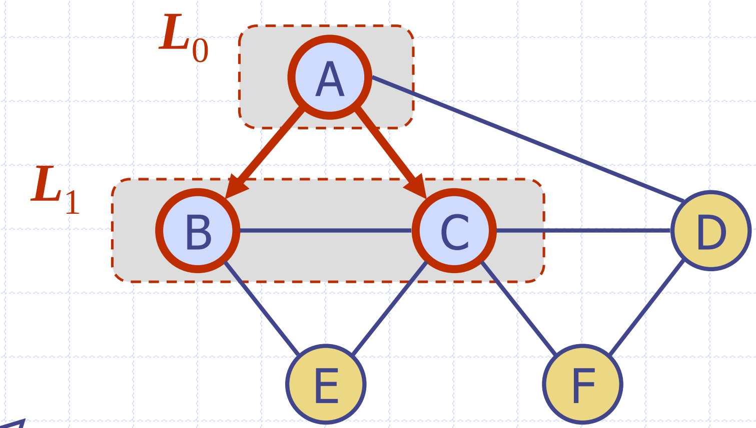
Spanning tree

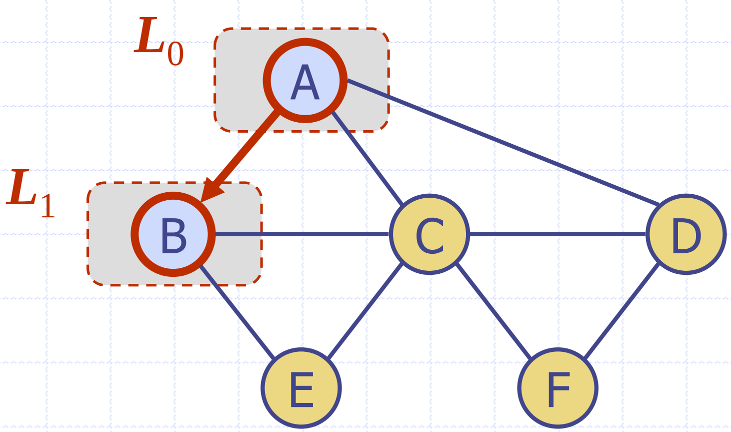
Breadth-First Search



Example

- unexplored vertex
- visited vertex
- unexplored edge
- discovery edge
- cross edge





Example

- unexplored vertex
- visited vertex
- unexplored edge
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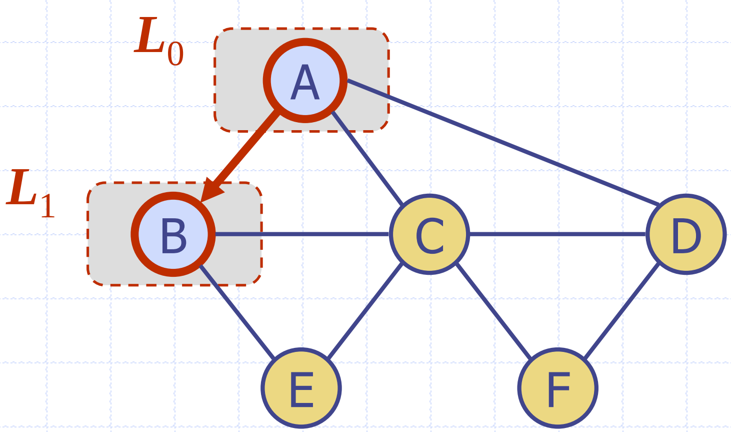
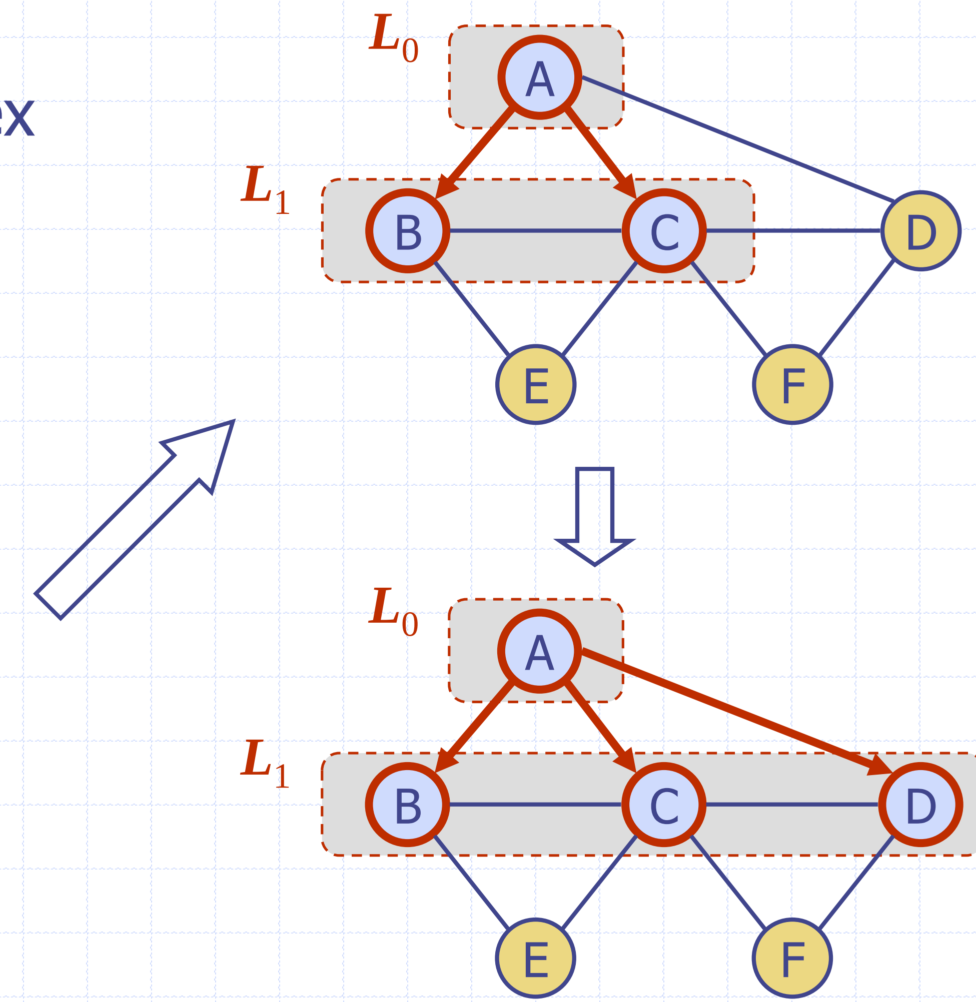
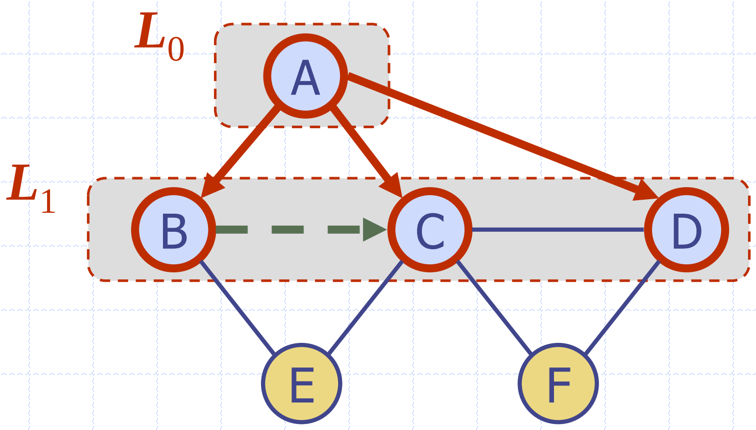
- cross edge

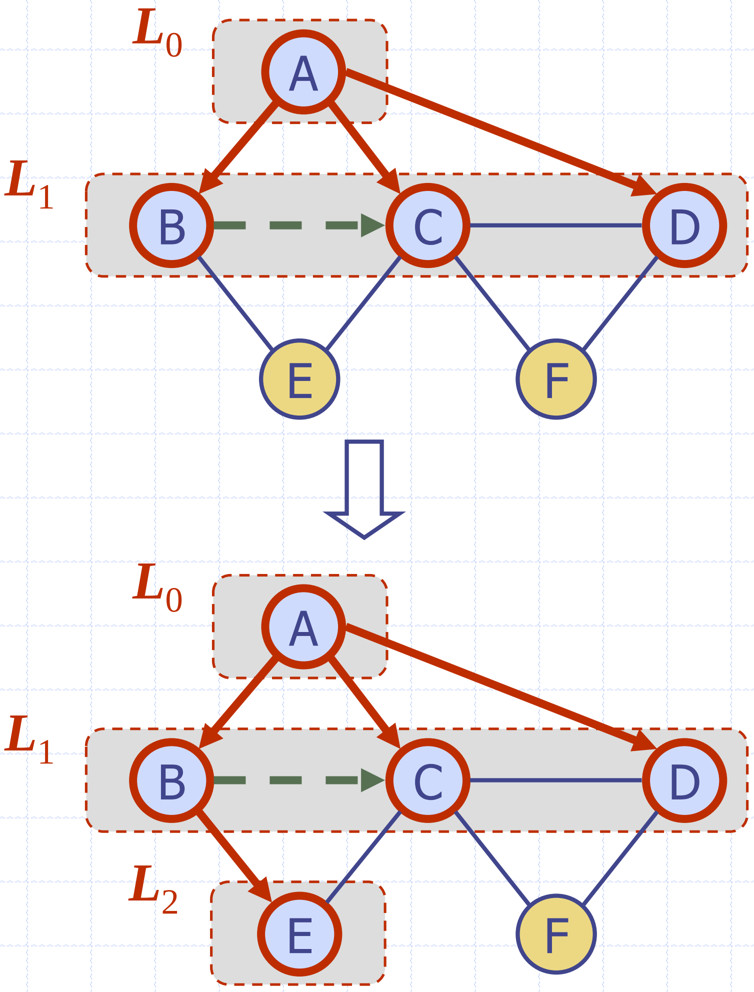




Example (cont.)



Example (cont.)



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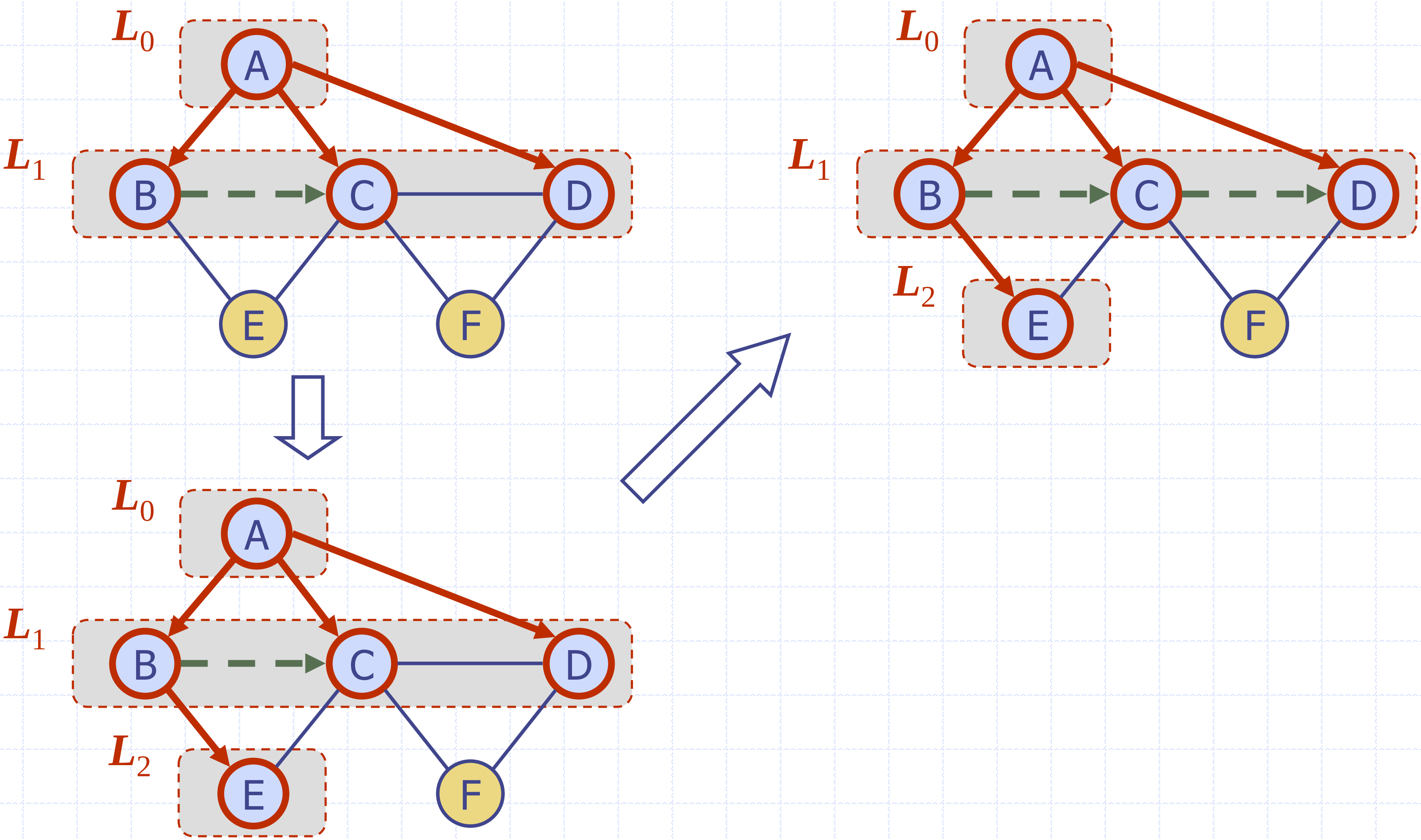

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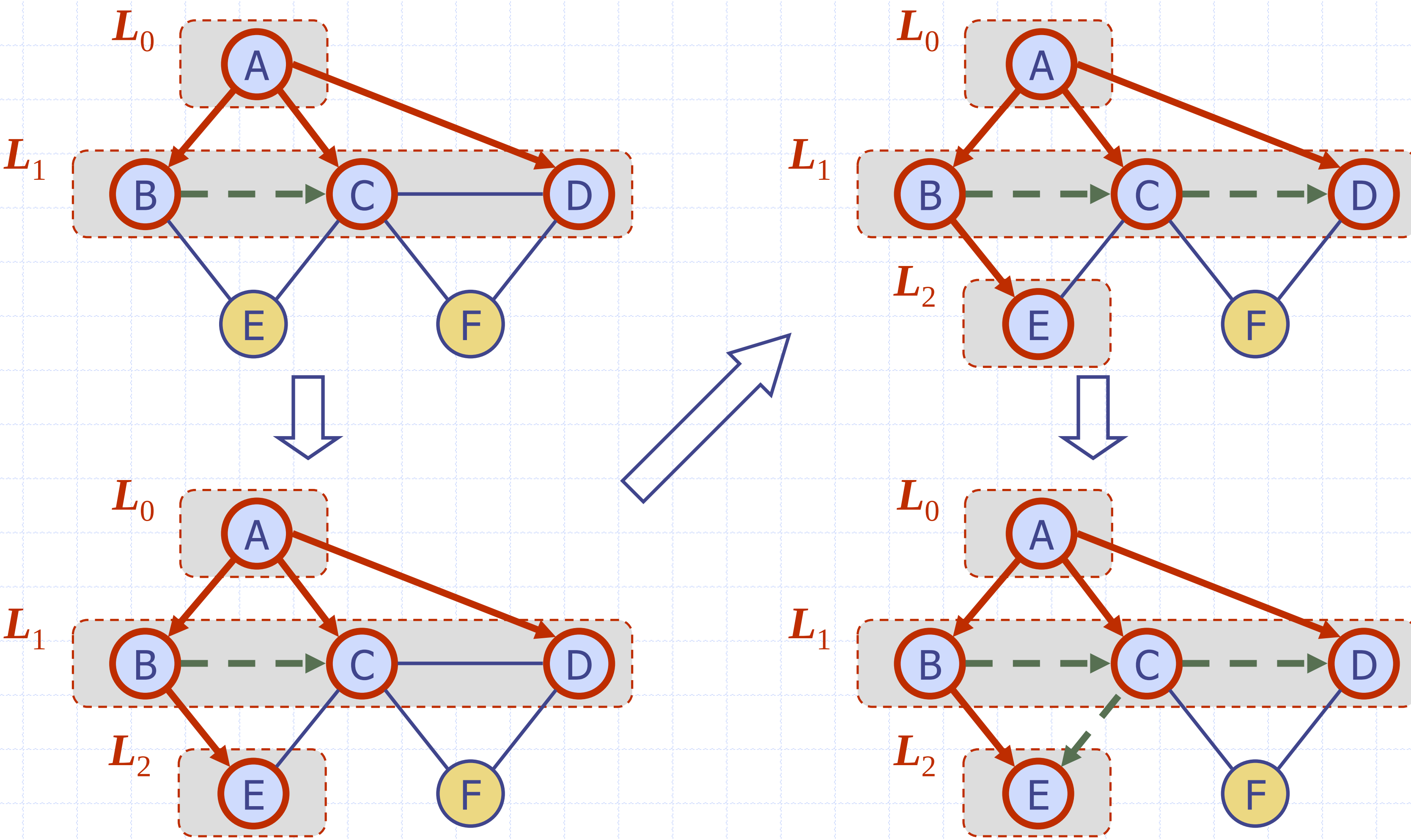

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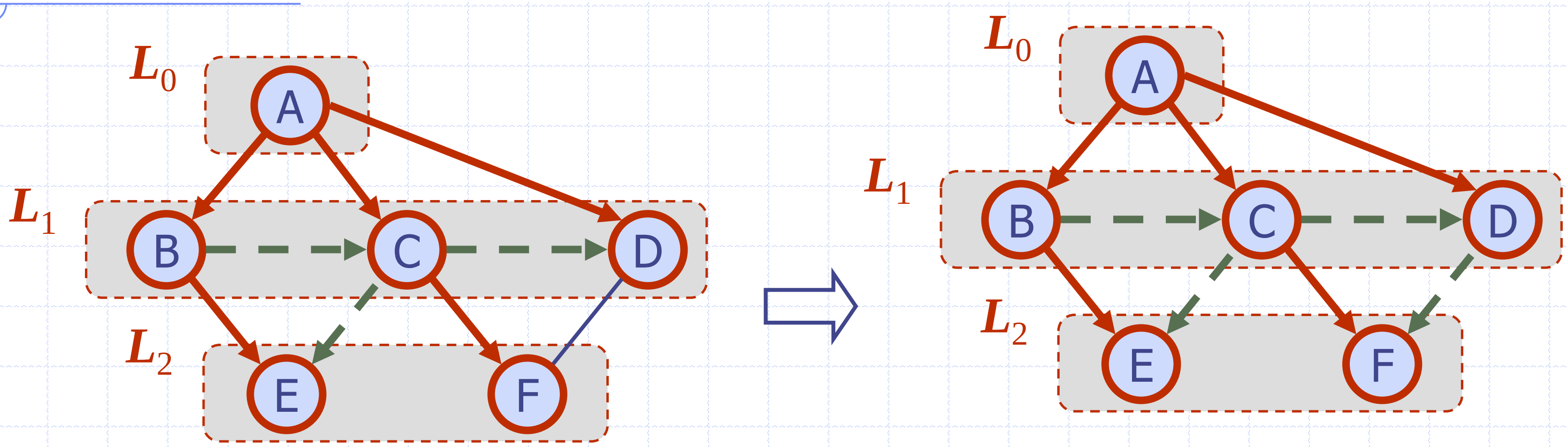
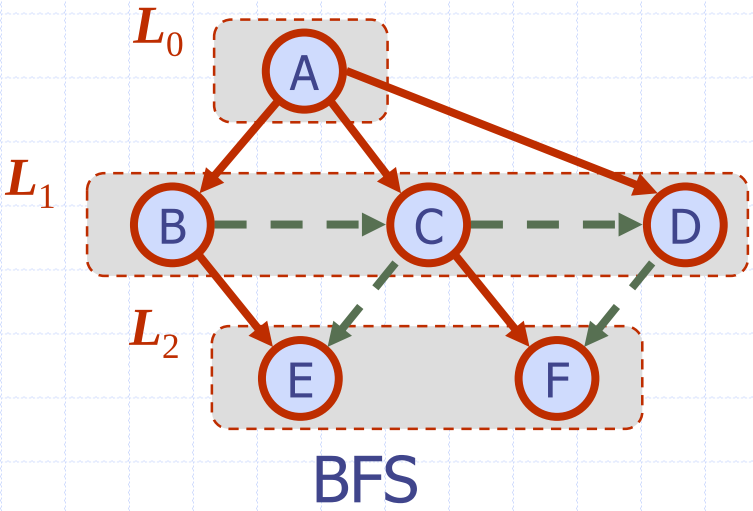


BFS Levels

When actually implemented, the levels are normally merged into a single list/queue



BFS Algorithm

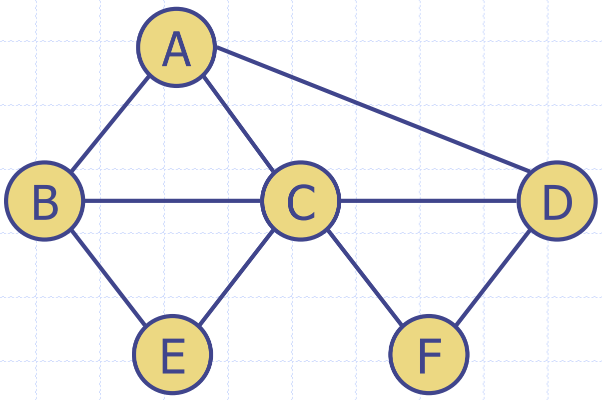
- The BFS algorithm using a single list/sequence/Queue

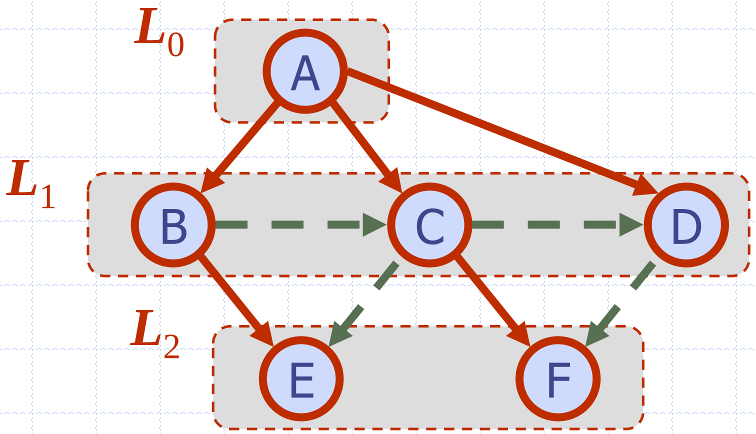
Algorithm *BFS*(*G*)
    Input graph *G*
    Output labeling of the edges and partition of the vertices of *G*
  for all $u \in$ *G.vertices*()
    setLabel(*u, UNEXPLORED*)
  for all $e \in$ *G.edges*()
    setLabel(*e, UNEXPLORED*)
  for all $v \in$ *G.vertices*()
    if *getLabel*(*v*) = *UNEXPLORED*
      BFScomponent(*G, v*)\

Algorithm *BFScomponent*(*G, s*)
  Q ← new empty Queue
  Q.enqueue(*s*)
  setLabel(*s, VISITED*)
  while ¬*Q.isEmpty*()
    v ← *Q.dequeue* ()
    for all $e \in$ *G.incidentEdges*(*v*)
      if *getLabel*(*e*) = *UNEXPLORED* **then**
        w ← *G.opposite*(*v,e*)
        if *getLabel*(*w*) = *UNEXPLORED*
        then
          setLabel(*e, DISCOVERY*)
          setLabel(*w, VISITED*)
          Q.enqueue(*w*)
        else
          setLabel(*e, CROSS*)

Properties

- Notation
  - G_s: connected component of s
- Property 1
  - $BFS(G, s)$ visits all the vertices and edges of G_s
- Property 2
  - The discovery edges labeled by $BFS(G, s)$ form a spanning tree T_s of G_s
- Property 3
  - For each vertex v in L_i
    - The path of T_s from s to v has i edges
    - Every path from s to v in G_s has at least i edges





Analysis

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or CROSS
- Each vertex is inserted once into a sequence L_i
- Method incidentEdges is called once for each vertex
- BFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

Breadth-First Search

- Breadth-first search (BFS) is a general technique for traversing a graph
- A BFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected
  - Computes the connected components of G
  - Computes a spanning forest of G

Breadth-First Search

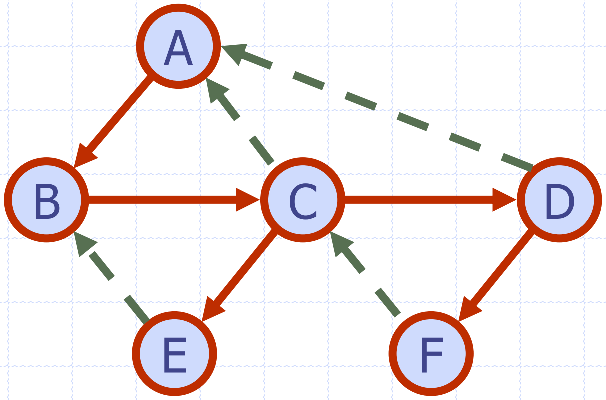
- BFS on a graph with n vertices and m edges takes $O(n + m)$ time
- BFS can be further extended to solve other graph problems
  - Find and report a path with the minimum number of edges between two given vertices
  - Find a simple cycle, if there is one

Applications

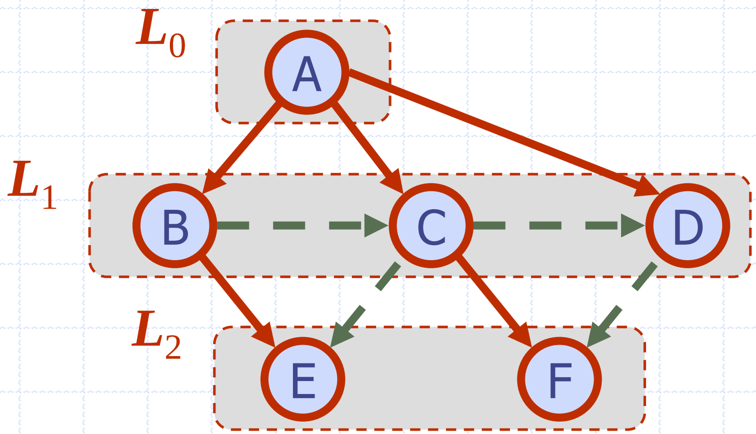
- Using the template method pattern, we can specialize the BFS traversal of a graph G to solve the following problems in $O(n + m)$ time
  - Compute the connected components of G
  - Compute a spanning forest of G
  - Find a simple cycle in G, or report that G is a forest
  - Given two vertices of G, find a path in G between them with the minimum number of edges, or report that no such path exists

DFS vs. BFS

Applications	DFS	BFS
Spanning forest, connected components, paths, cycles	√	√
Shortest paths		√
Biconnected components	√	



DFS



BFS

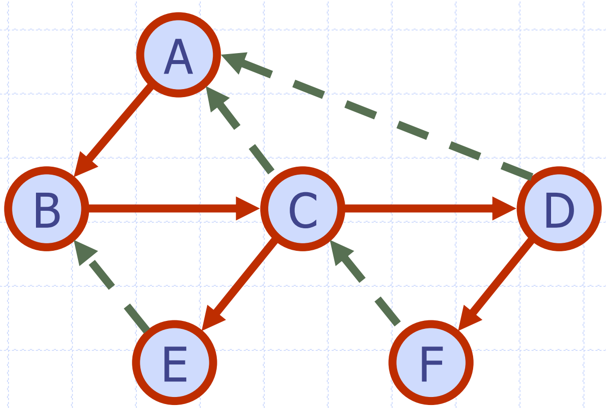
DFS vs. BFS (cont.)

Back edge (v,w)

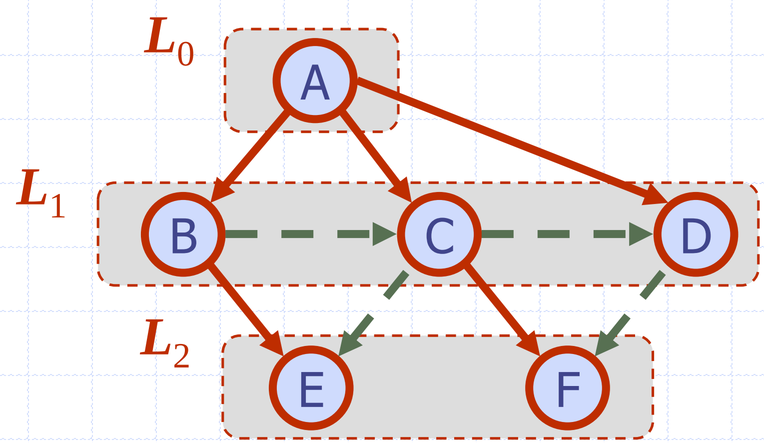
- w is an ancestor of v in the tree of discovery edges

Cross edge (v,w)

- w is in the same level as v or in the next level in the tree of discovery edges



DFS



BFS

Main Point

1. During breadth-first search of a graph, the search repeatedly takes one step in all directions until all vertices and edges are visited. This is a bit like searching for fulfillment in waking state, i.e., floating on the surface of the mind or through one's daily activity.
   Science of Consciousness: In contrast, Transcendental Meditation takes the mind immediately and effortlessly to the deepest levels where true fulfillment can be gained.

Template Method Pattern

Depth-first search is to graphs what the Euler tour is to binary trees

Example of the Template Method Pattern in JavaScript

- Generic algorithm that can be specialized by redefining certain steps
- Implemented by means of an abstract JavaScript class
- Visit methods that can be redefined by subclasses
- Template method eulerTour
  - Recursively called on the left and right children
  - A result array r with elements result[0], result[2] and result[1] keeps track of the output of the recursive calls to eulerTour

```
class EulerTour {
    visitExternal(T, p, result) { }
    visitPreOrder(T, p, result) { }
    visitInOrder(T, p, result) { }
    visitPostOrder(T, p, result) { }
    eulerTour(T, p) {
        let result = new Array(3);
        if (T.isExternal(p)) {
                this. visitExternal(T, p, result);
        } else {
                this.visitPreOrder(T, p, result);
                result[0] = this.eulerTour(T, T.leftChild(p));
                this.visitInOrder(T, p, result);
                result[2] = eulerTour(T.rightChild(p));
                this.visitPostOrder(T, p, result);
        }

        return result[1];
}
```

Euler Tour Template (pseudo-code)

Algorithm ***EulerTour***(T, ***v***)
    if T.***isExternal*** (***v***) **then**
        visitExternal(T, v, result)
    else
        visitPreOrder(T, v, result)
        result[0] ← ***EulerTour***(T, T.***leftChild***(***v***))
        visitInOrder(T, v, result)
        result[2] ← ***EulerTour***(T, T.***rightChild***(***v***))
        visitPostOrder(T, v, result)

    return result[1]

Exercise

- Using the template, give a pseudo code algorithm height(T) to calculate the height of a given tree T.

Specialization (Subclass) of EulerTour

- We show how to specialize class EulerTour to evaluate an arithmetic expression
- Assumptions
  - External nodes store Integer objects
  - Internal nodes store Operator objects supporting method operation (Integer, Integer)

```
public class Height extends EulerTour {

    visitExternal(T, p, result) {
        result[1] = 0;
    }

    visitPostOrder(T, p, result) {
        result[1] = 1 + Math.max(result[0], result[2]);
    }

    height(T) {

        return this.eulerTour(T, T.root());
    }

    …

}
```

Template Version of DFS

```
Algorithm DFS(G)
  Input graph G
  Output the edges of G are
         labeled as discovery edges
         and back edges

  initResult( G )
  for all  u ∈ G.vertices()
     setLabel(u, UNEXPLORED)
     preInitVertex(u)
  for all  e ∈ G.edges()
     setLabel(e, UNEXPLORED)
     preInitEdge(e)
  for all  v ∈ G.vertices()
     if  getLabel(v) = UNEXPLORED
        preComponentVisit(G, v)
        DFScomponent(G, v)
        postComponentVisit(G, v)

  return result( G )
```

```
Algorithm DFScomponent(G, v)
  setLabel(v, VISITED)
  startVertexVisit(G, v)
  for all  e ∈ G.incidentEdges(v)
     if  getLabel(e) = UNEXPLORED
        w ← opposite(v,e)
        if getLabel(w) = UNEXPLORED
           setLabel(e, DISCOVERY)
           preDiscoveryTraversal(G, v, e, w)
           DFS(G, w)
           postDiscoveryTraversal(G, v, e, w)
        else
           setLabel(e, BACK)
           backTraversal(G, v, e, w)
  finishVertexVisit(G, v)
```

Path Finding
Override hook operations

```
Algorithm DFScomponent(G, v)
    setLabel(v, VISITED)
    startVertexVisit(G, v)
    for all  e ∈ G.incidentEdges(v)
        if  getLabel(e) = UNEXPLORED
            w ← opposite(v,e)
            if getLabel(w) = UNEXPLORED
                setLabel(e, DISCOVERY)
                preDiscoveryTraversal(G, v, e, w)
                DFScomponent(G, w)
                postDiscoveryTraversal(G, v, e, w)
            else
                setLabel(e, BACK)
                backEdgeVisit(G, v, e, w)
    finishVertexVisit(G, v)
```

```
Algorithm pathDFS(G, v, z, S)
    setLabel(v, VISITED)
    S.push(v)
    if  v = z  then
        path ← S.elements()
    for all  e ∈ G.incidentEdges(v)  do
        if  getLabel(e) = UNEXPLORED  then
            w ← opposite(v, e)
            if getLabel(w) = UNEXPLORED then
                setLabel(e, DISCOVERY)
                S.push(e)
                pathDFS(G, w, z)
                S.pop()        { e gets popped }
            else
                setLabel(e, BACK)

    S.pop()                { v gets popped }
```

Overriding hook methods in a subclass FindSimplePath

Algorithm ***findPath***(***G, u, v***)
  S ← new empty stack {S is a subclass field}
  z ← v {z is a subclass field & is the target vertex}
  path ← ∅ {path is a subclass field & is the path from u to v}
  for all *u* ∈ *G.vertices*()
    setLabel(*u, UNEXPLORED*)
  for all *e* ∈ *G.edges*()
    setLabel(*e, UNEXPLORED*)
  DFScomponent(*G, u*)
  return(*path*)

Algorithm ***startVertexVisit***(***G, v***)
  S.push(v)
  if v=z then {z is a subclass field & is the target}
    path ← S.elements() {path is a subclass field & is the result}

Algorithm ***preDiscoveryTraversal***(***G, v , e, w***)
  S.push(e)

Algorithm ***postDiscoveryTraversal***(***G, v, e, w***)
  S.pop() {pop *e* off the stack}

Algorithm ***finishVertexVisit***(***G, v***)
  S.pop() {pop *v* off the stack}

Template Version of DFS (v2)

```
Algorithm DFS(G)
    Input graph G
    Output the edges of G are
            labeled as discovery edges
            and back edges

    initResult( G )
    for all u ∈ G.vertices()
        setLabel(u, UNEXPLORED)
        preInitVertex(u)
    for all e ∈ G.edges()
        setLabel(e, UNEXPLORED)
        preInitEdge(e)
    for all v ∈ G.vertices()
        if isNextComponent(G, v)
            preComponentVisit(G, v)
            DFScomponent(G, v)
            postComponentVisit(G, v)

    return result( G )

Algorithm isNextComponent(G, v)
    return getLabel(v) = UNEXPLORED
```

```
Algorithm DFScomponent(G, v)
    setLabel(v, VISITED)
    beginVertexVisit(G, v)
    for all e ∈ G.incidentEdges(v)
        if getLabel(e) = UNEXPLORED
            w ← opposite(v,e)
            preEdgeTraversal(G, v, e, w)
            if getLabel(w) = UNEXPLORED
                setLabel(e, DISCOVERY)
                preDiscoveryTraversal(G, v, e, w)
                DFScomponent(G, w)
                postDiscoveryTraversal(G, v, e, w)
            else
                setLabel(e, BACK)
                backTraversal(G, v, e, w)
    finishVertexVisit(G, v)
```

Overriding hook methods in a subclass FindSimplePath (v2)

Algorithm ***findPath*****(*****G, u, v*****)**
    S ← new empty stack   {S is a subclass field}
    start ← u   {start is a subclass field & is the starting vertex}
    dest ← v   {dest is a subclass field & is the destination vertex}
    path ← ∅   {path is a subclass field & is the path from u to v}
    return DFS(G)

Algorithm ***result*****(*****G*****)**
    return(path)

Algorithm ***isNextComponent*****(*****G, v*****)**
    return v=start   {start the component traversal at vertex start}

Algorithm ***beginVertexVisit*****(*****G, v*****)**
    S.push(v)
    if v=dest then   {dest is a subclass field & is the destination vertex}
        path ← S.elements() {path is a subclass field & is the result}

Algorithm ***preDiscoveryTraversal*****(*****G, v , e, w*****)**
    S.push(e)

Algorithm ***postDiscoveryTraversal*****(*****G, v, e, w*****)**
    S.pop()   {pop *e* off the stack}

Algorithm ***finishVertexVisit*****(*****G, v*****)**
    S.pop()   {pop *v* off the stack}

Exercise: Cycle Finding Override hook operations

```
Algorithm DFScomponent(G, v)
    setLabel(v, VISITED)
    startVertexVisit(v)
    for all  e ∈ G.incidentEdges(v)
        if  getLabel(e) = UNEXPLORED
            w ← opposite(v,e)
            preEdgeTraversal(G, v, e, w)
            if getLabel(w) = UNEXPLORED
                setLabel(e, DISCOVERY)
                preDiscoveryTraversal(G, v, e, w)
                DFScomponent(G, w)
                postDiscoveryTraversal(G, v, e, w)
            else
                setLabel(e, BACK)
                backEdgeVisit(G, v, e, w)
    finishVertexVisit(G, v)
```

```
Algorithm cycleDFS(G, v)
    setLabel(v, VISITED)
    if  cycle ≠ null  then return
    S.push(v)
for all  e ∈ G.incidentEdges(v)
    if  getLabel(e) = UNEXPLORED
        w ← opposite(v,e)
        if getLabel(w) = UNEXPLORED
            setLabel(e, DISCOVERY)
            S.push(e)
            cycleDFS(G, w)
            S.pop()
        else
            setLabel(e, BACK)
            S.push(e)
            cycle ← new empty sequence
            o ← w
            do
                cycle.insertLast(o)
                o ← S.pop()
            while o ≠ w
S.pop()
```

Overriding template methods in subclass FindCycles

Algorithm ***startVertexVisit***(G, v)
if ¬ cycleFound then S.push(v)

Algorithm ***finishVertexVisit***(G, v)
if ¬ cycleFound then S.pop()

Algorithm ***preDiscoveryTraversal***(G, v, e)
if ¬ cycleFound then S.push(e)

Algorithm ***postDiscoveryTraversal***(G, v, e, w)
if ¬ cycleFound then S.pop()

Algorithm ***backEdgeVisit*** (G, v, e, w)
if ¬ cycleFound then
    S.push(e)
    cycle ← new empty sequence
    o ← ***w***
    do
        cycle.insertLast(***o***)
        o ← ***S.pop***()
    while ***o*** ≠ ***w***
    cycleFound ← true {cycleFound is a subclass field, initially false}

- What additional method(s) do we need to create or need to override?
- We need the findcycle(G) method that calls BFS(G)
- Otherwise the hook methods are not executed

Template Version of DFS

```
Algorithm DFS(G)
  Input graph G
  Output the edges of G are
         labeled as discovery edges
         and back edges

   initResult( G )
   for all u ∈ G.vertices()
      setLabel(u, UNEXPLORED)
      preInitVertex(u)
   for all e ∈ G.edges()
      setLabel(e, UNEXPLORED)
      preInitEdge(e)
   for all v ∈ G.vertices()
      if getLabel(v) = UNEXPLORED
         preComponentVisit(G, v)
         DFScomponent(G, v)
         postComponentVisit(G, v)

   return result( G )
```

```
Algorithm DFScomponent(G, v)
  setLabel(v, VISITED)
  startVertexVisit(G, v)
  for all e ∈ G.incidentEdges(v)
     if getLabel(e) = UNEXPLORED
        w ← opposite(v,e)
        preEdgeTraversal(G, v, e, w)
        if getLabel(w) = UNEXPLORED
           setLabel(e, DISCOVERY)
           preDiscoveryTraversal(G, v, e, w)
           DFS(G, w)
           postDiscoveryTraversal(G, v, e, w)
        else
           setLabel(e, BACK)
           backTraversal(G, v, e, w)
  finishVertexVisit(G, v)
```

Main Point

2. The Template Method Pattern implements the changing and non-changing parts of an algorithm in the superclass; it then allows subclasses to override certain (changeable) steps of an algorithm without modifying the basic structure of the original algorithm.

   Science of Consciousness: The changing and non-changing aspects of creation are unified in the field pure intelligence that we experience every day during our TM program.

Recursive Programs

- The call structure can be described as a depth-first search of a rooted tree
  - Each non-root vertex corresponds to a recursive call
  - A tree is a logical construct, not an explicit data structure

Connecting the Parts of Knowledge with the Wholeness of Knowledge

1. Almost any algorithm for solving a problem on a graph or digraph requires examining or processing each vertex or edge.
2. Depth-first and breadth-first search are two particularly useful and efficient search strategies requiring linear time if implemented using adjacency lists.

3. **Transcendental Consciousness** is the goal of all searches, the field of complete fulfillment.
4. **Impulses within Transcendental Consciousness**: The dynamic natural laws within this unbounded field govern all activities and evolution of the universe.
5. **Wholeness moving within itself:** In Unity Consciousness, one experiences that the self-referral activity of the unified field gives rise to the whole breadth and depth of the universe.