数学笔记

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Part I

知识整理

Chapter 1

代数拓扑

1.1 Homotopy Theory

1.1.1 Mapping Cylinder and Mapping Path Space

In the homotopy theory, every map $f : A \to B$ can be viewed as an inclusion or a fibration up to homotopy.

Inclusion: The mapping cylinder of f is defined as

$$M_f = (A \times I) \sqcup B \big/ (a, 1) \sim f(a).$$

We have an inclusion $j : A \hookrightarrow M_f$ defined by

$$\begin{aligned} j : A &\hookrightarrow M_f \\ a &\mapsto (a, 0), \end{aligned}$$

an inclusion $i : B \hookrightarrow M_f$ defined by

$$\begin{aligned} i : B &\hookrightarrow M_f \\ b &\mapsto b, \end{aligned}$$

and a retraction $r : M_f \to B$ defined by

$$\begin{aligned} r : M_f &\to B \\ (a, t) &\mapsto f(a); \\ b &\mapsto b. \end{aligned}$$

Note that $r \circ i = \mathrm{id}_B$ and $i \circ r \simeq \mathrm{id}_{M_f}$ through the homotopy

$$H : M_f \times I \to M_f$$

$$((a,t),s) \mapsto (a,(1-s)t+s);$$
$$(b,s) \mapsto b.$$

Geometrically, the homotopy H can be viewed as the process of continuously collapsing the cylinder $A \times I$ in M_f down onto the image of f in B.

Since $r \circ j = f$, we have a commutative diagram

$$\begin{tikzcd} & M_f \arrow[dr, "\simeq"] & \\ A \arrow[ur, hook, "j"] \arrow[rr, "f"] & & B \end{tikzcd}$$

in this sense, f is homotopic to the inclusion j.

Fibration: The mapping path space of f is defined as

$$P_f = \left\{(a,\gamma) \in A \times \mathcal{P}(B) \,\middle|\, \gamma(1) = f(a)\right\}.$$

where

$$\mathcal{P}(B) = \{\gamma : I \to B\}$$

is the path space of B having arbitrary starting point and ending point.

We have a fibration $p : P_f \to B$ defined by

$$p : P_f \to B$$
$$(a,\gamma) \mapsto \gamma(0).$$

The fiber of p over a point $b \in B$ is given by

$$\Omega_b^A := p^{-1}(b) = \left\{(a,\gamma) \in A \times \mathcal{P}(B) \,\middle|\, \begin{array}{l} \gamma(0) = b \\ \gamma(1) = f(a) \end{array}\right\}.$$

We have an inclusion $j : A \hookrightarrow P_f$ defined by

$$i : A \hookrightarrow P_f$$
$$a \mapsto (a, c_{f(a)}),$$

where $c_{f(a)}$ is the constant path at $f(a)$, and a retraction $r : P_f \to A$ defined by

$$r : P_f \to A$$
$$(a,\gamma) \mapsto a.$$

Note that $r \circ i = \mathrm{id}_A$ and $i \circ r \simeq \mathrm{id}_{P_f}$ through the homotopy

$$\begin{aligned} H : P_f \times I &\to P_f \\ ((a,\gamma), s) &\mapsto (a, \tilde{\gamma}_s). \end{aligned}$$

where $\tilde{\gamma}_s(t) = \gamma(s + (1-s)t)$ for $t \in I$ is a path in B that starts at $\gamma(s)$ and ends at $\gamma(1)$. Geometrically, the homotopy H can be viewed as the process of continuously collapsing the path γ in P_f down onto the constant path $c_{f(a)}$.

Since $r \circ j = f$, we have a commutative diagram

$$\begin{tikzcd} & P_f \arrow[dr, two heads, "p"] & \\ A \arrow[ur, "\simeq"] \arrow[rr, "f"] & & B \end{tikzcd}$$

in this sense, f is homotopic to the fibration p.

注. *The mapping cylinder can be constructed as a pushout of the diagram*

$$\begin{tikzcd} A \arrow[r, hook, "\iota_0"] \arrow[d, "f"'] & A \times I \arrow[d] \arrow[dl, phantom, "\ulcorner", very near end] \\ B \arrow[r] & M_f \end{tikzcd}$$

where $\iota_0 : A \hookrightarrow A \times I$ *is the inclusion map defined by* $\iota_0(a) = (a, 0)$.

The mapping path space can be constructed as a pullback of the diagram

$$\begin{tikzcd} P_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ \mathcal{P}(B) \arrow[r, two heads, "\mathrm{ev}"] & B \end{tikzcd}$$

where $\mathrm{ev} : \mathcal{P}(B) \to B$ *is the evaluation map defined by* $\mathrm{ev}(\gamma) = \gamma(0)$.

1.1.2 Postnikov tower and Whitehead tower

We know that the homotopy type of a space X can be captured by its homotopy groups $\pi_q = \pi_q(X)$. In homotopy theory, the simplest spaces are the Eilenberg-MacLane spaces $K(A, n)$. So there is a natural question:

Can we decompose X as a "product" of a sequence of Eilenberg-MacLane spaces?

The directly product of Eilenberg-MacLane spaces is not enough to capture the homotopy type of X in general, i.e.,

$$X \not\simeq \prod_q K(\pi_q(X), q).$$

So we need to introduce a more general construction, the Postnikov tower, and the Whitehead tower.

定义 1.1.1 (Postnikov tower)**.** *Given a CW complex X and an integer n, the Postnikov tower of X is a sequence of fibrations*

$$\begin{tikzcd}
K(\pi_n, n) \arrow[r] & Y_n \arrow[d] & \\
& \vdots \arrow[d] & \\
K(\pi_2, 2) \arrow[r] & Y_2 \arrow[d] & \\
K(\pi_1, 1) \arrow[r, equal] & Y_1 & X \arrow[l] \arrow[lu] \arrow[luuu]
\end{tikzcd}$$

satisfying the following properties:

- $$\pi_k(Y_q) = \begin{cases} 0 & , k > q; \\ \pi_k(X) & , k \leq q. \end{cases}$$
- *the fibration $Y_q \to Y_{q-1}$ has fiber $K(\pi_q, q)$ and commutes with the maps $X \to Y_q$;*
- *the map $X \to Y_q$ induces isomorphisms*
  $$\pi_k(X) \xrightarrow{\cong} \pi_k(Y_q), \quad k \leq q.$$

Construction: The space Y_q is the space obtained from X by truncating its homotopy groups above degree q.

Step 1: Construct Y_n from X

To construct Y_n, we have to kill the homotopy groups $\pi_k(X)$ for $k > n$. The method is attaching cells. To be more precise, if $\alpha : S^{n+1} \to X$ is a generator of $\pi_{n+1}(X)$, we can attach a cell e^{n+2} to X, i.e.,

$$X \cup_\alpha e^{n+2} = X \sqcup e^{n+2} / x \sim \alpha(x)$$

to kill the generator α in $\pi_{n+1}(X)$, Rrepeat this process for all generators of $\pi_{n+1}(X)$, we get X_1. Note that this process does not change the homotopy groups $\pi_k(X)$ for $k \leq n$. Then we can repeat this process for X_1 to get Y_n.

$$Y_n = X \cup_\alpha e^{n+2} \cup \cdots$$

we have a natural inclusion $X \hookrightarrow Y_n$.

Step 2: Inductively construct Y_{q-1} from Y_q

To construct Y_{q-1} from Y_q, we have to kill $\pi_q(Y_q)$. The method is similar to the previous step, i.e., attaching cells.

$$Y_{q-1} = Y_q \cup e^{q+1} \cup \cdots$$

therefore we have a natural inclusion $Y_q \hookrightarrow Y_{q-1}$. It can be viewed as a fibration

$$\begin{tikzcd} F_q \arrow[r] & Y_q \arrow[d] \\ & Y_{q-1} \end{tikzcd}$$

Step 3: Check F_q is $K(\pi_q, q)$

The fibration induces a long exact sequence of homotopy groups

$$\cdots \to \pi_{k+1}(Y_q) \to \pi_{k+1}(Y_{q-1}) \to \pi_k(F_q) \to \pi_k(Y_q) \to \pi_k(Y_{q-1}) \to \cdots$$

- $k \geq q+1$: $\pi_{k+1}(Y_{q-1}) = \pi_k(Y_q) = 0$, so $\pi_k(F_q) = 0$.
- $k = q$: $\pi_{q+1}(Y_{q-1}) = \pi_q(Y_{q-1}) = 0$, so $\pi_q(F_q) = \pi_q(Y_q) = \pi_q(X)$.
- $k = q-1$: $\pi_q(Y_{q-1}) = 0$, $\pi_{q-1}(Y_q) \cong \pi_{q-1}(X)$, so $\pi_{q-1}(F_q) = 0$.
- $k \leq q-2$: $\pi_{k+1}(Y_q) \cong \pi_{k+1}(Y_{q-1})$, so $\pi_k(F_q) = 0$.

Therefore, we have

$$\pi_k(F_q) = \begin{cases} \pi_q(X) & , k = q; \\ 0 & , k \neq q. \end{cases}$$

So $F_q \simeq K(\pi_q, q)$. $\square$

定义 1.1.2 (Whitehead tower). *Given a CW complex X, the Whitehead tower of X is a sequence of fibrations*

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
K(\pi_n, n-1) \arrow[r] & X_n \arrow[d] \\
K(\pi_{n-1}, n-2) \arrow[r] & X_{n-1} \arrow[d] \\
 & \vdots \arrow[d] \\
K(\pi_1, 0) \arrow[r] & X_1 \arrow[d] \\
 & X
\end{tikzcd}$$

satisfying the following properties:

- $$\pi_k(X_n) = \begin{cases} 0 & , k \leq n; \\ \pi_k(X) & , k > n. \end{cases}$$
- *the fibration $X_n \to X_{n-1}$ has fiber $K(\pi_n, n-1)$.*

Construction: The space X_1 is the universal cover of X in the homotopy sense. The space X_n is the generalization of X_1 to higher dimensions.

Step 1: Construct X_1 from X

To construct X_1, we have to kill $\pi_1(X)$. First kill the higher homotopy groups of X to get

$$Y_1 = X \cup e^3 \cup \cdots = K(\pi_1(X), 1)$$

X is naturally included in Y_1. We define

$$X_1 = \Omega_*^X = \left\{ (x, \gamma) \in X \times \mathcal{P}(Y_1) \,\middle|\, \begin{matrix} \gamma(0) = * \\ \gamma(1) = x \end{matrix} \right\}.$$

The projection $p : X_1 \to X$ is a fibration with fiber

$$p^{-1}(*) = \Omega Y_1 = K(\pi_1, 0).$$

We can show that

$$\pi_k(X_1) = \begin{cases} \pi_k(X) & ,k \geq 2; \\ 0 & ,k \leq 1. \end{cases}$$

So X_1 is the universal cover of X in the homotopy sense.

Step 2: Inductively construct X_n from X_{n-1}

Firstly, kill the higher homotopy groups of X_{n-1} to get

$$Y_n = X_{n-1} \cup e^{n+2} \cup \cdots = K(\pi_n, n).$$

Then we can repeat the process for X_n to get

$$X_n = \Omega_*^{X_{n-1}}$$

The projection $p : X_n \to X_{n-1}$ gives a fibration

$$\begin{tikzcd} K(\pi_n, n-1) = \Omega Y_n \arrow[r] & X_n \arrow[d] \\ & X_{n-1} \end{tikzcd}$$

Step 3: Check $\pi_k(X_n)$

The fibration induces a long exact sequence of homotopy groups

$$\cdots \to \pi_{k+1}(X_{n-1}) \xrightarrow{\delta} \pi_k(\Omega Y_n) \to \pi_k(X_n) \to \pi_k(X_{n-1}) \xrightarrow{\delta} \pi_{k-1}(\Omega Y_n) \to \cdots$$

- $k \geq n+1$: $\pi_k(\Omega Y_n) = \pi_{k-1}(\Omega Y_n) = 0$. So $\pi_k(X_n) \cong \pi_k(X_{n-1}) = \pi_k(X)$.
- $k = n, n-1$:

$$\pi_n(\Omega Y_n) \to \pi_n(X_n) \to \pi_n(X_{n-1}) \xrightarrow{\delta} \pi_{n-1}(\Omega Y_n) \to \pi_{n-1}(X_n) \to \pi_{n-1}(X_{n-1})$$

  The connecting homomorphism $\delta : \pi_n(X_{n-1}) \to \pi_{n-1}(\Omega Y_n)$ is an isomorphism, $\pi_n(\Omega Y_n) = \pi_{n+1}(Y_n) = 0$, $\pi_{n-1}(X_{n-1}) = 0$. So $\pi_n(X_n) = 0$.

- $k \leq n-2$: $\pi_k(\Omega Y_n) = \pi_k(X_{n-1}) = 0$. So $\pi_k(X_n) = 0$.

Therefore, we have

$$\pi_k(X_n) = \begin{cases} \pi_k(X) & ,k \geq n+1; \\ 0 & ,k \leq n. \end{cases}$$

Why δ is an isomorphism?

The connecting homomorphism δ is defined as follows:

$$\begin{aligned}\delta : \pi_n(X_{n-1}) &\to \pi_{n-1}(\Omega Y_n)\\ [\alpha] &\mapsto [\tilde{\alpha}],\end{aligned}$$

where $\alpha : (I^n, \partial I^n) \to (X_{n-1}, *)$ represents $[\alpha]$. $\tilde{\alpha} : (I^{n-1}, \partial I^{n-1}) \to (\Omega Y_n, c_*)$ is defined by

$$\tilde{\alpha}(s)(t) = \alpha(s,t), \quad (s,t) \in I^{n-1} \times I = I^n.$$

This is essentially treating the last coordinate of α as the time parameter of the path. Note that the connecting homomorphism

$$\begin{aligned}\delta' : \pi_n(Y_n) &\xrightarrow{\cong} \pi_{n-1}(\Omega Y_n)\\ [\alpha] &\mapsto [\tilde{\alpha}],\end{aligned}$$

is defined in the same way. δ is excatly the composition of two isomorphisms:

$$\begin{tikzcd} \pi_n(X_n) \arrow[r, "\cong", "\iota^*"'] \arrow[rr, bend left, "\delta"] & \pi_n(Y_n) \arrow[r, "\cong", "\delta'"'] & \pi_{n-1}(\Omega Y_n) \end{tikzcd}$$

Therefore, δ is an isomorphism. □

注. *Postnikov tower* 将空间 X 从小空间一点一点拼装起来; *Whitehead tower* 则是将空间 X 从基本群开始逐步解开.

在 *Postnikov tower* 中, 每一项 Y_n 具有特殊的胞腔分解

$$Y_n = X \cup e^{n+2} \cup \cdots$$

所以我们能得到 Y_n 的同调群信息.

在 *Whitehead tower* 中, 每一项 X_n 都是 n-连通空间, 所以我们能得到 X_n 的同调群信息.

1.1.3 The rational homotopy groups of spheres

The rational homotopy groups of spheres are given by the following theorem:

定理 1.1.3 (Sere theorem)**.** *If n is odd, then*

$$\pi_q(S^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & , q = 0, n; \\ 0 & , \textit{otherwise}. \end{cases}$$

If n is even, then

$$\pi_q(S^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & , q = 0, n, 2n-1; \\ 0 & , \textit{otherwise}. \end{cases}$$

Before proving this theorem, we need two lemmas.

引理 1.1.4. *All $\pi_q(S^n)$ are finitely generated abelian groups.*

The proof of this lemma is based on Serre's $\mathscr{C}$-class. We skip the details here.

引理 1.1.5.

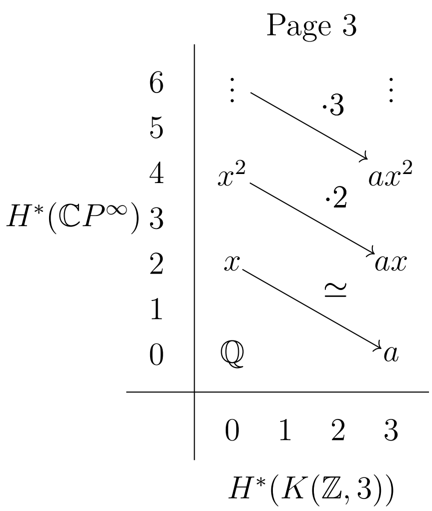
- *For any finite abelian group A, $H^*(K(A,n);\mathbb{Q}) = 0$.*
- *The cohomology ring $H^*(K(\mathbb{Z},n);\mathbb{Q})$ is free on one generator of degree n, i.e.,*

$$H^*(K(\mathbb{Z},n);\mathbb{Q}) = \begin{cases} \Lambda(x), \dim x = n, & n \text{ is odd;} \\ \mathbb{Q}[x], \dim x = n, & n \text{ is even.} \end{cases}$$

证明. For $n=1$, $K(\mathbb{Z},1) \simeq S^1$, so $H^*(K(\mathbb{Z},1);\mathbb{Q}) \cong \Lambda(x),\ \dim x = 1$.
For $n=2$, $K(\mathbb{Z},2) \simeq \mathbb{C}P^\infty$, so $H^*(K(\mathbb{Z},2);\mathbb{Q}) \cong \mathbb{Q}[x],\ \dim x = 2$.
For $n \geq 3$, we use the fibration

$$\begin{tikzcd} K(\mathbb{Z},2) \arrow[r] & \mathcal{P}K(\mathbb{Z},3) \arrow[d] \\ & K(\mathbb{Z},3) \end{tikzcd}$$

Its associated Serre spectral sequence has E_2-term



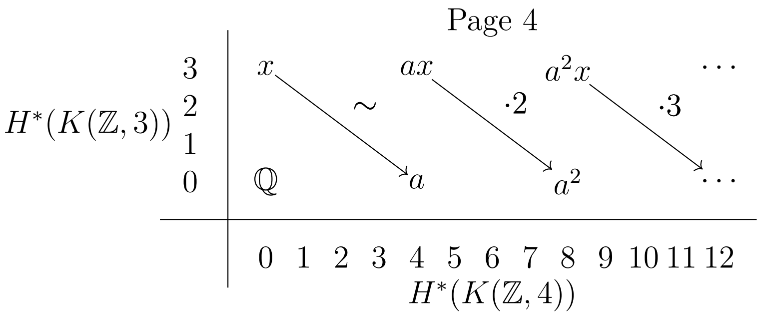
To cancel x, we have to set $d_3 x = a$ to be an isomorphism. The above diffrential map is given by product structure. Since we are working with rational coefficients, they are all isomorphisms. So we have $E_4 = E_\infty$. Therefore, we have

$$H^*(K(\mathbb{Z},3);\mathbb{Q}) = \Lambda(x),\ \dim x = 3.$$

For $n \geq 4$, we can use the fibration

$$\begin{tikzcd} K(\mathbb{Z},3) \arrow[r] & \mathcal{P}K(\mathbb{Z},4) \arrow[d] \\ & K(\mathbb{Z},4) \end{tikzcd}$$

Its associated Serre spectral sequence has E_2-term



Therefore, we have

$$H^*(K(\mathbb{Z},4);\mathbb{Q}) = \mathbb{Q}[x],\ \dim x = 4.$$

By induction, we can get the result for all $n \geq 5$. □

Now we can prove the Sere theorem. There are two ways to prove it, one is using the Postnikov tower, the other is using the Whitehead tower.

Proof of Sere theorem using Postnikov tower. Since S^n is n-connected, we have the Postnikov tower

$$\begin{tikzcd} & \vdots \arrow[d] \\ K(\pi_{n+2}, n+2) \arrow[r] & Y_{n+2} \arrow[d] \\ K(\pi_{n+1}, n+1) \arrow[r] & Y_{n+1} \arrow[d] \\ & K(\mathbb{Z}, n) \end{tikzcd}$$

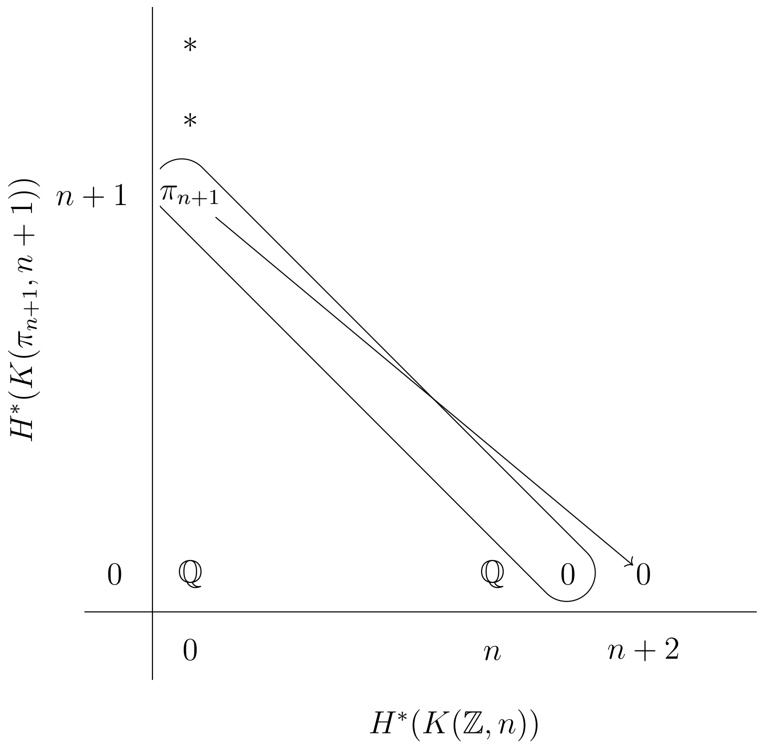
By the construction of the Postnikov tower, we have

$$Y_q = S^n \cup e^{q+2} \cup \cdots, \quad q \geq n+1.$$

Hence the cohomology groups of Y_q satisfies

$$H^q(Y_q; \mathbb{Q}) = H^{q+1}(Y_q; \mathbb{Q}) = 0.$$

If n is odd:
∘ dim $= n+1$: The fibration $Y_{n+1} \to K(\mathbb{Z}, n)$ has spectral sequence



Since $K(\pi_{n+1}, n+1)$ is n-connected, by Hurewicz theorem, we have

$$H^{n+1}(K(\pi_{n+1}, n+1); \mathbb{Q}) = \pi_{n+1} \otimes \mathbb{Q}.$$

By the construction of the Postnikov tower,

$$Y_{n+1} = S^n \cup e^{n+2} \cup \cdots$$

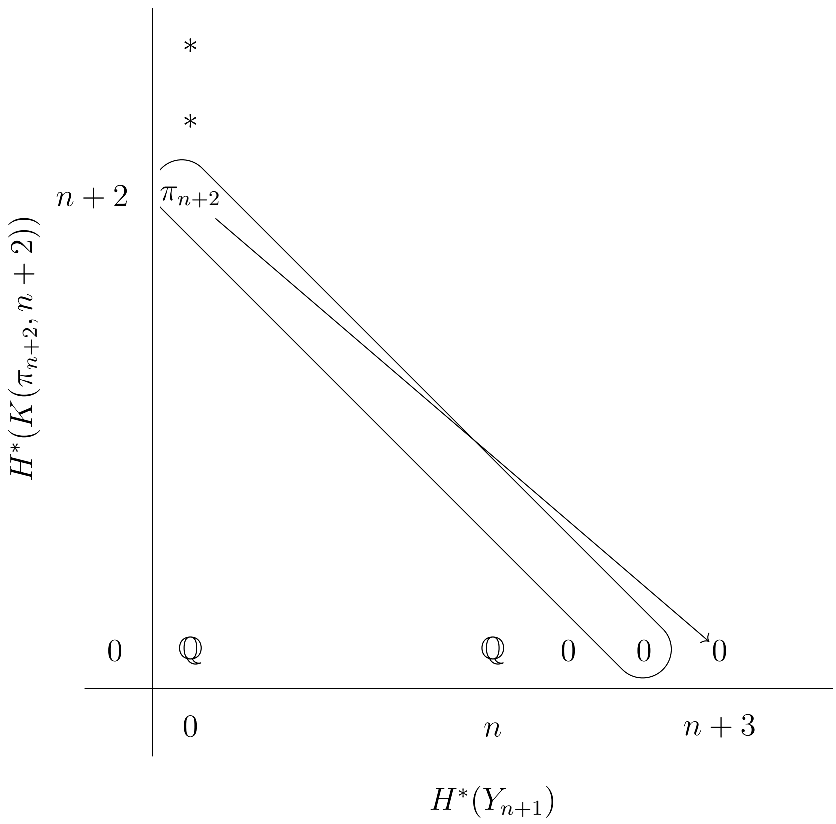
Therefore

$$H^{n+1}(Y_{n+1}; \mathbb{Q}) = 0 \Longrightarrow \pi_{n+1} \otimes \mathbb{Q} = 0.$$

∘ dim $= n+2$: By lemma 1.1.5, $K(\pi_{n+1}, n+1)$ has trivial rational cohomology, so

$$H^*(Y_{n+1};\mathbb{Q}) \cong H^*(K(\mathbb{Z},n);\mathbb{Q}).$$

The fibration $Y_{n+2} \to Y_{n+1}$ has spectral sequence



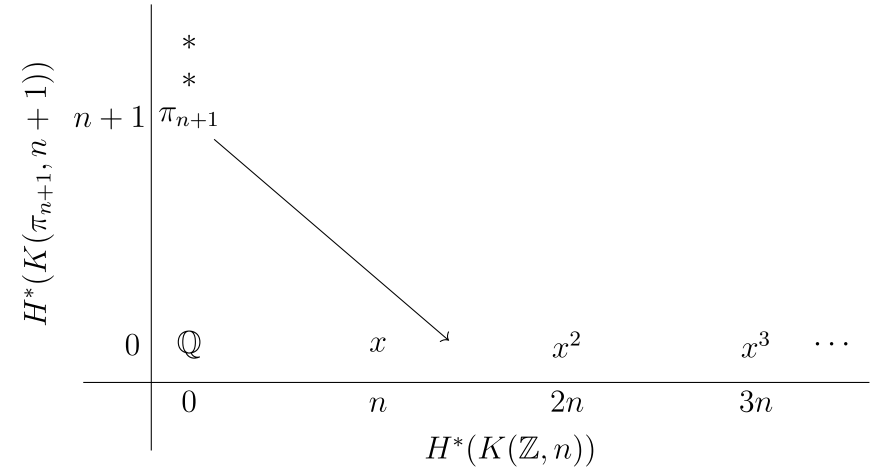
By the same argument as above, we have

$$H^{n+2}(Y_{n+2};\mathbb{Q}) = 0 \Longrightarrow \pi_{n+2}\otimes\mathbb{Q} = 0.$$

∘ dim $\geq n+3$: By induction.

If n is even:

∘ dim $= n+1$: The fibration $Y_{n+1} \to K(\mathbb{Z},n)$ has spectral sequence



By the construction of the Postnikov tower,

$$Y_{n+1} = S^n \cup e^{n+2} \cup \cdots$$

Therefore,

$$H^{n+1}(Y_{n+1}; \mathbb{Q}) = 0 \Longrightarrow \pi_{n+1} \otimes \mathbb{Q} = 0.$$

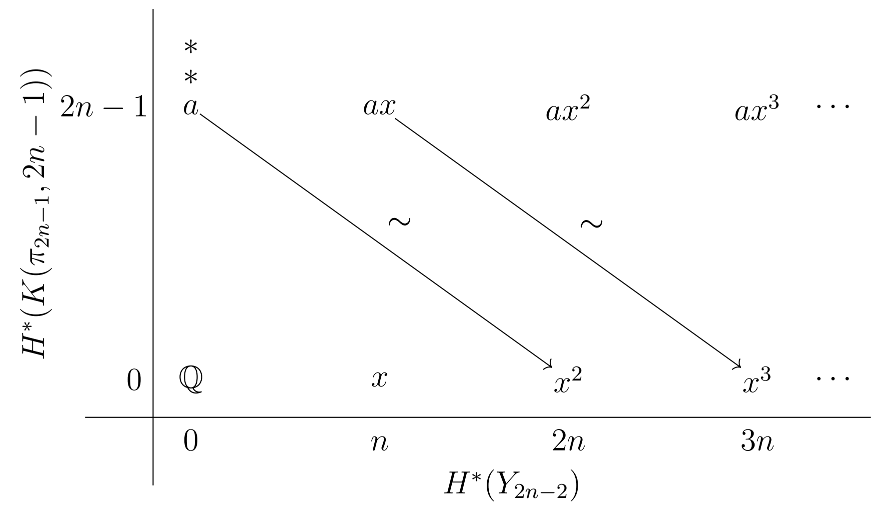
∘ $\dim = n+2$: Since $K(\pi_{n+1}, n+1)$ has trivial rational cohomology,

$$H^*(Y_{n+1}; \mathbb{Q}) \cong H^*(K(\mathbb{Z}, n); \mathbb{Q}) = \mathbb{Q}[x], \ \dim x = n.$$

The rest analysis is similar to $n+1$ case.

$\cdots\cdots$

∘ $\dim = 2n-1$: The fibration $Y_{2n-1} \to Y_{2n-2}$ has spectral sequence



By the same argument as above, we have

$$H^{2n-1}(Y_{2n-1};\mathbb{Q}) = H^{2n}(Y_{2n-1};\mathbb{Q}) = 0.$$

Therefore, $d_{2n} : \pi_{2n-1} \otimes \mathbb{Q} \to H^{2n}(Y_{2n-2};\mathbb{Q})$ is an isomorphism. So

$$\pi_{2n-1} \otimes \mathbb{Q} = \mathbb{Q}.$$

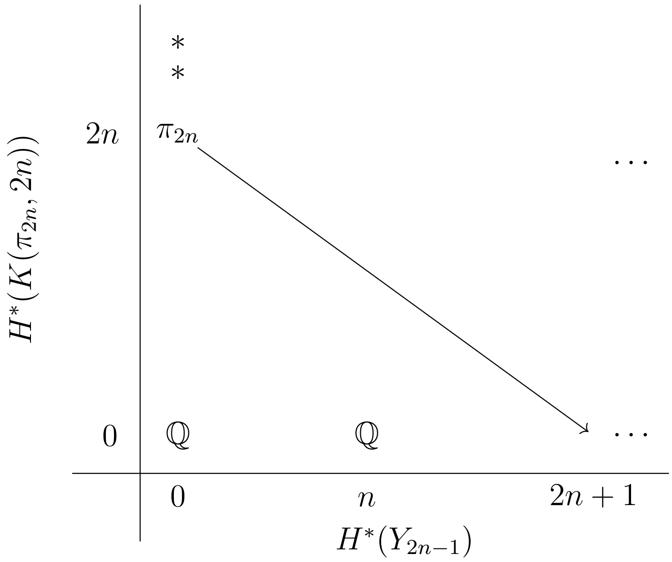
∘ $\dim = 2n$: By lemma 1.1.5,

$$H^*(K(\pi_{2n-1}, 2n-1);\mathbb{Q}) = \Lambda(x),\ \dim x = 2n-1.$$

Thus the unknown terms in the first column are all zero. So

$$H^*(Y_{2n-1};\mathbb{Q}) = \begin{cases} \mathbb{Q}, & ,\dim = 0, n; \\ 0 & ,\text{otherwise.} \end{cases}$$

The fibration $Y_{2n} \to Y_{2n-1}$ has spectral sequence



By the same argument as above, we have

$$H^{2n}(Y_{2n};\mathbb{Q}) = 0 \Longrightarrow \pi_{2n} \otimes \mathbb{Q} = 0.$$

∘ $\dim = 2n+1$: Since $K(\pi_{2n}, 2n)$ has trivial rational cohomology

$$H^*(Y_{2n};\mathbb{Q}) \cong H^*(Y_{2n-1};\mathbb{Q}).$$

The rest analysis is similar to $2n$ case.
∘ $\dim \geq 2n+2$: By induction. □

Proof of Sere theorem using Whitehead tower. The Whitehead tower of S^n is given by the following fibration

$$\begin{tikzcd}
& \vdots \arrow[d] \\
K(\pi_{n+1}, n) \arrow[r] & X_{n+1} \arrow[d] \\
K(\mathbb{Z}, n-1) \arrow[r] & X_n \arrow[d] \\
& S^n
\end{tikzcd}$$

By the construction of the Whitehead tower, we have

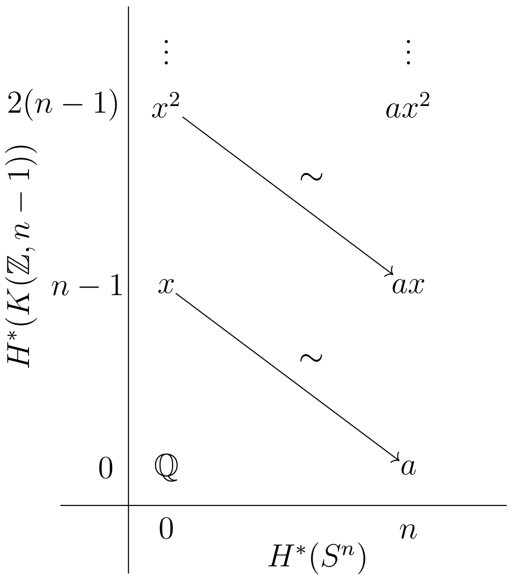
$$\pi_k(X_q) = \begin{cases} \pi_k(S^n) & , k \geq q+1; \\ 0 & , k \leq q. \end{cases}$$

By Hurewicz theorem, the cohomology groups of X_n satisfies

$$H^q(X_q; \mathbb{Q}) = 0, \quad H^{q+1}(X_q; \mathbb{Q}) = \pi_{q+1}(S^n) \otimes \mathbb{Q}.$$

If n is odd:

∘ $\dim = n+1$: The fibration $X_{n+1} \to K(\mathbb{Z}, n)$ has spectral sequence



By lemma 1.1.5,

$$H^*(K(\mathbb{Z}, n-1); \mathbb{Q}) = \mathbb{Q}[x], \quad \dim x = n-1.$$

By the construction of the Whitehead tower, X_n is n-connected.

$$H^{n-1}(X_n; \mathbb{Q}) = H^n(X_n; \mathbb{Q}) = 0.$$

So $d_n : \pi_{n-1} \otimes \mathbb{Q} \to H^n(S^n; \mathbb{Q})$ must be an isomorphism. The other differentials are all isomorphisms by the product structure and rational coefficients. Therefore,

$$H^*(X_n; \mathbb{Q}) \text{ is trivial } \implies \pi_{n+1} \otimes \mathbb{Q} = H^{n+1}(X_n; \mathbb{Q}) = 0.$$

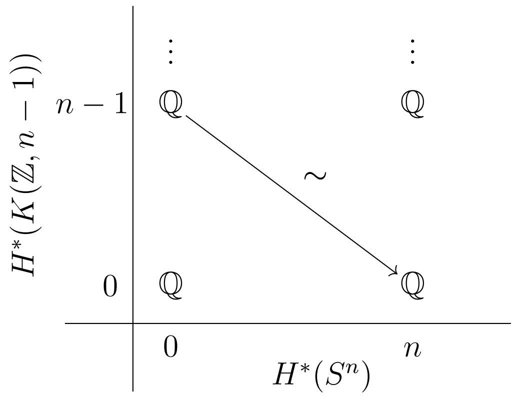
∘ $\dim = n+2$: In the fibration

$$\begin{tikzcd} K(\pi_{n+1}, n) \arrow[r] & X_{n+1} \arrow[d] \\ & X_n \end{tikzcd}$$

both $K(\pi_{n+1}, n)$ and X_n have trivial rational cohomology. Therefore,

$$H^*(X_{n+1}; \mathbb{Q}) \text{ is trivial } \implies \pi_{n+2} \otimes \mathbb{Q} = H^{n+2}(X_{n+1}; \mathbb{Q}) = 0.$$

∘ $\dim \geq n+3$: By induction.
If n is even:
∘ $\dim = n+1$: The fibration $X_{n+1} \to K(\mathbb{Z}, n-1)$ has spectral sequence



Since X_n is n-connected, the differential map shwon above must be an isomorphism. Therefore,

$$H^*(X_n; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \dim = 0, 2n-1; \\ 0, & \text{otherwise} \end{cases} \implies \pi_{n+1} \otimes \mathbb{Q} = H^{n+1}(X_n; \mathbb{Q}) = 0.$$

∘ $\dim = n+2$: In the fibration

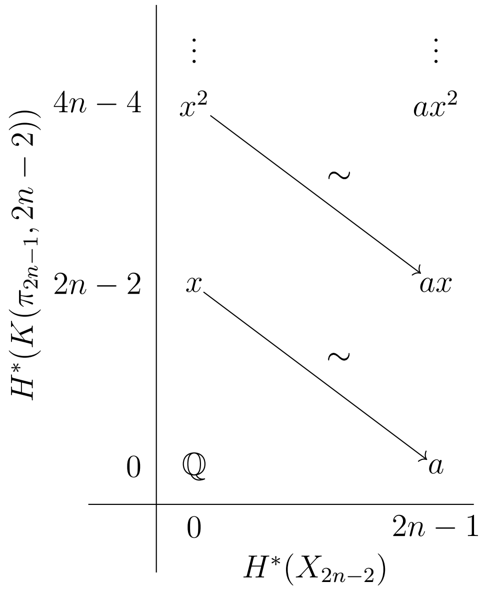
$$
\begin{tikzcd}
K(\pi_{n+1}, n) \arrow[r] & X_{n+1} \arrow[d] \\
& X_n
\end{tikzcd}
$$

$K(\pi_{n+1}, n)$ has trivial rational cohomology. Therefore,

$$H^*(X_{n+1}; \mathbb{Q}) \cong H^*(X_n; \mathbb{Q}) \Longrightarrow \pi_{n+2} \otimes \mathbb{Q} = H^{n+2}(X_{n+1}; \mathbb{Q}) = 0.$$

......

∘ dim $= 2n - 1$: The fibration $X_{2n-1} \to X_{2n-2}$ has spectral sequence



Since X_{2n-2} is $(2n-2)$-connected, the differential maps shown above must be isomorphisms. Therefore,

$$H^*(X_{2n-1}; \mathbb{Q}) \text{ is trivial } \Longrightarrow \pi_{2n-1} \otimes \mathbb{Q} = H^{2n-1}(X_{2n-1}; \mathbb{Q}) = 0.$$

∘ dim $= 2n$: Since $K(\pi_{2n-1}, 2n-1)$ has trivial rational cohomology.

$$H^*(X_{2n}; \mathbb{Q}) \cong H^*(X_{2n-1}; \mathbb{Q}) \text{ is trivial } \Longrightarrow \pi_{2n} \otimes \mathbb{Q} = 0.$$

∘ dim $\geq 2n + 1$: By induction. □

Bibliography

[1] 梅加强. 流形与几何初步. 科学出版社, 2013.