

Math 342W / 642 / RM742 Spring 2025

Midterm Examination One

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Full Name _____

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

signature

date

Instructions

This exam is 110 minutes (variable time per question) and closed-book. You are allowed **two** pages (front and back) of a “cheat sheet”, blank scrap paper (provided by the proctor) and a graphing calculator (which is not your smartphone). Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

Problem 1 This question relates to conceptual issues.

- (a) [8 pt / 8 pts] Circle the letters of all the following that are **true**. The notation below should be assumed the same from class.
- (a) A model g for a real-world, non-man-made phenomenon y which is a function of real-world measurements x_1, \dots, x_p could be absolutely true, (i.e. always have zero error when predicting in for future units).
 - (b) A model g for phenomenon y can be proved true analytically (using a mathematical deductive proof).
 - (c) A null model g_0 outputs a value in the set \mathcal{Y} .
 - (d) A null model g_0 can still be “useful” to the model’s user.
 - (e) Any model g which is a function of real-world measurements x_1, \dots, x_p *always* beats g_0 on predictive performance.
 - (f) A model g for phenomenon y which is of type nominal categorical with possible values within the set C_1, C_2, \dots, C_L must return a value within the set C_1, C_2, \dots, C_L .
 - (g) A model g for phenomenon y which is of type nominal categorical with possible values within the set C_1, C_2, \dots, C_L must have a proxy x_j which is also type nominal categorical with possible values within the set C_1, C_2, \dots, C_L .
 - (h) If x_j is of type nominal categorical with possible values within the set C_1, C_2, \dots, C_L , then it is reasonable when modeling that those categorical values should be replaced by $1, 2, \dots, L$.
 - (i) If x_j is of type nominal categorical with possible values within the set C_1, C_2, \dots, C_L , then it is reasonable when modeling that those categorical values should be replaced by $L - 1$ binary variables indicating the presence of absence of categories C_2, \dots, C_L .
 - (j) If x_j is of type ordinal with possible values within the set C_1, C_2, \dots, C_L , assuming that set is ordered from smallest to largest, then it is reasonable when modeling that those categorical values *can* be replaced by $1, 2, \dots, L$.
 - (k) If x_j is of type ordinal with possible values within the set C_1, C_2, \dots, C_L , then it is reasonable when modeling that those categorical values *can* be replaced by $L - 1$ binary variables indicating the presence of absence of categories C_2, \dots, C_L .
 - (l) In-sample performance metrics gives you an honest idea about how accurate your model is when using it for future prediction *always*.
 - (m) In-sample performance metrics gives you an honest idea about how accurate your model is when using it for future prediction *only* when n is much larger than p .
 - (n) Out-of-sample validation gives you an idea about how accurate your model is when using it for future prediction *always*.
 - (o) Out-of-sample validation gives you an idea about how accurate your model is when using it for future prediction *only* when n is much larger than p .
 - (p) Out-of-sample validation gives you an idea about how accurate your model is when using it for future prediction *only* when $\mathcal{Y} \subseteq \mathbb{R}$.

Problem 2 Consider the following data frame \mathbb{D} . Below is some R code (lines beginning with `>` are commands that were entered, other lines are output from those commands).

```

1 > skimr::skim(D)
2 ——— Data Summary ———
3
4 Name                               Values
5 Number of rows                     592
6 Number of columns                  4
7 Key                                NULL
8
9 Column type frequency:
10   character                        2
11   numeric                          2
12
13 Group variables                     None
14
15 ——— Variable type: factor ———
16   skim_variable n_missing complete_rate ordered n_unique top_counts
17 1 eye_color      0                1 FALSE          4 Bro: 220, Blu: 215,
18   Haz: 93, Gre: 64
19
20 2 hair_color     0                1 FALSE          4 Bro: 286, Blo: 127,
21   Bla: 108, Red: 71
22
23 ——— Variable type: numeric ———
24   skim_variable n_missing complete_rate mean    sd p0 p25 p50 p75 p100
25 1 is_male       0                1 0.471 0.500  0  0  0  1  1
26 2 is_not_male   0                1 0.529 0.500  0  0  1  1  1
27
28 > D$is_not_male = 1 - D$is_male
29 > D[sample(1 : nrow(D), 10), ]
30
31   is_male eye_color hair_color is_not_male
32 71       1   Brown   Brown      0
33 446      0   Hazel   Brown      1
34 474      0   Green   Brown      1
35 75       1   Brown   Brown      0
36 489      0   Brown   Red       1
37 107      1   Brown   Brown      0
38 473      0   Green   Brown      1
39 69       1   Brown   Brown      0
40 447      0   Hazel   Brown      1
41 137      1   Blue    Brown      0

```

(a) [1 pt / 9 pts] Circle one letter: the output on lines 28–37 is ...

- A) the first 10 rows of data frame \mathbb{D}
- B) a random 10 rows of data frame \mathbb{D}
- C) the last 10 rows of data frame \mathbb{D}

(b) [1 pt / 10 pts] What type of variable is `eye_color`? Be as specific as you can.

Let y be defined as the variable `is_male` and all other variables be considered features.

(c) [2 pt / 12 pts] What is the g_0 function equal to explicitly?

(d) [3 pt / 15 pts] Circle all of the following modeling procedures that would be appropriate for this modeling setting considering the response space.

A) OLS

B) Perceptron

C) SVM

D) SVM with Vapnik objective function (and a numeric choice for hyperparameter λ)

E) KNN with the Euclidean distance function

F) KNN with a non-Euclidean, but still valid distance function

(e) [3 pt / 18 pts] Regardless of your answer in (c), how many columns will the full-rank design matrix X have? Hint: make sure you include the $\mathbf{1}_n$ as the first column in X .

(f) [2 pt / 20 pts] Regardless of your answer in (c), consider using the SVM to fit g . What would the in-sample misclassification error for g be?

(g) [2 pt / 22 pts] Regardless of your answer in (c), consider using the OLS to fit g . What would the in-sample misclassification error for R^2 be?

Now we will change the modeling target. Let y be defined as the variable `eye_color` and all other variables be considered features.

- (h) [2 pt / 24 pts] What is the g_0 function equal to explicitly?
- (i) [3 pt / 27 pts] Circle all of the following modeling procedures that would be appropriate for this modeling setting considering the response space.
- A) OLS
 - B) Perceptron
 - C) SVM
 - D) SVM with Vapnik objective function (and a numeric choice for hyperparameter λ)
 - E) KNN with the Euclidean distance function
 - F) KNN with a non-Euclidean, but still valid distance function
- (j) [3 pt / 30 pts] Regardless of your answer in (g), how many columns will the full-rank design matrix X have? Hint: make sure you include the $\mathbf{1}_n$ as the first column in X .

Now we will change the modeling target again. Let y be defined as the variable `hair_color` and we only use the feature `eye_color`.

- (k) [5 pt / 35 pts] Let the reference level of `eye_color` be Green. For the subset of the data shown on rows 28–37 of the code, provide the same 10-row subset of the full-rank design matrix X . Name the columns of X appropriately.

- (1) [2 pt / 37 pts] Assume you use KNN with the Euclidean distance function to fit g . Of the three sources of error, which is likely the largest source of error in g 's predictions?

Problem 3 Here are some theoretical questions. For all notation used, assume it has the same as it did in class.

- (a) [4 pt / 41 pts] Assume $\mathcal{A} = \text{OLS}$ (linear regression). Prove that $\mathbf{b} = R^{-1}Q^\top \mathbf{y}$.

- (b) [10 pt / 51 pts] Assume $\mathcal{A} = \text{OLS}$ (linear regression). For each of the following, if the operation is illegal, write “illegal”. If the operation is legal and the expression allows for simplification, simplify as best you can after an “=” sign. If you cannot simplify, then leave the line *blank*. If your answer is the zero or one vector, please indicate the dimension as a subscript to get full credit.

- XQ

- XR

- $X\mathbf{b}$

- HQ

- QH
- Qb
- $\mathbf{y}^\top H$
- $\mathbf{e}^\top H$
- $QQ^\top \mathbf{1}_n$
- $\text{rank}[H] / \text{rank}[X]$

Problem 4 Consider the Galton dataset we examined in lab. Below is some R code (lines beginning with `>` are commands that were entered, other lines are output from those commands).

```

1 > skimr::skim(HistData::Galton)
2 ——— Data Summary ———
3
4 Name                               Values
5 Number of rows                     928
6 Number of columns                   2
7
8 Column type frequency:
9   numeric                           2
10
11 ——— Variable type: numeric ———
12   skim_variable n_missing complete_rate mean    sd    p0    p25    p50    p75    p100
13 1 parent         0           1 68.3  1.79  64    67.5  68.5  69.5  73
14 2 child          0           1 68.1  2.52  61.7  66.2  68.2  70.2  73.7

```

(a) [1 pt / 52 pts] What is the sample size of this dataset?

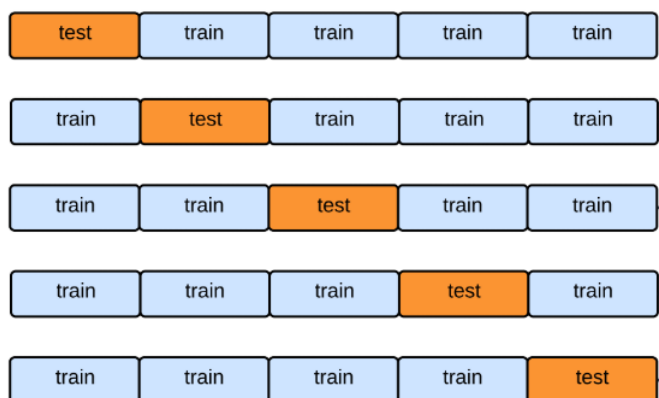
As in the lab, consider the dependent variable **child** which measures the height of a child and consider the independent variable **parent** which measures the average height of the mother and father of the child.

- (b) [1 pt / 53 pts] What is p in this modeling context?
- (c) [3 pt / 56 pts] “The unit is one ___” (fill in the blank below by describing the unit).
- (d) [2 pt / 58 pts] Regardless of the \mathcal{A} employed to create a model, which will likely be the greatest of the three sources of error?
- (e) [2 pt / 60 pts] If $\mathcal{A} = \text{OLS}$, what are the contents of the set \mathcal{H} ?
- $\mathcal{H} =$
- (f) [3 pt / 63 pts] If $\mathcal{A} = \text{OLS}$, provide one specific $\langle x, y \rangle$ point on the OLS line (i.e. the g model).

- (g) [5 pt / 68 pts] Prove numerically that there is “regression to the mean” for the phenomenon of interest in this modeling setting. Assume that the correlation coefficient between the two variables is 0.46.

- (h) [5 pt / 73 pts] If $\mathcal{A} = \text{OLS}$, calculate the RMSE to two decimals.

Problem 5 Consider the following illustration of cross-validation for \mathbb{D} with $n = 200$.

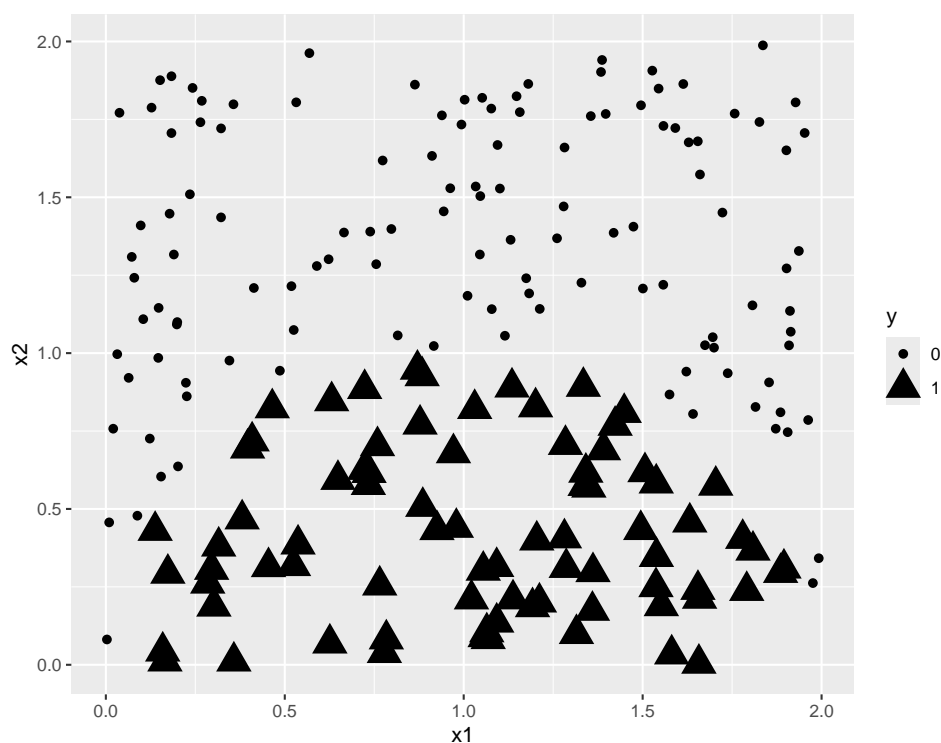


- (a) [1 pt / 74 pts] What is the value of K ?
- (b) [1 pt / 75 pts] How many units will be in \mathbb{D}_{train} in the first split?
- (c) [1 pt / 76 pts] If you calculate oos residuals in the second split, how many oos residuals will there be?
- (d) [2 pt / 78 pts] If g was overfit on \mathbb{D}_{train} , then how would oosRMSE be expected to compare with in-sample RMSE?
- (e) [3 pt / 81 pts] If you calculate the oosRMSE from the residuals in (c), this oosRMSE is expected to be greater than, less than or equal to the oosRMSE you expect in the future when predicting on g_{final} which is fit with all \mathbb{D} ?
- (f) [3 pt / 84 pts] If you aggregate all the residuals from all K folds together and calculate the oosRMSE, this oosRMSE is expected to be greater than, less than or equal to the oosRMSE you expect in the future when predicting on g_{final} which is fit with all \mathbb{D} ?

Problem 6 Consider the following \mathbb{D} illustrated within the questions below.

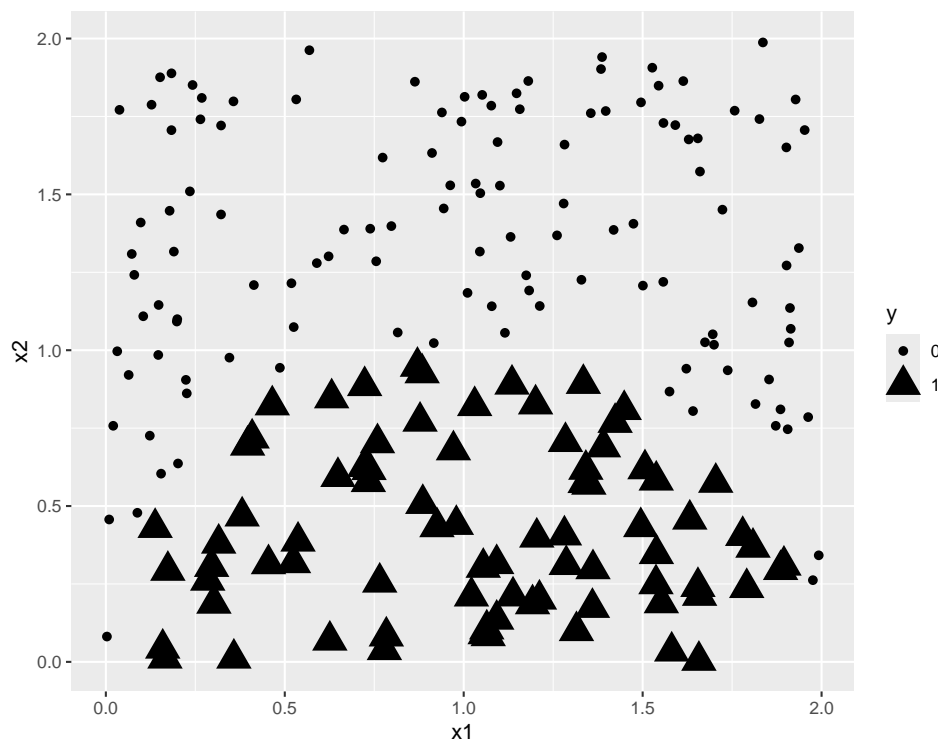
- (a) [2 pt / 86 pts] Examine the plot of \mathbb{D} in the following question. If y is the response variable and the other two variables are considered features, what is the formal name of the type of modeling scenario most likely depicted?

- (b) [2 pt / 88 pts] If $\mathcal{A} = \text{perceptron}$ with the \mathcal{H} we considered thus far in class, draw a possible g that outputs from this algorithm after 10,000 iterations on the illustration below:



- (c) [1 pt / 89 pts] If $\mathcal{A} = \text{perceptron}$ with the \mathcal{H} we considered thus far in class, will the algorithm eventually converge? Circle one: yes / no

- (d) [4 pt / 93 pts] If $\mathcal{A} = \text{SVM}$ with the \mathcal{H} we considered thus far in class and the Vapnik function with hinge errors and $\lambda = 0.001$, draw the g that most likely outputs from this algorithm on the illustration below:



- (e) [3 pt / 96 pts] If $\mathcal{A} = \text{SVM}$ with the \mathcal{H} we considered thus far in class and the Vapnik function with hinge errors and $\lambda = 0.001$, what is the most likely of the three sources of error in g ?
- (f) [2 pt / 98 pts] If $\mathcal{A} = \text{KNN}$ with the Euclidean distance function and $K = 1$, what would be $g(0,0)$?
- (g) [2 pt / 100 pts] If $\mathcal{A} = \text{KNN}$ with the Euclidean distance function and $K = 10$, what would be $g(0,0)$?