Celestini 2019

Section - E

i) 100 balls are tossed independently and at random into 50 bins. Let X be a random variable representing the number of empty boxes. Find the expected value of random variable X, i.e. E(X).

Answer:

We need to find the expected number of empty bins

If x is the total no. of bins that ends with zero balls

then

$$E(x) = E(x_1 + x_2 + x_3 \cdots x_{49} + x_{50})$$

$$= E(x_1) + E(x_2) + E(x_3) \cdots \cdots E(x_{50})$$

Let $x_i = 1$ if bin i has zero balls

$x_i = 0$ if otherwise

Thus

$$P(x_i = 1) = \left(\frac{49}{50}\right)^{100}$$

So
$$E(x) = E(x_1) + E(x_2) + \cdots E(x_{49}) + E(x_{50})$$

$$= \boxed{50 \times \left(\frac{49}{50}\right)^{100}}$$

$$\boxed{\text{ANSWER}: 50 \times \left(\frac{49}{50}\right)^{100}}$$

ii) Binary communication channel

The per bit error rate over a certain binary communication channel is 10^{-10}. No other statistics are known about the channel or the data.

ii) a) What is the expected number of erroneous bits in a block of 1000 bits?

Answer:

According to the ques, let

$$x_i = \begin{cases} 1 & \text{with probability} = 10^{-10} \\ 0 & \text{with probability} = 1-10^{-10} \end{cases}$$

Hence $E(x_i) = 1 \times 10^{-10} + 0 \times (1-10^{-10})$

$= \boxed{10^{-10}}$

Let N = no. of errors in a block of 1000 bits

$E(N) = 1000 \times E(x_i) = 1000 \times 10^{-10}$

$= \boxed{10^{-7}}$

$\boxed{\text{Answer : } 10^{-7}}$

ii) b) Find an upper bound on the probability that a block of 1000 bits has 10 or more erroneous bits.

Answer:

The upperbound can be found using Markov's inequality which gives an upperbound for the probability that a non negative function of a random variable is greater than or equal to some positive constant.

$$P(x \geq a) \leq \frac{E(x)}{a}$$

So, $P\{N \geq 10\} \leq \frac{E(N)}{10} = \boxed{10^{-8}}$

$\boxed{\text{Answer : } 10^{-8}}$

iii) Alina gambles against Gina. Each night Alina draws a card from a deck (with replacement). If it is a spade or a queen, Alina wins $4. If not, Alina loses $1. What is Alina's total expected winnings after 30 nights?

Answer:

Probability of drawing a spade or queen

$$= \frac{13+4-1}{52} = \frac{16}{52} = \boxed{\frac{4}{13} = 0.307}$$

Probability of losing $= 1 - \frac{4}{13} = \boxed{\frac{9}{13} = 0.692}$

Expected value for the game

$$= \$4 \times \text{win} + \$(-1) \times \text{lose}$$

$$= 4 \times 0.307 - 1 \times 0.692$$

$$= \boxed{\$0.538} \text{ perday}$$

So for 30 nights, expected winnings

$$= 30 \times 0.538$$

$$\boxed{= \$16.163}$$

$\boxed{\text{ANSWER: } 16.163}$