数学笔记

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Contents

Part I

知识整理

Chapter 1

复几何、多复变

1.1 实线性空间与复线性空间

流形 $\mathbb{C}^n$ 中每点的坐标为 $(z^1,\ldots,z^n)$, 记 $z^j=x^j+iy^j$, 于是 $x^1,\ldots,x^n$ 和 $y^1,\ldots,y^n$ 是 $\mathbb{C}^n$ 的实坐标. 对 $\forall p\in\mathbb{C}^n$, 切空间 $T_p\mathbb{C}^n$ 有一组基 $\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}$, $\frac{\partial}{\partial y^1},\ldots,\frac{\partial}{\partial y^n}$; 余切空间 $T_p^*\mathbb{C}$ 有一组基 $\mathrm{d}x^1,\ldots,\mathrm{d}x^n$, $\mathrm{d}y^1,\ldots,\mathrm{d}y^n$.

定义 $J_p:T_p\mathbb{C}^n\to T_p\mathbb{C}^n$

$$J_p\left(\frac{\partial}{\partial x^j}\right)=\frac{\partial}{\partial y^j},\qquad J_p\left(\frac{\partial}{\partial y^j}\right)=-\frac{\partial}{\partial x^j}$$

它的对偶变换为 $J_p^*:T_p^*\mathbb{C}^n\to T_p^*\mathbb{C}^n$

$$J_p^*\left(\mathrm{d}x^j\right)=-\mathrm{d}y^j,\qquad J_p^*\left(\mathrm{d}y^j\right)=\mathrm{d}x^j$$

因为 $(J_p)^2=-\mathrm{id}$, 所以 $(J_p^*)^2=-\mathrm{id}$, 于是 J_p 和 J_p^* 均可对角化且特征值均为 $\pm i$.

计算可发现 J_p^* 的属于 i 的特征子空间的一组基为 $\mathrm{d}z^1,\ldots,\mathrm{d}z^n$, 属于 $-i$ 的特征子空间的一组基为 $\mathrm{d}\bar{z}^1\ldots,\mathrm{d}\bar{z}^n$, 其中

$$\mathrm{d}z^j=\mathrm{d}x^j+i\mathrm{d}y^j,\qquad \mathrm{d}\bar{z}^j=\mathrm{d}x^j-i\mathrm{d}y^j$$

于是对应地它的对偶空间 $T_p\mathbb{C}^n\otimes\mathbb{C}$ 的属于 i 的特征子空间的一组基为 $\frac{\partial}{\partial z^1},\ldots\frac{\partial}{\partial z^n}$, 属于 $-i$ 的特征子空间的一组基为 $\frac{\partial}{\partial\bar{z}^1},\ldots,\frac{\partial}{\partial\bar{z}^n}$, 其中

$$\frac{\partial}{\partial z^j}=\frac{1}{2}\left(\frac{\partial}{\partial x^j}-i\frac{\partial}{\partial y^j}\right),\qquad \frac{\partial}{\partial\bar{z}^j}=\frac{1}{2}\left(\frac{\partial}{\partial x^j}+i\frac{\partial}{\partial y^j}\right)\tag{1.1}$$

回顾 Cauchy-Riemann 方程: 设 $f=u+iv$ 是全纯函数, 则 $u(x,y),v(x,y)$ 关于 x,y 的偏导数满足:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

代入 $\frac{\partial f}{\partial \bar{z}}$ 可得

$$\frac{\partial f}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)=\frac{1}{2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+i\frac{1}{2}\left(\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial y}\right)=0$$

于是 f 全纯当且仅当 $\frac{\partial f}{\partial \bar{z}}\equiv 0$, 推广到多元复变量即: $f(z^1,\ldots,z^n)$ 是多元全纯函数当且仅当每个 $\frac{\partial f}{\partial \bar{z}^j}\equiv 0$.

1.2 实可微与复可微

1.2.1 一维情形: 复导数是复数

我们知道一个复变量复值 (连续) 函数 $f(z)$ 可以视作一个实二元向量值 (连续) 函数, 即有 $1-1$ 对应 $\mathcal{C}(\mathbb{C},\mathbb{C})\leftrightarrow\mathcal{C}(\mathbb{R}^2,\mathbb{R}^2)$:

$$f(z)=u(x+\sqrt{-1}y)+\sqrt{-1}v(x+\sqrt{-1}y)\leftrightarrow\begin{pmatrix}u(x,y)\\ v(x,y)\end{pmatrix}.$$

回顾向量值函数可微性的定义, 如果

$$\begin{pmatrix}u(x+\Delta x,y+\Delta y)\\ v(x+\Delta x,y+\Delta y)\end{pmatrix}-\begin{pmatrix}u(x,y)\\ v(x,y)\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}+\begin{pmatrix}o(\sqrt{(\Delta x)^2+(\Delta y)^2})\\ o(\sqrt{(\Delta x)^2+(\Delta y)^2})\end{pmatrix}, \tag{1.2}$$

则称 $(u,v)'$ 在点 $(x,y)'$ 处可微, 此时 $\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}=:\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}$.

复变量函数可微性定义如下, 若

$$f(z+\Delta z)-f(z)=w\cdot\Delta z+o(\Delta z),$$

其中 $w=p+\sqrt{-1}q\in\mathbb{C}$, 则称 f 在点 z 处复可微.

这里为什么要强调 w 是一个复数呢? 这是为了提醒我们可以将 Jacobi 矩阵 $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ 看成一个复数, 那什么时候能把一个实 2×2 矩阵看成一个复数呢?

事实上我们能很自然地给一个 $\mathbb{C}$ 到 $\mathbb{R}^{2\times 2}$ 的嵌入:

$$\iota:\mathbb{C}\hookrightarrow\mathbb{R}^{2\times 2}$$
$$z=a+\sqrt{-1}b\mapsto\begin{pmatrix}a & -b\\ b & a\end{pmatrix}.$$

注. 该嵌入是通过将数乘一个复数看成 $\mathbb{C}(\cong\mathbb{R}^2)$ 到自身的线性变换, 再取一组实线性无关的基 1、$\sqrt{-1}$, 这个线性变换在这组基下的矩阵就是映射的像.

因此能将 $\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}$ 视作一个复数当且仅当存在 $p, q \in \mathbb{C}$ 使得

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$$

从而

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = p \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = q \end{cases}$$

此即 Cauchy-Riemann 方程.

1.2.2 高维情形: 复微分是复线性变换

一维情形下复线性变换只有数乘, 这种特殊性会让我们错失更一般的图景. 设 $f(z_1, \ldots, z_n) = (f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))'$ 为 $\mathbb{C}^n$ 上的 (连续) 函数. 设 $f_i = u_i + \sqrt{-1} v_i$, $z_j = x_j + \sqrt{-1} y_j$, 则 f 和 $\mathbb{R}^{2n}$ 到自身的 (连续) 映射一一对应:

$$f(z_1, \ldots, z_n) \leftrightarrow \begin{pmatrix} u_1(x_1, \cdots, x_n, y_1, \cdots, y_n) \\ \vdots \\ u_n(x_1, \cdots, x_n, y_1, \cdots, y_n) \\ v_1(x_1, \cdots, x_n, y_1, \cdots, y_n) \\ \vdots \\ v_n(x_1, \cdots, x_n, y_1, \cdots, y_n) \end{pmatrix}$$

当这个映射可微时, 我们也有它的 Jacobi 矩阵:

$$\mathrm{D}f := \begin{pmatrix} \frac{\mathrm{D}(u_1, \cdots, u_n)}{\mathrm{D}(x_1, \cdots, x_n)} & \frac{\mathrm{D}(u_1, \cdots, u_n)}{\mathrm{D}(y_1, \cdots, y_n)} \\ \\ \frac{\mathrm{D}(v_1, \cdots, v_n)}{\mathrm{D}(x_1, \cdots, x_n)} & \frac{\mathrm{D}(v_1, \cdots, v_n)}{\mathrm{D}(y_1, \cdots, y_n)} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} & \frac{\partial u_1}{\partial y_1} & \cdots & \frac{\partial u_1}{\partial y_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} & \frac{\partial u_n}{\partial y_1} & \cdots & \frac{\partial u_n}{\partial y_n} \\ \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial x_n} & \frac{\partial v_1}{\partial y_1} & \cdots & \frac{\partial v_1}{\partial y_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial v_n}{\partial x_1} & \cdots & \frac{\partial v_n}{\partial x_n} & \frac{\partial v_n}{\partial y_1} & \cdots & \frac{\partial v_n}{\partial y_n} \end{pmatrix}$$

Df 是一个实线性变换, 我们将说明当 f 复可微 (等价于满足高维 C-R 方程) 的时候 Df 是复线性的.

我们先做一些线性代数的准备, $\mathbb{C}^n$ 可看成 $\mathbb{R}^{2n}$, 因此复线性变换能对应一个实线性变换:

$$\begin{aligned} \iota : \mathbb{C}^{n\times n} &\hookrightarrow \mathbb{R}^{2n\times 2n} \\ A + \sqrt{-1} B &\mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \end{aligned} \tag{1.3}$$

特别地, 数乘一个复数 $w = p + \sqrt{-1}q$ (记为 λ_w) 对应的矩阵为

$$\lambda_w \mapsto \begin{pmatrix} pI_n & -qI_n \\ qI_n & pI_n \end{pmatrix} = p\begin{pmatrix} I_n & \\ & I_n \end{pmatrix} + q\begin{pmatrix} & -I_n \\ I_n & \end{pmatrix} =: pI_{2n} + qJ$$

反过来, 一个 $\mathbb{R}^{2n}$ 上的线性变换 φ 什么时候是复线性的呢? 将复线性的定义转化为实的语言即为:

$$\varphi(\lambda_w(v)) = \lambda_w(\varphi(v)), \quad \forall\, v \in \mathbb{R}^{2n},\, w \in \mathbb{C}.$$

由 φ 实线性, 实际上只需验证:

$$\varphi(\lambda_{\sqrt{-1}}(v)) = \lambda_{\sqrt{-1}}(\varphi(v)), \quad \forall\, v \in \mathbb{R}^{2n}.$$

写成矩阵形式即要求 φ 对应的矩阵 M_φ 满足:

$$M_\varphi \circ J = J \circ M_\varphi. \tag{1.4}$$

设 $M_\varphi = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$, 其中 $A, B, C, D \in \mathbb{R}^{n\times n}$ 则

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix}\begin{pmatrix} & -I_n \\ I_n & \end{pmatrix} = \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}\begin{pmatrix} A & C \\ B & D \end{pmatrix} \Rightarrow \begin{cases} A = D \\ B = -C \end{cases}.$$

定理 1.2.1 (复可微的含义)**.** 设 f 是 $\mathbb{C}^n$ 到 $\mathbb{C}^n$ 的连续映射, f 作为 $\mathbb{R}^{2n}$ 到 $\mathbb{R}^{2n}$ 的映射是光滑的, 则 f 复线性当且仅当其微分 $\mathrm{D}f$ 是复线性的.

证明. 由前文的讨论知 $\mathrm{D}f$ 复线性当且仅当它满足

$$\mathrm{D}f \circ J = J \circ \mathrm{D}f \Leftrightarrow \begin{cases} \dfrac{\mathrm{D}(u_1, \cdots, u_n)}{\mathrm{D}(x_1, \cdots, x_n)} = \dfrac{\mathrm{D}(v_1, \cdots, v_n)}{\mathrm{D}(y_1, \cdots, y_n)} \\ \dfrac{\mathrm{D}(v_1, \cdots, v_n)}{\mathrm{D}(x_1, \cdots, x_n)} = -\dfrac{\mathrm{D}(u_1, \cdots, u_n)}{\mathrm{D}(y_1, \cdots, y_n)} \end{cases} \Leftrightarrow \mathrm{D}f \text{满足 C-R 方程}$$

□

注. 由此可以看出复函数的全纯性确实比光滑性更强, 它要求函数的实 Jacobian 有对称性 (1.4).

1.2.3 复导数存在与复线性

回到 1 维情形, 复函数 $w = f(z)$ 在点 z 处的定义为

$$f'(z) := \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{1.5}$$

现假设 f 光滑 (不一定全纯), 则在点 $z=x+\sqrt{-1}y$ 处有 (1.2) 成立, 观察到在 (1.2) 两侧乘以行向量 $\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}$ 能得到

$$\Delta f=\Delta u+\sqrt{-1}\Delta v=\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}+o\left(\sqrt{(\Delta x)^2+(\Delta y)^2}\right)$$

为了配出导数定义式 (1.5) 的除法我们需要存一个复数 $w=p+\sqrt{-1}q$ 使得

$$\begin{aligned}\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}&=w\cdot\Delta z=(p+\sqrt{-1}q)(\Delta x+\sqrt{-1}\Delta y)\\ &=\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}\end{aligned}$$

因此

$$\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}=\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}$$

因为此时

$$\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}J=\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}\begin{pmatrix} & -1\\ 1 & \end{pmatrix}=\begin{pmatrix}\sqrt{-1} & -1\end{pmatrix}=\sqrt{-1}\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}$$

所以

$$\begin{aligned}\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}J&=\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}J\\ &=\sqrt{-1}\cdot\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}\\ &=\sqrt{-1}\cdot\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}J\\ &=\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}J\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}\end{aligned}$$

所以 Jacobian 需要满足

$$\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}J=J\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}$$

这就与前面的复线性联系上了.

注. *或许可以直接用*

$$\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}=\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}=(p+\sqrt{-1}q)\cdot\begin{pmatrix}1 & \sqrt{-1}\end{pmatrix}$$

说明 $\frac{\mathrm{D}(u,v)}{\mathrm{D}(x,y)}$ *是复线性的?*

1.2.4 全纯部分与反全纯部分

前面我们考虑了 $\mathbb{C}^n$ 到 $\mathbb{R}^{2n}$ 的嵌入 ι (详见 (1.3)), 实 $2n$ 阶矩阵 $M \in \operatorname{Im}\iota$ 当且仅当 $MJ = JM$. 现在若 $MJ = -JM$, 可以算出 M 形如 $\begin{pmatrix} A & B \\ B & -A \end{pmatrix}$, 定义

$$\begin{aligned} \bar{\iota} : \mathbb{C}^{n\times n} &\hookrightarrow \mathbb{R}^{2n\times 2n} \\ A + \sqrt{-1}B &\mapsto \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \end{aligned} \tag{1.6}$$

命题 1.2.2. *$\mathbb{R}^{2n\times 2n} = \operatorname{Im}\iota \oplus \operatorname{Im}\bar{\iota}$, 分别记 $\mathbb{R}^{2n\times 2n}$ 到两个分量的投影为 ∂、$\bar{\partial}$, (我想把它们分别称为全纯部分和反全纯部分)*

证明. $\operatorname{Im}\iota \cap \operatorname{Im}\bar{\iota} = \emptyset$ 显然, 仅需证 $\mathbb{R}^{2n\times 2n} = \operatorname{Im}\iota + \operatorname{Im}\bar{\iota}$, 当然可以待定系数

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} A_1 & -B_1 \\ B_1 & A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ B_2 & -A_2 \end{pmatrix}$$

解出 A_1, B_1, A_2, B_2. 下面用 J 给出一个具体表达式, 设 $M = \partial M + \bar{\partial} M$, 则

$$JMJ = J\partial MJ + J\bar{\partial}MJ = -\partial M + \bar{\partial}M$$

因此

$$\begin{aligned} \partial M &= \frac{1}{2}(M - JMJ) \\ \bar{\partial} M &= \frac{1}{2}(M + JMJ) \end{aligned} \tag{1.7}$$

□

注. *最后算出来的表达式和 (1.1) 很像.*

注. *一般复光滑函数的实 Jacobian 也可以分解成这两部分, 这与复几何的 $T_{\mathbb{R}}X \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$ 有没有联系?*

1.3 复流形的例子

例 1.3.1 (复射影平面是黎曼球面)**.** *黎曼球面是 $\mathbb{R}^3$ 中的单位球面, 它有如下整体坐标:*

$$S^2 = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1 \right\}.$$

复射影平面 $\mathbb{C}\mathrm{P}$ 则为:

$$\mathbb{C}\mathrm{P} = \left\{ [u:v] \,\middle|\, u, v \text{不同时为零} \right\}.$$

我们知道 S^2 有一个常用的坐标覆盖: $S^2 \backslash \{N\}$, $S^2 \backslash \{S\}$, $\mathbb{C}\mathrm{P}$ 有一个常用的坐标覆盖 U_0, U_1. 可分块定义同构映射:

$$\begin{aligned} S^2\backslash\{N\} &\longrightarrow U_0 \\ (x,y,z) &\longmapsto \left[1:\tfrac{x+\sqrt{-1}y}{1-z}\right] \\ \left(\tfrac{2\mathrm{Re}w_0}{|w_0|^2+1},\tfrac{2\mathrm{Im}w_0}{|w_0|^2+1},\tfrac{|w_0|^2-1}{|w_0|^2+1}\right) &\longleftarrow [1:w_0] \end{aligned}$$

以及

$$\begin{aligned} S^2\backslash\{S\} &\longrightarrow U_1 \\ (x,y,z) &\longmapsto \left[\tfrac{x+\sqrt{-1}y}{1+z}:1\right] \\ \left(\tfrac{2\mathrm{Re}w_1}{|w_1|^2+1},\tfrac{2\mathrm{Im}w_1}{|w_1|^2+1},\tfrac{1-|w_1|^2}{1+|w_1|^2}\right) &\longleftarrow [w_1:1] \end{aligned}$$

注意到当 $(x,y,z)\in S^2$ 时, $\left[1:\frac{x+\sqrt{-1}y}{1-z}\right]=\left[\frac{x+\sqrt{-1}y}{1+z}:1\right]$, 因此同构映射 $S^2\backslash\{N,S\}\to U_0\cap U_1$ 是良定义的. 从而有 $\mathbb{C}\mathrm{P}^1\cong S^2$.

例 1.3.2 (齐次多项式零点定义的射影空间的子集)**.** 设 $f(z_0,\dots,z_n)$ 是 $\mathbb{C}^{n+1}$ 上的 d 次齐次多项式, 将其限制在 $\mathbb{C}^{n+1}\backslash\{0\}$ 上. 设 0 是限制后函数的正则值, 则 $Z(f):=f^{-1}(0)$ 是复流形, 且若 $\omega=(\omega_0,\dots,\omega_n)\in Z(f)$, 则对 $\forall\lambda\in\mathbb{C}^*$, $\lambda\omega=(\lambda\omega_0,\dots,\lambda\omega_n)\in Z(f)$. 于是有 $\mathbb{C}^*$ 在 $Z(f)$ 上的作用, 定义:

$$V(f):=Z(f)/\mathbb{C}^*\subset\left(\mathbb{C}^{n+1}\backslash\{0\}\right)/\mathbb{C}^*=\mathbb{C}\mathrm{P}^n.$$

下面说明 $V(f)$ 仍有复流形结构, 我们知道复射影空间有典范的坐标覆盖:

$$\mathbb{C}\mathrm{P}^n=\bigcup_{i=0}^n U_i=\left\{[z_0:\cdots:z_n]\,\middle|\,z_i\neq 0\right\}.$$

坐标映射为

$$\varphi_i:U_i\to\mathbb{C}^n,$$
$$[z_0:\cdots:z_n]=\left[\cdots:\frac{z_{i-1}}{z_i}:1:\frac{z_{i+1}}{z_i}:\cdots\right]\mapsto\left(\dots,\frac{z_{i-1}}{z_i},\frac{z_{i+1}}{z_i},\dots\right).$$

则

$$V(f)=\bigcup_{i=0}^n\left(U_i\cap V(f)\right),$$

且

$$U_i\cap V(f)=\left\{[z_0:\cdots:z_n]\,\middle|\,f(z_0,\dots,z_n)=0,\ z_i\neq 0\right\}$$

$$= \left\{[\cdots : z_{i-1} : 1 : z_{i+1} : \cdots] \,\middle|\, f(\ldots, z_{i-1}, 1, z_{i+1}, \ldots) = 0\right\}.$$

于是

$$\varphi_i(U_i \cap V(f)) = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \,\middle|\, f(z_1, \ldots, z_i, 1, z_{i+1}, \ldots, z_n) = 0\right\}.$$

令 $f_i(z_1, \ldots, z_n) = f(z_1, \ldots, z_i, 1, z_{i+1}, \ldots, z_n)$, 则 $\varphi_i(U_i \cap V(f))$ 是 f_i 的零点集, 且

$$\frac{\partial f_i}{\partial z_j}(z_1, \ldots, z_n) = \begin{cases} \frac{\partial f}{\partial z_{j-1}}(z_1, \ldots, z_i, 1, z_{i+1}, \ldots, z_n), & j \leqslant i, \\ \frac{\partial f}{\partial z_j}(z_1, \ldots, z_i, 1, z_{i+1}, \ldots, z_n), & j > i. \end{cases}$$

假设 0 不是 f_i 的正则值, 即存在 $\omega = (\omega_1, \ldots, \omega_n)$ 使得

$$\begin{cases} f_i(\omega) = 0, \\ \dfrac{\partial f_i}{\partial z_j}(\omega) = 0, \ \forall 1 \leqslant j \leqslant n. \end{cases}$$

因为 f 是 d 次齐次多项式, 于是

$$f(\lambda z_0, \ldots, \lambda z_n) = \lambda^d f(z_0, \ldots, z_n)$$

两侧对 λ 求导并令 $\lambda = 1$ 得

$$z_0 \frac{\partial f}{\partial z_0}(z) + \cdots + z_n \frac{\partial f}{\partial z_n}(z) = d \cdot f(z)$$

代入 $z = (\omega_1, \ldots, \omega_i, 1, \omega_{i+1}, \ldots, \omega_n)$, 由前文可知

$$\begin{cases} f(\omega_1, \ldots, \omega_i, 1, \omega_{i+1}, \ldots, \omega_n) = 0, \\ \dfrac{\partial f}{\partial z_j}(\omega_1, \ldots, \omega_i, 1, \omega_{i+1}, \ldots, \omega_n) = 0, \ j \neq i. \end{cases}$$

且

$$\begin{aligned} &\frac{\partial f}{\partial z_i}(\omega_1, \ldots, \omega_i, 1, \omega_{i+1}, \ldots, \omega_n) \\ =&d \cdot f(\omega_1, \ldots, \omega_i, 1, \omega_{i+1}, \ldots, \omega_n) - \sum_{j \neq i} \frac{\partial f}{\partial z_j}(\omega_1, \ldots, \omega_i, 1, \omega_{i+1}, \ldots, \omega_n) \\ =&0. \end{aligned}$$

则 $(\omega_1, \ldots, \omega_i, 1, \omega_{i+1}, \ldots, \omega_n)$ 不是 f 的正则点, 0 不是 f 的正则值, 矛盾!

因此每个 $\varphi_i(U_i \cap V(f))$ 都是复流形, 因此 $V(f)$ 是一个复流形, 且注意到 $V(f)$ 是 $\mathbb{CP}^n$ 的闭子集, 从而也是一个紧集.

1.4 Kähler 几何

定义 1.4.1. *Let M be an almost complex manifold, J be its almost complex structure. We say a metric g is a Hermitian metric if it satisfies*

$$g(X,Y) = g(JX, JY)$$

We then extend it to be a $\mathbb{C}$-bilinear tensor, i.e. $g_{\mathbb{C}} \in C^{\infty}(M, \bigwedge^2 T^*_{\mathbb{C}} M)$.

Now let M be a complex manifold, J be its natural complex structure. In a local coordinate $\{U; (z^1, \cdots z^n)\}$, let

$$g_{i\bar{j}} := \left\langle \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j} \right\rangle$$

It satisfies

$$\begin{aligned} g_{i\bar{j}} &= g_{\bar{j}i} \\ g_{j\bar{i}} &= \overline{g_{i\bar{j}}} \end{aligned}$$

The first equality comes from the symmetirc of g. The second one comes from:

$$\begin{aligned} g_{j\bar{i}} = \left\langle \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^i} \right\rangle &= \left\langle \frac{1}{2}\left(\frac{\partial}{\partial x^j} - \sqrt{-1}\frac{\partial}{\partial y^j}\right), \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}\right) \right\rangle \\ =&\frac{1}{4}\left[\left(\left\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i} \right\rangle + \left\langle \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i} \right\rangle\right) + \sqrt{-1}\left(\left\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^i} \right\rangle - \left\langle \frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^i} \right\rangle\right)\right] \\ =&\frac{1}{4}\left[\left(\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle + \left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle\right) - \sqrt{-1}\left(\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \right\rangle - \left\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^i} \right\rangle\right)\right] \\ =&\overline{\left\langle \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}\right), \frac{1}{2}\left(\frac{\partial}{\partial x^j} + \sqrt{-1}\frac{\partial}{\partial y^j}\right) \right\rangle} = \overline{\left\langle \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j} \right\rangle} = \overline{g_{i\bar{j}}} \end{aligned}$$

Part II

杂题集萃

Part III
易错知识

Part IV

亟待整理

1. 李群正合列的分裂问题

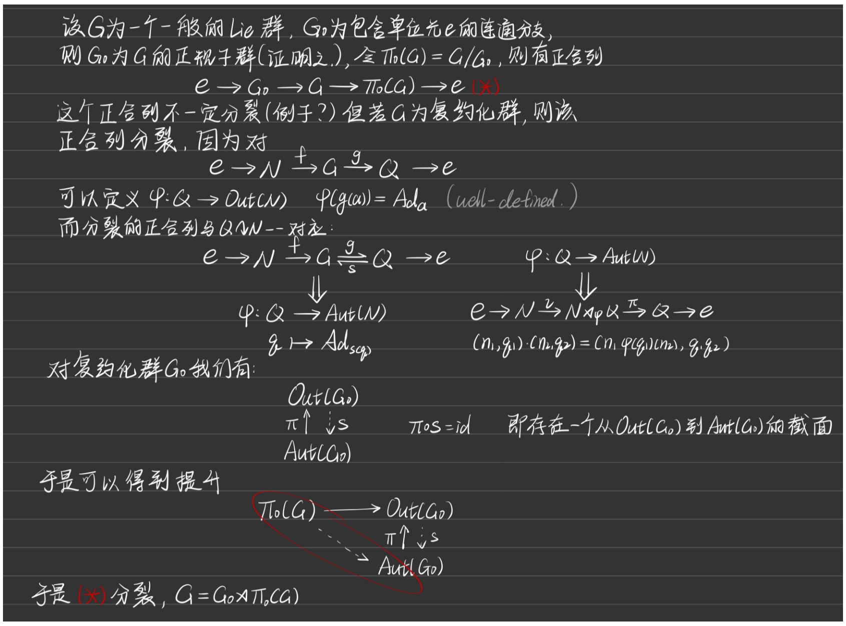

Figure 1.1: 李群的正合列何时分裂

2. Why is it called a twisting sheaf

3. $\mathfrak{UW}$